FRAMES OF IDEALS OF COMMUTATIVE f-RINGS

by

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Declaration

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I declare that *Frames of Ideals of Commutative f-Rings* is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

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Abstract

In his study of spectra of f-rings via pointfree topology, Banaschewski [6] considers lattices of ℓ-ideals, radical ℓ-ideals, and saturated ℓ-ideals of a given f-ring A. In each case he shows that the lattice of each of these kinds of ideals is a coherent frame. This means that it is compact, generated by its compact elements, and the meet of any two compact elements is compact. This will form the basis of our main goal to show that the lattice-ordered rings studied in [6] are coherent frames.

We conclude the dissertation by revisiting the d-elements of Martínez and Zenk [30], and characterise them analogously to d-ideals in commutative rings. We extend these characterisations to algebraic frames with FIP. Of necessity, this will require that we reappraise a great deal of Banaschewski's work on pointfree spectra, and that of Martínez and Zenk on algebraic frames.

Keywords: frame, compact normal frame, coherent frame, d-ideal, d-elements, ℓ -ideal, radical ideal, functor, f-ring, zero-dimensional, strongly normal.

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Contents

Chapter 1

Introduction and preliminaries

1.1 A brief history on frames in pointfree topology

Our interest is mainly ideals of pointfree function rings. Pointfree topology deals with particular complete lattices called frames. Our reference for the history of frames is the thesis by Martin Mugochi, supervised by Professor T.A. Dube, titled “Contribution to the theory of nearness in pointfree topology”[34].

The study of topological properties from a lattice-theoretic viewpoint was initiated by H. Wallman [40] in 1938. The term frame was introduced by C.H. Dowker in 1966 and brought to the fore in the article [14] co-authored with D. Papert. The dual notion locale was introduced by J.R. Isbell in 1972 in the ground-breaking paper titled “Atomless Parts Of Spaces”[20]. In the words of B. Banaschewski [11], Isbell was able to put the precise relationship between frames and spaces into categorical perspective.

Locales have sometimes been regarded as generalized topological spaces, and the terms pointless thinking and pointfree topology have been used in relation to the categories **Loc** (of locales) and **Frm** (of frames) respectively. Indeed there are those (like B. Banaschewski [11]) who maintain that **Frm** is the context in which the actual constructions of topological concepts are done, whereas others (like P.T. Johnstone [22]) maintain that frame theory is lattice theory applied to topology and locale theory is topology itself.

1.2 Synopsis of the dissertation

Our aim in this dissertation is to study lattices of various types of ideals of commutative f-rings with identity, and also f-rings with bounded inversion. The dissertation consists of four chapters. Chapter 1 is introductory. It is where we present the relevant definitions pertaining to frames and their frame homomorphisms. We also outline the requisite background for the ensuing chapters.

In Chapter 2 we show that the lattice, $\mathrm{RId}(A)$, of radical ideals of a commutative ring A is a coherent frame. In addition, we show that the association $A \mapsto \mathrm{RId}(A)$ is functorial from the category of commutative rings with identity to the category of coherent frames. We also prove some properties of onto ring homomorphisms which have not been proved in [4], such as, given an onto ring homomorphism $\phi : A \to B$, $\mathrm{RId}(B)$ is isomorphic to a closed quotient of $\mathrm{RId}(A)$.

In Chapter 3 we catalogue all that is known so far regarding lattices of ℓ-ideals, radical ℓ-ideals and saturated ℓ-ideals of commutative f-ring with identity. In each case we shall show that the lattice of each of these kinds of ideals is a coherent frame. We also investigate for which rings $\mathrm{RId}(A)$ is zero-dimensional and strongly normal. The particular rings considered here are regular rings, boolean rings, semiprime rings and exchange rings.

In Chapter 4 we revisit the d-elements of Martínez and Zenk [30], and characterise them analogously to d-ideals in commutative rings. There are several properties of the d-nucleus and of minimal prime elements that are not proved in [30] and are left to the reader. Here we prove some of those.

1.3 Frames and their homomorphisms

In this section we recall some definitions and results concerning frames that are used in the various parts of the dissertation. We start by defining basic terms which will be frequently used throughout the dissertation. Our main references for frames are [3], [21], [34] and [36]. Our references for categories and functors are [12] and [37].

Definition 1.3.1. A *partially ordered set* is a set X with a binary relation $\leq$ such that for every $x, y, z \in X$, it satisfies the following:

(i) Reflexivity: $x \leq x$,

(ii) Antisymmetry: if $x \leq y$ and $y \leq x$, then $x = y$,

(iii) Transitivity: if $x \leq y$ and $y \leq z$, then $x \leq z$.

Recall that a *lower bound* of a nonempty subset Y of a partially ordered set X is an element x of X such that $x \leq y$ for every $y \in Y$. Similarly, an *upper bound* of a subset Y of a partially ordered set X is an element x of X such that $x \geq y$ for every $y \in Y$. We say X is a *complete lattice* if every subset Y of X has a least upper bound (supremum or join) and a greatest lower bound (infimum or meet) in X.

Definition 1.3.2. Let X be a set. A collection $\mathcal{T}$ of subsets of X is called a *topology on* X if it satisfies the following:

(i) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,

(ii) $\mathcal{T}$ is closed under finite intersections, that is, for any $Y_1, Y_2 \in \mathcal{T}, Y_1 \cap Y_2 \in \mathcal{T}$,

(iii) $\mathcal{T}$ is closed under all unions, that is, for any $Y \subseteq \mathcal{T}, \bigcup Y \in \mathcal{T}$.

Furthermore, we say that a set X endowed with a topology $\mathcal{T}$ is called a *topological space*. The elements of X are called *points* of the space, and the subsets of X belonging to $\mathcal{T}$ are called *open*.

Definition 1.3.3. A *frame* is a complete lattice L in which binary meets distribute over arbitrary joins, that is,

$$x \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I} (x \wedge x_i)$$

for every $x \in L$ and $\{x_i \mid i \in I\} \subseteq L$, where $\wedge$ denotes binary meet and $\bigvee$ denotes arbitrary join. We denote the *bottom* or *zero* element of L by 0, and the *top* or *unit* element of L by 1.

