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FILTER EQUATION BY FRACTIONAL CALCULUS

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Aim of this paper is to represent a causal filter equation for any kind of linear system in the general form L=f(t), where f(t) is the forcing function, x(t) is the output and L is a summation of fractional operators. The exact form of the operator L is obtained by using Mellin transform in complex plane.

Keywords: Fractional calculus, Mellin transform, Filter equation, Non-anticipative filter

1 Introduction

In many cases of engineering interest the system may be considered as a black-box system and some informations are available from experimental data. The question is: what is the filter equation that exactly match the response for any kind of input? Such an example if the Power Spectral Density (PSD) function is measured or is assigned like in the case of wind velocity, or wave elevation or earthquake the PSD is already known and we search some differential equation enforced by a white noise whose output reproduces the PSD target. Discrete Auto-regressive (AR), Moving Average (MA) or their combination (ARMA) models are currently used for. Such an example in order to represent the wave elevation represented by the Pierson-Moskovitz spectrum (Pierson and Moskowitz, 1964) in Spanos (1983) the AR, MA and ARMA models have been widely discussed. In Spanos (1986) an analog filter representation is presented for the Jonswap spectrum (Hasselmann et al., 1973). The main disadvantage in the discrete filter representation is that the filter equation so obtained is able to reproduce only the given PSD (or the correlation function) but the differential equations governing the problem remain unknown and then powerful tools of the Itô calculus may not be applied. In Thampi (1999) a filter based on the Markov method has been proposed, however the filter is only suitable for modeling response of offshore structures.

Recently Cottone et al. (2010) proposed a continuous filter equation in the form $L[x(t)] = W(t)$ where $W(t)$ is a white noise process and $L[\cdot]$ is a linear fractional differential operator. The only problem is that the filter so obtained is non causal. This is due to the fact that the only knowledge of the PSD is not enough to represent both amplitude and phase of the given signal.

In order to overcome this problem, in this paper we assume that the black-box system is enforced by an impulse. Then we measure the impulse response function (or the transfer function), with this information the exact causal filter equation is readily found by using Mellin transform and fractional differential calculus (Podlubny, 1999; Samko et al., 1993). It is shown that the filter equation may be obtained in two different forms, a form involving a summation of Riesz integrals whose coefficients are readily found simply by evaluating the Mellin transform of the impulse response function.

2 Preliminary concepts and definitions

In this section some basic concepts on fractional calculus are introduced for clarity sake's as well as for introducing appropriate symbologies. Let us start with the definition of Riemann-Liouville fractional integral and derivative

$$({}_aI_t^\gamma f)(t) = \frac{1}{\Gamma(\gamma)}\int_a^t \frac{f(\tau)}{(t-\tau)^{1-\gamma}}d\tau \tag{1}$$

$$({}_aD_t^\gamma f)(t) = \frac{1}{\Gamma(n-\gamma)}\left(\frac{d}{dt}\right)^n\int_a^t \frac{f(\tau)}{(t-\tau)^{\gamma+1-n}}d\tau \tag{2}$$

with $\gamma = \rho + i\eta$, $\rho > 0$, $\eta \in \mathbb{R}$ and $\Gamma(\cdot)$ is the Euler Gamma function. Eqs. (1) and (2) are the *left fractional integral* and *derivative* respectively. It may be also defined the *right fractional integral* and *derivative*

$$({}_tI_b^\gamma f)(t) = \frac{1}{\Gamma(\gamma)}\int_t^b \frac{f(\tau)}{(\tau-t)^{1-\gamma}}d\tau \tag{3}$$

$$({}_tD_b^\gamma f)(t) = \frac{1}{\Gamma(n-\gamma)}\left(-\frac{d}{dt}\right)^n\int_t^b \frac{f(\tau)}{(\tau-t)^{\gamma+1-n}}d\tau \tag{4}$$

If $a \to -\infty$ in Eqs. (1) and (2) and $b \to \infty$ in Eqs. (3) and (4), they will be indicated as I_+^γ andD_+^γ and I_-^γ andD_-^γ, respectively. It may be easily demonstrated that the Fourier transform of such operator are (see Podlubny (1999); Samko et al. (1993))

$$\mathcal{F}\left\{(I_\pm^\gamma f)(t);\omega\right\} = (\mp i\omega)^{-\gamma} f_{\mathcal{F}}(\omega) \tag{5}$$

$$\mathcal{F}\left\{(D_\pm^\gamma f)(t);\omega\right\} = (\mp i\omega)^{\gamma} f_{\mathcal{F}}(\omega) \tag{6}$$

where $f_{\mathcal{F}}(\omega)$ is the Fourier transform of $f(t)$; Fourier transform of $f(t)$ and its inverse are defined as:

$$f_{\mathcal{F}}(\omega) = \mathcal{F}\{f(t);\omega\} = \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt \tag{7a}$$

$$f(t) = \mathcal{F}^{-1}\{f_{\mathcal{F}}(\omega);t\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_{\mathcal{F}}(\omega)e^{-i\omega t}d\omega \tag{7b}$$

