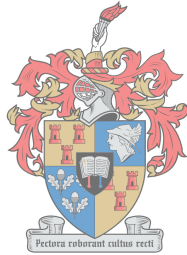# New multi-objective ranking and selection procedures for discrete stochastic simulation problems

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Dedicated to my parents, the two heroes of my life.

Declaration

By submitting this dissertation electronically, I declare that the entirety of the work contained therein is my own, original work; that I am the sole author thereof (save to the extent explicitly otherwise stated); that reproduction and publication thereof by Stellenbosch University will not infringe any third-party rights, and that I have not previously in its entirety or in part submitted it for obtaining any qualification.

This dissertation includes one original paper published in a peer-reviewed journal and three submitted, unpublished publications. The development and writing of the papers (published and unpublished) were the principal responsibility of myself and for each of the cases where this is not the case, a declaration is included in the dissertation indicating the nature and extent of the contributions of co-authors.

Date:. . . . .

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Abstract

In stochastic simulation optimisation, several system designs are considered. These designs are ranked in order and the best is selected based on one or more performance measures. Any ranking and selection (R&S) procedure must ensure that the correct system design is chosen, and this is a challenging task in the stochastic environment.

This dissertation discusses the design and development of a new multi-objective ranking and selection (MORS) procedure, called Procedure $\mathcal{MMY}$, and two variants of it, called Procedures $\mathcal{MMY}1$ and $\mathcal{MMY}2$.

Single-objective ranking and selection procedures endeavour to find the best system, *i.e.*, the system with the minimum or maximum output, out of a limited number of feasible solutions. There are two important approaches in the single-objective R&S area: the *indifference-zone* (IZ) approach and the *optimal computing budget allocation* (OCBA) framework. While the OCBA procedure has been extended to the multi-objective domain, an MORS procedure with the IZ approach has not yet appeared in the literature. The $\mathcal{MMY}$ family procedures have been developed in an attempt to fill this gap, therefore they take the IZ approach.

Indifference-zone procedures should guarantee that the probability of correct selection is at least a prespecified value P^*, denoted by $P(\text{CS}) \geq P^*$, where 'correct selection' denotes the event that the system with the minimum output is selected for a single-objective minimisation problem. In the multi-objective context, Pareto optimality is employed to define 'correct selection'.

The concept of *relaxed* Pareto optimality is proposed in this research to accommodate the indifference-zone concept properly in the multi-objective domain. Thus, Procedure $\mathcal{MMY}$ guarantees $P(\text{CS}) \geq P^*$ considering the event

of identifying a relaxed Pareto set as a correct selection. Procedure $\mathcal{MMY}1$ tries to find the normal Pareto optimal set while Procedure $\mathcal{MMY}2$ focuses on identifying Pareto optimal solutions with the IZ concept.

The statistical validity of the $\mathcal{MMY}$ family procedures is proved through rigorous mathematical analyses in this dissertation. A Bayesian probability model was used in the $P(\text{CS})$ formulation in the proofs. Using a Bayesian model in the $P(\text{CS})$ formulation in IZ R&S procedures is a novel approach even in the single-objective context. The researcher therefore proposed a new single-objective R&S procedure, called Procedure $\mathcal{MY}$, in addition to the multi-objective $\mathcal{MMY}$ family procedures. The $\mathcal{MY}$ procedure is discussed prior to the discussion of the $\mathcal{MMY}$ family procedures, verifying the effectiveness of the Bayesian model, thereby laying the theoretical foundation for employing it for the $\mathcal{MMY}$ family procedures.

The performance of the proposed $\mathcal{MMY}$ family procedures was demonstrated using four simulation case studies. These simulation case studies provided various types of test beds to understand the behaviour of the proposed procedures. In all four cases the estimated probability of correct selection was observed to be greater than P^* for all three procedures, proving the statistical validity of them empirically, too. In addition, the performance of the proposed $\mathcal{MMY}$ family procedures was compared to that of the MOCBA procedure, which is the only existing MORS procedure. The result showed the superiority of the $\mathcal{MMY}$ procedure over the MOCBA procedure in many cases.

Opsomming

In stogastiese simulasie-optimering word verskeie stelselontwerpe oorweeg. Hierdie ontwerpe word in rangorde rangskik en die beste gekies, gebaseer op een of meer prestasiemaatstawwe. Enige rangskik-en-kies prosedure moet verseker dat die korrekte stelselontwerp gekies word, en hierdie is 'n uitdagende taak in die stogastiese omgewing.

Hierdie proefskrif bespreek die ontwerp en ontwikkeling van 'n nuwe multi-doelwit rangskik-en-kies (MDRK) prosedure in stogastiese optimering. Die prosedure word $\mathcal{MMY}$ genoem, met twee variante genaamd $\mathcal{MMY}1$ en $\mathcal{MMY}2$.

Enkeldoelwit rangskik-en-kies prosedures (R&K) poog om die beste stelsel, dit wil sê, die stelsel met die minimum of maksimum afvoer, uit 'n beperkte aantal gangbare oplossings te vind. Daar is twee belangrike benaderings in die enkeldoelwit R&K area: die geen-verskilsone (GS) benadering en die optimum-rekenbegroting toedeling (ORBT) raamwerk. Hoewel die ORBT prosedure uitgebrei is na die multi-doelwitdomein, bestaan daar tans nie 'n MDRK prosedure in die GS domein nie. Die $\mathcal{MMY}$ familie van prosedures is geskep om hierdie gaping te vul, dus gebruik die prosedures die GS benadering tot R&K.

GS prosedures behoort te waarborg dat die waarskynlikheid van korrekte keuse 'n voorafgestelde waarde P^* bevredig, aangedui met $P(\text{CS}) \geq P^*$. Die term 'korrekte keuse' dui op die gebeurtenis dat die stelsels met die minimum uitsetwaarde gekies word in 'n enkeldoelwitoptimeringprobleem, terwyl Pareto-optimaliteit in die multi-doelwitkonteks gebruik word om 'korrekte keuse' te definieer.

Die konsep van verslapte Pareto-optimaliteit word in hierdie navorsing voorgestel om die geen-verskilkonsep voldoende in die multidoelwitdomein te akkommodeer. Prosedure $\mathcal{MMY}$ waarborg $P(\text{CS}) \geq P^*$ as 'n verslapte

Pareto-versameling as korrekte keuse aanvaar word. Prosedure $\mathcal{MMY}1$ poog om die streng-korrekte Paretostel te vind, terwyl Prosedure $\mathcal{MMY}2$ fokus op die vind van Pareto-optimale oplossings met die GS konsep.

Die statistiese geldigheid van die $\mathcal{MMY}$ familie van prosedures word in hierdie proefskrif bewys deur streng wiskundige analise. 'n Bayes-waarskynlikheidsmodel is gebruik in die formulering van $P(\text{CS})$ in die bewyse. Die gebruik van 'n Bayes-model in die formulering van $P(\text{CS})$ in GS R&K prosedures is uniek, selfs in die enkeldoelwit geval. Die navorser het dus 'n nuwe enkeldoelwit R&K prosedure, naamlik $\mathcal{MY}$, tesame met die multidoelwit $\mathcal{MMY}$ familie van prosedures voorgestel. Die $\mathcal{MY}$ prosedure word eerste aangebied en bespreek, en daardeur word die effektiwiteit van die Bayes-model bevestig. Sodoende is die teoretiese basis vir gebruik van die Bayes-model in die $\mathcal{MMY}$ familie van prosedures gelê.

Die prestasie van die $\mathcal{MMY}$ familie van prosedures word aan die hand van vier simulasiegevallestudies demonstreer. Hierdie gevallestudies verskaf verskillende tipes toetsplatforms wat bydra om die gedrag van die voorgestelde prosedures te verstaan. In al vier gevalle is die beraamde waarskynlikheid van korrekte keuse groter as P^* vir al drie prosedures, wat die statistiese geldigheid daarvan empiries ondersteun. Verder is die prestasie van die voorgestelde familie van $\mathcal{MMY}$ prosedures met die van die ORBT prosedure vergelyk, wat die enigste multidoelwit R&K prosedure tot op hede is. Die resultate toon dat die $\mathcal{MMY}$ prosedures in verskeie gevalle die ORBT prosedure oorheers.

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Chapter 1

Introduction

This dissertation presents a research on the topic of simulation optimisation. As an introduction this chapter offers some background of the research domain and the motivation for the study. The research aim and objectives are then presented, followed by the methodology employed in the research, and finally the structure of the dissertation is explained.

1.1 Background of the research domain

We often face moments where we have to make decisions the impacts of which are far too important to make them arbitrarily. One would like to consider all possible options, analyse the results from each option and compare them before making such decisions. Sometimes the options are so many that it is almost impossible to select the best after considering all of them; yet at other times there seems to be no option at all due to the constraints of the problem.

Optimisation is a research field that emerged to provide a scientific way to deal with such decision-making problems. It defines the 'options' as *decision variables*, and the 'result' of an option is formulated as a function of the decision variables, called *objective function* or *objective* for short. The 'constraints' are also, when they exist, designed as functions of decision variables. Optimisation then can be defined as a process of finding the best combination of decision variable values for the given objective and constraints. For example, in a classic single-commodity inventory problem of 'At what inventory level should a new order be made and how many?', the reorder level and the reorder

1

quantity serve as two decision variables of this problem. A typical objective of the problem would be to minimise the average total cost while the size of the warehouse could play the role of the constraint.

Formulating the objective (and constraints as well) as a closed-form function of decision variables is not an easy task. The correct design of the objective function would entail a thorough investigation of the system for reasonably predictable factors such as, in the aforementioned inventory problem, holding cost, operation/administration cost, *etc.* as well as assumptions of uncertain factors, for example, the frequency and the size of customer demands, again in the inventory problem. It is obvious that the effectiveness of a solution of an optimisation problem depends mostly on the modelling of the problem, *i.e.*, the comprehensive analysis of the system, followed by the proper definition of decision variables and the precise formulation of the objective function and constraints. In this regard, it can be said without exaggeration that the formulation of the problem is the most decisive step in solving optimisation problems.

Although existing optimisation algorithms have managed well to solve complex problems and have most certainly contributed to better decision-making in a vast number of applications of almost all types of real-life problems, it should be recognised that the exact formulation of the objective function is often impossible when the system is too complex or little is known about the system, and more importantly when there exists *uncertainty* in the system under consideration. *Simulation optimisation* (SO) steps in for this situation.

Simulation makes it possible to evaluate complex real-world systems where an analytic solution is out of the question due to the complexity and/or dynamic, stochastic nature of the system. In SO, the objective function values are not obtained analytically but they are *estimated* through computer simulation. This often offers a better evaluation of the system than an analytic solution, where the complex system should be simplified and a set of possibly unrealistic assumptions should be made in order to establish the closed-form analytical solution. In the above inventory problem, for example, while it is not straightforward to formulate the average total cost as a mathematical function of the two decision variables (the reorder level and reorder quantity), simulation software can easily produce the average total cost over any length of time after imitating, or *simulating*, the operation of the real system. In addition, the simulation model can take the stochastic nature of the system into account by using data

collected from observing the real system. The interarrival time between two customers, for example, can be modelled as an independently and identically distributed (i.i.d) exponential random variable with a mean of 2 minutes based on the observation of the real system over a fixed period of time. This way the uncertainty of the system is also modelled in simulation, thereby rendering it a more reliable model of the real system. See Law & Kelton (2000) for further discussion of simulation study and modelling.

There is a wide variety of terms used in referring to the inputs and outputs of a simulation optimisation problem (Fu, 2002). Inputs are normally referred to as 'decision variables' in optimisation, and 'scenarios', 'parameter settings', 'configurations', 'solutions', 'designs' or 'systems' are used in the simulation literature. Outputs are called 'objective functions' or 'objectives' in the optimisation context, and 'responses', 'performances' or 'performance measures' in simulation.

Note that due to the stochastic nature of the system, the output of a simulation run is merely a particular realisation of a random variable that may have a large variance (Law & Kelton, 2000). This means that different runs of the same simulation model produce different outputs for the same set of decision variable values, or for the same scenarios in a simulation-oriented term, due to the randomness inherent in the system. Therefore, in simulation studies typically a multiple number of simulation replications are performed for each scenario, and the performance of the system (the objective) is estimated (mostly) via sample means of the outputs. This contrasts with the *deterministic* optimisation case where the objective function value is uniquely determined by a set of decision variable values. The main concern of deterministic optimisation algorithms lies in identifying the best set of decision variable values from (typically) a vast number of feasible solutions within a realistic time limit. The algorithms focus on how to explore the large decision space in search for the optimal or near-optimal solutions. On the other hand, SO involves methods for obtaining accurate estimates of the objective function in addition to the identification of the best solution. This adds a fundamental complication to the simulation optimisation efforts.

When an optimisation problem involves more than one objective function, the task of finding one or more optimum solutions is known as *multi-objective optimisation* (MOO) (Deb, 2001). Finding the 'best' solution(s) in MOO is not trivial because the multiple objectives are often conflicting and non-commensurable. A good solution with respect to one objective could easily be a bad one in terms of other objectives.

For this reason MOO problems usually have a set of best solutions rather than a single best one. These solutions form the *Pareto optimal set*. A formal definition of Pareto optimality will follow in Section 4.1.3, but intuitively it is defined as follows: A solution to a minimisation MOO problem is Pareto optimal if there exists no other feasible solution which would decrease some objective function values without causing a simultaneous increase in at least one other objective (Coello Coello, 2006). Pareto optimal solutions are also called 'non-dominated' solutions as they are not dominated by any other solutions in the feasible set. The ultimate goal of any MOO problem involves determining the Pareto optimal set.

In this section, two important subfields of optimisation have been introduced: *simulation optimisation* (SO) and *multi-objective optimisation* (MOO). When SO is mentioned, normally a single objective is considered unless explicitly stated otherwise. Similarly, a deterministic environment is typically assumed when MOO is discussed. Combining these two forms the main subject of this research: *multi-objective simulation optimisation* (MOSO). In the next section, the motivation for the research is presented.

1.2 Motivation for the research

Both SO and MOO problems have been intensively studied for several decades (Fu, 2015; Miettinen, 2008). A preliminary literature study showed, however, that relatively little work has been done in the MOSO area (Xu *et al.*, 2015) compared to the two origins of the field, *i.e.*, SO and MOO. This has drawn the attention of the researcher. The literature study further identified a promising research topic in a subfield of MOSO that is called *multi-objective ranking and selection* (MORS).

When a simulation optimisation problem has a relatively small number of feasible solutions, the problem is classified as a *ranking and selection* (R&S) problem. Small-sized problems in deterministic optimisation can easily be solved by carrying out an exhaustive search, that is, by evaluating every possible solution and returning the optimal one (Burke & Kendall, 2005). However, in simulation optimisation, where the objective function value for each solution represents a random variable with variance, selecting the best even from a small number of possible solutions is not simple. For one thing, one can never be 100% sure that the selected solution is truly the best one even if the decision is made based on the results of a multiple number of simulation replications.

There is always a risk of making the wrong decision due to the stochastic nature of the system. One can reduce the risk by performing a large number of simulation replications, but simulation is often costly, therefore a trade-off exists between the quality of the output and the computational cost of simulation optimisation problems (Yoon & Bekker, 2017b). Ranking and selection procedures determine the way in which this trade-off is dealt with for small-sized SO problems.

The term 'ranking and selection' (R&S) comes from the statistics community, where researchers have been dedicated to identify the 'best' population, *i.e.*, with the largest (or smallest) mean, among k populations. The initial attempt of such is seen in Bechhofer (1954). His work was motivated by 'some deficiencies' of analysis of variance (ANOVA), one of the most popular statistical techniques in those days (and perhaps in these days as well). ANOVA tests if there is a significant difference among the means of k populations, often to identify the effects of k different treatments. In many instances, however, the interest of the experimenters would be to *rank* the treatments so that they can *select* the best treatment (Yoon & Bekker, 2017d). Bechhofer (1954) presented a procedure for ranking means of k normal populations with known variances as a solution to this kind of problem, which became the pioneering work of the vast amount of research in a new research field called *ranking and selection* in the statistics community. Interestingly, R&S has the same goal as simulation optimisation when the problem size is small. R&S and simulation optimisation have begun from different starting points (one from statistics and the other from optimisation), but they eventually met at the point where both are applied to identify the best solution among k alternatives when uncertainty exists.

There are two main approaches to ranking and selection: the *indifference-zone* (IZ) method and the *optimal computing budget allocation* (OCBA) framework (Lee *et al.*, 2010a). The formulations differ by whether the requirement is imposed on the evidence of correct selection, or on the simulation budget (Von Saint Ange, 2015). The former focuses on identifying the minimum number of simulation replications for each solution to meet the probability of correct selection requirement (predesignated by the decision-maker), while the latter is interested in efficiently allocating the limited simulation budget (often the total number of simulation replications available) in order to yield the maximum probability of correct selection given the budget.

R&S procedures that consider multiple objectives are called *multi-objective ranking and selection* (MORS) procedures. The OCBA approach has been extended to the multi-objective domain, resulting in the well-known *multi-objective optimal computing budget allocation* (MOCBA) algorithm by Lee *et al.* (2004, 2010b). However, there has not yet been an MORS procedure using the IZ method. This means that when an existing MORS procedure presents a Pareto optimal solution set, there is no way to assure the decision-maker of the 'quality' of the final solutions. Obviously one can be sure that the MOCBA algorithm presents the 'best' quality of solutions given the simulation budget, that is, the Pareto optimal solution set given by the MOCBA algorithm is as close to the true Pareto optimal solution set as possible under the limited simulation budget. However, one has no idea of what this 'best quality' means—it could mean a 90% probability of correct selection or 50%. On the other hand, an MORS procedure with the IZ approach, if it exists, would guarantee the quality of its final solution. That is, the probability of correct selection of this procedure would always be greater than or equal to the predesignated value because an IZ-based R&S procedure would not stop until it reaches the required quality no matter how many simulation replications are needed.

Whether to impose the requirement on the evidence of correct selection (the IZ approach) or on the simulation budget (the OCBA approach) should be the decision-makers' call. Under the current situation, however, they have no other option but to choose the latter because there is no MORS procedure with the IZ approach. This motivated the research, of which the aim and objectives can subsequently be stated.

1.3   Research aim and objectives

According to Muller (2008), a research aim means the macro purpose of the study, and research objectives are specific research tasks that need to be performed to achieve the aim. With this in mind, this section states the aim and objectives of this research.

> **The aim of this research is to develop a multi-objective ranking and selection procedure for stochastic systems with the indifference-zone approach.**

The procedure must provide evidence that the final solution has the required quality. The research objectives are as follows:

1. Review the literature.

2. Design a multi-objective ranking and selection procedure.

3. Prove mathematically that the procedure guarantees the required quality of its final solution.

4. Verify the statistical validity of the procedure through numerical experiments.

The first objective is an essential prerequisite of any research. The second objective states the main task of this research, and the third and fourth objectives respectively support the second objective theoretically and empirically.

1.4   Methodology

In this section, the methodology used in this research is introduced as follows:

0. The researcher has decided to use manuscripts as the foundation of the dissertation.

1. A thorough literature study on the topic of simulation optimisation, both single- and multi-objective areas, was followed. Two manuscripts were written and submitted as a result of the literature study. The first manuscript (Yoon & Bekker, 2017c) provides an overview of existing multi-objective simulation optimisation (MOSO) algorithms, classifying them based on the size of the feasible solution space and the method of dealing with the multiple objectives. The discussion includes multi-objective ranking and selection (MORS) procedures as well as large-scale MOSO algorithms. The second one (Yoon & Bekker, 2017d) focuses on SO algorithms with small-sized solution spaces, *i.e.*, ranking and selection (R&S) procedures. It discusses single- and multi-objective ranking and selection procedures from a historical point of view. This forms the first part of the dissertation (Chapter 2).

2. At an early stage in the research process, the researcher found that (single-objective) IZ procedures are often conservative, meaning the probability of correct selection, denoted by $P(\text{CS})$, tends to be higher than the required value P^*. This

is mostly due to the fact that IZ procedures assume the least favourable configuration (LFC), which will be discussed in detail in Section 3.1, and therefore do not consider sample mean information in the decision-making process. In addition, the researcher learnt during the literature study that there have been many efforts to eliminate the LFC assumption in existing IZ procedures, however none of them succeeded in developing such a procedure with a rigorous mathematical analysis that assures the $P(\text{CS})$ guarantee. This encouraged the researcher to delve for methods to design such a procedure with the $P(\text{CS})$ guarantee, which led to the development of Procedure $\mathcal{MY}$ in the single-objective R&S domain. Procedure $\mathcal{MY}$ takes advantage of sample mean information, thereby less conservative. Also, the statistical validity of the procedure is proved mathematically by using a Bayesian inference model. Some numerical experiments were performed to validate the procedure and to compare the performance with other existing R&S procedures. This forms the second part of this study (Chapter 3 of the dissertation), and the result was submitted for publication (Yoon & Bekker, 2017b).

3. In the next stage of the research, a new multi-objective ranking and selection (MORS) procedure, called Procedure $\mathcal{MMY}$, was developed based on the single-objective $\mathcal{MY}$ procedure. In addition, two variants of Procedure $\mathcal{MMY}$, called $\mathcal{MMY}1$ and $\mathcal{MMY}2$, were also established. These procedures are novel MORS procedures that use the indifference-zone approach. The statistical validity of these procedures is provided again based on the Bayesian inference model through rigorous mathematical proofs. In addition, four simulation case studies were carried out to verify the effectiveness of the proposed procedures. This forms the third part of the study and corresponds to Chapters 4 and 5 of the dissertation. The manuscript of this last part of the research is still under development (Yoon & Bekker, 2017a).

1.5   Structure of the dissertation

This chapter introduced the concept of optimisation, simulation optimisation (SO), multi-objective optimisation (MOO), and ranking and selection (R&S) to lead the reader into the main research field of this study: multi-objective ranking and selection

(MORS). The research motivation is then described, followed by the research aim, objectives and the methodology.

The remainder of the dissertation is structured as follows: In Chapter 2, a literature study on the single- and multi-objective ranking and selection area is presented. Chapter 3 provides a theoretical basis of the main work of this research by introducing a new single-objective ranking and selection procedure that uses a Bayesian inference model. The development of the multi-objective ranking and selection procedures is then described in Chapter 4 along with the mathematical proof of their statistical validity. The proposed procedures are assessed using a simple example in Chapter 4 and further through a few dynamic, stochastic simulation case studies in Chapter 5. Finally, Chapter 6 concludes the dissertation with a short summary, the contribution of the work to the body of knowledge, and recommendations for future work.

Chapter 2

Ranking and selection procedures: Literature study

This chapter presents an overview of scholarly literature of ranking and selection (R&S) procedures, first in the single-objective domain (Section 2.1), followed by the multi-objective domain (Section 2.2). The main focus of this research lies in multi-objective ranking and selection (MORS), therefore the scope in this chapter is restricted to R&S, which is a subfield of simulation optimisation (SO) where the number of feasible solutions is relatively small. In small-sized SO problems, one can simulate *all* solutions and select the best based on the complete enumeration of all solutions. The problem then boils down to how to guarantee that the selected best system is truly the best one in the presence of the stochastic nature of the problem (Yoon & Bekker, 2017c), which is the main concern of R&S procedures.

Large-scale simulation optimisation algorithms have a fundamentally different approach to solving the problems. Because the solution space is too large to simulate all solutions, the algorithm needs a strategic search mechanism to explore the vast solution space in addition to the problem of accurately estimating the system. See Pasupathy & Ghosh (2013), Amaran *et al.* (2014), Fu (2015) and Xu *et al.* (2015) for a comprehensive literature survey on the more general topic of SO, including small- and large-scale SO problems.

This chapter is largely based on Yoon & Bekker (2017d) and Yoon & Bekker (2017c). The former discusses single- and multi-objective ranking and selection from a historical point of view, and the latter gives an overview of multi-objective simulation optimi-

sation (MOSO) literature including multi-objective ranking and selection (MORS) as well as large-scale MOSO problems.

The researcher does not present a list of symbols in the following discussions because they are self-explanatory in this chapter.

2.1   Single-objective ranking and selection procedures

R&S procedures are statistical methods specifically developed to select the best system from a set of k competing alternatives (Goldsman & Nelson, 1994). There are two basic approaches in single-objective R&S: the *indifference-zone* (IZ) method and the *optimal computing budget allocation* (OCBA) approach. These are subsequently discussed.

2.1.1   Indifference-zone procedures

Indifference-zone procedures determine the number of simulation replications to be allocated to each system with the aim of guaranteeing the quality of the final solution to a certain level P^*. This level is decided by the decision-maker before the procedure begins. The decision-maker also determines the indifference-zone (IZ) value δ^*, which is defined as the smallest value that is 'worth detecting' (Bechhofer, 1954).

Suppose we have three different designs of a system, and the performance measure of each system design follows the three distributions labelled I, II and III, respectively, as shown in Figure 2.1. Suppose further that the decision-maker would like to find the system with the smallest performance measure, *i.e.*, design I is the best solution. However, this information is unknown, and the R&S procedures estimate the true means by using sample means. It can be concluded easily that design III is not the best solution, while design II can often be mistakenly selected as the best solution due to its close performance to design I. A smart procedure would take more samples from designs I and II to avoid this mistake; and more observations are needed as the difference between the two performances (marked as δ_1 in Figure 2.1) becomes smaller. An IZ procedure would take as many observations as needed to guarantee that the probability of selecting design I is greater than or equal to the required level of P^*. However, if $\delta_1 < \delta^*$, designs I and II are equally good to the decision-maker, and the
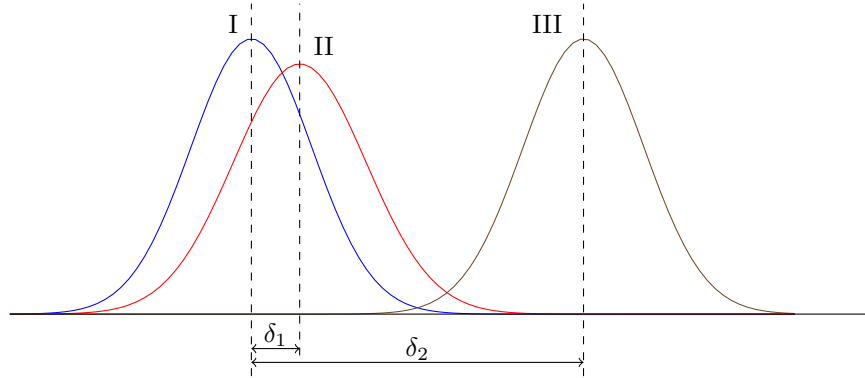
Figure 2.1: Different performance measure distributions of three system designs (Yoon & Bekker, 2017d)

decision-maker would be indifferent to either of them (hence the term *indifference-zone*). In this case, the IZ procedure stops trying to distinguish designs I and II, but presents either of them as the final solution.

In a more general form, suppose that there are k designs, of which the performance measure is associated with a distribution of mean μ_i $(i = 1, \ldots, k)$. Suppose further, without loss of generality, $\mu_b \leq \mu_i$ $(i = 1, \ldots, k; i \neq b)$ so that design b is the best system in a minimisation problem. Under these assumptions, an IZ procedure guarantees that the probability of correct selection, denoted by $P(\text{CS})$, is at least P^*, that is,

$$P(\text{CS}) = P\left[\text{select design } b \mid \mu_i - \mu_b \geq \delta^*, \forall i\, (i \neq b)\right] \geq P^*. \tag{2.1}$$

As mentioned in Chapter 1, the work of Bechhofer (1954) is the origin of R&S procedures. Having established not only the concept of indifference-zone δ^* but also that of the probability of correct selection $P(\text{CS})$, Bechhofer is considered as the 'father' of the field (Fu, 1994).

Bechhofer (1954) presented the procedure in a very general way, that is, the purpose of the procedure is to find (among k *normal* populations) the k_s best populations (or systems), the k_{s-1} second best populations, the k_{s-2} third best populations, *etc.*, and finally the k_1 worst populations, where $k_1, k_2, \ldots, k_s$ are positive integers such that $\sum_{i=1}^{s} k_i = k$. This general goal is given in (6) in Bechhofer (1954, p. 19). In the following discussions, however, the researcher restricted the case to identifying the best system among k systems, hence $s = 2$ and $k_2 = 1$ were used.

Let X_{ij} $(i = 1, \ldots, k; \ j = 1, 2, \ldots,)$ be normally and independently distributed variables with mean μ_i and variance σ_i^2. Also suppose system k has the largest true mean. Then for a maximisation problem, the probability of correct selection $P(\text{CS})$ can be written as

$$P(\text{CS}) = P[\max\{\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_{k-1}\} < \overline{X}_k] \tag{2.2}$$

$$= P[Y_1 > 0, Y_2 > 0, \ldots, Y_{k-1} > 0], \tag{2.3}$$

where $Y_i = \overline{X}_k - \overline{X}_i$ $(i = 1, \ldots, k-1)$. The random variables Y_i have a $(k-1)$-variate normal distribution, and the discussion in Bechhofer (1954) continues to express the $P(\text{CS})$ as a volume under the multivariate normal surface.

Bechhofer (1954) also expresses $P(\text{CS})$ as iterated integrals, which he states 'for certain purposes [...] is more convenient' (Bechhofer, 1954, p. 21). This was confirmed by subsequent studies that use the same principle of establishing $P(\text{CS})$ as iterated integrals, among which Dudewicz & Dalal (1975) and Rinott (1978) are important. In this approach, Bechhofer (1954) assumed the least favourable configuration (LFC), which is

$$\mu_1 = \mu_2 = \ldots = \mu_{k-1} = \mu_k - \delta^*. \tag{2.4}$$

For a maximisation problem where one would like to select the system with the largest mean μ_k, while ignoring the differences smaller than δ^*, the above configuration (2.4) is certainly the most difficult case to determine system k, thus is called the 'least favourable configuration'. Bechhofer (1954) also assumed known, equal variances $\sigma_i^2 = \sigma^2$ and $N_i = N$. Then, the probability of correct selection can be written as follows[1]:

$$P(\text{CS}) = P[\overline{X}_1 < \overline{X}_k, \ \ \overline{X}_2 < \overline{X}_k, \ \ldots, \ \ \overline{X}_{k-1} < \overline{X}_k]$$

$$= \int_{-\infty}^{\infty} P[\overline{X}_1 < \overline{X}_k, \ \ldots, \ \ \overline{X}_{k-1} < \overline{X}_k] f(\overline{X}_k) \mathrm{d}\overline{X}_k \tag{2.5}$$

$$= \int_{-\infty}^{\infty} P(\overline{X}_1 < \overline{X}_k) \times \ldots \times P(\overline{X}_{k-1} < \overline{X}_k) f(\overline{X}_k) \mathrm{d}\overline{X}_k \tag{2.6}$$

$$= \int_{-\infty}^{\infty} \Phi\left(\frac{\overline{X}_k - \mu_1}{\frac{\sigma}{\sqrt{N}}}\right) \times \ldots \times \Phi\left(\frac{\overline{X}_k - \mu_{k-1}}{\frac{\sigma}{\sqrt{N}}}\right) f(\overline{X}_k) \mathrm{d}\overline{X}_k \tag{2.7}$$

$$= \int_{-\infty}^{\infty} \left[\Phi\left(\frac{\overline{X}_k - \mu_1}{\frac{\sigma}{\sqrt{N}}}\right)\right]^{k-1} f(\overline{X}_k) \mathrm{d}\overline{X}_k \tag{2.8}$$

---

[1]The equations here were reformulated by the researcher for the case of $s = 2$ and $k_s = 1$ based on the general discussion in Bechhofer (1954, p. 21–22).

$$= \int_{-\infty}^{\infty} \left[\Phi \left(y + \frac{\sqrt{N}\delta^*}{\sigma} \right) \right]^{k-1} \phi(y) \mathrm{d}y, \tag{2.9}$$

where f denotes the probability density function (p.d.f) of the normal distribution $N(\mu_k, \frac{\sigma^2}{N})$, Φ and ϕ are the cumulative distribution function (c.d.f) of the standard normal distribution and its p.d.f, respectively, and y is a transformation of the variable $\overline{X}_k$:

$$y = \frac{\overline{X}_k - \mu_k}{\sigma/\sqrt{N}}. \tag{2.10}$$

The equality in (2.5) holds because $\overline{X}_k \sim N(\mu_k, \frac{\sigma^2}{\sqrt{N}})$ due to the central limit theorem, and (2.6) is based on the independence of the observations X_{ij} $(i = 1, \ldots, k; j = 1, 2, \ldots,)$. (2.7) follows because

$$P(\overline{X}_i < \overline{X}_k) = P \left(\frac{\overline{X}_i - \mu_i}{\frac{\sigma}{\sqrt{N}}} < \frac{\overline{X}_k - \mu_i}{\frac{\sigma}{\sqrt{N}}} \right) \tag{2.11}$$

and

$$\frac{\overline{X}_i - \mu_i}{\frac{\sigma}{\sqrt{N}}} \sim N(0, 1). \tag{2.12}$$

The equality in (2.8) is from the LFC, where $\mu_1 = \mu_2 = \ldots = \mu_{k-1}$, and (2.9) follows from the transformation of (2.10).

The probability of correct selection $P(\text{CS})$ is now expressed as a function of k, N, δ^* and σ (see (2.9)). For fixed k, δ^* and σ, the smallest N which will guarantee a specified probability of P^* can be obtained by solving the integral equation

$$P(\text{CS}) = \int_{-\infty}^{\infty} \left[\Phi \left(y + \frac{\sqrt{N}\delta^*}{\sigma} \right) \right]^{k-1} \phi(y) \mathrm{d}y = P^*. \tag{2.13}$$

Bechhofer's procedure is applicable to limited cases—*i.e.*, R&S problems with populations of *known and equal* variances. It is also a 'single-sample' procedure, which means the sampling occurs only one time in the procedure. The assumption of known variances made this possible. Dudewicz (1971) showed that a single-stage procedure cannot satisfy the requirement of (2.1) when the variances are unknown. A procedure needs at least two stages of sampling to deal with unknown variances, first to estimate the unknown variances and then to secure the $P(\text{CS})$ guarantee.

The two-stage procedure of Dudewicz & Dalal (1975), called Procedure $\mathbb{P}_E$, was the first to appear in the literature that assumed unknown, unequal variances (Yoon &

Bekker, 2017d). The procedure takes an initial sample of size n_0 from each population (this is the first stage of sampling), calculates the sample variances S_i^2 for each design i, and identifies the required sample size N_i based on the value of S_i^2, δ^* and a critical value h, which plays a crucial role in the proof of the required $P(\text{CS}) \geq P^*$. In the second stage of sampling, the procedure takes the remaining $N_i - n_0$ observations and identifies the best system based on the total N_i observations.

The drawback of Procedure $\mathbb{P}_E$ is that it uses weighted sample means $\widetilde{\overline{X}}_i$ (defined in (4.5) in Dudewicz & Dalal (1975, p. 37)), which is not as intuitive as ordinary sample means. It seems that they would have liked to develop a procedure that uses ordinary sample means, which is intuitively appealing, but that they failed to prove that such a procedure guaranteed the desired confidence P^*. Instead, they proposed another procedure, called Procedure $\mathbb{P}_R$, which is similar to Procedure $\mathbb{P}_E$, except that ordinary sample means $\overline{X}_i$ are used (instead of the weighted sample means $\widetilde{\overline{X}}_i$) in the final step when the best system is selected. Procedure $\mathbb{P}_R$ uses the same critical value h as in Procedure $\mathbb{P}_E$, which is calculated to guarantee $P(\text{CS}|\mathbb{P}_E) \geq P^*$ when Procedure $\mathbb{P}_E$ is followed, thus there is no guarantee that $P(\text{CS}|\mathbb{P}_R) \geq P^*$. However, Dudewicz & Dalal (1975, p. 40) proved that, in Theorem 4.2, $P(\text{CS}|\mathbb{P}_R) \geq P(\text{CS}|\mathbb{P}_E)$ when $k = 2$. In the case of $k > 2$, they also doubt that one could lose much, if anything, in $P(\text{CS})$ by using ordinary sample means $\overline{X}_i$ instead of the weighted sample means $\widetilde{\overline{X}}_i$. In summary, it is not proved that $P(\text{CS}|\mathbb{P}_R) \geq (\text{CS}|\mathbb{P}_E)$ when $k > 2$, but it is conjectured so. Chen (2011) discusses this issue.

Rinott (1978) developed a procedure that guarantees $P(\text{CS}) \geq P^*$ using ordinary sample means. It is named Procedure $\mathbb{P}_R(h^*)$. The procedure has almost the same structure as Procedure $\mathbb{P}_E$, except for the definition of the critical value h^* and the use of ordinary sample means $\overline{X}_i$. Procedure $\mathbb{P}_R(h^*)$ is considered to be one of the most important contributions in early R&S research, and became the cornerstone of the IZ procedures that followed. Many IZ procedures that were developed after this have been based on this procedure, with the goal of improving it. The focus was mainly on reducing the sample size N_i to achieve the same probability of correct selection P^* (Yoon & Bekker, 2017d). See for example Chen & Kelton (2000), Nelson *et al.* (2001), Chick & Inoue (2001), Chen & Kelton (2005) and Yoon & Bekker (2017b).

Nelson *et al.* (2001) proposed an IZ procedure that combines a subset selection procedure (to screen out non-competitive systems) with an IZ selection (to select the

best from among the survivors of the screening). A full algorithm of this combined procedure is illustrated in Nelson *et al.* (2001, p. 953–954). They proposed a subset selection procedure (Nelson *et al.*, 2001, Section 3) for problems with unknown, unequal variances for the initial screening, and used Rinott's procedure (Rinott, 1978) for the IZ selection.

Paulson (1964) proposed another IZ procedure that differs from the other procedures discussed above, in two ways: Firstly, it is a fully sequential procedure. This means that the procedure goes through as many sampling stages as needed, taking only one observation at each stage. After each sampling stage, the procedure searches for the evidence of the inferiority of each solution, and eliminates inferior solutions from further consideration. The procedure continues until only one solution is left, which becomes the best solution. Secondly, and more importantly, the $P(\text{CS})$ bound in this procedure is controlled by an idea borrowed from Brownian motion processes (Kim & Nelson, 2006b). More specifically, the procedure approximates the partial sum of differences between two systems as a Brownian motion process and uses a triangular continuation region to determine the stopping time of the selection process (Hong & Nelson, 2005). The procedure solves R&S problems with known or unknown, but equal variances.

Inspired by Paulson (1964) and Fabian (1974), Kim & Nelson (2001) developed a fully sequential R&S procedure, called the $\mathcal{KN}$ procedure, for problems with unknown and unequal variances. Procedure $\mathcal{KN}$ follows the same principle of using a Brownian motion process for its $P(\text{CS})$ bound, of which the details are not discussed in this document. Interested readers are referred to Fabian (1974) and Kim & Nelson (2001, p. 254–257). The approach used in proving $P(\text{CS}) \geq P^*$, however, is worth mentioning here: The $P(\text{CS})$ bound is given when only two systems (out of the total k feasible systems) are considered in isolation, and the overall $P(\text{CS})$ bound is established by combining all these isolated cases using the Bonferroni inequality. More specifically, let ICS_i be the event that an incorrect selection is made when the true best system b and system i ($i \neq b$) are considered. The $\mathcal{KN}$ procedure bounds the probability of an incorrect selection for this case, $P(\text{ICS}_i) \leq \beta$, where $\beta = \frac{(1-P^*)}{k-1}$. The overall probability of incorrect selection $P(\text{ICS})$ is then guaranteed as

$$P(\text{ICS}) = P(\bigcup_{i=1}^{k-1} \text{ICS}_i) \leq \sum_{i=1}^{k-1} P(\text{ICS}_i) \leq (k-1)\beta = 1 - P^*, \qquad (2.14)$$

where the first inequality follows from the Bonferroni inequality. The procedure thereby guarantees $P(\text{CS}) = 1 - P(\text{ICS}) \geq P^*$.

Kim & Nelson (2006a) also developed procedures for steady-state simulation (Procedures $\mathcal{KN}+$ and $\mathcal{KN}++$). These $\mathcal{KN}$ family procedures are considered state-of-the-art among the IZ procedures (Branke *et al.*, 2007), and are widely used in practice, having been incorporated into many commercial simulation software programs (Yoon & Bekker, 2017d).

2.1.2 Optimal computing budget allocation procedures

Optimal computing budget allocation (OCBA) procedures have a completely different approach in solving R&S problems. They do not guarantee $P(\text{CS}) \geq P^*$ nor use the IZ concept δ^*, but attempt to allocate a finite computing budget across systems so as to maximise the probability of correct selection. More precisely, OCBA procedures wish to choose the best numbers of simulation observations for each system such that $P(\text{CS})$ is maximised (Chen *et al.*, 2000). The problem is formulated as

$$
\max_{N_1,\ldots,N_k} \quad P(\text{CS}) \tag{2.15}
$$
$$
\text{subject to} \quad N_1 + N_2 + \ldots + N_k = N_{total},
$$
$$
N_i \geq 0,
$$

where N_i denotes the number of simulation replications for system i, and N_{total} represents the limited total computing budget. To solve the problem in (2.15), the $P(\text{CS})$ should be expressed as a function of N_i $(i = 1, \ldots, k)$. Chen (1996) proposed an approximation of $P(\text{CS})$ based on a Bayesian model, and Chen *et al.* (2000) formulated the approximated $P(\text{CS})$ as a function of N_i $(i = 1, \ldots, k)$ as follows:

$$
\text{Approximated } P(\text{CS}) = 1 - \sum_{i=1,\,i\neq b}^{k} \int_{-\frac{\delta_{b,i}}{\sigma_{b,i}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \, \mathrm{d}t, \tag{2.16}
$$

where b denotes the observed best system, $\delta_{b,i} = \overline{X}_b - \overline{X}_i$ and $\sigma_{b,i}^2 = \frac{\sigma_b^2}{N_b} + \frac{\sigma_i^2}{N_i}$. Furthermore, by solving the nonlinear programming optimisation problem in (2.15) using the Karush-Kuhn-Tucker (KKT) condition (Kuhn & Tucker, 1951), Chen *et al.* (2000) showed that the approximated $P(\text{CS})$ in (2.16) is asymptotically maximised when the

relationship between N_i and N_j is

$$\frac{N_i}{N_j} = \left(\frac{\sigma_i/\delta_{b,i}}{\sigma_j/\delta_{b,j}}\right)^2, \quad i,j \in \{1,2,\ldots,k\} \text{ and } i \neq j \neq b, \tag{2.17}$$

and the number of simulation replications for the best system is given as

$$N_b = \sigma_b \sqrt{\sum_{i=1,i\neq b}^{k} \frac{N_i^2}{\sigma_i^2}}. \tag{2.18}$$

An analysis of (2.17) brings some insights: Systems with larger variances and systems with closer performance to the best system are allocated more samples. This corresponds with one's intuition as more observations would not only reduce the uncertainty caused by the large variance but also help distinguishing the best system from those with close performance. Chen & Lee (2010) explain $\frac{\delta_{b,i}}{\sigma_i}$ as a signal to noise ratio for system i as compared with the best system b. A large value of this ratio means either the performance of system i is much worse than the best system or the estimation noise is small. In either case, it means that one can be confident in differentiating system i from the best system b, hence no more sampling is required for system i.

