# Pro-$l$ Fundamental Groups of Generically Ordinary Semi-stable Fibrations with Low Slope

*Minette D'Lima*

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Department of Mathematics

University College London

I, Minette D'Lima confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

# Abstract

Let $k$ be an algebraically closed field of characteristic $p > 0$ and $l$ a prime that is distinct from $p$. Let $f : S \to C$ be a generically ordinary, semi-stable fibration of a projective smooth surface $S$ to a projective smooth curve $C$ over $k$. Let $F$ be a general fibre of $f$, which is a smooth curve of genus $g \geq 2$. We assume that $f$ is generically strongly $l$-ordinary, by which we mean that every cyclic étale covering of degree $l$ of the generic fibre of $f$ is ordinary. Suppose that $f$ is not locally trivial and is relatively minimal. Then $\deg f_* \omega_{S/C} > 0$, where $\omega_{S/C}$ is the sheaf associated to the relative canonical divisor $K_{S/C} = K_S - f^* K_C$. Hence the slope of $f$, $\lambda(f) = K^2_{S/C} / \deg f_* \omega_{S/C}$ is well-defined. Consider the push-out square

$$\begin{array}{ccccc} \pi_1(F) & \longrightarrow & \pi_1(S) & \longrightarrow & \Pi(C) \to 1 \\ \downarrow & & \downarrow & & \\ \pi_1^l(F) & \xrightarrow{\alpha} & \Pi & & \end{array}$$

where $\pi_1$ is the algebraic fundamental group and $\pi_1^l$ is the pro-$l$ fundamental group. When $f$ is non-hyperelliptic and $\lambda(f) < 4$, we show that the morphism $\pi_1^l(F) \xrightarrow{\alpha} \Pi$ is trivial.

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# Contents

## Chapter 1

# Introduction

## 1.1 History

*If algebraic curves were God's creation, then algebraic surfaces were the Devils mischief.*

-Federigo Enriques 1949.

This is a quote by Enriques while giving an account of the classification of complex algebraic surfaces. The classification problem has been the guiding principle of the theory of algebraic surfaces. The foundations of the theory were laid by A. Clebsh and M. Noether (1870), who defined the first important invariants of algebraic surfaces: the geometric genus and the canonical class. At the beginning of the 20th century, mathematicians such as Castelnuovo, Enriques, Severi and many others, from the Italian school, had succeeded in creating an impressive and essentially geometric theory of birational classification of algebraic surfaces. Around 1960, after the development of the modern language of schemes and sheaves, Kodaira ([22], [23]) extended Enriques classification to the Enriques-Kodaira classification of all compact, complex surfaces [4].

Over the next decade and a half, Mumford explored what he termed "pathologies of positive characteristic": well-behaved results in characteristic zero that fail in positive characteristic. Subsequently Mumford and Bomberi ([30], [8], [7]), in three fundamental papers, written between 1969 and 1976, extended the Enriques-Kodaira classification of smooth projective surfaces from the case of complex surfaces to surfaces defined over an algebraically closed field of positive characteristic.

In 1966, Shafarevich ([37]) was the first to open up the possibility of a serious consideration of an "arithmetic surface" as a geometric object. The results of the theory of surfaces could be used in the study of algebraic curves over function fields. The methods and underlying ideas employed in the classification problem are intrinsically interesting, especially as we consider extensions to characteristic $p$ and the consequent tie-up with arithmetic problems. In this thesis we extend a theorem on complex algebraic surfaces by Xiao to characteristic $p$.

The motivation for Xiao's theorem was a particular "geographical problem" posed by Reid ([34], conjecture 4) in 1978.

**Conjecture 1.** Let $S$ be a (smooth, projective) minimal surface of general type such that $K_S^2 < 4\chi(S)$, where $K_S^2$ is the self-intersection of the canonical divisor on $S$, and $\chi(S)$ is the Euler-Poincare characteristic of $S$, then

(i) there is a finite étale cover $\tilde{S}$ such that $\pi_1(\tilde{S})$ equals the $\pi_1$ of a smooth curve of genus equal to $q(\tilde{S})$; or in a weaker form

(ii) $S$ is fibered over a smooth curve of genus $q(S)$.

Motivated by this conjecture, Xiao wrote the foundational paper [12] on surfaces fibered over a curve. In this paper, he introduced a new and useful technique for studying complex fibered surfaces. His method has been successfully used and generalised in several works to the study of projective varieties over a curve ([3]). This thesis adapts his technique to the study of fibered algebraic surfaces defined over an algebraically closed field of positive characteristic and proves a characteristic $p$ analogue of Xiao's theorem .

## 1.2 A technique for studying fibered surfaces

Let $S$ be a smooth, projective minimal surface of general type over $\mathbb{C}$, with a given fibration $f : S \to C$ (that is a morphism with connected fibres onto a smooth curve). Assume further that $S$ is relatively minimal with respect to $f$: that is, there is no $(-1)$ rational curve contained in the fibres and that $f$ is non-isotrivial. Let $F$ be a general fibre of $f$, which is a smooth curve of genus $g \geq 2$. We then have the following natural exact sequence of groups

$$\pi_1(F) \xrightarrow{\alpha} \pi_1(S) \to \Pi \to 1$$

where $\pi_1$ is the algebraic fundamental group and $\Pi$ is determined by $\pi_1(C)$ and the multiple fibres of $f$. Xiao proved the following theorem:

**Theorem 1.1** ([12], Theorem 1)**.** *In the above situation, suppose*

$$K_S^2 < 4\chi(O_S) + 4(g(C) - 1)(g - 1)$$

*and $f$ is non-hyperelliptic. Then the image of $\alpha$ is trivial.*

The method that he developed to prove this theorem, involved using numerical data of linear systems on fibres of $f$, along with the Harder-Narasimhan filtration of locally free sheaves. The key technique may be briefly summarised as follows:

Let $D$ be a divisor on $S$ such that $\mathcal{E} = f_* O_S(D)$ is a locally free sheaf. If $\mathcal{H}$ is a subsheaf of $\mathcal{E}$, define the fixed part of $\mathcal{H}$ to be the fixed part of the linear subsystem of $|D + f^*\mathcal{A}|$ induced by sections of $\mathcal{H} \otimes \mathcal{A}$, where $\mathcal{A}$ is a sufficiently ample divisor on $C$. Next consider the Harder-Narsimhan filtration of $\mathcal{E}$. Then there exists a filtration of subsheaves

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset ... \subset \mathcal{E}_n = \mathcal{E}$$

such that

(i) $\mathcal{E}_i/\mathcal{E}_{i-1}$ is semistable for every $i$.

(ii) $\mu(\mathcal{E}_1/\mathcal{E}_0) > \mu(\mathcal{E}_2/\mathcal{E}_1)... > \mu(\mathcal{E}_n/\mathcal{E}_{n-1})$, where $\mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ is the slope of the locally free sheaf $\mathcal{E}_i/\mathcal{E}_{i-1}$.

Now consider the special divisor $K_{S/C}$, where $K_{S/C}$ is the relative canonical divisor $K_S - f^*K_C$ and $\omega_{S/C}$ is the sheaf associated to it. We then have the following semi-positivity theorem:

**Theorem 1.2** ([11], Theorem 1.1.). *If $f : S \to C$ is a fibration of a proper smooth surface to a proper smooth curve over $\mathbb{C}$, then all the quotient bundles of $f_*\omega_{S/C}$ are of non-negative degree.*

Furthermore if $\eta$ is a torsion element in Pic($S$), then all the quotient bundles of $f_*\omega_{S/C} \otimes \eta$ are also of non-negative degree. Take $D = D_\eta$, a divisor with class $\omega_{S/C} \otimes \eta$. Then $\mathcal{E} = f_*\mathcal{O}_S(D)$ is a locally free sheaf. For each subsheaf $\mathcal{E}_i$ in the Harder-Narasimhan filtration of $\mathcal{E}$, we associate an effective divisor $Z_i$, the fixed part of $\mathcal{E}_i$ as above. Thus we get a sequence of effective divisors

$$Z_1 \geq Z_2 \geq ... \geq Z_n \geq Z_{n+1} = 0.$$

If we denote the slopes of filtration by $\mu_i = \mu(\mathcal{E}_i/\mathcal{E}_{i-1})$, the semi-positivity theorem implies that all $\mu_i$ are non-negative, and we have a non-negative sequence of rational numbers

$$\mu_1 > \mu_2 > ... > \mu_n > \mu_{n+1} = 0.$$

He then showed that the divisor $N_i := D - Z_i - \mu_i F$ is numerically effective, and hence $D^2 = K^2_{S/C}$ is bounded below by,

$$K^2_{S/C} \geq \sum_{i=1}^{n} (d_i + d_{i+1})(\mu_i - \mu_{i+1}),$$

where $d_i = N_i \cdot F$, for a general fibre $F$. The inequality obtained by this method gives a new relation involving numerical invariants of the fibration.

In the proof of his main theorem, Xiao showed that if the image of the fundamental group of the fibre is *not* trivial, then there must exist a cyclic étale cover $\psi : \tilde{S} \to S$ which corresponds to a torsion element in Pic($S$), such that $\eta^i|_F$ is not trivial whenever $\eta^i$ is not trivial. Then using the above lower bound on $K^2_{S/C}$ and Clifford's theorem on the dimension of special linear systems, Xiao showed that the existence of such a torsion contradicts the hypothesis that the fibration has low slope or equivalently the condition that $K^2_{S/C} < 4\chi(\mathcal{O}_S) + 4(g(C) - 1)(g - 1)$.

## 1.3 A characteristic $p$-version

We now assume that $k$ is an algebraically closed field of characteristic $p > 0$ and adapt the above technique to prove an analogous result in characteristic $p$. The method used above cannot be directly employed when the base field has positive characteristic. A pivotal result which fails in positive characteristic is Xiao's theorem of semi-positivity. However in a recent result, Jang has proved that for a semi-stable fibration over a field of positive characteristic, if the generic fibre is ordinary, then the semi-positivity theorem holds. The statement of the theorem is as follows:

**Theorem 1.3** ([21], Theorem 1). *Let $k$ be a perfect field of positive characteristic $p$ and let $X$ be a proper smooth surface over $k$. Assume $X$ admits a generically ordinary semi-stable fibration $f : X \to C$ to a smooth proper curve $C$ over $k$. Then the maximal Harder-Narasimhan slope of $R^1 f_*(O_X)$ is non-positive.*

Since $R^i f_* O_X$ is dual to $f_*\omega_{X/C}$, the Harder-Narasimhan slopes of $f_*\omega_{X/C}$ are non-negative. This semi-positivity result enables us to imitate Xiao's methods in positive characteristic, provided the fibration is generically ordinary and semi-stable. However there are some side effects of the assumption of generic ordinarity. Consider an étale Galois cover $\psi : \tilde{S} \to S$. By Stein Factorisation, $f$ induces a fibration $\tilde{f} : \tilde{S} \to \tilde{C}$, with an induced Galois cover $\psi : \tilde{C} \to C$.

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{f}} & \tilde{C} \\ \downarrow{\scriptstyle\psi} & & \downarrow \\ S & \xrightarrow{f} & C \end{array}$$

Ideally we would like the induced covering fibration $\tilde{f}$ to also be generically ordinary. However this is not always true. To see this, we factor the morphism $\tilde{S} \to S$ into two steps

$$\tilde{S} \to \hat{S} \to S,$$

where $\hat{S}$ is the minimal desingularisation of the normalization of $S \times_C \tilde{C}$, and the two morphisms are both étale and Galois. The first map is determined by the quotient group of $\pi_1(F)$ related to the restriction of $\psi$ on $F$, and the second map is determined by changing the base. Since the base-change of a generically ordinary fibration is generically ordinary, the induced morphism $\hat{S} \to \tilde{C}$ is generically ordinary. Therefore the generic ordinarity of the morphism $\tilde{f} : \tilde{S} \to \tilde{C}$ depends on the restriction

$$\psi : \tilde{F} \to F,$$

to general fibres of $\tilde{f}$ and $f$. In general, although $F$ is ordinary, $\psi : \tilde{F} \to F$ need not be an ordinary cover. It has been shown by Michel Raynaud in [33], that given a proper, smooth, connected curve of genus $g \geq 2$, it is possible to construct a finite Galois étale cover which is not ordinary. Therefore covers of generically ordinary fibrations are not necessarily generically ordinary.

As we would like to prove a characteristic $p$ analogue of Xiao's theorem, we note that we are particularly interested in cyclic étale coverings $\tilde{S}$ of $S$ corresponding to torsion elements in $\mathrm{Pic}(S)$. If $\eta \in \mathrm{Pic}(S)[n]$ then $\tilde{S} = \mathbf{Spec}(\oplus_{i=0}^{n-1} \eta^{\otimes i})$. This simple description of cyclic coverings plays an important role in Xiao's proof. In characteristic $p$, cyclic étale coverings have this structure if $n$ is coprime to $p$.

Therefore we need a sufficient condition for degree $n$ (g.c.d $(n, p) = 1$) cyclic covers of generically ordinary fibrations to be generically ordinary or equivalently for degree $n$ cyclic covers of ordinary curves to be ordinary. In [9], Bouw discusses the question of ordinary covers. We have the following general result: if $X_{gen}$ is the *generic curve* of genus $g$, that is the curve corresponding to the generic point of the moduli space of genus $g$ curves $\mathcal{M}_g$ then $X_{gen}$ is ordinary and further all abelian étale covers of $X_{gen}$ are ordinary (See [41]).

**Proposition 1.4.** *Let $f : Y \to X_{gen}$ be an étale cover whose Galois group G is abelian. Then Y is ordinary.*

The above observations leads us to define the notion of *strong l-ordinarity*:

**Definition 1.5.** Let $l$ be a prime distinct from $p$. We define an ordinary curve $B$ to be *strongly l-ordinary*, when every $l$-cyclic étale cover of $B$ is also ordinary.

Therefore if we assume that the generic fibre of our fibration is *strongly l-ordinary*, all $l$-cyclic étale covers of $S$ will be generically ordinary. In chapter 4, we argue that generic strong $l$-ordinarity is a plausible condition. We are now ready to state the theorem:

**Theorem 1.6.** *Let $k$ be an algebraically closed field of characteristic $p > 0$ and $l$ a prime that is distinct from $p$. Let $S$ be a smooth and projective surface over $k$. Assume that $S$ admits a* *generically ordinary semi-stable fibration $f : S \to C$ to a smooth and projective curve C over $k$ that is not isotrivial, and that $S$ is relatively minimal with respect to $f$. Let $F$ be a general fibre of $f$, which is a smooth curve of genus $g \geq 2$. We assume that $f$ is generically strongly $l$-ordinary. Let $K_{S/C} = K_S - f^*K_C$ be a relative canonical divisor. We have the following commutative diagram of profinite groups,*

$$\begin{array}{ccccc} \pi_1(F) & \longrightarrow & \pi_1(S) & \longrightarrow & \Pi(C) \to 1 \\ \downarrow & & \downarrow & & \\ \pi_1^l(F) & \longrightarrow & \Pi & & \end{array}$$

*where $\pi_1$ is the étale fundamental group, $\pi_1^l$ is the pro-$l$ fundamental group and $\Pi$ is the pushout group. In the above situation, if $K_{S/C}^2 < 4\deg(f_*\omega_{S/C})$ and $f$ is non-hyperelliptic, the image $\pi_1^l(F) \to \Pi$ is trivial.*

The proof of the theorem is roughly similar to Xiao's proof of Theorem 1.2. When a characteristic zero result breaks down in positive characteristic, we have been able to provide alternate proofs to overcome the problem. For example the proof given by Xiao for nefness of the divisor $N_i$ relies on Hartshorne's ([18] V.2.21) proof for ampleness of a divisor on a ruled surface, which is only valid if characteristic of the base field $k$ is zero. We suggest an alternate proof for nefness in positive characteristic using a result by Lange [24].

## 1.4 Extending to arithmetic surfaces

Given an algebraic curve $X$ defined over a number field $E$, one can construct a (minimal) arithmetic surface $f : \chi \to \operatorname{Spec} O_E$ which has $X$ as generic fibre ($\operatorname{Spec} O_E$ denotes the ring of integers of $E$). For arithmetic surfaces, it is not possible to use classical intersection theory as was defined over algebraically closed fields, since this would give a theory which is not well defined for all divisor classes. In [2], Arakelov solved this problem by adding some analytic data in order to "compactify" the base scheme and to "complete" the arithmetic surface. He defined an intersection theory for arithmetic divisors and reformulated everything in the language of Hermetian line bundles. Arakelov intersection theory allows us to define the self-intersection of all arithmetic divisors. In particular we can define $\omega^2_{\chi,Ar}$, where $\omega_{\chi,Ar}$ is the line bundle $\omega_{\chi/\operatorname{Spec}\mathbb{Z}} = \omega_{\chi/\operatorname{Spec}O_E} \otimes_{O_\chi} f^*\omega_{\operatorname{Spec}O_E/\operatorname{Spec}\mathbb{Z}}$ equipped with the Arakelov metric: which is a number of great importance. Szpiro ([40]) has shown by the Kodaira-Parshin construction that an upper bound for $\omega^2_{\chi,Ar}$ for certain families of morphisms of arithmetic surfaces $\{\chi_P \to \mathcal{Y}\}_{P\in Y(\bar{E})}$ would imply bounds for the height of rational points of the curve $Y$, hence it would yield an effective version of Mordell's conjecture.

Extending the theorem to arithmetic surfaces would provide a useful way to relate the Galois action of the fundamental group to the self-intersection number of the dualizing sheaf, which may have interesting implications.

## 1.5 Outline

Chapter 2 contains preliminaries for the study of divisors, linear systems, fibrations, the relative dualizing sheaf and the fundamental group. Chapter 3 is an exposition of Xiao's paper ([12]) . In Chapter 4 we define the problem in the positive characteristic case and present a proof of the theorem.

# Chapter 2

# Definitions and background information

## 2.1 Notation

We will always be working with an algebraically closed field $k$. A scheme will be a noetherian, integral, separated scheme of finite type over $k$.

We use the following standard notation of sheaves and schemes. We will denote the structure sheaf of a scheme $X$ by $\mathcal{O}_X$ and the sheaf associated to the presheaf of total quotient rings of $\mathcal{O}_X$ by $\mathcal{K}_X$. Let $U$ be an open subset of a scheme $X$ and $\mathcal{F}$ a sheaf on $X$ then $\Gamma(U, \mathcal{F})$ denotes the sections of the sheaf $\mathcal{F}$ on the open set $U$. The global sections of the sheaf $\mathcal{F}$ are denoted by $\Gamma(X, \mathcal{F})$ or $H^0(X, \mathcal{F})$. When $X$ is a projective scheme over $k$ and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module, then $H^0(X, \mathcal{F})$ is a finite-dimensional $k$-vector space. We often denote the dimension of this vector space by $h^0(\mathcal{F})$.

A surface will be a noetherian, integral scheme $X$ of dimension 2 over $k$, endowed with a projective flat morphism onto a base scheme $S$ that is regular, connected, of dimension 1. We can regard $X$ as a family of curves parameterised by $S$ (more geometric point of view) or as an extension of the generic fibre $X_K$ into a scheme over $S$ (more arithmetic point of view).

Let $S$ be a Dedekind scheme. We call an integral, projective, flat $S$-scheme $\pi : X \to S$ of dimension 1, a fibered suface over $S$ . The generic point of $S$ wil be denoted by $\eta$ and the fibre over $\eta$ will be denoted by $X_\eta$ and is called the generic fibre. The fibre over a closed point $s \in S$ is called a closed fibre. We will say that $X$ is a normal (resp. regular) fibered surface, if $X$ is normal (resp. regular).

## 2.2 Divisors

The notion of a divisor provides a very useful tool for studying the intrinsic geometry of a scheme. We also introduce the classical notion of linear systems, which are just the set of effective divisors linearly equivalent to a divisor. We also define ampleness and nefness for divisors. The definitions and results presented in this section can be found in [18], II. Sect. 6 and 7 and [27]

### 2.2.1 Weil and Cartier divisors

Weil divisors

This is a more intuitive and geometrical definition of divisors as linear combinations of co-dimension one subschemes on regular noetherian schemes. More generally, let $X$ be a scheme such that all of its local rings are regular in co-dimension one, we can then define the notion of Weil divisors. A prime divisor on $X$ is a closed integral subscheme $Y$ of co-dimension one.

**Definition 2.1.** A Weil divisor is an element of the free abelian group $\operatorname{WDiv} X$ generated by the prime divisors. We write a divisor a $D = \sum n_i Y_i$, where the $Y_i$ are prime divisors, the $n_i$ are integers, and only finitely many $n_i$ are different from zero. If all the $n_i \geq 0$, we say that $D$ is effective.

Let $\eta$ be the generic point of a prime divisor $Y$, then the local ring $\mathcal{O}_{\eta,X}$ is a discrete valuation ring with quotient field $K$, the function field of $X$, since $X$ is regular. Let $\nu_Y$ denote the valuation in $\mathcal{O}_{\eta,X}$. If $f \in K^*$ be any nonzero rational function on $X$. Then $\nu_Y(f)$ is an integer. If it is positive, we say that $f$ has a zero along $Y$ of that order; if it is negative, we say that $f$ has a pole along $Y$ of order $-\nu_Y(f)$.

**Definition 2.2.** Let $f \in K^*$. We define the divisor of $f$, denoted by $(f)$, by

$$(f) = \sum \nu_Y(f) Y$$

This is well defined since $\nu_Y(f) = 0$ for all except finitely many $Y$ on a regular noetherian scheme. Such a divisor is called a principal Weil divisor.

Two Weil divisors $D_1$ and $D_2$ are said to be linearly equivalent if $D_1 - D_2$ is principal.

**Definition 2.3.** The degree of a Weil divisor $D = \sum n_i Y_i$ is defined to be $\sum n_i$.

## Cartier divisors

Cartier divisors extend the above notion of divisor to an arbitrary scheme. The motivating idea is that a divisor should be something which locally looks like the divisor of a rational function.

**Definition 2.4.** Let $X$ be a scheme. We denote the group $\Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$ by $\mathrm{Div}(X)$. The elements of $\mathrm{Div}(X)$ are called *Cartier divisors* on $X$. Therefore a divisor $D$ on $X$ is the collection of local data $\{(U_i, f_i)\}$, where the open sets $\{U_i\}$ form an open cover of $X$ and for each $i$, $f_i \in \Gamma(U_i, \mathcal{K}_X^*)$ such that for each $i, j$, $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}_X^*)$.

**Definition 2.5.** A Cartier divisor on a scheme $X$ is called *effective*, if it is in the image of the map $\Gamma(X, \mathcal{O}_X \cap \mathcal{K}_X^*) \to \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*)$. We denote an effective divisor by writing $D \geq 0$.

*Remark.* (i) Let $D_1, D_2$ be two Cartier divisors represented by $\{(U_i, f_i)\}_i$ and $\{(V_j, g_j)\}_j$. Then $D_1 + D_2$ is represented by $\{(U_i \cap V_j, f_i g_j)\}_{i,j}$.

(ii) We have $D \geq 0$ if and only if it can be represented by $\{(U_i, f_i)\}_i$ with $f_i \in \mathcal{O}_X(U_i)$. It is *principal* if it can be represented by $(X, f)$.

**Definition 2.6.** Let $f \in \Gamma(X, \mathcal{K}_X^*)$, then its image in $\mathrm{Div}(X)$ is called a *principal Cartier divisor*. We say that two Cartier divisors $D_1$ and $D_2$, are *linearly equivalent* if $D_1 - D_2$ is principal (Note we use addition to express the group operation in the group $\mathrm{Div}(X)$, mostly for historical reasons, even though the operation in $\mathcal{K}_X^*/\mathcal{O}_X^*$ is multiplicative). We then write $D_1 \sim D_2$. The group of Cartier divisors modulo linear equivalence is denoted by CaCl

For a general Cartier divisor, we use definition of the length of module to determine the degree.

**Definition 2.7.** Let $A$ be a noetherian ring of dimension 1 and $f \in A$ a regular element. Then the length of $A/fA$ as an $A$- module is finite. Therefore we can define a map $f \mapsto \mathrm{length}_A(A/fA)$, which can be extended to a map $\mathrm{Frac}(A)^* \to \mathbb{Z}$ and whose kernel contains the invertible elements of $A$. We therefore obtain a group homomorphism

$$\mathrm{mult}_A : \mathrm{Frac}(A)^*/A^* \to \mathbb{Z}.$$

Let $X$ be a locally Noetherian scheme. Let $D \in \mathrm{Div}(X)$ be a Cartier divisor. For any point $x \in X$ of codimension 1, the stalk of $D$ at $x$ belongs to $(\mathcal{K}^*/\mathcal{O}_X^*)_x = \mathrm{Frac}(\mathcal{O}_{X,x})^*/\mathcal{O}_{X,x}^*$. We can therefore define

$$\mathrm{mult}_x(D) := \mathrm{mult}_{\mathcal{O}_{X,x}}(D_x)$$

as above.

Let $U$ be an open and everywhere dense subset of $X$ such that $D|_U = 0$. Then any $x \in X$ of codimension 1 such that $\mathrm{mult}_x(D) \neq 0$ is a generic point of $X \backslash U$. This implies that in any affine open subset of $X$, there are only a finite number of points $x$ of codimension 1 such that $\mathrm{mult}_x(D) \neq 0$.

**Definition 2.8.** Let $X$ be a smooth curve over a field $k$. Let $D \in \mathrm{Div}(X)$ be a Cartier divisor. The degree of $D$ is defined to be

$$\deg D = \sum \mathrm{mult}_x(D)[k(x) : k]$$

Let $\phi : X \to Y$ be a morphism of schemes. We would like to inverse image and direct image of divisors across a morphism.

**Definition 2.9.** If we want to define the inverse image of a Cartier divisor on $Y$, it suffices that we have a natural morphism $\mathcal{K}_Y \to \phi_*\mathcal{K}_X$. If $f$ is a flat morphism, then we have a natural morphism $\mathcal{K}_Y \to \phi_*\mathcal{K}_X$ derived from the natural morphism $\mathcal{O}_Y \to \phi_*\mathcal{O}_X$. In this case, we have a morphism from $\mathcal{K}_Y^*/\mathcal{O}_Y^* \to \phi_*(\mathcal{K}_X^*/\mathcal{O}_X^*)$ We then define the inverse image of a divisor $D \in H^0(\mathcal{K}_Y^*/\mathcal{O}_Y^*)$ by a morphism $\phi$ as its image in $H^0(\mathcal{K}_X^*/\mathcal{O}_X^*)$.

### Invertible sheaves

**Definition 2.10.** Let $X$ be a scheme. A sheaf $\mathcal{F}$ is said to be locally free over $X$, if there exists an open covering $\{U_i\}$ such that $\mathcal{F}|_{U_i}$ is a free $\mathcal{O}_X|_{U_i}$-module. The rank of $\mathcal{F}$ on $U_i$ is the number $r \in \mathbb{Z}^+$, where $\mathcal{F}|_{U_i} \sim \mathcal{O}_X|_{U_i}^r$. If $X$ is connected, then $r$ is constant for all $U_i$ and is called the rank of the locally free sheaf. A locally free sheaf $\mathcal{F}$ of rank 1 is called an *invertible sheaf*.