Other useful definitions for the ensuing derivations are *Riesz fractional integral* and *derivative*, denoted as I^γ and D^γ, respectively, defined as

$$(I^\gamma f)(t) = \frac{1}{2\nu_c(\gamma)}\int_{-\infty}^{\infty}\frac{f(\tau)}{|t-\tau|^{1-\gamma}}d\tau = \frac{1}{2\cos(\frac{\gamma\pi}{2})}[(I_+^\gamma f)(t) + (I_-^\gamma f)(t)] \tag{8}$$

$$(D^\gamma f)(t) = \frac{1}{2\nu_c(-\gamma)}\int_{-\infty}^{\infty}\frac{f(t-\tau)-f(t)}{|\tau|^{\gamma+1}}d\tau = -\frac{1}{2\cos(\frac{\gamma\pi}{2})}[(D_+^\gamma f)(t) + (D_-^\gamma f)(t)] \tag{9}$$

where $\nu_c(\gamma) = \Gamma(\gamma)\cos(\gamma\pi/2)$. Moreover we may define the *complementary Riesz fractional integral* and *derivative*, denoted as $\tilde{I}^\gamma$ and $\tilde{D}^\gamma$, in the form

$$(\tilde{I}^\gamma f)(t) = \frac{1}{2\nu_s(\gamma)}\int_{-\infty}^{\infty}\frac{f(\tau)\,\mathrm{sgn}(t-\tau)}{|t-\tau|^{1-\gamma}}d\tau = \frac{1}{2\sin(\frac{\gamma\pi}{2})}[(I_+^\gamma f)(t) - (I_-^\gamma f)(t)] \tag{10}$$

$$(\tilde{D}^\gamma f)(t) = \frac{1}{2\nu_s(-\gamma)}\int_{-\infty}^{\infty}\frac{f(t-\tau)-f(t)}{|\tau|^{\gamma+1}}\,\mathrm{sgn}(t-\tau)d\tau = -\frac{1}{2\sin(\frac{\gamma\pi}{2})}[(D_+^\gamma f)(t)-(D_-^\gamma f)(t)] \quad (11)$$

where $\nu_s(\gamma) = \Gamma(\gamma)\sin(\gamma\pi/2)$ and $\mathrm{sgn}(\cdot)$ is the signum function. Eqs. (8) to (11) are valid provided $\rho > 0$, $\rho \neq 1, 3, \ldots$.

Fourier transform of Riesz operators can be expressed as follows

$$\mathcal{F}\{(I^\gamma f)(t);\omega\} = |\omega|^{-\gamma} f_{\mathcal{F}}(\omega) \quad (12a)$$

$$\mathcal{F}\{(D^\gamma f)(t);\omega\} = |\omega|^{\gamma} f_{\mathcal{F}}(\omega) \quad (12b)$$

$$\mathcal{F}\left\{(\tilde{I}^\gamma f)(t);\omega\right\} = i\,\mathrm{sgn}(\omega)|\omega|^{-\gamma} f_{\mathcal{F}}(\omega) \quad (13a)$$

$$\mathcal{F}\left\{(\tilde{D}^\gamma f)(t);\omega\right\} = -i\,\mathrm{sgn}(\omega)|\omega|^{\gamma} f_{\mathcal{F}}(\omega) \quad (13b)$$

For the fractional operators above defined the following rules

$$D_\pm^\gamma(I_\pm^\gamma f)(t) = f(t), \qquad D^\gamma(I^\gamma f)(t) = f(t), \qquad \tilde{D}^\gamma(\tilde{I}^\gamma f)(t) = f(t) \quad (14)$$

hold true. Finally for Fourier transformable function and $0 < \rho < 1$

$$(I^\gamma f)(t) = (D^{-\gamma} f)(t) \qquad (\tilde{I}^\gamma f)(t) = (\tilde{D}^{-\gamma} f)(t) \quad (15)$$