There has been active research since the OCBA approach was first proposed by Chen *et al.* (2000, 1997), and many variants of OCBA have been developed. Having the same OCBA framework, these variants focus on different issues: correlated sampling (Fu *et al.*, 2007); non-normal distributions (Fu *et al.*, 2004; Glynn & Juneja, 2004); different objective functions (Chick & Wu, 2005; He *et al.*, 2007; Trailović & Pao, 2004); subset selection (Chen *et al.*, 2008; Xiao & Lee, 2014); complete ranking (Xiao *et al.*, 2014); constraints (Lee *et al.*, 2012; Pujowidianto *et al.*, 2009); and multiple objectives (Lee *et al.*, 2004, 2010b). Lee *et al.* (2010a) provide an excellent review of these OCBA procedures.

2.2 Multi-objective ranking and selection procedures

This section reviews multi-objective ranking and selection (MORS) procedures that appear in the literature. Although there has not been a great deal of research in the MORS area, the MORS research can be classified into three sections: the famous multi-objective optimal computing budget allocation (MOCBA) procedure, and MORS procedures before and after the MOCBA procedure. They are discussed in the following three sections in chronological order.

2.2.1 Multivariate indifference-zone approach

There have been early attempts to extend the IZ procedures to the multi-objective domain using a multivariate concept (Dudewicz & Taneja, 1978, 1981; Hyakutake, 1988). The purpose of these procedures is, as single-objective IZ procedures, to select the best population out of k populations $(\pi_1, \ldots, \pi_k)$ with the probability of correct selection of at least P^*. To accommodate the concept of 'multi-objective', each population π_i is assumed to follow a multivariate normal distribution with $p \geq 1$ component variates, mean vector $\boldsymbol{\mu}_i$ and covariance matrix $\boldsymbol{\Sigma}_i$. This is usually abbreviated by saying π_i is $N_p(\boldsymbol{\mu}_i, \boldsymbol{\Sigma}_i)$ (Dudewicz & Taneja, 1978).

It is remarkable that these procedures did not employ the concept of Pareto optimality. Instead, Dudewicz & Taneja (1978) proposed an experimenter-specified function $g(\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_k)$ with possible values of $1, 2, \ldots, k$ such that

$$g(\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_k) = j \tag{2.19}$$

if and only if, given a choice among $\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_k$, the experimenter would prefer $\boldsymbol{\mu}_j$. In order to establish a probability of correct selection requirement similar to (2.1), Dudewicz & Taneja (1978) introduced some new concepts such as

- the set of true mean vectors $\boldsymbol{\mu} = \{\boldsymbol{\mu}_1, \ldots, \boldsymbol{\mu}_k\}$,

- disjoint preference sets $P_1, \ldots, P_k$, where P_j $(j = 1, \ldots, k)$ is defined as

$$P_j = \{\boldsymbol{\mu} \mid g(\boldsymbol{\mu}) = j\}, \text{ and} \tag{2.20}$$

- the distance from $\boldsymbol{\mu}$ to the boundary of $P_{g(\boldsymbol{\mu})}$

$$d_B(\boldsymbol{\mu}) = \inf\{\mathrm{d}(\boldsymbol{\mu}, \boldsymbol{b}) \mid \boldsymbol{b} \notin P_{g(\boldsymbol{\mu})}\}, \tag{2.21}$$

where $\mathrm{d}(\boldsymbol{\mu}, \boldsymbol{b})$ denotes the usual Euclidean distance.

The probability of correct selection requirement is then stated as

$$P(\mathrm{CS}) = P(\text{select design } g(\boldsymbol{\mu}) \mid d_B(\boldsymbol{\mu}) \geq \delta^*) \geq P^*. \tag{2.22}$$

Based on these concepts introduced in Dudewicz & Taneja (1978), Dudewicz & Taneja (1981) developed a multivariate procedure that achieves the requirement (2.22).

This procedure is in essence the multivariate version of Procedure $\mathbb{P}_E$ by Dudewicz & Dalal (1975). Hyakutake (1988) later worked on this procedure to make it more efficient and easier to use in practice.

This line of research, however, is not found further in the literature, probably due to the fact that the procedures do not employ the Pareto optimality concept. The MORS problems were left untouched for more than a decade until the advent of the famous multi-objective optimal computing budget (MOCBA) procedure (Yoon & Bekker, 2017d).

2.2.2   Multi-objective optimal computing budget allocation procedure

As the name shows, the multi-objective optimal computing budget (MOCBA) procedure (Lee *et al.*, 2004, 2010b) is the multi-objective version of the OCBA procedure. Therefore it has a problem formation similar to (2.15). Instead of maximising the probability of correct selection, however, the MOCBA procedure attempts to minimise Type I and Type II errors, which involve the concept of Pareto optimality. They are defined as follows:

- A Type I error occurs when at least one truly dominated system is observed as non-dominated, and

- a Type II error occurs when at least one truly non-dominated system is observed as dominated.

The purpose of the MOCBA procedure is to find the best numbers of simulation samples for each system such that the probabilities of these two errors (denoted by e_1 and e_2) are minimised. In order to formulate the objective function as a function of N_i (the number of samples for each system), Lee *et al.* (2010b) proposed to approximate the probability of these two errors, resulting in ae_1 and ae_2, and further provided upper bounds for them (ub_1 and ub_2). They showed $e_1 \leq ae_1 \leq ub_1$ and $e_2 \leq ae_2 \leq ub_2$ in Lemma 3 (Lee *et al.*, 2010b, p. 660). The problem formulation is then

$$\min_{N_1,\ldots,N_n} \quad ub_1 \tag{2.23}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} N_i \leq N_{total},$$

$$N_i \geq 0, \quad i = 1, 2, \ldots, n,$$

or

$$\min_{N_1,\ldots,N_n} \quad ub_2 \tag{2.24}$$

$$\text{s.t.} \quad \sum_{i=1}^{n} N_i \leq N_{total},$$

$$N_i \geq 0, \quad i = 1, 2, \ldots, n.$$

Following the same approach as with OCBA procedures, Lee *et al.* (2010b) formulated the objective function ub_1 and ub_2 as functions of N_i ($i = 1, \ldots, n$) based on a Bayesian model, and provided the allocation rule by solving the nonlinear programming optimisation problems (2.23) and (2.24) using the KKT condition. The solution to (2.23) is presented here as an example. Before that, some definitions for the problem are required first: Suppose there are n systems with p objectives. The performance of the ith system for the kth objective is defined as a normal random variable with mean μ_{ik} and variance σ_{ik}^2. Let $S = \{1, \ldots, n\}$ be the set of all feasible solutions, S_p the true Pareto set and $\bar{S}_p$ the true non-Pareto set. Also, let j_i denote the system that dominates system i with the highest probability, and let $k_{j_i}^i$ denote the objective of j_i that dominates the corresponding objective of system i with the lowest probability. Define $\delta_{ijk} = \mu_{jk} - \mu_{ik}$ and $\sigma_{ijk}^2 = (\sigma_{ik}^2/N_i) + (\sigma_{jk}^2/N_j)$, and α_i is the fraction of N_{total} to be allocated to system i.

Lee *et al.* (2010b, p. 661) presented in Lemma 4 the solution to (2.23) as follows: As $N_{total} \to \infty$, with known true Pareto set S_p and true non-Pareto set $\bar{S}_p$, the upper bound of the Type I error (ub_1) can be asymptotically minimised when $\alpha_i = \beta_i / \sum_{s \in S} \beta_s$ for any system $i \in S$, where

$$\beta_l \equiv \frac{\left(\sigma_{lk_{j_l}^l}^2 + \sigma_{j_l k_{j_l}^l}^2/\rho_l\right)/\delta_{lj_l k_{j_l}^l}^2}{\left(\sigma_{mk_{j_m}^m}^2 + \sigma_{j_m k_{j_m}^m}^2/\rho_m\right)/\delta_{mj_m k_{j_m}^m}^2}, \tag{2.25}$$

for any system $l \in \bar{S}_p$ and given that m is any fixed system in $\bar{S}_p$ and $\rho_i \equiv \alpha_{j_i}/\alpha_i$; and

$$\beta_d \equiv \sqrt{\sum_{i \in \Omega_d} \frac{\sigma_{dk_d^i}^2}{\sigma_{ik_d^i}^2} \beta_i^2}, \tag{2.26}$$

for any system $d \in S_p$ and given that $\Omega_d \equiv \{\text{system } i \mid i \in \bar{S}_p, j_i = d\}$.

This solution is very complex and requires hard work to understand. The solution to (2.24), which is presented in Lemma 5 in Lee *et al.* (2010b, p. 661), is even more

complicated and harder to comprehend. Nevertheless, the MOCBA procedure has definitely been the most dominant method in the MORS area for more than a decade with a wide range of applications, because it has virtually been the only method applicable in MORS.

OCBA procedures (including the MOCBA procedure) do not consider the indifference-zone concept. This could lead to a huge waste of simulation budget when there exist two (or more) systems whose performances are similar to each other. If the difference in the performance of two systems becomes smaller than δ^*, IZ procedures would stop the effort to distinguish them while OCBA procedures would still take more samples from them trying to identify the better one even though the difference is so small that the decision-maker is indifferent to them. Recognising this problem, Teng *et al.* (2010) integrated the IZ concept to the MOCBA framework. They redefined the dominance relationship of two systems incorporating the IZ concept, and eventually reconstructed Pareto optimality based on the redefined dominance relationships. The concept of Pareto optimality with IZ becomes one of the cornerstones in this research, and will therefore be discussed further in Section 4.1.4.

2.2.3   New attempts in multi-objective ranking and selection

The MOCBA procedure has practically been the only procedure available for MORS problems for more than a decade. Very recently, however, some new attempts were made to develop other MORS procedures. One such attempt is seen in Hunter & Feldman (2015) and Feldman *et al.* (2015), where the same line of research is presented in the bi- and multi-objective context, respectively. Their research is based on Pasupathy *et al.* (2014), which introduced the SCORE (Sampling Criteria for Optimisation using Rate Estimators) framework to develop an asymptotically optimal sample allocation rule in the single-objective domain with stochastic constraints. The final goal of Hunter & Feldman (2015) and Feldman *et al.* (2015) is to derive the SCORE allocation rule in the multi-objective context. Similar to the MOCBA procedure, they construct the probability of misclassification and try to find the best ratio α_i $(i = 1, \ldots, n)$, the proportion of the total sampling budget given to system i, that maximises the decay of the probability of misclassification. The study is still in progress: Having formulated the probability of misclassification as a function of α_i $(i = 1, \ldots, n)$ and having established

the problem as a concave maximisation problem $\mathcal{Q}$, they are investigating techniques to solve Problem $\mathcal{Q}$ under the heavy computational burden.

Another new line of research in MORS is found in Branke & Zhang (2015). Inspired by the idea of the small-sample EVI (the Expected Value of Information) procedure (Chick *et al.*, 2010), they proposed a very simple yet efficient MORS method called the myopic multi-objective budget allocation algorithm (M-MOBA). This algorithm considers the following question: If τ more samples were allocated to system i, how would it change the current Pareto set in the myopic sense of looking only one step ahead? Suppose n_i samples have been allocated to system i $(i = 1, \ldots, n)$, resulting in sample means of $\bar{x}_{ik}$ $(i = 1, \ldots, n,\ k = 1, \ldots, p)$ for system i in objective k. If τ more samples were to be added to system i, the overall sample mean of system i in objective k, denoted by z_{ik}, is calculated as

$$z_{ik} = \frac{n_i \bar{x}_{ik} + \tau \bar{y}_{ik}}{n_i + \tau}, \tag{2.27}$$

where $\bar{y}_{ik}$ is the mean of the new τ samples of system i in objective k. Instead of actually running the additional τ simulation replications to obtain the value of z_{ik}, the algorithm *predicts* the result, observing that Z_{ik} is a random variable that follows a Student's t-distribution (Branke & Zhang, 2015), that is,

$$Z_{ik} \sim St\left(\bar{x}_{ik}, \frac{n_i(n_i + \tau)}{\tau \sigma_{ik}^2}, n_i - 1\right), \tag{2.28}$$

where $St(\mu, \kappa, \nu)$ denotes Student's t-distribution with mean μ, precision κ and ν degrees of freedom. Based on this predicted result, the algorithm calculates P_i, the probability that the current Pareto set will change if τ samples are allocated to system i $(i = 1, \ldots, n)$, then allocates the τ samples to the system with the largest P_i. The underlying idea of the M-MOBA algorithm is that if the additional τ sampling does not lead to a change to the current Pareto set, then it is considered of less use to the purpose of identifying the Pareto optimal set. On the other hand, it is deemed useful if the additional sampling does cause a change to the current Pareto set. The M-MOBA algorithm is presently in an early stage, having derived P_i only when two objectives are considered. The probability model for the case of more than two objectives is currently under development.

None of these new procedures is based on the IZ approach. In fact, there has been only a single attempt to develop an MORS procedure under the IZ framework with

the concept of Pareto optimality: Chen & Lee (2009) proposed a two-phase Pareto set selection procedure (TSP). In the first phase, the procedure considers the p objectives separately, and treats the MORS problem as if it were p individual single-objective R&S problems, taking only one objective into account in each single-objective problem. It solves these $p \times$ single-objective R&S problems, using one of the single-objective IZ procedures by Chen (2007), which results in p (or less than p, say $m_p \leq p$ systems, in case of duplication) systems that are the best for each objective. These systems are undoubtedly non-dominated, because they are the best systems for at least one of the p objectives. However, they form an *incomplete* Pareto set as there may be systems that are not best for any objective, yet non-dominated. In the second phase, the procedure searches for these additional non-dominated systems to make the incomplete Pareto set complete. This work, however, remains an empirical study, not guaranteeing the probability of correct selection requirement $P(\text{CS}) \geq P^*$ for the final Pareto optimal set.

2.3   Conclusion: Chapter 2

In this chapter, the researcher reviewed ranking and selection (R&S) procedures in literature both in the single- and multi-objective domain. In Section 2.1, the two important approaches in single-objective R&S, the indifference-zone (IZ) method and the optimal computing budget allocation (OCBA) framework, were introduced, and important procedures in each approach were discussed. It was also explained that the focus of IZ procedures is to guarantee the probability of correct selection requirement $P(\text{CS}) \geq P^*$, while the purpose of OCBA procedures is to maximise $P(\text{CS})$ given the limited simulation budget.

In Section 2.2, multi-objective ranking and selection (MORS) procedures were reviewed, although there are not many of such procedures. The multivariate approach was introduced as an attempt that first appeared in the late 1970s to extend the IZ procedure to the multi-objective domain. These procedures, however, did not consider Pareto optimality. The multi-objective optimal computing budget allocation (MOCBA) procedure was discussed in detail as the most important procedure in this area, followed by some new procedures that begin to appear in the literature recently.

It was pointed out that while the OCBA approach was extended to the multi-objective domain, resulting in the MOCBA procedure, there does not yet exist an MORS procedure that follows the IZ approach with the concept of Pareto optimality. This gap is illustrated in Figure 2.2. The TSP procedure by Chen & Lee (2009) can be categorised as one, but $P(\text{CS}) \geq P^*$ is not guaranteed in their work, leaving the procedure merely an empirical one. This motivated the present study, of which the aim is to develop an MORS IZ procedure that presents a Pareto optimal set as the final solution and guarantees the quality of it, *i.e.*, $P(\text{CS}) \geq P^*$. The result of this research, therefore, if it succeeds, would fill the gap shown in Figure 2.2.

Single-objective R&S   Multi-objective R&S

IZ $\longrightarrow$

OCBA $\longrightarrow$ MOCBA
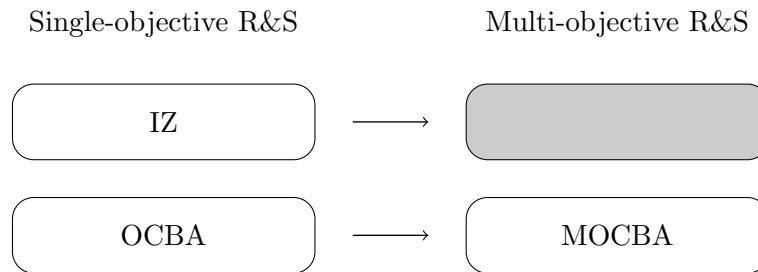
Figure 2.2: A diagram that shows the non-existence of an MORS IZ procedure

Chapter 3

A new single-objective ranking and selection procedure

This chapter provides a theoretical background of this research, by presenting a new *single-objective* ranking and selection procedure, called the $\mathcal{MY}$ procedure. As mentioned in Section 1.4, the $\mathcal{MY}$ procedure was developed in an attempt to improve existing single-objective R&S procedures, and the statistical validity of it is proved using a Bayesian inference model. Although the ultimate goal of this research is to develop a *multi-objective* ranking and selection procedure with the indifference-zone concept, the $\mathcal{MY}$ procedure is first discussed in this chapter to verify the Bayesian inference model, which forms a theoretical basis of the present study.

This chapter discusses the $\mathcal{MY}$ procedure along with its theoretical background and the statistical proof of its validity. Especially, Rinott's procedure (Rinott, 1978) is examined in detail, followed by the Bayesian inference model. Also, the results of some numerical experiments are presented to show the effectiveness of the procedure. This chapter is mainly based on Yoon & Bekker (2017b).

3.1   Motivation of the development of Procedure $\mathcal{MY}$

As briefly mentioned in Section 1.4, IZ procedures are well known to be conservative. Most empirical studies of IZ procedures show that the estimated $P(\mathrm{CS})$ is far greater than the required value P^*, which means more simulation budget was spent than actually needed to secure $P(\mathrm{CS}) \geq P^*$. Wang & Kim (2013) examined this matter and

identified a few sources of the conservativeness of the $\mathcal{KN}$ family procedures. They also recognised through a quantitative analysis the assumption of the slippage configuration (SC), also known as the least favourable configuration (LFC), was the most critical source of the conservativeness.

As explained in Section 2.1.1, the least favourable configuration occurs when the true mean of the best system is exactly δ^* apart from all other systems; see (2.4) for example. Almost all existing IZ procedures with a proven guarantee of $P(\text{CS}) \geq P^*$ assume the LFC, including Bechhofer (1954), Paulson (1964), Dudewicz & Dalal (1975), Rinott (1978) and Kim & Nelson (2001). They show (2.1) under the LFC assumption, that is

$$P(\text{CS}) \geq P_{\text{LFC}}(\text{CS}) \geq P^*, \tag{3.1}$$

which renders these procedures less efficient.

Recognising this problem, some researchers tried to develop IZ procedures without the LFC assumption. Chen & Kelton (2000) proposed an enhanced two-stage procedure (ETSS), which is essentially the non-LFC version of Rinott's two-stage procedure (Rinott, 1978). They greatly increased the efficiency of Rinott's procedure by eliminating the LFC assumption and instead using the sample mean information. This was further improved to a sequential R&S procedure (called SRS) by Chen & Kelton (2005). Interestingly, Chen & Kelton (2005) remark that the ratio of the sample sizes of two systems $\frac{N_i}{N_j}$ in the SRS procedure (and ETSS) is observed to be the same as in the OCBA procedure, which confirms the validity of this approach. However, they failed to prove the statistical validity of these procedures using a mathematical analysis as in Rinott's procedure.

Wang & Kim (2013) also proposed two non-LFC procedures, called $\mathcal{WK}$ and $\mathcal{WK}++$, based on the $\mathcal{KN}$ and $\mathcal{KN}++$ procedures, respectively. In these procedures the statistical validity is shown in an asymptotical sense as $\delta^* \to 0$. Although the asymptotic analysis provides insight into the effectiveness of the procedure in practice (Kim & Nelson, 2006a, p.478), it is limited in theory because $N_i \to \infty$ as $\delta^* \to 0$.

In summary, an R&S procedure that does not assume the LFC and yet proves its statistical validity with a rigorous mathematical analysis has not yet been seen in literature. This motivated the first part of the study, which resulted in the development of the $\mathcal{MY}$ procedure. The following section provides the theoretical background of the procedure.

3.2 Theoretical background of Procedure $\mathcal{MY}$

This section provides notation and assumptions, followed by two important concepts of the $\mathcal{MY}$ procedure: Rinott's procedure and the Bayesian approach.

3.2.1 Notation and assumptions for Procedure $\mathcal{MY}$

Throughout this chapter, *i.e.*, in the single-objective context, for both Rinott's procedure and Procedure $\mathcal{MY}$, the following notation is used.

Table 3.1: Notation for single-objective ranking and selection problems

k	the number of systems in the problem;
I	the set of systems that are still in competition;
X_{ij}	the jth observation from system i;
N_i	the total number of simulation replications assigned to system i;
μ_i	the unknown true mean of system i;
σ_i^2	the unknown variance of system i;
$\overline{X}_i(N_i)$	the sample mean of system i based on N_i observations;
$S_i^2(N_i)$	the sample variance of system i based on N_i observations, *i.e.*, $S_i^2(N_i) = \frac{1}{N_i-1} \sum_{j=1}^{N_i} (X_{ij} - \overline{X}_i(N_i))^2$;
n_0	the number of simulation replications at the first stage;
δ^*	the indifference-zone value;
P^*	the minimum required value for $P(\text{CS})$.

Also, it is assumed that there are k systems, and X_{ij} $(i = 1, \ldots, k; j = 1, 2, \ldots, N_i)$ are independently and identically distributed (i.i.d) random variables following a normal distribution with unknown means μ_i and unknown variances σ_i^2, that is,

$$X_{ij} \sim N\left(\mu_i, \sigma_i^2\right). \tag{3.2}$$

In addition, the researcher considers a minimisation problem, *i.e.*, the goal is to select the system with the smallest true mean. The notation and assumptions are applied to both Rinott's procedure (Section 3.2.2) and the $\mathcal{MY}$ procedure (Section 3.3). In fact, they are applicable throughout this dissertation whenever the problem is discussed in the single-objective context.

3.2.2 Rinott's procedure

Rinott's procedure (Rinott, 1978) forms the basis of Procedure $\mathcal{MY}$, hence the researcher recalls it in this section for the purpose of completeness. Besides the assumptions mentioned in the previous section, the procedure assumes the following LFC:

$$\mu_1 + \delta^* = \mu_2 = \ldots = \mu_k, \tag{3.3}$$

so that system 1 is the (unknown) best system for this problem. The procedure is then as follows:

Algorithm 1 Rinott's two-stage procedure (Rinott, 1978)

1: Select the probability requirement P^*, the indifference-zone value δ^*, and the first-stage sample size $n_0 \geq 2$.
2: Run n_0 simulations for each system i $(i = 1, \ldots, k)$.
3: Calculate sample variances $S_i^2(n_0)$ $(i = 1, \ldots, k)$.
4: Let $N_i = \max\left\{n_0, \lceil (h^* S_i(n_0)/\delta^*)^2 \rceil \right\}$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x, and h^* is the solution to (3.4).
5: Run additional $N_i - n_0$ simulation replications for system i $(i = 1, \ldots, k)$.
6: Compute the overall sample means $\overline{X}_i(N_i)$ $(i = 1, \ldots, k)$ and present system b as the best system, where $b = \arg\min_i \overline{X}_i(N_i)$.

The constant h^* in Step 4 is the solution to the following double integral equation:

$$\int_0^\infty \left[\int_0^\infty \frac{h^*}{\sqrt{(n_0-1)(\frac{1}{x}+\frac{1}{y})}} \, f(x) \, \mathrm{d}x \right]^{k-1} f(y) \, \mathrm{d}y = P^*, \tag{3.4}$$

where f denotes the probability density function (p.d.f) of the χ^2 distribution with $n_0 - 1$ degrees of freedom.

In what follows the researcher examines how $P(\mathrm{CS}) \geq P^*$ is achieved in Rinott's procedure. Because μ_1 (the true mean of system 1) is the smallest from the assumption in (3.3), it is clear that if $\overline{X}_1$ is observed as the smallest, that is, $b = \arg\min_i \overline{X}_i(N_i) = 1$, then the procedure has selected the correct system. Therefore the probability of correct selection is expressed as

$$P(\mathrm{CS}) = P[\overline{X}_1(N_1) < \overline{X}_i(N_i), \quad i = 2, \ldots, k]. \tag{3.5}$$

Note that, due to the assumption $X_{ij} \sim N(\mu_i, \sigma_i^2)$ and the central limit theorem,

$$\overline{X}_i(N_i) \sim N\left(\mu_i, \frac{\sigma_i^2}{N_i}\right). \tag{3.6}$$

Therefore

$$P(\text{CS}) = P\left[\overline{X}_1(N_1) < \overline{X}_i(N_i), \quad i = 2, \ldots, k\right]$$

$$= P\left[\frac{\overline{X}_1(N_1) - \overline{X}_i(N_i) - (\mu_1 - \mu_i)}{\sqrt{\frac{\sigma_1^2}{N_1} + \frac{\sigma_i^2}{N_i}}} < \frac{\mu_i - \mu_1}{\sqrt{\frac{\sigma_1^2}{N_1} + \frac{\sigma_i^2}{N_i}}}, \quad i = 2, \ldots, k\right].$$

Denote $Z_i = \frac{\overline{X}_1(N_1) - \overline{X}_i(N_i) - (\mu_1 - \mu_i)}{\sqrt{\frac{\sigma_1^2}{N_1} + \frac{\sigma_i^2}{N_i}}}$, then Z_i $(i = 2, \ldots, k)$ are $k-1$ independent random variables following the standard normal distribution. Also, under the LFC assumption (3.3), $\mu_i - \mu_1 = \delta^*$ for all $i \neq 1$. Thus,

$$P(\text{CS}) = P\left[Z_i < \frac{\mu_i - \mu_1}{\sqrt{\frac{\sigma_1^2}{N_1} + \frac{\sigma_i^2}{N_i}}}, \quad i = 2, \ldots, k\right] \tag{3.7}$$

$$= P\left[Z_i < \frac{\delta^*}{\sqrt{\frac{\sigma_1^2}{N_1} + \frac{\sigma_i^2}{N_i}}}, \quad i = 2, \ldots, k\right] \tag{3.8}$$

$$> \prod_{i=2}^{k} P\left[Z_i < \frac{\delta^*}{\sqrt{\frac{\sigma_1^2}{N_1} + \frac{\sigma_i^2}{N_i}}}\right] \tag{3.9}$$

$$= \prod_{i=2}^{k} \Phi\left(\frac{\delta^*}{\sqrt{\frac{\sigma_1^2}{N_1} + \frac{\sigma_i^2}{N_i}}}\right) \tag{3.10}$$

$$\geq \prod_{i=2}^{k} \Phi\left(\frac{\delta^*}{\sqrt{\frac{\sigma_1^2}{(h^* S_1(n_0)/\delta^*)^2} + \frac{\sigma_i^2}{(h^* S_i(n_0)/\delta^*)^2}}}\right) \tag{3.11}$$

$$= \prod_{i=2}^{k} \Phi\left(\frac{h^*}{\sqrt{\frac{\sigma_1^2}{S_1^2(n_0)} + \frac{\sigma_i^2}{S_i^2(n_0)}}}\right), \tag{3.12}$$

where Φ denotes the cumulative distribution function (c.d.f) of the standard normal distribution. The inequality in (3.9) is due to Slepian's inequality (Slepian, 1962) and because Z_i $(i = 2, \ldots, k)$ are positively correlated. The equality in (3.10) holds because Z_i $(i = 2, \ldots, k)$ follow $N(0, 1)$. The inequality in (3.11) comes from $N_i \geq (h^* S_i(n_0)/\delta^*)^2$ due to Step 4 in Rinott's procedure. Now, the variables Y_i $(i = 1, \ldots, k)$, defined by $Y_i = (n_0 - 1)\frac{S_i^2(n_0)}{\sigma_i^2}$, are k independent χ^2 variables with $n_0 - 1$ degrees of

freedom (DeGroot, 1970), and from (3.12),

$$
\begin{aligned}
P(\text{CS}) &\geq \prod_{i=2}^{k} \Phi \left(\frac{h^*}{\sqrt{\frac{\sigma_1^2}{S_1^2(n_0)} + \frac{\sigma_i^2}{S_i^2(n_0)}}} \right) \\
&= \prod_{i=2}^{k} \Phi \left(\frac{h^*}{\sqrt{(n_0 - 1)\frac{1}{Y_1} + (n_0 - 1)\frac{1}{Y_i}}} \right) \\
&= E \left[\prod_{i=2}^{k} \Phi \left(\frac{h^*}{\sqrt{(n_0 - 1)(\frac{1}{Y_1} + \frac{1}{Y_i})}} \right) \Bigg| Y_1 \right] \\
&= \int_0^\infty \left[\prod_{i=2}^{k} \Phi \left(\frac{h^*}{\sqrt{(n_0 - 1)(\frac{1}{x} + \frac{1}{Y_i})}} \right) \right] f(x)\mathrm{d}x \\
&= \int_0^\infty \left[\int_0^\infty \Phi \left(\frac{h^*}{\sqrt{(n_0 - 1)(\frac{1}{x} + \frac{1}{y})}} \right) f(y)\mathrm{d}y \right]^{k-1} f(x)\mathrm{d}x \\
&= P^*,
\end{aligned}
\tag{3.13}
$$

where f is the p.d.f of the χ^2 distribution with $n_0 - 1$ degrees of freedom. The equality in (3.13) is due to the definition of h^* in (3.4). Note that h^* is determined by n_0, k and P^*. Note also that in Rinott's procedure, the sample size N_i is determined by

$$
N_i = \max \left\{ n_0, \left\lceil \left(\frac{h^* S_i(n_0)}{\delta^*} \right)^2 \right\rceil \right\},
\tag{3.14}
$$

where only sample variances $S_i^2(n_0)$ $(i = 1, \dots, k)$ are used from the first-stage simulation result, wasting the information of sample means $\overline{X}_i(n_0)$ $(i = 1, \dots, k)$.

This concludes the discussion of Rinott's procedure. In the next section, the Bayesian inference model is introduced.

3.2.3   Bayesian approach

The LFC (3.3) is used in the proof of Rinott's procedure in (3.8) where the term $\mu_i - \mu_1$ is replaced by δ^*. This makes it possible to formulate the probability of correct selection free from the unknown true mean parameters (Wang & Kim, 2013). The unknown true mean parameters appear in the formulation of the probability of correct selection (3.7) because Rinott's procedure assumes (3.6). In order to identify the minimum N_i to

guarantee $P(\text{CS}) \geq P^*$, the procedure needs to take care of the unknown true means as well as the unknown variances from the right-hand side of (3.7). Rinott's procedure assumes the LFC to get rid of the term $\mu_i - \mu_1$ while the unknown variances are cleverly replaced by Y_i ($i = 1, \ldots, k$).

Rinott's procedure as well as other IZ procedures follow a theory of probability called *frequentist*, which references the probability of *observed* data, *e.g.*, (3.6). It was shown in the above discussion that this eventually forces the procedure to use the LFC assumption, which, in turn, prevents the procedure from taking advantage of the sample mean information. Optimal computing budget allocation (OCBA) procedures, on the other hand, follow another probability theory called the *Bayesian approach*, which enables the use of the sample mean information, and thus making these procedures the most efficient (Branke *et al.*, 2007).

A Bayesian probability model references the probability of parameters that are of true interest to researchers (Hacking, 2001), *e.g.*, the unknown true means. Unlike the frequentist probability model, where parameters are assumed to have fixed population values (such as $\theta = 0$), and the data, denoted by y, is assumed to carry uncertainty, Bayesian probability refers directly to the parameter θ itself (Zyphur & Oswald, 2015) by using the concept of *the posterior distribution*. A posterior distribution $p(\theta|y)$ refers to one's belief that θ is true, having observed dataset y, and is defined via Bayes' rule as follows (Hoff, 2009):

$$p(\theta|y) = \frac{p(\theta)p(y|\theta)}{\int_{\Theta} p(\tilde{\theta})p(y|\tilde{\theta})\mathrm{d}\tilde{\theta}} \propto p(\theta)p(y|\theta), \tag{3.15}$$

where $p(\theta)$ denotes the belief in θ before any sample is taken, known as the *prior distribution*, and $p(y|\theta)$ refers to one's belief in the sample y if one knew θ to be true, also called the *sample distribution*, and Θ represents the parameter space. Bayes' rule implies that the belief in θ before observation ($p(\theta)$) can be updated after observing data y, in proportion to the information contributed by the observation ($p(y|\theta)$), resulting in hopefully the more probable posterior distribution ($p(\theta|y)$) (Yoon & Bekker, 2017b).

Regarding the normal conjugate model, Hoff (2009, p.70–71) shows that, if $\theta \sim N(\mu_0, \tau_0^2)$ and y_i ($i = 1, \ldots, n$) are n independent and identical observations from $N(\theta, \sigma^2)$ (σ^2 is assumed to be known), then the posterior distribution $p(\theta|y_1, \ldots, y_n)$ is

also a normal distribution with mean μ_n and variance τ_n^2, where

$$\mu_n = \frac{\frac{1}{\tau_0^2}\mu_0 + \frac{n}{\sigma^2}\bar{y}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} \quad \text{and} \quad \frac{1}{\tau_n^2} = \frac{1}{\tau_0^2} + \frac{n}{\sigma^2}. \tag{3.16}$$

In particular, if no prior knowledge is available before the observation, that is, if the prior distribution is non-informative, then the posterior distribution is approximately as if $\tau_0 = \infty$ (Gelman *et al.*, 2013, p. 52):

$$\theta|y_1, \ldots, y_n \sim N(\bar{y}, \frac{\sigma^2}{n}). \tag{3.17}$$

In the context of the R&S problem under the discussion, with the assumption of $X_{ij} \sim N(\mu_i, \sigma_i^2)$, this means that the posterior distribution of the unknown true mean μ_i $(i = 1, \ldots, k)$ after N_i observations is

$$\mu_i \sim N\left(\overline{X}_i(N_i), \frac{\sigma_i^2}{N_i}\right), \tag{3.18}$$

as explained in Chen & Lee (2010, p. 30–31).

Having directly established the probability model of the unknown true means as in (3.18), the probability of correct selection can now be formulated as

$$P(\text{CS}) = P\left[\,\text{the observed best system } b \text{ is actually the best system}\,\right]$$

$$= P\left[\,\mu_b < \mu_i, \quad i = 1, \ldots, k, \ i \neq b\,\right] \tag{3.19}$$

$$= P\left[\frac{\mu_b - \mu_i - (\overline{X}_b - \overline{X}_i)}{\sqrt{\frac{\sigma_i^2}{N_i} + \frac{\sigma_b^2}{N_b}}} < \frac{\overline{X}_i - \overline{X}_b}{\sqrt{\frac{\sigma_i^2}{N_i} + \frac{\sigma_b^2}{N_b}}} \ i = 1, \ldots, k, \ i \neq b\right]$$

$$= P\left[Z_i < \frac{\overline{X}_i - \overline{X}_b}{\sqrt{\frac{\sigma_i^2}{N_i} + \frac{\sigma_b^2}{N_b}}} \ i = 1, \ldots, k, \ i \neq b\right], \tag{3.20}$$

which is exactly the same $P(\text{CS})$ formulation as in OCBA procedures (Chen *et al.*, 2000). The $\mathcal{MY}$ procedure follows the Bayesian approach, thus takes (3.19) for its probability of correct selection formulation. Note that the unknown true means μ_i $(i = 1, \ldots, k)$ do not appear in the right-hand side of (3.20), and the sample mean information is used instead. Note also that the unknown true variances σ_i^2 $(i = 1, \ldots, k)$ are still used in (3.20), which will be dealt with by the $\mathcal{MY}$ procedure at a later stage.

3.3   The $\mathcal{MY}$ procedure

This section describes the $\mathcal{MY}$ procedure and shows its validity on the probability of correct selection. First, the procedure is presented in Algorithm 2.

Algorithm 2 The $\mathcal{MY}$ procedure

1: Select the probability requirement $P^* = 1 - \alpha$, the indifference-zone value δ^*, and the first-stage sample size $n_0 \geq 2$. Let $I = \{1, 2, \dots, k\}$ be the set of systems in competition, and let $\beta = \frac{\alpha}{k-1}$.

2: Simulate n_0 replications for all k systems, and calculate sample means $\overline{X}_i(n_0)$ and sample variances $S_i^2(n_0)$. Let $N_i = n_0$ $(i = 1, \dots, k)$, and let $b = \arg \min_i \overline{X}_i(N_i)$.

3: Delete system i $(i \neq b)$ from I if

$$N_i \geq \left\lceil \left(\frac{h\, S_i(N_i)}{\delta_i} \right)^2 \right\rceil \text{ and } N_b \geq \left\lceil \left(\frac{h\, S_b(N_b)}{\delta_i} \right)^2 \right\rceil, \tag{3.21}$$

and delete system b from I if

$$N_b \geq \left\lceil \left(\frac{h\, S_b(N_b)}{\delta_i} \right)^2 \right\rceil \text{ for all } i \neq b, \tag{3.22}$$

where $\delta_i = \max\{\delta^*, \overline{X}_i(N_i) - \overline{X}_b(N_b)\}$, and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x, and h is the solution of the following equation:

$$\int_0^\infty \left[\int_0^\infty \frac{h}{\sqrt{(N_i - 1)\frac{1}{x} + (N_b - 1)\frac{1}{y}}}\, f_1(x)\, \mathrm{d}x \right] f_2(y)\, \mathrm{d}y = 1 - \beta, \tag{3.23}$$

where f_1 and f_2 denote the p.d.f of the χ^2 distribution with $N_i - 1$ and $N_b - 1$ degrees of freedom, respectively.

4: If $|I| = 0$, then stop and present system b as the best system. Otherwise, go to Step 5.

5: Take one additional observation X_{i,N_i+1} from each system $i \in I$, and set $N_i \leftarrow N_i + 1$ $(\forall i \in I)$. Set $I = \{1, 2, \dots, k\}$ and update $\overline{X}_i(N_i)$, $S_i^2(N_i)$ and $b = \arg \min_i \overline{X}_i(N_i)$, go to Step 3.

The proof on the probability of correct selection of Procedure $\mathcal{MY}$

The proof follows the same scheme used in the $\mathcal{KN}$ procedure (see Section 2.1.1, especially (2.14)). Let CS_i (ICS_i) be the event of correct (incorrect) selection when only system b and system i are considered. If it is shown that

$$P(\mathrm{CS}_i) \geq 1 - \beta, \tag{3.24}$$

which is equivalent to $P(\mathrm{ICS}_i) = 1 - P(\mathrm{CS}_i) \leq \beta$, for all $i = 1, \ldots, k$ ($i \neq b$), the probability of correct selection, from (3.19), is then shown to be greater than or equal to P^*:

$$
\begin{aligned}
P(\mathrm{CS}) &= P\left[\mu_b < \mu_i, \quad i = 1, \ldots, k, \ i \neq b\right] \\
&= P(\bigcap_{i=1, \, i \neq b}^{k} \mathrm{CS}_i) \\
&\geq 1 - \sum_{i=1, \, i \neq b}^{k} P(\mathrm{ICS}_i) \geq 1 - (k-1)\beta = 1 - \alpha = P^*.
\end{aligned} \tag{3.25}
$$

The first inequality in (3.25) follows from the Bonferroni inequality while the second one comes from the fact that $P(\mathrm{ICS}_i) \leq \beta$.