Definition 1.3.4. A *frame homomorphism* is a map $h : L \to M$ between frames L and M preserving finite meets and arbitrary joins, including the top element and the bottom element.

An important class of frames arises from topology. For any topological space X, the lattice $\mathfrak{O}X$ of open subsets of X is a frame. For any topological spaces X and Y, a continuous map $f : X \to Y$ induces a frame homomorphism $\mathfrak{O}f : \mathfrak{O}Y \to \mathfrak{O}X$ which takes $U \in \mathfrak{O}Y$ to $f^{-1}(U) \in \mathfrak{O}X$.

Remark 1.3.5. Associated with any frame homomorphism $h : L \to M$ is a map $h_* : M \to L$, called the *right adjoint* of h, which is not necessarily a frame homomorphism, but preserves arbitrary meets. It is given by

$$h_*(y) = \bigvee\{x \in L \mid h(x) \leq y\}.$$

The following property holds for every $x \in L$ and every $y \in M$,

$$h(x) \leq y \qquad \Longleftrightarrow \qquad x \leq h_*(y).$$

Definition 1.3.6. Let $h : L \to M$ be a frame homomorphism. Then:

(1) h is *dense* if for every $x \in L$, $h(x) = 0$ implies $x = 0$.

(2) h is *codense* if for every $x \in L$, $h(x) = 1$ implies $x = 1$.

(3) h is *onto* if and only if $hh_* = id_M$.

Definition 1.3.7. The *pseudocomplement* of an element x of a frame L is the element

$$x^* = \bigvee\{y \in L \mid x \wedge y = 0\} \text{ in } L.$$

We note that $x \wedge x^* = 0$, however $x \vee x^* = 1$ does not hold in general. In the case where $x \vee x^* = 1$ holds, we say x is *complemented*. We say an element $x \in L$ is dense if $x^* = 0$.

Definition 1.3.8. An element x of a frame L is *rather below* an element a, written $x \prec a$, if there exists an element $y \in L$, called a *separating element*, such that $x \wedge y = 0$ and $y \vee a = 1$. Furthermore, $x \prec a$ also means that $a \vee x^* = 1$. We say L is *regular* if for every $a \in L$, $a = \bigvee\{x \in L \mid x \prec a\}$.

Below are some properties of the rather below relation:

(i) $y \leq x \prec a \leq b \implies y \prec b$.

(ii) $x, y \prec a \implies x \vee y \prec a$.

(iii) $x \prec a, b \implies x \prec a \wedge b$.

Definition 1.3.9. A frame L is said to be *spatial*, if and only if it has enough points in the sense that for every pair of distinct elements of the frame L, there exists a point of L separating the elements, for example, more precisely:

$$a, b \in L, a < b \implies \text{ there exists } \phi \in \Sigma L \text{ such that } \phi(a) = 0 < 1 = \phi(b).$$

Definition 1.3.10. We call $D \subseteq L$ a *downset* if $x \in D$ and $y \leq x$ implies $y \in D$, and we call $U \subseteq L$ an *upset* if $u \in U$ and $u \leq v$ implies $v \in U$. For any $a \in L$, we write

$$\downarrow a = \{x \in L \,|\, x \leq a\}, \text{ which is a downset,}$$

and

$$\uparrow a = \{y \in L \,|\, a \leq y\}, \text{ which is an upset.}$$

We note that $\downarrow a$ is a frame whose bottom element is $0 \in L$ and top element is a. Similarly, $\uparrow a$ has $1 \in L$ as its top element and a as its bottom element.

We say that a subset S of a frame L *generates* L if for every element $x \in L$,

$$x = \bigvee \{s \in S \mid s \leq x\}.$$

Definition 1.3.11. An element x of a frame L is *compact* if for any $S \subseteq L$, $x \leq \bigvee S$ implies $x \leq \bigvee T$, for some finite $T \subseteq S$. We say L is *compact* if its top element is compact, and we denote the set of all compact elements of a frame L by $\mathfrak{k}(L)$.

Definition 1.3.12. A frame L generated by its compact elements is *algebraic*. We say L is *coherent* if it is a compact algebraic frame such that $x \wedge y \in \mathfrak{k}(L)$ for every $x, y \in \mathfrak{k}(L)$.

Definition 1.3.13. A frame homomorphism $h : L \to M$ between coherent frames is *coherent* provided it maps $\mathfrak{k}(L)$ into $\mathfrak{k}(M)$.

Definition 1.3.14. A frame L is *normal* if for any $x, y \in L$, $x \vee y = 1$ in L implies there exists $a, b \in L$ such that $x \vee a = 1 = y \vee b$ and $a \wedge b = 0$.

Definition 1.3.15. A *category* $\mathbf{C}$ consists of:

(1) A collection of *objects* of $\mathbf{C}$,

(2) For every ordered pair of objects X, Y of $\mathbf{C}$ we associate a collection $\mathbf{C}(X, Y)$ of *morphisms* (also called *maps*) $f : X \to Y$,

(3) For every ordered triple X, Y, Z of objects there is a *composition map*

$$m_{XYZ} : \mathbf{C}(Y, Z) \times \mathbf{C}(X, Y) \to \mathbf{C}(X, Z)$$

that sends (g, f) to $g \circ f = gf$,

subject to the following conditions:

(C1) Composition is associative:

$$h \circ (g \circ f) = (h \circ g) \circ f,$$

for all objects $X, Y, Z, W \in \mathbf{C}$ and all morphisms $f : X \to Y$, $\ g : Y \to Z$ and $h : Z \to W$,

(C2) For every object $X \in \mathbf{C}$, there is an *identity morphism* $id_X \in \mathbf{C}(X, X)$ which acts as an identity under composition. That is, for every $f : X \to Y$ we have

$$id_Y \circ f = f = f \circ id_X.$$

Example 1.3.16. (1) $\mathbf{C} = \mathbf{Set}$: the objects in this category are sets and morphisms are functions.