Now let us introduce the Mellin transform operator as

$$\mathcal{M}\{f(t),\gamma\} = \int_0^\infty t^{\gamma-1} f(t)dt = f_{\mathcal{M}}(\gamma); \qquad \gamma = \rho + i\eta \ \in \mathbb{C} \quad (16)$$

where ρ has to be chosen in an interval $-p < \rho < -q$ called *Fundamental strip* (FS) of the Mellin transform; p and q depend of the asymptotic behavior of $f(t)$ at $t \to 0$ and $t \to \infty$:

$$\lim_{t\to 0} f(t) = \mathcal{O}(t^p); \qquad \lim_{t\to\infty} f(t) = \mathcal{O}(t^q) \quad (17)$$

where $\mathcal{O}(\cdot)$ is the order of the term in parenthesis. From the knowledge of $f_{\mathcal{M}}(\gamma)$ we may reconstruct the function $f(t)$ in the whole domain by the inverse Mellin transform namely

$$f(t) = \mathcal{M}^{-1}\{f_{\mathcal{M}}(\gamma),t\} = \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} f_{\mathcal{M}}(\gamma)t^{-\gamma}d\gamma = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_{\mathcal{M}}(\gamma)t^{-\gamma}d\eta \quad (18)$$

Integrals in Eq. (18) are independent of the value of ρ selected, provided they belongs to the FS, because the Mellin transform of $f(t)$ is holomorph in the FS.
In discretized form Eq. (18) may be written as

$$f(t) \simeq \frac{\Delta\eta}{2\pi}\sum_{k=-m}^{m} f_{\mathcal{M}}(\gamma_k)t^{-\gamma_k} = \frac{1}{b}e^{-\rho\xi}\sum_{k=-m}^{m} f_{\mathcal{M}}(\gamma_k)e^{-ik\frac{\pi}{b}\xi} \quad (19)$$

where $\gamma_k = \rho + ik\Delta\eta$, $b = \pi/\Delta\eta$ and $\xi = \ln t$. The number m of the summation has to be chosen in such a way that contribution of terms order $n > m$ do not produce sensible variations on $f(t)$. Discretization produces a divergence phenomenon in $t = 0$ (unless $f(0) = 0$). This pathological behavior is however confined in the range $-e^{-b} \div e^{-b}$ then it may be easily dropped out by assuming the value in e^{-b} as the value in zero.
The Mellin transform operates for $f(t)$ defined in the range $0 \leq t < \infty$. For ensuing derivations we have to work with $f(t)$ defined in $-\infty < t < \infty$.

Then we divide $f(t)$ into a symmetric and anti-symmetric function $u(t)$ and $v(t)$, respectively

$$f(t) = u(t) + v(t) \tag{20a}$$

$$u(t) = \frac{f(t) + f(-t)}{2}; \qquad v(t) = \frac{f(t) - f(-t)}{2} \tag{20b}$$

In this way, the Mellin transform of $f(t)$ can be written as

$$\mathcal{M}\{f(t), \gamma\} = \int_0^\infty (u(t) + v(t))t^{\gamma-1}dt = u_{\mathcal{M}}(\gamma) + v_{\mathcal{M}}(\gamma) = f_{\mathcal{M}}(\gamma) \tag{21}$$

where $u_{\mathcal{M}}(\gamma)$, $v_{\mathcal{M}}(\gamma)$ are Complex Fractional Moments (CFMs) of order $\gamma - 1$ of $u(t)$ and $v(t)$, respectively. Inverse Mellin transform defined in Eqs. (18) or in its discretized form (19), now restitutes $u(t)$ and $v(t)$ for $t > 0$, and by deifnition of even and odd function the whole $f(t)$ may be defined also for $t < 0$. Moreover by taking into account Eqs. (12), (13) and Eq. (15) we can write

$$(I^\gamma f)(t) = (D^{-\gamma} f)(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\omega|^{-\gamma} f_{\mathcal{F}}(\omega) e^{-i\omega t} d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\omega|^{-\gamma}[u_{\mathcal{F}}(\omega)\cos(\omega t) - i\, v_{\mathcal{F}}(\omega)\sin(\omega t)]d\omega \tag{22a}$$

$$(\tilde{I}^\gamma f)(t) = -(\tilde{D}^{-\gamma} f)(t) = -\frac{i}{2\pi}\int_{-\infty}^{\infty} \operatorname{sgn}(\omega)|\omega|^{-\gamma} f_{\mathcal{F}}(\omega) e^{-i\omega t} d\omega = -\frac{i}{2\pi}\int_{-\infty}^{\infty} |\omega|^{-\gamma}\operatorname{sgn}(\omega)[v_{\mathcal{F}}(\omega)\cos(\omega t) - i\, u_{\mathcal{F}}(\omega)\sin(\omega t)]d\omega \tag{22b}$$