**Definition 2.11.** Let $X$ be a scheme. Given an invertible sheaf $\mathcal{L}$, we can define its dual sheaf, $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$ such that $\mathcal{L} \otimes L^\vee \cong \mathcal{O}_X$. We define the group $\mathrm{Pic}(X)$ to be the group of isomorphism classes of invertible sheaves on $X$, under the operation $\otimes$.

**Definition 2.12.** To a Cartier divisor $D = \{(U_i, f_i)\}$ we associate an invertible subsheaf $\mathcal{O}_X(D) \subset \mathcal{K}_X$, defined by $\mathcal{O}_X(D)(U_i) = f_i^{-1}\mathcal{O}_X(U_i)$. We therefore get a map $\mathrm{Div}(X) \to \mathrm{Pic}(X)$. Further by this association, given two divisors $D_1$ and $D_2$ on a scheme $X$, the sheaf associated to the sum of two divisors $D_1 + D_2$ is the tensor product of the sheaves $\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$.

**Proposition 2.13** ([18], II.6.15)**.** *If $X$ is an integral scheme, then $Pic(X) \cong CaCl(X)$.*

**Definition 2.14.** Let $\eta$ be an invertible sheaf on $X$. Then $\eta$ is a *torsion element* of $\mathrm{Pic}(X)$ if there exists a positive integer $k$ such that $\eta^{\otimes k} \cong \mathcal{O}_X$.

The next theorem gives the useful condition under which we can relate the two different notions of divisors.

**Theorem 2.15** ([18] II.6.11)**.** *Let $X$ be an integral, separated noetherian scheme, all of whose local rings are unique factorization domains (locally factorial). Then the group of Weil divisors on $X$ is isomorphic to the group of Cartier divisors on $X$ and furthermore the principal Weil divisors correspond to the principal Cartier divisors under this isomorphism.*

*Remark.* Since a regular local ring is a unique factorization domain, this theorem applies in particular to any regular integral separated noetherian scheme.

### 2.2.2 Linear systems

Given a divisor $D$ on a $X$ with non-zero global sections, we will see that the set of global sections of $\mathcal{O}_X(D)$ correspond to effective divisors all linearly equivalent to $D$. This leads to the notion of linear systems which were the historical way of studying divisors, especially on non-singular projective varieties. It is a certain set of effective divisors all linearly equivalent to each other.

We will assume that $X$ is a non-singular projective variety for the rest of this subsection. Therefore the notion of Weil divisor and Cartier divisor are equivalent. We also have a one-to-one correspondence between $\mathrm{Pic}(X)$, the isomorphism class of invertible sheaves and $\mathrm{CaCl}(X)$, the linear equivalence of divisors. Finally for any invertible sheaf $\mathcal{L}$ on $X$, the global sections $\Gamma(X, \mathcal{L})$ form a finite-dimensional $k$-vector space ([18] II,5.19).

**Definition 2.16.** Let $\mathcal{L}$ be an invertible sheaf on $X$ with a non-zero global section $s \in \Gamma(X, \mathcal{L})$. We define the divisor of zeros $(s)_0$ associated to $s$ in the following way: Take a covering of $X$ such that $\mathcal{L}$ has a local trivialisation. Then for each open affine subset $U_i \subset X$, $s$ maps to a regular function $f_i$ on $U_i$. We take the effective divisor $\{(U_i, f_i)\}$ defined in this manner to be divisor of zeros $(s)_0$.

The following proposition gives the correspondence between global sections of an invertible sheaf and effective divisors that are linearly equivalent on a variety.

**Proposition 2.17** ([18], II.7.7)**.** *Let $X$ be a nonsingular projective variety over the algebraically closed field $k$. Let $D_0$ be a divisor on $X$ and let $\mathcal{L} = \mathcal{O}_X(D_0)$ be the corresponding invertible sheaf. Then:*

*(a) for each nonzero $s \in \Gamma(X, \mathcal{L})$, the divisor of zeros $(s)_0$ is an effective divisor linearly equivalent to $D_0$;*

*(b) every effective divisor linearly equivalent to $D_0$ is $(s)_0$ for some $s \in \Gamma(X, \mathcal{L})$; and*

*(c) two sections $s, s' \in \Gamma(X, \mathcal{L})$ have the same divisor of zeros if and only if there is a $\lambda \in k^*$ such that $s' = \lambda s$.*

**Definition 2.18.** Let $X$ be a non-singular projective scheme. By a divisor on a scheme we mean a Cartier divisor. For a divisor $D$ on a scheme $X$, we denote by $|D|$ the set of all effective divisors on $X$ linearly equivalent to $D$. This is called a *complete linear system*. From the above proposition, we see that every non-vanishing global section of $\mathcal{O}_X(D)$ defines an element of $|D|$, namely its divisor of zeros, and conversely every element of $|D|$ is the divisor of zeros of a non-vanishing section of $\mathcal{O}_X(D)$, well defined up to scalar multiplication. Thus $|D|$ can be naturally identified with the projective space associated to the vector space $\Gamma(X, \mathcal{O}_X(D))$. We denote $\dim H^0(X, \mathcal{L}(D))$ by $l(D)$, so that the dimension of $|D|$ is $l(D) - 1$.

**Definition 2.19.** A linear subset $L$ of $|D|$ is called a *linear system* on $X$, when $L$ corresponds to a vector subspace $V \subset \Gamma(X, \mathcal{O}_X(D))$, where $V = \{s \in \Gamma(X, \mathcal{O}_X(D)|(s)_0 \in L\} \cup \{0\}$. The dimension of the linear system $L$ is its dimension as a linear projective variety. Hence $\dim L = \dim V - 1$.

**Definition 2.20.** The *support* of a divisor $D$, Supp $D$, is defined to be the set of points $x \in X$ at which the local equation $f_i$ is not a unit in the stalk $\mathcal{O}_{X,x}$. It is the union of the prime divisors of $D$.

**Definition 2.21.** A point $P$ is a base point of a linear system $L(\subset |D|)$ if $P \in \operatorname{Supp} D$ for all $D \in L$.

**Definition 2.22.** We define the base locus $Bs(L)$ of a linear system $L$ to be the maximal closed subscheme of $X$ contained in Supp $D$, for all $D \in L$.

For a surface $X$, the base locus $Bs(L)$ consists of zero-dimensional and one-dimensional components. The *fixed part* of $L$ is the one-dimensional locus of $Bs(L)$. We denote it by $F$. Then $L - F := \{D' - F : D' \in L\}$ is a linear system with no fixed part. We call $L - F$ the variable part of $L$. Note that the finite number of base points forming the zero-dimensional locus of $Bs(L)$ is the base locus $Bs(L - F)$ of $L - F$. We call these points the isolated fixed points of $L$.

**Lemma 2.23** ([18], II.7.8)**.** *Let $L$ be a linear system on $X$ corresponding to the subspace $V \subset \Gamma(X, \mathcal{L})$. Then a point $p \in X$ is a base point of $L$ if and only if $s_p \in m_p\mathcal{L}_p$ for all $s \in V$. In particular, $L$ is base point free if and only if $\mathcal{L}$ is generated by global sections.*

**Definition 2.24.** Let $i : Y \hookrightarrow X$ be a closed immersion of nonsingular projective varieties over $k$. If $L$ is a linear system on $X$, we define the *restriction* of $L$ on $Y$, denoted $L|_Y$ as follows. The linear system $L$ corresponds to an invertible sheaf $\mathcal{L}$ on $X$, and a sub-vector space $V \subset \Gamma(X, \mathcal{L})$. We take the invertible sheaf $i^*\mathcal{L} = \mathcal{L} \otimes \mathcal{O}_Y$ on $Y$, and we let $W \subset \Gamma(Y, i^*\mathcal{L})$ be the image of $V$ under the natural map $\Gamma(X, \mathcal{L}) \to \Gamma(Y, i^*\mathcal{L})$. Then $i^*\mathcal{L}$ and $W$ define the linear system $L|_Y$.

Note that even if $L$ is a complete linear system $L|_Y$ may not be complete.

### 2.2.3 Cyclic coverings

The cyclic cover trick is a powerful technique, which we will use often use in this thesis. The general theory on cyclic covers in arbitrary characteristic was given in [16] (Section 3.5, pg. 22). We discuss here the construction of such covers.

**Theorem 2.25.** *Let $X$ be a variety defined over $k$, and $\mathcal{L}$ a line bundle on $X$. Suppose given a positive integer $m \geq 1$, (where if char $k$ is $p > 0$ then $p \nmid m$) plus a non-zero section*

$$s \in \Gamma(X, \mathcal{L}^{\otimes m})$$

*defining a divisor $D$ on $X$ which is either effective or zero. Then there exists a finite flat cyclic covering*

$$\phi : Y \to X,$$

*where $Y$ is a scheme having the property tha the pullback $\mathcal{L}' = \phi^*\mathcal{L}$ of $\mathcal{L}$ carries a section*

$$s' \in \Gamma(Y, \mathcal{L}') \qquad \textit{with} \qquad (s')^m = \phi^* s.$$

*The divisor $D' = (s')_0$ maps isomorphically to $D$. Moreover if $X$ and $D$ are non-singular, then so too are $Y$ and $D'$.*

For proof see [25] or [16]]. For a local description of this covering we can take an affine variety $X$ and a non-zero regular function $s \in \Gamma(X, \mathcal{O}_X)$ which defines a divisor $D$ on $X$. Start with the product $X \times \mathbb{A}^1$ of $X$ and the affine line. Taking $t$ for the coordinate on $\mathbb{A}^1$, consider the subvariety $Y \subseteq X \times \mathbb{A}^1$ defined by taking the equation $t^m - s = 0$:

$$\begin{array}{ccc} \{t^m - s = 0\} = Y & \subseteq & X \times \mathbb{A}^1 \\ & \overset{\phi}{\searrow} \quad \swarrow & \\ & X & \end{array}$$

The natural mapping $\phi : Y \to X$ is a cyclic covering branched along $D$. Setting $s' = t|_Y$, one has the equality

$$(s')^m = \phi^* s.$$

One can globalize the above argument to obtain a global covering, or a more elegant solution can be obtained by taking

$$Y = \mathbf{Spec} \oplus_{j=0}^{n-1} \mathcal{L}^{-j},$$

where $\mathbf{Spec}\,\mathcal{A}$ of a quasi-coherent $\mathcal{O}_X$-algebra is as described in [18] II.5.

If $D \neq 0$ and reduced, $Y$ is an $m$-cyclic covering of $X$ that is branched along $D$. If $D = 0$, we must take $m$ minimal (i.e $\mathcal{L}$ is exactly of order $m$ in $\mathrm{Pic}(X)$), to obtain a connected unramified cyclic covering $Y$ of $X$.

## 2.3 Relative dualizing sheaf

In this section we define a very useful and extremely important divisor called the canonical divisor, and its associated sheaf, the canonical sheaf, of a scheme $X$. We then define the relative dualizing sheaf of $X$ over $S$, and note that it coincides with the canonical sheaf when $f : X \to S$ is a flat morphism between projective varieties (see [27]). It is a very important invariant as there are many fundamental results associated to it, such as: the adjunction formula, Serre duality and the Riemann-Roch formula. We are particularly interested in this divisor as we use it extensively in the proof of our main theorem.

### 2.3.1 Modules of relative differential forms

**Definition 2.26.** Let $B$ be an $A$-algebra. The module of differential forms of $B$ over $A$ is a $B$-module $\Omega^1_{B/A}$ endowed with an $A$-derivation $d : B \to \Omega^1_{B/A}$ having the following universal property:

For any $B$-module $M$ and any $A$-derivation $d' : B \to M$, there exists a unique homomorphism of $B$-modules $\phi : \Omega^1_{B/A} \to M$ s.t. $d' = \phi \circ d$

$$\begin{array}{ccc} B & \xrightarrow{d'} & M \\ \downarrow d & \nearrow \phi & \\ \Omega^1_{B/A]} & & \end{array}$$

Here is a construction of a module of relative differential forms. Let $F$ be a free $B$-module generated by the symbols $db$, $b \in B$. Let $E$ be generated by $\{da, a \in A\} \cup \{d(b_1 + b_2) - db_1 - db_2\} \cup \{db_1b_2 - b_1db_2 - b_2db_1\}$. Then

$$\begin{aligned} \Omega^1_{B/A} &= F/E \\ d : B &\to \Omega^1_{B/A} \\ b &\mapsto db \end{aligned}$$

### 2.3.2 Sheaves of relative differentials (of degree 1)

**Definition 2.27.** Let $f : X \to Y$ be a morphism of schemes. There exists a unique quasi-coherent sheaf $\Omega^1_{X/Y}$ on $X$ such that for any affine open subset $V$ of $Y$, any affine open subset $U$ of $f^{-1}(V)$ and any $x \in U$, we have

$$\begin{aligned} \Omega^1_{X/Y}|_U &\cong (\Omega^1_{O_X(U)/O_Y(V)})^{\sim} \\ (\Omega^1_{X/Y})_x &\cong \Omega^1_{O_{X,x}/O_{Y,f(x)}} \end{aligned}$$

We can construct $\Omega^1_{X/Y}$ in the following manner:

$\Delta : X \to X \times_Y X$ is a locally closed immersion. $\Delta(X)$ is closed in an open subset $U$ of $X \times_Y X$. Let $\mathcal{I} = \ker \Delta^\#$ be the sheaf of ideals defining the closed subset $\Delta(X)$ in $U$. Then

$$\Omega^1_{X/Y} \cong \Delta^*(\mathcal{I}/\mathcal{I}^2)$$

### 2.3.3 The canonical sheaf

**Definition 2.28.** Let $f : X \to Y$ be an immersion into a scheme. Let $V$ be an open subscheme of $Y$ such that $f$ factors through a closed immersion $i : X \to V$ and let $\mathcal{J}$ be the sheaf of ideals defining $i$. The *conormal sheaf* is defined as follows,

$$\mathcal{C}_{X/Y} := i^*(\mathcal{J}/\mathcal{J}^2).$$

We also define the normal sheaf, denoted by $\mathcal{N}_{X/Y}$ as follows:

$$\mathcal{N}_{X/Y} := \mathcal{C}_{X/Y}^\vee.$$

Let $f : X \to Y$ be a morphism. For any $r \geq 1$, the quasi-coherent sheaf $\Omega^r_{X/Y} := \bigwedge^r \Omega^1_{X/Y}$ is the sheaf of differentials of order $r$.

**Definition 2.29.** Let $Y$ be a locally noetherian scheme and $f$ a quasi-projective locally complete intersection. $i : X \to Z$ an immersion into a scheme $Z$ that is smooth over $Y$. We define the *canonical sheaf* of the morphism $X \to Y$ to be the invertible sheaf

$$\omega_{X/Y} := \mathrm{Det}(\mathcal{C}_{X/Z})^\vee \otimes_{\mathcal{O}_X} i^*(\mathrm{Det}(\Omega^1_{Z/Y}))$$

This definition is independent of the immersion into $Z$.

**Theorem 2.30** ([27], 6.4.9)**.** *Let $f : X \to Y$, $g : Y \to Z$ be quasi-projective locallly complete intersections*

*(i) (Adjunction Formula) We have a canonical isomorphism*

$$\omega_{X/Z} \cong \omega_{X/Y} \otimes_{\mathcal{O}_X} f^*\omega_{Y/Z}$$

*(ii) (Base Change) Let $Y' \to Y$ be a morphism. Let $X' := X \times_Y Y'$ and let $p : X' \to X$ be the first projection. If either $Y' \to Y$ or $X \to Y$ is flat, then $X' \to Y'$ is a local complete intersection and we have a canonical isomorphism*

$$\omega_{X'/Y'} \cong p^*\omega_{X/Y}$$

*Remark.* For a smooth morphism $f : X \to Y$ of relative dimension $d$,

$$\omega_{X/Y} = \bigwedge^d \Omega_{X/Y}.$$

In particular, if $X$ is a non-singular variety over $k$ of dimension $n$, then we define the canonical sheaf of $X$, denoted by $\omega_X$, to be

$$\omega_X = \bigwedge^n \Omega_{X/k}.$$

### 2.3.4 Relative dualizing sheaf

Let $f : X \to Y$ be a proper morphim to a locally noetherian scheme $Y$, with fibres of dimension $\leq r$. let $\mathcal{F}$, $\mathcal{G}$ b quasi-coherent sheaves on $X$. For any affine open subset $V$ of $Y$, each homomorphism $\phi : \mathcal{F}|_{f^{-1}(V)} \to \mathcal{G}|_{f^{-1}(V)}$ induces a homomorphism

$$H^r(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)}) \stackrel{H^r(\phi)}{\to} H^r(f^{-1}(V), \mathcal{G}|_{f^{-1}(V)}).$$

This defines a canonical bilinear map

$$f_*\mathcal{H}om_{O_X}(\mathcal{F}, \mathcal{G}) \times R^r f_*\mathcal{F} \to R^r f_*\mathcal{G}.$$

**Definition 2.31.** We define the *dualizing sheaf* (or $r$-dualizing sheaf) for $f$ to be a quasi-coherent sheaf $\omega_f$ on $X$, endowed with a homomorphism of $O_Y$-modules

$$\mathrm{tr}_f : R^r f_*\omega_f \to O_Y$$

such that for any quasi-coherent sheaf $\mathcal{F}$ on $X$, the natural bilinear map

$$f_*\mathcal{H}om_{O_X}(\mathcal{F}, \omega_f) \times R^r f_*\mathcal{F} \to R^r f_*\omega_f \stackrel{\mathrm{tr}_f}{\to} O_Y$$

induces an isomorphism

$$f_*\mathcal{H}om_{O_X}(\mathcal{F}, \omega_f) \cong \mathcal{H}om_{O_Y}(R^r f_*\mathcal{F}, O_Y).$$

The existence of the dualizing sheaf is a difficult question, but the next theorem proves the existence of the dualizing sheaf for projective morphisms.

**Theorem 2.32** ([27], 6.4.30)**.** *Let $f : X \to Y$ be a projective morphism to a locally noetherian scheme $Y$, with fibres of dimension $\leq r$. Then the $r$-dualizing sheaf exists.*

The final theorem in this section proves the isomorphism between the dualizing sheaf and the canonical sheaf when $f$ is flat.

**Theorem 2.33** ([27], 6.4.32)**.** *Let $Y$ be a locally noetherian scheme, and let $f : X \to Y$ be a flat projective locally compete intersection of relative dimension $r$. Then the $r$-dualizing sheaf $\omega_f$ is isomorphic to $\omega_{X/Y}$. In particular if $f$ is smooth, then $\omega_f \cong \Omega^r_{X/Y}$.*

*Remark.* Let $f : X \to S$ be a fibered surface. Then by the above theorem, $\omega_{S/C}$ is the relative dualizing sheaf of $f$. Further taking $\mathcal{F}$ to be the structure sheaf on $S$, we get

$$f_*\mathcal{H}om_{O_X}(O_S, \omega_{S/C}) \cong \mathcal{H}om_{O_C}(R^1 f_* O_S, O_C).$$

or equivalently

$$f_*\omega_{S/C} \cong R^1 f_* O_S^\vee.$$

## 2.4 Fibered surfaces

Fibered surfaces are relative curves over a Dedekind scheme (we assume that $S$ is of $k$-dimension 1). In this section we recall some properties of fibered surfaces, intersections on fibered surfaces [27], and of semi-stable and generically ordinary fibrations [21].

### 2.4.1 Properties of fibres

**Definition 2.34.** Let $S$ be a Dedekind scheme. We call an integral, projective, flat $S$-scheme $\pi : X \to S$ of dimension 2, a fibered suface over $S$. The generic point of $S$ wil be denoted by $\eta$ and the fibre over $\eta$ will be denoted by $X_\eta$ and is called the generic fibre. The fibre over a closed point $s \in S$ is called a closed fibre. We will say that $X$ is a normal (resp. regular) fibered surface, if $X$ is normal (resp. regular).

**Definition 2.35.** We will call a regular fibered surface $X \to S$ over a Dedekind scheme $S$ of dimension 1 an arithmetic surface.

**Definition 2.36.** Let $\pi : X \to S$ be a fibered surface or a fibration with connected fibres from a smooth surface $X$ to a smooth projective curve $S$. We say that $\pi$ is *smooth* if all the fibres are smooth, *isotrivial* if all the smooth fibers are mutually isomorphic, and locally trivial if it is smooth and isotrivial.

We now list some elementary properties of fibres. A detailed discussion of fibered surfaces can be found in [27], section 8.3. A few of these results are recalled here to remind ourselves of the properties we need.

**Lemma 2.37** ([27] 4.4.16)**.** *Let $S$ be a Dedekind scheme with generic point $\eta$, and let $X \to S$ be a dominant morphism of finite type with $X$ irreducible. Then for any $s \in S$ such that $X_s \neq 0$, we have $X_s$ equidimensional of dimension $\dim X_\eta$.*

**Lemma 2.38** ([27] 8.3.3)**.** *Let $S$ be a Dedekind scheme, with generic point $\eta$. Let $X \to S$ be a fibered (resp. normal fibered) surface. Then $X_\eta$ is an integral (resp. normal) curve over $K(S)$. For any $s \in S$, $X_s$ is a projective curve over $k(s)$.*

**Proposition 2.39** ([27] 8.3.4)**.** *Let $\pi : X \to S$ be a fibered surface over a Dedekind scheme of dimension 1.*

*(a) Let $x$ be a closed point of the generic fibre $X_\eta$. Then $\bar{\{x\}}$ is an irreducible closed subset of $X$, finite and surjective to $S$.*

*(b) Let $D$ be an irreducible closed susbset of $X$. If $\dim D = 1$, then either $D$ is an irreducible component of a closed fibre, or $D = \bar{\{x\}}$ where $x$ is a closed point of $X_\eta$.*

*(c) Let $x_0$ be a closed point of $X$, then $\dim O_{X,x_0} = 2$.*

**Definition 2.40.** Let $\pi : X \to S$ be a fibered surface over a Dedekind scheme $S$. Let $D$ be an irreducible Weil divisor. We say that $D$ is horizontal if $\dim S = 1$ and if $\pi|_D : D \to S$ is surjective. If $\pi(D)$ is reduced to a point, we say that $D$ is vertical. More generally we will say that an arbitrary Weil divisor is horizontal (resp. vertical) if its components are horizontal (resp. vertical). We say that a cartier divisor is horizontal or vertical if the associated Weil divisor is horizontal or vertical.

**Corollary 2.41** ([27] 8.3.6)**.** *Let $\pi : X \to S$ be a fibered surface over a Dedekind scheme $S$ of dimension 1. Let $s \in S$. then the following properties are true.*

*(a) The fibre $X_s$ is a projective curve over $k(s)$, and we have the equality of arithmetic genera $p_a(X_x) = p_a(X_\eta)$.*

*(b) If $X_\eta$ is geometrically connected, then the same holds for $X_s$.*

*(c) If $X_\eta$ is geometrically integral, then the canonical homomorphism $O_S \to \pi_* O_X$ is an isomorphism.*

*(d) Let us suppose that $X$ is a regular scheme. Then the morphisms $X \to S$ and $X_s \to$ Spec $k(s)$ are local complete intersections, and we have the relation $\omega_{X_s/k(s)} = \omega_{X/S}|_{X_s}$ between the dualizing sheaves.*

**Proposition 2.42.** *Let $f : X \to S$ be a fibered surface from a smooth, projective surface $X$ onto a smooth projective curve $S$ over an algebraically closed field $k$ with connected fibres. Then $O_S \to \pi_* O_X$ is an isomorphism.*

*Proof.* From 2.38, we see that $X_\eta$ is an integral curve over $k$. Therefore by 2.41 (c), we get the result. □

**Proposition 2.43** ([27] 8.3.11)**.** *Let $\pi : X \to S$ be a fibered surface over a Dedekind scheme $S$. We suppose that the generic fibre $X_\eta$ is smooth. Then there exists a non-empty open susbset $V$ of $S$ such that $\pi^{-1}(V) \to V$ is smooth. In other words, $X_s$ is smooth over $k(s)$ except maybe for a finite number of $s$.*

The following lemma proves that base change of a regular fibered surface by a finite morphism that is unramified on the singular fibres is a regular fibered surface.

**Lemma 2.44.** *Let $f : X \to S$ be a generically smooth fibration from a proper smooth surface $X$ to a proper smooth curve $S$. If $S'$ is a smooth curve and $\psi : S' \to S$ is a flat and finite morphism, that is unramified on the singular fibres of $f$, then $X' = X \times_S S'$ is smooth over $k$.*

*Proof.* Consider the fibre product

$$\begin{array}{ccc} X' = X \times_S S' & \xrightarrow{f'} & S' \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\ X & \xrightarrow{f} & S. \end{array}$$

Let $y \in X'$ be a point.

case(i) $\phi(y)$ lies on a smooth fibre of $f$.

Define $T = \{x \in X | X_{f(x)}$ is a smooth fibre over $k(f(x))\}$. Then $T$ is open in $X$, and $f|_T : T \to S$ is smooth. The base change of a smooth morphism is smooth and $y \in T \times_S S'$. Therefore $f'$ is smooth at $y$ and $S' \to k$ is smooth, hence $X'$ is smooth at $y$.

case(ii) $\phi(y)$ lies on a singular fibre of $f$.

In this case, $\psi$ is unramified over $f(\phi(y))$. Therefore there exists open neighbourhoods $U' \subset S'$ and $U \subset S$ such that $\psi|_{U'} : U' \to U$ is étale and hence smooth. Again since the base change of a smooth morphism is smooth, $X \times_U U' \to X$ is smooth and $y \in X \times_U U'$. Therefore $\phi$ is smooth at $y$ and $X \to k$ is smooth, hence $X'$ is smooth at $y$.

□

The next result is an important proposition that we will use repeatedly in our thesis. It proves the existence of an intermediate scheme such that a morphism can be expressed as one with connected fibres, followed by a finite morphism.

**Proposition 2.45** (Stein Factorization, [18] III Cor 11.5.)**.** *Let $f : X \to Y$ be a projective morphism to a locally Noetherian scheme $Y$.*

*(a) $f$ factors into $g : X \to Z$ and $h : Z \to Y$ such that $g$ has only geometrically connected fibres and $h$ is finite.*

*(b) We can factor $f$ as in (a) with moreover $\mathcal{O}_Z \cong g_*\mathcal{O}_X$. This implies that $Z$ is unique up to isomorphism.*

### 2.4.2 Intersection of two divisors on a surface

We are interested in studying the internal geometry of a surface and like in the case of curves, use divisors to do so. We assume that $X$ is a nonsingular projective surface over an algebraically closed field $k$. There is a unique symmetric bilinear pairing $\operatorname{Pic} X \times \operatorname{Pic} X \to \mathbb{Z}$, which is normalized by requiring that for any two irreducible nonsingular curves $C, D$ meeting transversally, $C.D$ is just the number of intersection points of $C$ and $D$.