(2) $\mathbf{C} = \mathbf{Group}$: the objects in this category are groups and morphisms are group homomorphisms.

Definition 1.3.17. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A *functor* $\mathbf{F} : \mathbf{C} \to \mathbf{D}$ is an assignment which maps objects of $\mathbf{C}$ to objects of $\mathbf{D}$ and arrows of $\mathbf{C}$ to arrows of $\mathbf{D}$ such that for X, Y and f, g in $\mathbf{C}$ we have:

(1) For every object $X \in \mathbf{C}$ there exists an object $F(X) \in \mathbf{D}$,

(2) For every morphism $f : X \to Y$ in $\mathbf{C}$ there exists a morphism $F(f) : F(X) \to F(Y)$ in $\mathbf{D}$,

subject to the following conditions:

(C1) Given morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathbf{C}$, $F(g \circ f) = F(g) \circ F(f)$,

(C2) For every object $X \in \mathbf{C}$, $F(id_X) = id_{F(X)}$.

Note that functors exist in both covariant and contravariant types. A *covariant functor* preserves the directions of arrows, that is, every morphism $f : X \to Y$ is mapped to a morphism $F(f) : F(X) \to F(Y)$. Whereas a *contravariant functor* reverses the directions of arrows, that is, every morphism $f : X \to Y$ is mapped to a morphism $F(f) : F(Y) \to F(X)$.

Example 1.3.18. (1) If $\mathbf{C}$ is a category then the *identity functor* $id_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ is defined by

$$id_{\mathbf{C}}(X) = X \text{ for all objects } X,$$

and

$$id_{\mathbf{C}}(f) = f \text{ for all morphisms } f.$$

(2) The relationship between the category **Top** of topological spaces and continuous maps, and the category **Frm** of frames and their homomorphisms, constitutes a contravariant functor as illustrated below [34].

$$\mathbf{Top} \xrightarrow{\mathfrak{O}} \mathbf{Frm}$$

$$\begin{tikzcd} X \arrow[d, "f"] & \mathfrak{O}X \arrow[d, "f^{-1}=\mathfrak{O}f"] \\ Y & \mathfrak{O}Y \end{tikzcd}$$

Below we introduce the spectrum of a frame L, which will be used later in the dissertation. Some of the material covered below can be found in Banaschewski's papers [3] and [6], and Johnstone's book [21].

Definition 1.3.19. We define $\Sigma : \mathbf{Frm} \to \mathbf{Top}$. The spectrum of a frame L is the space

$$\Sigma L = \{\xi : L \to 2 \,|\, \xi \text{ is a frame homomorphism}\},$$

with open sets $\Sigma_a = \{\xi \in \Sigma L \,|\, \xi(a) = 1\}$ for every $a \in L$.

Lemma 1.3.20. $(\Sigma L, \{\Sigma_a \,|\, a \in L\})$ *is a topological space.*

Proof. For the bottom element we have

$$\Sigma_0 = \{\xi \in \Sigma L \,|\, \xi(0) = 1\} = \emptyset.$$

Similarly, for the top element we have

$$\Sigma_1 = \{\xi \in \Sigma L \,|\, \xi(1) = 1\} = \Sigma L.$$

For binary meets we must show that $\Sigma_a \cap \Sigma_b = \Sigma_{a \wedge b}$. Let $\phi \in \Sigma_a, \Sigma_b$. Then

$$\begin{aligned}
\Sigma_a \cap \Sigma_b &= \{\phi \in \Sigma L \mid \phi(a) = 1\} \cap \{\phi \in \Sigma L \mid \phi(b) = 1\} \\
&= \{\phi \in \Sigma L \mid \phi(a) = 1 \text{ and } \phi(b) = 1\} \\
&= \{\phi \in \Sigma L \mid \phi(a \wedge b) = 1\} \\
&= \Sigma_{a \wedge b}, \text{ thus } \phi \in \Sigma_{a \wedge b}.
\end{aligned}$$

For arbitrary joins we must show that $\bigcup \Sigma_{a_i} = \Sigma_{\bigvee a_i}$. Let $\alpha \in \bigcup \Sigma_{a_i}$ for some $i \in I$. Then

$$\begin{aligned}
\bigcup \Sigma_{a_i} &= \bigcup \{\alpha \in \Sigma L \mid \alpha(a_i) = 1 \text{ for some } i \in I\} \\
&= \{\alpha \in \Sigma L \mid \bigvee_{i \in I} \alpha(a_i) = 1 \text{ for some } i \in I\} \\
&= \{\alpha \in \Sigma L \mid \alpha(\bigvee_{i \in I} a_i) = 1 \text{ for some } i \in I\} \\
&= \Sigma_{\bigvee a_i} \text{ for some } i \in I.
\end{aligned}$$

Therefore, $(\Sigma L, \{\Sigma_a \,|\, a \in L\})$ is a topological space. $\square$

1.4 Rings and f-rings

In this section we recall some basic definitions concerning rings and f-rings, which will be frequently used throughout the dissertation. Our reference is the book titled "Advanced modern Algebra" [37]. Throughout this dissertation all rings considered will be commutative with identity.

Definition 1.4.1. Let $(A, +, \cdot)$ be a non-empty set equipped with two binary operations. Then $(A, +, \cdot)$ is called a *ring with identity* if it satisfies the following:

(1) $(A, +)$ is an abelian group with identity, which we denote by 0. The element 0 is called the *additive identity*.

(2) For each $x, y, z \in A$ we have

  (i) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$,

  (ii) $x \cdot (y + z) = x \cdot y + x \cdot z$,

  (iii) $(y + z) \cdot x = y \cdot x + z \cdot x$.

(3) There exists an element $1 \in A$ called a *multiplicative identity* with respect to the operation $\cdot$ such that for every $x \in A, 1 \cdot x = x = x \cdot 1$. We assume $1 \neq 0$.

Example 1.4.2. $(\mathbb{Z}, +, \cdot)$, $(\mathbb{Q}, +, \cdot)$, $(M_n(\mathbb{R}), +, \cdot)$, $(\mathbb{Z}_n, +, \cdot)$ are examples of rings.