where $u_{\mathcal{F}}(\omega)$ and $v_{\mathcal{F}}(\omega)$ are the Fourier transforms of $u(t)$ and $v(t)$ respectively. Since $u(t)$ is a real even function $u_{\mathcal{F}}(\omega)$ is real and even, while since $v(t)$ is a real odd function $v_{\mathcal{F}}(\omega)$ is imaginary and odd. Then Eqs. (22) evaluated in $t = 0$ reveal that the Riesz integrals and its complementary are related to the CFMs of order $-\gamma$ of $u_{\mathcal{F}}(\omega)$ and $v_{\mathcal{F}}(\omega)$, namely

$$(I^\gamma f)(0) = (D^{-\gamma} f)(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\omega|^{-\gamma} u_{\mathcal{F}}(\omega) d\omega \tag{23a}$$

$$(\tilde{I}^\gamma f)(0) = -(\tilde{D}^{-\gamma} f)(0) = -\frac{i}{2\pi}\int_{-\infty}^{\infty} \operatorname{sgn}(\omega)|\omega|^{-\gamma} v_{\mathcal{F}}(\omega) d\omega \tag{23b}$$

On the other hand from Eq. (8) and (10), by letting $t = 0$ in Eq. (8) and by taking into account Eq. (20a) we get

$$(I^\gamma f)(0) = \frac{1}{2\nu_c(\gamma)}\int_{-\infty}^{\infty} \frac{f(\tau)}{|\tau|^{1-\gamma}} d\tau = \frac{1}{\nu_c(\gamma)}\int_0^\infty \tau^{\gamma-1} u(\tau) d\tau \tag{24a}$$

$$(\tilde{I}^\gamma f)(0) = \frac{1}{2\nu_s(\gamma)}\int_{-\infty}^{\infty} \frac{f(\tau)\operatorname{sgn}(\tau)}{|\tau|^{1-\gamma}} d\tau = \frac{1}{\nu_s(\gamma)}\int_0^\infty \tau^{\gamma-1} v(\tau) d\tau \tag{24b}$$

From Eqs. (23) and (24) it may be stated that the Riesz fractional integrals are related to the Mellin transform of $u(t)$ and $v(t)$ (Di Paola and Cottone, 2009; Di Paola et al., 2010; Di Paola and Pinnola, 2012). From the previous considerations it may be asserted that the quantities $u_{\mathcal{M}}(\gamma)$ and $v_{\mathcal{M}}(\gamma)$ are able to represent both $f(t)$ and its Fourier transform $f_{\mathcal{F}}(\omega)$ in the form

$$f(t) \simeq \frac{\Delta\eta}{2\pi}\sum_{k=-m}^{m} [u_{\mathcal{M}}(\gamma_k) + v_{\mathcal{M}}(\gamma_k)\operatorname{sgn}(t)]\, |t|^{-\gamma_k} \tag{25a}$$

$$f_{\mathcal{F}}(\omega) \simeq \frac{\Delta\eta}{2\pi}\sum_{k=-m}^{m} [\nu_c(1-\gamma_k)u_{\mathcal{M}}(\gamma_k) + i\operatorname{sgn}(\omega)\nu_s(1-\gamma_k)v_{\mathcal{M}}(\gamma_k)]\, |\omega|^{\gamma_k - 1} \tag{25b}$$

Other informations may be found in Di Paola and Cottone (2009); Di Paola et al. (2010); Di Paola and Pinnola (2012).

3 Fractional linear non anticipative filter

In this section we suppose that we have a black-box linear system enforced by an impulse at $t = 0$. In time domain direct measure of the response (displacement, or current, or any other physical quantity) gives the impulse response function $h(t)$, that due to causality condition is zero fo $t < 0$. A measure in frequency domain gives the transfer function $H(\omega)$. $h(t)$ and $H(\omega)$ are related each another by the Fourier transform operator, that is

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{i\omega t}dt = \int_{0}^{\infty} h(t)e^{i\omega t}dt \quad (26)$$

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)e^{-i\omega t}d\omega \quad (27)$$

Let us now suppose that the filter equation is ruled by a linear differential operator $\mathcal{L}[y(t)]$ and then during the test, the equation is

$$\mathcal{L}[y(t)] = \delta(t) \quad (28)$$

where $\mathcal{L}[\cdot]$ is an unknown linear operator that describes the black-box system whose impulse response function is $h(t)$ and $\delta(t)$ is the impulse at $t = 0$. We know the function $h(t)$ (or $H(\omega)$) and we want identify the operator $\mathcal{L}[\cdot]$.