Now the researcher shows $P(\mathrm{CS}_i) \geq 1 - \beta$. Recall that at the end of the $\mathcal{MY}$ procedure system i ($i = 1, \ldots, k$) has been allocated N_i simulation replications and system $b = \arg\min_i \overline{X}_i(N_i)$ is presented as the best system. Also recall $\mu_i \sim N\left(\overline{X}_i(N_i), \frac{\sigma_i^2}{N_i}\right)$ under the Bayesian inference model with a non-informative prior distribution, after N_i observations. Then $P(\mathrm{CS}_i) \geq 1 - \beta$ can be proved as follows:

$$
\begin{aligned}
P(\mathrm{CS}_i) &= P(\mu_b < \mu_i) \\
&= P\left[\frac{\mu_b - \mu_i - (\overline{X}_b(N_b) - \overline{X}_i(N_i))}{\sqrt{\frac{\sigma_i^2}{N_i} + \frac{\sigma_b^2}{N_b}}} < \frac{\overline{X}_i(N_i) - \overline{X}_b(N_b)}{\sqrt{\frac{\sigma_i^2}{N_i} + \frac{\sigma_b^2}{N_b}}}\right] \\
&= P\left[Z_i < \frac{\overline{X}_i(N_i) - \overline{X}_b(N_b)}{\sqrt{\frac{\sigma_i^2}{N_i} + \frac{\sigma_b^2}{N_b}}}\right] \\
&= \Phi\left(\frac{\overline{X}_i(N_i) - \overline{X}_b(N_b)}{\sqrt{\frac{\sigma_i^2}{N_i} + \frac{\sigma_b^2}{N_b}}}\right)
\end{aligned} \tag{3.26}
$$

$$\geq \Phi \left(\frac{\overline{X}_i(N_i) - \overline{X}_b(N_b)}{\sqrt{\frac{\sigma_i^2}{\left(\frac{h\,S_i(N_i)}{\delta_i}\right)^2} + \frac{\sigma_b^2}{\left(\frac{h\,S_b(N_b)}{\delta_i}\right)^2}}} \right) \tag{3.27}$$

$$= \Phi \left(\frac{h}{\sqrt{\frac{\sigma_i^2}{S_i(N_i)^2} + \frac{\sigma_b^2}{S_b(N_b)^2}}} \right), \tag{3.28}$$

where $Z_i = \frac{\mu_b - \mu_i - (\overline{X}_b(N_b) - \overline{X}_i(N_i))}{\sqrt{\frac{\sigma_i^2}{N_i} + \frac{\sigma_b^2}{N_b}}}$ and Φ denotes the c.d.f of the standard normal distribution. The equality (3.26) holds because Z_i follows $N(0,1)$. The inequality in (3.27) comes from the condition in (3.21) and $\overline{X}_i(N_i) - \overline{X}_b(N_b) \geq 0$. The equality in (3.28) holds if it is assumed that $\delta_i = \max\{\delta^*, \overline{X}_i(N_i) - \overline{X}_b(N_b)\} = \overline{X}_i(N_i) - \overline{X}_b(N_b)$ (discussions for the case of $\delta_i = \delta^*$ are to follow). Now, let $Y_i = (N_i - 1)\frac{S_i^2(N_i)}{\sigma_i^2}$ and $Y_b = (N_b - 1)\frac{S_b^2(N_b)}{\sigma_b^2}$, then Y_i and Y_b are two independent random variables following the χ^2 distribution with $N_i - 1$ and $N_b - 1$ degrees of freedom, respectively. Replacing $\frac{\sigma_i^2}{S_i(N_i)^2} = (N_i - 1)\frac{1}{Y_i}$ and $\frac{\sigma_b^2}{S_b(N_b)^2} = (N_b - 1)\frac{1}{Y_b}$ from (3.28) leads to

$$P(\text{CS}_i) \geq \Phi \left(\frac{h}{\sqrt{\frac{\sigma_i^2}{S_i(N_i)^2} + \frac{\sigma_b^2}{S_b(N_b)^2}}} \right)$$

$$= \Phi \left(\frac{h}{\sqrt{(N_i - 1)\frac{1}{Y_i} + (N_b - 1)\frac{1}{Y_b}}} \right)$$

$$= E \left[\Phi \left(\frac{h}{\sqrt{(N_i - 1)\frac{1}{Y_i} + (N_b - 1)\frac{1}{Y_b}}} \right) \Big| Y_b \right]$$

$$= \int_0^\infty \left[\Phi \left(\frac{h}{\sqrt{(N_i - 1)\frac{1}{Y_i} + (N_b - 1)\frac{1}{y}}} \right) \right] f_2(y)\mathrm{d}y$$

$$= \int_0^\infty \left[\int_0^\infty \Phi \left(\frac{h}{\sqrt{(N_i - 1)\frac{1}{x} + (N_b - 1)\frac{1}{y}}} \right) f_1(x)\mathrm{d}x \right] f_2(y)\mathrm{d}y$$

$$= 1 - \beta, \tag{3.29}$$

where f_1 and f_2 denote the p.d.f of the χ^2 distribution with $N_i - 1$ and $N_b - 1$ degrees of freedom, respectively. The equality in (3.29) comes from the definition of h in (3.23).

Discussion on the $\mathcal{MY}$ procedure

Procedure $\mathcal{MY}$ is a fully sequential procedure, where only one observation is added at each iteration and the procedure does not stop until it is assured of the quality of the final solution. At each iteration, the procedure repeats investigating (Step 3) and sampling (Step 5). In the investigation stage, it examines whether system i is believed to be inferior to system b with a confidence level of $1 - \beta$, that is, if $P(\text{CS}_i) \geq 1 - \beta$. This is done by checking whether the conditions in (3.21) are satisfied or not, for this leads to satisfying (3.24) as proved in the previous paragraph. These systems, once concluded with sufficient evidence as not being the best system, are excluded from further sampling. The rest of the systems are assigned one more simulation replication in the sampling stage. By assigning only one more simulation replication to the remaining systems, the procedure is taking a very cautious step to avoid allocating an unnecessarily large number of simulation replications to any system (Yoon & Bekker, 2017b). After the sampling and updating of relevant information, the procedure goes back to Step 3 for the next iteration.

However, the systems found to be inferior at an iteration are not excluded from further sampling in the following iterations. The set of systems that are still in competition I is restored in Step 5 to be the whole solution set $I = \{1, 2, \ldots, k\}$, and all k systems go through the investigating process once again with the updated information in the next iteration. This is fundamentally different from the $\mathcal{KN}$ procedure, where a system cannot come back to competition once deleted from I.

Restoring I in every iteration is possible because the constant h is dynamically determined by solving the double integral equation in (3.23), where different numbers of simulation replications ($N_i \neq N_b$) are allowed. It is distinguished from Rinott's constant h^* or the constant h^2 of the $\mathcal{KN}$ procedure, which are a function of n_0, k and P^*. Therefore they remain a constant throughout the whole process. In the $\mathcal{MY}$ procedure, on the other hand, h is calculated every iteration using the most recent information available, *i.e.*, information based on N_i observations, not n_0. The value of Rinott's constant h^* is available in Wilcox (1984) for several parameter settings of n_0, k and P^*. Nowadays, however, with advanced computing power, one can solve the complex double integral equations such as (3.4) or (3.23) dynamically using, for

example, Matlab, albeit somewhat costly in terms of computation time (Yoon & Bekker, 2017b).

The following condition was assumed in proving (3.24):

$$\delta_i = \max\{\delta^*, \overline{X}_i(N_i) - \overline{X}_b(N_b)\} = \overline{X}_i(N_i) - \overline{X}_b(N_b). \tag{3.30}$$

This is equivalent to the IZ prerequisite of frequentist's IZ procedures

$$\mu_i - \mu_b \geq \delta^*. \tag{3.31}$$

Frequentist's IZ procedures guarantee $P(\text{CS}) \geq P^*$ under the assumption that (3.31) is satisfied for all i $(i = 1, \ldots, k;\ i \neq b)$ (See (2.1)). If there exists a system $b'(\neq b)$ whose true mean is within δ^* apart from μ_b $(\mu_{b'} - \mu_b < \delta^*)$, then the procedures consider system b' indifferent to system b, thus select system b or b' with the probability of at least P^*. Because Procedure $\mathcal{MY}$ follows a Bayesian inference model, it does not impose the IZ prerequisite on the unknown true means as in (3.31), but applies it to sample means as in (3.30). That is, the probability of correct selection is guaranteed to be greater than or equal to P^* in Procedure $\mathcal{MY}$ if (3.30) is satisfied for all i $(i = 1, \ldots, k;$ $i \neq b)$. In case of $\delta_{b'} = \max\{\delta^*, \overline{X}_{b'} - \overline{X}_b\} = \delta^*$, that is when $\overline{X}_i - \overline{X}_b < \delta^*$ for $i = b'(\neq b)$, the procedure assigns $N_{b'}$ by

$$N_{b'} = \left\lceil \left(\frac{h\,S_{b'}}{\delta_{b'}}\right)^2 \right\rceil = \left\lceil \left(\frac{h\,S_{b'}}{\delta^*}\right)^2 \right\rceil \leq \left\lceil \left(\frac{hS_{b'}}{\overline{X}_{b'} - \overline{X}_b}\right)^2 \right\rceil. \tag{3.32}$$

The procedure does not assign the larger sample size $N_{b'} = \left\lceil \left(\frac{h\,S_{b'}}{\overline{X}_{b'} - \overline{X}_b}\right)^2 \right\rceil$ to system b', which means there is a danger of the difference between system b' and system b not being distinguished. But this is exactly what one wants from IZ procedures—one does not want to distinguish the difference if it is smaller than δ^*.

3.4 Experiments with Procedure $\mathcal{MY}$

This section describes some numerical experiments conducted to verify the statistical validity of the $\mathcal{MY}$ procedure. Other procedures were also implemented and used for the same experiments to compare the results. They are Rinott's procedure (Rinott, 1978), the ETSS procedure (Chen & Kelton, 2000), the SRS procedure (Chen & Kelton, 2005), the $\mathcal{KN}$ procedure (Kim & Nelson, 2001), and the $\mathcal{WK}$ and $\mathcal{WK}$++ procedures (Wang & Kim, 2013).

3.4.1 Experimental setup for Procedure $\mathcal{MY}$

The experimental setup was borrowed from Chen & Kelton (2005), of which the details are briefly explained here: 10 systems are assumed ($k = 10$), and $X_{ij} \sim N(\mu_i, \sigma_i^2)$. For Experiments 1 to 3, the true means are assumed as $\mu_i = i$ ($i = 1, \ldots, k$); and for Experiment 4, $\mu_1 + \delta^* = \mu_2 = \ldots = \mu_{10}$, thus Experiment 4 represents the LFC case. Equal variances are used for Experiments 1 and 4 ($\sigma_i^2 = 6^2$); increasing and decreasing variances are assumed for Experiments 2 ($\sigma_i^2 = (6 + (i - 1)/2)^2$) and 3 ($\sigma_i^2 = (6 - (i - 1)/2)^2$), respectively. In Experiments 1 to 3, $\delta^* = 0.9$ is used while $\delta^* = 1$ is assumed for Experiment 4. See Table 3.2 for a summarised experimental setup. For all experiments, the interest is to select the system with the smallest true mean, that is, system 1.

Table 3.2: Experimental settings for Procedure $\mathcal{MY}$

	mean	variance	IZ value
Exp. 1	$\mu_i = i$	$\sigma_i^2 = 6^2$	$\delta^* = 0.9$
Exp. 2	$\mu_i = i$	$\sigma_i^2 = (6 + (i - 1)/2)^2$	$\delta^* = 0.9$
Exp. 3	$\mu_i = i$	$\sigma_i^2 = (6 - (i - 1)/2)^2$	$\delta^* = 0.9$
Exp. 4	$\mu_1 + \delta^* = \mu_2 = \ldots = \mu_{10}$	$\sigma_i^2 = 6^2$	$\delta^* = 1$

3.4.2 Experimental results of Procedure $\mathcal{MY}$

The results of the four experiments are discussed in this section. Each experiment was done repeatedly and independently 1 000 times for each procedure, and the estimated probability of correct selection $\hat{P}(\text{CS})$ was calculated by the number of experiments with the correct answer (*i.e.*, system 1 is the best system), divided by 1 000. The average of the total number of simulation replications $\overline{N}_{total} = \frac{1}{1\,000} \sum_{R=1}^{1\,000} \sum_{i=1}^{k} N_{i,R}$, where $N_{i,R}$ denotes the number of simulation replications assigned to system i in the Rth run of the experiment, was also presented to see the efficiency of each procedure. $P^* = 0.9$ was assumed for all experiments, and three different values were used for the initial number of simulation replications, namely, $n_0 = 10$, $n_0 = 20$, and $n_0 = 30$.

Experiments 1 to 3

Tables 3.3 to 3.5 show the results of Experiments 1 to 3. The 'Procedure' column lists the procedures in the order that they were developed, first the procedures that are based on Rinott's procedure, then the $\mathcal{KN}$ family procedures that follow the principle of Paulson's procedure (Paulson, 1964).

Table 3.3: Estimated $P(\text{CS})$ and total number of simulations for Experiment 1

Procedure	$n_0 = 10$		$n_0 = 20$		$n_0 = 30$	
	$\hat{P}(\text{CS})$	$\overline{N}_{total}$	$\hat{P}(\text{CS})$	$\overline{N}_{total}$	$\hat{P}(\text{CS})$	$\overline{N}_{total}$
Rinott	0.986	5 273	0.994	5 247	0.995	5 266
ETSS	0.889	1 139	0.951	1 217	0.974	1 280
SRS	0.984	1 242	0.993	1 231	0.990	1 287
$\mathcal{MY}$	0.980	817	0.989	868	0.993	923
$\mathcal{KN}$	0.991	1 330	0.994	1 084	0.995	1 023
$\mathcal{WK}$	0.970	789	0.986	706	0.987	731
$\mathcal{WK}$++	0.961	505	0.978	587	0.980	658

Table 3.4: Estimated $P(\text{CS})$ and total number of simulations for Experiment 2

Procedure	$n_0 = 10$		$n_0 = 20$		$n_0 = 30$	
	$\hat{P}(\text{CS})$	$\overline{N}_{total}$	$\hat{P}(\text{CS})$	$\overline{N}_{total}$	$\hat{P}(\text{CS})$	$\overline{N}_{total}$
Rinott	0.988	10 245	0.994	10 306	0.997	10 199
ETSS	0.853	1 490	0.935	1 512	0.967	1 596
SRS	0.980	1 644	0.989	1 553	0.994	1 548
$\mathcal{MY}$	0.993	969	0.992	987	0.995	1 052
$\mathcal{KN}$	0.993	1 756	0.997	1 431	0.993	1 345
$\mathcal{WK}$	0.966	890	0.980	774	0.988	802
$\mathcal{WK}$++	0.968	602	0.981	674	0.984	720

In all procedures the estimated $P(\text{CS})$ appears to be greater than the required value $P^* = 0.9$, with the exception of the ETSS procedure with $n_0 = 10$ in Experiments 1 and 2. This is because the ETSS procedure depends only on the first-stage sample means and variances, and the initial sample size $n_0 = 10$ is not large enough to carry accurate information. For all three experiments, the efficiency of the procedures is improved from Rinott's to ETSS/SRS to Procedure $\mathcal{MY}$, spending a smaller number of simulation replications and at the same time achieving $P(\text{CS}) \geq P^*$ (except for

Table 3.5: Estimated $P(\text{CS})$ and total number of simulations for Experiment 3

Procedure	$n_0 = 10$		$n_0 = 20$		$n_0 = 30$	
	$\hat{P}(\text{CS})$	$\overline{N}_{total}$	$\hat{P}(\text{CS})$	$\overline{N}_{total}$	$\hat{P}(\text{CS})$	$\overline{N}_{total}$
Rinott	0.985	2 345	0.998	2 372	0.993	2 356
ETSS	0.906	908	0.946	1 006	0.961	1 102
SRS	0.985	994	0.995	1 034	0.995	1 094
$\mathcal{MY}$	0.992	749	0.985	808	0.988	863
$\mathcal{KN}$	0.995	1 023	0.994	860	0.991	828
$\mathcal{WK}$	0.982	680	0.978	632	0.984	676
$\mathcal{WK}$++	0.964	463	0.977	538	0.986	613

the aforementioned cases of ETSS with $n_0 = 10$). While the improvements between Rinott's and ETSS/SRS are due to the elimination of the LFC assumption, those between ETSS/SRS and the $\mathcal{MY}$ procedure are on account of the fact that Procedure $\mathcal{MY}$ is a fully sequential one as well as that in the $\mathcal{MY}$ procedure the value of h is calculated at each iteration to keep N_i as small as possible. The ETSS procedure and the SRS procedure show similar efficiency. However, the SRS procedure achieves a greater $\hat{P}(\text{CS})$ with a similar or a slightly larger value of $\overline{N}_{total}$. Therefore, the SRS procedure is still considered as an improvement of the ETSS procedure.

In all three experiments the efficiency of the $\mathcal{KN}$ family procedures are much better than others with regard to the number of total simulation replications, among which the $\mathcal{WK}$++ procedure improves on $\mathcal{KN}$ and $\mathcal{WK}$. As briefly mentioned in Section 3.1, the $\mathcal{WK}$ procedure eliminates the LFC assumption from $\mathcal{KN}$, which leads to a significant saving on the simulation effort. The $\mathcal{WK}$++ procedure further decreases the total number of simulation replications by using the updated sample variance information (Wang & Kim, 2013).

It is not clear how the initial number of simulation replications n_0 affects the performance of the procedures. In principle, for procedures that depend mainly on the first-stage information (Rinott, ETSS and $\mathcal{KN}$), a large value of n_0 means more reliable information at the early stage of the procedure, which could thus attribute to decreasing the total number of simulation replications (Procedure $\mathcal{KN}$) or improving $P(\text{CS})$ (ETSS).

However, for fully sequential procedures that do not depend much on the first-stage information, a large value of n_0 could sometimes be a waste, especially when the sample

Table 3.6: The average number of simulation replications assigned to each system by Procedure $\mathcal{MY}$ for Experiment 1

system	1	2	3	4	5	6	7	8	9	10
$n_0 = 10$	266.124	270.551	116.347	55.682	32.089	21.954	17.039	13.906	12.119	11.073
$n_0 = 20$	271.889	276.284	116.816	58.593	35.137	26.467	22.335	20.700	20.158	20.033
$n_0 = 30$	272.729	276.767	120.244	59.178	40.285	32.692	30.598	30.100	30.002	30.000

mean difference $(\overline{X}_i(n_0) - \overline{X}_b(n_0))$ is large and the procedure is able to conclude the inferiority of a system with a small value of n_0 (Yoon & Bekker, 2017b). See Table 3.6, for example, which shows the number of simulation replications assigned to each system by Procedure $\mathcal{MY}$ for Experiment 1. When $n_0 = 10$, the average number of simulation replications for systems 8 to 10 is not much greater than the initial value of $n_0 = 10$. This means that the procedure was able to decide shortly after the initial sampling of $n_0 = 10$ that these systems were not the best. In other words, $N_{10} = 11$ was enough, for example, to exclude system 10 from the candidates of the best system. If one uses $n_0 = 20$ or 30 in this case, 9 or 19 simulation replications are being wasted; and this could lead to a rise in the total number of simulation replications. This phenomenon is remarkable in Procedure $\mathcal{MY}$ and the $\mathcal{WK}++$ procedure.

Experiment 4

The result of Experiment 4 is quite different from the other experiments, because Experiment 4 assumes the LFC. Table 3.7 shows that many procedures do not satisfy the $P(\mathrm{CS}) \geq P^*$ requirement, particularly the ETSS procedure. This is because, as discussed in the previous section, the ETSS procedure depends only on the first-stage information, which is highly unreliable in the LFC. This is supported by the observation that the $\hat{P}(\mathrm{CS})$ increases as n_0 grows. Rinott's procedure and the $\mathcal{KN}$ procedure also depend on the first-stage information, however they are designed to achieve $P(\mathrm{CS}) \geq P^*$ when the LFC is assumed, therefore they show acceptable $P(\mathrm{CS})$ in most cases.

It is remarkable that the $\mathcal{WK}$ and $\mathcal{WK}++$ procedures do not meet the $P(\mathrm{CS})$ requirement, while Procedure $\mathcal{MY}$ does. The validity of $\mathcal{WK}$ family procedures is proven only in an asymptotical sense, that is, as $\delta^* \to 0$ (Wang & Kim, 2013). Therefore in this LFC case with $\delta^* = 1$, these procedures failed to achieve $P(\mathrm{CS}) \geq P^*$. On the other

hand, the statistical validity of Procedure $\mathcal{MY}$ is proven based on a solid mathematical analysis, as shown in Section 3.3, regardless of the value of δ^*.

Table 3.7: Estimated $P(\text{CS})$ and total number of simulations for Experiment 4

Procedure	$n_0 = 10$		$n_0 = 20$		$n_0 = 30$	
	$\hat{P}(\text{CS})$	$\overline{N}_{total}$	$\hat{P}(\text{CS})$	$\overline{N}_{total}$	$\hat{P}(\text{CS})$	$\overline{N}_{total}$
Rinott	0.898	4274	0.929	4250	0.937	4276
ETSS	0.538	1414	0.674	1849	0.771	2111
SRS	0.903	3174	0.934	3271	0.935	3235
$\mathcal{MY}$	0.941	2894	0.929	2951	0.932	2940
$\mathcal{KN}$	0.917	2592	0.933	2174	0.935	2022
$\mathcal{WK}$	0.843	2190	0.892	1758	0.890	1708
$\mathcal{WK}++$	0.811	1318	0.878	1426	0.883	1511

3.4.3 Additional experiments

Two more experiments were added besides the four experiments discussed in the previous two sections, to further compare the performance of the $\mathcal{WK}++$ procedure and the $\mathcal{MY}$ procedure. The setup of true means and variances in Experiment 5 is the same as in Experiment 4, but the indifference-zone parameter was set to a smaller value $\delta^* = 0.5$. Therefore, Experiment 5 does not assume the LFC any more; and the problem is easier in Experiment 5 than in Experiment 4. In Experiment 6, a more challenging condition is assumed, *i.e.*, the LFC prevails with $\delta^* = 0.5$. Table 3.8 summarises the settings of Experiments 5 and 6. The settings of Experiment 4 are also presented in the table for the purpose of comparison.

The results of Experiments 4 to 6 are presented in Table 3.9. Again the results of Experiment 4 for Procedure $\mathcal{WK}++$ and Procedure $\mathcal{MY}$ are presented for a convenient comparison. The estimated $P(\text{CS})$ increases in Experiment 5, in all three cases with $n_0 = 10$, 20 and 30; and for both procedures. This is because Experiment 5 assumes

Table 3.8: Experimental settings for additional experiments

	mean	variance	IZ value	LFC
Exp. 4	$\mu_1 + 1 = \mu_2 = \ldots = \mu_{10}$	$\sigma_i^2 = 6^2$	$\delta^* = 1$	Yes
Exp. 5	$\mu_1 + 1 = \mu_2 = \ldots = \mu_{10}$	$\sigma_i^2 = 6^2$	$\delta^* = 0.5$	No
Exp. 6	$\mu_1 + 0.5 = \mu_2 = \ldots = \mu_{10}$	$\sigma_i^2 = 6^2$	$\delta^* = 0.5$	Yes

Table 3.9: Estimated $P(\mathrm{CS})$ and total number of simulations for additional experiments

Procedure	$n_0 = 10$		$n_0 = 20$		$n_0 = 30$	
	$\hat{P}(\mathrm{CS})$	$\overline{N}_{total}$	$\hat{P}(\mathrm{CS})$	$\overline{N}_{total}$	$\hat{P}(\mathrm{CS})$	$\overline{N}_{total}$
Experiment 4						
$\mathcal{MY}$	0.941	2 894	0.929	2 951	0.932	2 940
$\mathcal{WK}$++	0.811	1 318	0.878	1 426	0.883	1 511
Experiment 5						
$\mathcal{MY}$	0.998	5 033	0.998	4 846	0.996	5 104
$\mathcal{WK}$++	0.847	2 541	0.911	2 615	0.925	2 724
Experiment 6						
$\mathcal{MY}$	0.933	11 718	0.920	11 713	0.926	11 782
$\mathcal{WK}$++	0.726	4 476	0.765	4 775	0.797	4 897

an easier problem to solve than Experiment 4. It also corresponds to the claim that the asymptotic validity of Procedure $\mathcal{WK}$++ is proven as $\delta^* \to 0$. However, both procedures spend more simulation replications in Experiment 5, now that they should distinguish the difference to the precision of $\delta^* = 0.5$. Nevertheless, Procedure $\mathcal{WK}$++ still shows an undesirable estimated $P(\mathrm{CS})$ of 0.847 when $n_0 = 10$.

The performance of the $\mathcal{WK}$++ procedure becomes even worse in Experiment 6, where the estimated $P(\mathrm{CS})$ could not exceed 0.8 for all three cases of $n_0 = 10$, 20 and 30; dropping to the value of 0.726 when $n_0 = 10$. In contrast, Procedure $\mathcal{MY}$ shows $P(\mathrm{CS}) \geq P^*$ all the time, demonstrating its validity empirically, too.

One might think that the $\mathcal{WK}$++ procedure spends a considerably smaller amount of simulation budget compared to Procedure $\mathcal{MY}$, hence maybe a better choice. However, it should be remembered that the purpose of IZ procedures is to *guarantee* the probability of correct selection requirement given in (2.1) no matter how many simulation replications are required, therefore the most important measure of performance for IZ procedures is the probability of correct selection, as argued in Kim & Nelson (2006a).

3.5   Conclusion: Chapter 3

In this chapter, a new single-objective R&S procedure, called Procedure $\mathcal{MY}$, was presented with a rigorous mathematical proof of its statistical validity. The motivation

was explained that indifference-zone procedures are well known to be conservative due to the LFC assumption, and there have been many efforts to remove the LFC assumption from existing IZ procedures; however, these efforts remained only to develop IZ procedures without a solid mathematical proof of $P(\text{CS}) \geq P^*$. Procedure $\mathcal{MY}$ not only successfully eliminates the LFC assumption, hence improving the efficiency of existing IZ procedures, but also provides the proof based on a Bayesian inference model. The validity of the Bayesian approach was supported by numerical experiments, which showed the superiority of Procedure $\mathcal{MY}$ over existing IZ procedures in terms of the probability of correct selection.

The next chapter advances to discuss multi-objective R&S procedures, which are the main work of this research.

Chapter 4

The developed multi-objective ranking and selection procedures

This chapter leads to the main work of this research: the development of a new multi-objective ranking and selection (MORS) procedure and its variants. First a few important concepts in multi-objective optimisation (MOO) are discussed in Section 4.1 to provide a theoretical background. The new MORS procedure, named Procedure $\mathcal{MMY}$, is then proposed in Section 4.2, followed by the proposal of two variants of this new MORS procedure named $\mathcal{MMY}1$ and $\mathcal{MMY}2$.

4.1 Theoretical background for the $\mathcal{MMY}$ family procedures

The $\mathcal{MMY}$ family procedures share the same principle of following a Bayesian inference model as Procedure $\mathcal{MY}$. Being *multi-objective* ranking and selection procedures, however, they require more theoretical background. This section discusses multi-objective optimisation (MOO) and the concept of Pareto optimality as a theoretical foundation for the work presented in this dissertation.

4.1.1 Multi-objective optimisation

Traditionally decision-making problems with multiple, conflicting objectives have been considered in a research area called *multiple criteria decision making* (MCDM). As discussed in Section 1.1, there is no unique solution to MOO problems but a set of

non-dominated solutions is identified. These solutions are called the *Pareto optimal solutions*. Despite the existence of multiple Pareto optimal solutions, in practice, usually only one of these solutions is to be chosen (Branke *et al.*, 2008). Typically, in the MCDM literature, this decision-making task for choosing a single most preferred solution has been treated as importantly as mathematical programming techniques for identifying the Pareto optimal solutions. The former normally requires preference information from a decision-maker.

In Miettinen (1999), MOO algorithms are classified into four categories, *i.e.*, no-preference, *a priori*, *a posteriori* and interactive methods, according to the role of the decision-maker in the solution process. In *a priori* methods, for example, the decision-maker articulates preference information related to the multiple objectives *before* the solution process begins, and the MOO problem is then formulated based on the preference information given by the decision-maker. All these four categories, however, can be considered as 'interactive approach' in the sense that mathematical programming techniques and the decision-making task have been used in an intertwined manner, and the ultimate aim of solving an MOO problem has been characterised as supporting the decision-maker in finding the solution that best fits the decision-maker's preferences (Branke *et al.*, 2008). For details on MCDM algorithms and methods, see Chankong & Haimes (1983), Miettinen (1999) and especially the first two chapters of Branke *et al.* (2008).

Meanwhile, another approach to MOO problems based on evolutionary algorithms emerged independently in the beginning of the 1990s. This soon formed a very popular research field called *evolutionary multi-objective optimisation* (EMO). Unlike the interactive approach in MCDM, EMO algorithms focus on identifying Pareto optimal solutions efficiently. These algorithms solve the MOO problem independently from the preference information of the decision-maker, who is then to choose a final solution from the already obtained Pareto solutions. This is similar to *a posteriori* methods of MCDM, in the sense that the Pareto optimal solutions are first found and then the decision-maker's preference information is considered to pick up a single solution. However, *a posteriori* methods of MCDM and EMO algorithms are fundamentally different for the former usually requires repetitive application of a single-objective algorithm with different values of parameters to identify a Pareto optimal solution set while the latter finds a Pareto set with a single application of an EMO algorithm. For more

information on the topic of EMO, the reader is referred to Deb (2001, 2005, 2008) and Coello Coello *et al.* (2002).

The interest in this research is closer to that of EMO, *i.e.*, identifying a Pareto optimal solution set as accurately and efficiently as possible is the ultimate goal of the MORS procedures proposed in this research. Therefore, the decision-making task of selecting a single preferred solution out of a number of Pareto optimal solutions is not considered in this study. Unlike the EMO algorithms, however, of which the main task is to often identify Pareto optimal solutions out of a huge number of feasible solutions in a deterministic environment, the MORS problem domain assumes a relatively *small* number of feasible solutions in a *stochastic* environment. The values of the stochastic solutions are usually estimated with computer simulation.

A 'relatively small' size of solution space means that the number of feasible solutions is small enough for a complete enumeration of *all* solutions (see Yoon & Bekker (2017c) for more discussion on the solution space size of simulation optimisation problems). Even though the evaluation of all solutions is possible, selecting the best system is still challenging due to the stochastic nature of the simulation optimisation (SO) problems, thus the research field of ranking and selection (R&S) was established as previously mentioned in Section 1.2.

In summary, the purpose of the proposed MORS procedures in this research is to identify a Pareto optimal solution set and to guarantee the quality of the Pareto set with a probability of correct selection greater than or equal to a prespecified value of P^*. Before advancing the discussion further, the following section presents the notation and assumptions used in the proposed MORS procedures.

4.1.2 Notation and assumptions for the $\mathcal{MMY}$ family procedures

Table 4.1 provides the notation used in $\mathcal{MMY}$ family procedures. These are for MORS problems, hence different from those presented in Section 3.2.1. The notation and assumptions given in this section are used throughout this dissertation so far as the MORS procedures are concerned.

Table 4.1: Notation for multi-objective ranking and selection problems

M	the number of systems in the problem;
S	the feasible solution set, $i.e.,$ $S = \{1, \ldots, M\}$;
I	the set of systems that are still in competition;
H	the number of objectives;
K	the objective set, $i.e.,$ $K = \{1, \ldots, H\}$;
μ_{ik}	the unknown true mean of system i for objective k;
σ_{ik}^2	the unknown variance of system i for objective k;
N_i	the total number of simulation replications assigned to system i;
X_{ikl}	the lth observation from system i for objective k;
$\overline{X}_{ik}(N_i)$	the sample mean of system i for objective k based on N_i observations;
$S_{ik}^2(N_i)$	the sample variance of system i for objective k based on N_i observations, $i.e.,$ $S_{ik}^2(N_i) = \dfrac{1}{N_i - 1} \sum_{l=1}^{N_i} (X_{ikl} - \overline{X}_{ik}(N_i))^2$;
Q	the true Pareto set based on μ_{ik} ($i \in S$ and $k \in K$);
Q^c	the true non-Pareto set based on μ_{ik} ($i \in S$ and $k \in K$);
Q_{IZ}	the true Pareto set with IZ;
Q_{IZ}^c	the true non-Pareto set with IZ;
Q_R	the true relaxed Pareto set;
$\mathbb{Q}_{\mathbb{T}}$	the set of all possible true relaxed Pareto sets, defined in (4.9);
S_p	the observed Pareto set based on $\overline{X}_{ik}$ ($i \in S$ and $k \in K$);
S_p^c	the observed non-Pareto set based on $\overline{X}_{ik}$ ($i \in S$ and $k \in K$);
S_{IZ}	the observed Pareto set with IZ;
S_{IZ}^c	the observed non-Pareto set with IZ;
n_0	the number of simulation replications at the first stage;
δ_k^*	the indifference-zone value for objective k;
P^*	the minimum required value for $P(\text{CS})$.

The observations X_{ikl} ($l = 1, \ldots, N_i$) are assumed to be i.i.d random variables following a normal distribution with unknown mean μ_{ik} and unknown variance σ_{ik}^2:

$$X_{ikl} \sim N(\mu_{ik}, \sigma_{ik}^2). \tag{4.1}$$

It is also assumed that the observations across the M systems are independent, as well as observations across the objectives in a system. In other words, the observations X_{ikl} are independent of all other responses for each $i \in S$, $k \in K$, and for $l = 1, \ldots, N_i$. This

is the same condition given in the MOCBA algorithm (Lee *et al.*, 2010b). From (4.1) and the Bayesian inference model discussed in Section 3.2.3, the posterior distribution of the unknown true mean μ_{ik} after N_i observations is obtained as

$$\mu_{ik} \sim N\left(\overline{X}_{ik}(N_i), \frac{\sigma_{ik}^2}{N_i}\right). \tag{4.2}$$

This becomes the foundation for the construction of the probability of correct selection in Sections 4.2, 4.3 and 4.4.

In the following two sections, a formal definition of Pareto optimality is presented, first without the IZ concept and then with it.

4.1.3 Introduction to Pareto optimality

Coello Coello (2009) gives a formal definition of Pareto optimality for a minimisation problem as follows:

Definition 1: Given two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, we say $\boldsymbol{u} \leq \boldsymbol{v}$ if $u_i \leq v_i$ for $i = 1, 2, \ldots, n$, and that $\boldsymbol{u} < \boldsymbol{v}$ if $\boldsymbol{u} \leq \boldsymbol{v}$ and $\boldsymbol{u} \neq \boldsymbol{v}$.

Definition 2: Given two vectors $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, we say $\boldsymbol{u}$ *dominates* $\boldsymbol{v}$ (denoted by $\boldsymbol{u} \prec \boldsymbol{v}$) iff $\boldsymbol{u} < \boldsymbol{v}$.

Definition 3: A vector of decision variables $\boldsymbol{x}^* \in S$ is *Pareto optimum* if there does not exist another $\boldsymbol{x} \in S$ such that $f(\boldsymbol{x}) \prec f(\boldsymbol{x}^*)$.

Definition 4: The Pareto optimal set S_p is defined by $S_p = \{\boldsymbol{x} \in S \mid \boldsymbol{x} = \boldsymbol{x}^*\}$. The vectors in S_p are also called *non-dominated*.

These definitions are based on an optimisation point of view in a deterministic environment. In the following two subsections the concept of Pareto optimality is discussed in a simulation context, where the stochastic environment is usually assumed.

4.1.3.1 The dominance relationship of true means

Suppose there are M systems with H objectives, and let μ_{ik} be the true mean of the performance measure associated with the kth objective of system i ($i \in S = \{1, \ldots, M\}$; and $k \in K = \{1, \ldots, H\}$). Then the above definitions can be converted into the following, in a simulation context:
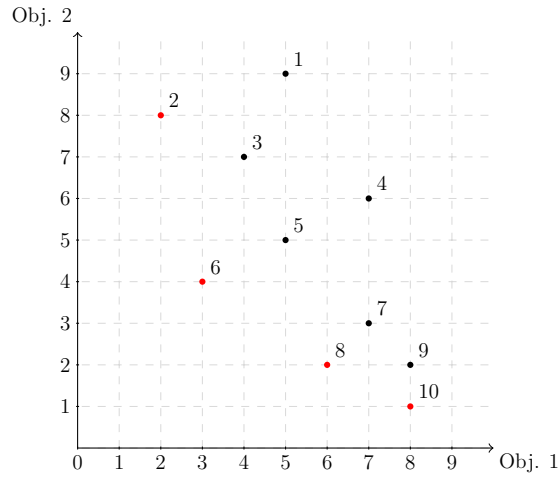
Figure 4.1: An example of Pareto optimal solutions for two minimised objectives

- System j dominates system i, denoted by $j \prec i$, if $\mu_{jk} \leq \mu_{ik}$ for $\forall k \in K$, and $\mu_{jk} < \mu_{ik}$ for $\exists k \in K$.

- System $i \in S$ is Pareto optimum if there does not exist another system $j \in S$ such that $j \prec i$.

- The Pareto optimal set Q is defined by

$$Q = \{i \in S \mid \nexists j \in S \text{ such that } j \prec i\}. \tag{4.3}$$

Figure 4.1 shows an example of Pareto optimal solutions (represented by red dots) with two objectives both to be minimised. It is clear, for example, that the performance of system 2 is better than system 1 for both objectives, thus system 2 dominates system 1. However, systems 2, 6, 8 and 10 are not dominated by any other system in the solution set $S = \{1, \ldots, 10\}$, thus $Q = \{2, 6, 8, 10\}$ in this case.

4.1.3.2 The dominance relationship of sample means

In the previous section the dominance relationship of *true means* was discussed. However, in stochastic simulation problems, true means are unknown and therefore estimated using sample means $\overline{X}_{ik}$ ($i \in S$, $k \in K$). The Pareto optimal solution set constructed from these sample means are called 'observed Pareto set' in this dissertation. An observed Pareto set can only approximate the 'true' Pareto optimal solution

set Q. To highlight this discrepancy, the 'observed Pareto set' is denoted using a different symbol in this dissertation, *i.e.*, S_p. The dominance relationship of sample means is then as follows:

- System j dominates system i in observation, denoted by $j\hat{\prec}i$, if $\overline{X}_{jk} \leq \overline{X}_{ik}$ for $\forall k \in K$, and $\overline{X}_{jk} < \overline{X}_{ik}$ for $\exists k \in K$.

- System $i \in S$ is Pareto optimum in observation if there does not exist another system $j \in S$ such that $j\hat{\prec}i$.

- The observed Pareto set S_p is defined by

$$S_p = \{i \in S \mid \nexists j \in S \text{ such that } j\hat{\prec}i\}.$$

The hat (ˆ) symbol indicates that the dominance relationship is based on observation (sample means).

In this section (Section 4.1.3), a formal definition of Pareto optimality was considered in a simulation context, where the stochastic environment is assumed. The true Pareto solution set Q and the observed Pareto solution set S_p were defined based on true means and sample means, respectively. The indifference-zone concept was not considered in constructing either Q nor S_p. Considering the IZ concept, however, is important in a stochastic environment; otherwise the procedures would spend a huge amount of simulation budget trying to distinguish insignificantly small differences in performance. This becomes the subject of the following section.

4.1.4   Pareto optimality with the indifference-zone concept

In this section, Pareto optimality is redefined based on the IZ concept. The dominance relationship of true means is contemplated first, followed by that of sample means. These new definitions of Pareto optimality with the IZ concept are based on Teng *et al.* (2010).

4.1.4.1   The dominance relationship of true means with the IZ concept

When the IZ concept is involved, three relationships exist between two systems for a given objective k, as follows:

4.1 Theoretical background for the $\mathcal{MMY}$ family procedures
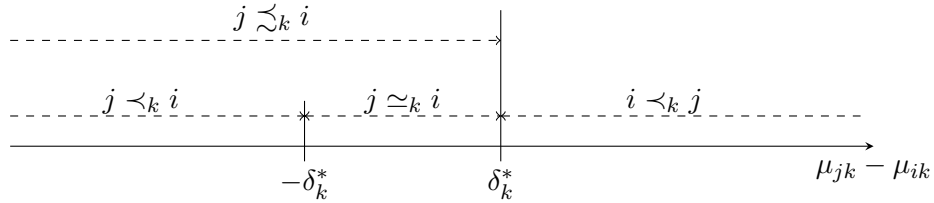
Figure 4.2: Three relationships between systems i and j for objective k with IZ

- System j is better than system i in objective k, denoted by $j \prec_k i$, if $\mu_{jk} - \mu_{ik} < -\delta_k^*$.

- System j is indifferent to system i in objective k, denoted by $j \simeq_k i$, if $|\mu_{jk} - \mu_{ik}| \leq \delta_k^*$.

- System j is worse than system i in objective k, denoted by $i \prec_k j$, if $\mu_{jk} - \mu_{ik} > \delta_k^*$.

Sometimes, the first and second relationships are combined as follows:

- System j is better than or is indifferent to system i in objective k, denoted by $j \precsim_k i$, if $\mu_{jk} - \mu_{ik} \leq \delta_k^*$.

Figure 4.2 shows these relationships graphically. Based on these three relationships between two systems for a given objective k, now the relationship between two systems in general, *i.e.*, considering all H objectives, can be defined as follows when the indifference-zone concept is considered:

- System j dominates or is indifferent to system i, denoted by $j \precsim_{IZ} i$, if

$$\mu_{jk} - \mu_{ik} \leq \delta_k^*, \ \forall k \in K;$$

and the probability that system j dominates or is indifferent to system i is

$$P(j \precsim_{IZ} i) = \bigcap_{k=1}^{H} P(\mu_{jk} - \mu_{ik} \leq \delta_k^*).$$

- System j is indifferent to system i, denoted by $j \simeq_{IZ} i$, if

$$|\mu_{jk} - \mu_{ik}| \leq \delta_k^*, \ \forall k \in K; \tag{4.4}$$

and the probability that system j is indifferent to system i is

$$P(j \simeq_{IZ} i) = \bigcap_{k=1}^{H} P(|\mu_{jk} - \mu_{ik}| \leq \delta_k^*).$$

- System j dominates system i, denoted by $j \prec_{IZ} i$, if

$$\mu_{jk} - \mu_{ik} \leq \delta_k^*, \ \forall k \in K, \ \text{and} \ \mu_{jk} - \mu_{ik} < -\delta_k^*, \ \exists k \in K; \tag{4.5}$$

and the probability that system j dominates system i is defined as

$$\begin{aligned} P(j \prec_{IZ} i) &= P(j \precsim_{IZ} i) - P(j \simeq_{IZ} i) \\ &= \bigcap_{k=1}^{H} P(\mu_{jk} - \mu_{ik} \leq \delta_k^*) - \bigcap_{k=1}^{H} P(|\mu_{jk} - \mu_{ik}| \leq \delta_k^*). \end{aligned} \tag{4.6}$$

- System $i \in S$ is Pareto optimum (with the IZ concept) if there does not exist another system $j \in S$ such that $j \prec_{IZ} i$.