The main tool in proving this theorem is Bertini's Theorem, which allows us to move any two divisors in their linear equivalence class, so that they become differences of irreducible nonsingular curves meeting transversally.

**Theorem 2.46** (Bertini's Theorem, [18] II.8.18.). *Let $X$ be a nonsingular closed subvariety of $\mathbb{P}^n_k$, where $k$ is an algebraically closed field. Then there exists a hyperplane $H \subset \mathbb{P}^n_k$, not containing $X$, and such that the scheme $H \cap X$ is regular at every point. (In fact, if $\dim X \geq 2$, then $H \cap X$ is connected, hence irreducible, and so $H \cap X$ is a non singular variety.) Furthermore, the set of hyperplanes with this property forms an open dense subset of the complete linear system $|H|$, considered as a projective space.*

**Lemma 2.47** ([18] V.1.2). *Let $C_1, ..., C_r$ be irreducible curves on the surface $X$, and let $D$ be a very ample divisor. Then almost all curves $D'$ in the complete linear system $|D|$ are irreducible, nonsingular, and meet each of the $C_i$ transversally.*

**Lemma 2.48** ([18] V.1.3). *Let $C$ be an irreducible nonsingular curve on $X$, and let $D$ be any curve meeting $C$ transversally. Then*

$$\#(C \cap D) = \deg_C(\mathcal{O}_X(D) \otimes \mathcal{O}_C).$$

The intersection pairing is defined by the following properties:

**Theorem 2.49** ([18] V.1.1). *There is a unique pairing $\operatorname{Div} X \times \operatorname{Div} X \to \mathbb{Z}$, denoted by $C.D$ for any two divisors $C, D$, such that*

(1) *if $C$ and $D$ are nonsingular curves meeting transversally, then $C.D = \#(C \cap D)$, the number of points of $C \cap D$,*

(2) *it is symmetric: $C.D = D.C$,*

(3) *it is additive: $(C_1 + C_2).D = C_1.D + C_2.D$, and*

(4) *it depends only on the linear equivalence classes: if $C_1 \sim C_2$ then $C_1.D = C_2.D$.*

**Definition 2.50.** If $D$ is any divisor on the surface $X$, we can define the self-intersection number $D.D$, usually denoted by $D^2$.

**Definition 2.51.** Let $X$ be a projective nonsingular surface over an algebraically closed field $k$. Let $\omega_{X/k}$ be the canonical sheaf on the surface. Any divisor $K$ in the linear equivalence class corresponding to $\omega_{X/k}$ is called a canonical divisor. Then $K^2$, the self-intersection of $K$ is a number dependent only on $X$ and is an invariant of $X$.

The following result gives a method of calculating the intersection number of two curves $C, D$ with no common irreducible component. If $P \in C \cap D$, then we define the intersection multiplicity $(C.D)_P$ of $C$ and $D$ at $P$ to be the length of $O_{P,X}/(f, g)$, where $f$ and $g$ are local equations of $C$ and $D$ at $P$.

**Proposition 2.52** ([18] V.1.4)**.** *IF $C$ and $D$ are curves on $X$ having no common irreducible component, then*

$$C.D = \sum_{P \in C \cap D} (C.D)_P.$$

### 2.4.3 Intersection and morphisms

We study the behaviour of intersection numbers with respect to morphisms of fibered surfaces.

#### 2.4.3.1 Projection formula

Let $X$, $Y$ two schemes, and let $f : X \to Y$ be a proper morphism. For any prime cycle $Z$ on $X$, we set $W = f(Z)$ and

$$f_*Z = \begin{cases} [K(Z) : K(W)]W, & \text{if } K(Z) \text{ is finite over } K(W) \\ 0, & \text{otherwise} \end{cases}$$

By linearity we define a homomorphism $f_*$ from the group of cycles on $X$ to the group of cycles on $Y$.

**Proposition 2.53** ([27] 9.2.11)**.** *Let $f : X \to Y$ be a surjective projective mophism of Noetherian integral schemes. We suppose that $[K(X) : K(Y)] = n$ is finite. Then for any Cartier divisor $D$ on $Y$, we have*

$$f_*[f^*D] = n[D].$$

**Theorem 2.54** ([27] 9.2.12)**.** *Let $f : X \to Y$ be a dominant morphism of regular fibered surfaces over $S$. Let $C$ (resp. $D$) be a divisor on $X$ (resp, on $Y$). Then the following properties are true.*

*(a) For any divisor $E$ on $X$ such that $f(\text{Supp } E)$ is finite, we have $E \cdot f^*D = 0$.*

*(b) Suppose that $C$ or $D$ is vertical. Then*

$$C \cdot f^*D = f_*C \cdot D \qquad \textit{Projection Formula},$$

*where $f_*C$ is a Cartier divisor on $Y$.*

*(c) The extension $K(X)/K(Y)$ is finite. Let $C$ be a vertical divisor on $Y$. Then $f^*C$ is vertical and we have*

$$f^*C \cdot f^*D = [K(X) : K(Y)]C \cdot D.$$

*Note.* There is a unique pairing $\text{Div}_{\mathbb{Q}} X \times \text{Div}_{\mathbb{Q}} X \to \mathbb{Q}$, denoted by $C.D$ for any two divisors $C, D$, which is defined via extension of scalars from the analogous product on $\text{Div}(X)$.

**Definition 2.55.** A $\mathbb{Q}$−divisor $D$ is called numerically effective (nef), if $D.C \geq 0$ for all irreducible curves $C$ on $X$.

### 2.4.4 Relatively minimal surfaces

In this section we include a brief overview of minimal surfaces.

**Definition 2.56.** Let $X \to S$ be a regular fibered surface. A prime divisor $E$ on $X$ is called an *exceptional divisor (or (-1)-curve)* if there exists a regular fibered surface $Y \to S$ and a morphism $f : X \to Y$ of $S$-schemes such that $f(E)$ is reduced to a point, and that $f : X \backslash E \to Y \backslash f(E)$ is an isomorphism. In other words, an exceptional divisor is an integral curve that can be contracted to a regular point. Let us note that as $f(E)$ is a closed point , its image in $S$ is also a closed point. Hence $E$ is a vertical divisor.

**Definition 2.57.** We say that a regular fibered surface $X \to S$ is *relatively minimal* if it does not contain an exceptional divisor. We say that $X \to S$ is minimal if every birational map of regular fibered $S$-surfaces $Y \dashrightarrow X$ is a birational morphism. A minimal fibered surface is relatively minimal.

Under sufficiently general conditions, we have the following result about the existence of a minimal surface in a class of birational surfaces.

**Theorem 2.58** ([27] 9.3.24)**.** *Let $X \to S$ be a relatively minimal arithmetic surface, with generic fibre $X_\eta$ verifying $p_a(X_\eta) \geq 1$. Then $X$ is minimal.*

**Theorem 2.59** ([27] 9.3.26)**.** *Let $X \to S$ be an arithmetic surface with $p_a(X_\eta) \geq 1$. Let $K_{X/S}$ be a canonical divisor, then $X \to S$ is minimal if and only if $K_{X/S}$ is numerically effective.*

We also have the additional nice property that being minimal is preserved under étale base change.

**Proposition 2.60** ([27] 9.3.28)**.** *Let $X \to S$ be an arithmetic surface such that $p_a(X_\eta) \geq 1$. Let $S' \to S$ be a morphism. Let us suppose that $S' \to S$ is étale surjective, or that $S$ is the spectrum of a discrete valuation ring $R$ and that $S' = \operatorname{Spec} \hat{R}$. Then $X \to S$ is minimal if and only if $X' = X \times S' \to S'$ is minimal.*

### 2.4.5 Minimal desingularization

**Definition 2.61.** Let $X$ be a reduced locally Noetherian scheme. A proper birational morphism $\pi : Z \to X$ with $Z$ regular is called a *desingularization* of $X$. If $\pi$ is an isomorphism above every regular point of $X$, we say that it is a *desingularization in the strong sense*.

**Theorem 2.62** ([27] 8.3.51)**.** *Let $X \to S$ be a fibered surface. Let us suppose that $\dim S = 1$ and that $X$ has a smooth generic fibre. Then $X$ admits a desingularization in the strong sense.*

**Definition 2.63.** Let $Y$ be a Noetherian scheme. We call a desingularization morphism $Z \to Y$ such that every other desingularization morphism $Z' \to Y$ factors throught $Z' \to Z \to Y$ a *minimal desingularization of $Y$*. If $Y$ is already regular then it is its own minimal desingularization.

**Proposition 2.64** ([27] 9.3.32)**.** *Let $Y \to S$ be a normal fibered surface. If $Y$ admits a desingularization, then it admits a minimal desingularization. More precisely, if $X \to Y$ is a desingularization such that no exceptional divisor of $X$ is contained in the exceptional locus of $X \to Y$, then it is a minimal desingularization.*

### 2.4.6 Semi-stable fibrations

In the main theorem of this thesis, we are mostly concerned with generically smooth semi-stable fibrations. In this section we give a quick introduction to semi-stable curves and its properties.

**Definition 2.65.** A proper curve $S$ over $k$ is called *semi-stable* if

(a) $S$ is connected and reduced.

(b) Any singular point of $S$ is an ordinary double point.

**Definition 2.66.** Let $f : X \to S$ be a morphism of finite type to a scheme $S$. We say that $f$ is *semi-stable*, or that $X$ is a *semi-stable curve over $S$*, if

(a) $f$ is flat.

(b) The fibre $X_s$ is a semi-stable curve over $k(s)$.

**Proposition 2.67** ([27] 10.3.15)**.** *Let $f : X \to S$ be a semi-stable curve over a scheme $S$.*

*(a) Let $S' \to S$ be a morphism. Then $X \times S' \to S'$ is semi-stable.*

*(b) If $S$ is locally Noetherian, then $X \to S$ is a locally complete intersection.*

*(c) If $S$ is a Dedekind domain and if the generic fibre of $X \to S$ is normal, then $X$ is normal.*

### 2.4.7 Generic ordinarity

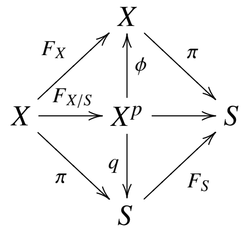
We are interested in generically ordinary semi-stable fibrations $\pi : X \to S$, since [21] has recently shown that for such fibrations, all the quotient bundles of $\pi_* \omega_{X/S}$ are of non-negative degree. This is a crucial component of our main theorem, and here we present a brief introduction to such fibrations. Assume that $k$ is a perfect field of positive characteristic $p$. In the case that $S$ is an 1-dimensional non-singular scheme over $k$, we always assume that a semistable curve $X \to S$ is generically smooth and $X$ is smooth over $k$. The definitions in this section can be found in [21].

**Definition 2.68.** Let $X$ be a scheme over $k$. We call the morphism $F_X : X \to X$ induced by the ring homomorphism $O_X \to O_X : a \mapsto a^p$ the absolute Frobenius of $X$.

Let $S$ be a scheme over $k$ and $\pi : X \to S$ an $S$-scheme. We let $X^p$ denote the fibered product $X \times_S S$, where the second factor $S$ is endowed with the structure of an $S$-scheme via $F_S : S \to S$. We will endow $X^p$ with the structure of an $S$-scheme given by the second projection $q : X \times_S S \to S$. Let us denote the first projection by $\phi : X^p \to X$. We have a communicative diagram

$$\begin{array}{ccc} X & \xrightarrow{F_X} & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{F_S} & S \end{array}$$

There therefore exists a morphism of $S$-schemes $F_{X/S} : X \to X^p$ making the folllowing diagram commutative:




**Definition 2.69.** We call the morphism of $S$-schemes $F_{X/S} : X \to X^p$ as above the *relative Frobenius* or the *standard Frobenius*.

Let $X$ be a smooth and proper variety defined over $k$. Denote by $\Omega^{\cdot}_{X/k}$ the DeRham complex of $X/k$. Then $F_{X/k_*}(\Omega^{\cdot}_{X/k})$ is a $O_{X^p}$-linear complex of coherent $O_{X^p}$-modules. The image of $F_{X/k_*}\Omega^{i-1}_{X/k} \to F_{X/k_*}\Omega^{i}_{X/k}$ is denoted by $B^i\Omega_{X/k}$ or $B^i\Omega$ when there is no risk of confusion. Each $B^i\Omega$ is a vector bundle on $X^p$.

**Definition 2.70.** $X$ is ordinary (Bloch-Kato ordinary) if $H^iB^j\Omega_{X/k} = 0$ for all $i$ and $j$.

We can extend the definition of ordinarity to any proper smooth morphism of schemes of characteristic $p$.

**Definition 2.71.** Let $f : X \to S$ be a proper smooth morphism of schemes of characteristic $p$. The image $B^i_{X/S}$ of $F_{X/S_*}\Omega^{i-1}_{X/S} \to F_{X/S_*}\Omega^{i}_{X/S}$ is a vector bundle on $X^p$. We define $X/S$ to be ordinary if $R^if_*(B^j_{X/S}) = 0$ for all $i$ and $j$

This definition of ordinarity can be further extended to a semi-stable morphism over a discrete valuation ring of characteristic $p > 0$ as follows [19] [20]. Let $A$ be a discrete valuation ring. Let $S = \operatorname{Spec} A$ and $s \in S$ is the closed point. Let $f : X \to S$ be a semi-stable fibration. Let $U \subset X$ be he smooth locus of $f : X \to S$ and $u : U \hookrightarrow X$ be the inclusion. Then $X \backslash U$ is of codimension at least 2, hence

$$\omega^{\cdot}_{X/S} = u_*\Omega^{\cdot}_{U/S}$$

is a complex of locally free sheaves on $X$ and $\omega^i_{X/S} = \wedge^i\omega^1_{X/S}$. When $X$ is given as $\operatorname{Spec} A[x_1, \dots, x_n]/(x_1, \dots, x_r - t)$ étale locally, $\omega^1_{X/S}$ is the free module of rank $n-1$, generated by

$$dx_1/x_1, \dots dx_r/x_r, dx_{r+1}, \dots, dx_n$$

with the relation

$$\sum_{i=1}^{r} dx_i/x_i = 0.$$

If the relative dimension of $f$ is $d$, the highest wedge product $\omega^d_{X/S}$ is the relative dualizing sheaf of $f : X \to S$. Now assume $A$ is of characteristic $p > 0$ and $X^p$ is the base change of $X$ by the Frobenius morphism of $S$. Let $F : X \to X^p$ be the relative Frobenius morphism. Then $F_*\omega^\cdot_{X/S}$ is an $O_{X^p}$-linear complex and the usual Cartier isomorphism $C^{-1} : \Omega^i_{U^p/S} \to \mathcal{H}^i F_*\Omega^\cdot_{U/S}$ extends to an isomorphism [19]

$$C^{-1} : \omega^i_{U^p/S} \to \mathcal{H}^i F_*\Omega^\cdot_{U/S}.$$

Here $\omega^i_{X^p/S} = F^*(\omega^i_{X/S})$. Note that $\omega^d_{X^p/S}$ is the dualizing sheaf for $X^p \to S$ when $d$ is the relative dimension. The image and the kernel of the differentials of the complex $F_*\omega^\cdot X/S$ are denoted by $B^i\omega_{X/S}$, and $Z^i\omega_{X/S}$ respectively. They are $O_{X^p}$-coherent sheaves and flat over $S$. In particular the Cartier isomorphism at $i = 0$ induces

$$0 \to O_{X^p} \to F_*O_X \to B^1\omega_{X/S}.$$

**Definition 2.72.** A proper semi-stable morphism $f : X \to S$ is ordinary if $H^j(B^i\omega_{X/S}) = 0$ for all $i$ and $j$.

*Remark.* (a) Since $B^i\omega_{X/S}$ are flat over $S$, $f$ is ordinary if and only if $H^j(X_s, B^i\omega_{X/S}|_{X_s}) = 0$ for all $i, j$ when $X_s$ is the special fibre. This definition depends on entire $X \to S$, and not only on the special fibre.

(b) But if $f$ is smooth, $f$ is ordinary if and only if the special fibre is ordinary.

(c) If the relative dimension of $f$ is 1 and the residue field is perfect, $f$ is ordinary if and only if the Frobenius morphism on $H^1(O_{X_s})$ is bijective, hence the ordinarity depends only on the special fibre.

Since all the above arguments are local on the base $S$, all the results are still valid if we replace the base $S$ by a smooth curve over a field of positive characteristic. Let $C$ be a smooth proper curve over a perfect field $k$ of positive characteristic and $\pi : X \to C$ be a proper semi-stable morphism. Since each $B^i\omega_{X/C}$ is flat over $C$, by semi-continuity, the set of points $s \in C$ satisfying $X \otimes_{O_C} O_{X_s}$ is ordinary forms an open set in $C$.

**Definition 2.73.** Let $\pi : X \to C$ be a proper semi-stable morphism. We say that $\pi$ is generically ordinary or that the generic fibre of $\pi$ is ordinary if at least one closed fibre of $\pi$ is ordinary.

**Proposition 2.74** ([19], Pro 1.2)**.** *Generic ordinarity is preserved under base change.*

## 2.5 Algebraic fundamental group

The algebraic fundamental group is a generalization of Galois theory and the topological fundamental group. We first define finite étale covers which is a good generalization of finite separable extensions of a field and finite covering maps in topology. Then we define the notion of geometric points and the geometric fibre functor, and finally the algebraic fundamental group.

### 2.5.1 Finite étale coverings

**Definition 2.75.** A morphism $f : A \to B$ between local rings $(A, m)$ and $(B, n)$ is etale if $f$ is flat (i.e. $B$ is a flat $A$- module), and $B/f(m)B \to A/m$ is a finite separable field extension. Let $f : Y \to X$ be a morphism between noetherian schemes. Then $f$ is etale if $f$ is locally etale at every point of $Y$.

**Example.** Let $k$ be a field and $X = \operatorname{Spec}(k)$. A finite etale morphism to $X$ is the spectrum of a $k$-algebra, $R$, where $R$ is a finite product of finite separable field extensions of $k$. i.e. $R = \operatorname{Spec}(\prod_i L_i)$. We will call $R$ an etale $k$-algebra.

*Remark.* A finite morphism $\phi : Y \to X$ etale if $\phi$ is flat and the fibre $Y_p$ over every point $p \in X$ is the spectrum of a finite etale $k(p)$-algebra.

**Proposition 2.76.** *The following properties are true for étale morphisms.*

*(a) The composite of two etale morphisms is etale.*

*(b) Any base change of an etale morphism is etale.*

*(c) If $\phi \circ \psi$ and $\phi$ are etale, then so is $\psi$.*

**Proposition 2.77** ([38] 5.2.7)**.** *Let $\phi : X \to S$ be a finite and locally free morphism. The following are equivalent.*

*(a) The morphism $\phi$ is étale.*

*(b) The sheaf of relative differentials $\Omega^1_{X/S}$ is 0.*

*(c) The diagonal morphism $\Delta : X \to X \times_S X$ coming from $\phi$ is an isomorphism of $X$ onto an open and closed subscheme of $X \times_S X$.*

**Definition 2.78.** Given a morphism of schemes $\phi : X \to S$, define $\mathrm{Aut}(X|S)$ to be the group of scheme automorphisms of $X$ preserving $\phi$.

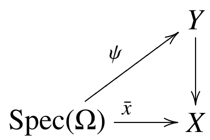
**Definition 2.79.** Let $\phi : X \to S$ be an affine surjecive morphism of schemes, and $G \subset \mathrm{Aut}(X|S)$ a finite subgroup. Define a ringed space $G\backslash X$ and a morphism $\pi : X \to G\backslash X$ of ringed spaces as follows. The underlying topological space of $G\backslash X$ is to be the quotient of $X$ by the action of $G$, and the continuous map of $\pi$ the natural projection. Then define the structure of $G\backslash X$ as the subsheaf $(\pi_* O_X)^G$ of $G$-invariant elements in $\pi_* O_X$.

**Proposition 2.80** ([38] 5.3.6)**.** *The ringed space $G\backslash X$ constructed above is a scheme, the morphism $\pi$ is affine and surjective, and $\phi$ factors as $\phi = \psi \circ \pi$ with an affine morphism $\psi : G\backslash X \to S$. The scheme $G\backslash X$ over $S$ is the quotient of $X$ by $G$.*

**Proposition 2.81** ([38] 5.3.7)**.** *Let $\phi : X \to S$ be a connected finite étale cover, and $G \subset$ Aut$(X|S)$ a finite group of $S$-automorphisms of $X$. Then $X \to G\backslash X$ is a finite étale cover of $G\backslash X$, and $G\backslash X$ is a finite étale cover of $S$.*

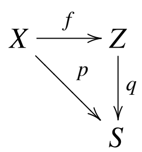
**Definition 2.82.** A geometric point of a scheme $X$ is defined as a morphism $\bar{x} : \mathrm{Spec}(\Omega) \to X$ where $\Omega$ is a separably closed field. In other words a geometric point is given by a topological point $x \in X$ and an embedding of the residue field $k(x)$ in $\Omega$

**Definition 2.83.** Given a morphism $\phi : Y \to X$ and a geometric point $\bar{x} : \mathrm{Spec}(\Omega) \to X$, the geometric fibre $X_{\bar{x}}$ of $\phi$ over $\bar{x}$ is defined to be the set of lifts $\psi$




**Definition 2.84.** We define a connected finite étale cover $X \to S$ to be *Galois* if its $S$-automorphism group acts transitively on geometric fibres.

**Proposition 2.85** ([38] 5.3.8)**.** *Let $\phi : X \to S$ be a finite étale Galois cover. If $Z \to X$ is a connected finite étale cover fitting into a commutative diagram*

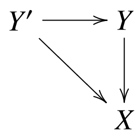



*then $f : X \to Z$ is a finite étale Galois cover, and actually $Z \simeq H/X$ with some subgroup $H$ of $G = Aut(X|S)$. In this way we get a bijection between subgroups of G and intermediate covers Z as above. The cover $q : Z \to S$ is Galois if and only if H is a normal subgroup of G, in which case $Aut(Z|S) \simeq G/H$.*

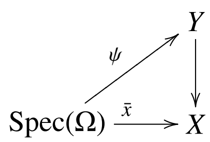
**Proposition 2.86** ([38] 5.3.9)**.** *Let $\phi : X \to S$ be a connected finite étale cover. Then there is a morphism $\pi : P \to X$ such that $\phi \circ \pi : P \to S$ is a finite Galois cover, and moreover every $S$-morphism from a Galois cover to $S$ factors through $P$.*

### 2.5.2 Fundamental group

**Definition 2.87.** Let $Fet_X$ be the category whose objects are finite etale covers $(Y \to X)$ of the scheme $X$ and the morphisms are morphisms of schemes over $X$,

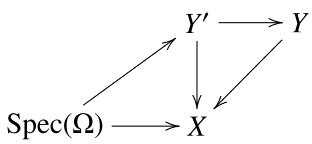


Given a morphism $\phi : Y \to X$ and a geometric point $\bar{x} : \mathrm{Spec}(\Omega) \to X$, we defined the geometric fibre $X_{\bar{x}}$ of $\phi$ over $\bar{x}$ is defined to be the set of lifts $\psi$



or equivalently it is defined to be the fibre product $Y \times_X \mathrm{Spec}(\Omega)$ induced by $\bar{x} : \mathrm{Spec}(\Omega) \to X$.

**Definition 2.88.** Let $Fet_X$ be the category of finite etale covers of the scheme $X$. Fix a geometric point $\bar{x} : \mathrm{Spec}(\Omega) \to X$. For an object $Y$ of $Fet_X$, we consider the geometric fibre $Y_{\bar{x}}$ as defined above and denote it by $Fib_{\bar{x}}(Y)$. Given a morphism $Y' \to Y$ in $Fet_X$ there is an induced morphism from $Fib_{\bar{x}}(Y') \to Fib_{\bar{x}}(Y)$ which comes from



Therefore we can define a functor $Fib_{\bar{x}}$ from the category $Fet_X$ to the category of sets, that we call the fibre functor at the geometric point $\bar{x}$.

Let us now define a morphism of functors. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F$ and $G$ be two covariant functors from $\mathcal{C}$ to $\mathcal{D}$. A morphism of functors $\phi : F \to G$ is a rule where to each object $X$ of $\mathcal{C}$ we associate a morphism $\phi_X : F(X) \to G(X)$ such that for any morphism $f : X \to Y$ between objects of $\mathcal{C}$, the following diagram is commutative.

$$\begin{array}{ccc} F(X) & \xrightarrow{\phi_X} & G(X) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\phi_Y]{} & G(Y) \end{array}$$

An automorphism of a functor $F$ is a morphism of functors with a two sided inverse.

**Definition 2.89.** Given a scheme $X$ and a geometric point $\bar{x} : \mathrm{Spec}(\Omega) \to X$ we define the *algebraic fundamental group* $\pi_1(X, \bar{x})$ as the automorphism group of the fibre functor $Fib_{\bar{x}}$ on $Fet_X$.

*Remark.* The category $Fet_X$ forms an inverse system and we may view it as an indexing set for the image of the fibre functor. As $Y$ ranges over all finite etale coverings of $X$, the fibre sets $Fib_{\bar{x}}(Y)$ over the geometric point $\bar{x}$ form a projective system of finite sets. Given an automorphism $\sigma \in Aut(Fib_{\bar{x}})$ and a morphism $f : Y' \to Y$ the following diagram is commutative

$$\begin{array}{ccc} Fib_{\bar{x}}(Y') & \xrightarrow{\sigma} & Fib_{\bar{x}}(Y') \\ {\scriptstyle Fib_{\bar{x}}(f)}\downarrow & & \downarrow{\scriptstyle Fib_{\bar{x}}(f)} \\ Fib_{\bar{x}}(Y) & \xrightarrow[\sigma]{} & Fib_{\bar{x}}(Y) \end{array}$$

So an automorphism of the fibre functor is the group of compatible system of (auto)morphisms $\{Fib_{\bar{x}}(Y) \to Fib_{\bar{x}}(Y)\}$ indexed by the elements of the category $Fet_X$.

### 2.5.3 Properties of the algebraic fundamental group

The algebraic fundamental group is a profinite group

We define a profinite group as an inverse limit of an inverse system of finite groups with discrete topology. For a prime number $p$, a *pro* $-$ $p$ group is an inverse limit of finite $p$-groups.