Let us denote as $h_\mu(\gamma)$ the Mellin transform of $h(t)$, that is

$$\mathcal{M}\{h(t);\gamma\} = h_\mu(\gamma) = \int_{0}^{\infty} h(t)t^{\gamma-1}dt \quad (29)$$

ant its inverse Mellin transform returns $h(t)$ in the form

$$h(t) = \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} h_\mu(\gamma)t^{-\gamma}d\gamma \simeq \frac{\Delta\eta}{2\pi}\sum_{k=-m}^{m} h_\mu(\gamma_k)t^{-\gamma_k}; \quad t > 0 \quad (30)$$

Fourier transform of $h(t)$ described by the discretized form of Eq. (30) gives

$$H(\omega) \simeq \frac{\Delta\eta}{2\pi}\sum_{k=-m}^{m} h_\mu(\gamma_k)|\omega|^{\gamma_k-1}\left[\nu_c(1-\gamma_k) + i\nu_s(1-\gamma_k)\,\mathrm{sgn}(\omega)\right] \quad (31)$$

On the other hand the transfer function $H(\omega)$ has an even real part and an odd imaginary part that will be labeled as $A(\omega)$ and $B(\omega)$, respectively. Then we may write

$$H(\omega) = A(\omega) + iB(\omega) \quad (32)$$

where $A(\omega) = A(-\omega)\in\mathbb{R}$, $B(\omega) = -B(-\omega)\in\mathbb{R}$. It follows that because of Eq. (31) these two functions are given as

$$A(\omega) \simeq \frac{\Delta\eta}{2\pi}\sum_{k=-m}^{m} \nu_c(1-\gamma_k)h_\mu(\gamma_k)|\omega|^{\gamma_k-1} \quad (33)$$

$$B(\omega) \simeq \frac{\Delta\eta}{2\pi}\,\mathrm{sgn}(\omega)\sum_{k=-m}^{m} \nu_s(1-\gamma_k)h_\mu(\gamma_k)|\omega|^{\gamma_k-1} \quad (34)$$

From Eqs. (32) to (34) it may be stated that from the knowledge of the Mellin transform of the impulse response function also $H(\omega)$ may be directly evaluated. The causality condition ($h(t) = 0\ \forall t < 0$) implies that

$$A(\omega) = \mathcal{H}\{B(\omega)\} \tag{35}$$

where $\mathcal{H}\{\cdot\}$ is the Hilbert transform operator defined as

$$\mathcal{H}\{B(\omega)\} = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{B(\bar{\omega})}{\omega - \bar{\omega}} d\bar{\omega} \tag{36}$$

and $\mathcal{P}$ means principal value. Condition (36) proves to be fulfilled by making the Hilbert transform of $B(\omega)$ given in Eq. (34).
Now let us suppose that, from the experimental test performed with the impulse response function, we directly obtain $A(\omega)$ and $B(\omega)$, namely the transfer function is given then we may directly operate in frequency domain by making the Mellin transform of $A(\omega)$ and $B(\omega)$, that is

$$\mathcal{M}\{A(\omega); \bar{\gamma}\} = A_\mu(\bar{\gamma}) = \int_0^\infty A(\omega)\omega^{\bar{\gamma}-1} d\omega \tag{37}$$

$$\mathcal{M}\{B(\omega); \bar{\gamma}\} = B_\mu(\bar{\gamma}) = \int_0^\infty B(\omega)\omega^{\bar{\gamma}-1} d\omega \tag{38}$$

where $\bar{\gamma} = \bar{\rho} + i\eta$ and then $A(\omega)$ and $B(\omega)$ are restored in the form

$$A(\omega) = \frac{\Delta\eta}{2\pi}\sum_{k=-m}^{m} A_\mu(\bar{\gamma}_k)\omega^{-\bar{\gamma}_k};$$
$$B(\omega) = \frac{\Delta\eta}{2\pi}\sum_{k=-m}^{m} B_\mu(\bar{\gamma}_k)\omega^{-\bar{\gamma}_k}; \quad \omega > 0 \tag{39}$$

on the other hand, due to the simmetry and antisimmetry of $A(\omega) = A(-\omega)$, $B(\omega) = -B(-\omega)$, Eq. (39) may be also rewritten as

$$A(\omega) = \frac{\Delta\eta}{2\pi}\sum_{k=-m}^{m} A_\mu(\bar{\gamma}_k)|\omega|^{-\bar{\gamma}_k};$$
$$B(\omega) = \frac{\Delta\eta \operatorname{sgn}(\omega)}{2\pi}\sum_{k=-m}^{m} B_\mu(\bar{\gamma}_k)|\omega|^{-\bar{\gamma}_k} \tag{40}$$