Definition 1.4.3. Let A and B be rings. A *ring homomorphism* is a function $\phi : A \to B$ such that

(i) $\phi(1) = 1$,

(ii) $\phi(x + y) = \phi(x) + \phi(y)$ for every $x, y \in A$,

(iii) $\phi(xy) = \phi(x)\phi(y)$ for every $x, y \in A$.

Furthermore, if $x \cdot y = y \cdot x$ for every $x, y \in A$ then the ring A is said to be *commutative.* A ring homomorphism that is also a bijection (both one-to-one and onto) is called an *isomorphism*, and rings A and B are called *isomorphic*, denoted by $A \cong B$.

Definition 1.4.4. An *idempotent* of a ring is an element a such that $a^2 = a$. Furthermore, a ring in which all elements are idempotent is called a *Boolean ring.*

For any ring A, $\operatorname{Idp} A$ will denote the Boolean algebra of its idempotents, with the following properties:

(i) $u' = 1 - u$,

(ii) $u \wedge v = uv$,

(iii) $u \vee v = u + v - uv$,

for any $u, v \in \operatorname{Idp} A$. For any frame L, its Boolean part BL will be the Boolean algebra of its complemented elements with its lattice operation induced from L.

Definition 1.4.5. A ring A is a *regular ring* if for every $a \in A$ there exists $b \in A$ such that $a = aba$.

Definition 1.4.6. Let A be a commutative ring with identity and suppose $I \subseteq A$ is a nonempty subset. Then I will be called an *ideal* in A if it satisfies the following:

(i) $0 \in I$,

(ii) $b \in I$ and $a \leq b$ implies $a \in J$,

(iii) $a, b \in J$ implies $a \vee b \in J$.

Example 1.4.7. For each integer n, $n\mathbb{Z}$ is an ideal in $\mathbb{Z}$.

The material covered below can also be found in the readings [6] and [21].

Definition 1.4.8. A *lattice-ordered ring (ℓ-ring)* is a ring A together with a lattice structure on its underlying set such that

$$a, b \geq 0 \implies a + b \geq 0 \text{ and } ab \geq 0,$$

and that in any lattice-ordered ring A one defines

$$a^+ = a \vee 0, \quad a^- = (-a) \vee 0, \quad |a| = a \vee (-a)$$

for which one then shows that:

(i) $a = a^+ - a^-$,

(ii) $|a| = a^+ + a^-$,

(iii) $a^+ \wedge a^- = 0$,

(iv) $|a + b| \leq |a| + |b|$,

(v) $|ab| \leq |a||b|$.

Definition 1.4.9. An *f-ring* A is a lattice-ordered ring such that

$$(a \wedge b)c = (ac) \wedge (bc)$$

holds for every $a, b \in A$ and $c \in A^+ = \{x \in A \mid x \geq 0\}$. It has *bounded inversion* if every element $a \geq 1$ in A is invertible in A.

One of the special property of f-rings that we note is, for all $a, b \in A$,

$$a^2 \geq 0 \text{ and } |ab| = |a||b|.$$

Chapter 2

Radical ideals and coherent frames

Our aim in this chapter is to show that the lattice, $\mathrm{RId}(A)$, of radical ideals of a commutative ring A is a coherent frame. Our main result here is to show that the association $A \mapsto \mathrm{RId}(A)$ is functorial from the category of commutative rings with identity to the category of coherent frames. Before that, we first consider some frame properties and theorems which are of particular importance in the present context. We denote the category of coherent frames and coherent homomorphisms by **CohFrm**. Our main reference here will be Banaschewski's papers [4], [7] and [9]. As stated in the Introduction, all rings are assumed to be commutative rings with identity.

2.1 Radical ideals

Definition 2.1.1. The *radical of an ideal* I of a ring A, denoted by $\sqrt{I}$, is the ideal

$$\sqrt{I} = \{x \in A \,|\, x^n \in I \text{ for some } n \in \mathbb{N}\}.$$

If $I = \sqrt{I}$, then I is called a *radical ideal.* We denote the lattice of radical ideals of a ring A by $\mathrm{RId}(A)$.

Example 2.1.2. Consider any ring $\mathbb{Z}$ of integers:

(1) The radical ideal of $8\mathbb{Z}$ is $2\mathbb{Z}$.

(2) The radical ideal of $12\mathbb{Z}$ is $6\mathbb{Z}$.

(3) The radical ideal of $7\mathbb{Z}$ is $7\mathbb{Z}$.

(4) In general, the radical ideal of $n\mathbb{Z}$ is given by $r\mathbb{Z}$ where r is the product of all distinct prime factors of n.

Below we present some well known properties of radical ideals that will be very useful throughout this chapter.

Proposition 2.1.3. *Let I, J be ideals of a ring A. Then we have the following.*

1. $I \subseteq \sqrt{I}$.
2. *If* $I \subseteq J$ *then* $\sqrt{I} \subseteq \sqrt{J}$.
3. $\sqrt{\sqrt{I}} = \sqrt{I}$.

Proof. (1) Let $x \in I$. Then $x^1 = x \in I$, and so $x \in \sqrt{I}$. Thus, I is contained in its radical.

(2) Let $x \in \sqrt{I}$. Then there exists $n \in \mathbb{N}$ such that $x^n \in I$. Since $I \subseteq J$, $x^n \in J$, and so $x \in \sqrt{J}$. Thus, $\sqrt{I} \subseteq \sqrt{J}$, and hence a radical of an ideal preserves inclusion.