**Theorem 2.90** (Grothendieck, [38] 5.4.2)**.** *Let $S$ be a connected scheme, and $\bar{s} : \mathrm{Spec}(\Omega) \to S$ a geometric point.*

1. *The group $\pi_1(S, \bar{s})$ is profinite, and its action on $Fib_{\bar{s}}(X)$ is continuous for every $X$ in $Fet_S$.*

2. *The functor $Fib_{\bar{s}}$ induces an equivalence of $Fet_S$ with the category of finite continuous left $\pi_1(S, \bar{s})$-sets. Here connected covers correspond to sets with transitive $\pi_1(S, \bar{s})$-action, and Galois covers to finite quotients of $\pi_1(S, \bar{s})$.*

[Szamuely]

**Definition 2.91.** Let $X$ be a connected scheme with geometric point $\bar{x}$. The $p$-part of the fundamental group $\pi_1(X, \bar{x})$ is defined as the inverse limit of finite quotients of $p$-power order.

### Functoriality w. r. t. base point preserving morphisms

This property follows quite naturally from the definition of the fundamental group using base points. Let $S$ and $S'$ be connected schemes equipped with geometric points $\bar{s} : \mathrm{Spec}(\Omega) \to S$ and $\bar{s'} : \mathrm{Spec}(\Omega) \to S'$. Let $\phi : S' \to S$ with $\phi \circ \bar{s'} = \bar{s}$. Then $\phi$ induces a functor $Fet_S$ to $Fet_{S'}$ by mapping objects $X$ to the base change $X \times_S S'$.

Here we observe that the universal property of fibre products gives a set bijection between $Fib_{\bar{s}}(X)$ and $Fib_{\overline{(s')}}(X \times_S S')$. Each lift $\psi' : \mathrm{Spec}\,\Omega \to X \times_S S'$ of $\bar{s}' : \mathrm{Spec}\,\Omega \to S'$ corresponds to a lift $\psi : \mathrm{Spec}\,\Omega \to X$, since

$$\begin{array}{ccccc} & \overset{\psi'}{\nearrow} & X \times_S S' & \longrightarrow & X \\ & & \downarrow & & \downarrow \\ \mathrm{Spec}(\Omega) & \longrightarrow & S' & \longrightarrow & S. \end{array}$$

Conversely given a lift

$$\begin{array}{ccc} & \nearrow & X \\ & & \downarrow \\ \mathrm{Spec}(\Omega) & \longrightarrow & S \end{array}$$

and the geometric point $\bar{s} : \mathrm{Spec}(\Omega) \to S'$, the universal property of fibre products gives a lift

$$\begin{array}{ccc} & \nearrow & X \times_S S' \\ & & \downarrow \\ \mathrm{Spec}(\Omega) & \longrightarrow & S' \end{array}$$

Further we observe that given a set-automorphism from $Fib_{\overline{(s')}}(X \times_S S') \to Fib_{\overline{(s')}}(X \times_S S')$ we get an set-automorphism $Fib_{\bar{s}}(X) \to Fib_{\bar{s}}(X)$. The definition of the functor induced by $\phi$ from $Fet_S$ to $Fet_{S'}$ causes the bijection to be functorial in X. Therefore we have a map

$$\phi_* : \pi_1(S', \bar{s}') \to \pi_1(S, \bar{s})$$

It is a continuous homomorphism of profinite groups.

**Proposition 2.92** ([38] 5.5.4)**.** *Given the homomorphism $\phi_*$ as above,*

*(a) The map $\phi_*$ is injective if and only if for every connected finite étale cover $X' \to S'$ there exists a finite étale cover $X \to S$ and a morphism $X_i \to X'$ over $S'$, where $X_i$ is a connected component of $X \times_S S'$.*

*In particular, if every connected finite étale cover $X' \to S'$ is of the form $X \times_S S' \to S'$ for a finite étale cover $X \to S$, then $\phi$ is injective.*

*(b) The map $\phi_*$ is surjective if and only if for every connected finite étale cover $X \to S$ the base change $X \times_S S'$ is connected as well.*

### Exactness of the fundamental sequence

**Theorem 2.93** ([38] 5.6.1)**.** *Let X be a quasi-compact and geometrically connected scheme over a perfect field k, and let $\bar{x}$ be a geometric point of $\bar{X}$. Then the sequence of profinite groups*

$$1 \to \pi_1(\bar{X}, \bar{x}) \to \pi_1(X, \bar{x}) \to Gal(k) \to 1$$

*induced by the maps $\bar{X} \to X$ and $X \to \mathrm{Spec}(k)$ is exact.*

[Szamuely, pg 149]

The short exact sequence is considered to be an analogue to the fibre exact sequence of homotopy groups. That is $X \to \mathrm{Spec}\, k$ is a family, and $\bar{X}$ is a fibre above $\mathrm{Spec}\, \bar{K} \to \mathrm{Spec}\, K$.

**Theorem 2.94** ([38] 5.6.4)**.** *Let $f : X \to S$ be a proper separable morphism of connected schemes, with Y locally Noetherian. Let $x \to X$ be a geometric point, with image $s \to S$. Then* *the following sequence of homomorphisms of groups is exact:*

$$\pi_1(\bar{X}_s, x) \to \pi_1(X, x) \to \pi_1(S, s) \to 1$$

If $f$ is not separable, then the fibres may not be reduced, i.e. there may be multiple fibres...

## 2.6 Vector bundles

In this section we examine vector bundles and locally free sheaves on schemes. There is a one-to-one correspondence between vector bundles and locally free sheaves. We will further define stable and semi-stable bundles and Harder Narsimhan filtration.

### 2.6.1 Vector bundles and locally free sheaves

**Definition 2.95.** A vector bundle of rank $r$ over a curve $C$ is a variety $E$ together with a morphism $\pi : E \to C$ such that there exists an open affine covering $U_i$ of $C$ and isomorphisms

$$\phi_i : (\pi)^{-1}(U_i) \to U_i \times \mathbb{A}^r$$

where $\mathbb{A}^r$ denotes the affine space of dimension $r$ and such that in the intersections $U_i \cap U_j$, the composition

$$\phi_j(\phi_i)^{-1}|_{U_i \cap U_j} = (Id, \phi_{i,j}), \phi_{i,j} \in GL(r)$$

is given by linear maps.

**Theorem 2.96** ([6] 1.7)**.** *There is a natural one to one correspondence between vector bundles and locally free sheaves over a curve.*

*Note.* As a result of this theorem, we will interchangebly use the terms vector bundles and locally free sheaves.

**Definition 2.97.** The determinant bundle of a vector bundle $E$ of rank $r$ (or a locally free sheaf of rank $r$), is defined as the $r^{th}$ - wedge product of the bundle or the sheaf.

**Definition 2.98.** Let $X$ be a projective scheme over a field $k$, and let $\mathcal{F}$ be a coherent sheaf on $X$. We define the *Euler characteristic* of $\mathcal{F}$ by

$$\chi(\mathcal{F}) = \sum(-1)^i \dim_k H^i(X, \mathcal{F}).$$

**Definition 2.99.** The degree of a vector bundle $E$ of rank $r$ on a curve $C$ of genus $g$ is defined as the degree of the associated locally free sheaf which is in turn defined as

$$\deg(\mathcal{E}) = \chi(\mathcal{E}) - r\chi(\mathcal{O}_C) = \chi(E) - r(1-g)$$

The degree of a locally free sheaf can also be defined as the degree of the determinant bundle.

**Lemma 2.100** ([6] 1.16)**.** *If $\mathcal{E}_1$, $\mathcal{E}_2$ are locally free sheaves of ranks $r_1$, $r_2$, then $deg(\mathcal{E}_1 \otimes \mathcal{E}_2) = r_1 deg(\mathcal{E}_2) + r_2 deg(\mathcal{E}_1)$.*

**Lemma 2.101** ([6] 1.18)**.** *If $\mathcal{E}$ is a locally free sheaf on a curve $C$, there exists a positive divisor $D$ on $C$ such that $\mathcal{E}(D)$ is generated by global sections.*

### 2.6.2 Semi-stable bundles and Harder-Narasimhan filtration

**Definition 2.102.** The slope of a vector bundle or a locally free sheaf $E$ on a curve $C$ is defined as $\mu(E) = \frac{\deg E}{\operatorname{rank} E}$.

**Definition 2.103.** A locally free sheaf or bundle $E$ is said to be semi-stable if for every locally free subsheaf or subbundle $E'$ of $E$, the inequality

$$\mu(E') \leq \mu(E)$$

holds (and it is called stable if the above inequality is strict).

*Remark.* It is sufficient in this definition to look at subbundles $E' \subset E$, i.e. locally free subsheaves such that the quotient $E/E'$ is also locally free.

**Lemma 2.104** ([6] 3.4)**.** *Let $C$ be a smooth projective curve. Then the following hold.*

*(a) If $\mathcal{E}$ is semi-stable bundle of negative degree then $\mathcal{E}$ has no global sections.*

*(b) If $\mathcal{E}$ is semi-stable then so is its dual $\mathcal{E}^\vee$.*

*(c) If $\mathcal{E}$ is semi-stable then so is $\mathcal{E} \otimes \mathcal{L}$, for every invertible sheaf $\mathcal{L}$*

The next theorem is perhaps the most important result of this section, and crucial to for the proof of our main theorem.

**Theorem 2.105** ([26] 5.4.2). *Every locally free sheaf $\mathcal{E}$ has a unique Harder-Narasimhan filtration. This is a filtration of locally free subbundles*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_n = \mathcal{E}$$

*such that*

*(a)* $\mathcal{E}_i/\mathcal{E}_{i-1}$ *is semi-stable for every* $i = 1, \ldots n$*, and*

*(b) the slopes* $\mu_i = \mu(\mathcal{E}_i/\mathcal{E}_{i-1})$ *form a decreasing sequence*

$$\mu_1 > \ldots > \mu_n.$$

# Chapter 3

# A technique for studying fibered surfaces

In [12], Xiao developed a new and useful method to study *complex* fibered algebraic surfaces. This chapter is devoted to understanding the techniques used in the above paper with a view towards extending the result to characteristic $p$.

We assume that $S$ is a smooth, projective, minimal surface of general type, with a fibration $f : S \to C$ onto a smooth curve $C$ over $\mathbb{C}$. Therefore $f$ has connected fibres. We assume that $f$ is not isotrivial. Let $F$ be a general fibre of $f$, which is a smooth curve of genus $g \geq 2$. We also assume that $f$ is relatively minimal, namely that there is no $-1$-curve contained in the fibres of $f$. The inclusion $F \to S$ induces the following natural exact sequence of groups,

$$\pi_1(F) \xrightarrow{\alpha} \pi_1(S) \to \Pi \to 1.$$

where $\pi_1$ is the algebraic fundamental group and $\Pi$ is determined by $\pi_1(C)$ and the multiple fibres of $f$. We can now state Xiao's main theorem:

**Theorem** ([12], Theorem 1)**.** *In the above situation, suppose*

$$K_S^2 < 4\chi(O_S) + 4(g(C) - 1)(g - 1)$$

*and $f$ is non-hyperelliptic. Then the image of $\alpha$ is trivial.*

## 3.1 Definitions and technical results

In this section we will review results in Xiao's paper that are required in the proof of the main theorem. These are essentially properties of the relative dualising sheaf, and some numerical results.

### 3.1.1 Properties of the relative dualising sheaf

Recall the relative dualizing sheaf $\omega_{S/C}$, which is the sheaf associated to the relative canonical divisor $K_{S/C} := K_S - f^*K_C$. All the Harder-Narasimhan slopes of $f_*\omega_{S/C}$ are non-negative. This semi-positivity theorem was proved by Xiao in an earlier paper [11].

**Theorem 3.1** ([11] Theorem 1.1)**.** *Let $S$ be a surface that is smooth and projective over $\mathbb{C}$,*

$$f : S \to C$$

*a fibration. Then the direct image of the relative dualizing sheaf, $f_*\omega_{S/C}$, is locally free of rank equal to the genus $g$ of a general fibre of $f$, and for all locally free quotients $\mathcal{E}$ of $f_*\omega_{S/C}$, we have* $\deg(\mathcal{E}) \geq 0$*. In particular* $\deg(f_*\omega_{S/C}) \geq 0$.

**Proposition 3.2** ([12])**.** *Let $f$ be a fibration as in the theorem above. We then have:*

$$\deg f_*\omega_{S/C} = \chi(\mathcal{O}_S) - (g(C) - 1)(g - 1).$$

*Proof.* By Serre duality we have $(f_*\omega_{S/C})^\vee \cong R^1 f_*\mathcal{O}_S$, hence

$$\deg(f_*\omega_{S/C}) = -\deg(R^1 f_*\mathcal{O}_S).$$

$$
\begin{aligned}
\deg(R^1 f_* \mathcal{O}_S) &= \chi(R^1 f_* \mathcal{O}_S) - \operatorname{rank}(R^1 f_* \mathcal{O}_S)\chi(\mathcal{O}_C), \quad [6]\ 1.14 \\
&= \chi(R^1 f_* \mathcal{O}_S) - g(1 - g(C)) \\
\deg(f_* \omega_{S/C}) &= -\chi(R^1 f_* \mathcal{O}_S) - g(g(C) - 1) \\
&= -(h^0(C, R^1 f_* \mathcal{O}_S) - h^1(C, R^1 f_* \mathcal{O}_S)) - (g(C) - 1) \\
&\quad + (g(C) - 1) - g(g(C) - 1) \\
&= h^1(C, R^1 f_* \mathcal{O}_S) - h^0(C, R^1 f_* \mathcal{O}_S) - g(C) + 1 - (g - 1)(g(C) - 1) \\
&= h^1(C, R^1 f_* \mathcal{O}_S) - h^0(C, R^1 f_* \mathcal{O}_S) - h^1(C, f_* \mathcal{O}_S) + h^0(S, \mathcal{O}_S) \\
&\quad - (g - 1)(g(C) - 1) \\
&= h^2(S, \mathcal{O}_S) - h^1(S, \mathcal{O}_S) + h^0(S, \mathcal{O}_S) - (g - 1)(g(C) - 1) \\
&= \chi(\mathcal{O}_S) - (g - 1)(g(C) - 1)
\end{aligned}
$$

□

We are particularly interested in the properties of the push-forward of the relative dualising sheaf twisted by a non-vertical torsion element in $\operatorname{Pic}(S)$. If $\eta$ is a torsion element in $\operatorname{Pic}(S)$ such that $\eta^i|_F$ is not trivial whenever $\eta^i$ is not trivial, we call $\eta$ a non-vertical torsion.

**Lemma 3.3** ([12]). *Let $\eta$ be a non-trivial non-vertical torsion in* $\operatorname{Pic}(S)$. *Then $f_*(\omega_{S/C} \otimes \eta)$ is also locally free and has rank $g - 1$ and degree equal to $f_* \omega_{S/C}$. It also has non-negative quotients.*

*Proof.* Let $\eta$ be a $n$-torsion element in $\operatorname{Pic}(S)$ such that $\eta^i|_F$ is not trivial whenever $\eta^i$ is not trivial. Then $\eta$ corresponds to an étale cover $\psi : \tilde{S} \to S$ (Theorem 2.25). By Stein Factorization (Theorem 2.45) $f$ induces a fibration,

$$\tilde{f} : \tilde{S} \to \tilde{C},$$

where $\tilde{C} = \mathbf{Spec}((f \circ \psi)_* \mathcal{O}_{\tilde{S}})$. But

$$(f \circ \psi)_* \mathcal{O}_{\tilde{S}} = f_*(\oplus_{i=0}^{n-1} \eta^i) = \oplus_{i=0}^{n-1} f_* \eta^i$$

Since $\eta$ is locally free of rank one, flat over $C$, and $H^0(F, \eta|_F) = 0$, by Grauert's theorem ([18] III 12.9), we have $f_*\eta^i = 0, \forall 1 \leq i < n$. We have also assumed that $f_*\mathcal{O}_S = \mathcal{O}_C$, therefore $\tilde{C} = \mathbf{Spec}(\oplus_{i=0}^{n-1} f_*\eta^i) = \mathbf{Spec}(\mathcal{O}_C) = C$. We then have

$$\tilde{f}_*\omega_{\tilde{S}/C} = f_*(\oplus_{i=1}^{n}(\omega_{S/C} \otimes \eta^{\otimes i})) = \oplus_{i=1}^{n} f_*(\omega_{S/C} \otimes \eta^{\otimes i}).$$

Since $f_*(\omega_{S/C} \otimes \eta)$ is a direct summand of a locally free sheaf, it is locally free.

**Claim:.** rank $f_*(\omega_{S/C} \otimes \eta) = g - 1$.

We have,

$$\begin{aligned} \operatorname{rank} f_*(\omega_{S/C} \otimes \eta) &= \dim H^0(F, (\omega_{S/C} \otimes \eta)|_F), \quad \text{-Grauert's Theorem} \\ &= H^0((\eta^\vee)|_F) - \deg((\eta^\vee)|_F) + g - 1 \quad \text{-Riemann Roch} \\ &= g - 1 \end{aligned}$$

**Claim:.** $\deg f_*(\omega_{S/C} \otimes \eta) = \deg f_*(\omega_{S/C})$.

We first show that $\chi(S, \omega_{S/C} \otimes \eta) = \chi(C, f_*\omega_{S/C} \otimes \eta)$ and then use Riemann Roch to prove the claim.

$$\begin{aligned} (R^1 f_*\omega_{S/C} \otimes \eta)^\vee &= \mathcal{H}om_{\mathcal{O}_C}(R^1 f_*\omega_{S/C} \otimes \eta, \mathcal{O}_C) \\ &= f_*(\mathcal{H}om_{\mathcal{O}_S}(\omega_{S/C} \otimes \eta, \omega_{S/C})), \quad \text{definition of dualising sheaf, [27] 6.4.18} \\ &= f_*\eta^\vee, \quad \text{since the canonical map is locally an isomorphism} \\ &= 0, \quad \text{since } f_*\eta = 0 \text{ and } \eta \text{ is locally free.} \end{aligned}$$

We thus have $R^i f_*(\omega_{S/C} \otimes \eta) = 0$ for all $i > 0$. Therefore

$$H^i(S, \omega_{S/C} \otimes \eta) \cong H^i(C, f_*\omega_{S/C} \otimes \eta)$$

and hence

$$\chi(S, \omega_{S/C} \otimes \eta) = \chi(C, f_*\omega_{S/C} \otimes \eta).$$

Applying Riemann Roch to the locally free sheaf $f_*(\omega_{S/C} \otimes \eta)$ on $C$, we have

$$\chi(f_*(\omega_{S/C} \otimes \eta)) = \deg(f_*(\omega_{S/C} \otimes \eta) + (g-1)(1-g(C)).$$

Now applying Riemann Roch on $S$, we get

$$\chi(\omega_{S/C} \otimes \eta) = \frac{1}{2}D.(D-K_S) + \chi(O_S),$$

where $D$ has class $\omega_{S/C} \otimes \eta$. Therefore we get,

$$\deg(f_*(\omega_{S/C} \otimes \eta)) = \chi(O_S) + (g-1)(g(C)-1) + \frac{1}{2}D.(D-K_S).$$

By Lemma 3.2 we are done if we show that

$$\frac{1}{2}D.(D-K_S) = -2(g-1)(g(C)-1).$$

Take $D = K_{S/C} + \eta = K_S - f^*K_C + \eta$, by abusing notation and using $\eta$ to represent it, in its equivalence class of divisors, we get,

$$\begin{aligned}\frac{1}{2}D.(D-K_S) &= \frac{1}{2}(K_{S/C}+\eta)(K_S - f^*K_C + \eta - K_S)\\ &= \frac{1}{2}(K_{S/C}+\eta)(\eta - f^*K_C)\\ &= \frac{1}{2}(-K_{S/C}\cdot f^*K_C + K_{S/C}\cdot\eta - f^*K_C\cdot\eta + \eta\cdot\eta)\\ &= -2(g-1)(g(C)-1),\end{aligned}$$

since the degree of $\eta$ restricted to any curve is zero and the degree of $K_C$ and is $2g(C)-2$ and the degree of the restriction of $K_{S/C}$ to a fibre $F$ is $2g-2$.

**Claim:.** *$f_*(\omega_{S/C} \otimes \eta)$ has non-negative quotients.*

Finally we note that $f_*(\omega_{S/C} \otimes \eta)$ is a direct summand of $\tilde{f}_*(\omega_{\tilde{S}/C})$. Then since $\tilde{f}_*(\omega_{\tilde{S}/C})$ has non-negative quotients, it implies that $f_*(\omega_{S/C} \otimes \eta)$ must also have non-negative quotients. □

### 3.1.2 Slope of a fibration

The condition $K_S^2 < 4\chi(\mathcal{O}_S) + 4(g(C) - 1)(g - 1)$ in Theorem 3, can be reinterpreted using the notion of a slope of a fibration. We define the slope as follows: assume $f$ is non-isotrivial.

**Definition 3.4.** Let $f : S \to C$ be a fibration with the above properties. Let $\Delta(f) = \deg(f_*\omega_{S/C})$ and let $K_{S/C} \equiv K_S - f^*K_C$ be a relative canonical divisor. We then define the slope of the fibration to be

$$\lambda(f) = \frac{K_{S/C}^2}{\Delta(f)}.$$

*Note.* The definition is well defined since $\deg f_*\omega_{S/C} > 0$ by [4] Theorem III.17.3.

We also have,

$$\begin{aligned}
K_{S/C}^2 &= K_{S/C} \cdot K_{S/C} \\
&= (K_S - f^*K_C) \cdot (K_S - f^*K_C) \\
&= K_S^2 - 2K_S \cdot f^*K_C + (f^*K_C)^2 \\
&= K_S^2 - 2\deg(K_C)K_S \cdot F \\
&= K_S^2 - 8(g(C) - 1)(g - 1) \\
\Delta(f) &= \chi(\mathcal{O}_S) - (g(C) - 1)(g - 1)
\end{aligned}$$

Therefore the condition $K_S^2 < 4\chi(\mathcal{O}_S) + 4(g(C) - 1)(g - 1)$ is equivalent to

$$\lambda(f) < 4$$

**Definition 3.5.** Let $\tilde{S}$ be another surface, and $\psi : \tilde{S} \to S$ be a surjective morphism. Then $\psi$ is called *admissible* if $\omega_{\tilde{S}/\tilde{C}} = \psi^*\omega_{S/C}$ , where $\tilde{S} \to \tilde{C}$ is the morphism induced by Stein factorisation.

**Lemma 3.6** ([12])**.** *Let $\psi : \tilde{S} \to S$ be an admissible morphism, then $\lambda(\tilde{f}) = \lambda(f)$.*

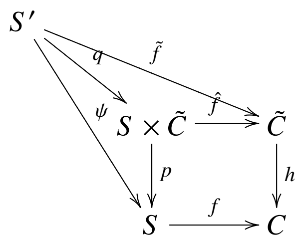
*Proof.*

**Claim:.** $K^2_{\tilde{S}/\tilde{C}} = \deg(\psi) K_{S/C} \cdot K_{S/C}$.

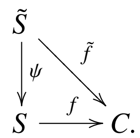
$$\begin{aligned} K^2_{\tilde{S}/\tilde{C}} &= \psi^* K_{S/C} \cdot \psi^* K_{S/C} \\ &= \psi_*(\psi^* K_{S/C}.K_{S/C}), \quad \text{Projection formula} \\ &= [K(\tilde{S}) : K(S)] K_{S/C} \cdot K_{S/C} \\ &= \deg(\psi) K_{S/C} \cdot K_{S/C} \end{aligned}$$

**Claim:.** $\deg(\tilde{f}_* \omega_{\tilde{S}/\tilde{C}}) = \deg(\psi) \deg(f_* \omega_{S/C})$.

Let $S'$ be the relatively minimal resolution of the singularities of $S \times_C \tilde{C}$.




The result is true for the morphism $S' \to S$, by Lemma 3 in [39]. We may therefore assume that $\tilde{C} = C$, and




The generalized Riemann-Roch theorem of Grothendieck ([18], Appendix A,Theorem 5.3) gives,

$$\mathrm{ch}(\sum (-1)^i R^i f_* \omega_{S/C}) = f_*(\mathrm{ch}(\omega_{S/C}) \cdot \mathrm{td}(\mathcal{T}_f))$$

where $\mathcal{T}_f = \ker(\mathcal{T}_S \to \mathcal{T}_C)$ is the relative tangent sheaf of $f$.

Since $R^1 f_* \omega_{S/C} = \mathcal{O}_C$, we have

$$\deg\left(\mathrm{ch}(\sum (-1)^i R^i f_* \omega_{S/C})\right)_1 = \deg(f_* \omega_{S/C}).$$

On the other hand,

$$
\begin{aligned}
\tilde{f}_*(\mathrm{ch}(\omega_{\tilde{S}/\tilde{C}}) \cdot \mathrm{td}(\mathcal{T}_{\tilde{f}})) &= f_* \circ \psi_*(\mathrm{ch}(\psi^*\omega_{S/C}) \cdot \mathrm{td}(\mathcal{T}_{\tilde{f}})) \\
&= f_* \circ \psi_*(\psi^*\mathrm{ch}(\omega_{S/C}) \cdot \mathrm{td}(\mathcal{T}_{\tilde{f}})) \\
&= f_*(\mathrm{ch}(\omega_{S/C}) \cdot \psi_*\mathrm{td}(\mathcal{T}_{\tilde{f}})), \text{ Projection Formula} \\
&= f_*(\mathrm{ch}(\omega_{S/C}) \cdot \psi_*\mathrm{td}(\psi^*\mathcal{T}_f)) \\
&= f_*(\mathrm{ch}(\omega_{S/C}) \cdot \psi_*(\psi^*(\mathrm{td}(\mathcal{T}_f))) \\
&= f_*(\mathrm{ch}(\omega_{S/C}) \cdot \deg(\psi)\mathrm{td}(\mathcal{T}_f)) \\
&= \deg(\psi) f_*(\mathrm{ch}(\omega_{S/C}) \cdot \mathrm{td}(\mathcal{T}_f))
\end{aligned}
$$

□

Let $\tilde{S}$ be another surface, and $\psi : \tilde{S} \to S$ be a surjective morphism as above.

**Lemma 3.7** ([12]). *Let $S$ be a fibered surface $f : S \to C$. Let $\tilde{S} \to S$ be a surjective morphism and $\tilde{C} \to C$ the morphism induced by Stein Factorisation. We assume one of the following is true.*

*(a) $\tilde{C} = C$ and $\psi$ is unramified.*

*(b) The natural morphism $\tilde{C} \to C$ is unramified on images of non-semistable fibres of $f$, and $\tilde{S}$ equals the minimal desingularisation of the pull-back of $f$ by $\psi$.*

*Then $\psi$ is admissible, i.e. $\omega_{\tilde{S}/\tilde{C}} = \psi^*\omega_{S/C}$.*

*Proof.* (a) By the adjunction formula, [27],Theorem 6.4.9, we have

$$\omega_{\tilde{S}/C} = \omega_{\tilde{S}/S} \otimes \psi^*\omega_{S/C}$$

and

$$\omega_{\tilde{S}/S} = \omega_{\tilde{S}} \otimes (\psi^*\omega_S)^{-1}.$$

Since $\psi$ is unramified, $\omega_{\tilde{S}} = \psi^*\omega_S$, which implies $\omega_{\tilde{S}/S} = \mathcal{O}_{\tilde{S}}$ and hence the result.