By letting $\bar{\gamma}_k = 1 - \gamma_k$, Eq. (40) may be rewritten as

$$A(\omega) = \frac{\Delta\eta}{2\pi}\sum_{k=-m}^{m} A_\mu(1-\gamma_k)|\omega|^{\gamma_k - 1};$$
$$B(\omega) = \frac{\Delta\eta \operatorname{sgn}(\omega)}{2\pi}\sum_{k=-m}^{m} B_\mu(1-\gamma_k)|\omega|^{\gamma_k - 1} \tag{41}$$

Direct comparison with Eqs. (33) and (34) gives

$$A_\mu(1-\gamma_k) = \nu_c(1-\gamma_k)h_\mu(\gamma_k); \quad B_\mu(1-\gamma_k) = \nu_s(1-\gamma_k)h_\mu(\gamma_k) \tag{42}$$

From Eq. (42) we may conclude that in the Mellin domain $A_\mu(1-\gamma_k)$, $B_\mu(1-\gamma_k)$ and $h_\mu(\gamma_k)$ are strictly related each another. With this results in mind we can proceed to define the filter equation.

4 Filter equation

Due to the linearity of the system, once $h(t)$ (or $H(\omega)$) are obtained in analytical form, the response of the system (28) may be easily obtained by invoking the Duhamell superposition integral in the form

$$y(t) = \int_0^t h(t-\tau)f(\tau)d\tau \cong \frac{\Delta\eta}{2\pi}\sum_{k=-m}^{m} h_\mu(\gamma_k)\int_0^t (t-\tau)^{-\gamma_k} f(\tau)d\tau \tag{43}$$

that is valid for quiescent system at $t = 0$.
Inspection of Eqs. (1) and (43) allow us to rewrite $y(t)$ as a summation of RL fractional integrals in the form

$$y(t) \cong \frac{\Delta\eta}{2\pi} \sum_{k=-m}^{m} h_\mu(\gamma_k)\Gamma(1-\gamma_k)(I_{0^+}^{1-\gamma_k} f)(t) \tag{44}$$

The exact filter equation may then be obtained by letting $\Delta\eta \to 0$ and $m \to \infty$, so obtaining

$$y(t) = \frac{1}{2\pi i} \int_{\rho-i\infty}^{\rho+i\infty} h_\mu(\gamma)\Gamma(1-\gamma)(I_{0^+}^{1-\gamma} f)(t) d\gamma \tag{45}$$

From Eq. (44) we get the solution of the equation

$$\mathcal{L}[y(t)] = f(t) \tag{46}$$

where $f(t)$ is the forcing function acting on the black-box system. From Eq. (44) we may state that the inverse operator of Eq. (46) is a linear combination of RL fractional integrals.

5 Filtered white noise process

Let us now suppose that $f(t)$ is a white noise process, labeled as $W(t)$. In this case $y(t)$ is a process and will be denoted, as customary, with a capital letter $Y(t)$. Then Eq. (28) is rewritten as

$$\mathcal{L}[Y(t)] = W(t) \tag{47}$$

The filter equation, for each sample function of $W(t)$ remain valid and then Eq. (44) is rewritten in the form

$$Y(t) \cong \frac{\Delta\eta}{2\pi} \sum_{k=-m}^{m} h_\mu(\gamma_k)\Gamma(1-\gamma_k)(I_{0^+}^{1-\gamma_k} W)(t) \tag{48}$$

Now let us first suppose that $W(t)$ is represented as a Poisson white noise labeled as $W_P(t)$

$$W_P(t) = \sum_{s=1}^{N(t)} R_s \delta(t - T_s) \tag{49}$$

where R_s is the realization of a random variable with assigned distribution $P_R(r)$, T_s is the realization of random times distributed according to Poisson law (independent of R) and $N(t)$ is a Poisson counting process giving the number of spikes in $0 \div t$. The compound Poisson process labeled as $C_P(t)$ is the integral of $W_P(t)$, that is

$$C_P(t) = \sum_{j=1}^{N(t)} R_s U(t - T_s) \tag{50}$$

and increment $dC_P(t)$ are characterized in probabilistic setting by

$$E[(dC_P(t))^k] = \lambda E[R^k] dt \tag{51}$$

where $E[\cdot]$ means mathematical expectation and λ is the mean number of impulses per unit time.
As we insert Eq. (49) in Eq. (48) we get