(3) Since $\sqrt{I}$ preserves inclusions, it is immediate $\sqrt{I} \subseteq \sqrt{\sqrt{I}}$. On the other hand, we note that if $x \in \sqrt{\sqrt{I}}$, then $x^n \in \sqrt{I}$ for some $n \in \mathbb{N}$. So $x^{nm} = (x^n)^m \in AI$ for some $m \in \mathbb{N}$, which implies $x \in \sqrt{I}$. Thus, $\sqrt{\sqrt{I}} = \sqrt{I}$. □

Remark 2.1.4. Let $\{I_\alpha\}$ be a family of radical ideals of A partially ordered by inclusion. We define

$$\Sigma_\alpha I_\alpha = \{x \in A \mid x^n = a_{\alpha_1} + \cdots + a_{\alpha_m}\},$$

for some $n \in \mathbb{N}$ and $a_{\alpha_k} \in I_{\alpha_k}$ for some indices $\alpha_1, \ldots, \alpha_m$. For the join in $\mathrm{RId}(A)$, we have

$$\bigvee_\alpha I_\alpha = \sqrt{\Sigma_\alpha I_\alpha},$$

that is, $x \in \bigvee_\alpha I_\alpha$ if and only if $x^n = a_{\alpha_1} + \cdots + a_{\alpha_m}$ for some $n \in \mathbb{N}$ and $a_{\alpha_k} \in I_{\alpha_k}$ for some indices $\alpha_1, \ldots, \alpha_m$.

Lemma 2.1.5. *If $\{I_\alpha\}$ is a family of radical ideals of A, then $\sqrt{\Sigma_\alpha I_\alpha}$ is the smallest radical ideal of A containing every I_α.*

Proof. Let J be a radical ideal of A containing every I_α. Then $\Sigma_\alpha I_\alpha \subseteq J$. So $\sqrt{\Sigma_\alpha I_\alpha} \subseteq \sqrt{J}$, but $\sqrt{J} = J$ since J is a radical ideal itself. Thus $\sqrt{\Sigma_\alpha I_\alpha} \subseteq J$, so $\sqrt{\Sigma_\alpha I_\alpha}$ is the smallest radical ideal containing every I_α. $\square$

Definition 2.1.6. For every $a \in A$, the *principal radical ideal* generated by a is denoted by

$$[a] = \{x \in A \,|\, x^n \in Aa \text{ for some } n\}.$$

We note that $a \in [a]$. We shall denote by $[a_1, \ldots, a_n]$ the radical ideal of A generated by the finitely many elements $a_1, \ldots, a_n$ in A, that is, the ideal consisting of all $x \in A$ such that $x^k = c_1a_1 + \cdots + c_na_n$ for some $k \in \mathbb{N}$ and some $c_1, \ldots, c_n \in A$.

As mentioned in the Introduction, we plan to show that $\mathrm{RId}(A)$ is a coherent frame, but before that we first show that it is a frame.

Theorem 2.1.7. *For any commutative ring A with identity, the lattice $\mathrm{RId}(A)$ is a frame.*

Proof. $\mathrm{RId}(A)$ preserves the bottom element and the top element, that is,

$$0_{\mathrm{RId}(A)} = \{0\} \quad \text{and} \quad 1_{\mathrm{RId}(A)} = A.$$

Let $\{I_\alpha\}$ be a family of radical ideals of A, and consider any $x \in A$ such that $x^n \in \bigcap_\alpha I_\alpha$ for some $n \in \mathbb{N}$. This implies that $x^n \in I_\alpha$ for every α. Thus, $x \in I_\alpha$ for every α since every I_α is a radical ideal. So $x \in \bigcap_\alpha I_\alpha$, showing that $\bigcap_\alpha I_\alpha$ is a radical ideal. Therefore the intersection of radical ideals is again a radical ideal, and so $\mathrm{RId}(A)$ is closed under arbitrary meets. Thus $\mathrm{RId}(A)$ is a complete lattice. As observed in Remark 2.1.4, for the join in $\mathrm{RId}(A)$ we have $x \in \bigvee_\alpha I_\alpha$ if and only if $x^n = a_{\alpha_1} + \cdots + a_{\alpha_m}$, for some $n \in \mathbb{N}$ and $a_{\alpha_k} \in I_{\alpha_k}$ for some indices $\alpha_1, \ldots, \alpha_m$. To prove that binary meets distributes over arbitrary joins, let $x \in I \cap \bigvee_\alpha I_\alpha$ with $x^n = a_{\alpha_1} + \cdots + a_{\alpha_m}$ so that

$$\begin{aligned} x^{n+1} &= x \cdot x^n \\ &= x\left(a_{\alpha_1} + \cdots + a_{\alpha_m}\right) \\ &= xa_{\alpha_1} + \cdots + xa_{\alpha_m}. \end{aligned}$$

This shows that $xa_{\alpha_k} \in I \cap I_{\alpha_k}$ for every $k = 1, \dots, m$. Thus,

$$x^{n+1} \in (I \cap I_{\alpha_1}) + \cdots + (I \cap I_{\alpha_m}) \subseteq \sum_{\alpha} (I \cap I_\alpha)$$

which implies $x \in \sqrt{\sum_\alpha (I \cap I_\alpha)} = \bigvee_\alpha (I \cap I_\alpha)$. Therefore $I \cap \bigvee_\alpha I_\alpha \subseteq \bigvee_\alpha (I \cap I_\alpha)$. Hence the frame law holds since the other inclusion holds anyway. $\square$

We shall now show that, in fact, $\mathrm{RId}(A)$ is a coherent frame. For this we need to have a good description of the compact elements of this frame.

Lemma 2.1.8. *The compact elements of* $\mathrm{RId}(A)$ *are precisely the ideals* $[a_1, \dots, a_n]$ *for some* $a_1, \dots, a_n \in A$.

Proof. We begin by showing that $[a_1] \vee \cdots \vee [a_n] = [a_1, \dots, a_n]$ for all $a_1, \dots, a_n \in A$. Now let $x \in [a_1, \dots, a_n]$, which implies that $x \in \sqrt{[a_1, \dots, a_n]}$. Then there exists $k \in \mathbb{N}$ such that $x^k \in [a_1, \dots, a_n]$. So $x^k = u_1 a_1 + \cdots + u_n a_n \in [a_1] \vee \cdots \vee [a_n]$. For the reverse implication, we observe that $[a_1] \subseteq [a_1, a_2, \dots, a_n]$, $[a_2] \subseteq [a_1, a_2, \dots, a_n], \dots,$ $[a_n] \subseteq [a_1, a_2, \dots, a_n]$, and so $[a_1] \vee \cdots \vee [a_n] \subseteq [a_1, \dots, a_n]$. Thus, the desired equality holds.