(b) Let $\hat{S} = S \times_C \tilde{C}$. Then since $\psi$ is flat, by [27],Theorem 6.4.9,

$$\omega_{\hat{S}/\tilde{C}} = \psi^* \omega_{S/C}.$$

By 2.44, the singularities of $\hat{S}$ only occur when a relative singular point of a semi-stable fibre maps to a point on $C$ which ramifies for $\psi : \tilde{C} \to C$. Then $\psi$ is admissible by [39], Lemma 3.b.

□

### 3.1.3 Fixed and moving parts

**Definition 3.8.** Let $D$ be a divisor on $S$ such that $\mathcal{E} = f_* O_S(D)$ is a locally free sheaf on $C$ and $\mathcal{H}$ is a subbundle of $\mathcal{E}$. We define the fixed and moving parts of $\mathcal{H}$ in the following manner: Choose a sufficiently ample sheaf $\mathcal{A}$, so that $\mathcal{H}_1 = \mathcal{H} \otimes \mathcal{A}$ is generated by global sections. Then the inclusion $\mathcal{H} \subset \mathcal{E}$ induces the homorphisms

$$f^*\mathcal{H}_1 \to f^*((f_* O_S(D)) \otimes *A) \to O_S(D) \otimes f^* A.$$

Let $\mathfrak{D}$ be the linear subsystem of $|D + f^*\mathcal{A}|$ corresponding to the global sections of $\mathcal{H}_1$. We then define the fixed part of $\mathcal{H}$, $Z(\mathcal{H})$, to be the fixed part of the linear system $\mathfrak{D}$ on $S$. We also define the moving part of $\mathcal{H}$, $M(\mathcal{H}) = D - Z(\mathcal{H})$, and $N(\mathcal{H}) = D - Z(\mathcal{H}) - \mu_f(\mathcal{H})F$, where $\mu_f(\mathcal{H})$ is final Harder-Narasimhan slope of $\mathcal{H}$.

*Remark.* We note that the fixed part of $\mathcal{H}$ is independent of the choice of $\mathcal{A}$. $x$ is base point of $\mathfrak{D}$, if all its global sections vanish at $x$. However the behaviour of these global sections at $x$, are determined by the local sections of $\mathcal{H}_1$ at $f(x)$. Since $\mathcal{A}$ is a very ample sheaf, and $f$ is flat, the sections of $f^*\mathcal{A}$ do not contribute any base points.

**Definition 3.9.** On a fibre $F'$, the restriction of $\mathcal{H}$ to $F'$ corresponds to a sublinear system, $\mathfrak{D}|_{F'}$, of dimension equal to the rank of $\mathcal{H}$. This system consists of all divisors $D'|_{F'}$, where $D' \in \mathfrak{D}$ is a divisor whose support does not contain $F'$.

We can define the fixed part of the restriction of $\mathcal{H}$ to $F'$ as the fixed part of the linear system $\mathfrak{D}|_{F'}$, and the moving part of $\mathcal{H}$ on $F'$, as $D|_{F'} - Z(\mathcal{H}|_{F'})$.

**Lemma 3.10.** *The restriction of $Z(\mathcal{H})$ and $M(\mathcal{H})$ to $F'$ is just the fixed and moving part of sublinear system corresponding to the restriction of $\mathcal{H}$ to $F'$.*

*Proof.* If $Z = Z(\mathcal{H})$ is the fixed part of $\mathcal{H}$, then $Z \leq D'$ for all $D' \in \mathfrak{D}$. By the definition of fixed part of a linear system, $Z|_{F'} \leq Z(H|_{F'})$, as the fixed part is the unique largest effective divisor contained in every divisor of the linear system.

On the other hand if $p \in \mathrm{Supp}(Z(H|_{F'}))$, then $p \in \mathrm{Supp}(F')$ and $p \in \mathrm{Supp}(D')$ for all $D' \in \mathfrak{d}$ that do not contain $F'$. This implies that $p \in \mathrm{Supp}(D')$ for all $D' \in \mathfrak{D}$. Hence $p$ is a base point of $\mathfrak{D}$. Then either $p \in \mathrm{Supp}(Z)$ or $p$ is a point in the zero-dimension part of the base locus of $\mathfrak{D}$.

In the latter case as there are only finitely many isolated base points, we can always choose a fibre $F'$ which does not meet any of the isolated base points of $\mathfrak{D}$. □

**Definition 3.11.** We call a finite base extension $\psi : \tilde{C} \to C$ *good* if it is unramified on images of singular fibres of $f$.

**Lemma 3.12.** *Let $\psi : \tilde{C} \to C$ be a good base extension. We denote the pull-back of an object by $\psi$ by writing $\tilde{}$ over the object, except $\tilde{F}$ means a general fibre of $\tilde{f}$. Then*

*(a) $\tilde{S} = S \times_C \tilde{C}$ is smooth.*

*(b) $R^i \tilde{f}_*(O_{\tilde{S}}(\tilde{D})) \cong R^i \widetilde{f_*(O_S(D))}$. In particular, if $\mathcal{E} = f_* O_S(D)$ then $\tilde{E} = \tilde{f}_* O_S(\tilde{D})$.*

*(c) Let $\mathcal{H}$ be a sheaf on $C$, then the pull-back of the fixed and moving parts of $\mathcal{H}$ is just the fixed and moving parts of $\tilde{\mathcal{H}}$.*

*(d) Furthermore since the pull-back of a semi-stable vector bundle is semi-stable and by uniqueness of the Harder filtration, the pull-back of the Harder filtration of $\mathcal{E}$ is the Harder filtration of $\tilde{\mathcal{E}}$, with*

$$\mu(\tilde{E}_i/\tilde{E}_{i-1}) = \mu(E_i/E_{i-1}) \cdot \deg \psi.$$

*In particular, $\mu_f(\tilde{E}) = \mu(E) \cdot \deg \psi$.*

*(e) $N(\mathcal{H})$ commutes with good base extensions.*

*Proof.* (a) Shown in lemma 2.44

(b) Since $\psi$ is flat, it follows from [18], III.9.3.

(c) By (b), and viewing $\tilde{H}$ as a subsheaf of $\tilde{E}$, we get the result.

(d) Let $\mathcal{F}$ be a locally free sheaf on $C$. Then $\mu(\tilde{\mathcal{F}}) = \deg(\psi)\mu(\mathcal{F})$, since

$$\deg(\tilde{\mathcal{F}}) = \deg(\det(\psi^*\mathcal{F})) = \deg(\psi^*(\det\mathcal{F})) = \deg(\psi)\deg(\mathcal{F}),$$

and for a finite separable morphism between two smooth curves

$$\text{rank}(\tilde{\mathcal{F}}) = \text{rank}(\mathcal{F}).$$

Therefore

$$\mu(\tilde{E}_i/\tilde{E}_{i-1}) = \mu(E_i/E_{i-1}) \cdot \deg\psi,$$

and in particular, $\mu_f(\tilde{E}) = \mu(E) \cdot \deg\psi$.

Further since $\psi$ is a separable, finite morphism, the pull back of the Harder-Narsimhan filtration of $E$ is the Harder-Narsimhan filtration of $\tilde{E}$, (Y. Miyaoka, Chern Classes and Kodaira Dimension 3.2.) i.e. if

$$0 \subset \mathcal{E}_1 \subset ... \subset \mathcal{E}_n = \mathcal{E}$$

is the Harder-filtration of $\mathcal{E}$ then

$$0 \subset \tilde{\mathcal{E}}_1 \subset ... \subset \tilde{\mathcal{E}}_n = \tilde{\mathcal{E}}$$

is the Harder-Narsimhan filtration of $\tilde{\mathcal{E}}$.

(e)

$$\begin{aligned} N(\tilde{\mathcal{H}}) &= \tilde{D} - Z(\tilde{\mathcal{H}}) - \mu_f(\tilde{\mathcal{H}})\tilde{F} \\ &= \tilde{D} - \widetilde{Z(\mathcal{H})} - \deg(\psi)\mu_f(\mathcal{H})\tilde{F} \\ &= \widetilde{N(\mathcal{H})} \end{aligned}$$

□

**Lemma 3.13** ([12]). *Let $D$, $\mathcal{E}$, $\mathcal{H}$ be defined as above. Then $N(\mathcal{H})$ is a nef $\mathbb{Q}$ divisor.*

The proof of this lemma relies quite heavily on the following result. Xiao has implicitly proved it in theorem 1, in [11]. I will state it as a lemma here and reproduce the proof.

**Lemma 3.14** ([11]). *Let $f : S \to C$ be a fibration as above. Let $\mathcal{E}$ be a locally free sheaf of positive degree on $C$. Then modulo a good base extension, $\mathcal{E}$ has a global section.*

*Proof.* The theorem is proved in the following three steps:

(i) *If rank($\mathcal{E}$) = 1, then modulo a good base extension, $h^0(\mathcal{E}) \neq 0$.*

Since $\deg(\mathcal{E}) > 0$ and $\mathcal{E}$ is ample, there exists an integer $n >> 0$ such that the linear system $|\mathcal{E}'^{\otimes n}|$ contains a reduced divisor $D$. $D$ corresponds to an injection $|\mathcal{E}^{\otimes n}| \to O_C$, which defines a ring structure on the sheaf

$$O_C \oplus \mathcal{E}^{\otimes -1} \oplus \mathcal{E}^{\otimes -2} \oplus ... \oplus \mathcal{E}^{\otimes -n+1}$$

By hypothesis, $\mathrm{Spec}(O_C \oplus ... \oplus \mathcal{E}^{\otimes -n+1})$ is a smooth curve $\tilde{C}$, with a cyclic cover $\pi : \tilde{C} \to C$ of degree $n$ which is ramified along $D$. Let $\tilde{f} : \tilde{S} \to \tilde{C}$ be the pull-back of $f$ by $\pi$. We can choose $D$ such that the fibres of $f$ above $D$ are smooth. In this case $\tilde{S}$ is a smooth surface, and $\tilde{\mathcal{E}} = \pi^*\mathcal{E}$ is an invertible sheaf. But by construction, we have

$$h^0(\tilde{\mathcal{E}}) \neq 0.$$

(ii) *Let $\mathcal{E}$ be a locally free sheaf of rank 2 over $\mathbb{C}$, and $\deg(\mathcal{E}) > 0$. There exists a finite covering $\pi : \tilde{C} \to C$ which is etale over the images of the singular fibres of $f$, such that the pull-back $\tilde{\mathcal{E}}$ of $\mathcal{E}$ has an invertible subsheaf of positive degree. Then we are done by case(i).*

Let $P = \mathbb{P}(\mathcal{E})$, and

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

a filtration of $\mathcal{E}$ such that the degree of $\mathcal{E}_1$ is maximal. $P$ is a regular surface over $C$ having a section $C_0$ corresponding to the filtration of $\mathcal{E}$ such that $C_0^2 = \deg(\mathcal{E}_2) - \deg(\mathcal{E}_1)$. We can assume

$$\deg(\mathcal{E}_1) \leq 0 < \deg(\mathcal{E}_2).$$

Let $a, b$ be two positive integers such that

$$-\deg(\mathcal{E}_1) < \frac{b}{a} < \frac{1}{2}C_0^2 \tag{1}$$

By the Nakai Criterion on the amplitude ([18][1,V.2.21]), the divisor $aC_0 - bF$in $P$ is ample, where $F$ is a fibre of $P$ over $C$. Hence for an integer $n >> 0$, there is a smooth and irreducible divisor $D$ in the system $|naC_o - nbF|$. Let $\pi : \tilde{C} \to C$ be a Galois cover which is the factorisation due to the projection of $D$ on $C$. (Since $\deg(\pi) = kn$, $k \in \mathbb{N}^+$). We can choose $D$ and $\pi$ such that over the ramified points of $\pi$ over $C$, the fibres of $f$ are smooth.

Let $\tilde{\mathcal{E}} = \mathcal{E} \times_C \tilde{C}$, $\tilde{P} = P \times_C \tilde{C}$, with $\pi : \tilde{P} \to P$ the cover induced by $\pi$. We have $\tilde{P} = \mathbb{P}(\tilde{\mathcal{E}})$. It is well known that $\pi^*(D)$ is composed of the sections $C_1, ..., C_{na}$ over $\tilde{C}$ which are in the same numerical class of $\mathrm{Pic}(\tilde{P})$.

Since $(\pi^* D)^2 = knaD^2$, we have

$$C_1^2 = (\frac{1}{na}\pi^* D)^2 = \frac{k}{na}D^2.$$

If

$$0 \to \tilde{\mathcal{E}_1} \to \tilde{\mathcal{E}} \to \tilde{\mathcal{E}_2} \to 0$$

is the filtration of $\tilde{\mathcal{E}}$ corresponding to the section $C_1$, we have

$$\begin{aligned}\deg(\tilde{\mathcal{E}_2}) - \deg(\tilde{\mathcal{E}_1}) = C_1^2 &= \frac{k}{na}(n^2a^2C_0^2 - 2n^2ab) \\ &= kn(a.\deg(\mathcal{E}_2) - a.\deg(\mathcal{E}_1) - 2b).\end{aligned}$$

On the other hand,

$$\begin{aligned}\deg(\tilde{\mathcal{E}_2}) + \deg(\tilde{\mathcal{E}_1}) = \deg(\tilde{\mathcal{E}}) &= kna.\deg(\mathcal{E}) \\ &= kna(\deg(\mathcal{E}_1) + \deg(\mathcal{E}_2)).\end{aligned}$$

which gives

$$\deg(\tilde{\mathcal{E}_1}) = kn(a.\deg(\mathcal{E}_1) + b) > 0, \qquad \text{(by (1))}$$

Thus $\tilde{\mathcal{E}_1}$ is the subsheaf we are looking for.

(iii) *Let $\mathcal{E}$ be a locally free sheaf of positive degree over $C$. There exists a finite covering $\pi : \tilde{C} \to C$ étale above the images of the singular fibres of $f$, such that the pull-back $\tilde{\mathcal{E}}$ of $\mathcal{E}$ has an invertible subsheaf of positive degree.*

We will use recursion on the rank of $\mathcal{E}$. Let $\operatorname{rank}\mathcal{E} \geq 3$.

Let $\lambda(\mathcal{E}) = \deg(\mathcal{E})/\operatorname{rank}(\mathcal{E})$ the slope of $\mathcal{E}$. Modulo a good base change, we may assume that $\lambda(\mathcal{E})$ is an integer. Let $\alpha$ be a real number defined in the following manner:

$$\alpha = \operatorname{Sup}\begin{cases} \frac{\deg(\tilde{\mathcal{E}_1})}{\lambda(\tilde{\mathcal{E}})}; \tilde{\mathcal{E}} \text{ pull-back of } \mathcal{E} \text{ by a suitable base change,} \\ \tilde{\mathcal{E}_1} \text{ an invertible subsheaf of } \tilde{\mathcal{E}} \end{cases}$$

It suffices to find a contradiction to the assumption that $\alpha \leq 0$.

Let $\alpha_1$ be the largest integer strictly less that $\alpha$, $\alpha_2 = \alpha_1 + 1$ (then $\alpha_2 \geq \alpha$). Modulo a suitable base change, we can suppose that $\mathcal{E}$ has an invertible subsheaf $\mathcal{E}_1$ such that $\deg(\mathcal{E}_1) > \alpha_1\lambda(\mathcal{E})$. By hypothesis $\deg(\mathcal{E}_1) \leq 0 < \lambda(\mathcal{E})$, then $\lambda(\frac{\mathcal{E}}{\mathcal{E}_1}) > \lambda(\mathcal{E})$. Since $\operatorname{rank}(\frac{\mathcal{E}}{\mathcal{E}_1}) = \operatorname{rank}(\mathcal{E}) - 1$, by the hypothesis applied to $\frac{\mathcal{E}}{\mathcal{E}_1} \otimes \mathcal{L}$, where $\mathcal{L}$ is an invertible sheaf of degree $-\lambda(\mathcal{E})$, we can assume, after a suitable base change, that $\frac{\mathcal{E}}{\mathcal{E}_1}$ has an invertible sub-fibre $\mathcal{E}_2$ with $\deg(\mathcal{E}_2) > \lambda(\mathcal{E})$. Let $\mathcal{E}'$ be the inverse image of $\mathcal{E}_2$ in $\mathcal{E}$. We have a filtration

$$0 \to \mathcal{E}_1 \to \mathcal{E}' \to \mathcal{E}_2 \to 0$$

then

$$\deg(\mathcal{E}') = \deg(\mathcal{E}_1) + \deg(E_2) > (\alpha_1 + 1)\lambda(\mathcal{E}) = \alpha_2\lambda(\mathcal{E})$$

Now by case (ii) applied to the sheaf $\mathcal{E}' \otimes \mathcal{L}'$ where $\mathcal{L}'$ is invertible with $\deg(\mathcal{L}') = -\frac{1}{2}\alpha_2\lambda(E)$, we find a suitable base change $\pi : \tilde{C} \to C$ such that $\pi^*(\mathcal{E}')$ has an invertible subsheaf $\tilde{\mathcal{E}}_1'$ with $\deg(\tilde{\mathcal{E}}_1') > \frac{1}{2}\alpha_2\lambda(\tilde{\mathcal{E}}) \geq \alpha\lambda(\tilde{\mathcal{E}})$, where $\tilde{\mathcal{E}} = \pi^*(\mathcal{E})$, contradicting the definition of $\alpha$.

□

We now return to the proof of the lemma that $N(\mathcal{H})$ is nef.

*Proof for nefness of $N(\mathcal{H})$:* We show that $N(\mathcal{H})$ is nef by viewing it modulo good base extensions as a limit of nef and effective divisors. We will show that $N(\mathcal{H}) + \epsilon F$ is a nef $\mathbb{Q}$-divisor modulo good base extensions, for all rational numbers $r$. Note that given a good base extension $\psi : \tilde{C} \to C$ of degree $d$, $N(\mathcal{H}) + \epsilon F$ is nef if $N(\tilde{\mathcal{H}}) + \epsilon d\tilde{F}$ is nef. In this proof ˜will always denote the pull-back, except in the case $\tilde{F}$ which will be a general fibre of $\tilde{f}$.

Let $\epsilon > 0$. Consider the sheaf $\mathcal{H}' = \mathcal{H} \otimes O_C((\epsilon - \mu_f(\mathcal{H}))p)$ We may assume that modulo a good base extension, $\mu_f(\mathcal{H}) - \epsilon$ is an integer. Also

$$\mu_f(\mathcal{H}') = \mu_f(h) + \mu_f(O_C(\epsilon p)) + \mu_f(O_C(-\mu_f(\mathcal{H}))) = \epsilon > 0,$$

which implies that $\deg \mathcal{H}' > 0$. Hence by the above lemma, we may assume that $\mathcal{H}'$ has a global section.

Let $B$ be the image of this global section in $|N(\mathcal{H}) + \epsilon F|$. We want to show $B$ is nef modulo a good base extension. To this end, it is sufficient to show that for every component $A$ of $B$, there exists a good base extension $\psi : \tilde{C} \to C$ such that $H^0(\mathcal{H}')$ has another global section $B'$ such that it does not intersect $A$. The case where $A$ is contained in a fibre of $f$ has trivial self-intersection, and so we may assume that $A$ is not contained in a fibre. We may also assume that the inverse image of $A$ under a good base extension is irreducible, as we can always split the components of $B$ by base extensions till they cannot be split any further.

Consider the subsheaf $\mathcal{G}$ of $\mathcal{H}'$ which consists of all local sections, whose divisors in the moving part contain $A$. Then for every open set $U$ in $C$,

$$\mathcal{G}(U) = \{s \in \mathcal{H}'(U) | (f^*s)_A \in \mathfrak{m}_A \mathcal{L}_A\},$$

where $\mathcal{L} = O_S(N(\mathcal{H}) + \epsilon F)$ and $A$ denotes the generic point of $A$ in $S$. Since the moving part has no fixed part, there must exist local sections of $\mathcal{H}'$ which do not contain $A$, and hence $\mathcal{G}$ must be a proper subsheaf of $\mathcal{H}'$.

Now $B'$ contains $A$ if and only if the corresponding section in $H^0(\tilde{\mathcal{H}}')$ lies in $\tilde{\mathcal{G}}$. Therefore the lemma is reduced to showing that:

For any proper subbundle $\mathcal{G}$ of $\mathcal{H}'$, there is a good base extension $\psi : \tilde{C} \to C$ such that $H^0(\tilde{\mathcal{H}}')$ contains a section of positive degree not lying in $\tilde{G}$.

We show this by the above lemma with a minor modification. In fact, the cases of $\mathrm{rank}(\mathcal{H}') = 1$ or 2 go without significant change. For the general case, we first suppose that there is a section $s$ of positive degree in $H^0(\mathcal{H}')$, using the above lemma. If $s$ does not lie in $\mathcal{G}$ we are done; otherwise $s$ generates a subbundle $\mathcal{G}_1$ of $\mathcal{G}$. Then the image of $\mathcal{G}$ in $\mathcal{H}'/\mathcal{G}_1$ is a proper subbundle, hence by induction hypothesis, we get an invertible subbundle $\mathcal{H}_1$ of positive degree in $\mathcal{H}'/\mathcal{G}_1$ (modulo base extensions) which is not contained in the image og $\mathcal{G}$. Now use the proof of the rank 2 case to the inverse image of $\mathcal{H}_1$ in $\mathcal{H}'$, and the lemma is shown. □

### 3.1.4 Technical lemmas

We need two additional technical lemmas for the main theorem

**Lemma 3.15** ([12]). *Let $f : S \to C$ as before, with a general fibre $F$. Let $D$ be a divisor on $S$, and suppose that there is a sequence of effective divisors*

$$Z_1 \geq Z_2 \geq ... \geq Z_n \geq Z_{n+1} = 0$$

*and a sequence of rational numbers*

$$\mu_1 > \mu_2 > ... > \mu_n > \mu_{n+1} = 0$$

*such that for every $i$, $N_1 = D - Z_i - \mu_i F$ is a nef $\mathbb{Q}$-divisor. Then*

$$D^2 \geq \sum_{i=1}^{n} (d_i + d_{i+1}(\mu_i - \mu_{i+1}),$$

*where $d_i = N_i F$.*

*Proof.* We have $N_i^2 \geq 0$ since by assumption the $N_i$ are nef $\mathbb{Q}$-divisors. For $i \geq 2$,

$$
\begin{aligned}
N_i^2 &= N_i(D - Z_i - \mu_i F) \\
&= N_i(N_{i-1} + Z_{i-1} + \mu_{i-1}F - Z_i - \mu_i F) \\
&= N_i(N_{i-1} + (Z_{i-1} - Z_i) + (\mu_{i-1} - \mu_i)F) \\
&\geq N_i(N_{i-1} + (\mu_{i-1} - \mu_i)F), \text{ since } Z_{i-1} \geq Z_i \text{ and } N_i \text{ is nef} \\
&\geq (N_{i-1} + (Z_{i-1} - Z_i) + (\mu_{i-1} - \mu_i)F)(N_{i-1} + (\mu_{i-1} - \mu_i)F) \\
&\geq N_{i-1}^2 + (Z_{i-1} - Z_i)N_{i-1} + (\mu_{i-1} - \mu_i)N_{i-1}F + (\mu_{i-1} - \mu_i)N_{i-1}F + (\mu_{i-1} - \mu_i)(Z_{i-1} - Z_i)F + (\mu_{i-1} - \mu_i)^2 \\
&\geq N_{i-1}^2 + 2(\mu_{i-1} - \mu_i)N_{i-1}F + (\mu_{i-1} - \mu_i)(Z_{i-1} - Z_i)F \\
&\geq N_{i-1}^2 + 2(\mu_{i-1} - \mu_i)(2N_{i-1}F + (Z_{i-1} - Z_i)F \\
&\geq N_{i-1}^2 + (d_{i-1} + d_i)(\mu_{i-1} - \mu_i)
\end{aligned}
$$

□

**Lemma 3.16** ([12]). *Let $D_1$, $D_2$ be two numerically equivalent divisors on $S$ such that $N_i = D_i - Z_i - \mu_i F$, $i = 1, 2$, is nef, where $\mu_1, \mu_2$ are rational numbers with $\mu_1 \geq \mu_2$, and $Z_1, Z_2$ are effective divisors. Let $Z$ be the common part of $Z_1$ and $Z_2$ (i.e., $Y_i = Z_i - Z$) are two effecive divisors without common component). Then*

$$N = D_2 - Z - \mu_2 F$$

*is nef.*

*Proof.* For any curve $B$ on $S$, there is a $Y_i$ which does not contain $B$. Hence

$$NB = N_i B + Y_i B + (\mu_i - \mu_2)FB \geq N_i B \geq 0$$

Furthermore,

$$N^2 + (N_2 + Y_2)^2 = N_2^2 + N_2 Y_2 + NY_2 \geq N_2^2 \geq 0$$

□

## 3.2 Deconstruction of the main theorem

Let $p \in C$ such that $f(F) = p$ and $k(p)$ be the residue field of $p$. We have the following fibre product.

$$\begin{array}{ccc} F & \longrightarrow & \mathrm{Spec}(k(p)) \\ \downarrow & & \downarrow \\ S & \xrightarrow{f} & C) \end{array}$$

The injection of $F$ into $S$ induces the following natural exact sequence of groups

$$\pi_1(F) \overset{\alpha}{\to} \pi_1(S) \to \pi \to 1$$

The image of $\alpha$, is a normal subgroup of $\pi_1(S)$. The quotient $\pi_1(S)/\mathrm{Im}(\alpha)$ is a quotient of $\pi_1(C-\Delta)$ (see [13], Section 2), where $\Delta$ is the subset of $C$ composed of images of multiple fibre of $f$; when $f$ has no multiple fibres, $\pi_1(S)/\mathrm{Im}(\alpha)$ is canonically isomorphic to $\pi_1(C)$.

$$\begin{array}{ccccc} & & & & \pi_1(C-\Delta) \\ & & & \nearrow & \downarrow \\ \pi_1(F) & \xrightarrow{\alpha} & \pi_(S) & \longrightarrow & \Pi \\ & & & & \downarrow \\ & & & & \pi_1(C) \end{array}$$

Given an étale Galois cover $\psi : \tilde{S} \to S$, $f$ induces a fibration $\tilde{f} : \tilde{S} \to \tilde{C}$, with an induced Galois cover $\psi : \tilde{C} \to C$, by Stein Factorisation. Now if $\hat{S}$ is the minimal desingularization of the normalization of $S \times_C \tilde{C}$, $\psi$ factors into two steps:

$$\tilde{S} \to \hat{S} \to S$$

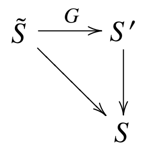
where the composing maps are both étale and Galois. The first map is determined by the quotient group of $\pi_1(F)$ related to the restriction of $\psi$ on $f$ and the second is the morphism arising from the universal properties of desingularization and normalization.

The statement of the theorem is as follows.