$$Y(t) \cong \frac{\Delta\eta}{2\pi} \sum_{k=-m}^{m} h_\mu(\gamma_k) \sum_{s=1}^{N(t)} R_s (t-T_s)^{-\gamma_k} U(t-T_s) \tag{52}$$

where $U(\cdot)$ is the unit step function. Eq. (52) may be used to generate a filtered Poisson white noise. On the other hand it is well known that if $E[R] = 0$, $\lambda \to \infty$ and $\lambda E[Y^2]$ remains a finite quantity then the Poisson white noise process reverts to the normal white noise and the Compound Poisson process reverts to the Brownian motion $B(t)$.
In order to generate the sample functions of the normal white noise we may subdivide the interval $0 \div t$ into small intervals of amplitude Δt and we suppose that λ in Eq. (51) is $1/\Delta t$, in this way in each interval we have one impulse (in mean). Moreover we assume that the amplitudes R in

each interval is normally distributed with a variance $E[R^2] = \sigma^2\Delta t$ then $\lambda E[R^2] = \sigma^2$. As we assume that in each interval only a spike occurs ($\lambda = 1/\Delta t$), then we may suppose that at the current time $t = r\Delta t$ we have r impulses in each of them acts the realization of a normal random variable with variance $\sigma^2\Delta t$, moreover because small changes of the exact position of the spike occurrence into Δt does not produce sensible variations on $Y(t)$, then we may suppose that each spike occours at the end (or at the beginning) of each interval. If $\Delta t \to 0$ then we get a true white noise, if Δt is small we generate a band limited white noise, and the smaller Δt the wider the bandwidth is. It follows that Eq. (52) may be rewritten as

$$Y(r\Delta t) \cong \frac{\Delta\eta}{2\pi} \sum_{k=-m}^{m} h_\mu(\gamma_k) \sum_{s=1}^{r} R_s((r-s)\Delta t)^{-\gamma_k} U(r-s) \quad (53)$$

where R_s is the s-th realization of a zero mean normal random variable having $\sigma^2\Delta t$ variance. It is obvious that Eq. (53) restitutes a non stationary process and $Y(r\Delta t)$ for $r = 0$ is zero with probability one. If a stationary process Y has to be simulated it may be selected as a starting time a value of r so large that the steady-state is already reached.
Statistics of $Y(r\Delta t)$ generated by Eq. (53) is readily found in time domain by taking account that $E[R_s R_p] = 0$ if $s \neq p$. Then

$$E[Y(r\Delta t)Y(q\Delta t)] = \frac{(\Delta\eta)^2}{4\pi^2} \sum_{k=-m}^{m} \sum_{s=-m}^{m} h_\mu(\gamma_k) h_\mu(\gamma_j) \sum_{s=1}^{|r-q|} \sigma^2\Delta t[(r-s)\Delta t]^{-\gamma_k}[(q-s)\Delta t]^{-\gamma_j} \quad (54)$$

Eq. (54) gives the correlation function of the process $Y(r\Delta t)$. The correlation of the process $Y(t)$ depends of $r\Delta t = t_1$ and $q\Delta t = t_2$. This s becuase the response process generated by Eq. (48) is quiescent in $t = 0$. Then in order to get a steady state correlation it is enough to assume r very large and $q < r$.

6 Numerical examples

6.1 Transfer function and impulse response function

In this section we want to validate the procedure described in previous section. Firstly we suppose to test a mechanical system and to measure its impulse response function $h(t)$; we suppose that the function has the form of the impulse response function of the single degree of freedom (SDOF) system and that we don't know the filter equation (that in this case is already known). In this case the impulse response function target is

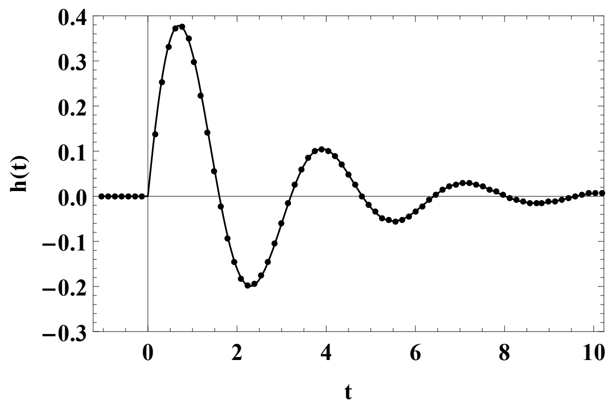

Figure 1: Impulse response function h(t): Eq. (55) (continuous line) vs. Eq. (30) (dotted line).

$$h(t) = \frac{1}{\omega_d} \exp(-\zeta\omega_0 t)\sin(\omega_d t) \quad (55)$$

and its Fourier transform is given as

$$H(\omega) = \frac{(\omega_0^2 - \omega^2) + 2i\zeta\omega\omega_0}{(\omega_0^2 - \omega^2)^2 + 4\zeta^2\omega^2\omega_0^2} \quad (56)$$

where ω_0 is the natural frequency of the undamped system, while ω_d is the damped frequency $\omega_d = \omega_0\sqrt{1-\zeta^2}$ and ζ is the percentage of the critical damping.