- The Pareto optimal set with the IZ concept Q_{IZ} is defined by

$$Q_{IZ} = \{i \in S \mid \nexists j \in S \text{ such that } j \prec_{IZ} i\}. \tag{4.7}$$

Introducing the IZ concept changes the dominance relationship between two systems, therefore could lead to a different Pareto set. Refer to Figure 4.3 for example, to see the difference in Q and Q_{IZ} for the same set of μ_{ik} ($i \in S$; $k \in K$). These true means have the same value as in Figure 4.1, except for $\boldsymbol{\mu}_9$, which was changed to (7.7, 1.7) to highlight the difference in the two Pareto sets: one without the IZ concept and one with it. The non-dominated systems are marked in red. In this example, system 9 is non-dominated if the IZ concept is not considered as shown in Figure 4.3(a). However, the IZ values $\delta_1^* = \delta_2^* = 0.5$ are considered, according to (4.5), system 9 is dominated not only by system 8 but also by system 10. Figure 4.3(b) shows this concept.

Figure 4.4 provides another example, where the value of $\boldsymbol{\mu}_9$ was changed again to (8.3, 1.3). In this example, system 9 is dominated by system 10 when the IZ concept is not considered (Figure 4.4(a)). However, if the IZ concept is applied, system 9 is indifferent to system 10 by (4.4), thereby non-dominated (Figure 4.4(b)).

(a) Pareto set without IZ: Q

(b) Pareto set with IZ: Q_{IZ}
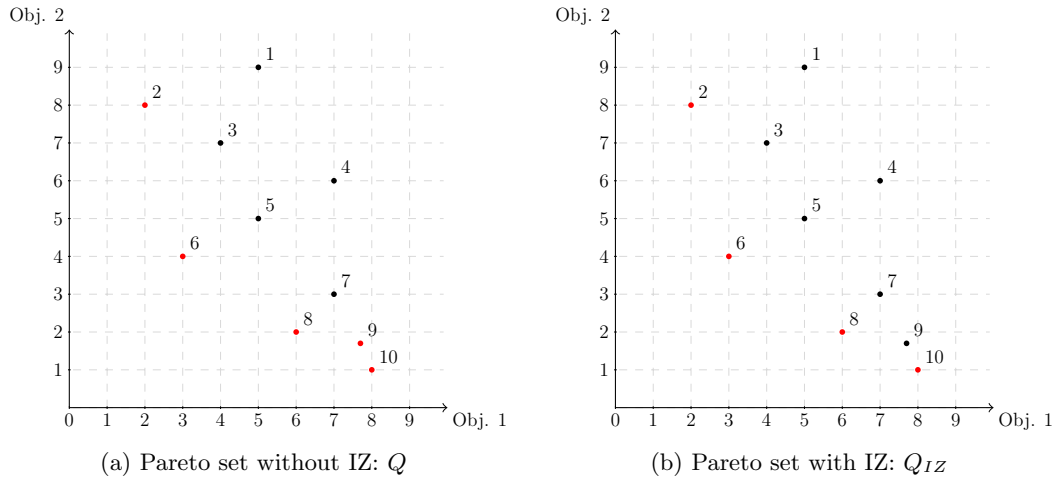
Figure 4.3: Pareto set examples: Example 1

(a) Pareto set without IZ: Q

(b) Pareto set with IZ: Q_{IZ}
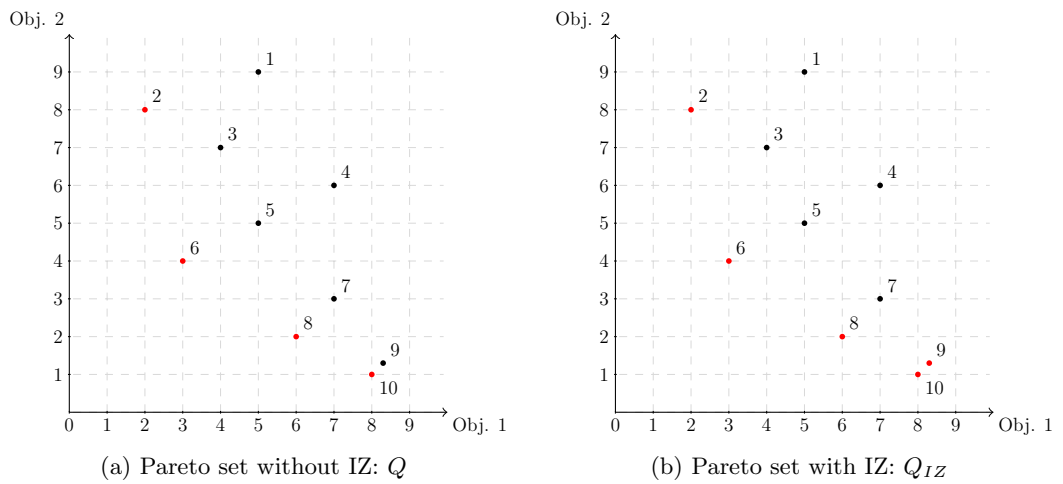
Figure 4.4: Pareto set examples: Example 2

4.1.4.2   The dominance relationship of sample means with the IZ concept

In the previous section the dominance relationship of *true means* was discussed when the IZ concept was considered. This section presents the dominance relationship of *sample means*, which is similar to what was discussed in the previous section. System j dominates system i by observation with IZ, denoted by $j\hat{\prec}_{IZ}i$, if

$$\overline{X}_{jk} - \overline{X}_{ik} \le \delta_k^*, \ \forall k \in K, \text{ and } \overline{X}_{jk} - \overline{X}_{ik} < -\delta_k^*, \ \exists k \in K. \qquad (4.8)$$

Note that the hat symbol (ˆ) is used to indicate that the dominance relationship is based on observation (sample means). The Pareto optimality concept is then defined as follows:

- System $i \in S$ is Pareto optimum (with the IZ concept and by observation) if there does not exist another system $j \in S$ such that $j\hat{\prec}_{IZ}i$.

- The observed Pareto optimal set with the IZ concept S_{IZ} is defined by

$$S_{IZ} = \{i \in S \mid \nexists j \in S \text{ such that } j\hat{\prec}_{IZ}i\}.$$

4.1.5   Relaxed Pareto set

The discussion given in Sections 4.1.3.1 and 4.1.4.1 regarding the dominance relationship with and without the IZ concept raises a series of tricky questions: Which kind of Pareto optimal set would a decision-maker prefer? Suppose the (unknown) true means are dispersed as given in Figure 4.3. Would the decision-maker like a Pareto set without the IZ concept (Figure 4.3(a)) or with IZ (Figure 4.3(b))? One might prefer the set *without* IZ, as it provides a more extensive set of Pareto solutions than the other. However, in Example 2 given in Figure 4.4, the Pareto set *with* IZ (Figure 4.4(b)) is more comprehensive. Furthermore, what if the procedure presents either one of the two indifferent systems (systems 9 and 10) as a Pareto solution in Example 2, that is, $S_p = \{2, 6, 8, 9\}$ or $S_p = \{2, 6, 8, 10\}$? Would these cases be counted as a correct selection or an incorrect selection? Teng *et al.* (2010) proposed to use the Pareto set with IZ (Q_{IZ}) to accommodate the IZ concept in the multi-objective domain. However, simply using Q_{IZ} instead of Q does not seem proper when these questions are considered.

   In this study, the researcher defined a 'relaxed' Pareto set in order to reply to these tricky questions more properly, as follows:

4.1 Theoretical background for the $\mathcal{MMY}$ family procedures

- Consider a true Pareto solution set Q based on true means without the IZ concept defined in (4.3); and a true Pareto solution set Q_{IZ} based on true means with the IZ concept defined in (4.7).

- Construct the union of these two Pareto sets: $Q_U = Q \cup Q_{IZ}$.

- Consider a subset of Q_{IZ}, called Q_S, which contains all systems that have indifferent systems.

$$Q_S = \{i \in Q_{IZ} \mid \exists j \in Q_{IZ} \text{ such that } i \simeq_{IZ} j\}$$

- Divide Q_S into m sets, each consisting of systems indifferent to each other:

$$Q_{S,k} = \{i, j \in Q_S, \mid i \simeq_{IZ} j\} \ (k = 1, \dots, m).$$

Note that $Q_{S,k} \ (k = 1, \dots, m)$ are m subsets of Q_S whose union is Q_S.

- A set of Pareto optimal solutions is called a 'relaxed Pareto set', denoted by Q_R, if it is a subset of Q_U, containing all solutions in $Q_{IZ} - Q_S$ and at least one solution from each $Q_{S,k} \ (k = 1, \dots, m)$.

Note that there can be a multiple number of relaxed Pareto sets that qualify the above-mentioned conditions. Let $\mathbb{Q}_{\mathbb{T}}$ be the set of all possible relaxed Pareto sets, then $\mathbb{Q}_{\mathbb{T}}$ is defined as

$$\mathbb{Q}_{\mathbb{T}} = \{Q_R \subset Q_U \mid Q_{IZ} - Q_S \subset Q_R, \ |Q_R \cap Q_{S,k}| \geq 1, \ \forall k \ (k = 1, \dots, m)\}. \qquad (4.9)$$

Example 3 given in Figure 4.5 helps to understand the concept of Q_R. Suppose $M = 10$, $H = 2$ and $\delta_1^* = \delta_2^* = 0.5$. The Pareto optimal set without IZ is then constructed as $Q = \{2, 6, 7, 8, 10\}$ by the definition given in (4.3). This is displayed in red in Figure 4.5(a). Also, according to (4.7), the Pareto optimal set with IZ is defined as $Q_{IZ} = \{2, 5, 6, 8, 9, 10\}$ as illustrated in Figure 4.5(b). To obtain the relaxed Pareto set Q_R, consider the following sets besides Q and Q_{IZ}:

- $Q_U = \{2, 5, 6, 7, 8, 9, 10\}$.

- $Q_S = \{5, 6, 9, 10\}$.

- $Q_{S,1} = \{5, 6\}$ and $Q_{S,2} = \{9, 10\}$.

4.1 Theoretical background for the $\mathcal{MMY}$ family procedures

(a) Pareto set without IZ: Q (b) Pareto set with IZ: Q_{IZ} (c) Relaxed Pareto set: Q_R
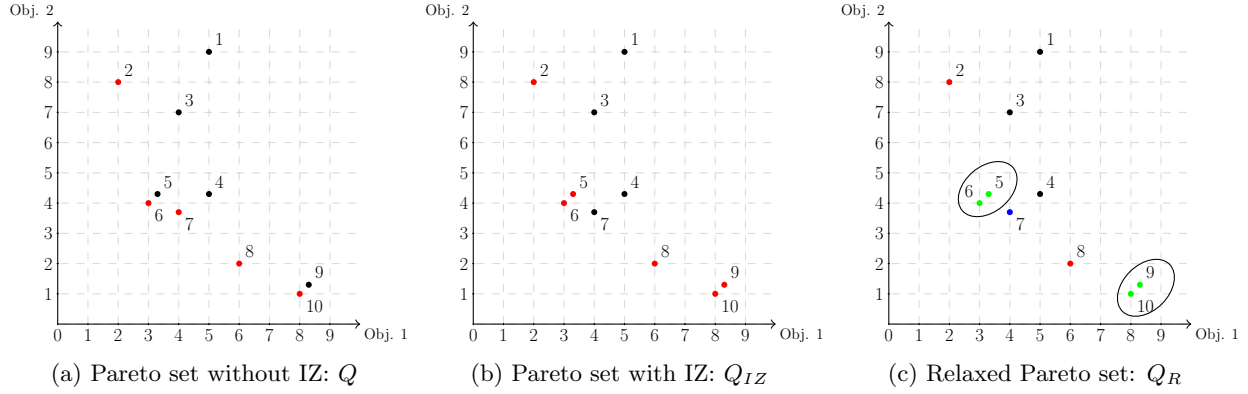
Figure 4.5: Pareto set examples: Example 3

- $Q_{IZ} - Q_S = \{2, 8\}$.

Then a relaxed Pareto set Q_R should be a subset of Q_U that contains $Q_{IZ} - Q_S = \{2, 8\}$ and at least one solution from $Q_{S,1} = \{5, 6\}$ and $Q_{S,2} = \{9, 10\}$. It may or may not include system 7, which is a member of $Q_U - Q_{IZ}$. In summary, a relaxed Pareto set Q_R contains all non-dominated solutions that do not have indifferent systems ($Q_{IZ} - Q_S = \{2, 8\}$), at least one system from a group of indifferent systems ($Q_{S,1}$ and $Q_{S,2}$); and regarding members of $Q_U - Q_{IZ}$, which are classified as non-dominated without the IZ concept, but considered dominated under the IZ regime, Q_R may or may not contain. This concept is illustrated in Figure 4.5(c). Systems that must be contained in Q_R are marked in red, those that may or may not be contained are displayed in blue. Systems marked in green in a circle represent that at least one of the systems in the circle must be included in Q_R. According to these rules, the relaxed Pareto set Q_R for Example 3 can be any one of the following 18 subsets of Q_U:

$Q_{R,1} = \{2, 5, 8, 9\}$, $Q_{R,2} = \{2, 5, 8, 10\}$, $Q_{R,3} = \{2, 5, 8, 9, 10\}$,
$Q_{R,4} = \{2, 6, 8, 9\}$, $Q_{R,5} = \{2, 6, 8, 10\}$, $Q_{R,6} = \{2, 6, 8, 9, 10\}$,
$Q_{R,7} = \{2, 5, 6, 8, 9\}$, $Q_{R,8} = \{2, 5, 6, 8, 10\}$, $Q_{R,9} = \{2, 5, 6, 8, 9, 10\}$,
$Q_{R,10} = \{2, 5, 7, 8, 9\}$, $Q_{R,11} = \{2, 5, 7, 8, 10\}$, $Q_{R,12} = \{2, 5, 7, 8, 9, 10\}$,
$Q_{R,13} = \{2, 6, 7, 8, 9\}$, $Q_{R,14} = \{2, 6, 7, 8, 10\}$, $Q_{R,15} = \{2, 6, 7, 8, 9, 10\}$,
$Q_{R,16} = \{2, 5, 6, 7, 8, 9\}$, $Q_{R,17} = \{2, 5, 6, 7, 8, 10\}$, $Q_{R,18} = \{2, 5, 6, 7, 8, 9, 10\}$.

The set of all possible relaxed Pareto sets $\mathbb{Q}_{\mathbb{T}}$ is then $\mathbb{Q}_{\mathbb{T}} = \{Q_{R,1}, \ldots, Q_{R,18}\}$ in Example 3. Note that $Q = Q_{R,14}$ and $Q_{IZ} = Q_{R,9}$ are also included in $\mathbb{Q}_{\mathbb{T}}$. The set $\mathbb{Q}_{\mathbb{T}}$ can be understood as a multi-objective counterpart of 'the set' mentioned by Hong & Nelson (2009, p. 77) in the single-objective context as follows:

> 'If there is *a set* of solutions whose objective values are within δ^* to the objective value of the best solution, then all solutions in *the set* are acceptable [as a best solution]. Then R&S procedures [...] typically select one of the solutions from *the set* with a probability at least P^*.'

The researcher believes that $\mathbb{Q}_{\mathbb{T}}$ provides comprehensively and not mistakenly all options of Pareto solution sets that a decision-maker would want, and that the decision-maker would be indifferent to which member of $\mathbb{Q}_{\mathbb{T}}$ is selected as a final solution set. Procedure $\mathcal{MMY}$ is therefore designed to regard any one of the relaxed Pareto sets from $\mathbb{Q}_{\mathbb{T}}$ as a correct selection, and guarantees the probability of correct selection of at least P^* in this sense. For the decision-maker who wants for some reason an exact Pareto optimal solution set with and without the IZ concept (Q_{IZ} and Q), two additional procedures were developed as well. Procedure $\mathcal{MMY}1$ ($\mathcal{MMY}2$) guarantees the final solution to be exactly the same as Q (Q_{IZ}) with a probability of at least P^*.

4.2 The $\mathcal{MMY}$ procedure

In this section the $\mathcal{MMY}$ procedure is presented along with the proof of its statistical validity.

4.2.1 The $\mathcal{MMY}$ procedure steps

The steps of the $\mathcal{MMY}$ procedure are given in Algorithm 3. Following are some definitions used in the procedure. Let

$$\delta_{ijk} = \max\{\delta_k^*, \overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)\} \tag{4.10}$$

and $\lceil x \rceil$ denotes the smallest integer greater than x. Consider a pair of systems (i,j) where system i is observed as non-dominated and system j can be any other system in S. This pair (i,j) ($i \in S_p$ and $j \in S$, $j \neq i$) is relevant to Steps 4 and 5 in Algorithm 3.

Algorithm 3 The $\mathcal{MMY}$ procedure

1: Select the probability of correction requirement $P^* = 1 - \alpha$, the indifference-zone value δ_k^* for each objective $k \in K$, and the first-stage sample size $n_0 \geq 2$. Set $I = \{1, 2, \ldots, M\}$ and $\beta = \dfrac{\alpha}{M}$.

2: Simulate n_0 replications for all M systems, and calculate sample means $\overline{X}_{ik}(n_0)$ and sample variances $S_{ik}^2(n_0)$ ($i \in S$ and $k \in K$). Let $N_i = n_0$.

3: Observe the Pareto set S_p and the non-Pareto set S_p^c based on the sample means $\overline{X}_{ik}(N_i)$ ($i \in S$ and $k \in K$) without the indifference-zone concept.

4: For each system $i \in S_p$ and $j \in S$ ($j \neq i$) with $K_1 = K$, check if the following two conditions are met:

$$N_i > \left\lceil \max_k \left(\frac{h_1\, S_{ik}(N_i)}{\delta_{ijk}} \right)^2 \right\rceil \ \text{ and } \ N_j > \left\lceil \max_k \left(\frac{h_1\, S_{jk}(N_j)}{\delta_{ijk}} \right)^2 \right\rceil, \qquad (4.11)$$

where h_1 is the solution to (4.16), and K_1 is defined in (4.14).

5: For each system $i \in S_p$ and $j \in S$ ($j \neq i$) with $K_1 \neq K$, check if the following two conditions are met:

$$N_i > \left\lceil \left(\frac{h_2\, S_{ik'}(N_i)}{\delta_{ijk'}} \right)^2 \right\rceil \ \text{ and } \ N_j > \left\lceil \left(\frac{h_2\, S_{jk'}(N_j)}{\delta_{ijk'}} \right)^2 \right\rceil, \qquad (4.12)$$

where k' is defined in (4.15) and h_2 is the solution to (4.17).

6: Delete system i from I if conditions (4.11) or (4.12) are satisfied for all $j \in S$ ($j \neq i$).

7: For each system $j \in S_p^c$, find system $i \in S_p$ as defined in (4.21). Check if the following two conditions are met:

$$N_i > \left\lceil \max_k \left(\frac{h_3\, S_{ik}(N_i)}{\delta_{ijk}} \right)^2 \right\rceil \ \text{ and } \ N_j > \left\lceil \max_k \left(\frac{h_3\, S_{jk}(N_j)}{\delta_{ijk}} \right)^2 \right\rceil, \qquad (4.13)$$

where h_3 is the solution to (4.22).

8: Delete system j from I if conditions in (4.13) are satisfied.

9: If $|I| = 0$, then stop and present the current Pareto set S_p as the final solution set. Otherwise, for each system $i \in S_p \cap I$, that is, systems in S_p that were not deleted from I in Step 6, add system $j \in S$ ($j \neq i$) to I if it does not satisfy conditions (4.11) or (4.12). Similarly, for each system $j \in S_p^c \cap I$, that is, systems in S_p^c that were not deleted from I in Step 8, add the corresponding system $i \in S_p$ to I if it does not satisfy (4.13). Go to Step 10.

10: Take one additional observation X_{i,k,N_i+1} from each system $i \in I$, and set $N_i \leftarrow N_i + 1$ ($\forall i \in I$). Set $I = \{1, 2, \ldots, M\}$ and update $\overline{X}_{ik}(N_i)$ and $S_{ik}^2(N_i)$ for all $i \in S$ and $k \in K$. and go to Step 3.

For each pair (i, j) $(i \in S_p$ and $j \in S$, $j \neq i)$, let

$$K_1 = \{k \mid |\overline{X}_{jk} - \overline{X}_{ik}| \leq \delta_k^*, k \in K\} \quad (4.14)$$

and

$$k' = \arg\max_{k \in K} \Phi\left(\frac{\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)}{\sqrt{\dfrac{S_{ik}^2(N_i)}{N_i} + \dfrac{S_{jk}^2(N_j)}{N_j}}}\right), \quad (4.15)$$

where Φ denotes the c.d.f of the standard normal distribution. Note that K_1 and k' should be defined for every pair of (i, j) $(i \in S_p$ and $j \in S$, $j \neq i)$. Step 4 in Algorithm 3 deals with (i, j) pairs when $K_1 = K$, that is, systems i and j are observed to be indifferent to each other, while Step 5 considers the case when $K_1 \neq K$.

The constants h_1 (in Step 4) and h_2 (in Step 5) are the solution to the following equations respectively:

$$\left[\int_0^\infty \left[\int_0^\infty \Phi\left(\frac{h_1}{\sqrt{(N_i - 1)\dfrac{1}{x} + (N_j - 1)\dfrac{1}{y}}}\right) f_1(x)\,\mathrm{d}x\right] f_2(y)\,\mathrm{d}y\right]^H = 1 - \gamma \quad (4.16)$$

and

$$\int_0^\infty \left[\int_0^\infty \Phi\left(\frac{h_2}{\sqrt{(N_i - 1)\dfrac{1}{x} + (N_j - 1)\dfrac{1}{y}}}\right) f_1(x)\,\mathrm{d}x\right] f_2(y)\,\mathrm{d}y = 1 - \gamma, \quad (4.17)$$

where $\gamma = \frac{\beta}{M-1}$, and f_1 and f_2 denote the p.d.f of the χ^2 distribution with $N_i - 1$ and $N_j - 1$ degrees of freedom, respectively.

Now the researcher considers systems observed as dominated, *i.e.*, $j \in S_p^c$ (Step 7 in Algorithm 3). Observe that for each system $j \in S_p^c$, there exists at least one system $i \in S_p$, such that $i \stackrel{\hat{}}{\prec} j$, otherwise system j would not have been observed as dominated. Find such system $i \in S_p$ for system $j \in S_p^c$. If there exists more than one system in S_p that dominates j, choose $i = \arg\max_{i' \in S_p} P(i' \prec j)$. That is, system i is defined as the system in the observed Pareto set S_p that dominates system j in truth with the maximum probability. From (4.2), one knows that $\mu_{i'k} \sim N\left(\overline{X}_{i'k}(N_{i'}), \dfrac{\sigma_{i'k}^2}{N_{i'}}\right)$ and

$\mu_{jk} \sim N\left(\overline{X}_{jk}(N_j), \dfrac{\sigma_{jk}^2}{N_j}\right)$, hence the probability $P(i' \prec j)$ is defined as

$$P(i' \prec j) = P(\mu_{i'k} \leq \mu_{jk}, \, \forall k \in K) \tag{4.18}$$

$$= P\left[\frac{\mu_{i'k} - \mu_{jk} - (\overline{X}_{i'k}(N_{i'}) - \overline{X}_{jk}(N_j))}{\sqrt{\dfrac{\sigma_{i'k}^2}{N_{i'}} + \dfrac{\sigma_{jk}^2}{N_j}}} \leq \frac{\overline{X}_{jk}(N_j) - \overline{X}_{i'k}(N_{i'})}{\sqrt{\dfrac{\sigma_{i'k}^2}{N_{i'}} + \dfrac{\sigma_{jk}^2}{N_j}}}, \, \forall k \in K\right]$$

$$= P\left[Z_{i'jk} \leq \frac{\overline{X}_{jk}(N_j) - \overline{X}_{i'k}(N_{i'})}{\sqrt{\dfrac{\sigma_{i'k}^2}{N_{i'}} + \dfrac{\sigma_{jk}^2}{N_j}}}, \, \forall k \in K\right]$$

$$= \prod_{k=1}^{H} \Phi\left(\frac{\overline{X}_{jk}(N_j) - \overline{X}_{i'k}(N_{i'})}{\sqrt{\dfrac{\sigma_{i'k}^2}{N_{i'}} + \dfrac{\sigma_{jk}^2}{N_j}}}\right), \tag{4.19}$$

where $Z_{ijk} = \dfrac{\mu_{ik} - \mu_{jk} - (\overline{X}_{ik}(N_i) - \overline{X}_{jk}(N_j))}{\sqrt{\frac{\sigma_{ik}^2}{N_i} + \frac{\sigma_{jk}^2}{N_j}}}$ and Φ denotes the c.d.f of the standard normal distribution. Strictly speaking, the probability that system i dominates system j in general is defined as

$$P(i \prec j) = P(\mu_{ik} \leq \mu_{jk}, \, \forall k \in K) - P(\mu_{ik} = \mu_{jk}, \, \forall k \in K) \tag{4.20}$$

according to the definition given in Section 4.1.3.1. However, since the probability that an unknown parameter is any single value is always equal to zero (Zyphur & Oswald, 2015, p. 392), the second term in the right-hand side of (4.20) can be ignored, *i.e.*, the probability can be defined as in (4.18). The same rule was applied throughout the dissertation.

The equality in (4.19) holds because $Z_{i'jk}$ are i.i.d random variables following $N(0, 1)$. However, one cannot calculate the exact value of (4.19) since the true variances σ_{ik}^2 ($i \in S$, $k \in K$) are not known. Instead, it can be estimated by using sample variances $S_{ik}^2(N_i)$ ($i \in S$, $k \in K$). This is supported by the statement 'when n is large, replacing the true standard deviation σ by the sample standard deviation S has little effect on the distribution of $Z = \dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}}$,' (Montgomery & Runger, 2010, p. 265). Even

if this is not true, it would not undermine the proof for the reason that will be discussed later in Section 4.2.2.2. Then the system that dominates system j with the maximum probability can be found as follows:

$$i = \arg\max_{i' \in S_p} P(i' \prec j)$$

$$\approx \arg\max_{i' \in S_p} \prod_{k=1}^{H} \Phi\left(\frac{\overline{X}_{jk}(N_j) - \overline{X}_{i'k}(N_{i'})}{\sqrt{\dfrac{S_{i'k}^2(N_i)}{N_{i'}} + \dfrac{S_{jk}^2(N_j)}{N_j}}}\right). \tag{4.21}$$

Note that such i should be defined for every $j \in S_p^c$. This pair of systems (i, j) $(i \in S_p, j \in S_p^c, i \hat{\prec} j)$ is considered in Step 7 in Algorithm 3. The constant h_3 in the same step of the algorithm is the solution to the following equation:

$$\left[\int_0^\infty \left[\int_0^\infty \Phi\left(\frac{h_3}{\sqrt{(N_i - 1)\dfrac{1}{x} + (N_j - 1)\dfrac{1}{y}}}\right) f_1(x)\, dx\right] f_2(y)\, dy\right]^H = 1 - \beta, \tag{4.22}$$

where $\beta = \frac{\alpha}{M}$, and f_1 and f_2 denote the p.d.f of the χ^2 distribution with $N_i - 1$ and $N_j - 1$ degrees of freedom, respectively.

4.2.2 Proof of $P(\mathrm{CS})$ for Procedure $\mathcal{MMY}$

In this section, it is shown how Procedure $\mathcal{MMY}$ guarantees the probability of correct selection requirement $P(\mathrm{CS}) \geq P^* = 1 - \alpha$. Let CS_i be the event that system i is observed correctly, and ICS_i be the event that system i is observed incorrectly. Then, if it is shown that either

$$P(\mathrm{CS}_i) \geq 1 - \beta \tag{4.23}$$

or

$$P(\mathrm{ICS}_i) \leq \beta, \tag{4.24}$$

where $\beta = \dfrac{\alpha}{M}$, then the overall probability of correct selection is guaranteed to be greater than or equal to P^*, as follows:

$$P(\mathrm{CS}) = \bigcap_{i=1}^{M} P(\mathrm{CS}_i) \geq 1 - \sum_{i=1}^{M} P(\mathrm{ICS}_i) \geq 1 - \sum_{i=1}^{M} \beta = 1 - \alpha = P^*. \tag{4.25}$$

The first inequality in (4.25) holds due to the Bonferroni inequality, and the second follows from (4.24). Now the researcher focuses on how to show (4.23) or (4.24) in the following two sections.

4.2.2.1 Proof for observed non-dominated systems in Procedure $\mathcal{MMY}$

In this section, (4.23) is proved for all systems that are observed as non-dominated. The discussion in this section is therefore relevant to Steps 4 and 5 in Algorithm 3. Consider a system $i \in S_p$. For this system, $P(\text{CS}_i)$ is by definition the probability that the observed non-dominated system i is truly non-dominated, *i.e.*, $P(\text{CS}_i) = P\left(i \text{ is not dominated by any other solution in } S \text{ in truth}\right)$. Let $P(\text{CS}_{ij})$ be the probability that the observed non-dominated system i is, in truth, not dominated by system j and $P(\text{ICS}_{ij})$ the probability that the observed non-dominated system i is, in truth, dominated by system j. Let $\gamma = \dfrac{\beta}{M-1}$, and if either

$$P(\text{CS}_{ij}) = P(j \nprec i) \geq 1 - \gamma \qquad (4.26)$$

or

$$P(\text{ICS}_{ij}) = P(j \prec i) \leq \gamma \qquad (4.27)$$

is shown for all $j \in S$ $(j \neq i)$, then, by the Bonferroni inequality and (4.27),

$$P(\text{CS}_i) = \bigcap_{j=1,\, j \neq i}^{M} P(\text{CS}_{ij}) \geq 1 - \sum_{j=1,\, j \neq i}^{M} P(\text{ICS}_{ij}) \geq 1 - (M-1)\gamma = 1 - \beta, \qquad (4.28)$$

thereby (4.23) holds.

Now, it will be shown that (4.27) holds for a pair (i,j) $(i \in S_p$ and $j \in S$, $j \neq i)$ when $K_1 \neq K$. This case corresponds to Step 5 in Algorithm 3. The probability of incorrect selection in this case can be formulated as

$$
\begin{aligned}
P(\text{ICS}_{ij}) = P(j \prec i) &= \prod_{k=1}^{H} P(\mu_{jk} \leq \mu_{ik}) \\
&\leq \min_{k} P(\mu_{jk} \leq \mu_{ik}) \\
&= \min_{k} \left[1 - P(\mu_{ik} \leq \mu_{jk}) \right] \\
&= 1 - \max_{k} P(\mu_{ik} \leq \mu_{jk}).
\end{aligned}
\qquad (4.29)
$$

Therefore if it is shown that

$$\max_k P(\mu_{ik} \leq \mu_{jk}) \geq 1 - \gamma,$$

then (4.27) holds, thereby $P(\mathrm{CS}_i) \geq 1 - \beta$ is proved.

From (4.2), it is known that $\mu_{ik} \sim N\left(\overline{X}_{ik}(N_i), \dfrac{\sigma_{ik}^2}{N_i}\right)$ and $\mu_{jk} \sim N\left(\overline{X}_{jk}(N_j), \dfrac{\sigma_{jk}^2}{N_j}\right)$, therefore

$$
\begin{aligned}
\max_k P(\mu_{ik} \leq \mu_{jk}) &= \max_k P\left[\frac{\mu_{ik} - \mu_{jk} - (\overline{X}_{ik}(N_i) - \overline{X}_{jk}(N_j))}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}} \leq \frac{\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}}\right] \\
&= \max_k P\left[Z_{ijk} \leq \frac{\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}}\right] \\
&= \max_k \Phi\left(\frac{\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}}\right),
\end{aligned}
\tag{4.30}
$$

where $Z_{ijk} = \dfrac{\mu_{ik} - \mu_{jk} - (\overline{X}_{ik}(N_i) - \overline{X}_{jk}(N_j))}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}}$ and Φ denotes the c.d.f of the standard normal distribution. From similar reasons given in the discussion related to (4.19) in Section 4.2.1, the objective k that maximises $P(\mu_{ik} \leq \mu_{jk})$ can be estimated by using sample variances $S_{ik}^2(N_i)$ and $S_{jk}^2(N_j)$ instead of the unknown true variances σ_{ik}^2 and σ_{jk}^2 in (4.30). Even if this brings about an approximated result, the proof is still valid as any objective k satisfies the inequality given in (4.29). The researcher prefers, however, the objective k that minimises $P(\mu_{jk} \leq \mu_{ik})$, or maximises $P(\mu_{ik} \leq \mu_{jk})$, in order to minimise the difference between the right- and left-hand sides of (4.29). For this reason

the objective k' defined in (4.15) is chosen in this proof, which leads to

$$
\max_k P(\mu_{ik} \leq \mu_{jk}) = \max_k \Phi\left(\frac{\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}} \right)
$$

$$
\approx \Phi\left(\frac{\overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i)}{\sqrt{\dfrac{\sigma_{ik'}^2}{N_i} + \dfrac{\sigma_{jk'}^2}{N_j}}} \right)
$$

$$
\geq \Phi\left(\frac{\overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i)}{\sqrt{\dfrac{\sigma_{ik'}^2}{\left(\dfrac{h_2\, S_{ik'}(N_i)}{\overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i)}\right)^2} + \dfrac{\sigma_{jk'}^2}{\left(\dfrac{h_2\, S_{jk'}(N_j)}{\overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i)}\right)^2}}} \right) \tag{4.31}
$$

$$
= \Phi\left(\frac{h_2}{\sqrt{\dfrac{\sigma_{ik'}^2}{S_{ik'}(N_i)^2} + \dfrac{\sigma_{jk'}^2}{S_{jk'}(N_j)^2}}} \right). \tag{4.32}
$$

In (4.31), N_i and N_j were replaced by the right-hand side of corresponding conditions given in (4.12) assuming $\delta_{ijk'} = \max\{\delta_{k'}^*, \overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i)\} = \overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i)$. Because system i is observed as non-dominated, there exists at least one objective $k \in K$ such that $\overline{X}_{jk} - \overline{X}_{ik} > 0$, otherwise system i would not have been observed as non-dominated. And certainly for objective k' defined in (4.15), $\overline{X}_{jk'} - \overline{X}_{ik'} > 0$ is true, which makes the input value of Φ in (4.31) positive. Therefore the inequality in (4.31) holds. Let

$$
Y_{ik} = (N_i - 1)\frac{S_{ik}^2(N_i)}{\sigma_{ik}^2}, \tag{4.33}
$$

then $Y_{ik'}$ and $Y_{jk'}$ are two χ^2 variables with $N_i - 1$ and $N_j - 1$ degrees of freedom,

respectively. The proof follows from (4.32) as

$$
\max_k P(\mu_{ik} \le \mu_{jk}) \ge \Phi\left(\frac{h_2}{\sqrt{\dfrac{\sigma_{ik'}^2}{S_{ik'}(N_i)^2} + \dfrac{\sigma_{jk'}^2}{S_{jk'}(N_j)^2}}}\right)
$$

$$
= \Phi\left(\frac{h_2}{\sqrt{(N_i-1)\dfrac{1}{Y_{ik'}} + (N_j-1)\dfrac{1}{Y_{jk'}}}}\right) \tag{4.34}
$$

$$
= E\left[\Phi\left(\frac{h_2}{\sqrt{(N_i-1)\dfrac{1}{Y_{ik'}} + (N_j-1)\dfrac{1}{Y_{jk'}}}}\right)\Bigg| Y_{jk'}\right]
$$

$$
= \int_0^\infty \left[\Phi\left(\frac{h_2}{\sqrt{(N_i-1)\dfrac{1}{Y_{ik'}} + (N_j-1)\dfrac{1}{y}}}\right)\right] f_2(y)\mathrm{d}y
$$

$$
= \int_0^\infty \left[\int_0^\infty \Phi\left(\frac{h_2}{\sqrt{(N_i-1)\dfrac{1}{x} + (N_j-1)\dfrac{1}{y}}}\right) f_1(x)\mathrm{d}x\right] f_2(y)\mathrm{d}y
$$

$$
= 1 - \gamma, \tag{4.35}
$$

where $\gamma = \frac{\beta}{M-1}$, and f_1 and f_2 denote the p.d.f of the χ^2 distribution with $N_i - 1$ and $N_j - 1$ degrees of freedom, respectively. The equality in (4.34) holds due to (4.33), and the equality in (4.35) comes from the definition of h_2 in (4.17). Thus, it has been shown that (4.30) is true, thereby proving (4.26) for a pair (i,j) ($i \in S_p$ and $j \in S$, $j \ne i$) when $K_1 \ne K$.

The proof, however, proceeded with the assumption of $\delta_{ijk'} = \max\{\delta_{k'}^*, \overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i)\} = \overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i)$. This means $P(\mathrm{CS}_{ij}) \ge 1 - \gamma$ is *not* proved if $\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i) < \delta_k^*$ for some k. Intuitively speaking, the procedure needs at least

$$
N_i > \left\lceil\left(\frac{h_2\, S_{ik'}(N_i)}{\overline{X}_{jk'} - \overline{X}_{ik'}}\right)^2\right\rceil \quad \text{and} \quad N_j > \left\lceil\left(\frac{h_2\, S_{jk'}(N_j)}{\overline{X}_{jk'} - \overline{X}_{ik'}}\right)^2\right\rceil \tag{4.36}
$$

simulation replications in order to prove $P(\mathrm{CS}_{ij}) = P(j \nprec i) \ge 1 - \gamma$. In other words, for the procedure to show that system i is truly not dominated by system j with

the probability of at least $1 - \gamma$, the sample sizes given in (4.36) are needed as seen in the proof above. Obviously the sample sizes N_i and N_j approach infinity as the difference in the performance between the two systems in objective k' comes closer to zero ($\overline{X}_{jk'} - \overline{X}_{ik'} \to 0$). In order to avoid spending an unnecessarily large amount of simulation budget to distinguish an insignificant difference $|\overline{X}_{jk'} - \overline{X}_{ik'}| < \delta_{k'}^*$, the term $\overline{X}_{jk'} - \overline{X}_{ik'}$ in (4.36) is replaced with $\delta_{k'}^*$ in the procedure if $\overline{X}_{jk'} - \overline{X}_{ik'} < \delta_{k'}^*$, which leads to the conditions given in (4.12). The procedure thus spends less simulation budget with conditions (4.12) than with (4.36), at the cost of not being able to distinguish the difference between system i and system j.

Therefore $P(\text{CS}_{ij}) \geq 1 - \gamma$ is not proved in this case, *i.e.*, system i is not shown to be not dominated by system j in truth. However, limiting the maximum number of simulation replications by conditions (4.12) is still important, because by replacing $\overline{X}_{jk'} - \overline{X}_{ik'}$ with $\delta_{k'}^*$, the procedure confirms that system i and system j are close to each other. In other words, even though following the conditions (4.12) does not guarantee $P(\text{CS}_{ij}) = P(j \not\prec i) \geq 1 - \gamma$ when $\delta_{ijk'} = \max\{\delta_{k'}^*, \overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i)\} = \delta_{k'}^*$, it assures $P(i \simeq_{k'} j) \geq 1 - \gamma$, which is sufficient for the procedure to guarantee $P(\text{CS}) \geq P^*$ as the aim of Procedure $\mathcal{MMY}$ is to find one of the relaxed Pareto sets (discussed in Section 4.1.5), not the exact Pareto optimal set Q.

Now the discussion considers a pair (i, j) ($i \in S_p$ and $j \in S$, $j \neq i$) when $K_1 = K$ (Step 4 in Algorithm 3). This is the case when systems i and j are observed to be indifferent to each other. The probability of correct selection is thus defined as

$$P(\text{CS}_{ij}) = P(\text{system } i \text{ and system } j \text{ are indifferent to each other in truth})$$
$$= P(i \simeq_{IZ} j). \tag{4.37}$$

Let

$$K_2 = \{k \mid \overline{X}_{jk} - \overline{X}_{ik} < 0, \ k \in K\}$$

and

$$K_3 = \{k \mid \overline{X}_{ik} - \overline{X}_{jk} < 0, \ k \in K\}.$$

The researcher does not consider the case $\overline{X}_{ik} = \overline{X}_{jk}$ as it is extremely unlikely for any two sample means to be observed exactly the same as each other. Therefore K_2 and K_3 are two disjoint subsets of K, and an objective $k \in K$ belongs to either K_2 or

K_3. Then, the probability of correct selection for any pair of (i, j) $(i, j \in S, i \neq j)$ in general can be formulated as follows:

$$P(\mathrm{CS}_{ij}) = P(\, \mu_{jk} - \mu_{ik} < 0, \ \forall k \in K_2 \ \text{ and } \ \mu_{ik} - \mu_{jk} < 0, \ \forall k \in K_3). \qquad (4.38)$$

For a given pair of (i, j) $(i, j \in S, i \neq j)$, the probability given in (4.38) represents the following two events:

- system j is truly better than system i in objective k $(\mu_{jk} - \mu_{ik} < 0)$ when observed so, *i.e.*, when $k \in K_2$.

- system i is truly better than system j in objective k $(\mu_{ik} - \mu_{jk} < 0)$ when observed so, *i.e.*, when $k \in K_3$.