Now we show that for any $a \in A$, $[a]$ is a compact element in $\mathrm{RId}(A)$, which will establish that $[a_1, \dots, a_n]$ is compact because the join of finitely many compact elements is compact in any frame. Suppose that

$$[a] \leq \bigvee_\alpha I_\alpha = \sqrt{\sum_\alpha I_\alpha}, \text{ where } I_\alpha \in \mathrm{RId}(A) \text{ for all } \alpha.$$

Then $a \in \sqrt{\sum_\alpha I_\alpha}$, which implies $a^m \in \sum_\alpha I_\alpha$ for some $m \in \mathbb{N}$. So $a^m \in I_{\alpha_1} + \cdots + I_{\alpha_n}$ for some indices $\alpha_1, \dots, \alpha_n$. Thus $a \in \sqrt{I_{\alpha_1} + \cdots + I_{\alpha_n}} = I_{\alpha_1} \vee \cdots \vee I_{\alpha_n}$. Since $I_{\alpha_1} \vee \cdots \vee I_{\alpha_n}$ is a radical ideal containing a, it follows that $[a] \subseteq I_{\alpha_1} \vee \cdots \vee I_{\alpha_n}$. So $[a]$ is compact, and hence $[a_1, \dots, a_n]$ is compact.

Lastly, we show that the compact elements of $\mathrm{RId}(A)$ are precisely the ideals $[a_1, \dots, a_n]$. To achieve this, we need to show that for any $J \in \mathrm{RId}(A)$, $J = \bigvee\{[a] \mid a \in J\}$. To prove that $J \subseteq \bigvee\{[a] \mid a \in J\}$, let $x \in J$, observe that $x \in [x] \subseteq \bigvee\{[a] \mid a \in J\}$. So $J \subseteq \bigvee\{[a] \mid a \in J\}$. On the other hand, if $a \in J$ then $[a] \subseteq J$, hence $\bigvee\{[a] \mid a \in J\} \subseteq J$. Thus, for any $J \in \mathrm{RId}(A)$,

$J = \bigvee\{[a] \mid a \in J\}$. So if J is compact, then there are finitely many $a_1, \dots, a_n \in J$ such that $J = [a_1] \vee \cdots \vee [a_n] = [a_1, \dots, a_n]$. □

Theorem 2.1.9. $\mathrm{RId}(A)$ *is a coherent frame.*

Proof. The top element of $\mathrm{RId}(A)$ is the ideal $[1]$, which is compact by Lemma 2.1.8. Thus $\mathrm{RId}(A)$ is a compact algebraic frame. Now we need to show that the meet of any two compact elements is a compact element. For this, we firstly show that for any $a, b \in A$, $[a] \wedge [b] = [ab]$. Let $x \in [a] \cap [b]$. Then by the definition of radical ideals, $x^n = ra$ and $x^m = sb$ for some $n, m \in \mathbb{N}$ and some $r, s \in A$. So $x^{n+m} = (ra)(sb) = (rs)(ab) \in A(ab)$, which implies that $x \in [ab]$. It follows therefore that $[a] \cap [b] \subseteq [ab]$.

Conversely, let $x \in [ab]$. So for some $n \in \mathbb{N}$, $x^n = rab$ for some $r \in A$. Then $x^n = (ra)b \in Ab$, and so $x \in [b]$. Also, $x \in [a]$ since $x^n = (rb)a \in Aa$. Thus, $x^n \in A(ab)$, which implies $x \in [a] \cap [b]$. We conclude that for any $a, b \in A$, $[a] \wedge [b] = [ab]$. Now consider any two compact elements $I, J \in \mathrm{RId}(A)$ such that $I = [a_1] \vee \cdots \vee [a_n]$ and $J = [b_1] \vee \cdots \vee [b_m]$ for some $a_i \in A$ and $b_i \in A$. Then

$$\begin{aligned} I \cap J &= \left(\bigvee_i [a_i]\right) \cap \left(\bigvee_j [b_j]\right) \\ &= \bigvee_{i,j} ([a_i] \cap [b_j]) \\ &= \bigvee_{i,j} [a_i b_j], \text{ since } [a] \cap [b] = [ab], \end{aligned}$$

proving that $I \cap J$ is compact. Hence, $\mathrm{RId}(A)$ is a coherent frame. □

2.2 The RId functor

In this section we plan to show that RId is a functor from **CRng** to **CohFrm**, where **CRng** is the category of commutative rings with identity and ring homomorphisms that preserve the identity. Since we know how RId maps objects, we need to describe how it maps morphisms. But firstly, we recall the following notation. Our main references are [4] and [26].

If A is a ring and $S \subseteq A$, we denote by $\langle S \rangle$, the ideal of A generated by S. In the case of a singleton $\{a\}$, we abbreviate $\langle \{a\} \rangle$ by $\langle a \rangle$. To this end we formulate the following definition.

Definition 2.2.1. Let $\phi : A \to B$ be a ring homomorphism in **CRng**. If I is a radical ideal of A then $\phi[I]$ is an ideal of B. We define

$$\mathrm{RId}(\phi) : \mathrm{RId}(A) \to \mathrm{RId}(B) \quad \text{by} \quad \mathrm{RId}(\phi)(I) = \sqrt{\langle\, \phi[I] \,\rangle},$$

that is, $y \in B$ lies in $\mathrm{RId}(\phi)(I)$ if and only if there exists $m \in \mathbb{N}$ such that

$$y^m = b_1\phi(u_1) + \cdots + b_k\phi(u_k)$$

for some $k \in \mathbb{N}$, $b_i \in B$ and $u_i \in I$. Thus, $\mathrm{RId}(\phi)(I)$ is the radical of the ideal of B generated by $\phi[I]$.