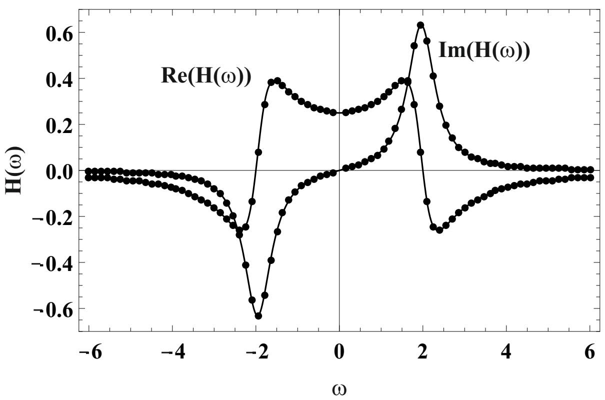

Figure 2: Transfer function, real and imaginary part $\Re[H(\omega)]$ and $\Im[H(\omega)]$; Eq. (56) (continuous line) vs. Eq. (31) (dotted line).

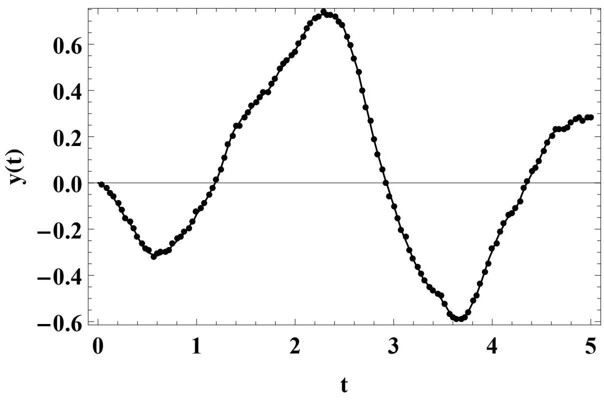

Figure 3: Sample of the process $Y(t)$; step-bystep integration (continuous line) vs. Eq. (53) (dotted line).

Eq. (55) and (56) are our target. By evaluating $h_\mu(\gamma)$ defined in Eq. (27) we may reconstruct both $h(t)$ and $H(\omega)$ by using Eqs. (30) and (31). In Fig. 1 and 2 the target $h(t)$ and $H(\omega)$ are plotted and contrasted with results obtained by Eq. (30) and (31).

The parameters selected are $\omega_0 = 2rad/sec$, $\zeta = 0.2$, $\omega_d = \omega_0\sqrt{1-\zeta^2} \simeq 1.96rad/sec$, $\Delta\eta = 0.5$, $m = 40$ and consequently $\eta_c = 20$; Figs. 1 and 2 show the contrasts between $h(t)$, $\Re[H(\omega)]$ and $\Im[H(\omega)]$ respectively and their counterparts evaluated with CFMs.

6.2 Response to white noise process

In this section we want to validate the procedure discussed in Sec. 5. The equation of motion is that descibed in Eq. (47) with $W(t)$ a normal white noise process. The impulse response function is that described in Eq. (55) that is represented in terms of $h_\mu(\gamma_k)$how it has been made in Sec. 6.1. The sample function of the response is plotted in continuous line in Fig. 3. Then Eq. (53) is applied for the case in exam and results are plotted in dotted line.

With the results obtained in this section we may generate any stochastic process with a general form of filter.

7 Conclusions

In this paper a method to obtain an exact and an approximated filter equation for any linear system is proposed. The procedure starts from the knowledge of the impulse response function or the transfer function of the system. Once one of this quantity is known, we can reconstruct both the impulse response function and the transfer function simply by calculating the Mellin transform of the known fucntion, namely its complex fractional moments. The filter equation can be then constructed simply by performing the Duhamell integral of the impulse response function written in terms of complex fractional moments, that renstitute a summation of Riemann-Liouville integral. The main advantage of this filter is that it is non anticipative. It has been shown that with a limited number of complex fractional moments it is possible to construct a general form of the filter and to generate filtered processes with the same accuracy of existing methods.

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