Now, $P(\mathrm{CS}_{ij}) \geq 1 - \gamma$ is shown for a pair (i, j) $(i \in S_p$ and $j \in S, j \neq i)$ with $K_1 = K$ (Step 4) using the probability of correct selection defined in (4.38), *i.e.*,

$$P(\mathrm{CS}_{ij}) = P(\, \mu_{jk} - \mu_{ik} < 0, \ \forall k \in K_2 \ \text{ and } \ \mu_{ik} - \mu_{jk} < 0, \ \forall k \in K_3) \geq 1 - \gamma \quad (4.39)$$

will be shown in the following discussion.

$$
\begin{aligned}
P(\mathrm{CS}_{ij}) &= P(\, \mu_{jk} - \mu_{ik} < 0, \ \forall k \in K_2 \ \text{ and } \ \mu_{ik} - \mu_{jk} < 0, \ \forall k \in K_3) \\
&= \prod_{k \in K_2} P(\mu_{jk} - \mu_{ik} < 0) \times \prod_{k \in K_3} P(\mu_{ik} - \mu_{jk} < 0) \\
&= \prod_{k \in K_2} P\left[\frac{\mu_{jk} - \mu_{ik} - (\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i))}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}} < \frac{\overline{X}_{ik}(N_i) - \overline{X}_{jk}(N_j)}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}} \right]
\end{aligned}
$$
$$(4.40)$$

$$
\times \prod_{k \in K_3} P\left[\frac{\mu_{ik} - \mu_{jk} - (\overline{X}_{ik}(N_i) - \overline{X}_{jk}(N_j))}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}} < \frac{\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}} \right]
$$
$$(4.41)$$

$$
= \prod_{k=1}^{H} P\left[Z_{ijk} < \frac{|\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)|}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}} \right]
$$
$$(4.42)$$

$$= \prod_{k=1}^{H} \Phi \left(\frac{|\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)|}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}} \right). \tag{4.43}$$

Note that the product symbol in (4.42) has all $k \in K$, combining the two terms (4.40) and (4.41) using the absolute value $|\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)|$. The equality in (4.43) holds because Z_{ijk} are i.i.d random variables following $N(0,1)$. Now suppose

$$\delta_{ijk} = \max\{\delta_k^*, \overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)\} = \overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i), \tag{4.44}$$

which is not true in this case as $|\overline{X}_{jk} - \overline{X}_{ik}| < \delta_k^*$ for all $k \in K$. However, (4.44) is assumed to be true for now, and the case of $\delta_{ijk} = \delta_k^*$ will be discussed later. The proof follows from (4.43):

$$P(\text{CS}_{ij}) = \prod_{k=1}^{H} \Phi \left(\frac{|\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)|}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}} \right)$$

$$\geq \prod_{k=1}^{H} \Phi \left(\frac{|\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)|}{\sqrt{\dfrac{\sigma_{ik}^2}{\left(\dfrac{h_1 \, S_{ik}(N_i)}{\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)}\right)^2} + \dfrac{\sigma_{jk}^2}{\left(\dfrac{h_1 \, S_{jk}(N_j)}{\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)}\right)^2}}} \right) \tag{4.45}$$

$$= \prod_{k=1}^{H} \Phi \left(\frac{h_1}{\sqrt{\dfrac{\sigma_{ik}^2}{S_{ik}(N_i)^2} + \dfrac{\sigma_{jk}^2}{S_{jk}(N_j)^2}}} \right) \tag{4.46}$$

$$= \prod_{k=1}^{H} \Phi \left(\frac{h_1}{\sqrt{(N_i - 1)\dfrac{1}{Y_{ik}} + (N_j - 1)\dfrac{1}{Y_{jk}}}} \right) \tag{4.47}$$

$$
\begin{aligned}
&= \prod_{k=1}^{H} E \left[\Phi \left(\frac{h_1}{\sqrt{(N_i - 1)\frac{1}{Y_{ik}} + (N_j - 1)\frac{1}{Y_{jk}}}} \right) \middle| Y_{jk} \right] \\
&= \prod_{k=1}^{H} \int_{0}^{\infty} \left[\Phi \left(\frac{h_1}{\sqrt{(N_i - 1)\frac{1}{x} + (N_j - 1)\frac{1}{y}}} \right) \right] f_2(y) \, \mathrm{d}y \\
&= \prod_{k=1}^{H} \left[\int_{0}^{\infty} \left[\int_{0}^{\infty} \Phi \left(\frac{h_1}{\sqrt{(N_i - 1)\frac{1}{x} + (N_j - 1)\frac{1}{y}}} \right) f_1(x) \, \mathrm{d}x \right] f_2(y) \, \mathrm{d}y \right] \\
&= \left[\int_{0}^{\infty} \left[\int_{0}^{\infty} \Phi \left(\frac{h_1}{\sqrt{(N_i - 1)\frac{1}{x} + (N_j - 1)\frac{1}{y}}} \right) f_1(x) \, \mathrm{d}x \right] f_2(y) \, \mathrm{d}y \right]^{H} \\
&= 1 - \gamma.
\end{aligned}
\tag{4.48}
$$

The inequality in (4.45) holds from the conditions in (4.11) and because of (4.44). The equality in (4.47) follows due to (4.33), *i.e.*, Y_{ik} ($k \in K$) are H independent χ^2 variables with $N_i - 1$ degrees of freedom; and Y_{jk} ($k \in K$) are also H independent χ^2 variables with $N_j - 1$ degrees of freedom. The equality in (4.48) holds from the definition of h_1 given in (4.16). Therefore (4.39) is proved.

The proof was carried out with the assumption in (4.44). This means for any pair of systems (i, j) ($i, j \in S$, $i \neq j$), when one applies the conditions given (4.11), $P(\mathrm{CS}_{ij}) \geq 1 - \gamma$ is guaranteed if (4.44) is true. However, in Step 4 in Algorithm 3, these conditions (4.11) are applied for pairs of systems (i, j) ($i \in S_p$ and $j \in S$, $j \neq i$) when $K_1 = K$, which means the assumption in (4.44) is *not* true in this case, as $|\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)| < \delta_k^*$ for all $k \in K$. Therefore $P(\mathrm{CS}_{ij}) \geq 1 - \gamma$ is not proved in this case because the inequality in (4.45) does not hold. According to the proof above, the procedure needs in this case at least

$$
N_i > \left\lceil \max_k \left(\frac{h_1 \, S_{ik}(N_i)}{|\overline{X}_{jk} - \overline{X}_{ik}|} \right)^2 \right\rceil \quad \text{and} \quad N_j > \left\lceil \max_k \left(\frac{h_1 \, S_{jk}(N_j)}{|\overline{X}_{jk} - \overline{X}_{ik}|} \right)^2 \right\rceil
\tag{4.49}
$$

simulation replications in order to prove (4.39). However, this involves a risk of having $N_i \to \infty$ and $N_i \to \infty$ if the performances of the two systems are close to each other,

i.e., $|\mu_{jk} - \mu_{ik}| \to 0$ for some k. By replacing the term $|\overline{X}_{jk} - \overline{X}_{ik}|$ in (4.49) with δ_k^* if $|\overline{X}_{jk} - \overline{X}_{ik}| < \delta_k^*$, the procedure essentially limits the effort to the point where it can assure $|\mu_{jk} - \mu_{ik}| \leq \delta_k^*$, not exerting to the extent where it can actually identify the exact relations of μ_{ik} and μ_{jk} as specified in (4.39). Therefore the procedure does not prove (4.39); however, by spending the simulation budget as much as given in condition (4.11), it still guarantees that $P(i \simeq_{IZ} j) \geq 1 - \gamma$, which is exactly what is required according to (4.37).

4.2.2.2 Proof for observed dominated systems in Procedure $\mathcal{MMY}$

In the previous section, the researcher proved (4.23) for all observed non-dominated systems $i \in S_p$, by proving (4.26) for all pairs of (i,j) $(i \in S_p, j \in S, i \neq j)$. The proof was relevant to Steps 4 and 5 in Algorithm 3. In this section, (4.23) is proved for all systems that are observed as dominated. The discussion in this section is therefore relevant to Step 7 in Algorithm 3.

Consider a system $j \in S_p^c$. For this system, $P(\mathrm{CS}_j)$ is by definition the probability that the observed dominated system j is truly dominated, *i.e.*, $P(\mathrm{CS}_j) = P(j$ is dominated by at least one solution in S_p in truth). If it is shown that there exists at least a system i, such that $i \prec j$ with the probability at least $1 - \beta$, then system j is dominated (by system i) and $P(\mathrm{CS}_j) \geq 1 - \beta$, thus (4.23) holds. In Step 7, the procedure has chosen such system i as defined in (4.21), *i.e.*, system i was selected so that the probability of system j being dominated by system i in truth is to be maximised. It does not threaten the validity of the proof, however, even if the procedure chose a wrong system, because any system $i \in S_p$ that is observed to dominate system j will do to show $P(\mathrm{CS}_j) \geq 1 - \beta$, although obtaining system i as defined in (4.21) would render the procedure efficient. The probability of correct selection for system $j \in S_p^c$ is then constructed as

$$
\begin{aligned}
P(\mathrm{CS}_j) &= P(j \text{ is in truth dominated by at least one solution in } S_p) \\
&\geq \max_{i' \in S_p} P(i' \prec j) \approx P(i \prec j) \\
&= P(\mu_{ik} \leq \mu_{jk}, \quad \forall k \in K),
\end{aligned}
\tag{4.50}
$$

where i is defined in (4.21). From (4.2), it is known that $\mu_{ik} \sim N\left(\overline{X}_{ik}(N_i), \dfrac{\sigma_{ik}^2}{N_i}\right)$ and

$\mu_{jk} \sim N\left(\overline{X}_{jk}(N_j), \dfrac{\sigma_{jk}^2}{N_j}\right)$, therefore from (4.50)

$$P(\mathrm{CS}_j) \geq P(\mu_{ik} \leq \mu_{jk}, \quad \forall k \in K)$$

$$= P\left[\frac{\mu_{ik} - \mu_{jk} - (\overline{X}_{ik}(N_i) - \overline{X}_{jk}(N_j))}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}} \leq \frac{\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}}, \quad \forall k \in K\right]$$

$$= P\left[Z_{ijk} \leq \frac{\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}}, \quad \forall k \in K\right]$$

$$= \prod_{k=1}^{H} \Phi\left(\frac{\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}}\right) \tag{4.51}$$

$$\geq \prod_{k=1}^{H} \Phi\left(\frac{\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)}{\sqrt{\dfrac{\sigma_{ik}^2}{\left(\dfrac{h_3\, S_{ik}(N_i)}{\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)}\right)^2} + \dfrac{\sigma_{jk}^2}{\left(\dfrac{h_3\, S_{jk}(N_j)}{\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)}\right)^2}}}\right) \tag{4.52}$$

$$= \prod_{k=1}^{H} \Phi\left(\frac{h_3}{\sqrt{\dfrac{\sigma_{ik}^2}{S_{ik}(N_i)^2} + \dfrac{\sigma_{jk}^2}{S_{jk}(N_j)^2}}}\right). \tag{4.53}$$

The equality in (4.51) holds because Z_{ijk} are i.i.d random variables following $N(0,1)$. The inequality in (4.52) holds from the fact that the two conditions in (4.13) are met and if it is assumed, again, (4.44). $\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)$ is observed as a positive value for all $k \in K$ by definition (because i is chosen as $i \,\hat{\prec}\, j$). The proof continues from

(4.53) as follows:

$$
P(\text{CS}_j) \geq \prod_{k=1}^{H} \Phi \left(\frac{h_3}{\sqrt{\dfrac{\sigma_{ik}^2}{S_{ik}(N_i)^2} + \dfrac{\sigma_{jk}^2}{S_{jk}(N_j)^2}}} \right)
$$

$$
= \prod_{k=1}^{H} \Phi \left(\frac{h_3}{\sqrt{(N_i - 1)\dfrac{1}{Y_{ik}} + (N_j - 1)\dfrac{1}{Y_{jk}}}} \right) \tag{4.54}
$$

$$
= \prod_{k=1}^{H} E \left[\Phi \left(\frac{h_3}{\sqrt{(N_i - 1)\dfrac{1}{Y_{ik}} + (N_j - 1)\dfrac{1}{Y_{jk}}}} \right) \bigg| Y_{jk} \right]
$$

$$
= \prod_{k=1}^{H} \int_0^\infty \left[\Phi \left(\frac{h_3}{\sqrt{((N_i - 1)\dfrac{1}{x} + (N_j - 1)\dfrac{1}{y})}} \right) \right] f_2(y) \, \mathrm{d}y
$$

$$
= \prod_{k=1}^{H} \left[\int_0^\infty \left[\int_0^\infty \Phi \left(\frac{h_3}{\sqrt{((N_i - 1)\dfrac{1}{x} + (N_j - 1)\dfrac{1}{y})}} \right) f_1(x) \, \mathrm{d}x \right] f_2(y) \, \mathrm{d}y \right]
$$

$$
= \left[\int_0^\infty \left[\int_0^\infty \Phi \left(\frac{h_3}{\sqrt{((N_i - 1)\dfrac{1}{x} + (N_j - 1)\dfrac{1}{y})}} \right) f_1(x) \, \mathrm{d}x \right] f_2(y) \, \mathrm{d}y \right]^H
$$

$$
= 1 - \beta. \tag{4.55}
$$

The definition given in (4.33) was applied in (4.54), and the equality in (4.55) comes from the definition of h_3 in (4.22). Thus, (4.23) is proved for observed dominated systems $j \in S_p^c$ when (4.44) is true.

However, $P(\text{CS}) \geq 1 - \beta$ is not guaranteed if $\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i) < \delta_k^*$ for some k, because the inequality in (4.52) holds only if (4.44) is true. In this regard, the discussion in the last paragraph of Section 4.2.2.1 is applicable in this context, too. For a given $j \in S_p^c$ and $i \in S_p$ obtained by (4.21), Procedure $\mathcal{MMY}$ tries to guarantee that system j is in truth dominated by system i with the probability of at least $1 - \beta$, *i.e.*, $P(\text{CS}_j) = P(i \prec j) \geq 1 - \beta$. However, by using δ_{ijk} instead of $\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)$ in

(4.13), it guarantees $P(i \prec_{IZ} j) \geq 1 - \beta$ instead of $P(i \prec j) \geq 1 - \beta$. By doing so, the procedure avoids wasting simulation effort on differentiating two systems with similar performance and at the same time it delivers equally good Pareto optimal solutions (Q_R instead of Q).

This concludes the proof on the probability of correct selection for Procedure $\mathcal{MMY}$. The researcher proved $P(\mathrm{CS}) \geq P^*$ by proving (4.23) for all observed non-dominated systems $i \in S_p$ in Section 4.2.2.1, and for all observed dominated systems $j \in S_p^c$ in Section 4.2.2.2. The next section discusses Procedure $\mathcal{MMY}1$.

4.3 The $\mathcal{MMY}1$ procedure

As mentioned previously, Procedure $\mathcal{MMY}$ was designed to find a relaxed Pareto set, of which the concept was discussed in Section 4.1.5. In this and the following section, two other procedures are presented, called Procedure $\mathcal{MMY}1$ and Procedure $\mathcal{MMY}2$. Procedure $\mathcal{MMY}1$ and Procedure $\mathcal{MMY}2$ provide an exact Pareto optimal set, Q and Q_{IZ}, respectively, and guarantee that the probability of correct selection is greater than or equal to a prespecified value P^*.

Procedure $\mathcal{MMY}1$ is the same as Procedure $\mathcal{MMY}$ except that it uses $\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)$ wherever δ_{ijk} was used in Procedure $\mathcal{MMY}$. The following was explained in the last paragraph of Section 4.2.2.1 and in the second-last paragraph of Section 4.2.2.2:

1. For the procedure to be able to distinguish two systems, it requires the sample size to be inversely proportional to the square of the difference $\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)$; see (4.36) for example.

2. Procedure $\mathcal{MMY}$, however, uses δ_{ijk} instead of $\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)$.

3. The principle behind it is that the procedure distinguishes the two systems to the extent where the difference of the two systems for objective k becomes equal to the indifference-zone value δ_k^*; and this results in Procedure $\mathcal{MMY}$ delivering a relaxed Pareto set Q_R instead of the exact Pareto optimal set Q.

Therefore the exact Pareto optimal set Q can be obtained by using $\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)$ instead of δ_{ijk}. Procedure $\mathcal{MMY}1$ is given in Algorithm 4 for the sake of completeness. The proof is similar to that used in Procedure $\mathcal{MMY}$, hence it is omitted.

Algorithm 4 The $\mathcal{MMY}1$ procedure

1: Select the probability requirement $P^* = 1 - \alpha$, the indifference-zone value δ_k^* for each objective $k \in K$, and the first-stage sample size $n_0 \geq 2$. Set $I = \{1, 2, \ldots, M\}$ and $\beta = \dfrac{\alpha}{M}$.

2: Simulate n_0 replications for all M systems, and calculate sample means $\overline{X}_{ik}(n_0)$ and sample variances $S_{ik}^2(n_0)$ ($i \in S$ and $k \in K$). Let $N_i = n_0$.

3: Observe the Pareto set S_p and the non-Pareto set S_p^c based on the sample means $\overline{X}_{ik}(N_i)$ ($i \in S$ and $k \in K$) without the indifference-zone concept.

4: For each system $i \in S_p$ and $j \in S$ ($j \neq i$) with $K_1 = K$, check if the following two conditions are met:

$$N_i > \left\lceil \max_k \left(\frac{h_1\, S_{ik}(N_i)}{\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)} \right)^2 \right\rceil \;\; \text{and} \;\; N_j > \left\lceil \max_k \left(\frac{h_1\, S_{jk}(N_j)}{\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)} \right)^2 \right\rceil, \tag{4.56}$$

where h_1 is the solution to (4.16), and K_1 is defined in (4.14).

5: For each system $i \in S_p$ and $j \in S$ ($j \neq i$) with $K_1 \neq K$, check if the following two conditions are met:

$$N_i > \left\lceil \left(\frac{h_2\, S_{ik'}(N_i)}{\overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i)} \right)^2 \right\rceil \;\; \text{and} \;\; N_j > \left\lceil \left(\frac{h_2\, S_{jk'}(N_j)}{\overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i)} \right)^2 \right\rceil, \tag{4.57}$$

where k' is defined in (4.15) and h_2 is the solution to (4.17).

6: Delete system i from I if conditions (4.56) or (4.57) are satisfied for all $j \in S$ ($j \neq i$).

7: For each system $j \in S_p^c$, find system $i \in S_p$ as defined in (4.21). Check if the following two conditions are met:

$$N_i > \left\lceil \max_k \left(\frac{h_3\, S_{ik}(N_i)}{\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)} \right)^2 \right\rceil \;\; \text{and} \;\; N_j > \left\lceil \max_k \left(\frac{h_3\, S_{jk}(N_j)}{\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)} \right)^2 \right\rceil, \tag{4.58}$$

where h_3 is the solution to (4.22).

8: Delete system j from I if conditions in (4.58) are satisfied.

9: If $|I| = 0$, then stop and present the current Pareto set S_p as the final solution set. Otherwise, for each system $i \in S_p \cap I$, that is, systems in S_p that were not deleted from I in Step 6, add system $j \in S$ ($j \neq i$) to I if it does not satisfy conditions (4.56) or (4.57). Similarly, for each system $j \in S_p^c \cap I$, that is, systems in S_p^c that were not deleted from I in Step 8, add the corresponding system $i \in S_p$ to I if it does not satisfy (4.58). Go to Step 10.

10: Take one additional observation X_{i,k,N_i+1} from each system $i \in I$, and set $N_i \leftarrow N_i + 1$ ($\forall i \in I$). Set $I = \{1, 2, \ldots, M\}$ and update $\overline{X}_{ik}(N_i)$ and $S_{ik}^2(N_i)$ for all $i \in S$ and $k \in K$. and go to Step 3.

4.4 The $\mathcal{MMY}2$ procedure

In this section, Procedure $\mathcal{MMY}2$ is presented along with the proof of its statistical validity. Procedure $\mathcal{MMY}2$ aims to identify the exact Pareto set with the indifference-zone concept (Q_{IZ}) with the probability of correct selection greater than or equal to a prespecified value P^*.

4.4.1 The $\mathcal{MMY}2$ procedure steps

Algorithm 5 presents the steps of Procedure $\mathcal{MMY}2$. Because Procedure $\mathcal{MMY}2$ delivers Q_{IZ}, it uses the observed Pareto set with IZ (S_{IZ}) instead of S_p in Step 3. As in Procedure $\mathcal{MMY}$, observed non-dominated systems ($i \in S_{IZ}$) are considered first and then observed dominated systems ($j \in S_{IZ}^c$).

Consider a pair of systems (i, j) ($i \in S_{IZ}$ and $j \in S$, $j \neq i$). Let

$$K_4 = \{k \mid \overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i) > \delta_k^*, \ k \in K\} \tag{4.59}$$

and

$$k' = \underset{k \in K_4}{\arg \max} \ \Phi \left(\frac{-\delta_k^* + \overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)}{\sqrt{\dfrac{S_{ik}^2(N_i)}{N_i} + \dfrac{S_{jk}^2(N_j)}{N_j}}} \right), \tag{4.60}$$

where Φ denotes the c.d.f of the standard normal distribution. Note that K_4 and k' should be defined for every pair of (i, j) ($i \in S_{IZ}$ and $j \in S$, $j \neq i$) as in Procedure $\mathcal{MMY}$. Because system i is observed to be non-dominated with IZ, it is observed $j \hat{\nprec}_{IZ} i$ for any $j \in S$ ($j \neq i$). Reversing the statement in (4.8), it is known that at least one of the following two conditions is true if $j \hat{\nprec}_{IZ} i$:

$$\overline{X}_{jk} - \overline{X}_{ik} > \ \delta_k^*, \ \exists k \in K, \tag{4.61}$$

$$\overline{X}_{jk} - \overline{X}_{ik} \geq -\delta_k^*, \ \forall k \in K. \tag{4.62}$$

The set K_4 contains all objectives that satisfy the first condition (4.61), hence if $K_4 = \varnothing$, this means that the second condition (4.62) should be met for this pair of (i, j). Therefore there are two cases: the first one with $K_4 = \varnothing$, and the second case with $K_4 \neq \varnothing$. These cases are dealt with in Steps 4 and 5 in Algorithm 5, respectively.

Algorithm 5 The $\mathcal{MMY}2$ procedure

1: Select the probability requirement $P^* = 1 - \alpha$, the indifference-zone value δ_k^* for each objective $k \in K$, and the first-stage sample size $n_0 \geq 2$. Set $I = \{1, 2, \ldots, M\}$ and $\beta = \dfrac{\alpha}{M}$.

2: Simulate n_0 replications for all M systems, and calculate sample means $\overline{X}_{ik}(n_0)$ and sample variances $S_{ik}^2(n_0)$ ($i \in S$ and $k \in K$). Let $N_i = n_0$.

3: Observe the Pareto set S_{IZ} and the non-Pareto set S_{IZ}^c based on the sample means $\overline{X}_{ik}(N_i)$ ($i \in S$ and $k \in K$) with the indifference-zone concept.

4: For each system $i \in S_{IZ}$ and $j \in S$ ($j \neq i$) with $K_4 = \varnothing$, check if the following two conditions are met:

$$N_i > \left\lceil \max_k \left(\frac{h_1 \, S_{ik}(N_i)}{\delta_k^* + \overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)} \right)^2 \right\rceil \text{ and } N_j > \left\lceil \max_k \left(\frac{h_1 \, S_{jk}(N_j)}{\delta_k^* + \overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)} \right)^2 \right\rceil,$$
$$(4.63)$$

where h_1 is the solution to (4.16), and K_4 is defined in (4.59).

5: For each system $i \in S_{IZ}$ and $j \in S$ ($j \neq i$) with $K_4 \neq \varnothing$, check if the following two conditions are met:

$$N_i > \left\lceil \left(\frac{h_2 \, S_{ik'}(N_i)}{-\delta_k^* + \overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i)} \right)^2 \right\rceil \text{ and } N_j > \left\lceil \left(\frac{h_2 \, S_{jk'}(N_j)}{-\delta_k^* + \overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i)} \right)^2 \right\rceil,$$
$$(4.64)$$

where k' is defined in (4.60) and h_2 is the solution to (4.17).

6: Delete system i from I if conditions (4.63) or (4.64) are satisfied for all $j \in S$ ($j \neq i$).

7: For each system $j \in S_{IZ}^c$, find system $i \in S_{IZ}$ as defined in (4.21). Check if the following two conditions are met:

$$N_i > \max_k N_{ik} \text{ and } N_j > \max_k N_{jk},$$
$$(4.65)$$

where N_{ik} is defined in (4.66) and (4.67); and N_{jk} is defined in (4.68) and (4.69).

8: Delete system j from I if conditions in (4.65) are satisfied.

9: If $|I| = 0$, then stop and present the current Pareto set S_{IZ} as the final solution set. Otherwise, for each system $i \in S_{IZ} \cap I$, that is, systems in S_{IZ} that were not deleted from I in Step 6, add system $j \in S$ ($j \neq i$) to I if it does not satisfy conditions (4.63) or (4.64). Similarly, for each system $j \in S_{IZ}^c \cap I$, that is, systems in S_{IZ}^c that were not deleted from I in Step 8, add the corresponding system $i \in S_{IZ}$ to I if it does not satisfy (4.65). Go to Step 10.

10: Take one additional observation X_{i,k,N_i+1} from each system $i \in I$, and set $N_i \leftarrow N_i + 1$ ($\forall i \in I$). Set $I = \{1, 2, \ldots, M\}$ and update $\overline{X}_{ik}(N_i)$ and $S_{ik}^2(N_i)$ for all $i \in S$ and $k \in K$. and go to Step 3.

Now consider a system j that is observed as dominated with IZ, *i.e.*, $j \in S_{IZ}^c$. For this system, there exists at least one system $i \in S_{IZ}$ such that $i \hat{\prec}_{IZ} j$, otherwise it would not have been observed as dominated. Find such system $i \in S_{IZ}$ for system $j \in S_{IZ}^c$. If there exists more than one system in S_{IZ} that dominates j by observation, choose $i = \arg\max_{i' \in S_{IZ}} P(i' \prec_{IZ} j)$. Such system i can be obtained in a similar manner described in (4.21). Now, because system i is observed to dominate system j, from (4.8), both (4.61) and (4.62) are true for this pair (i, j). For system i, let

$$N_{ik} = \left\lceil \left(\frac{h_3 \, S_{ik}(N_i)}{\delta_k^* + (\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i))} \right)^2 \right\rceil \quad \text{for } k \in K, k \neq k' \qquad (4.66)$$

$$N_{ik} = \left\lceil \left(\frac{h_3 \, S_{ik'}(N_i)}{-\delta_{k'}^* + (\overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i))} \right)^2 \right\rceil \quad \text{for } k', \qquad (4.67)$$

where $\lceil x \rceil$ denotes the smallest integer greater than x; k' is defined in (4.60); and h_3 is the solution to (4.22). Also, similarly for system j, let

$$N_{jk} = \left\lceil \left(\frac{h_3 \, S_{jk}(N_j)}{\delta_k^* + (\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i))} \right)^2 \right\rceil \quad \text{for } k \in K, k \neq k' \qquad (4.68)$$

$$N_{jk} = \left\lceil \left(\frac{h_3 \, S_{jk'}(N_j)}{-\delta_{k'}^* + (\overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i))} \right)^2 \right\rceil \quad \text{for } k'. \qquad (4.69)$$

4.4.2   Proof of $P(\text{CS})$ for Procedure $\mathcal{MMY}2$

This section provides a mathematical proof that Procedure $\mathcal{MMY}2$ guarantees the probability of correct selection requirement $P(\text{CS}) \geq P^* = 1 - \alpha$. Note that 'correct selection' for Procedure $\mathcal{MMY}2$ denotes the case when the final solution set is exactly the same as Q_{IZ} defined in (4.7). Let CS_i be the event that system i is observed correctly, and ICS_i be the event that system i is observed incorrectly. The proof follows the same strategy in Procedure $\mathcal{MMY}$, *i.e.*, (4.23) will be shown for all $i \in S$, then by (4.25) $P(\text{CS}) \geq P^* = 1 - \alpha$ is guaranteed.

4.4.2.1   Proof for observed non-dominated systems in Procedure $\mathcal{MMY}2$

First the researcher considers observed non-dominated systems when the indifference-zone concept is accounted for, *i.e.*, (4.23) will be shown for all $i \in S_{IZ}$ in this sec-

tion. For these systems, $P(\text{CS}_i)$ is by definition the probability that the observed non-dominated system i is truly non-dominated when the IZ concept is considered both in observation and in truth, *i.e.*, $P(\text{CS}_i) = P(i$ is not dominated by any other solution in S in truth with the IZ concept). Let $P(\text{CS}_{ij})$ be the probability that the observed non-dominated system i (with IZ) is, in truth, not dominated by system j (with IZ); and $P(\text{ICS}_{ij})$ the probability that the observed non-dominated system i (with IZ) is, in truth, dominated by system j (with IZ). As in Procedure $\mathcal{MMY}$, for every pair (i,j) $(i \in S_{IZ}$ and $j \in S, j \neq i)$,

$$P(\text{CS}_{ij}) = P(j \nprec_{IZ} i) \geq 1 - \gamma$$

or

$$P(\text{ICS}_{ij}) = P(j \prec_{IZ} i) \leq \gamma, \tag{4.70}$$

will be shown where $\gamma = \dfrac{\beta}{M-1}$, then again by (4.28), (4.23) is proved.

In Step 5 in Algorithm 5, the pair (i,j) $(i \in S_{IZ}$ and $j \in S, j \neq i)$ is considered when $K_4 \neq \varnothing$, that is, the condition in (4.61) is satisfied. From (4.70) and (4.6),

$$
\begin{aligned}
P(\text{ICS}_{ij}) = P(j \prec_{IZ} i) &= P(j \precsim_{IZ} i) - P(j \simeq_{IZ} i) \\
&= \bigcap_{k=1}^{H} P(\mu_{jk} - \mu_{ik} \leq \delta_k^*) - \bigcap_{k=1}^{H} P(|\mu_{jk} - \mu_{ik}| \leq \delta_k^*) \\
&\leq \bigcap_{k=1}^{H} P(\mu_{jk} - \mu_{ik} \leq \delta_k^*) \\
&= \prod_{k=1}^{H} P(\mu_{jk} - \mu_{ik} \leq \delta_k^*) \\
&\leq \min_k P(\mu_{jk} - \mu_{ik} \leq \delta_k^*) \\
&= \min_k \left[1 - P(\mu_{jk} - \mu_{ik} \geq \delta_k^*)\right] \\
&= \min_k \left[1 - P(\mu_{ik} - \mu_{jk} \leq -\delta_k^*)\right] \\
&= 1 - \max_k P(\mu_{ik} - \mu_{jk} \leq -\delta_k^*).
\end{aligned}
$$

Therefore, if

$$\max_k P(\mu_{ik} - \mu_{jk} \leq -\delta_k^*) \geq 1 - \gamma$$

is shown, then (4.70) holds.

From (4.2), it is known that $\mu_{ik} \sim N\left(\overline{X}_{ik}(N_i), \dfrac{\sigma_{ik}^2}{N_i}\right)$ and $\mu_{jk} \sim N\left(\overline{X}_{jk}(N_j), \dfrac{\sigma_{jk}^2}{N_j}\right)$, therefore

$$
\begin{aligned}
&\max_k P(\mu_{ik} - \mu_{jk} \le -\delta_k^*) \\[2mm]
&= \max_k P\left[\frac{\mu_{ik} - \mu_{jk} - (\overline{X}_{ik}(N_i) - \overline{X}_{jk}(N_j))}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}} \le \frac{-\delta_k^* + \overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}}\right] \\[2mm]
&= \max_k P\left[Z_{ijk} \le \frac{-\delta_k^* + \overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}}\right] \\[2mm]
&= \max_k \Phi\left(\frac{-\delta_k^* + \overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}}\right).
\end{aligned}
\tag{4.71}
$$

Based on a similar logic to the discussion related to (4.19), the objective k' defined in (4.60) is used in the proof from (4.71).

$$
\begin{aligned}
&\max_k P(\mu_{ik} - \mu_{jk} \le -\delta_k^*) \\[2mm]
&= \Phi\left(\frac{-\delta_{k'}^* + \overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i)}{\sqrt{\dfrac{\sigma_{ik'}^2}{N_i} + \dfrac{\sigma_{jk'}^2}{N_j}}}\right) \\[4mm]
&\ge \Phi\left(\frac{-\delta_{k'}^* + \overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i)}{\sqrt{\dfrac{\sigma_{ik'}^2}{\left(\dfrac{h_2\, S_{ik'}(N_i)}{-\delta_{k'}^* + \overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i)}\right)^2} + \dfrac{\sigma_{jk'}^2}{\left(\dfrac{h_2\, S_{jk'}(N_j)}{-\delta_{k'}^* + \overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i)}\right)^2}}}\right)
\end{aligned}
\tag{4.72}
$$

$$= \Phi \left(\frac{h_2}{\sqrt{\dfrac{\sigma^2_{ik'}}{S_{ik'}(N_i)^2} + \dfrac{\sigma^2_{jk'}}{S_{jk'}(N_j)^2}}} \right). \tag{4.73}$$

The inequality in (4.72) follows from the fact that the conditions in (4.64) are met and $-\delta^*_k + \overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i) > 0$ by the definition of k' given in (4.60). The right-hand side of (4.73) is exactly the same as (4.32), hence the proof repeats the steps from (4.32), assuring finally $\max_k P(\mu_{ik} - \mu_{jk} \leq -\delta^*_k) \geq 1 - \gamma$. Thus, it has been shown that for a pair (i,j) ($i \in S_{IZ}$ and $j \in S$, $j \neq i$) when $K_4 \neq \varnothing$, $P(\mathrm{CS}_{ij}) \geq 1 - \gamma$ is satisfied when following the $\mathcal{MMY}2$ procedure in Step 5.

Now the researcher considers a pair (i,j) ($i \in S_{IZ}$ and $j \in S$, $j \neq i$) when $K_4 = \varnothing$ (Step 4 in Algorithm 5). As discussed in Section 4.4.1, for any system $i \in S_{IZ}$ and $j \in S$ ($j \neq i$), at least one of the two conditions given in (4.61) and (4.62) is true; and $K_4 = \varnothing$ suggests (4.61) is not satisfied therefore (4.62) is true. The probability of correct selection in this case is then constructed as follows:

$$
\begin{aligned}
P(\mathrm{CS}_{ij}) &= P(j \not\succ_{IZ} i) \\
&= P(\mu_{jk} - \mu_{ik} > \delta^*_k, \exists k \in K) \cup P(\mu_{jk} - \mu_{ik} \geq -\delta^*_k, \forall k \in K) \\
&\geq P(\mu_{jk} - \mu_{ik} \geq -\delta^*_k, \forall k \in K) \\
&= P(\mu_{ik} - \mu_{jk} \leq \delta^*_k, \forall k \in K).
\end{aligned}
$$

Again, from (4.2),

$$P(\mathrm{CS}_{ij}) \geq (\mu_{ik} - \mu_{jk} \leq \delta^*_k, \forall k \in K)$$

$$= P \left[\frac{\mu_{ik} - \mu_{jk} - (\overline{X}_{ik}(N_i) - \overline{X}_{jk}(N_j))}{\sqrt{\dfrac{\sigma^2_{ik}}{N_i} + \dfrac{\sigma^2_{jk}}{N_j}}} \leq \frac{\delta^*_k + \overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)}{\sqrt{\dfrac{\sigma^2_{ik}}{N_i} + \dfrac{\sigma^2_{jk}}{N_j}}}, \quad \forall k \in K \right]$$

$$= P \left[Z_{ijk} \leq \frac{\delta^*_k + \overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)}{\sqrt{\dfrac{\sigma^2_{ik}}{N_i} + \dfrac{\sigma^2_{jk}}{N_j}}}, \quad \forall k \in K \right]$$

$$= \prod_{k=1}^{H} \Phi \left(\frac{\delta_k^* + \overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}} \right)$$

$$\geq \prod_{k=1}^{H} \Phi \left(\frac{\delta_k^* + \overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)}{\sqrt{\dfrac{\sigma_{ik}^2}{\left(\dfrac{h_1 \, S_{ik}(N_i)}{\delta_k^* + \overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)} \right)^2} + \dfrac{\sigma_{jk}^2}{\left(\dfrac{h_1 \, S_{jk}(N_j)}{\delta_k^* + \overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)} \right)^2}}} \right) \tag{4.74}$$

$$= \prod_{k=1}^{H} \Phi \left(\frac{h_1}{\sqrt{\dfrac{\sigma_{ik}^2}{S_{ik}^2(N_i)} + \dfrac{\sigma_{jk}^2}{S_{jk}^2(N_j)}}} \right). \tag{4.75}$$

The inequality in (4.74) holds because the conditions in (4.63) are satisfied, and $\delta_k^* + \overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i) > 0$ for all $k \in K$ due to (4.62). Again (4.75) is the same as (4.46), therefore the following steps of the proof are the same as for Procedure $\mathcal{MMY}$.

4.4.2.2 Proof for observed dominated systems in Procedure $\mathcal{MMY}2$

In this section, the researcher shows (4.23) for all systems that are observed as dominated with IZ, *i.e.*, for all $j \in S_{IZ}^c$. For these systems, the probability of correct selection is defined as $P(\text{CS}_j) = P(j$ is, in truth, dominated with IZ by at least one system in $S_{IZ})$. If it is shown that there exists at least a system $i \in S_{IZ}$, such that $i \prec_{IZ} j$ with the probability of at least $1 - \beta$, then system j is dominated (by system i) and $P(\text{CS}_j) \geq 1 - \beta$, thus (4.23) holds. Finding such system $i \in S_{IZ}$ for system $j \in S_{IZ}^c$ follows in a similar manner as described in (4.21). The probability of correct selection for system $j \in S_{IZ}^c$ is then constructed as

$$P(\text{CS}_j) = P(j \text{ is, in truth, dominated with IZ by at least one solution in } S_{IZ})$$

$$\geq \max_{i' \in S_{IZ}} P(i' \prec_{IZ} j)$$

$$\approx P(i \prec_{IZ} j)$$

$$= \bigcap_{k=1}^{H} P(\mu_{ik} - \mu_{jk} \leq \delta_k^*) - \bigcap_{k=1}^{H} P(|\mu_{ik} - \mu_{jk}| \leq \delta_k^*) \tag{4.76}$$

$$\geq \left[\bigcap_{k=1,\, k \neq k'}^{H} P(\mu_{ik} - \mu_{jk} \leq \delta_k^*) \right] P(\mu_{ik'} - \mu_{jk'} \leq -\delta_{k'}^*), \tag{4.77}$$

where k' is defined in (4.60). The equality in (4.76) is straightforward from (4.6). The inequality in (4.77) holds because

$$\bigcap_{k=1}^{H} P(\mu_{ik} - \mu_{jk} \leq \delta_k^*) - \bigcap_{k=1}^{H} P(|\mu_{ik} - \mu_{jk}| \leq \delta_k^*)$$

$$- \left[\bigcap_{k=1,\, k \neq k'}^{H} P(\mu_{ik} - \mu_{jk} \leq \delta_k^*) \right] P(\mu_{ik'} - \mu_{jk'} \leq -\delta_{k'}^*)$$

$$= \left[\bigcap_{k=1,\, k \neq k'}^{H} P(\mu_{ik} - \mu_{jk} \leq \delta_k^*) \right] \left[P(\mu_{ik'} - \mu_{jk'} \leq \delta_{k'}^*) - P(\mu_{ik'} - \mu_{jk'} \leq -\delta_{k'}^*) \right]$$

$$- \bigcap_{k=1}^{H} P(|\mu_{ik} - \mu_{jk}| \leq \delta_k^*)$$

$$= \left[\bigcap_{k=1,\, k \neq k'}^{H} P(\mu_{ik} - \mu_{jk} \leq \delta_k^*) \right] P(|\mu_{ik'} - \mu_{jk'}| \leq \delta_{k'}^*) - \bigcap_{k=1}^{H} P(|\mu_{ik} - \mu_{jk}| \leq \delta_k^*)$$

$$= P(|\mu_{ik'} - \mu_{jk'}| \leq \delta_{k'}^*)$$

$$\times \left[\bigcap_{k=1,\, k \neq k'}^{H} P(\mu_{ik} - \mu_{jk} \leq \delta_k^*) - \bigcap_{k=1,\, k \neq k'}^{H} P(|\mu_{ik} - \mu_{jk}| \leq \delta_k^*) \right] > 0. \tag{4.78}$$

For each $k \in K, k \neq k'$, $P(\mu_{ik} - \mu_{jk} \leq \delta_k^*)$ is greater than $P(|\mu_{ik} - \mu_{jk}| \leq \delta_k^*)$, therefore the term inside the big bracket in (4.78) remains positive.