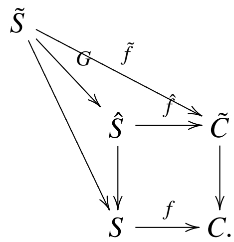
**Theorem 3.17.** *If $f : S \to C$ is a non-locally trivial (relatively minimal) fibration with $\lambda < 4$, Im($\alpha$) is trivial when $f$ is non-hyperelliptic.*

*Proof.* In his proof, Xiao first translates the question to the existence of non-vertical torsion elements in $\mathrm{Pic}(S)$ in the following manner.

Since $\alpha$ is a continuous map between profinite groups, $\mathrm{Im}(\alpha)$ is a closed and normal subgroup of $\pi_1(S)$. If $\mathrm{Im}(\alpha)$ has a finite quotient $G \cong \mathrm{Im}(\alpha)/K$, then by basic properties of profinite groups (proposition 2.1.4, [35]), there exists open normal subgroups of $\pi_1(S)$ of the form $\mathrm{Im}(\alpha)U$ and $KU$. These correspond to finite étale covers of $S$ which corresponds to the following commutative diagram between finite étale maps.




Now by Stein Factorisation, the morphism $\tilde{S} \to S$ can be factored into $\psi \circ \tilde{f}$ such that $\tilde{f} : \tilde{S} \to \tilde{C}$ is a fibration and $\psi : \tilde{C} \to C$ is an étale morphism. If $\hat{S}$ is the minimal desingularization of $S \times_C \tilde{C}$, then we can assume that the morphism $\tilde{S} \to \hat{S}$ is the one corresponding to $G$ and we get the following commutative diagram,




Since $\lambda(\hat{f}) \leq \lambda(f) < 4$, we can replace $f$ with $\hat{f}$. If $G$ has a cyclic quotient, then we can take the corresponding étale cover $\bar{S} \to S$ as a non-vertical torsion. Else we are reduced to the case where $G$ is non-cylic simple group. In this case we set up a non-vertical torsion in $\mathrm{Pic}(S)$ by taking $\hat{S}$ to be the quotient of $S$ by a cyclic subgroup $H$ of odd order of $G$. We have reduced to the case that if $\mathrm{Im}(\alpha)$ is non-trivial then $f$ is a fibration of slope $< 4$ having a non-vertical torsion $\eta$ of odd order.

We will now see that this is impossible using the lemmas proved above. Let $D_\eta$ be an effective divisor on $S$ having class $\omega_{S/C} \otimes \eta$ and $\mathcal{E} = f_*\omega_{S/C} \otimes \eta$ have a Harder filtration

$$0 = \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_n = \mathcal{E}$$

Take

$$r_i = \mathrm{rank}(\mathcal{E}_i)$$

$$\mu_i = \mu_f(\mathcal{E}_i) = \mu(\mathcal{E}_i/\mathcal{E}_{i=1})$$

$$Z_i = Z(\mathcal{E}_i)$$

$$N_i = N(\mathcal{E}_i)$$

$$d_i = N_i F$$

Let also $N_{n+1} = D_\eta$, $r_{n+1} = g-1$, $Z_{n+1} = 0$, $\mu_{n+1} = 0$, $d_{n+1} = D_\eta F$. $N_{n+1}$ is a nef divisor, by [5]. Then by lemma 3.13 and semi-positivity of the slopes of the Harder-Narasimhan filtration, we get the sequences

$$Z_1 \geq Z_2 \geq Z_3 \geq \ldots \geq Z_{n+1}$$

$$\mu_1 > \mu_2 > \mu_3 > \ldots > \mu_{n+1}.$$

These satisfy the condition of lemma 3.15, which gives

$$K^2_{S/C} = D^2_\eta \geq \sum_{i=1}^{n} (d_i + d_{i+1})(\mu_i - \mu_{i+1}) \tag{3.1}$$

On the other hand,

$$\Delta(f) = \deg \mathcal{E} = \sum_{i=1}^{n} r_i(\mu_i - \mu_{i+1}) \tag{3.2}$$

Combining 3.1 and 3.2, we see that we will have $\lambda \geq 4$ if

$$d_i + d_{i+1} \geq 4r_i \tag{3.3}$$

Now $d_i$ is defined to be the degree of $N_iF$. Since $N_i$ is nef, we may assume $N_iF$ is effective and also $N_iF \leq N_{n+1}F = D_\eta F = 2g-2$ hence $N_iF$ is a special divisor provided $d_i \neq 0$. Further the dimension of the restriction of the linear system $N_i$ to $F$ is $r_i - 1$. Now Clifford's Lemma states that, if $D$ is an effective special divisor on a curve $X$, then

$$\dim|D| \leq \frac{1}{2} \deg D$$

with equality if and only if either $D = 0$ or $D = K_F$ when $X$ is non-hyperelliptic. Since $N_nF \neq 0$ or $N_nF \neq K_F$,

$$r_i - 1 < \frac{1}{2}d_i$$

$$2r_i - 2 < d_i$$

$$2r_i - 1 \leq d_i$$

Thus we have $d_i \geq 2r_i - 1$, $i = 1, ..., n$, except if $d_1 = 0$, (and $r_1 = 1$). We therefore only need to handle the exceptional case $d_1 = 0, r_1 = 1$. We require condition 3.3 to be satisfied for all $i$ to prove the theorem. For $2 \leq i \leq n-1$ we have

$$\begin{aligned} d_i + d_{i+1} &\geq 2r_i - 1 + 2r_{i+1} - 1 \\ &\geq 2r_i + 2(r_i + 1) - 2 \\ &\geq 4r_i \end{aligned}$$

implying condition 3.3 is satisfied.

However the cases $i = 1$ and $i = n$ are more delicate. In the case $i = 1$, $d_1 + d_2 \geq 0 + 2r_2 - 1$. If $r_2 = 2$ then $d_1 + d_2 \geq 3$. Therefore if $d_2 = 3$ condition 3.3 will not be satisfied. In the case $i = n$, we have $d_n \geq 2r_n - 1 = 2(g-1) - 1 = 2g - 3$ and $d_{n+1} = 2g - 2$, therefore

$$\begin{aligned} d_n + d_{n+1} &\geq 2(g-1) - 1 + 2g - 2 \\ &\geq 4(g-1) - 1. \end{aligned}$$

Thus if $d_n = 2g - 3$ again condition 3.3 is not satisfied. Hence the theorem is proved except if $d_n = 2g - 3$ or $d_1 = 0, d_2 = 3$ (hence $r_1 = 1, r_2 = 2$).

We will first try to modify $N_n$ so that $d_n > 2g - 3$. Now $d_n$ is the degree of the linear system $|N_n|_F|$ and $d_n = N_nF = M(f_*\omega_{S/C} \otimes \eta)F = 2g - 3$. $N_n = D_\eta - Z(f_*\omega_{S/C} \otimes \eta) - \mu_nF$ and $Z(f_*\omega_{S/C} \otimes \eta)$ is the fixed part of $|D_\eta|$. We have defined the restriction of $Z(f_*\omega_{S/C} \otimes \eta)$ to $F$ to be the fixed part of the linear system $\mathcal{E} = f_*\omega_{S/C} \otimes \eta$ restricted to $F$, which will be the fixed part of the linear system $|D_\eta|$ restricted to $F$. Therefore

$$N_n|_F = M(\mathcal{E})|_F = D_\eta|_F - Z(\mathcal{E})|_F.$$

We may assume that for every torsion $\eta$, $d_n = 2g - 3$ or equivalently the linear system $|D_\eta|_F|$ has a base point $p_\eta$. We fix an $\eta$ and let $p = p_\eta$. Let $D$ be a divisor in $|M(\mathcal{E})|_F|$.

$$\begin{aligned} 2g - 3 &= \deg(D) \\ &= \deg(D_\eta|_F - Z(\mathcal{E})|F) \\ &= \deg(K_F) + \deg(\eta|_F - Z(\mathcal{E})|_F) \\ &= 2g - 2 - \deg(Z(\mathcal{E})|_F) \end{aligned}$$

Therefore $\deg(Z(\mathcal{E})|_F) = 1$ which implies $Z(\mathcal{E})|_F = p$, since we may assume $Z(\mathcal{E})|_F$ is effective. Applying Riemann Roch to the divisor $D$, we get

$$\begin{aligned} h^0(F, \mathcal{O}_F(D)) - h^0(F, \mathcal{L}(K_F - D)) &= \deg D + 1 - g \\ g - 1 - h^0(F, \mathcal{O}_F(K_F - D)) &= 2g - 3 + 1 - g \\ h^0(F,)_F(K_F - D)) &= g - 1 - 2g + 3 - 1 + g \\ h^0(F, \mathcal{O}_F(K_F - D)) &= 1 \end{aligned}$$

Now $K_F - D$ is linearly equivalent to $-\eta|_F + p$ (we abuse notation to denote a divisor in the class $\eta|_F$ by $\eta|_F$ itself) . Since $h^0(F, \mathcal{O}_F(K_F - D)) = 1$, the complete linear system $| - \eta|_F + p|$ is nonempty. Hence $| - \eta|_F + p|$ is linearly equivalent to some effective divisor $q$ of degree 1. Therefore $q$ is a point of $F$ such that $\eta|_F = \mathcal{O}_F(p - q)$. Further note that $\eta$ cannot have order two. If it does then $2p \equiv 2q$. However $|2q|$ is a linear system of degree 2 and dimension 1. This contradicts the hypothesis that $f$ is non-hyperelliptic, ([18] IV.5). Consequently $\eta|_F^2 = \mathcal{O}_F(p' - q')$ with $(p' \neq q')$. Note also that $p \neq p'$ else $2p - 2q \equiv p - q'$, or $2q \equiv p + q'$, again contradicting the condition that $F$ is non-hyperelliptic. We now assume that

$$\mu_f(f_* \omega_{S/C} \otimes \eta) \leq \mu_f(f_* \omega_{S/C} \otimes \eta^2)$$

Let $Z'_1 = Z(f_*\omega_{S/C} \otimes \eta)$, $Z'_2 = Z(f_*\omega_{S/C} \otimes \eta^2)$. Since $Z'_i F = 1$, there is a unique section $C_i$ in $Z'_i$ such that $C_1 \cap F = \{p\}$, $C_2 \cap F = \{p'\}$, in particular $C_1 \neq C_2$. Now we can apply Lemma 3.16 and let $N_n = D_\eta - Z - \mu_n F$, where $Z$ is the common part of $Z'_1$ and $Z'_2$. Then $d_n = 2g - 2$.

Next we consider the case $d_1 = 0$, $d_2 = 3$. We want to show

$$K^2_{S/C} \geq 4\Delta(f).$$

We will do this by considering two cases; $d_3 = 5$ and $d_3 \geq 6$. First we consider the case $d_3 = 5$. Since $r_3 = 3$ we have $3 = r_3 < r_n = g - 1$, hence $g \geq 5$. The linear system $|N_3|_F|$ is a $g^2_5$ and hence defines a map $\phi$ of $F$ onto $\mathbb{P}^2$. The image of $F$ is a plane curve $B$ which is not contained in any hyperplane. The degree of $B$ is equal to $B.H$ where $H$ is a divisor in the class $\mathcal{O}(1)$. However $\deg(\phi^*\mathcal{O}(1)) = 5$ which implies $B.H = 5$. Hence $B$ is a curve of degree 5. $|N_3|_F|$ has a sublinear system $|N_2|_F|$ which is a $g^1_3$. This implies that $B$ has a double point. Then by a Riemann-Hurwitz type formula for singular curves ([14]), we have $g \leq 5$. Therefore $g = 5$, and $n = 4$ and $d_4 = 8$. Now if $\mu_1 - \mu_2 \leq \mu_3 - \mu_4$, 3.1 and 3.2 give

$$\begin{aligned} K^2_{S/C} &\geq 3(\mu_1 - \mu_2) + 8(\mu_2 - \mu_3) + 13(\mu_3 - \mu_4) + 16\mu_4 \\ &\geq 4(\mu_1 - \mu_2) + 8(\mu_2 - \mu_3) + 12(\mu_3 - \mu_4) + 16\mu_4 \\ &\geq 4\Delta(f) \end{aligned}$$

On the other hand if $\mu_1 - \mu_2 \geq \mu_3 - \mu_4$, we use Lemma 3.15 on the sequence $\{Z_1, Z_4, 0\}$,$\{\mu_1, \mu_4, 0\}$ to get

$$K^2_{S/C} \geq 8(\mu_1 - \mu_4) + 16\mu_4 \geq 4\Delta(f)$$

Now we consider the case when $d_3 \geq 6$. We now have two possibilities: if $\mu_1 - \mu_2 \leq \mu_2 - \mu_3$, then by 3.1 and 3.2,

$$\begin{aligned} K^2_{S/C} &\geq 3(\mu_1 - \mu_2) + 9(\mu_2 - \mu_3) + \sum_{i=3}^{n}(d_i + d_{i+1})(\mu_i - \mu_{i+1}) \\ &\geq 4(\mu_1 - \mu_2) + 8(\mu_2 - \mu_3) + 4\sum_{i=3}^{n}(\mu_i - \mu_{i+1}) \\ &= 4\Delta(f) \end{aligned}$$

otherwise use Lemma 3.15 on the sequences

$$\{Z_1, Z_3, Z_4, ..., Z_n, 0\}, \{\mu_1, \mu_3, \mu_4, ..., \mu_n, 0\},$$

to get

$$\begin{aligned} K^2_{S/C} &\geq 6(\mu_1 - \mu_3) + \sum_{i=3}^{n}(d_i + d_{i+1})(\mu_i - \mu_{i+1}) \\ &\geq 4(\mu_1 - \mu_2) + 8(\mu_2 - \mu_3) + 4\sum_{i=3}^{n}(\mu_i - \mu_{i+1}) \\ &= 4\Delta(f) \end{aligned}$$

□

## Chapter 4

# Fibered surfaces in characteristic $p$.

## 4.1 Introduction

In this chapter, we prove our main theorem which extends Xiao Gang's technique for studying fibered algebraic surfaces to characteristic $p$. We have described Xiao Gang's technique over the complex numbers in detail in the last chapter. For the rest of the section we will assume that the base field $k$ is an algebraically closed field of positive characteristic $p$, and $l$ is a prime not equal to $p$. We assume $f : S \to C$ over $k$, is a fibration with $F$ a general fibre of genus $g \geq 2$. Given $f : S \to C$ and the induced exact sequence

$$\pi_1(F) \to \pi_1(S) \to \Pi(C) \to 1,$$

is it possible to say something about the image of the homomorphism $\pi_1(F) \to \pi_1(S)$?

## 4.2 Hiccups in characteristic $p$

We examine the classical results of the previous section that fail either mildly or spectacularly in characteristic $p$. As we attempt to deal with these pathologies, we are able to formulate an answer to the above question.

### 4.2.1 Semi-positivity

A pivotal result in the classical case is the semi-positivity result of the relative dualising sheaf. It implies that all the Harder-Narasimhan slopes of $f_*\omega_{X/C}$ are of non-negative degree.

**Theorem 4.1** ([11], Theorem1.1)**.** *If $f : X \to C$ is a fibration of a proper smooth surface to a proper smooth curve over $\mathbb{C}$, then all the quotient bundles of $f_*\omega_{X/C}$ are of non-negative degree.*

Moret-Bailly [29], p. 137 showed that this result is false in positive characteristic. In a counterexample, he constructed a non-isotrivial semi-stable fibration of fiber genus 2, $\pi : X \to \mathbb{P}^1$ such that

$$R^1\pi_*O_X = O_C(-p) \oplus O_C(1).$$

The failure of the semi-positivity theorem is the first problem in extending Xiao's theorem to positive characteristic. However a recent result on semi-positivity by Jang suggests the possibility of overcoming the problem by studying generically ordinary fibrations. His result can be stated as:

**Theorem 4.2** ([21], Theorem 1)**.** *Let k be a perfect field of positive characteristic p and X a proper smooth surface over k. Assume X admits a generically ordinary semi-stable fibration $f : X \to C$ to a smooth proper curve C over k. Then the maximal Harder-Narasimhan slope of $R^1 f_*(O_X)$ is non-positive.*

Therefore if we restrict our attention to generically ordinary semi-stable fibrations, all the Harder-Narasimhan slopes of $f_*\omega_{X/C}$ are of non-negative degree. We next examine some consequences of these new assumptions.

### 4.2.2 Generically ordinary covers

We must check whether it is reasonable to restrict our attention to generically ordinary semi-stable fibrations. For instance, are étale covers of generically ordinary semi-stable fibrations, generically ordinary?

Let $f : X \to C$ be a generically ordinary semi-stable fibration and $\psi : \tilde{S} \to S$ a finite covering. We first factor the morphism $\tilde{S} \to C$ using Stein Factorisation into a fibration $\tilde{S} \to \tilde{C}$

and a Galois cover $\tilde{C} \to C$.

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{f}} & \tilde{C} \\ \downarrow{\scriptstyle \psi} & & \downarrow \\ S & \xrightarrow{f} & C \end{array}$$

Ideally we would like the induced covering fibration $\tilde{f}$ to also be generically ordinary. However this is not always true. To see this, we factor the morphism $\tilde{S} \to S$ into two parts; the part that is determined by the restriction of the map to a fibre and, the second part that is determined its image in $C$, as follows:

$$\tilde{S} \to \hat{S} \to S,$$

where $\hat{S}$ is the minimal desingularization of the normalization of $S \times_C \tilde{C}$. Since the base change of a generically ordinary fibration is generically ordinary, the induced morphism $\hat{S} \to \tilde{C}$ is generically ordinary. Therefore the generic ordinarity of the morphism $\tilde{S} \to \tilde{C}$ depends on the restriction of $\psi$ to $F$, a general fibre of $f$. i.e.

$$\psi : \tilde{F} \to F,$$

In general $\psi : \tilde{F} \to F$ need not be an ordinary cover, even if $F$ is ordinary. It has been shown by Raynaud ([33]Theorem 2) that given a proper, smooth, connected curve of genus $g \geq 2$, it is possible to construct a finite Galois étale cover, which is not ordinary. Madore discusses this construction in [28] (Section 4, page 285).

**Theorem 4.3** ([28], Theorem 4.1)**.** *Let $k$ be an algebraically closed field of characteristiic $p > 0$, and let $X$ be the generic curve of any genus $g \geq 2$ over $k$. Then there exists a Galois cover of $X$ with solvable Galois group of order prime to $p$ that is not ordinary.*

We thus need to find a suitable condition under which coverings of generically ordinary fibrations are also generically ordinary. By the above discussion, this is equivalent to finding a suitable condition for covers of ordinary curves to be ordinary. We examine a couple of these results in turn, to determine a reasonable hypothesis for our theorem (c.f. [9], [31], [41], [33], [10]).

### $p$-covers

A result about $p$-covers states that étale covers of an ordinary curve with Galois group a $p$-group, are ordinary.

**Theorem 4.4** ([10], Corollary 1.8.3)**.** *Let $Y$ be a complete nonsingular connected curve over an algebraically closed field of characteristic $p$ and let $Y \to X$ be a finite étale galois covering of degree a power of $p$. Then $X$ is ordinary if and only if $Y$ is ordinary.*

This neat result suggests that we should perhaps restrict our attention to étale $p$-covers of $F$, which would lead to a statement involving the pro-$p$ fundamental group of the fibre $F$. However we are keen to adapt Xiao's method for the study of the fundamental group in relation to the intersection number of the relative dualising sheaf. As was seen in 3.3 and 3.17, a key element of Xiao's classical technique takes advantage of the simple structure of a cyclic covering, namely if $Y \to X$ is a cyclic étale cover, then there exists a torsion element $\mathcal{L}$ in $\mathrm{Pic}(X)$ of order $n$ such that $Y = \mathbf{Spec}(\oplus_{i=0}^{n-1}\mathcal{L}^{\otimes i})$.

Unfortunately over a field of characteristic $p > 0$, cyclic covers $\psi : Y \to X$ with Galois group a $p$-group do not have such a nice description. For example if $\deg\psi = p$ and $\psi$ is an Artin-Schreier cover that is locally given by an equation $x^p - x = a$ where $a$ is a regular function on $X$, [36]. Therefore although $p$-covers are always ordinary, they do not have the simple structure we need, and so we cannot use them to adapt Xiao's technique to characteristic $p$. This sort of degeneration occurs only when the $p$ divides the degree of the morphism.

So if $l$ is a prime that is distinct from $p$, and the degree of $\psi : Y \to X$ is equal to $l$, then there exists an invertible sheaf $\mathcal{L}$ on $X$, of order $l$, such that $Y = \mathbf{Spec}(\oplus_{i=0}^{l-1}\mathcal{L}^{\otimes i})$, [16], 3.10. This observation leads us to the study of the ordinarity $l$-cyclic covers.

### $l$-cyclic covers

Let $l$ be a prime distinct from $p$. In this section we introduce the notion of "strongly $l$-ordinary curve". We also show that "general" curves of given genus with level-$l$ structure are strongly $l$-ordinary. Let $X$ be a smooth curve of genus $g$ defined over $k$. We denote the Jacobian of $X$ by

Jac($X$). It is an abelian group scheme. Given a prime $l$ distinct from $p$, the $l$-torsion Jac($X$)[$l$] of the Jacobian is the kernel of the multiplication-by-$l$ morphism, and,

$$\mathrm{Jac}(X)[l] \cong (\mathbb{Z}/l\mathbb{Z})^{2g}, \qquad [32]\ 4.6$$

Further there is a bijection between $l$-torsion points of Jac($X$)[$l$] and unramified $\mathbb{Z}/l\mathbb{Z}$-Galois covers of $X$. Finite quotients of $\pi_1(X, \mathbb{Z}/l\mathbb{Z})$ correspond to unramified $\mathbb{Z}/l\mathbb{Z}$-Galois covers of $X$, and by ([32] 4.7), we have

$$\pi_1(X, \mathbb{Z}/l\mathbb{Z}) \cong H^1(X, \mathbb{Z}/l\mathbb{Z}) \cong \mathrm{Jac}(X)[l]$$

Let us consider the moduli space $\mathcal{M}_g[l]$ of smooth stable genus $g$ curves with full level $l$ structure ([17], pg 37-38). A full level $l$ structure on a curve $X$ of genus $g$ is a symplectic basis $\{\alpha_1, \ldots, \alpha_g, \beta'_1 \ldots, \beta_g\}$ for $H^1(X, \mathbb{Z}/l\mathbb{Z})$: here symplectic means that, in terms of basis, the intersection pairing on $H^1(X, \mathbb{Z}/l\mathbb{Z})$ has matrix of the form

$$\begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$$

This data is equivalent to the choice of a basis $(L_1, \ldots, L_{2g})$ of the space Jac($C$)[$l$] of $l$-torsion points in the Jacobian of $C$ that is symplectic with respect to the Weil pairing of the space. The rigidity provided by the level structure on $\mathcal{M}_g[l]$ makes it a fine moduli space.

**Theorem 4.5** ([1] Theorem 13.2)**.** *For $m \geq 3$ and $2g - 2 + n \geq 0$ ($n \in \mathbb{Z}_{\geq 0}$), there exists a fine moduli space scheme $\mathcal{M}_{g,n}[m]$ for smooth stable n-pointed curves with level m structure. In particular there exists a universal curve with level structure over $\mathcal{M}_{g,n}[m]$. This moduli scheme is smooth over* Spec($\mathbb{Z}[1/m]$).

It follows that $\mathcal{M}_{g,0}[l]$ (or $\mathcal{M}_g[l]$) is a fine moduli scheme when $l \geq 3$ and $g \geq 1$, and hence there exists a universal curve

$$C \to \mathcal{M}_g[l]$$

with level $l$-structure representing the functor of smooth curves of genus $g$ with such levels ([15] 1.7). Further $\mathcal{M}_g[l]$ is smooth over Spec $\mathbb{Z}[1/l]$, and for every field $k$ of characteristic not dividing $l$ the fiber $\mathcal{M}_g[l] \otimes \operatorname{Spec} k$ is a regular variety.

**Definition 4.6.** Let $X$ be an ordinary curve defined over $k$. We say that $X$ has *strong l-ordinarity* or is *strongly l-ordinary* when every connected étale cyclic covering of $Y \to X$ of degree $l$ is ordinary.

**Proposition 4.7.** *For $l \geq 3$, there exists a non-empty open set in $\mathcal{M}_g[l]$ of strongly l-ordinary curves defined over an algebraically closed field k.*

*Proof.* As we have seen above $\mathcal{M}_g[l]$ is a fine moduli space. Let $C$ be the universal curve of $\mathcal{M}_g[l]$. Then it follows that $C \to \mathcal{M}_g[l]$ is a proper smooth morphism. Let $\eta$ be the geometric generic point of $\mathcal{M}_g[l]$, and denote the algebraic curve above $\eta$ by $C_\eta$. Then $C_\eta$ is the generic curve of genus $g$ with level $l$ structure.

Let $V$ be the set of points $s \in \mathcal{M}_g[l]$ such that $C_s$ is ordinary. By [31], we know that $V$ is open, and non-empty, since the generic curve, $C_\eta$ is ordinary,

Now let $\mathcal{D}$ be the universal $l$-cover of $C$, i.e. $\mathcal{D}$ has the universal property that it factors through any degree $l$ covering of $C$. Then $\mathcal{D}$ is the pull-back of the universal $l$-cover of Jac($X$). Therefore $\mathcal{D} \to C$ is also a proper smooth morphism with Galois group isomorphic to $(\mathbb{Z}/l\mathbb{Z})^{2g}$.

Every étale abelian cover of $C_\eta$ is ordinary ([41], Theorem 3.1). In particular, the covering $\mathcal{D}_\eta \to C_\eta$ being abelian, is ordinary. Consider the proper smooth morphism $\mathcal{D} \to \mathcal{M}_g[l]$. By ([19], Pro. 1.2), the set of points in $s \in \mathcal{M}_g[l]$, whose fibres $\mathcal{D}_s$ are ordinary, is an open set $U$ in $\mathcal{M}_g[l]$. We know that $U$ is non-empty as $\eta \in U$.

Let $W$ be the intersection of $U$ and $V$. Then $W$ is a non-empty open set. Let $s \in W$. We want to show that $C_s$ is strongly $l$-ordinary. $\mathcal{D}_s$ is an ordinary curve, and factors through every étale $l$-cyclic covering $T$ of $C_s$. Since the quotient of an ordinary curve is ordinary ([9], pg 275), it follows that every such $T$ is ordinary. Therefore $C_s$ is strongly $l$-ordinary. □

Let $B$ be a smooth projective scheme over $k$, and $f : X \to B$ a stable family of genus $g$

curves. Then there exists a natural map $B \to \mathcal{M}_g$, which is uniquely determined by $f$. This map send a point $b \in B$, to the moduli point corresponding to the fibre $X_b$ of $X$ above $b$. By restricting to the smooth locus of $f$, we regard $f$ as a smooth morphism. Then there exists an étale base change $B' \to B$, such that the pull-back of the family $f' : X' \to B'$ has full level-$l$ structure. Therefore it is reasonable for us to assume that a general family of semi-stable stable curves of genus $g$, has a generic fibre that is strongly $l$-ordinary.