In the following proposition we prove that $\mathrm{RId}(\phi)\colon \mathrm{RId}(A) \to \mathrm{RId}(B)$ is a coherent map for every ring homomorphism $\phi\colon A \to B$. To show that it is a frame homomorphism we shall use the familiar fact that if a map $h\colon L \to M$ between frames preserves 0, 1, binary meets and arbitrary joins, then it is a frame homomorphism.

Proposition 2.2.2. *For any ring homomorphism $\phi : A \to B$ in* **CRng***, the mapping $\mathrm{RId}(\phi) : \mathrm{RId}(A) \to \mathrm{RId}(B)$ is a coherent frame homomorphism and its right adjoint is given by the map $J \mapsto \phi^{-1}[J]$, where $J \in \mathrm{RId}(B)$.*

Proof. We begin by showing that $\mathrm{RId}(\phi)$ preserves the bottom element and the top element. For the bottom element we show that $\mathrm{RId}(\phi)([0_A]) = [0_B]$. Let $y \in \mathrm{RId}(\phi)([0_A])$ then there exists $m \in \mathbb{N}$ such that $y^m \in \sqrt{\langle\, \phi[0_A] \,\rangle}$. But,

$$\begin{aligned}\sqrt{\langle\, \phi[0_A] \,\rangle} &= \sqrt{\langle\, [0_B] \,\rangle}\\ &= \sqrt{[0_B]}\\ &= [0_B], \text{ since } [0_B] \text{ is a radical ideal.}\end{aligned}$$

It follows that $y^m \in [0_B]$, which implies $(y^m)^n \in A0_B = \{0_B\}$ for some $n \in \mathbb{N}$. Thus, $y \in [0_B]$, and so $\mathrm{RId}(\phi)([0_A]) \subseteq [0_B]$. Conversely, since $[0_B]$ is the bottom element of $\mathrm{RId}(B)$, it follows that $\mathrm{RId}(\phi)\,([0_A]) = [0_B]$.

Now for the top element, we show that $\mathrm{RId}(\phi)([1_A]) = [1_B]$.

$$\begin{aligned}\mathrm{RId}(\phi)([1_A]) =& \sqrt{\langle\, \phi[1_A]\,\rangle}\\ =& \sqrt{\langle\, [1_B]\,\rangle}\\ =& \sqrt{[1_B]}\\ =& [1_B], \text{ since } [1_B] \text{ is a radical ideal.}\end{aligned}$$

Secondly, we prove that $\mathrm{RId}(\phi)$ preserves joins. Let $I, J \in \mathrm{RId}(A)$ and we show that

$$\sqrt{\langle\, \phi[I \vee J]\,\rangle} \;=\; \sqrt{\langle\, \phi[I]\,\rangle} \vee \sqrt{\langle\, \phi[J]\,\rangle}.$$

Since $\mathrm{RId}(\phi)$ preserves order, it is immediate that $\sqrt{\langle\, \phi[I]\,\rangle} \vee \sqrt{\langle\, \phi[J]\,\rangle} \subseteq \sqrt{\langle\, \phi[I \vee J]\,\rangle}$. To establish the reverse implication, we let $y \in \sqrt{\langle\, \phi[I \vee J]\,\rangle}$. Then there exists $m \in \mathbb{N}$ such that $y^m \in \langle\, \phi[I \vee J]\,\rangle = \langle\, \phi[\sqrt{I+J}]\,\rangle$, since $I \vee J = \sqrt{I+J}$. This implies $y^m = b_1\phi(x_1) + \cdots + b_n\phi(x_n)$ where $b_i \in A$ and $x_i \in I + J$. By the definition of sum of ideals, there exists $u_i \in I$ and $v_i \in J$ with $x_i = u_i + v_i$. Since ϕ is a ring homomorphism, $\phi(u_i + v_i) = \phi(u_i) + \phi(v_i)$ for some i. Then

$$y^m = \Big(b_1\phi(u_1) + \cdots + b_n\phi(u_n)\Big) + \Big(b_1\phi(v_1) + \cdots + b_n\phi(v_n)\Big),$$

which implies that $y^m \in \langle\phi[I]\rangle + \langle\phi[J]\rangle \subseteq \sqrt{\langle\phi[I]\rangle} + \sqrt{\langle\phi[J]\rangle}$. Therefore

$$y \in \sqrt{\sqrt{\langle\phi[I]\rangle} + \sqrt{\langle\phi[J]\rangle}} = \sqrt{\langle\phi[I]\rangle} \vee \sqrt{\langle\phi[J]\rangle}, \text{ as required.}$$

Thus $\mathrm{RId}(\phi)\,(I \vee J) = \mathrm{RId}(\phi)\,(I) \vee \mathrm{RId}(\phi)\,(J)$.

Now we prove that $\mathrm{RId}(\phi)$ preserves binary meets. Let $I, J \in \mathrm{RId}(A)$ and we show that

$$\sqrt{\langle\, \phi[I \cap J]\,\rangle} \;=\; \sqrt{\langle\, \phi[I]\,\rangle} \cap \sqrt{\langle\, \phi[J]\,\rangle}.$$

Since $\mathrm{RId}(\phi)$ preserves order, it follows that $\sqrt{\langle\, \phi[I \cap J]\,\rangle} \subseteq \sqrt{\langle\, \phi[I]\,\rangle} \cap \sqrt{\langle\, \phi[J]\,\rangle}$. To establish the reverse implication, let $z \in \sqrt{\langle\, \phi[I]\,\rangle} \cap \sqrt{\langle\, \phi[J]\,\rangle}$ so that $z^m \in \langle\, \phi\,[I]\,\rangle$ and $z^k \in \langle\, \phi\,[J]\,\rangle$ for some $m, k \in \mathbb{N}$. Hence, $z^m = b_1\phi(u_1) + \cdots + b_n\phi(u_n)$ and $z^k = s_1\phi(j_1) + \cdots + s_l\phi(j_l)$ for some $b_t, s_v \in B$, $u_t \in I$ and $j_v \in J$. Then

$$\begin{aligned}z^m z^k &= (b_1\phi(u_1) + \cdots + b_n\phi(u_n))\,(s_1\phi(j_1) + \cdots + s_l\phi(j_l))\\ &= b_1 s_1\phi(u_1)\phi(j_1) + \cdots + b_n s_l\phi(u_n)\phi(j_l)\\ &= \sum_{t,v} b_t s_v \phi(u_t j_v).\end{aligned}$$

So $z^{m+k} = z^m z^k \in \langle\, \phi[I \cap J]\, \rangle$, which implies that $z \in \sqrt{\langle\, \phi[I \cap J]\, \rangle}$ since $u_t j_v \in I \cap J$. Thus $\mathrm{RId}(\phi)\,(I \cap J) = \mathrm{RId}(\phi)\,(I) \cap \mathrm{RId}(\phi)\,(J)$.