From (4.77), the probability of correct selection $P(\mathrm{CS}_j)$ is formulated as follows:

$$P(\mathrm{CS}_j) \geq \left[\bigcap_{k=1,\, k \neq k'}^{H} P(\mu_{ik} - \mu_{jk} \leq \delta_k^*) \right] P(\mu_{ik'} - \mu_{jk'} \leq -\delta_{k'}^*) \tag{4.79}$$

$$= \left[\bigcap_{k=1,\, k \neq k'}^{H} P \left(Z_{ijk} \leq \frac{\delta_k^* + (\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i))}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}} \right) \right]$$

$$\times P\left(Z_{ijk'} \leq \frac{-\delta_{k'}^* + (\overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i))}{\sqrt{\dfrac{\sigma_{ik'}^2}{N_i} + \dfrac{\sigma_{jk'}^2}{N_j}}}\right) \tag{4.80}$$

$$= \left[\prod_{k=1,\,k\neq k'}^{H} \Phi\left(\frac{\delta_k^* + (\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i))}{\sqrt{\dfrac{\sigma_{ik}^2}{N_i} + \dfrac{\sigma_{jk}^2}{N_j}}}\right)\right]\Phi\left(\frac{-\delta_{k'}^* + (\overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i))}{\sqrt{\dfrac{\sigma_{ik'}^2}{N_i} + \dfrac{\sigma_{jk'}^2}{N_j}}}\right)$$

$$\geq \left[\prod_{k=1,\,k\neq k'}^{H} \Phi\left(\frac{\delta_k^* + (\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i))}{\sqrt{\dfrac{\sigma_{ik}^2}{\left(\dfrac{h_3\, S_{ik}(N_i)}{\delta_k^* + (\overline{X}_{jk} - \overline{X}_{ik})}\right)^2} + \dfrac{\sigma_{jk}^2}{\left(\dfrac{h_3\, S_{jk}(N_j)}{\delta_k^* + (\overline{X}_{jk} - \overline{X}_{ik})}\right)^2}}}\right)\right]$$

$$\times \Phi\left(\frac{-\delta_{k'}^* + (\overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i))}{\sqrt{\dfrac{\sigma_{ik'}^2}{\left(\dfrac{h_3\, S_{ik'}(N_i)}{-\delta_{k'}^* + (\overline{X}_{jk'} - \overline{X}_{ik'})}\right)^2} + \dfrac{\sigma_{jk'}^2}{\left(\dfrac{h_3\, S_{jk'}(N_j)}{-\delta_{k'}^* + (\overline{X}_{jk'} - \overline{X}_{ik'})}\right)^2}}}\right) \tag{4.81}$$

$$= \prod_{k=1}^{H} \Phi\left(\frac{h_3}{\sqrt{\dfrac{\sigma_{ik}^2}{S_{ik}(N_i)^2} + \dfrac{\sigma_{jk}^2}{S_{jk}(N_j)^2}}}\right). \tag{4.82}$$

The inequality in (4.79) was proved in (4.78), and the equality in (4.80) holds again due to (4.2). The inequality in (4.81) follows from the fact that the two conditions in (4.65) are met and $\delta_k^* + (\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i))$ is observed to be positive for all $k \in K$ ($k \neq k'$) by (4.62); also, $-\delta_{k'}^* + (\overline{X}_{jk'}(N_j) - \overline{X}_{ik'}(N_i))$ is positive by the definition of k' given in (4.59) and (4.60). The right-hand side of (4.82) is the same as (4.53), therefore the proof follows in the same manner given in Section 4.2.2.2, resulting eventually in $P(\mathrm{CS}_j) \geq 1 - \beta$.

This section (Section 4.4) presented Procedure $\mathcal{MMY}$2 and proved its statistical validity. The following section demonstrates the effectiveness of the $\mathcal{MMY}$ family procedures empirically using some numerical experiments.

4.5 Experiments with the $\mathcal{MMY}$ family procedures

In this section, some experiments are presented to verify the statistical validity of the $\mathcal{MMY}$ family procedures.

4.5.1 Experimental setup for the $\mathcal{MMY}$ family procedures

An MORS problem with known true means—Example 3 given in Figure 4.5—was used in this experiment. The purpose of the experiment is to demonstrate that the final solution set presented by Procedure $\mathcal{MMY}$ is a relaxed Pareto set, *i.e.*, a member of $\mathbb{Q}_{\mathbb{T}} = \{Q_{R,1}, \ldots, Q_{R,18}\}$ in Example 3, with the probability of at least $P^* = 0.9$. Also, it will be shown that Procedure $\mathcal{MMY}$1 and Procedure $\mathcal{MMY}$2 present $Q = Q_{R,14}$ and $Q_{IZ} = Q_{R,9}$, respectively, with the probability of at least $P^* = 0.9$.

In order to compare the $\mathcal{MMY}$ family procedures with the multi-objective computing budget allocation (MOCBA) family procedures, the same MORS problem was solved also by the MOCBA procedure due to Lee *et al.* (2010b) and by the MOCBA_IZ procedure due to Teng *et al.* (2010). As discussed in Section 2.2.2, these procedures have a different approach to solving MORS problems, *i.e.*, they begin the procedure with a simulation budget—the number of simulation replications available for the problem—and allocate them across the systems so that the probability of correct selection is maximised. This makes it difficult to compare these procedures directly with the $\mathcal{MMY}$ family procedures. Therefore, the researcher compared them in the following way: The $\mathcal{MMY}$ family procedures are performed first with the probability of correct selection requirement $P^* = 0.9$. From the result, one obtains the average total number of simulation replications to secure $P(\text{CS}) \geq P^*$ for each procedure $\mathcal{MMY}$, $\mathcal{MMY}$1 and $\mathcal{MMY}$2. The MOCBA family procedures are then performed using this value as the simulation budget B. Hence, they are performed with three different simulation budgets, *i.e.*, the average total number of simulation replications acquired from Procedure $\mathcal{MMY}$, $\mathcal{MMY}$1 and $\mathcal{MMY}$2. Other parameters in the MOCBA family procedures were set as follows: $\Delta = 10$, $\tau = 5$. See Section 6 in Lee *et al.* (2010b) for detail information on these

parameters. Also, while implementing the MOCBA family procedures, the researcher encountered a number of cases where the set S_p^A is observed to be empty, *i.e.*, $S_p^A = \varnothing$, which leads to $\beta_i = 0$ for all $i \in S$, and eventually makes the algorithm run indefinitely without solving the problem. As no explicit direction is provided in Lee *et al.* (2010b) for this case, the researcher assigned one additional simulation replication to all systems to avoid the endless loop. The parameter settings and the approach to the case of $S_p^A = \varnothing$ described in this paragraph, were consistently applied in all implementations of the MOCBA family procedures in this research.

Table 4.2 shows the values of true means and variances in Example 3. $\delta_1^* = \delta_2^* = 0.5$ were used for IZ values, and $n_0 = 10$ was used in all cases.

Table 4.2: Experimental settings for Example 3

System	1	2	3	4	5	6	7	8	9	10
True mean	(5, 9)	(2, 8)	(4, 7)	(5, 4.3)	(3.3, 4.3)	(3, 4)	(4, 3.7)	(6, 2)	(8.3, 1.3)	(8, 1)
True variance	(1, 1)	(1, 1)	(1, 1)	(1, 1)	(1, 1)	(1, 1)	(1, 1)	(1, 1)	(1, 1)	(1, 1)

4.5.2 Experimental results of the $\mathbb{MMY}$ family procedures

Table 4.3 presents the result of this experiment. The $\mathbb{MMY}$ family procedures were independently performed 1 000 times each, and the total number of simulation replications was averaged over these 1 000 repetitions, *i.e.*, $\overline{N}_{total} = \frac{1}{1\,000} \sum_{R=1}^{1\,000} \sum_{i=1}^{M} N_{i,R}$, where $N_{i,R}$ denotes the number of simulation replications assigned to system i in the Rth run of the procedure. The MOCBA family procedures were performed with three different simulation budgets, *i.e.*, $B = 594$ (from the result of Procedure $\mathbb{MMY}$), $B = 1\,651$ (from the result of Procedure $\mathbb{MMY}1$) and $B = 4\,186$ (from the result of Procedure $\mathbb{MMY}2$), also 1 000 times independently for each case.

The estimated probability of correct selection $\hat{P}(\text{CS})$ was obtained by the number of correct selections out of the 1 000 repetitions. Recall that the $\mathbb{MMY}$ procedure acknowledges any of the 18 members of the set $\mathbb{Q}_{\mathbb{T}}$ as a correct selection. On the other hand, Procedure $\mathbb{MMY}1$ and the MOCBA procedure consider only $Q = Q_{R,14}$ correct, while Procedure $\mathbb{MMY}2$ and the MOCBA_IZ procedure count only $Q = Q_{R,9}$ as a correct selection. Accordingly, three different types of estimated probability of correct

Table 4.3: Experimental results for Example 3

Procedure	$\overline{N}_{total}$	$\hat{P}(\text{CS})$		
		Q_R	$Q_{R,14}$	$Q_{R,9}$
$\mathcal{MMY}$	593.93	100.0%	87.6%	0.0%
MOCBA (B=594)	599.41	99.8%	90.9%	0.0%
MOCBA_IZ (B=594)	601.61	99.8%	0.1%	43.8%
$\mathcal{MMY}1$	1650.50	100.0%	100.0%	0.0%
MOCBA ($B = 1\,651$)	1655.80	100.0%	98.3%	0.0%
MOCBA_IZ ($B = 1\,651$)	1657.50	99.8%	0.0%	58.8%
$\mathcal{MMY}2$	4185.80	100.0%	0.0%	99.9%
MOCBA ($B = 4\,186$)	4191.50	100.0%	99.7%	0.0%
MOCBA_IZ ($B = 4\,186$)	4195.80	100.0%	0.0%	99.8%

selection $\hat{P}(\text{CS})$ were given in Table 4.3: The third column (titled Q_R) presents $\hat{P}(\text{CS})$ when any relaxed Pareto set was counted as correct selection, while the fourth and fifth columns show $\hat{P}(\text{CS})$ respectively when only the exact Pareto optimal set without IZ ($Q = Q_{R,14}$) and with IZ ($Q_{IZ} = Q_{R,9}$) was considered as correct selection. Therefore, only one of the three columns of $\hat{P}(\text{CS})$ is relevant for each procedure. Those are highlighted with a grey background. Others were presented only for reference.

The result in Table 4.3 shows that the statistical validity of the $\mathcal{MMY}$ family procedures is verified as $\hat{P}(\text{CS})$ is observed to be greater than the requirement $P^* = 90\%$ for all important cases (highlighted with a grey background). The MOCBA procedure also shows good performance counting $Q = Q_{R,14}$ as a correct selection. Note that the $\hat{P}(\text{CS})$ from the MOCBA procedure increases as the simulation budget increases. The MOCBA_IZ procedure, however, shows poor performance in the first two cases. It exhibits only 43.8% and 58.8% of $\hat{P}(\text{CS})$, considering only $Q_{IZ} = Q_{R,9}$ as a correct selection, when the simulation budget is given as $B = 594$ and $B = 1\,651$, respectively. It achieves $\hat{P}(\text{CS}) \geq 90\%$ only with a larger simulation budget of $B = 4\,186$. This does not necessarily mean that the MOCBA_IZ procedure is inferior to the $\mathcal{MMY}$ family procedures or the MOCBA procedure. (In fact, the performances of all procedures are similar to each other if relaxed Pareto sets are considered, *i.e.*, $\hat{P}(\text{CS})$ given in the third column shows similar performances among procedures given similar simulation budgets.) However, it does show a critical property of the MOCBA family procedures:

The decision-maker has no idea of the reliability of the final solution set of the MOCBA family procedures.

4.6   Discussion of the $\mathcal{MMY}$ family procedures

In this section, the researcher discusses some features of the $\mathcal{MMY}$ family procedures. The greater part of the discussions in what follows is given in the context of Procedure $\mathcal{MMY}$. The discussions are valid, however, for the $\mathcal{MMY}1$ and $\mathcal{MMY}2$ procedures, too.

First of all, the $\mathcal{MMY}$ family procedures guarantee the probability of correct selection requirement $P(\text{CS}) \geq P^*$ based on a solid mathematical analysis using a Bayesian inference model. Therefore they take advantage of using the sample mean information instead of assuming the least favourable configuration. This is reflected in the denominators of the conditions, for example, in (4.11), (4.12) and (4.13).

In spite of this, however, the result in Table 4.3 indicates that the $\mathcal{MMY}$ family procedures are conservative, showing a much higher value of $\hat{P}(\text{CS})$ than the prespecified $P^* = 0.9$. This is due to the use of the Bonferroni inequality in (4.25) and again in (4.28). The principle is to divide the significance level (the level of overall risk of concluding incorrect selection) $\alpha = 1 - P^*$ into M smaller significance levels of $\beta = \frac{\alpha}{M}$, and show (4.23) for each system $i \in S$. For observed non-dominated systems, *i.e.*, systems in S_p, this is further decomposed into $M - 1$ smaller significance levels of $\gamma = \frac{\beta}{M-1}$ to show that system i is not dominated by system $j \in S$ ($j \neq i$), with the probability of at least $1 - \gamma$. This multiple-level application of the Bonferroni inequality renders the procedure very conservative.

The number of simulation replications assigned to a system by the $\mathcal{MMY}$ family procedures given in (4.11), (4.12) and (4.13), is proportional to the sample variance $S_{ik}^2(N_i)$ and inversely proportional to the square of the difference in sample means of two systems if $\delta_{ijk} = \max\{\delta_k^*, \overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)\} = \overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)$. Interestingly but not surprisingly, the same structure is seen in the MOCBA procedure (see Remark 2 in Lee *et al.* (2010b, p. 662)). Procedure $\mathcal{MMY}$ allows the size of the difference to decrease to the point of δ_k^*, from where all the smaller differences are ignored and the indifference-zone value is used instead. By doing this, the procedure avoids wasting an unnecessarily large amount of simulation budget to distinguish an insignificantly small difference, as discussed in Section 4.2.2. The MOCBA procedure, however, does not

use the concept of the indifference-zone value. This causes a huge amount of simulation budget spent on differentiating systems with close performances, which eventually led to the development of the MOCBA_IZ procedure by Teng *et al.* (2010).

Likewise, Procedure $\mathcal{MMY}1$ does not replace $\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)$ with δ_k^* even if the difference becomes smaller than the IZ value. This means that the procedure could assign an extremely large number of simulation replications if the true means of two systems for a given objective k is the same. See the conditions given in (4.56) for example. If $\mu_{ik} = \mu_{jk}$ for a certain pair of (i, j) $(i \neq j)$ and an objective k, $\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)$ would be observed as a very small value from the beginning of the procedure, and the procedure would assign more samples to systems i and j in an attempt to distinguish them. However, more samples would estimate the true means more accurately, leading to an even smaller value of $\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)$, which eventually makes N_i and N_j infinite.

The same situation occurs in Procedure $\mathcal{MMY}2$ when the difference of true means of two systems for an objective k is exactly the same as the IZ value for that objective, *i.e.*, when $|\mu_{ik} - \mu_{jk}| = \delta_k^*$. In this case, $|\overline{X}_{jk}(N_j) - \overline{X}_{ik}(N_i)|$ is observed to be close to δ_k^*, and the conditions given in (4.64) and (4.65) would ultimately require N_i and N_j to be infinite.

The MOCBA_IZ procedure suffered a similar problem (see the discussions in the last paragraph in Teng *et al.* (2010, p. 442)). This is why Teng *et al.* (2010) proposed to *approximate* the value $\delta_{ijk}^* = \delta_k^* - (\mu_{jk} - \mu_{ik})$ as follows:

$$\delta_{ijk}^* = \delta_k^* - (\mu_{jk} - \mu_{ik}) = \begin{cases} \delta_k^* & \text{if } |\mu_{jk} - \mu_{ik}| \leq \delta_k^* \\ -(\mu_{jk} - \mu_{ik}) & \text{otherwise} \end{cases}. \qquad (4.83)$$

This is ironic because the MOCBA_IZ procedure was designed in the first place to prevent the MOCBA procedure from spending too much simulation budget on differentiating two systems when $\mu_{ik} = \mu_{jk}$. Now the same kind of problem occurs in a different configuration, *i.e.*, when $|\mu_{jk} - \mu_{ik}| = \delta_k^*$, and the approximation (4.83) prevents δ_{ijk}^* becoming too small, thus the procedure avoids spending too much simulation budget. However, this inevitably impairs the performance of the procedure as seen in the results given in Table 4.3.

Applying the same principle of approximation given in (4.83) to Procedure $\mathcal{MMY}$2, *i.e.*, applying

$$\delta_k^* + (\overline{X}_{jk} - \overline{X}_{ik}) = \begin{cases} \delta_k^* & \text{if } |\overline{X}_{jk} - \overline{X}_{ik}| \leq \delta_k^* \\ \overline{X}_{jk} - \overline{X}_{ik} & \text{otherwise} \end{cases} \qquad (4.84)$$

or

$$-\delta_k^* + (\overline{X}_{jk} - \overline{X}_{ik}) = \begin{cases} -\delta_k^* & \text{if } |\overline{X}_{jk} - \overline{X}_{ik}| \leq \delta_k^* \\ \overline{X}_{jk} - \overline{X}_{ik} & \text{otherwise} \end{cases} \qquad (4.85)$$

to conditions in (4.63), (4.64) and (4.65) would help Procedure $\mathcal{MMY}$2 avoid assigning an infinity to N_i and N_j. However, this would let Procedure $\mathcal{MMY}$2 be exactly the same as Procedure $\mathcal{MMY}$, except that S_{IZ} is used in Procedure $\mathcal{MMY}$2 instead of S_p. Therefore, Procedure $\mathcal{MMY}$2 with the approximation given in (4.84) and (4.85) would not be able to find Q_{IZ} with the probability of at least P^*.

In this section, some features of the $\mathcal{MMY}$ family procedures were discussed. Especially, it was shown that it is not possible for Procedures $\mathcal{MMY}$1 and $\mathcal{MMY}$2 to identify the exact Pareto solution set Q and Q_{IZ}, respectively, when some extreme cases occur, *i.e.*, $|\mu_{ik} - \mu_{jk}| \to 0$ for Procedure $\mathcal{MMY}$1 and $|\mu_{ik} - \mu_{jk}| \to \delta_k^*$ for Procedure $\mathcal{MMY}$2. The MOCBA and MOCBA_IZ procedures also perform poorly in their corresponding extreme cases. In contrast, the $\mathcal{MMY}$ procedure can deal with these extreme cases by aiming at identifying a relaxed Pareto set Q_R, instead of Q or Q_{IZ}, which is equally good considering the indifference of the decision-maker.

The following section concludes this chapter.

4.7   Conclusion: Chapter 4

This chapter presented the core work of the research, *i.e.*, the design and development of a multi-objective ranking and selection procedure, called Procedure $\mathcal{MMY}$. This procedure employs the indifference-zone method, determining the minimum number of simulation replications for each system in order to meet the probability of correct selection requirement $P(\text{CS}) \geq P^*$. The necessity of the work is clear as there has not been an MORS procedure with the IZ approach before Procedure $\mathcal{MMY}$. Therefore, the work in this research fills the gap shown in Figure 2.2. The statistical validity of the procedure was shown through a rigorous mathematical analysis based on the Bayesian inference model.

Also, the researcher proposed in this chapter the concept of 'relaxed' Pareto set (Q_R). In the single-objective context, if the true mean of a system i is within δ^* from the true mean of the best system b, *i.e.*, $|\mu_i - \mu_b| \leq \delta^*$, the IZ procedures accept both systems as a correct selection. This idea was expanded to the multi-objective domain, resulting in the concept of the relaxed Pareto set. Therefore, Procedure $\mathcal{MMY}$ was designed to find a relaxed Pareto set Q_R, not the exact Pareto set Q (as the MOCBA procedure) or the exact Pareto set with IZ, Q_{IZ} (as the MOCBA_IZ procedure).

In addition, two variants of the procedure, called Procedure $\mathcal{MMY}$1 and Procedure $\mathcal{MMY}$2, were proposed. Procedure $\mathcal{MMY}$1 selects Q as a final solution set, therefore it can be considered as the IZ counterpart of the MOCBA procedure. Likewise, attempting to find Q_{IZ}, Procedure $\mathcal{MMY}$2 can be acknowledged as the counterpart of the MOCBA_IZ procedure. The statistical validity of these two procedures was also proved using the same mechanism as in Procedure $\mathcal{MMY}$.

The experiments given in Section 4.5 demonstrated the effectiveness of these three MORS procedures empirically by showing the estimated probability of correct selection is greater than P^* in all cases. It was also discussed that Procedures $\mathcal{MMY}$1 and $\mathcal{MMY}$2 are not applicable when some extreme cases occur. The MOCBA family procedures perform poorly in such cases. Procedure $\mathcal{MMY}$, however, is robust in those extreme cases too because it tries to find a relaxed Pareto set Q_R, not Q or Q_{IZ}. The superiority of Procedure $\mathcal{MMY}$ is shown even more clearly in the following chapter, where a few simulation case studies are used to validate the $\mathcal{MMY}$ family procedures.

Chapter 5

Simulation case studies

In this chapter, the multi-objective ranking and selection (MORS) procedures proposed in the previous chapter (Chapter 4) are validated through a variety of simulation case studies. First, a simple buffer allocation problem (BAP) is presented as a dynamic, stochastic MORS problem. The BAP problem was chosen because it is a classical simulation problem especially well used in production lines, of which the efficiency is one of the main concerns of Industrial Engineers. Next, the classical (s, S) inventory problem is considered. The inventory problem is also a well-established, typical optimisation problem often used in academics as well as in practice. See Cruz *et al.* (2008); Glasserman & Yao (1996); Vouros & Papadopoulos (1998) for buffer allocation problem applications and Bollapragada *et al.* (2004); Janssen *et al.* (1998); Moors & Strijbosch (2002); Rossi *et al.* (2010) for inventory problem applications. A gold mine problem extracted from Kelton *et al.* (2010) is then examined as an example of a typical simulation problem with sequential resources and continuous material handling. Finally a hospital management problem, designed and taught by the promoter of this research, is considered, which provides an example of simulation problems with categorical/Boolean variables. These four simulation case studies are discussed in four sections in this chapter (Sections 5.1 to 5.4).

In each simulation case study, a subsection is first assigned to describe the simulation model in detail, followed by two subsections that are dedicated to experimental setup and results, respectively. In the experimental setup section, as a preliminary step to validate Procedures $\mathcal{MMY}$, $\mathcal{MMY}1$ and $\mathcal{MMY}2$, the simulation model is run independently 10 000 times. The purpose of this simulation run is to estimate the unknown true

means of the objectives as accurately as possible. The sample means over these 10 000 simulation replications are considered to be very close to the unknown true means due to the strong law of large numbers (Law & Kelton, 2000, p. 292). Therefore they can be considered 'correct', and used in this study to identify 'true' Pareto optimal solutions. In particular, the researcher used these sample means based on the 10 000 simulation replications to identify the exact Pareto optimal solutions without and with IZ (Q and Q_{IZ}) as well as the true relaxed Pareto solutions, *i.e.*, Q_R.

Once these true Pareto optimal sets are defined for each simulation case study, the $\mathcal{MMY}$ family procedures are applied 1 000 times independently to each simulation problem. The experimental results section presents the results of these 1 000 applications of $\mathcal{MMY}$ family procedures, especially the proportion of the number of correct selections (based on Q, Q_{IZ} and Q_R) out of the 1 000 experiments. The statistical validity of the procedures will then be verified by showing that the proportion of the number of correct selections is greater than or equal to the prespecified value P^*. The MOCBA family procedures were also applied to solve the same problems, and the results are compared in the experimental results section.

5.1   Case study 1: Buffer allocation problem

The first simulation case study is a buffer allocation problem (BAP). The BAP is a typical decision-making problem in the design of production lines (Vouros & Papadopoulos, 1998), as well as telecommunication networks (Jouini *et al.*, 2009). Often the problem is to allocate finite queues (or buffer spaces) among m network centres (or machines) so as to maximise the throughput, *e.g.*, the number of calls attended to in a call-centre problem. The BAP is a classical NP-hard problem with a potentially very large set of feasible solutions, thus difficult to find the optimal solution even if the objective function is completely known (Alon *et al.*, 2005). The number of feasible solutions, for example, for a BAP in a serial production line with n buffer spaces and m machines, amounts to $\binom{n+m-2}{m-2}$.

5.1.1   Simulation model: Buffer allocation problem

The BAP model considered in this study is similar to what was described in Rubinstein & Kroese (2004, p. 207–208), which is repeated as follows: Consider a single production

line consisting of m machines. The machines are placed sequentially, and a product must be processed in each and every machine in sequence. Buffer space is needed between machines due to asynchronous processing times of the machines. The buffer spaces in front of the first machine and beyond the last machine are assumed to be infinite. Therefore, there exist $m-1$ buffer spaces to consider in this problem. See Figure 5.1 for an example design of this problem.
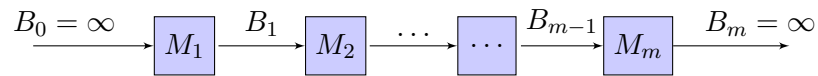

Figure 5.1: A single product line with m machines and $m-1$ buffers (Bekker, 2012)

The processing time of Machine M_i $(i = 1, \ldots, m)$ follows an exponential distribution with rate μ_i (hence the mean processing time is $1/\mu_i$). The machines are also subject to failures. In this study, operation-based failures (OBF) were considered rather than time-based failures as the former type of failures is more realistic than the latter (Yang et $al.$, 2000). Each machine therefore breaks down after a number of operations have been completed. This operation count is determined by a Poisson distribution with rate $\beta_i = 100$. The repair time for Machine M_i is exponentially distributed with rate $r_i = 2$.

When a part is finished at Machine M_i, it immediately proceeds to the next machine, M_{i+1}, if Machine M_{i+1} is not occupied and in working condition. Otherwise the part is stocked in the buffer between machine M_i and M_{i+1}. If the buffer is full then the part remains in Machine M_i until a space in the buffer becomes available. Therefore, an upstream machine becomes blocked when its successor has failed, while a downstream machine can eventually become starved if its predecessor has failed (Bekker, 2012). The first machine is fed as soon as it finishes processing the current part.

Let n be the total available buffer slots for the problem, and let B_i be the buffer size between Machine M_i and M_{i+1} $(i = 1, \ldots, m-1)$ so that $\sum_{i=1}^{m-1} B_i = n$. The MORS problem is then to allocate the n available buffer slots into the $m-1$ buffer spaces so that the multiple objectives are optimised. Two objectives are taken into account in this case study: the throughput and the average work-in-process (WIP). The throughput rate (T_R) is defined as the average number of products completed per

day, therefore calculated by

$$T_R = \frac{Total\ number\ of\ products\ completed\ during\ the\ simulation\ days}{Number\ of\ simulation\ days}.$$

Work-in-process represents partly finished products in the system. The existence of WIP is not recommended in production management for various reasons. For example, WIP not only incurs inventory cost but also carries the inherent risk of being damaged or causing hazard in the work place. Therefore minimising the average WIP, denoted by W_P, serves as the second objective in this case study.

The simulation model was built with $m = 5$ machines and $n = 6$ total available buffer slots. Other parameters used in the model are given in Table 5.1. The simulation runtime was set to 100 days.

Table 5.1: Parameters used in the BAP simulation model

Machine	1	2	3	4	5
Mean process time $1/\mu_i$ (min.)	60	55	50	46	43
Failure rate β_i	100	100	100	100	100
Mean repair time $1/r_i$ (min.)	120	120	120	120	120

5.1.2 Experimental setup: Buffer allocation problem

In this experiment, 10 different buffer allocations were used. These values are shown in Table 5.2. Table 5.3 shows the sample means of the two performance measures (the throughput rate and the average WIP) over 10 000 simulation runs, rounded to two decimal places. These values are considered as 'accurately estimated true means' due to the large sample size, hence treated in the discussion as true means and used to identify true Pareto optimal solution sets. One can see that, for example, when the buffer slots are allocated as $(B_1, B_2, B_3, B_4) = (1, 1, 1, 3)$ (system 1), an average of 16.42 units is completed each day in the production line, while having an average of 1.65 units of WIP all the time. Figures 5.2 to 5.4 show the same information graphically. For the IZ values, $\delta_1^* = 0.2$ (units) and $\delta_2^* = 0.12$ (units) were used.

Figure 5.2 shows that all solutions except for systems 5 and 8 are members of the exact Pareto optimal set without IZ, that is, $Q = \{1, 2, 3, 4, 6, 7, 9, 10\}$. However, when the IZ value $\boldsymbol{\delta}^* = [0.2, 0.12]$ is taken into account, the dominance relationship among

Table 5.2: Feasible solutions of the BAP

System	1	2	3	4	5
(B_1, B_2, B_3, B_4)	(1,1,1,3)	(1,1,2,2)	(1,2,1,2)	(1,2,2,1)	(2,1,1,2)
System	6	7	8	9	10
(B_1, B_2, B_3, B_4)	(2,1,2,1)	(2,2,1,1)	(3,1,1,1)	(1,3,1,1)	(1,1,3,1)

Table 5.3: Estimated true means in the BAP

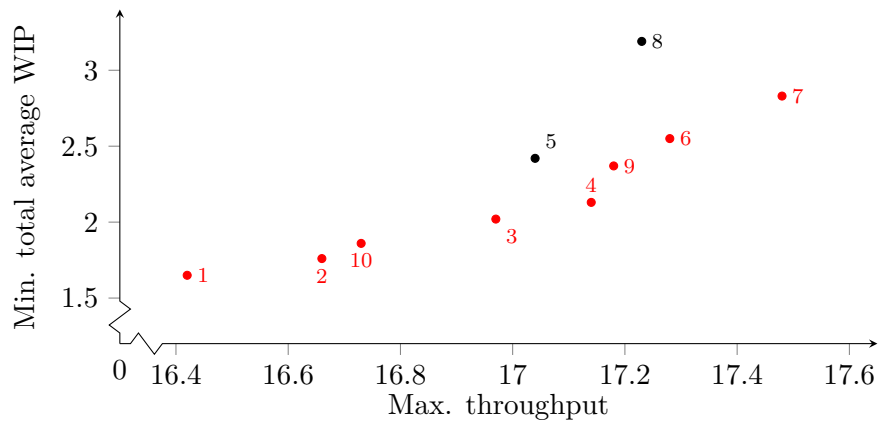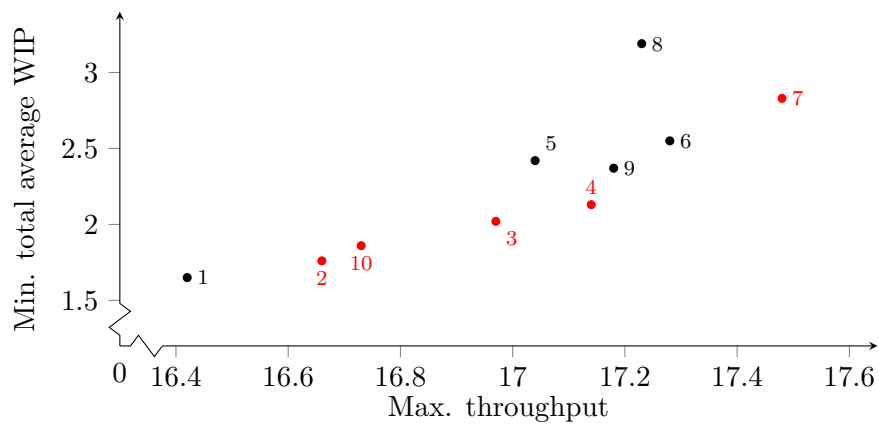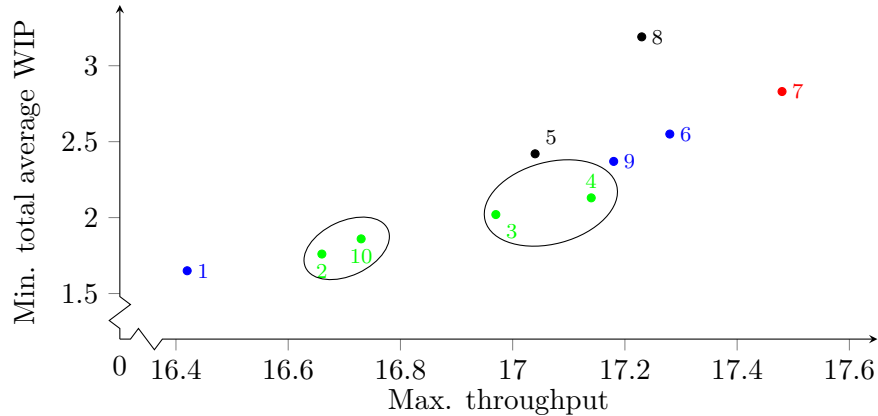| System	1	2	3	4	5
(T_R, W_P)	(16.42, 1.65)	(16.66, 1.76)	(16.97, 2.02)	(17.14, 2.13)	(17.04, 2.42)
System	6	7	8	9	10
(T_R, W_P)	(17.28, 2.55)	(17.48, 2.83)	(17.23, 3.19)	(17.18, 2.37)	(16.73, 1.86)

Figure 5.2: The true Pareto solution set Q: BAP

Figure 5.3: The true Pareto solution set with IZ Q_{IZ}: BAP

Figure 5.4: The true relaxed Pareto solution set Q_R: BAP

the solutions is changed: system 1 is dominated by system 2; system 4 dominates both systems 6 and 9. Therefore systems 1, 6 and 9 are excluded from Q_{IZ}, resulting in $Q_{IZ} = \{2, 3, 4, 7, 10\}$.

Figure 5.4 shows members of possible relaxed Pareto sets using the legends described in Section 4.1.5, *i.e.*, red for the solutions that must be included in Q_R, blue for the solutions that may or may not be included in Q_R; and green in a circle for each group of indifferent solutions of which at least one solution should be included in Q_R. The same legend will be used in all following figures in this dissertation as necessary. Note that there are two groups of systems that are indifferent to each other in this problem: systems 2 and 10, and systems 3 and 4. The relaxed Pareto set Q_R should contain system 7, and at least one solution from $Q_{S,1} = \{2, 10\}$ and from $Q_{S,2} = \{3, 4\}$. It may or may not contain systems 1, 6 and 9. Therefore there exist 72 possible relaxed Pareto solution sets (three options to choose from $Q_{S,1}$ (system 2 or 10 or both) × three options to choose from $Q_{S,2}$ (system 3 or 4 or both) × two options for system 1 (whether to include system 1 or not) × two options for system 6 (whether to include system 6 or not) × two options for system 9 (whether to include system 9 or not) = 72). They are listed in Table 5.4 as an example for this first case study. Note that $Q = Q_{R,72}$ and $Q_{IZ} = Q_{R,9}$ are included in these possible relaxed Pareto sets.

Procedures $\mathcal{MMY}$, $\mathcal{MMY}1$ and $\mathcal{MMY}2$ were then applied to solve this buffer allocation problem. They took the observations from the result of 10 000 simulation runs

Table 5.4: Possible relaxed Pareto solution sets: BAP

$Q_{R,1} = \{2,3,7\}$	$Q_{R,2} = \{2,4,7\}$	$Q_{R,3} = \{3,7,10\}$
$Q_{R,4} = \{4,7,10\}$	$Q_{R,5} = \{2,3,7,10\}$	$Q_{R,6} = \{2,4,7,10\}$
$Q_{R,7} = \{2,3,4,7\}$	$Q_{R,8} = \{3,4,7,10\}$	$Q_{R,9} = \{2,3,4,7,10\}$
$Q_{R,10} = \{1,2,3,7\}$	$Q_{R,11} = \{1,2,4,7\}$	$Q_{R,12} = \{1,3,7,10\}$
$Q_{R,13} = \{1,4,7,10\}$	$Q_{R,14} = \{1,2,3,7,10\}$	$Q_{R,15} = \{1,2,4,7,10\}$
$Q_{R,16} = \{1,2,3,4,7\}$	$Q_{R,17} = \{1,3,4,7,10\}$	$Q_{R,18} = \{1,2,3,4,7,10\}$
$Q_{R,19} = \{2,3,6,7\}$	$Q_{R,20} = \{2,4,6,7\}$	$Q_{R,21} = \{3,6,7,10\}$
$Q_{R,22} = \{4,6,7,10\}$	$Q_{R,23} = \{2,3,6,7,10\}$	$Q_{R,24} = \{2,4,6,7,10\}$
$Q_{R,25} = \{2,3,4,6,7\}$	$Q_{R,26} = \{3,4,6,7,10\}$	$Q_{R,27} = \{2,3,4,6,7,10\}$
$Q_{R,28} = \{2,3,7,9\}$	$Q_{R,29} = \{2,4,7,9\}$	$Q_{R,30} = \{3,7,9,10\}$
$Q_{R,31} = \{4,7,9,10\}$	$Q_{R,32} = \{2,3,7,9,10\}$	$Q_{R,33} = \{2,4,7,9,10\}$
$Q_{R,34} = \{2,3,4,7,9\}$	$Q_{R,35} = \{3,4,7,9,10\}$	$Q_{R,36} = \{2,3,4,7,9,10\}$
$Q_{R,37} = \{1,2,3,6,7\}$	$Q_{R,38} = \{1,2,4,6,7\}$	$Q_{R,39} = \{1,3,6,7,10\}$
$Q_{R,40} = \{1,4,6,7,10\}$	$Q_{R,41} = \{1,2,3,6,7,10\}$	$Q_{R,42} = \{1,2,4,6,7,10\}$
$Q_{R,43} = \{1,2,3,4,6,7\}$	$Q_{R,44} = \{1,3,4,6,7,10\}$	$Q_{R,45} = \{1,2,3,4,6,7,10\}$
$Q_{R,46} = \{1,2,3,7,9\}$	$Q_{R,47} = \{1,2,4,7,9\}$	$Q_{R,48} = \{1,3,7,9,10\}$
$Q_{R,49} = \{1,4,7,9,10\}$	$Q_{R,50} = \{1,2,3,7,9,10\}$	$Q_{R,51} = \{1,2,4,7,9,10\}$
$Q_{R,52} = \{1,2,3,4,7,9\}$	$Q_{R,53} = \{1,3,4,7,9,10\}$	$Q_{R,54} = \{1,2,3,4,7,9,10\}$
$Q_{R,55} = \{2,3,6,7,9\}$	$Q_{R,56} = \{2,4,6,7,9\}$	$Q_{R,57} = \{3,6,7,9,10\}$
$Q_{R,58} = \{4,6,7,9,10\}$	$Q_{R,59} = \{2,3,6,7,9,10\}$	$Q_{R,60} = \{2,4,6,7,9,10\}$
$Q_{R,61} = \{2,3,4,6,7,9\}$	$Q_{R,62} = \{3,4,6,7,9,10\}$	$Q_{R,63} = \{2,3,4,6,7,9,10\}$
$Q_{R,64} = \{1,2,3,6,7,9\}$	$Q_{R,65} = \{1,2,4,6,7,9\}$	$Q_{R,66} = \{1,3,6,7,9,10\}$
$Q_{R,67} = \{1,4,6,7,9,10\}$	$Q_{R,68} = \{1,2,3,6,7,9,10\}$	$Q_{R,69} = \{1,2,4,6,7,9,10\}$
$Q_{R,70} = \{1,2,3,4,6,7,9\}$	$Q_{R,71} = \{1,3,4,6,7,9,10\}$	$Q_{R,72} = \{1,2,3,4,6,7,9,10\}$

in a random order, whenever they needed to take observations (in Steps 2 and 10 in Algorithms 3 to 5). As mentioned earlier, the procedures were applied repeatedly and independently 1 000 times to obtain the proportion of correct selections, denoted by $\hat{P}(\text{CS})$. Then

$$\hat{P}(\text{CS}) \geq P^* \tag{5.1}$$

will be shown for Procedure $\mathcal{MMY}$ counting any of the 72 relaxed Pareto solution sets as correct selections. Similarly (5.1) will be shown for Procedures $\mathcal{MMY}1$ and $\mathcal{MMY}2$, taking only Q and Q_{IZ}, respectively, as correct selections. The probability of correct selection requirement was always set to $P^* = 0.9$, and the initial number of simulation replications was set to $n_0 = 2$ in all cases unless otherwise stated.

The MOCBA and the MOCBA_IZ procedures were also applied in the case study. As done in Section 4.5, the simulation budget for the MOCBA family procedures were set to the average total number of simulation replications, denoted by $\overline{N}_{total}$, obtained by the 1 000 applications of the $\mathcal{MMY}$ family procedures. The average total number of simulation replications is defined as

$$\overline{N}_{total} = \frac{1}{1\,000} \sum_{R=1}^{1\,000} \sum_{i=1}^{M} N_{i,R}, \tag{5.2}$$

where $N_{i,R}$ denotes the number of simulation replications assigned to system i in the Rth run of the procedure.

As discussed in Section 4.6, Procedures $\mathcal{MMY}1$ and $\mathcal{MMY}2$ could require an infinite number of simulation replications ($N_i \to \infty$ and $N_j \to \infty$) if $|\mu_{ik} - \mu_{jk}| \to 0$ (Procedure $\mathcal{MMY}1$) or $|\mu_{ik} - \mu_{jk}| \to \delta_k^*$ (Procedure $\mathcal{MMY}2$). In this experiment, it is considered that such a case occurs when a procedure requires $N_i > 10\,000$ for any system i. The procedure will then be terminated even though the termination criterion ($|I| = 0$) is not satisfied, and the result will be classified as 'not applicable' (NA).

The experimental settings described in this section apply in all simulation case studies in this dissertation unless otherwise stated. Thus they are not repeated in following sections.

5.1.3   Experimental results: Buffer allocation problem

This section discusses the results of the 1 000 applications of the $\mathcal{MMY}$ family procedures as well as the MOCBA family procedures to the buffer allocation problem.

Table 5.5 summarises the results, showing $\hat{P}(\text{CS})$ and $\overline{N}_{total}$ for each procedure. The first three rows present the results of Procedure $\mathcal{MMY}$ and the MOCBA family procedures using $B = 373$ (rounded, and obtained from the experiment of the $\mathcal{MMY}$ procedure) as the simulation budget. Note that the actual average number of total simulation replications $\overline{N}_{total}$ for the MOCBA family procedures differ slightly from the budget B due to other parameter settings. These first three rows of the result table will be called 'the first section' throughout the dissertation. The following three rows will be referred to as 'the second section', and they show the results from Procedure $\mathcal{MMY}1$ and the MOCBA family procedures with the simulation budget changed accordingly. Similarly the last three rows will be called 'the third' or 'the last section', which displays

Table 5.5: Experimental results: BAP

Procedure	$\overline{N}_{total}$	$\hat{P}(\text{CS})$		
		Q_R	$Q = Q_{R,72}$	$Q_{IZ} = Q_{R,9}$
$\mathcal{MMY}$	372.50	100.0%	98.5%	0.0%
MOCBA ($B = 373$)	378.99	99.7%	98.8%	0.0%
MOCBA_IZ ($B = 373$)	379.24	65.2%	0.0%	58.3%
$\mathcal{MMY}$1	2429.60	100%	100%	0.0%
MOCBA ($B = 2\,430$)	2435.40	100.0%	100.0%	0.0%
MOCBA_IZ ($B = 2\,430$)	2437.00	84.2%	0.0%	84.2%
$\mathcal{MMY}$2	NA	NA	NA	NA
MOCBA	NA	NA	NA	NA
MOCBA_IZ	NA	NA	NA	NA

the results from Procedure $\mathcal{MMY}$2 and the MOCBA family procedures with relevant simulation budget. Note also that because Procedures $\mathcal{MMY}$, $\mathcal{MMY}$1 and $\mathcal{MMY}$2 take different notions of correct selection, only one of Columns 3 to 5 is relevant to each procedure. Relevant columns are marked with grey background colour.