### 4.2.3 Theorem statement

We are now in a position to state a theorem that will extend Xiao's classical technique for studying fibered surfaces to characteristic $p$.

**Theorem 4.8.** *Let $k$ be a algebraically closed field of characteristic $p > 0$ and $l$ a prime that is distinct from $p$. Let $S$ be a smooth and projective surface over $k$. Assume that $S$ admits a generically ordinary semi-stable fibration $f : S \to C$ to a smooth and projective curve $C$ over $k$ that is not isotrivial, and that $S$ is relatively minimal with respect to $f$. We also assume that the generic fibre of $f$ is strongly $l$-ordinary. Let $F$ be a general fibre of $f$, which is a smooth curve of genus $g \geq 2$. Let $K_{S/C} = K_S - f^*K_C$ be a relative canonical divisor. We have the following commutative diagram of profinite groups,*

$$\begin{array}{ccccc} \pi_1(F) & \longrightarrow & \pi_1(S) & \longrightarrow & \Pi(C) \to 1 \\ \downarrow & & \downarrow & & \\ \pi_1^l(F) & \longrightarrow & \Pi & & \end{array}$$

*where $\pi_1$ is the étale fundamental group, $\pi_1^l$ is the pro-$l$ fundamental group and $\Pi$ is the pushout group.*

*If $K_{S/C}^2 < 4\deg(f_*\omega_{S/C})$ and $f$ is non-hyperelliptic, then the image $\pi_1^l(F) \to \Pi$ is trivial.*

We keep the above assumptions on the fibration $f : S \to C$ for the remainder of the thesis.

### 4.2.4 Definitions and technical lemmas

Now that we have a theorem statement that is very similar to Xiao's theorem in the classical case, we investigate whether the results and technical lemmas used in Xiao's technique extend to characteristic $p$. With the extra hypothesis of generic strong $l$-ordinariness, we are able to extend all the results with some modifications.

## Properties of the relative dualising sheaf

The first result that extends is the property that the direct image of the relative dualising sheaf is locally free of rank equal to the genus of a general fibre, has degree equal to $\deg f_*\omega_{S/C} = \chi(O_S) - (g(C)-1)(g-1)$ and has non-negative quotients.

**Proposition 4.9.** *Let $f : X \to S$ be a fibration, as in the statement of the theorem above. Then*

*(i)* $f_*\omega_{X/S}$ *is locally free of rank $g$ over $S$,*

*(ii)* $\deg f_*\omega_{S/C} = \chi(O_S) - (g(C)-1)(g-1)$

*(iii)* $f_*\omega_{X/S}$ *has non-negative quotients.*

*Proof.* (i) The question is local on $S$, so we can assume that $f : X \to S = \operatorname{Spec} A$, where $A$ is a finitely generated $k$-algebra. The relative dualising sheaf $\omega_{X/S}$ is an invertible sheaf that is locally free of rank 1 on $X$. Therefore it is flat over $X$, by ([18], Pro. III.9.2.e), that is, for every $x \in X$, ${\omega_{X/S}}_x$ is a flat $O_{X,x}$-module. Since $f$ is flat, for every $x \in X$, $O_{S,f(x)} \to O_{X,x}$ is a flat morphism. Therefore for every $x \in X$, ${\omega_{X/S}}_x$ is a flat $O_{S,f(x)}$-module, and hence $\omega_{S/C}$ is flat over $S$. Furthermore by ([18], II.5.20), $f_*\omega_{X/S}$ is a coherent $O_S$-module. Therefore $f_*\omega_{X/S}$ is locally free by [18], Pro. III.9.2.e.

It remains to show that the rank of $f_*\omega_{X/S}$ is $g$. We have the following pull-back square:

$$\begin{array}{ccc} X_\eta & \xrightarrow{g} & \operatorname{Spec} k(\eta) \\ \downarrow i & & \downarrow p \\ X & \xrightarrow{f} & \operatorname{Spec} A \end{array}$$

where $\eta$ is the generic point of $A$. Then the map $\operatorname{Spec} k(\eta) \to \operatorname{Spec} A$ is a flat map. Therefore by the flat base change theorem, we have $p^* f_*\omega_{X/S} \cong g_* i^*\omega_{X/S}$, or

$$H^0(S, f_*\omega_{X/S}) \otimes k(\eta) \cong H^0(X_\eta, \omega_\eta)$$

Since $H^0(X_\eta, \omega_{X_\eta}) \cong H^1(X_\eta, O_{X_\eta})$ and $\dim H^1(X_\eta, O_{X_\eta}) = g$, we conclude that the rank of $f_*\omega_{X/S}$ is $g$.

(ii) The proof that $\deg f_*\omega_{S/C} = \chi(O_S) - (g(C) - 1)(g - 1)$ can be extended from the classical case to positive characteristic without any change, see 3.2.

(iii) The proof of this statement can be found in [21], theorem 1.

□

**Lemma 4.10.** *Let $f : S \to C$ be a fibration as in the theorem statement. If $\eta$ is an $l$-order torsion element in* Pic($S$)*, where $l$ is a prime distinct from $p$, such that $\eta^i|_F$ is not trivial whenever $\eta^i$ is not trivial, then $f_*(\omega_{S/C} \otimes \eta)$ is also locally free and has rank $g - 1$ and degree equal to $f_*\omega_{S/C}$. It also has non-negative quotients.*

*Proof.* Since $p \nmid l$, we know that (2.25) $\eta$ corresponds an étale cyclic covering $\tilde{S} \to S$, where

$$\tilde{S} = \mathbf{Spec}(\oplus_{j=0}^{n-1} \eta^{-j}),$$

Then the proof that $f_*(\omega_{S/C} \otimes \eta)$ is locally free of rank $g - 1$ and has degree equal to degree of $f_*\omega_{S/C}$ can be extended from the classical case (see 3.3) to positive characteristic with no change. To see that $f_*\omega_{S/C} \otimes \eta$ also has non-negative quotients, we note that our hypothesis about a general fibre of $f$ implies that the induced fibration $\tilde{f} : \tilde{S} \to C$ is also generically ordinary, since $f$ is generically strongly $l$-ordinary. Then by [21] and following the argument in 3.3, we are done. □

## Slope of a fibration

It is possible to define the slope of a fibration $f$ in characteristic $p > 0$ just like the complex case, since Szpiro, ([39], Theorem 1), has shown that for a semi-stable and non-isotrivial fibration, the degree of $f_*\omega_{S/C}$ is strictly positive.

**Definition 4.11.** We therefore define the *slope* of the fibration $f$ as

$$\lambda(f) = \frac{K_{S/C}^2}{\deg f_*\omega_{S/C}}.$$

**Definition 4.12.** Let $f : S \to C$ be a semi-stable fibration. We then define a surjective morphism $\psi : \tilde{S} \to S$ to be an *admissible cover* if $\omega_{\tilde{S}/\tilde{C}} = \psi^* \omega_{S/C}$, where the fibration $\tilde{f} : \tilde{S} \to \tilde{C}$ and and the finite morphism $\phi : \tilde{C} \to C$ are the morphisms induced by Stein Factorisation of $\tilde{S} \to C$.

**Lemma 4.13.** *If $f : S \to C$ is a semi-stable fibration, and $\psi : \tilde{S} \to S$ is an admissible cover then, $\lambda(\tilde{f}) = \lambda(f)$, where $\tilde{f} : \tilde{S} \to \tilde{C}$ is the fibration induced by Stein Factorization.*

*Proof.* See 3.6. □

Therefore admissible covers do not alter the slope of the fibration. Admissible covers arise in the following situations. Note that since we restrict our attention to semi-stable fibrations in positive characteristic, the second condition is simpler than the classical case.

**Lemma 4.14.** *Let $f : S \to C$ be a semi-stable fibration. Let $\tilde{S} \to S$ be a surjective morphism and $\tilde{C} \to C$ the resulting finite affine morphism by Stein Factorization. Under either of the following conditions:*

*(a) $\tilde{C} = C$ and $\psi$ is unramified.*

*(b) $\tilde{S}$ equals the minimal desingularisation of the pull-back of $f$ by $\psi$,*

*$\psi$ is admissible, i.e. $\omega_{\tilde{S}/\tilde{C}} = \psi^* \omega_{S/C}$.*

*Proof.* (a) See 3.7 (a).

(b) $\psi$ is admissible by [39], Lemma 3.b.

□

### Fixed and moving parts

We now define the fixed and moving parts as in the complex case.

**Definition 4.15.** Let $D$ be a divisor on $S$ such that $\mathcal{E} = f_* \mathcal{O}_S(D)$ is a locally free sheaf on $C$ and $\mathcal{H}$ is a subbundle of $\mathcal{E}$. We define the fixed and moving parts of $\mathcal{H}$ as follows:

Choose a sufficiently ample divisor $A$, so that $\mathcal{H}_1 = \mathcal{H} \otimes A$ is generated by global sections. Then the inclusion $\mathcal{H} \subset \mathcal{E}$ induces the homorphisms

$$f^*\mathcal{H}_1 \to f^*((f_*\mathcal{O}_S(D)) \otimes A) \to \mathcal{O}_S(D) \otimes f^*A.$$

Let $\mathfrak{D}$ be the linear subsystem of $|D + f^*A|$ corresponding to the global sections of $\mathcal{H}_1$.

(i) Let $Z(\mathcal{H})$ be the fixed part of the linear system $\mathfrak{D}$ on $S$. We then define the *fixed part of* $\mathcal{H}$ to be $Z(\mathcal{H})$. This definition is independent of the choice of $A$.

(ii) We also define $M(\mathcal{H}) = D - Z(\mathcal{H})$ to be the *moving part* of $\mathcal{H}$, and

(iii) $N(\mathcal{H}) := D - Z(\mathcal{H}) - \mu_f(\mathcal{H})F$, where $\mu_f(\mathcal{H})$ is the final Harder-Narasimhan slope of $\mathcal{H}$.

**Definition 4.16.** On a fibre $F'$, the restriction of $\mathcal{H}$ to $F'$ corresponds to a sublinear system, $\mathfrak{D}|_{F'}$, of dimension equal to the rank of $\mathcal{H}$. This system consists of all divisors $D'|_{F'}$, where $D' \in \mathfrak{D}$ is a divisor whose support does not contain $F'$.

We can define the fixed part of the restriction of $\mathcal{H}$ to $F'$ as the fixed part of the linear system $\mathfrak{D}|_{F'}$, and the moving part of $\mathcal{H}$ on $F'$, as $D|_{F'} - Z(\mathfrak{D}|_{F'})$.

**Lemma 4.17.** *The restriction of $Z(\mathcal{H})$ and $M(\mathcal{H})$ to $F'$ is just the fixed and moving part of the sublinear system corresponding to the restriction of $\mathcal{H}$ to $F'$.*

*Proof.* See 3.10. □

**Definition 4.18.** We will call a finite, *separable* morphism $\pi : \tilde{C} \to C$ a *good base extension* for $f : S \to C$, if it is unramified on images of the singular fibres of $f$.

*Remark.* Note that in characteristic $p > 0$, we needed to modify the classical definition of a good base extension to include the condition that the morphism is also separable. This is required to ensure that the pull-back of the Harder-Narasimhan filtration of a sheaf on a curve $C$ is the Harder-Narasimhan of the pull-back of the sheaf to the extension $\tilde{C}$. We also note that good base extensions give rise to admissible covers. Further since ordinarity is preserved by base change (see 2.74) in general, the admissible covers resulting from good base extensions are also generically ordinary.

**Lemma 4.19.** *Let $\psi : \tilde{C} \to C$ be a good base extension. We denote the pull-back of an object by $\psi$ by writing ˜over the object, except $\tilde{F}$ means a general fibre of $\tilde{f}$. Then*

*(a)* $\tilde{S} = S \times_C \tilde{C}$ *is smooth.*

*(b)* $R^i\tilde{f}_*(O_{\tilde{S}}(\tilde{D})) \cong R^i f_*\widetilde{(O_S}(D))$. *In particular, if* $\mathcal{E} = f_* O_S(D)$ *then* $\tilde{\mathcal{E}} = \tilde{f}_* O_S(\tilde{D})$.

*(c) Let $\mathcal{H}$ be a sheaf on C, then the pull-back of the fixed and moving parts of $\mathcal{H}$ are just the fixed and moving parts of $\tilde{\mathcal{H}}$.*

*(d) Furthermore since the pull-back of a semi-stable vector bundle is semi-stable and by uniqueness of the Harder filtration, the pull-back of the Harder filtration of $\mathcal{E}$ is the Harder filtration of $\tilde{\mathcal{E}}$, with*

$$\mu(\tilde{\mathcal{E}}_i/\tilde{\mathcal{E}_{i-1}}) = \mu(\mathcal{E}_i/\mathcal{E}_{i-1}) \cdot \deg\psi.$$

*In particular,* $\mu_f(\tilde{\mathcal{E}}) = \mu(\mathcal{E}) \cdot \deg\psi$.

*(e)* $N(\mathcal{H})$ *commutes with good base extensions.*

*Proof.* See 3.12. □

### 4.2.5 Nefness of $N(\mathcal{H})$

Recall that the divisor $N(\mathcal{H})$ is formed by taking away the base locus of the linear subsystem corresponding to $\mathcal{H}$. The base locus of the linear subsytem contains the base locus of the complete linear system $|D|$. Although this does not imply that the divisor will be nef, it makes it more likely that it will be nef. We prove nefness of $N(\mathcal{H})$ by following Xiao' proof in the classical case to a large extent. We basically show that it is always possible to find a good base extension such that there are enough divisors in the moving part.

We first show that a locally free sheaf $\mathcal{E}$ over $C$ has global sections modulo a good base extension. Xiao indirectly proved it in [11], theorem 1.1, in the complex case. We cannot directly extend the proof to positive characteristic. In the second step of his proof, he needs the characteristic of the field to be zero to use Nakai's criterion to find an ample divisor on the ruled surface. We use an alternative method following [24] to find an ample divisor.

**Lemma 4.20.** *Let $f : S \to C$ be a fibration as above. Let $\mathcal{E}$ be a locally free sheaf of positive degree on $C$. Then modulo a good base extension, $\mathcal{E}$ has a global section.*

*Proof.* The theorem is proved in the following three steps:

(i) *If rank$(\mathcal{E}) = 1$, then modulo a good base extension, $h^0(\mathcal{E}) \neq 0$.*

Since $\deg(\mathcal{E}) > 0$ and $\mathcal{E}$ is ample, there exists an integer $n >> 0$, relatively prime to the characteristic of $k$, such that the linear system $|\mathcal{E}^{\otimes n}|$ contains a reduced divisor $D$. $D$ corresponds to an injection $|\mathcal{E}^{\otimes n}| \to O_C$, which defines a ring structure on the sheaf

$$O_C \oplus \mathcal{E}^{\otimes -1} \oplus \mathcal{E}^{\otimes -2} \oplus \ldots \oplus \mathcal{E}^{\otimes -n+1}$$

By hypothesis, $\mathrm{Spec}(O_C \oplus \mathcal{E} \oplus \ldots \oplus \mathcal{E}^{\otimes -n+1})$ is a smooth curve $\tilde{C}$, with a cyclic cover $\pi : \tilde{C} \to C$ of degree $n$ which is ramified along $D$. Let $\tilde{f} : \tilde{S} \to \tilde{C}$ be the pull-back of $f$ by $\pi$. We can choose $D$ such that the fibres of $f$ above $D$ are smooth. Then $\tilde{C} \to C$ is a good base extension and hence $\tilde{S}$ is a smooth surface. Further $\tilde{\mathcal{E}} = \pi^*\mathcal{E}$ is an invertible sheaf and by construction, we have

$$h^0(\tilde{\mathcal{E}}) \neq 0,$$

(ii) *Let $\mathcal{E}$ be a locally free sheaf of rank 2 over $\mathbb{C}$, and* $\deg(\mathcal{E}) > 0$*. There exists a finite separable covering $\pi : \tilde{C} \to C$ which is etale over the images of the singular fibres of $f$, such that the pull-back $\tilde{\mathcal{E}}$ of $\mathcal{E}$ has an invertible subsheaf of positive degree. Then we are done by case(i).*

We construct a good base extension such that the pull-back $\tilde{\mathcal{E}}$ of $\mathcal{E}$ admits an invertible subsheaf of positive degree, using a technique defined by Lange, [24]. Lange has showed that integral curves $D$ of degree $n$ on the projective bundle $\mathbb{P}(\mathcal{E})$ correspond in a natural and bijective way to finite coverings $\pi : \tilde{C} \to C$ of degree $n$, such that the normalization of $D$ is isomorphic to $\tilde{C}$. Furthermore he has shown that the resulting pull-back $\tilde{\mathcal{E}}$ of $\mathcal{E}$ contains an invertible sheaf $\mathcal{E}_1$ whose degree is uniquely determined by $\mathcal{E}$, $\pi$ and the arithmetic genus of the corresponding curve on $\mathbb{P}(\mathcal{E})$ in the following way:

$$\deg \mathcal{E}_1 = \frac{\deg \pi.(g(C) - 1) - (p_a(D) - 1)}{\deg \pi - 1} + \frac{1}{2} \deg \pi^* \mathcal{E}$$

Therefore it is sufficient to construct a smooth, connected curve $D$ on $\mathbb{P}(\mathcal{E})$ of degree $n$ over $C$ such that

$$\deg \mathcal{E}_1 = \frac{n.(g(C) - 1) - (p_a(D) - 1)}{n - 1} + \frac{1}{2} \deg \pi^* \mathcal{E} > 0 \qquad (*)$$

As there is a finite product $g : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1 \times C$ of elementary transformations ([18], pg 416), $\mathbb{P}(\mathcal{E})$ and $\mathbb{P}^1 \times C$ are isomorphic after removing a finite set of points. Therefore the genus and degree over $C$ of the proper transform of $D$ remains the same, and it is enough to construct a smooth connected curve $D$ on $\mathbb{P}^1 \times C$ of degree $n$ over $C$ satisfying (*).

Let $p_1$ and $p_2$ denote the projections of $\mathbb{P}^1 \times C$ onto $\mathbb{P}^1$ and $C$, $l_1 \in \mathrm{Pic}(\mathbb{P}^1)$ be of degree $n_1$, and $l_2 \in \mathrm{Pic}(C)$ of degree $n_2$. If $n_1$ and $n_2$ are high enough, $p_1^* l_1 \otimes p_2^* l_2$ is very ample, so by Bertinis theorem its linear systems contains a smooth connected curve $D$, that meets each of the fibres above the images of the non-singular fibres of $f$ transversally. This ensures that the resulting finite morphism $\psi : D \to C$ is unramified over the singular fibres of $f$. Moreover by the Adjunction formula, we have $g(D) = n_1 g(C) + (n_1 - 1)(n_2 - 1)$ and note that $\deg(\psi) = n_1$.

Therefore,

$$\begin{aligned}
\deg \mathcal{E}_1 &= \frac{\deg \pi.(g(C)-1) - (p_a(D)-1)}{\deg \pi - 1} + \frac{1}{2}\deg \pi^*\mathcal{E} \\
&= \frac{n_1.(g(C)-1) - ((n_1 g(C) + (n_1-1)(n_2-1)) - 1)}{n_1 - 1} + \frac{n_1 \deg \mathcal{E}}{2} \\
&= \frac{n_1.g(C) - n_1 - n_1 g(C) - (n_1-1)(n_2-1)) + 1}{n_1 - 1} + \frac{n_1 \deg \mathcal{E}}{2} \\
&= \frac{-n_1 - (n_1-1)(n_2-1)) + 1}{n_1 - 1} + \frac{n_1 \deg \mathcal{E}}{2} \\
&= \frac{-(n_1-1) - (n_1-1)(n_2-1))}{n_1 - 1} + \frac{n_1 \deg \mathcal{E}}{2} \\
&= -n_2 + \frac{n_1 \deg \mathcal{E}}{2}
\end{aligned}$$

So choosing $n_1 > \frac{2n_2}{\deg \mathcal{E}}$ and relatively prime to the characteristic of $k$, gives $\deg \mathcal{E}_1 > 0$ and hence an invertible sub bundle of positive degree modulo a good base extension.

(iii) *Let $\mathcal{E}$ be a locally free sheaf of positive degree over $C$. There exists a finite covering $\pi : \tilde{C} \to C$ étale above the images of the singular fibres of $f$, such that the pull-back $\tilde{\mathcal{E}}$ of $\mathcal{E}$ has an invertible sub-sheaf of positive degree.*

We will use induction on the rank of $\mathcal{E}$. Let $\operatorname{rank} \mathcal{E} \geq 3$.

Let $\lambda(\mathcal{E}) = \frac{\deg \mathcal{E}}{\operatorname{rank} \mathcal{E}}$ the slope of $E$. Modulo a good base change, we may assume that $\lambda(E)$ is an integer. Here we note that although the characteristic of the field is $p > 0$, if necessary, we can always construct a separable base extension of degree a power of $p$ by using Artin-Schreier covers that are unramified over the singular points of $f$, thereby getting a good base extension.

Let $\alpha$ be a real number defined in the following manner:

$$\alpha = \operatorname{Sup} \left\{ \begin{array}{l} \frac{\deg \tilde{\mathcal{E}}_1}{\lambda(\tilde{\mathcal{E}})}; \tilde{\mathcal{E}} \text{ pull-back of } \mathcal{E} \text{ by a good base change,} \\ \quad \tilde{\mathcal{E}}_1 \text{ an invertible sub-sheaf of } \tilde{\mathcal{E}} \end{array} \right.$$

We assume that $\alpha \leq 0$. Let $\alpha_1$ be the largest integer strictly less than $\alpha$, $\alpha_2 = \alpha_1 + 1$ (then $\alpha_2 \geq \alpha$). By the definition of $\alpha$, we can suppose that modulo a suitable base change, $\mathcal{E}$ has an invertible sub-sheaf $\mathcal{E}_1$ such that

$$\deg \mathcal{E}_1 > \alpha_1 \lambda(\mathcal{E}).$$

By our assumption $\deg \mathcal{E}_1 \leq 0 < \lambda(\mathcal{E})$, hence $\deg(\frac{\mathcal{E}}{\mathcal{E}_1}) = \deg(\mathcal{E}) - \deg(\mathcal{E}_1) > \deg(\mathcal{E})$. Also $\operatorname{rank}(\frac{\mathcal{E}}{\mathcal{E}_1}) = \operatorname{rank}(\mathcal{E}) - 1$. This implies that $\lambda(\frac{\mathcal{E}}{\mathcal{E}_1}) > \lambda(\mathcal{E})$.

We can apply the induction hypothesis to $\frac{\mathcal{E}}{\mathcal{E}_1} \otimes \mathcal{L}$, where $\mathcal{L}$ is an invertible sheaf of degree equal to $-\lambda(E)$, since

$$\begin{aligned}
\deg(\frac{\mathcal{E}}{\mathcal{E}_1} \otimes \mathcal{L}) &= \operatorname{rank} \mathcal{L} \deg(\frac{\mathcal{E}}{\mathcal{E}_1}) + \operatorname{rank}(\frac{\mathcal{E}}{\mathcal{E}_1}) \deg \mathcal{L} \\
&= \deg(\frac{\mathcal{E}}{\mathcal{E}_1}) - \operatorname{rank}(\frac{\mathcal{E}}{\mathcal{E}_1})\lambda(\mathcal{E}) \\
&= (\operatorname{rank}(\mathcal{E}) - 1)(\lambda(\frac{\mathcal{E}}{\mathcal{E}_1}) - \lambda(\mathcal{E})) \\
&> 0
\end{aligned}$$

We can assume, after a suitable base change, that $\frac{\mathcal{E}}{\mathcal{E}_1}$ has an invertible subsheaf $\mathcal{E}_2$ with $\deg \mathcal{E}_2 > \lambda(\mathcal{E})$.

Let $\mathcal{E}'$ be the inverse image of $\mathcal{E}_2$ in $\mathcal{E}$. We have a filtration

$$0 \to \mathcal{E}_1 \to \mathcal{E}' \to \mathcal{E}_2 \to 0$$

then

$$\deg \mathcal{E}' = \deg \mathcal{E}_1 + \deg \mathcal{E}_2 > (\alpha_1 + 1)\lambda(\mathcal{E}) = \alpha_2 \lambda(\mathcal{E})$$

Now we can apply step (ii) to the sheaf $\mathcal{E}' \otimes \mathcal{L}'$ where $\mathcal{L}'$ is invertible with $\deg \mathcal{L}' = -\frac{1}{2}\alpha_2 \lambda(\mathcal{E})$, since

$$\begin{aligned}
\deg(\mathcal{E}' \otimes \mathcal{L}') &= \operatorname{rank} \mathcal{L}' \deg(\mathcal{E}') + \operatorname{rank}(\mathcal{E}') \deg \mathcal{L}' \\
&= \deg(\mathcal{E}') - 2(\frac{1}{2}\alpha_2 \lambda(\mathcal{E}) \\
&> 0.
\end{aligned}$$

We find a suitable base change $\pi : \tilde{C} \to C$ such that $\pi^*(\mathcal{E}')$ has an invertible subsheaf $\tilde{\mathcal{E}}'_1$ with

$$\deg \tilde{\mathcal{E}}'_1 > \frac{1}{2}\alpha_2 \lambda(\tilde{\mathcal{E}}) \geq \alpha\lambda(\tilde{\mathcal{E}}),$$

where $\tilde{\mathcal{E}} = \pi^*(\mathcal{E})$, contradicting the definition of $\alpha$.

□

**Lemma 4.21.** *Let $D$, $\mathcal{E}$, $\mathcal{H}$ be defined as above. Then $N(\mathcal{H})$ is a nef $\mathbb{Q}$ divisor.*

*Proof:* We show that $N(\mathcal{H})$ is nef by viewing it modulo good base extensions as a limit of nef and effective divisors. We will show that $N(\mathcal{H}) + \epsilon F$ is a nef $\mathbb{Q}$-divisor modulo good base extensions, for all rational numbers $\epsilon$. Note that given a good base extension $\psi : \tilde{C} \to C$ of degree $d$, $N(\mathcal{H}) + \epsilon F$ is nef if $N(\tilde{\mathcal{H}}) + \epsilon d\tilde{F}$ is nef. In this proof $\tilde{}$ will always denote the pull-back, except $\tilde{F}$ will be a general fibre of $\tilde{f}$.