The first section in Table 5.5 shows that Procedure $\mathcal{MMY}$ succeeded in finding a relaxed Pareto set 100% with an average of total simulation replications $\overline{N}_{total} = 372.50$. Also, the procedure presented the exact Pareto optimal solution Q 98.5% of the 1 000 experiments. The MOCBA procedure also achieved 98.8% of $\hat{P}(\text{CS})$ when applied with the simulation budget $B = 373$. The second section also shows that Procedure $\mathcal{MMY}$1 and the MOCBA procedure work well to identify the exact Pareto optimal solution set Q though they spent more simulation replications. The MOCBA_IZ procedure, however, did not show good performance in both sections in Table 5.5: The estimated probability of correct selection became only $\hat{P}(\text{CS}) = 58.3\%$ when Q_{IZ} was counted as correct selection in the first section. It improved when more simulation budget ($B = 2\,430$) was assigned in the second section. The proportion of correct selection, however, was still low in this case ($\hat{P}(\text{CS}) = 84.2\%$).

An in-depth analysis revealed that most of the time when the MOCBA_IZ procedure presented a wrong result, the final solution set did not include system 7, which should be included in all three types of correct selections, *i.e.*, Q, Q_{IZ} or Q_R. This is due to the fact that the true mean of system 7 in the first objective is exactly $\delta_1^* = 0.2$ apart

from that of system 6, that is, $|\mu_{7,1} - \mu_{6,1}| = |17.48 - 17.28| = 0.2 = \delta_1^*$. Suppose it was observed slightly smaller than the true mean, *e.g.*, $\overline{X}_{7,1} = 17.46$. Suppose also, though very unlikely, that all other systems were observed exactly, *i.e.*, $\overline{X}_{i,k} = \mu_{i,k}$ ($i \in S$, $i \neq 7$, $k \in K$, $k \neq 1$), to make the problem tractable. The procedure would then classify system 7 as dominated (by system 6) with IZ ($6 \hat{\prec}_{IZ} 7$), for these values ($\overline{\boldsymbol{X}}_6 = [17.28, 2.55]$ and $\overline{\boldsymbol{X}}_7 = [17.46, 2.83]$) satisfy conditions given in (4.8)[1]. This would eventually lead the procedure to exclude system 7 from the final solution set as the MOCBA_IZ procedure constructs the Pareto set based on the IZ concept (Teng *et al.*, 2010), *i.e.*, it uses S_{IZ} instead of S_p. The same supposition on the observed value, however, does not make a difference in the other two procedures (Procedure $\mathcal{MMY}$ and the MOCBA procedure), which use S_p, not S_{IZ}. System 7 remains non-dominated in these procedures even if the observation was $\overline{\boldsymbol{X}}_6 = [17.28, 2.55]$ and $\overline{\boldsymbol{X}}_7 = [17.46, 2.83]$.

This explains also why Procedure $\mathcal{MMY}2$ is not applicable in this problem, as seen in the last section of Table 5.5. If the sample mean of system 7 for the first objective is observed slightly smaller than the true mean and all other systems were observed exactly, as supposed in the previous paragraph, system 7 is observed as dominated with IZ by system 6, therefore, for the procedure to stop, conditions (4.65) in Algorithm 5 must be satisfied. This requires (4.66) to be met for objective 1 in this case, and the denominator of the right-hand side of (4.66) has a value very close to zero when systems 6 and 7 are considered for systems i and j, respectively. On the other hand, if the sample mean of system 7 for the first objective is observed slightly larger than the true mean, *e.g.*, $\overline{X}_{7,1} = 17.50$, and again if all other systems were observed exactly, then system 7 is observed as non-dominated, thus conditions (4.64) in Algorithm 5 must be met. In this case, system 7 plays the role of system i in (4.64), and if system 6 is chosen for system j, objective 1 becomes k' in (4.64), and again the denominators in (4.64) approach zero. Therefore, in both cases the procedure requires N_6 and N_7 to be extremely large to guarantee what it observed regarding system 7 to be true, and eventually to identify Q_{IZ} with the probability of at least P^*. Therefore Procedure $\mathcal{MMY}2$ is not practically applicable for this particular problem, because the true means of systems 6 and 7 in

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[1]Note that the conditions in (4.8) assume a minimisation problem for all objectives, while the buffer allocation problem in this experiment requires the first objective to be maximised. Therefore the first element of the vectors $\overline{\boldsymbol{X}}_6$ and $\overline{\boldsymbol{X}}_7$ should be multiplied by -1 before the conditions (4.8) are applied. The same rule applied to all objectives that are to be maximised in the discussions in this chapter.

the first objective are exactly δ_1^* apart from each other, or rather the indifference-zone value was set such that $|\mu_{7,1} - \mu_{6,1}| = \delta_1^*$ unintentionally.

5.1.4   Conclusion: Buffer allocation problem

This section summarises the discussions in Section 5.1, where the buffer allocation problem was presented as a dynamic, stochastic simulation optimisation problem to verify the $\mathcal{MMY}$ family procedures. The details of the simulation model were discussed first in Section 5.1.1, followed by Section 5.1.2, where the result of 10 000 simulation replications of the BAP model was presented. This result was used to define the true Pareto solution sets Q, Q_{IZ} and Q_R, which then were used to verify the performance of the proposed $\mathcal{MMY}$ family procedures. The result in Section 5.1.3 showed that Procedures $\mathcal{MMY}$ and $\mathcal{MMY}1$ successfully identified Q_R and Q, respectively, with $\hat{P}(\text{CS}) = 100\%$. Procedure $\mathcal{MMY}2$, however, could not find the exact Pareto solution set with IZ (Q_{IZ}), due to the fact that the indifference-zone value for objective 1 was accidentally determined to be exactly the same as the difference of true means of two systems for that objective. Therefore Procedure $\mathcal{MMY}2$ was not applicable for this particular setting of the problem. It is applicable, of course, if the IZ value is changed, as will be seen in the following simulation case studies. However, deciding the IZ value to avoid such a situation is not an easy task because one does not know the true means beforehand.

This concludes the discussion of the buffer allocation problem. The following section introduces the second simulation case study.

5.2   Case study 2: Inventory problem

In this section, a single-commodity inventory problem is discussed. A typical inventory problem investigates alternative ordering policies, *i.e.*, reorder point (s) and reorder quantity (S), for an inventory system when customer demands and the lead time of an order are determined randomly. For a detailed description of the problem, see Bashyam & Fu (1998).

5.2.1   Simulation model: Inventory problem

In this section the specific inventory model considered in this study is described. A single, discrete commodity is sold to customers who arrive according to a Poisson process so that the interarrival times are exponentially distributed with a mean of 20 minutes. The customer demand X is a random, discrete variable, decided by a Weibull distribution with parameters $\alpha = 1$ and $\beta = 8$ and rounded up.

Every time the customer demands $X = x$ units of goods, the inventory level decreases by x if the current inventory level exceeds the customer demand; otherwise the customer is assumed to leave without purchasing any. The service level, one of the two objectives in this problem, is thus defined as the percentage of demand met:

$$S_L = \frac{Number\ of\ customers\ serviced}{Number\ of\ customers\ arrived}.$$

Once the inventory level drops below the reorder point, denoted by s, the management orders S units of the commodity. The delivery will take a random period of time from a triangular distribution with lower limit, mode and upper limit of 12, 14 and 20 hours, respectively. Figure 5.5 illustrates an example of the inventory level over time.

The other objective is the total cost (C_T) over n days, which consists of two factors: the inventory cost (C_I) and the delivery cost (C_D). The inventory cost involves the expenses incurred holding the stock, and is calculated as

$$C_I = \sum_{i=1}^{n} 1.1 \times\ the\ average\ number\ of\ units\ stocked\ in\ day\ i.$$

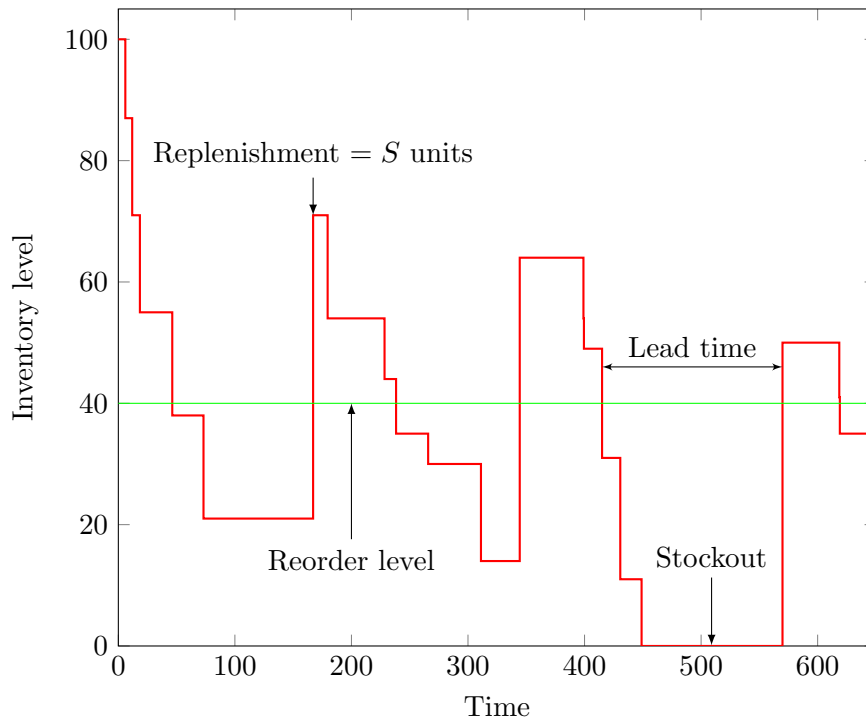Each delivery is assumed to cost ZAR100, therefore the delivery cost is

$$C_D = 100 \times d,$$

where d denotes the number of deliveries observed during the n days. The total cost is defined as the sum of these two costs:

$$C_T = C_I + C_D.$$

The simulation runtime was set to $n = 50$ days, and the initial inventory in the system at the beginning of the runtime was set to 200 units.

Figure 5.5: Some characteristics of the (s, S) inventory process (Bekker, 2012)

5.2.2 Experimental setup: Inventory problem

The experiment was performed considering the 18 ordering policies given in Table 5.6, and the aim was to find a Pareto optimal set to minimise the total cost over the 50 days and at the same time to maximise the service level.

Table 5.7 shows the estimated true means of the two objectives (the total cost C_T and the service level S_L) obtained by the results of 10 000 simulation runs. Figures 5.6 and 5.7 show the true Pareto optimal set without and with the indifference-zone con-

Table 5.6: Feasible solutions of the inventory problem

System	1	2	3	4	5	6
(s, S)	(200, 500)	(200, 600)	(200, 700)	(200, 800)	(200, 900)	(200, 1 000)
System	7	8	9	10	11	12
(s, S)	(300, 500)	(300, 600)	(300, 700)	(300, 800)	(300, 900)	(300, 1 000)
System	13	14	15	16	17	18
(s, S)	(400, 500)	(400, 600)	(400, 700)	(400, 800)	(400, 900)	(400, 1 000)

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Table 5.7: Estimated true means in the inventory problem

System	1	2	3	4
(C_T, S_L)	(14587.38, 77.48)	(16638.25, 80.33)	(18845.66, 82.53)	(21159.39, 84.28)

System	5	6	7	8
(C_T, S_L)	(23560.63, 85.71)	(26026.11, 86.90)	(17163.51, 87.65)	(19162.59, 89.38)

System	9	10	11	12
(C_T, S_L)	(21353.47, 90.68)	(23664.90, 91.68)	(26070.36, 92.49)	(28536.21, 93.14)

System	13	14	15	16
(C_T, S_L)	(21077.67, 95.63)	(22964.72, 96.25)	(25085.95, 96.70)	(27367.21, 97.04)

System	17	18		
(C_T, S_L)	(29738.86, 97.31)	(32186.81, 97.52)		
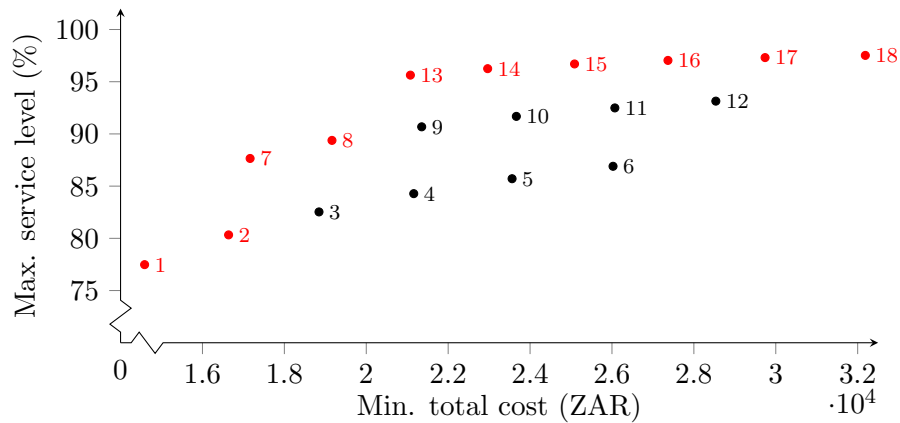
Figure 5.6: The true Pareto solution set Q: Inventory problem

cept, Q and Q_{IZ} respectively, with the Pareto optimal solutions marked in red. The indifference-zone value $\delta_1^* = 500$ (ZAR) was used for the first objective (total cost), and $\delta_2^* = 5$ (%) for the second objective (service level).

When the IZ concept was not considered, systems 1, 2, 7, 8, 13, 14, 15, 16, 17 and 18 are classified as non-dominated, *i.e.*, $Q = \{1, 2, 7, 8, 13, 14, 15, 16, 17, 18\}$. This is illustrated in Figure 5.6. One can also see from Figure 5.6 that increasing reorder quantity S while keeping the same reorder point s does not improve service level much, especially when $s = 400$ (systems 13 to 18). However, increasing reorder quantity by $S = 100$ would increase the cost more than ZAR 2 000 in almost all cases. Therefore, if one would like to keep the reorder point at $s = 400$, ordering a small quantity of
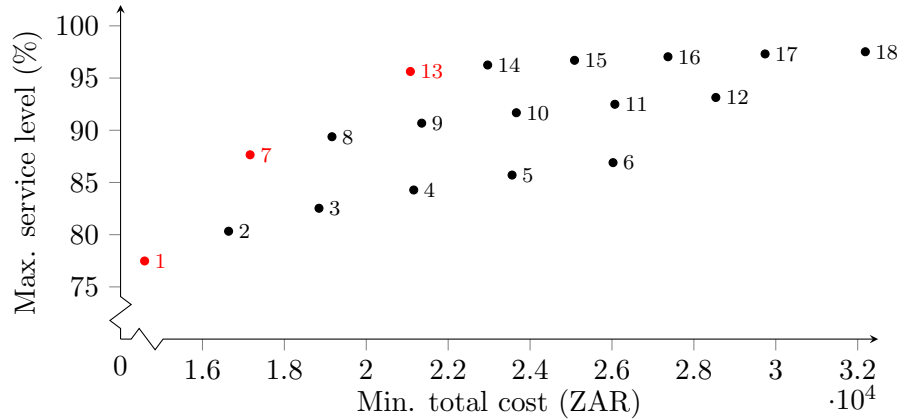
Figure 5.7: The true Pareto solution set with IZ Q_{IZ}: Inventory problem

goods more frequently is a smarter choice than ordering a large quantity and keeping the stock. On the other hand, keeping the reorder quantity at the same level, for example at $S = 500$, and increasing the reorder point by $s = 100$ raises the service level by almost 10% (systems 1, 7 and 13), though the total cost also increases. Therefore trade-offs exist in this case, which is the decision-maker's task to decide.

The same concept is manifested in Figure 5.7 when the IZ value $\boldsymbol{\delta^*} = [500, 5]$ is considered. In this case, system 2 is classified to be dominated by system 1 with IZ, *i.e.*, $1 \prec_{IZ} 2$, because the service level of these two systems does not show much difference ($|\mu_{1,2} - \mu_{2,2}| = |77.48 - 80.33| < \delta_2^* = 5$) while the total cost of system 1 is significantly smaller than the total cost of system 2. Similarly, system 8 is dominated by system 7 and systems 14 to 18 are all dominated by system 13. Therefore, the Pareto optimal set with IZ (Q_{IZ}) has only three members: systems 1, 7 and 13.

Figure 5.8 represents the relaxed Pareto solutions according to the same legend mentioned before. In this case, there are no two solutions that are indifferent to each other; systems 1, 7 and 13 must be included in Q_R; and any one of the seven systems marked in blue, *i.e.*, systems 2, 8, 14, 15, 16, 17 and 18, may or may not be included. Therefore, there exist $2^7 = 128$ relaxed Pareto sets in this case.

In the following section, the $\mathcal{MMY}$ family procedures were applied to the inventory problem to see if they could find correct Pareto sets with $\hat{P}(\text{CS}) \geq P^*$.
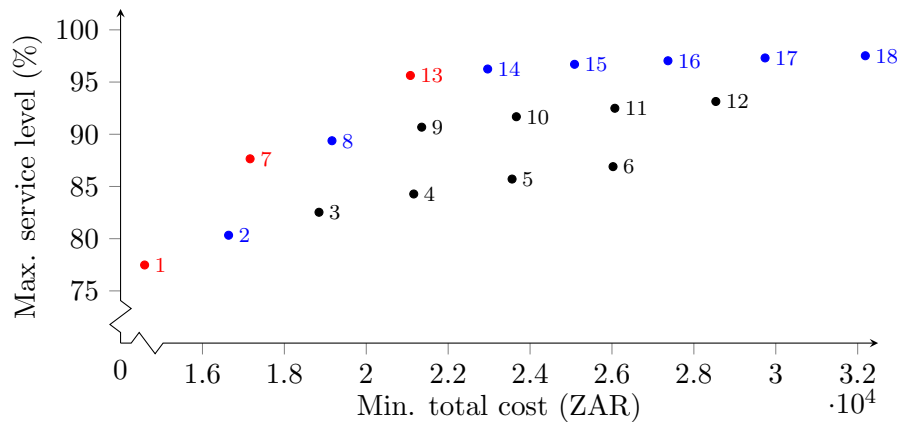
Figure 5.8: The true relaxed Pareto solution set Q_R: Inventory problem

5.2.3   Experimental results: Inventory problem

This section discusses the results of the experiments for the inventory problem. Table 5.8 presents the summary of the results.

The first section of Table 5.8 shows that Procedure $\mathcal{MMY}$ found relaxed Pareto solutions with $\hat{P}(\mathrm{CS}) = 100\%$ using an average of $\overline{N}_{total} = 143.77$ simulation replications, although the proportion of finding the exact Pareto optimal set Q was only 39.2%. This is not a problem with Procedure $\mathcal{MMY}$ as it is designed to find relaxed Pareto solutions. The MOCBA procedure, however, when applied with the same simulation budget $B = 144$ as Procedure $\mathcal{MMY}$, was not able to identify Q with an acceptable probability of correct selection. It showed even less performance ($\hat{P}(\mathrm{CS}) = 27.2\%$) than the $\mathcal{MMY}$ procedure, which was designed to identify Q_R, not Q. Furthermore, the probability of correct selection did not improve much when more simulation replications were used, as shown in the second and third section of Table 5.8. This is in sharp contrast to the results of the buffer allocation problem discussed in Section 5.1.3, where both Procedure $\mathcal{MMY}$ and the MOCBA procedure could identify Q with $\hat{P}(\mathrm{CS}) \geq 90\%$.

A further investigation showed that this was due to the fact that systems 13 to 18 have very close performance in the second objective, *i.e.*, the service level, while the dominance relationship in the first objective (total cost) is apparent. The true mean estimation in Table 5.7 shows that they are all non-dominated, however, it is very likely that any one of them (or more) is observed to be dominated due to the very similar performance in the second objective. Consider for example, systems 16 and 17.

Table 5.8: Experimental results: Inventory problem

Procedure	$\overline{N}_{total}$	$\hat{P}(\text{CS})$		
		Q_R	Q	Q_{IZ}
$\mathcal{MMY}$	143.77	100.0%	39.2%	0.0%
MOCBA ($B = 144$)	149.74	97.8%	27.2%	0.0%
MOCBA_IZ ($B = 144$)	152.18	99.5%	0.0%	99.3%
$\mathcal{MMY}1$	1 333.10	100.0%	100.0%	0.0%
MOCBA ($B = 1\,333$)	1 337.40	99.7%	51.6%	0.0%
MOCBA_IZ ($B = 1\,333$)	1 337.50	99.8%	0.0%	99.7%
$\mathcal{MMY}2$	211.56	100.0%	0.0%	100.0%
MOCBA ($B = 212$)	218.02	98.6%	35.1%	0.0%
MOCBA_IZ ($B = 212$)	220.05	99.8%	0.0%	99.8%

Suppose the sample mean of system 17 in objective 2 was observed slightly smaller than the true mean and all others were observed exactly, $i.e.$, $\overline{\boldsymbol{X}}_{\boldsymbol{16}} = [27367.21, 97.04]$ and $\overline{\boldsymbol{X}}_{\boldsymbol{17}} = [29738.86, 97.00]$. Then system 17 is observed to be dominated by system 16, while in truth it is non-dominated. When this happens, Procedure $\mathcal{MMY}$ tries to show that the observed dominated system (system 17) is truly dominated by showing the conditions (4.13) in Algorithm 3 are true. The conditions (4.13), however, have the term (4.10), which means the procedure endeavours to investigate the relationship of the two systems only to the extent that the difference in sample means in objective k becomes δ_k^*. Since in this case the difference in objective 2 (service level) is already within δ_k^*, the procedure uses $\delta_{16,17,2} = \delta_2^* = 5$ in conditions (4.13) instead of $\overline{X}_{17,2} - \overline{X}_{16,2} = -97.00 - (-97.04) = 0.04$, which leads the procedure to conclude with relatively small values of N_{16} and N_{17} that system 17 is indeed dominated by system 16. This explains the low $\hat{P}(\text{CS}) = 39.2\%$ of Procedure $\mathcal{MMY}$ in the first section of Table 5.8. Note that this is still a correct selection in terms of relaxed Pareto sets (Q_R), but it is not correct with regard to the exact Pareto set Q.

This case became even worse with the MOCBA procedure. As can be seen in the allocation rule given in (2.25) and (2.26), the MOCBA procedure inherently assigns more simulation replications to systems that are observed as non-dominated. When there exist two non-dominated systems whose performance is close to each other at least in one objective, the demands of these two systems overwhelm those of all other systems.
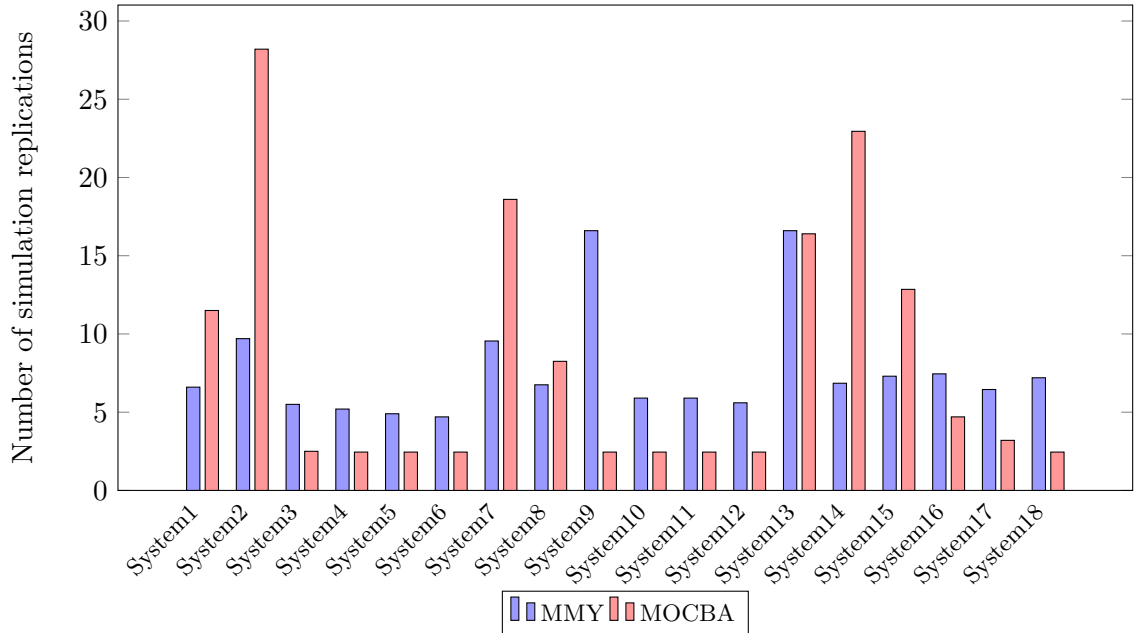
Figure 5.9: An example of simulation budget allocation: Inventory problem

This is what happened with the MOCBA procedure in the same situation given in the previous paragraph. When system 17 was wrongly observed to be dominated and all the other systems were observed correctly, instead of tending to system 17, the MOCBA procedure paid more attention to other systems, for example, systems 1 and 2; or systems 7 and 8; or systems 13 to 15. They are all observed non-dominated systems whose performances are close to each other in the second objective. Figure 5.9 illustrates this concept, which shows the simulation budget allocation among the 18 systems by Procedure $\mathcal{MMY}$ and the MOCBA procedure. To highlight the impact of the different behaviours of these two procedures, the information illustrated in Figure 5.9 was collected from an additional 20 experiments where the particular situation discussed in the previous paragraph occurred, *i.e.*, system 17 was observed to be dominated by system 16. Figure 5.9 clearly shows that the MOCBA procedure focused on the aforementioned non-dominated systems while neglecting system 17 after an average of $N_{17} = 3.2$ simulation replications. Procedure $\mathcal{MMY}$, on the other hand, assigned more simulation replications to system 17 than the MOCBA procedure, until it is assured of the genuineness of what it observed regarding system 17. In this case, $\overline{N}_{17} = 6.45$ was enough to conclude $16 \overset{\hat{}}{\succcurlyeq}_{IZ} 17$.
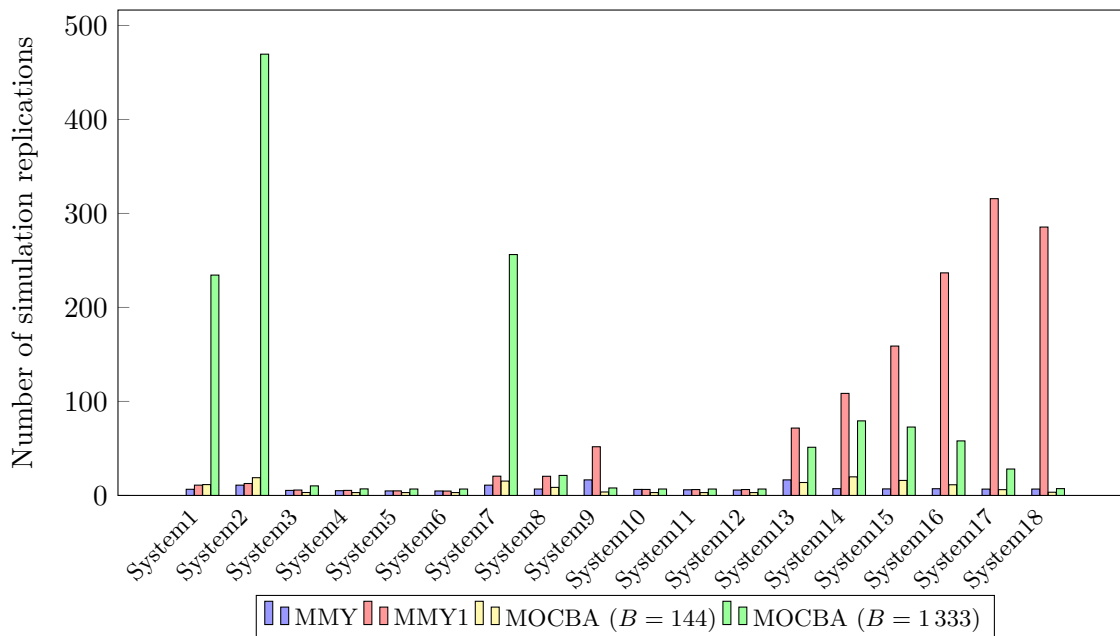
Figure 5.10: The average number of simulation replications assigned to each system by Procedures $\mathcal{MMY}$ and $\mathcal{MMY}1$, and the MOCBA procedure with simulation budget 144 and 1 333: Inventory problem

The second section of Table 5.8 shows that Procedure $\mathcal{MMY}1$ spent much more total simulation replications than Procedure $\mathcal{MMY}$, *i.e.*, $\overline{N}_{total} = 1\,333.10$, and as a result, it succeeded in identifying the exact Pareto set Q with 100% of the 1 000 experiments. However, the MOCBA procedure still showed only $\hat{P}(\text{CS}) = 51.6\%$, due to the behaviour discussed in the previous paragraph. Figure 5.10 shows how Procedures $\mathcal{MMY}$ and $\mathcal{MMY}1$ and the MOCBA procedure (with two different simulation budgets $B = 144$ and $B = 1\,333$) allocated simulation replications in the experiment of which the results are shown in the first and second sections in Table 5.8. As expected, Procedure $\mathcal{MMY}1$ spent most of the additional simulation replications in identifying the relationship between systems 13 to 18, which contributed for the procedure to correctly identify the exact Pareto solution set Q. On the other hand, one can see that the MOCBA procedure still focused on the aforementioned non-dominated systems when a larger simulation budget was given, therefore the probability of correct selection did not increase much even with the budget of $B = 1\,333$. This is a serious defect of the MOCBA procedure, which led to the development of the MOCBA_IZ procedure by

Teng *et al.* (2010).

The last section of Table 5.8 shows that Procedure $\mathcal{MMY}2$ could relatively easily find the exact Pareto solution set with IZ (Q_{IZ}). It achieved 100% of estimated probability of correct selection with a relatively small number of total simulation replications. As observed in the buffer allocation case, Procedure $\mathcal{MMY}2$ could suffer if the difference between the true means of two systems in an objective is close to the indifference-zone value for that objective, *i.e.*, $|\mu_{ik} - \mu_{jk}| \to \delta_k^*$. There was not such a complicated case in the inventory problem, however, therefore Procedure $\mathcal{MMY}2$ as well as the MOCBA_IZ procedure showed good performance.

5.2.4   Conclusion: Inventory problem

In this section, a classical (s, S) inventory problem was used to validate the $\mathcal{MMY}$ family procedures. Section 5.2.1 presented the details of the simulation model while Section 5.2.2 showed the result of 10 000 simulation replications of the model and identified Q, Q_{IZ} and Q_R. The results in Section 5.2.3 showed that the $\mathcal{MMY}$ family procedures worked well to find Q, Q_{IZ} and Q_R with 100% estimated probability of correct selection. However, the proportion of correct selection of Procedure $\mathcal{MMY}$, when Q was considered as correct selection, was remarkably low, and it became worse with the MOCBA procedure. This is not wrong in the case of the $\mathcal{MMY}$ procedure as it pursues to find Q_R, not Q. Yet curiosity made the researcher examine the problem further, which revealed that this was due to the close performances among systems 13 to 18 in the second objective. These close performances often led the procedure to construct an observed Pareto solution set S_p that is wrong in terms of Q, but still a correct selection with regard to Q_R. This eventually produced the result in the first row of Table 5.8. It was also discussed that the MOCBA procedure responded incorrectly in this situation by focusing too much on observed non-dominated systems whose performances are close to each other. It was mentioned that this is an example where the weakness of the MOCBA procedure manifests itself, which eventually led to the development of the MOCBA_IZ procedure.

The third simulation case study is presented in the following section.

5.3   Case study 3: Gold mine problem

In this case study, a gold mine problem from Kelton *et al.* (2010) is modified for an MORS problem. The management of a gold mine would like to increase the daily production of the mine and wants to investigate if changes on the transportation system would help it. Currently, the transportation system in the mine consists of a truck and a hoist. The truck delivers gold ore from the mining face to the bottom of the vertical shaft through the underground tunnel, the hoist then moves the ore up to the surface. Figure 5.11 shows a schematic picture of the mine. Operating more trucks would likely increase the production, however hiring more trucks would drive the cost up. The management has also a choice of increasing the speed of the hoist as a means to increase the daily production, though this also means additional operating cost. The management would particularly like to know the best combination of the number of trucks and the speed of the hoist to maximise the throughput and at the same time to minimise the operation cost.
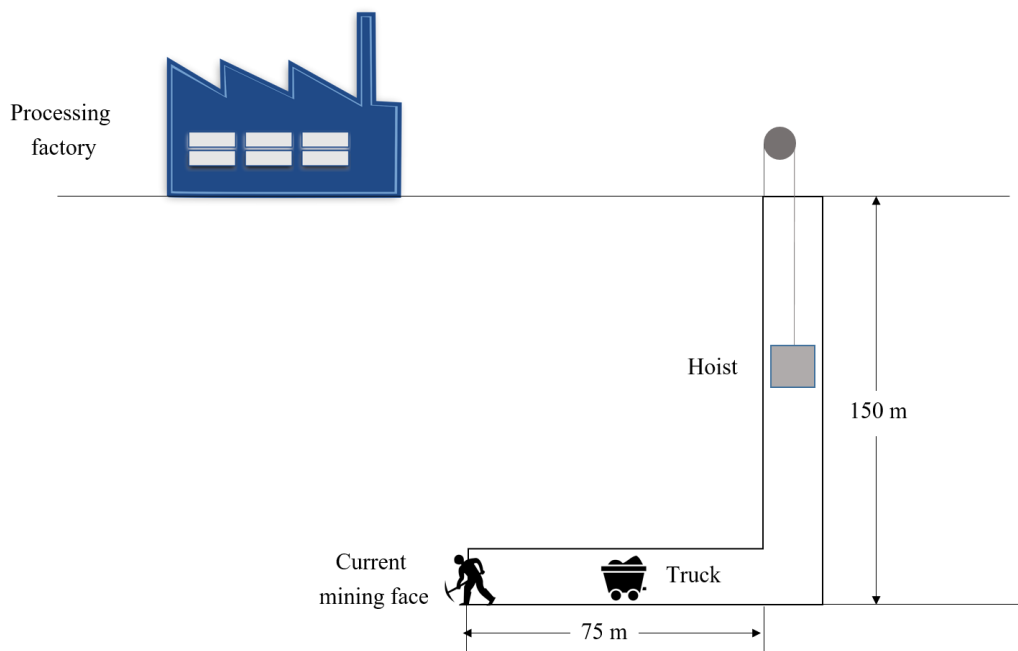

Figure 5.11: A schematic drawing of a mine system

5.3.1   Simulation model: Gold mine problem

In this section, the simulation model of the gold mine problem is described. As seen in Figure 5.11, the mining face is located 150 m below the surface and the underground tunnel is currently 75 m long. The underground space cannot accommodate more than four trucks. There are enough passing points in the tunnel where the trucks can pass each other, so interferences between trucks do not need to be taken into account (Kelton *et al.*, 2010). The time of drilling ore to fill a load of truck varies depending on the geographical conditions; it is a random variable determined by a triangular distribution with lower limit, mode and upper limit of 2, 7 and 12 (minutes), respectively. The ore is loaded as soon as an empty truck arrives at the mining face, and the loading time is exponentially distributed with a mean loading time of 6 minutes. Once loaded, the truck delivers the ore to the bottom of the mine shaft. The truck speed is initially set to 1.5 m/s, then decreases by 0.003 m/s after each delivery. This is to implement the fact that as more ore is mined, the travel distance for the trucks increases.

Once the truck arrives at the bottom of the vertical shaft, the ore is transferred from the truck to the hoist. However, the transfer does not occur if the hoist is not at the bottom of the vertical shaft. The transfer time from the truck to the hoist is exponentially distributed with a mean of 4 minutes. The hoist can accommodate one truckload each time. The truck moves back to the mining face as soon as it empties the load. Likewise, the hoist moves down to the bottom of the vertical shaft as soon as it unloads the ore on the surface. The hoist can be configured with three different speeds: 0.5, 1 and 1.5 m/s, though the management is currently keeping the speed at 0.5 m/s to minimise the operating cost.

The accounting department informed management that hiring a truck incurs a fixed cost of ZAR500/day and an operation cost of ZAR200/hour when the truck is in operation. The operation cost is ignorable when the truck is not in use, *i.e.*, when it is not moving or waiting for a load or the hoist. Also, a cost analysis indicated that the operation cost of a hoist is ZAR600/hour, ZAR800/hour or ZAR1 000/hour when the hoist speed is set to 0.5, 1 and 1.5 m/s, respectively. The cost is also applicable only when the hoist is in operation. Therefore, the total daily cost C_T is calculated as

follows:

$$C_T = 500 \times N_T \quad + \quad 200 \times N_T \times \ \textit{average truck utilisation} \tag{5.3}$$
$$+ \quad c_H \times \ \textit{average hoist utilisation} \ ,$$

where N_T denotes the number of trucks in operation, c_H denotes the coefficient representing the operation cost of a hoist per unit time ($c_H = 600$, 800 or 1000 according to the speed of the hoist). The simulation runtime was set to $n = 1$ day (24 hours). The objective was to find the best configurations of the number of trucks (N_T) and hoist speeds (S_H) to maximise the daily throughput as well as to minimise the total daily cost.

5.3.2 Experimental setup: Gold mine problem

Twelve different combinations of truck numbers and hoist speeds were considered in this experiment, which are listed in Table 5.9. Table 5.10 shows the estimated true means of the two objectives (the daily throughput T_R and the total cost C_T) obtained by 10 000 simulation runs. Figure 5.12 shows the true Pareto optimal solutions, which is $Q = \{9, 10, 11, 12\}$. This means that keeping the hoist speed at 1.5 m/s is always the best choice. In addition, it is observed that the higher hoist speed causes the lower total cost when the number of trucks is fixed. This is counter-intuitive as the operation cost of the hoist (c_H) increases as the speed rises. It turns out that with the higher speed the hoist is idle for longer time periods, which causes the decrease in average hoist utilisation. This is more than enough to compensate for the effect of the increased operation cost, which eventually leads to the decrease in the total cost. Increasing the number of trucks, however, brings out a trade-off situation: it contributes to produce more gold ore; however it incurs a higher cost.

Table 5.9: Feasible solutions of the gold mine problem

System	1	2	3	4	5	6
(N_T, S_H)	(1, 0.5)	(2, 0.5)	(3, 0.5)	(4, 0.5)	(1, 1)	(2, 1)

System	7	8	9	10	11	12
(N_T, S_H)	(3, 1)	(4, 1)	(1, 1.5)	(2, 1.5)	(3, 1.5)	(4, 1.5)

Table 5.10: Estimated true means in the gold mine problem

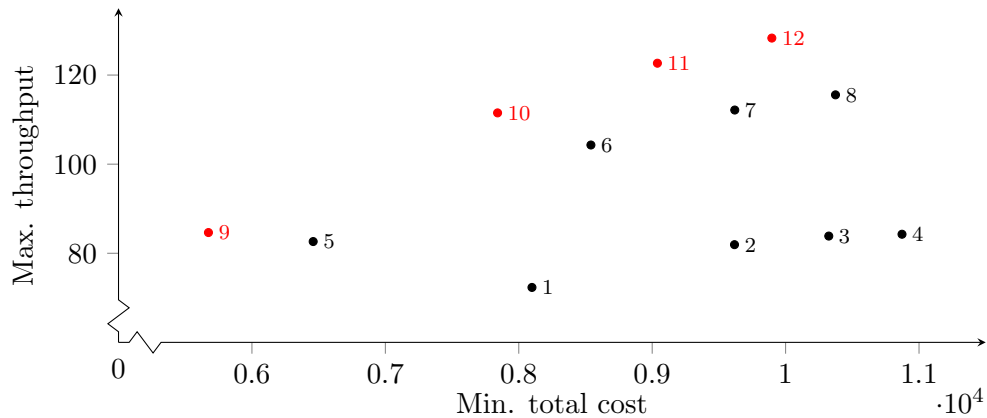| System	1	2	3	4
(C_T, T_R)	(8098.85, 72.33)	(9616.90, 81.93)	(10323.52, 83.86)	(10872.41,84.27)
System	5	6	7	8
(C_T, T_R)	(6459.02, 82.63)	(8540.69, 104.30)	(9618.88, 112.15)	(10373.92,115.56)
System	9	10	11	12
(C_T, T_R)	(5674.03, 84.65)	(7841.11, 111.51)	(9039.13, 122.66)	(9896.73, 128.30)

Figure 5.12: The true Pareto solution set Q: Gold mine problem

Figure 5.13 shows how the Pareto optimal solutions change if the IZ values $\delta_1^* = 1\,000$ (rands) and $\delta_2^* = 5$ (loads) are considered. Systems 5 and 9 are indifferent to each other, while system 11 is dominated by system 12. Based on the information given in Figures 5.12 and 5.13, Figure 5.14 illustrates relaxed Pareto solutions. The relaxed Pareto solution set must include systems 10 and 12; at least one of systems 5 and 9; and it may or may not include system 11. Therefore there exist six different relaxed Pareto solution sets, two of which are the exact Pareto sets without and with IZ: $Q = \{9, 10, 11, 12\}$ and $Q_{IZ} = \{5, 9, 10, 12\}$. These relaxed Pareto solution sets become a touchstone for judging 'correct selection' in the following section, where numerical experiments were performed to verify the $\mathcal{MMY}$ family procedures.

5.3.3   Experimental results: Gold mine problem

In this section the researcher discusses the results of the experiments for the gold mine problem. Table 5.11 shows the summary of the results. The first two sections of the
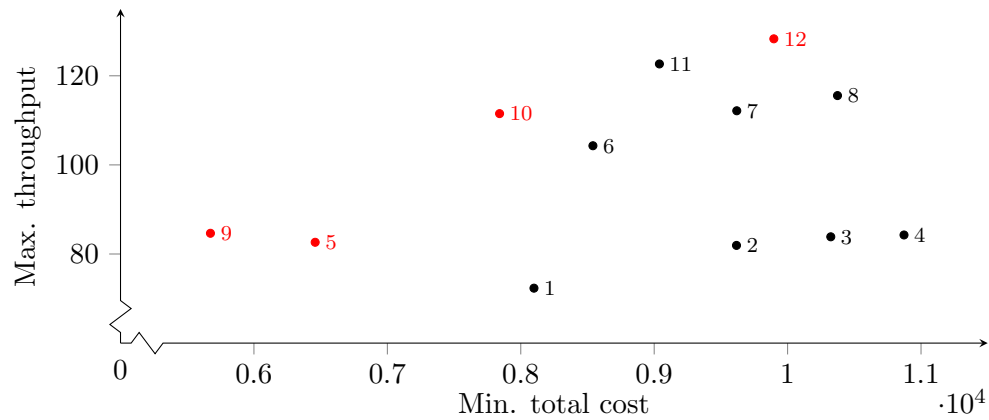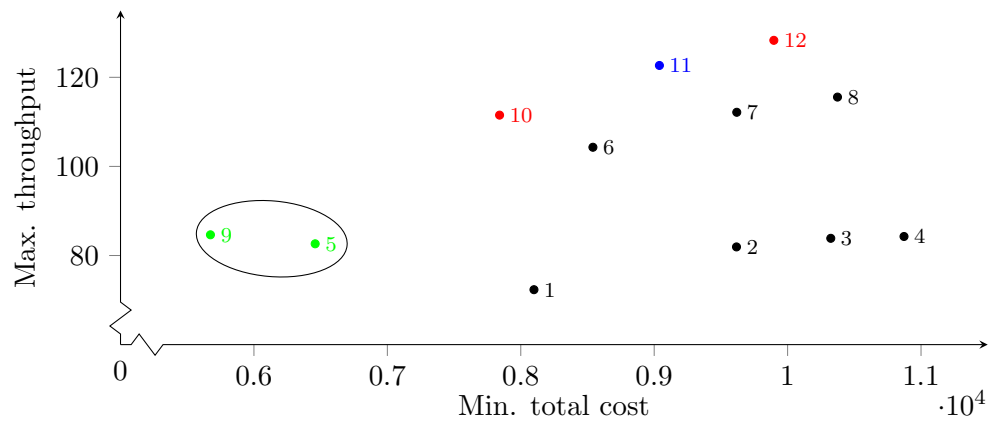
Figure 5.13: The true Pareto solution set with IZ Q_{IZ}: Gold mine problem

Figure 5.14: The true relaxed Pareto solution set Q_R: Gold mine problem

table indicate that Procedures $\mathcal{MMY}$ and $\mathcal{MMY}1$ competently solved the problem with $\hat{P}(\text{CS}) = 100\%$, using the average total simulation replications of $\overline{N}_{total} = 212.69$ and $\overline{N}_{total} = 530.22$, respectively. The MOCBA procedure also showed good performance: It identified the exact Pareto optimal set Q $\hat{P}(\text{CS}) = 99.4\%$ and 99.8% of the time with the simulation budget $B = 213$ and $B = 531$, respectively. The MOCBA_IZ procedure, however, struggled to identify Q_{IZ} with $\hat{P}(\text{CS}) \geq 90\%$ when the simulation budget was set to $B = 213$. It classified system 11 as non-dominated 12.6% of the time, which is wrong in terms of Q_{IZ}. This comes from a similar situation that was seen in the buffer allocation problem. The difference of true means of systems 11 and 12 in the second objective is close to the indifference-zone value for that objective, *i.e.*, $|\mu_{11,2} - \mu_{12,2}| = |122.66 - 128.30| = 5.64$, which is close to $\delta_2^* = 5$. However,

Table 5.11: Experimental results: Gold mine problem

Procedure	$\overline{N}_{total}$	$\hat{P}$(CS)		
		Q_R	Q	Q_{IZ}
$\mathcal{MMY}$	212.69	100.0%	100.0%	0.0%
MOCBA ($B = 213$)	218.55	99.8%	99.4%	0.0%
MOCBA_IZ ($B = 213$)	215.96	100.0%	0.0%	87.4%
$\mathcal{MMY}$1	530.22	100.0%	100%	0.0%
MOCBA ($B = 530$)	536.59	100.0%	99.8%	0.0%
MOCBA_IZ ($B = 530$)	535.34	99.9%	0.0%	97.2%
$\mathcal{MMY}$2	6 151.80	100.0%	0.0%	100.0%
MOCBA ($B = 6\,152$)	6 157.60	100.0%	99.8%	0.0%
MOCBA_IZ ($B = 6\,152$)	6 156.00	100.0%	0.0%	100.0%

the problem in this case study was not as serious as in the buffer allocation problem, where the difference was exactly the same as the indifference-zone value. Therefore Procedure $\mathcal{MMY}$2 could find Q_{IZ} with $\hat{P}$(CS) $= 100\%$ as shown in the last section of Table 5.11, although it required a large number of total simulation replications $\overline{N}_{total} = 6\,151.80$. The MOCBA_IZ procedure, too, succeeded in finding Q_{IZ} with $\hat{P}$(CS) $= 100\%$ when a similar amount of simulation budget $B = 6\,152$ was given.

5.3.4   Conclusion: Gold mine problem

In this section, a gold mine problem was considered as a simulation case study to apply the proposed $\mathcal{MMY}$ family procedures. The simulation model was described in detail in Section 5.3.1 while in Section 5.3.2 the results of 10 000 runs of the simulation model were presented to estimate true means of the two performance measures. Section 5.3.3 presented and discussed the results of the application of the $\mathcal{MMY}$ family procedures to the gold mine problem. Procedures $\mathcal{MMY}$, $\mathcal{MMY}$1 and the MOCBA procedure solved the problem with comparative ease, showing good performance while spending a relatively small amount of simulation budget. Identifying Q_{IZ} was more difficult because one of the IZ values was unknowingly set close to the difference in true means of two systems. However, the problem was not critical and both Procedure $\mathcal{MMY}$2 and the MOCBA_IZ procedure could solve the problem by spending more simulation budget.

This concludes the discussions of the gold mine problem. The next section advances to the last simulation case study: a trauma unit problem.

5.4   Case study 4: Trauma unit problem

In this section, a hospital management problem, designed and taught by the promoter of this research, is considered as the final simulation case study. The director of the Cure-You-Now hospital group, Dr I Fixu, is concerned about the workload in the trauma unit of the Bapalong branch. On Saturdays, starting at approximately 14:00, many casualties arrive, and the trend decreases about 24 hours later. To deal with this particularly high demand on the trauma personnel and equipment, the director considers increasing the staffing level as well as expanding facilities.

5.4.1   Simulation model: Trauma unit problem

In this section, the trauma unit problem is described in detail. Patients arrive randomly with an exponential distribution with a mean time of 5 minutes between arrivals, and they are screened at the reception according to severity of ailment. Although crude, the experienced staff at the reception does the screening quite effectively. Screening time is also exponentially distributed with a mean of 2 minutes.

The patients are classified into three categories: Category 1, 2 and 3. Patients with the most severe ailment are assigned to Category 1 while the least severe patients are classified to Category 3. Past records show that the proportion of patients assigned to each category is distributed according to Column 2 in Table 5.12. The table also shows the required number of doctors and nurses for each category, and the processing times. It is assumed that the doctors are homogeneous, *i.e.*, all doctors are the same in terms of performance. The nurses are also assumed to be identical.

Table 5.12: Parameters used in the trauma unit problem

Category	Percentage	Treating time (min)	Doctors	Nurses
1	25	Lognormal, $\mu = 28$, $\sigma = 12$	2	2
2	45	Lognormal, $\mu = 15$, $\sigma = 3$	1	2
3	30	Exponential, $\beta = 1/3$	1	1

There are three consulting rooms (CR1, CR2 and CR3), one for each category. Also, there is a rest room where doctors and nurses can rest when idle, and also scrub and prepare for the next patient. All personnel return to the sanitising section for a scrub in the rest room after a patient has been treated. Scrubbing takes a minute, with hardly any variation. When there is a need, the doctors and nurses who stayed in the rest room the longest are called on duty. The rest room is 6 m away from CR1, 14 m from CR2 and 24 m from CR3, and the doctors and nurses walk at an average speed of 1.67 m/s. Patients may be discharged after consultation, or assigned to a bed in the hospital. It is assumed that there are sufficient beds in the hospital, and that the scope of the problem is from patient arrival to end of treatment by the trauma personnel.

Currently 3 doctors and 4 nurses are working simultaneously within an 8-hour shift. The director would like to know the effect of the increase in the staffing level to the throughput, *i.e.*, the number of patients treated in the 24-hr period. In addition, she considers opening a fourth CR (CR4), which would share the workload of the slowest CR. This proposed new CR will also be 6 m from the rest room, on the opposite side of CR1. She wonders by how much the throughput would increase if an additional CR is built. Also, she would like to know to which category this new CR should be assigned, *i.e.*, Category 1 or Category 2. It is assumed that the consulting rooms cannot be used interchangeably for different categories due to the differing requirements of medical equipment. Category 1 patients have the longest average processing time but they are outnumbered by Category 2 patients.

Another important issue for the director to consider is the average waiting time of the patients classified as Category 1. While patients in other categories are assumed to be able to wait for consultation without causing fatal consequences, patients in Category 1 often need immediate treatment. Therefore, minimising the waiting time of Category 1 patients is considered as another objective in this problem.

5.4.2   Experimental setup: Trauma unit problem

Table 5.13 shows the 18 configurations used in the simulation case study. The variables N_D and N_N denote the number of doctors and nursing staff, respectively; $CR4$ represents whether the new CR should be built (True or False), and the variable CR indicates the category that the new CR should serve. Note that when the decision

Table 5.13: Feasible solutions of the trauma unit problem

System	1	2	3
$(N_D, N_N, CR4, CR)$	(3, 4, F, NA)	(3, 4, T, 1)	(3, 4, T, 2)
System	4	5	6
$(N_D, N_N, CR4, CR)$	(3, 5, F, NA)	(3, 5, T, 1)	(3, 5, T, 2)
System	7	8	9
$(N_D, N_N, CR4, CR)$	(4, 5, F, NA)	(4, 5, T, 1)	(4, 5, T, 2)
System	10	11	12
$(N_D, N_N, CR4, CR)$	(4, 6, F, NA)	(4, 6, T, 1)	(4, 6, T, 2)
System	13	14	15
$(N_D, N_N, CR4, CR)$	(5, 6, F, NA)	(5, 6, T, 1)	(5, 6, T, 2)
System	16	17	18
$(N_D, N_N, CR4, CR)$	(5, 7, F, NA)	(5, 7, T, 1)	(5, 7, T, 2)

variable $CR4$ is set to False, CR is set to 'Not available' accordingly. This problem differs from the first three simulation case studies because of this *categorical variables*.

Table 5.14 shows the estimated true means of the two performance measures, the average time spent in the queue of Category 1 patients (W_1) and the total number of patients treated by the trauma personnel for the 24-hr period (T_R), based on $10\,000$ simulation replications. Figures 5.15 and 5.16 show the Pareto optimal solutions based on the results given in Table 5.14. The IZ values of $\delta_1^* = 20$ (minutes) and $\delta_2^* = 5$ (patients) were used in Figure 5.16.

The result of this simulation case study reveals an interesting feature of this problem. Many pairs of configurations have very close performances in both objectives: systems 14 and 17; systems 8 and 11; systems 12, 15 and 18; systems 2 and 5; systems 1 and 4; systems 7, 10, 13 and 16. They are marked with green circles in Figure 5.15. This provides key information for the management of the hospital. For example, systems 8 and 11 have the same decision variable values except for the number of nurses, *i.e.*, 4 doctors, 5 or 6 nurses, and the new CR is built for Category 1 patients. The fact that the performance of these two systems is very close to each other means that assigning an additional nurse would not help in this case. Also, one can see that increasing the

Table 5.14: Estimated true means in the trauma unit problem

System	1	2	3	4
(W_1, T_R)	(227.87, 204.48)	(160.51, 202.06)	(332.19, 204.86)	(228.74, 204.01)

System	5	6	7	8
(W_1, T_R)	(161.18, 201.27)	(347.04, 217.76)	(194.83, 217.45)	(20.86, 197.73)

System	9	10	11	12
(W_1, T_R)	(289.05, 220.46)	(194.83, 217.45)	(20.86, 197.70)	(204.69, 230.14)

System	13	14	15	16
(W_1, T_R)	(194.82, 217.45)	(4.50, 223.13)	(205.00, 230.12)	(194.82, 217.45)

System	17	18		
(W_1, T_R)	(4.51, 222.97)	(191.74, 231.22)		

number of doctors from 4 to 5, with six nurses, would not make a difference if the new CR is not built or when it is built but assigned to Category 2 patients (compare systems 10 and 13; and systems 12 and 15). It improves the system, however, in terms of both objectives, if the new CR opens for Category 1 patients (see systems 11 and 14).

Figure 5.15 shows that the exact Pareto optimal solution set without IZ (Q) has two members, *i.e.*, $Q = \{14, 18\}$. Although the performances of systems 14 and 17 are extremely close to each other, system 14 still dominates system 17 when the IZ concept is not considered. However, when the IZ values $\delta_1^* = 20$ (minutes) and $\delta_2^* = 5$ (patients) are taken into account, the Pareto optimal solution set becomes $Q_{IZ} = \{12, 14, 15, 17, 18\}$, with two subsets containing indifferent systems, *i.e.*, $Q_{S,1} = \{14, 17\}$ and $Q_{S,2} = \{12, 15, 18\}$. Therefore the relaxed Pareto solution set can be any subset of $Q_U = \{12, 14, 15, 17, 18\}$ that contains at least one solution from $Q_{S,1}$ and $Q_{S,2}$ each. There are 21 such subsets, among which Q and Q_{IZ} are included. Figure 5.17 shows members of Q_R according to the legend described in Section 4.1.5.

These relaxed Pareto solution sets will be used as criteria for assessing the validity of the $\mathcal{MMY}$ family procedures in the following section, where the procedures are employed to solve the trauma unit problem.
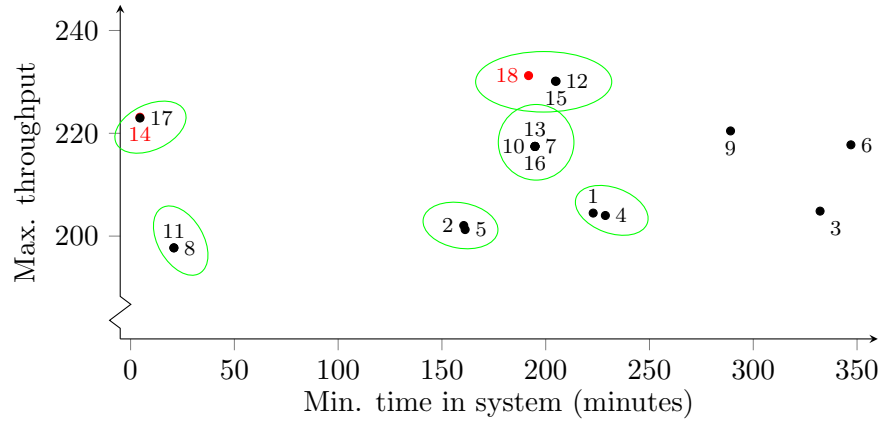
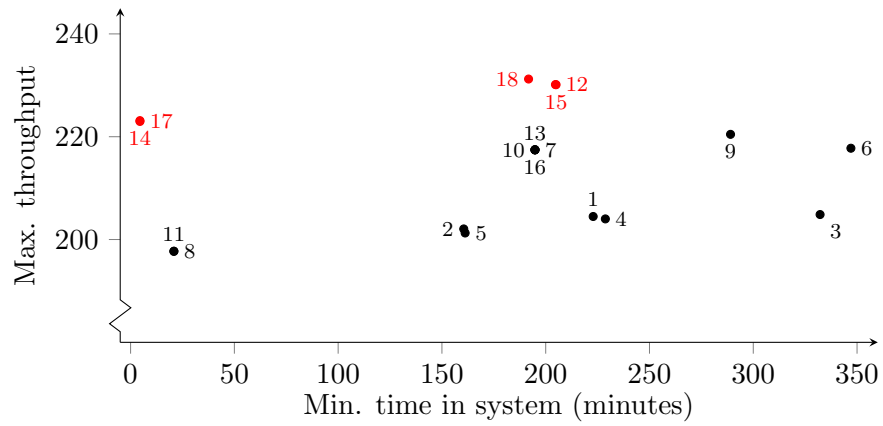Figure 5.15: The true Pareto solution set Q: Trauma unit problem

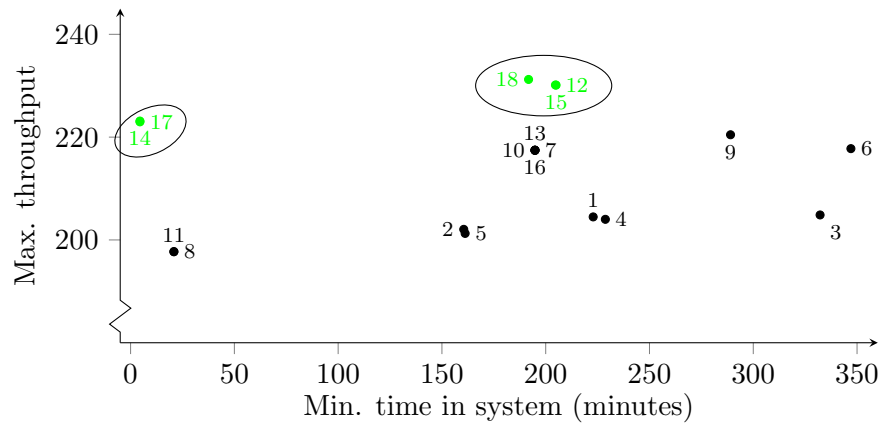Figure 5.16: The true Pareto solution set with IZ Q_{IZ}: Trauma unit problem

Figure 5.17: The true relaxed Pareto solution set Q_R: Trauma unit problem

5.4.3   Experimental results: Trauma unit problem

In this section, the results of $1\,000$ applications of the $\mathcal{MMY}$ family procedures to the trauma unit problem are discussed. The main results are shown in Table 5.15.

The first section of Table 5.15 shows that Procedure $\mathcal{MMY}$ needed an average of $\overline{N}_{total} = 1\,013.80$ total simulation replications to solve the trauma unit problem. This is a very large number, compared to the corresponding results from the previous three simulation case studies. This is due to the extremely close performances between non-dominated systems, *i.e.*, systems 14 and 17, and systems 12, 15 and 17. As mentioned in the previous section, there are other pairs of systems that have close performances, for example, systems 8 and 11; or systems 2 and 5. However, they did not affect the behaviour of the procedures, as they are obviously dominated systems. Systems 14 and 17; and systems 12, 15 and 18 are non-dominated systems that have at least one other system also in the non-dominated set that has very close performances to themselves. Figure 5.18 shows the importance of these systems in this case study. Procedure $\mathcal{MMY}$ spent 74.6% of the total simulation replications on these five systems. This proportion rose sharply to 96.4% in the MOCBA procedure. Especially the MOCBA procedure hardly spent more than an average of 3.9 simulation replications on other systems, except for these five systems.

Figure 5.18 also shows that while Procedure $\mathcal{MMY}$ focused more on systems 12, 15 and 18 than systems 14 and 17, the MOCBA procedures used more simulation replications on systems 14 and 17. The $\mathcal{MMY}$ procedure determines N_i for these five systems according to conditions (4.11) and since in this case the performances of systems 14 and 17; and systems 12, 15 and 18 are extremely close to each other, the procedure used $\delta_{ijk} = \delta_k^*$ in conditions (4.11) for all five systems. Therefore, the number of simulation replications assigned to each system N_i is determined in this case by sample standard deviations $S_{ik}(N_i)$. Because the estimated true standard deviations (obtained from the $10\,000$ simulation replications) of systems 12, 15 and 18 were much larger than those of systems 14 and 17, Procedure $\mathcal{MMY}$ reasonably assigned more simulation replications to systems 12, 15 and 18. The estimated true standard deviations of these systems, for both objectives, are presented in Table 5.16. The effect of h_1 was ignorable compared to that of standard deviations in this case.

Table 5.15: Experimental results: Trauma unit problem

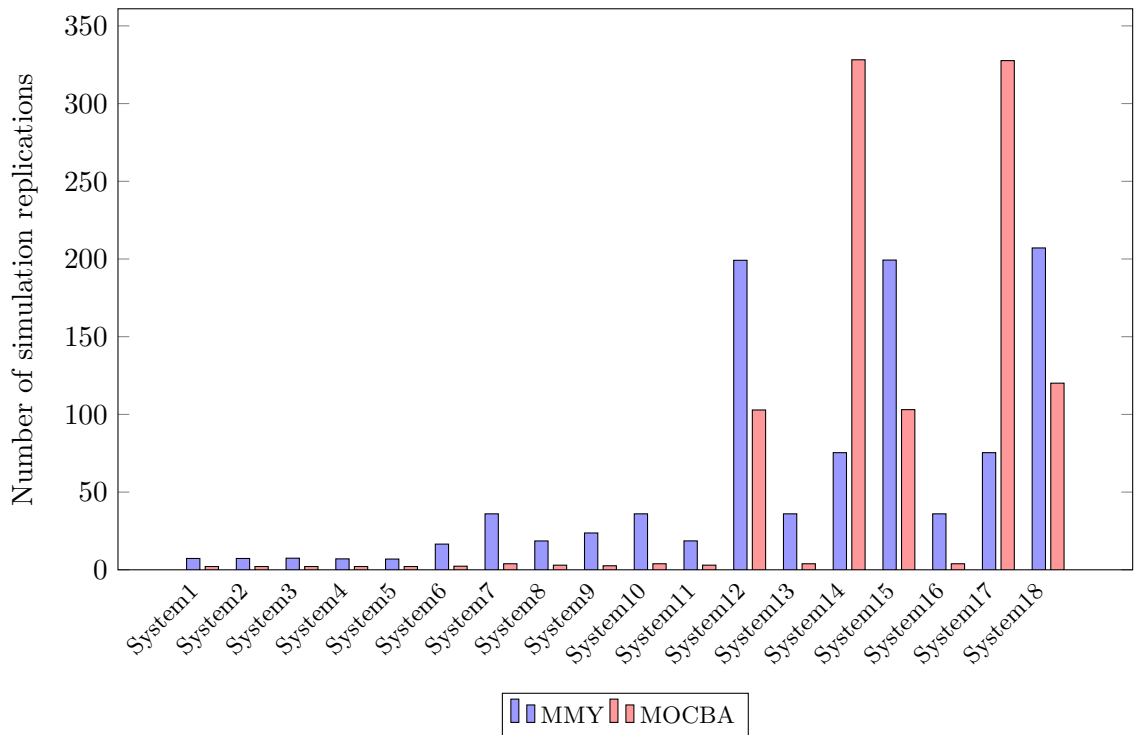| Procedure	$\overline{N}_{total}$	$\hat{P}$(CS)		
		Q_R	Q	Q_{IZ}
$\mathcal{MMY}$	1 013.80	100.0%	64.1%	0.0%
MOCBA ($B = 1\,014$)	1 018.90	99.8%	35.1%	1.1%
MOCBA_IZ ($B = 1\,014$)	1 025.20	100.0%	0.0 %	99.8%
$\mathcal{MMY}$1	NA	NA	NA	NA
MOCBA	NA	NA	NA	NA
MOCBA_IZ	NA	NA	NA	NA
$\mathcal{MMY}$2	6 284.00	100.0%	0.0%	100.0%
MOCBA ($B = 6\,284$)	6 289.40	99.7%	57.5%	0.3%
MOCBA_IZ ($B = 6\,284$)	6 298.10	100.0%	0.0%	100.0%

Figure 5.18: The average number of simulation replications assigned to each system by Procedure $\mathcal{MMY}$ and the MOCBA procedure ($B = 1\,014$)

Table 5.16: Estimated true standard deviations of certain systems in the trauma model

System	12	14	15	17	18
Estimated true standard deviation for W_1	54.99	3.03	55.03	3.05	53.91
Estimated true standard deviation for T_R	11.89	8.37	11.88	8.34	12.09

The MOCBA procedure, on the other hand, does not employ the IZ concept, hence uses the difference of sample means of two systems directly to calculate N_i, which led the procedure to determine $N_{14} \to \infty$ and $N_{17} \to \infty$ in this case. However, assigning an extremely large number of simulation replications to systems 14 and 17 did not help much for the MOCBA procedure to identify Q: It showed only 35.1% of estimated correct selection when the simulation budget was set to $B = 1\,014$, and 57.5% when $B = 6\,284$. In fact, identifying the exact Pareto set Q is impossible in this problem, where the true means of systems 14 and 17 are extremely close to each other. This is why Procedure $\mathcal{MMY}1$ was not applicable in this problem, as shown in the second section in Table 5.15.

It is also noticeable that the performance of the MOCBA procedure in the first section of Table 5.15 is much lower than the $\mathcal{MMY}$ procedure even though these two procedures spent almost the same amount of simulation budget. A further investigation showed that while the incorrect selection from Procedure $\mathcal{MMY}$ was $S_p = \{14, 17, 18\}$, *i.e.*, system 17 was included when it should not have been, the incorrect selection from the MOCBA procedure involved $S_p = \{12, 14, 18\}$ and $S_p = \{14, 15, 18\}$ in addition to $S_p = \{14, 17, 18\}$. This means in many cases the MOCBA procedure failed to distinguish systems 12 and 15 from system 18, because it paid more attention to the impossible task of distinguishing systems 14 and 17, as illustrated in Figure 5.18. This is another example where the MOCBA procedure does not work well.

5.4.4 Conclusion: Trauma unit problem

In this section, the trauma unit problem was used as the last simulation case study to validate the proposed $\mathcal{MMY}$ family procedures. This simulation model provided a unique result where there exist several pairs of systems with very close performances to each other. The result was very valuable not only from the hospital management point of view, but also in understanding the behaviour of the MORS procedures. In

particular, the problem revealed that when the performances of two non-dominated systems are very close to each other it is almost impossible to identify the exact Pareto solution set Q; however, it does not affect the behaviour of the procedures if those systems are observed to be dominated. Also, the categorical nature of some decision variables makes the problem difficult for the MOCBA procedure. It performed poorly in this extreme example. On the other hand, the $\mathcal{MMY}$ and $\mathcal{MMY}2$ procedures could find Q_R and Q_{IZ}, respectively, with $\hat{P}(\text{CS}) = 100\%$.

5.5 Conclusion: Chapter 5

In this chapter, the researcher employed four simulation case studies as subjects of experiments to verify the statistical validity of the $\mathcal{MMY}$ family procedures numerically. Those were a buffer allocation problem, a classical (s, S) inventory problem, a gold mine problem and a hospital management problem. These four simulation studies indeed provided various testing environments, which helped the researcher to understand at a deeper level the behaviours of the $\mathcal{MMY}$ family procedures. In summary, the $\mathcal{MMY}$ family procedures could identify relevant Pareto sets, *i.e.*, Q, Q_{IZ} and Q_R, in all four simulation case studies with the estimated probability of correct selection 100%, except when extreme situations happened, *i.e.*, $|\mu_{ik} - \mu_{jk}| \to 0$ or $|\mu_{ik} - \mu_{jk}| \to \delta_k^*$ for some $i, j \in S$ and $k \in K$. In these cases, Procedures $\mathcal{MMY}1$ or $\mathcal{MMY}2$ required $N_i \to \infty$, therefore were not applicable for the problem. This was predictable and discussed in Chapter 4 with the theoretical reasons of the phenomenon. On the other hand, the simulation case studies in this chapter demonstrated that Procedure $\mathcal{MMY}$ was always able to find relaxed Pareto sets with a reasonable total number of simulation replications, as the theory in Chapter 4 proved.

The next chapter concludes the dissertation.

Chapter 6

Conclusion

This chapter concludes the dissertation by presenting the summary of the work done in this research, the main contributions to the body of knowledge, and recommendations for future work.

6.1   Summary of the research

The aim of the research was to develop a *multi-objective ranking and selection* (MORS) procedure for stochastic systems with the *indifference-zone* (IZ) approach.

Multi-objective ranking and selection, as the name suggests, is the extension of a research field called *ranking and selection* (R&S) to the multi-objective optimisation domain. Ranking and selection procedures deal with *simulation optimisation* (SO) problems, where the performance of a system is obtained through simulation (not by a mathematical function) due to the complex, stochastic nature of the problem. Simulation usually involves what-if questions, which leads to analyses of different, multiple systems. The simulation analyst (or the decision-maker) often pursues finding the 'best' system among, say, k different systems. This forms the purpose of R&S procedures.

The 'best' system in this context means the system that yields the minimum or maximum output according to the objective of the problem. Because of the inherent stochastic nature of the problem, the output of a simulation run is only a realisation of a random variable, therefore not to be reliable unless it is the expected value (usually the sample mean) over a multiple number of simulation replications. The larger the number of simulation replications, the more reliable the simulation result becomes, therefore

there is a greater chance of selecting the best system correctly. This is often referred to as 'probability of correct selection', and denoted by $P(\mathrm{CS})$, in the R&S literature.

R&S procedures determine the number of simulation replications so that the balance between the number of simulation replications spent and the probability of correct selection obtained is controlled as the decision-maker wishes. In this regard, the decision-makers can choose from two strategies: they could require the procedure to determine the minimum number of simulation replications for each system to guarantee that the probability of correct selection is at least a prespecified value P^*, or the decision-maker could want to know how to allocate a limited simulation budget (the total number of simulation replications) among k systems so that the probability of correct selection is maximised. There are two types of R&S procedures: The *indifference-zone* (IZ) type of R&S procedures deal with the former requirement, *i.e.*, they identify the minimum number of simulation replications for each system to guarantee $P(\mathrm{CS}) \geq P^*$, while the *optimal computing budget allocation* (OCBA) methods respond to the latter, *i.e.*, they allocate a limited simulation budget among k systems so that $P(\mathrm{CS})$ is maximised.

It is worth mentioning that R&S procedures require the number of systems k to be relatively small, so that they can estimate all k systems with at least n_0 simulation replications each. If the solution space is too large for the procedure to estimate all feasible solutions, then a search mechanism must be employed to explore the vast solution space efficiently, which is beyond the scope of R&S procedures.

To fulfil the first research objective given in Section 1.3, an extensive literature study was conducted on the general topic of simulation optimisation (SO), both for the small- and large-sized SO problems; and both in the single- and multi-objective domains. Part of the results of the literature study, focusing on the multi-objective simulation optimisation domain, was developed into a manuscript and submitted for publication (Yoon & Bekker, 2017c). The literature was further studied intensively focusing on the small-sized SO problems, of which the result was published (Yoon & Bekker, 2017d).

The literature study revealed two important research opportunities. Firstly, it showed that most single-objective IZ procedures assumed the least favourable configuration (LFC), for example (3.3), in their proof of $P(\mathrm{CS}) \geq P^*$. This rendered the procedures conservative, which means they tend to spend more simulation replications than actually needed to guarantee $P(\mathrm{CS}) \geq P^*$. There have been many attempts

to develop more efficient IZ procedures by removing the LFC assumption, but none of them succeeded in developing such a procedure with a rigorous mathematical proof that guarantees the probability of correct selection requirement. This led the researcher to delve into the problem, which resulted in the development of Procedure $\mathcal{MY}$. Procedure $\mathcal{MY}$ is a single-objective R&S procedure based on Rinott's procedure (Rinott, 1978). It does not assume the LFC, therefore it is less conservative than any other R&S procedures that are based on Rinott's procedures. Moreover, the procedure proves the probability of correct selection guarantee using a solid mathematical proof based on a *Bayesian inference model*. Chapter 3 presented the procedure and its proof, along with the results of some numerical experiments. The researcher has also written a manuscript based on Chapter 3, which was submitted for publication (Yoon & Bekker, 2017b). In summary, Procedure $\mathcal{MY}$ is a single-objective R&S procedure with the IZ approach, and serves as an important basis for the main work done in this research in the MORS domain.

Secondly and more importantly, the literature study revealed that there does not yet exist an MORS procedure using the indifference-zone approach. In the single-objective R&S domain, both the IZ and the OCBA methods are well developed. The decision-maker could therefore choose freely from the two strategies mentioned earlier. In the MORS domain, however, the decision-maker does not have any other option but to choose the OCBA approach, because there is no MORS procedure with the IZ approach. This was a definite gap between the single- and multi-objective ranking and selection areas as shown in Figure 2.2. The researcher therefore set the research direction towards developing an MORS procedure with the IZ approach, which became the main work of this research.

Chapter 4 of the dissertation deals with the main work of the research. The work in this chapter fulfils the second, third and fourth research objectives given in Section 1.3. Chapter 4 first introduces the concept of Pareto optimality (Coello Coello, 2009) in Section 4.1.3, which is essential in any type of multi-objective optimisation research, followed by the concept of Pareto optimality when the indifference-zone value is taken into account (Teng *et al.*, 2010) in Section 4.1.4, which is necessary to accommodate the IZ concept in the multi-objective domain. These two concepts of Pareto optimality produce the exact Pareto optimal set without and with IZ, named Q and Q_{IZ}, respectively. The researcher soon realised, however, that these two types of Pareto optimal

sets are not enough to correctly integrate the IZ concept into the multi-objective domain. The main point of employing the IZ concept to Pareto optimality is to reflect the decision-maker's indifference to a small difference in the performances of two systems, that is, the IZ concept is employed to avoid $N_i \to \infty$ and $N_i \to \infty$ when $|\mu_{ik} - \mu_{jk}| \to 0$ in identifying the exact Pareto optimal set Q. One does avoid such cases when trying to identify Q_{IZ} instead of Q. However, there are other extreme cases, *i.e.*, when $|\mu_{ik} - \mu_{jk}| \to \delta_k^*$. The researcher therefore proposed the concept of *relaxed* Pareto optimality and relaxed Pareto sets (Q_R) in Section 4.1.5, which she believes correctly reflects the indifference-zone concept in the multi-objective domain.

Chapter 4 then proposes three multi-objective ranking and selection procedures, called Procedures $\mathcal{MMY}$, $\mathcal{MMY}1$ and $\mathcal{MMY}2$, each endeavours to find Q_R, Q and Q_{IZ}, respectively. Procedure $\mathcal{MMY}$, which finds the relaxed Pareto set Q_R, is the main procedure that serves the purpose of an IZ MORS procedure the best. In addition, for the decision-makers who would want to identify the exact Pareto optimal set Q and Q_{IZ}, Procedures $\mathcal{MMY}1$ and $\mathcal{MMY}2$ were also proposed. However, these two procedures sometimes do not work, because they require an infinite size of simulation replications if some extreme cases occur, *i.e.*, $|\mu_{ik} - \mu_{jk}| \to 0$ for Procedure $\mathcal{MMY}1$ and $|\mu_{ik} - \mu_{jk}| \to \delta_k^*$ for Procedure $\mathcal{MMY}2$. In pursuit of the third research objective, the statistical validity of all these three procedures was proved though rigorous mathematical analyses given in Sections 4.2 to 4.4. Chapter 4 also described some numerical experiments conducted to verify the proposed MORS procedures, of which the results demonstrated the effectiveness of them. This fulfils the fourth research objective.

The numerical experiments were extended to more realistic simulation case studies in Chapter 5, which also fulfilled the fourth research objective. The simulation case studies considered were the buffer allocation problem, the (s, S) inventory problem, the gold mine problem and the hospital management problem. These four simulation case studies provided various environments in terms of the unknown true mean distributions, including the extreme cases discussed in the previous paragraph. In all four simulation case studies, the proposed MORS procedures successfully found the relevant Pareto optimal sets Q, Q_{IZ} or Q_R, with the estimated probability of correct selection $\hat{P}(\text{CS}) = 100\%$, except for when the aforementioned extreme cases occurred, thereby demonstrating the effectiveness of the three procedures. These simulation case studies also revealed that the $\mathcal{MMY}$ family procedures generally outperform the MOCBA

family procedures in terms of the estimated probability of correct selection, especially when the extreme cases occurred.

The summary of the research in this section shows that the research aim and objectives presented in Section 1.3 were all achieved successfully. The next section presents the contribution of this research to the body of knowledge.

6.2 Contribution to the body of knowledge

The contribution of the work done in this research to the body of knowledge can be explained by the following three aspects:

1. The researcher developed **a multi-objective ranking and selection procedure and two of its variants with the indifference-zone approach**, filling the gap given in Figure 2.2. By doing this, she opened up an opportunity for the decision-makers to approach the MORS problem differently, *i.e.*, with the IZ method. Furthermore, she presented rigorous mathematical proofs that verify that the proposed MORS procedures guarantee $P(\mathrm{CS}) \geq P^*$, as expected of any R&S procedure with the IZ approach.

2. In the development process, the researcher also defined the concept of **relaxed Pareto optimality**, which relieved Procedure $\mathcal{MMY}$ from the danger of having to assign an infinite size of simulation replications in certain extreme cases. The concept of Pareto optimality with IZ (discussed in Section 4.1.4) and the MOCBA_IZ procedure were also proposed with the same purpose by Teng *et al.* (2010). However, the MOCBA_IZ procedure resulted in unexpectedly introducing another extreme case where the difference between two systems in an objective approaches the indifference-zone value of that objective, thus could not serve the purpose properly. In contrast to this, Procedure $\mathcal{MMY}$ avoids both kinds of extreme cases by using the proposed relaxed Pareto optimality, and yet produces equally good Pareto solutions based on the decision-maker's indifference-zone value.

3. In addition, the researcher developed **a single-objective ranking and selection procedure that does not employ the least favourable configuration**

assumption, yet proves its statistical validity through a solid mathematical analysis. This procedure not only showed better performance than existing IZ R&S procedures, but also became the theoretical foundation of the proposed MORS procedures that followed.

The following section provides suggestions for future research.

6.3   Recommended future work

Possible future research related to the work presented in this dissertation includes the following:

1. The proposed MORS procedures can be improved to be less conservative. The numerical experiments presented in this dissertation showed in almost all cases that the estimated probability of correct selection reached 100% while the required value was set to $P^* = 90\%$. This would entail identifying sources of such conservativeness as well as devising methods to eliminate them. A good starting point is to look at the Bonferroni inequalities used in the proposed MORS procedures.

2. The $\mathcal{MMY}$ family procedures were developed with the assumption that the individual observations X_{ikl} ($i \in S$, $k \in K$, and $l = 1, \ldots, N_i$) are independent of all other responses for each $i \in S$, $k \in K$ and $l = 1, \ldots, N_i$, as mentioned in Section 4.1.2. In many real-life problems, however, this is often not true. For example, the waiting time in a queue for a customer is definitely positively related to that of the previous customer, *i.e.*, the lth and the $(l + 1)$th observations can be related to each other. In this sense, research focused on developing an MORS procedure free from these assumptions would be a reasonable following step.

3. It was mentioned in Section 4.6 that the number of simulation replications identified by the $\mathcal{MMY}$ family procedures has the same structure as the one presented by the MOCBA procedure, *i.e.*, it is proportional to the sample variance $S_{ik}^2(N_i)$ and inversely proportional to the square of the difference in sample means of two systems. However, the MOCBA procedure applies this structure only to the observed dominated systems, and the number of simulation replications for

observed non-dominated systems are determined differently. It would be an interesting research topic to investigate how these different allocation strategies affect the performance of the two procedures.

4. Integrating Procedure $\mathcal{MMY}$ with a commercial software package such as Tecnomatix® can be a necessary future work in a practical sense.

The next section concludes the research with some personal thoughts.

6.4   Conclusion: Chapter 6

This chapter concluded the research by presenting a summary of the work done in this research, contribution to the body of knowledge and recommendations for future work. Following are some thoughts of the researcher in closing this research.

- Like many real-life problems one faces in one's daily life, the level of difficulties of the MORS problems in the simulation case studies depended on the unknown true mean distributions, which means one does not know beforehand how much effort the problem will require. The best way to deal with this is to simply set to work, and keep doing it until it solves the problem. And in many cases, even when the true mean distributions are not favourable, the problem is solved in the end though it might take long. However, staring at the problem taken by a sudden worry or unproven frustration, like the researcher did at some point of the research process, would not solve the problem at all.

- Extreme cases do happen in reality like in the buffer allocation problem or in the trauma unit problem where the procedures could not find the solution that satisfies the requirement of Q or Q_{IZ}. However, in these cases too, the procedure could find the solution when the requirement was slightly relaxed to Q_R. There is always a solution out there if one lowers one's expectation a little bit and decides to be satisfied with a seemingly lower-quality answer, which often turns out to be equally good as the other, if one thinks carefully.

- Trade-offs exist in almost all decision-making problems. Although MORS procedures help us finding optimal solutions, and Procedure $\mathcal{MMY}$ even guarantees the quality of them, this is the limit of what most algorithms can do. In the

end, one has to choose a single best solution from a number of optimal solutions, and must live with that. There are some algorithms designed to help with this decision (Jahanshahloo *et al.*, 2006), but the ultimate choice is always in our hands. Therefore, stopping what we are doing and humbly reevaluating our priority before making the final decision is well worth the effort if we know that our preference/priority can lead our life to a completely different path.

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