Let $\epsilon > 0$. Consider the sheaf $H' = H \otimes O_C((\epsilon - \mu_f(\mathcal{H}))b)$. We may assume that modulo a good base extension, $\mu_f(\mathcal{H}) - \epsilon$ is an integer. Then

$$\mu_f(\mathcal{H}') = \mu_f(\mathcal{H}) + \mu_f(O_C(\epsilon b)) + \mu_f(O_C(-\mu_f(\mathcal{H})b)) = \epsilon > 0,$$

which implies that $\deg \mathcal{H}' > 0$. Hence by the above lemma, we may assume that $\mathcal{H}'$ has a global section. Let $B$ be the image of this global section in $|N(\mathcal{H}) + \epsilon F|$. We want to show $B$ is nef modulo a good base extension. To this end, it is sufficient to show that for every component $A$ of $B$, there exists a good base extension $\psi : \tilde{C} \to C$ such that $H^0(\tilde{\mathcal{H}}')$ has another global section $\tilde{B}'$ which has no common component with the inverse image $\tilde{A}$ of $A$. The case where $A$ is contained in a fibre of $f$ has trivial self-intersection, and we can assume that $A$ is not contained in a fibre of $f$. We may also assume that the inverse image of $A$ under a good base extension is irreducible, as we can always split the components of $B$ by base extensions till they cannot be split any further.

Consider the subsheaf $\mathcal{G}$ of $\mathcal{H}'$ which consists of all local sections, whose divisors in the moving part contain $A$. Then for every open set $U$ in $C$,

$$\mathcal{G}(U) = \{s \in \mathcal{H}'(U) | (f^* s)_A \in \mathfrak{m}_A \mathcal{L}_A\},$$

where $\mathcal{L} = \mathcal{O}_S(N(\mathcal{H}) + \epsilon F)$ and $A$ denotes the generic point of $A$ in $S$. Since the moving part contines no common part, there must exist local sections of $\mathcal{H}'$ which do not contain $A$, and hence $\mathcal{G}$ must be a proper subbundle of $\mathcal{H}'$. Now $\tilde{B}'$ contains $\tilde{A}$ iff the section in $H^0(\tilde{\mathcal{H}}')$ corresponding to $\tilde{B}'$ lies in $\tilde{\mathcal{G}}$. Therefore the lemma is reduced to showing that:

For any proper subbundle $\mathcal{G}$ of $\mathcal{H}'$, there is a good base extension $\psi : \tilde{C} \to C$ such that $H^0(\tilde{\mathcal{H}}')$ contains a section of positive degree not lying in $\tilde{\mathcal{G}}$.

We show this by the above lemma with a minor modification. In fact, the cases of $\text{rank}(\mathcal{H}') = 1$ or 2 go without significant change. For the general case, we first suppose that there is a section $s$ of positive degree in $H^0(\mathcal{H}')$, using the above lemma. If $s$ does not lie in $\mathcal{G}$ we are done; otherwise $s$ generates a subbundle $\mathcal{G}_1$ of $\mathcal{G}$. Then the image of $\mathcal{G}$ in $\mathcal{H}'/\mathcal{G}_1$ is a proper subbundle, hence by induction hypothesis, we get an invertible subbundle $\mathcal{H}_1$ of positive degree in $\mathcal{H}'/\mathcal{G}_1$ (modulo base extensions) which is not contained in the image of $\mathcal{G}$. Now use the proof of the rank 2 case to the inverse image of $\mathcal{H}_1$ in $\mathcal{H}'$, and the lemma is shown. □

### 4.2.6 Technical lemmas

The following two technical lemmas can be extended directly from the classical case without any alteration to characteristic $p > 0$.

**Lemma 4.22.** *Let $f : S \to C$ as before, with a general fibre $F$. Let $D$ be a divisor on $S$, and suppose that there is a sequence of effective divisors*

$$Z_1 \geq Z_2 \geq \ldots \geq Z_n \geq Z_{n+1} = 0$$

*and a sequence of rational numbers*

$$\mu_1 > \mu_2 > \ldots > \mu_n > \mu_{n+1} = 0$$

*such that for every i, $N_i = D - Z_i - \mu_i F$ is a nef $\mathbb{Q}$-divisor. Then*

$$D^2 \geq \sum_{i=1}^{n} (d_i + d_{i+1})(\mu_i - \mu_{i+1}),$$

*where $d_i = N_i F$.*

*Proof.* See 3.15 □

**Lemma 4.23.** *Let $D_1$, $D_2$ be two numerically equivalent divisors on $S$ such that $N_i = D_i - Z_i - \mu_i F$, $i = 1, 2$, is nef, where $\mu_1, \mu_2$ are rational numbers with $\mu_1 \geq \mu_2$, and $Z_1, Z_2$ are effective divisors. Let $Z$ be the common part of $Z_1$ and $Z_2$ (i.e., $Y_i = Z_i - Z$ are two effective divisors without common component). Then*

$$N = D_2 - Z - \mu_2 F$$

*is nef.*

*Proof.* See 3.16 □

## 4.3 Proof of the main theorem

Let $f : S \to C$ over $k$ be a fibration with $F$ a general fibre of genus $g \geq 2$. Given $f : S \to C$ we have the following induced exact sequence,

$$\pi_1(F) \to \pi_1(S) \to \Pi(C) \to 1$$

Consider the pro-l quotient of $\pi_1(F)$ and the resulting push-out square of groups.

$$\begin{array}{ccccc} \pi_1(F) & \xrightarrow{\phi} & \pi_1(S) & \longrightarrow & \Pi(C) \\ \downarrow{\scriptstyle \psi} & & \downarrow & & \\ \pi_1^l(F) & \xrightarrow{\alpha} & \Pi & & \end{array}$$

$\Pi$ is defined as the push-out of the diagram given by the morphisms $\phi : \pi_1(F) \to \pi_1(S)$ and $\psi : \pi_1(F) \to \pi_1^l(F)$. It is the free product of $\pi_1^l(F)$ and $\pi_1(S)$ with amalgamation defined by the following relation:

$$\phi(\alpha)\psi^{-1}(\alpha) = e, \qquad \alpha \in \pi_1(F)$$

$$\Pi = \pi_1^l(F) * \pi_1(S)/N$$

where $N$ is the smallest normal subgroup containing words of the form $\phi(\alpha)\psi^{-1}(\alpha)$

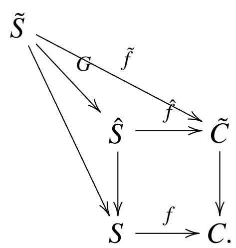
**Theorem 4.24.** *Let $k$ be an algebraically closed field of characteristic $p > 0$ and $l$ a prime that is distinct from $p$. Let $S$ be a smooth and projective surface over $k$. Assume that $S$ admits a generically ordinary semi-stable fibration $f : S \to C$ to a smooth and projective curve $C$ over $k$ that is not isotrivial, and that $S$ is relatively minimal with respect to $f$. We also assume that the generic fibre of $f$ is strongly $l$-ordinary. Let $F$ be a general fibre of $f$, which is a smooth curve of genus $g \geq 2$. Let $K_{S/C} = K_S - f^*K_C$ be a relative canonical divisor. We have the following commutative diagram of profinite groups,*

$$\begin{array}{ccccc} \pi_1(F) & \longrightarrow & \pi_1(S) & \longrightarrow & \Pi(C) \to 1 \\ \downarrow & & \downarrow & & \\ \pi_1^l(F) & \longrightarrow & \Pi & & \end{array}$$

*where $\pi_1$ is the étale fundamental group, $\pi_1^l$ is the pro-$l$ fundamental group and $\Pi$ is the pushout group.*

*If $K^2_{S/C} < 4\deg(f_*\omega_{S/C})$ and $f$ is non-hyperelliptic, then the image of $\pi_1^l(F) \to \Pi$ is trivial.*

*Proof.* We first show that if the image of $\alpha$ is non-trivial then there exists a non-vertical $l$-torsion element in $\mathrm{Pic}(S)$. Since $\alpha$ is a continuous map between profinite groups, $\mathrm{Im}(\alpha)$ is a closed and normal subgroup of $\Pi$. If $\mathrm{Im}(\alpha)$ has a finite quotient $G \cong \mathrm{Im}(\alpha)/K$, then by basic properties of profinite groups, $G$ is reflected in some étale covering $\tilde{S}$ of $S$. i.e. by Stein Factorization, the morphism $\tilde{S} \to S$ can be factored into $\psi \circ \tilde{f}$ such that $\tilde{f} : \tilde{S} \to \tilde{C}$ is a fibration and $\psi : \tilde{C} \to C$ is an finite morphism. If $\hat{S}$ is the minimal desingularization of $S \times_C \tilde{C}$, then we can assume that the morphism $\tilde{S} \to \hat{S}$ corresponds to $G$ and we get the following commutative diagram,



Since $\lambda(\hat{f}) \leq \lambda(f) < 4$ by 4.14 and 4.13, and generic ordinarity is preserved by base change, we can replace $f$ with $\hat{f}$. Since $G$ is a finite $l$-group with order $l^n$, it contains normal subgroups of order $l^i$ with $0 \leq i \leq n$, and hence by taking a quotient we may as well assume that $G$ is a cyclic group of degree $l$. Then $G$ corresponds to an étale $l$-cyclic cover $\tilde{S} \to S$ or equivalently a non-vertical $l$-torsion $\eta$ on $S$, such that $\tilde{S} := \mathbf{Spec}(\oplus_{i=0}^{l-1}\eta^{\otimes i})$.

We will now see that this is impossible using the lemmas proved above. Let $D_\eta$ be an effective divisor on $S$ having class $\omega_{S/C}\otimes\eta$. Then by 4.10, $\mathcal{E} = f_*(\omega_{S/C}\otimes\eta)$ is locally free of rank $g-1$ and has non-negative quotients. Let the Harder-Narasimhan filtration of $\mathcal{E} = f_*(\omega_{S/C} \otimes \eta)$ be given by

$$0 = \mathcal{E}_1 \subset \mathcal{E}_2 \subset \ldots \subset \mathcal{E}_n = \mathcal{E}.$$

Take

$$\begin{aligned} r_i &= \mathrm{rank}(\mathcal{E}_i) \\ \mu_i &= \mu_f(\mathcal{E}_i) = \mu(\mathcal{E}_i/\mathcal{E}_{i-1}) \\ Z_i &= Z(\mathcal{E}_i) \\ N_i &= N(\mathcal{E}_i) \\ d_i &= N_i F \end{aligned}$$

Let also $N_{n+1} = D_\eta$, $r_{n+1} = g-1$, $Z_{n+1} = 0$, $\mu_{n+1} = 0$, $d_{n+1} = D_\eta F$. Thus we have a sequence of effective divisors

$$Z_1 \geq Z_2 \geq Z_3 \geq \ldots \geq Z_{n+1} = 0$$

and by above, a sequence of non-negative rational numbers

$$\mu_1 > \mu_2 > \mu_3 > \ldots > \mu_{n+1} = 0$$

such that by 4.21 and [5], $N_i$ is a nef $\mathbb{Q}$-divisor. This satisfies the conditions of Lemma 4.22, which gives

$$K_{S/C}^2 = D_\eta^2 \geq \sum_{i=1}^{n}(d_i + d_{i+1})(\mu_i - \mu_{i+1}) \tag{4.1}$$

On the other hand,

$$\Delta(f) = \deg \mathcal{E} = \sum_{i=1}^{n} r_i(\mu_i - \mu_{i+1}) \tag{4.2}$$

Combining 4.1 and 4.2, we see that we will have $\lambda \geq 4$ if

$$d_i + d_{i+1} \geq 4r_i \tag{4.3}$$

Now $d_i$ is defined to be the degree of $N_iF$. Since $N_i$ is nef, we may assume $N_iF$ is effective and also $N_iF \leq N_{n+1}F = D_\eta F = 2g - 2$. Further the dimension of the restriction of the linear system $N_i$ to $F$ is $r_i - 1$. Clifford's Lemma states that,

$$\dim|D| \leq \frac{1}{2} \deg D$$

with equality if and only if either $D = 0$ or $D = K_F$ when $X$ is non-hyperelliptic. Since $N_nF \neq 0$ and $N_nF \neq K_F$,

$$\begin{aligned} r_i - 1 &< \frac{1}{2} d_i \\ 2r_i - 2 &< d_i \\ 2r_i - 1 &\leq d_i \end{aligned}$$

Thus we have $d_i \geq 2r_i - 1$, $i = 1, \ldots, n$, except if $d_1 = 0$, (and $r_1 = 1$). We require condition 4.3 to be satisfied for all $i$ to prove the theorem. For $2 \leq i \leq n - 1$ we have

$$\begin{aligned} d_i + d_{i+1} &\geq 2r_i - 1 + 2r_{i+1} - 1 \\ &\geq 2r_i + 2(r_i + 1) - 2 \\ &\geq 4r_i \end{aligned}$$

so that condition 4.3 is satisfied. However the cases $i = 1$ and $i = n$ are more delicate. In the case $i = 1$, $d_1 + d_2 \geq 0 + 2r_2 - 1$. If $r_2 = 2$ then $d_1 + d_2 \geq 3$. Therefore if $d_2 = 3$ condition 4.3 will not be satisfied. In the case $i = n$, we have $d_n \geq 2r_n - 1 = 2(g - 1) - 1 = 2g - 3$ and $d_{n+1} = 2g - 2$, therefore

$$\begin{aligned} d_n + d_{n+1} &\geq 2(g - 1) - 1 + 2g - 2 \\ &\geq 4(g - 1) - 1. \end{aligned}$$

Thus if $d_n = 2g - 3$ again condition 4.3 is not satisfied. Hence the theorem is proved except if $d_n = 2g - 3$ or $d_1 = 0, d_2 = 3$ (hence $r_1 = 1, r_2 = 2$).

We will first try to modify $N_n$ so that $d_n > 2g - 3$. Now $d_n$ is defined to be the degree of the linear system $|N_n|_F|$. So

$$d_n = N_n F = M(f_*\omega_{S/C} \otimes \eta)F = 2g - 3.$$

$N_n = D_\eta - Z(f_*\omega_{S/C} \otimes \eta) - \mu_n F$ and $Z(f_*\omega_{S/C} \otimes \eta)$ is the fixed part of $|D_\eta|$. We have defined the restriction of $Z(f_*\omega_{S/C} \otimes \eta)$ to $F$ to be the fixed part of the linear system $E = f_*\omega_{S/C} \otimes \eta$ restricted to $F$, which will be the fixed part of the linear system $|D_\eta|$ restricted to $F$. Therefore

$$N_n|_F = M(E)|_F = D_\eta|_F - Z(E)|_F.$$

We may assume that for every torsion $\eta$, $d_n = 2g - 3$ or equivalently the linear system $|D_\eta|_F|$ has a base point $P_\eta$. We fix an $\eta$ and let $P = P_\eta$. Let $D$ be a divisor in $|M(E)|_F|$.

$$\begin{aligned} 2g - 3 &= \deg(D) \\ &= \deg(D_\eta|_F - Z(E)|F) \\ &= \deg(K_F) + \deg(\eta|_F - Z(E)|_F) \\ &= 2g - 2 - \deg(Z(E)|_F) \end{aligned}$$

Therefore $\deg(Z(E)|_F) = 1$ which implies $Z(E)|_F = P$, since we may assume $Z(E)|_F$ is effective. Applying Riemann Roch to the divisor $D$, we get

$$\begin{aligned} h^0(F, \mathcal{O}_F(D)) - h^0(F, \mathcal{L}(K_F - D)) &= \deg D + 1 - g \\ g - 1 - h^0(F, \mathcal{O}_F(K_F - D)) &= 2g - 3 + 1 - g \\ h^0(F, \mathcal{O}_F(K_F - D)) &= g - 1 - 2g + 3 - 1 + g \\ h^0(F, \mathcal{O}_F K_F - D)) &= 1 \end{aligned}$$

Now $K_F - D$ is linearly equivalent to $-\eta|_F + P$ (we abuse notation to denote a divisor in the class $\eta|_F$ by $\eta|_F$ itself) . Since $h^0(F, \mathcal{O}_F(K_F - D)) = 1$, the complete linear system $| - \eta|_F + P|$

is nonempty. Hence $|-\eta|_F + P|$ is linearly equivalent to some effective divisor $Q$ of degree 1. Therefore $Q$ is a point of $F$ such that $\eta|_F = \mathcal{O}_F(P-Q)$. Further note that $\eta$ cannot have order two. If it does then $2P \equiv 2Q$. However $|2Q|$ is a linear system of degree 2 and dimension 1. This contradicts the hypothesis that $f$ is non-hyperellitptic, ([18] IV.5). Consequently $\eta|_F^2 = \mathcal{O}_F(P'-Q')$ with $(P' \neq Q')$. Note also that $P \neq P'$ else $2P-2Q \equiv P-Q'$, or $2Q \equiv P+Q'$, again contradicting the condition that $F$ is non-hyperelliptic. We assume

$$\mu_f(f_*\omega_{S/C} \otimes \eta) \leq \mu_f(f_*\omega_{S/C} \otimes \eta^2)$$

Let $Z_1' = Z(f_*\omega_{S/C} \otimes \eta)$, $Z_2' = Z(f_*\omega_{S/C} \otimes \eta^2)$. Since $Z_i'F = 1$, there is a unique section $C_i$ in $Z_i'$ such that $C_1 \cap F = \{P\}$, $C_2 \cap F = \{P'\}$, in particular $C_1 \neq C_2$. Now we can apply Lemma 4.23 and let $N_n = D_\eta - Z - \mu_n F$, where $Z$ is the common part of $Z_1'$ and $Z_2'$. Then $d_n = 2g-2$.

Next we consider the case $d_1 = 0$, $d_2 = 3$. We want to show

$$K_{S/C}^2 \geq 4\Delta(f).$$

We will do this by considering two cases; $d_3 = 5$ and $d_3 \geq 6$. First we consider the case $d_3 = 5$. Since $r_3 = 3$ we have $3 = r_3 < r_n = g-1$, hence $g \geq 5$. The linear system $|N_3|_F|$ is a $g_5^2$ and hence defines a map $\phi$ of $F$ to $\mathbb{P}^2$. The image of $F$ is a plane curve $B$ which is not contained in any hyperplane. The degree of $B$ is equal to $B.H$ where $H$ is a divisor in the class $\mathcal{O}(1)$. However $\deg(\phi^*\mathcal{O}(1)) = 5$ which implies $B.H = 5$. Hence $B$ is a curve of degree 5. $|N_3|_F|$ has a sublinear system $|N_2|_F|$ which is a $g_3^1$. This implies that $B$ has a double point. Then by a Riemann-Hurwitz type formula for singular curves ([14]), we have $g \leq 5$. Therefore $g = 5$, and $n = 4$ and $d_4 = 8$. If $\mu_1 - \mu_2 \leq \mu_3 - \mu_4$, 4.1 and 4.2 give

$$\begin{aligned} K_{S/C}^2 &\geq 3(\mu_1-\mu_2) + 8(\mu_2-\mu_3) + 13(\mu_3-\mu_4) + 16\mu_4 \\ &\geq 4(\mu_1-\mu_2) + 8(\mu_2-\mu_3) + 12(\mu_3-\mu_4) + 16\mu_4 \\ &\geq 4\Delta(f) \end{aligned}$$

On the other hand if $\mu_1 - \mu_2 \geq \mu_3 - \mu_4$, we use Lemma 4.22 on the sequence $\{Z_1, Z_4, 0\}$,$\{\mu_1, \mu_4, 0\}$

to get

$$K_{S/C}^2 \geq 8(\mu_1 - \mu_4) + 16\mu_4 \geq 4\Delta(f)$$

We finally consider the case when $d_3 \geq 6$. We then have two possibilities: if $\mu_1 - \mu_2 \leq \mu_2 - \mu_3$, then by 4.1 and 4.2,

$$\begin{aligned} K_{S/C}^2 &\geq 3(\mu_1 - \mu_2) + 9(\mu_2 - \mu_3) + \sum_{i=3}^{n}(d_i + d_{i+1})(\mu_i - \mu_{i+1}) \\ &\geq 4(\mu_1 - \mu_2) + 8(\mu_2 - \mu_3) + 4\sum_{i=3}^{n}(\mu_i - \mu_{i+1}) \\ &= 4\Delta(f) \end{aligned}$$

otherwise use Lemma 4.23 on the sequences

$$\{Z_1, Z_3, Z_4, \ldots, Z_n, 0\}, \{\mu_1, \mu_3, \mu_4, \ldots, \mu_n, 0\},$$

to get

$$\begin{aligned} K_{S/C}^2 &\geq 6(\mu_1 - \mu_3) + \sum_{i=3}^{n}(d_i + d_{i+1})(\mu_i - \mu_{i+1}) \\ &\geq 4(\mu_1 - \mu_2) + 8(\mu_2 - \mu_3) + 4\sum_{i=3}^{n}(\mu_i - \mu_{i+1}) \\ &= 4\Delta(f) \end{aligned}$$

□

# References


[1] D. Abramovich and F. Oort. Alterations and resolution of singularities. In *Resolution of singularities*, volume 181 of *Progress in mathematics*, pages 39–108. Birkauser, Basel, 1997.

[2] S.J. Arakelov. An intersection theory for divisors on an arithmetic surface. *Izv. Akad. Nauk SSSR Ser. Mat.*, 38:1179–1192, 1974.

[3] M.A. Barja. *On the slope and geography of fibred surfaces and threefolds*. PhD thesis, University of Barcelona, 1998.

[4] W. Barth. *Compact Complex Surfaces*, volume 4 of *A series of modern surveys in mathematics*. Springer, 2 edition, 2000.

[5] A. Beauville. L'inégalité $p_g \geq 2q - 4$ pour les surfaces de type général. *Bull. Soc. Math. Fr.*, 110:319–346, 1982.

[6] M. T. I. Bigas. Vector bundles on curves. http://www.tufts.edu/ mteixido/files/vectbund.pdf.

[7] E. Bombieri and D. Mumford. Enriques' classification of surfaces in characteristic $p$ III. *Invent. Math.*, 35:197–232, 1976.

[8] E. Bombieri and D. Mumford. Enriques' classification of surfaces in characteristic $p$ II. *Complex Analysis and Algebraic Geometry*, 1977.

[9] I. Bouw. The $p$-rank of curves and covers of curves. In *Courbes semistables et groupe fondamental en géométrie algébrique*, volume 187 of *Progress in mathematics*, pages 267–277. Birkauser Basel, 2000.

[10] R. Crew. étale $p$-covers in characteristic $p$. *Compositio Mathematica*, 52:31–45, 1984.

[11] X. Gang. Surfaces fibrées en courbes de genre deux. *Lecture Notes in Mathematics*, 1137, 1985.

[12] X. Gang. Fibered algebraic surfaces with low slope. *Math. Ann.*, 276(3):449–466, 1987.

[13] X. Gang. Hyperelliptic surfaces of general type with $k^2 < 4\chi$. *Manuscripta Mathematica*, 57:125–148, 1987.

[14] A. Garcia and R. F. Lax. Rational nodal curves with no smooth weierstrass points. *Proceedings of the American Mathematical Society*, 124(2), 1996.

[15] B Geemen and F. Oort. A compactification of a fine moduli space of curves. In H. Hauser, J. Lipman, F. Oort, and A. Quirós, editors, *Resolution of Singularities*, volume 181 of *Progress in mathematics*, pages 285–298. Birkauser, 1997.

[16] E. H. Esnault and Viehweg. *Lectures on vanishing theorems*, volume 20 of *DMV seminar*. Birkhauser, 1992.

[17] J. Harris and I. Morrison. *Moduli of Curves*, volume 187 of *Graduate texts in mathematics*. Springer, 1991.

[18] R. Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate texts in mathematics*. Springer, 1977.

[19] L. Illusie. Ordinarité des intersections complètes générales. *The Grothedieck Festschrift*, II:375–405, 1990.

[20] L. Illusie. Réduction semi-stable et décomposition de complexes de de rham à coefficients. *Duke Math. J.*, 60(1):139–185, 1990.

[21] J. Jang. Generic ordinarity for semi-stable fibrations. *Archive*, 2008.

[22] K. Kodaira. On compact analytic surfaces i,ii,iii. *Ann. of Math.*, 71,77,78:111–152, 563–626, 1–40, 1960,1963.

[23] K. Kodaira. On the structure of compact analytic surfaces I,II,III,IV. *Amer. J. Math*, 86, 88:751–798 , 682–721 , 55–83 , 1048–1066, 1964 , 1966.

[24] H. Lange. On stable and ample vector bundles of rank 2 on curves. *Math. Ann.*, 238(3):193–202, October 1978.

[25] R. Lazarsfeld. *Positivity in Algebraic Geometry*, volume 48 of *Graduate texts in mathematics*. Springer, 2000.

[26] J. Le Potier. *Lectures on Vector Bundles*, volume 54 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 1997.

[27] Q. Liu. *Algebraic Geometry and Arithmetic Curves*, volume 6 of *Oxford graduate texts in mathematics*. Oxford Science Publications, 2002.

[28] D. Madore. Theta divisors and the frobenius morphism. In *Courbes semistables et groupe fondamental en géométrie algébrique*, volume 187 of *Progress in mathematics*, pages 279–289. Birkauser Basel, 2000.

[29] L. Moret-Bailly. Families de courbes et de variétés abéliennes sur $\mathbb{P}^1$. *Astérisque*, 86:125–140, 1981.

[30] D. Mumford. Enriques' classification of surfaces in characteristic $p$ I. *Global Analysis (Papers in Honor of K. Kodaira)*, pages 325–339, 1969.

[31] S. Nakajima. On generalized hasse-witt invariants of an algebraic curve. In *Galois groups and their representations*, volume 2 of *Advanced Studies in Pure mathematics*, pages 69–88, 1983.

[32] R. Pries and K. Stevenson. A survey of galois theory of curves in characteristic $p$. *Fields Institute Communications*, 60, 2011. to be published, arXiv:1004.2267v1 [math.AG].

[33] M. Raynaud. Revêtements des courbes en caractéristique $p > 0$ et ordinarité. *Annales Scientifiques de lcole Normale Suprieure*, 32:87–126, 1999.

[34] M. Reid. $\pi_1$ for surfaces with small $c_1^2$. *Springer-Verlag Lecture Notes in Mathematics*, 732:534–544, 1978.

[35] L. Ribes and P. Zalesskii. *Profinite Groups*. A Series of Modern Surveys in Mathematics. Springer, Berlin, 2000.

[36] M. Saidi. On the degeneration of étale $F/pF$ and $F/p^2F$-torsors in equal characteristic. *Hiroshima Mathematical Journal*, 37(2):315–341, 2007.

[37] I.R Shafarevich. Lectures on minimal models and birational transformations of two dimensional schemes. *Tata Institute of fundamental research*, pages 1–175, 1966.

[38] T. Szamuely. *Galois groups and fundamental groups*, volume 117 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 2009.

[39] L. Szpiro. Propriétés numériques du dualisant-relatif. *Asterisque*, 86:44–78, 1981.

[40] L. Szpiro. Sur les propriétés numériques du dualisant-relatif d'une surface arithmétique. *The Grothedieck Festschrift*, III:229–246, 1990.

[41] B. Zhang. Revêtements étale abelians de courbes génériques et ordinarité. *Ann. Fac. Sci. Toulouse Sér.*, 6:133–138, 1992.