If I is a radical ideal of A generated by $X = [a_1, \ldots, a_n]$, then $\mathrm{RId}(\phi)(I)$ is the radical ideal of B generated by X. Thus, the map $\mathrm{RId}(\phi)$ is a coherent frame homomorphism. By the definition of inverse images, we have $\mathrm{RId}(\phi)(I) = \phi[I] \subseteq J$ if and only if $I \subseteq \phi^{-1}[J]$. Thus, the right adjoint of $\mathrm{RId}(\phi)$ is the map $J \mapsto \phi^{-1}[J]$. □

Proposition 2.2.3. RId *is a covariant functor.*

Proof. Associated with every ring morphism $\phi : A \to B$ is a morphism $\varphi : B \to C$. We begin by showing that the RId functor preserves composition. To achieve this, we plan to show that $\mathrm{RId}\,(\varphi \circ \phi) = \mathrm{RId}(\varphi) \circ \mathrm{RId}(\phi)$. Let I be a radical ideal of A. Clearly,

$$\varphi\,(\phi[I]) \subseteq \varphi\,(\mathrm{RId}(\phi)\,(I)) \subseteq \mathrm{RId}(\varphi)(\mathrm{RId}(\phi)\,(I)).$$

Thus $\mathrm{RId}\,(\varphi \circ \phi) \leq \mathrm{RId}(\varphi) \circ \mathrm{RId}(\phi)$. To prove the reverse inclusion, take any $J \in \mathrm{RId}(B)$ such that $\varphi(\phi[I]) \subseteq J$, then J contains all $\varphi(y)$ with $y^m = b_1\phi(u_1) + \cdots + b_n\phi(u_n)$ for some $m \in \mathbb{N}$, $b_k \in B$ and $u_k \in I$. Thus, $\varphi\,(\mathrm{RId}(\phi)(I)) \subseteq J$ and so $\mathrm{RId}(\varphi)(\mathrm{RId}(\phi)\,(I)) \subseteq J$. We therefore have $\mathrm{RId}(\varphi) \circ \mathrm{RId}(\phi) \leq \mathrm{RId}\,(\varphi \circ \phi)$, and hence equality.

Lastly, we show that for every ideal $I \in \mathrm{RId}(A)$, we have $\mathrm{RId}(id_A) = id_{\mathrm{RId}(A)}$.

$$\begin{aligned} \mathrm{RId}(id_A)(I) &= \sqrt{\langle\, id_A[I]\, \rangle} \\ &= \sqrt{\langle\, I\, \rangle} \\ &= \sqrt{I} = I, \text{ since } I \text{ is a radical ideal.} \end{aligned}$$

□

Definition 2.2.4. A frame homomorphism $h : L \to M$ is *closed* if for every $a, b \in L$ and $y \in M$, $h(b) \leq h(a) \vee y$ implies $b \leq a \vee h_*(y)$.

We remind the reader that a frame homomorphism $h : L \to M$ is onto if and only if $hh_*(m) = m$ for every $m \in M$. Since $hh_*(m) \leq m$ always, h is onto if and only if $m \leq hh_*(m)$ for every $m \in M$.

Proposition 2.2.5. *For every onto ring homomorphism $\phi : A \to B$,* $\mathrm{RId}(\phi)$ *is closed and onto* [26].

Proof. We first show that $\mathrm{RId}(\phi)$ is onto. Let $y \in J$, then there exists $x \in I$ such that $\phi(x) = y$. This implies that $x \in \phi^{-1}(y)$. But $\phi^{-1}(y) \subseteq \phi^{-1}[J]$, so $x \in \phi^{-1}[J]$. Thus,

$$y = \phi(x) \subseteq \phi(\phi^{-1}[J]) \subseteq \sqrt{\langle \phi(\phi^{-1}[J]) \rangle} = \mathrm{RId}(\phi)(\phi^{-1}[J]).$$

Therefore $\mathrm{RId}(\phi)$ is onto.

Now we show that $\mathrm{RId}(\phi)$ is closed, that is,

$$\mathrm{RId}(\phi)\,(I_1) \;\subseteq\; \mathrm{RId}(\phi)\,(I_2) \vee J \quad \Longrightarrow \quad I_1 \;\subseteq\; I_2 \vee \phi^{-1}\,[J] = \sqrt{I_2 + \phi^{-1}\,[J]}$$

where $I_1, I_2 \in \mathrm{RId}(A)$ and $J \in \mathrm{RId}(B)$. Assume that $\mathrm{RId}(\phi)\,(I_1) \subseteq \mathrm{RId}(\phi)\,(I_2) \vee J$. Let $r \in I_1$, then $\phi(r) \in \sqrt{\langle\, \phi[I_1]\,\rangle} = \mathrm{RId}(\phi)\,(I_1)$. Then for every $r \in I_1$, there exists l such that $\phi(r^l) \in \sqrt{\langle\, \phi[I_2]\,\rangle} + J$. So we have

$$\begin{aligned}
&\phi(r^l - s) \in J, \text{ for some } s \in I_2 \\
&\Rightarrow (r^l - s) \in \phi^{-1}[J] \\
&\Rightarrow r^l \in I_2 + \phi^{-1}[J] \\
&\Rightarrow r \in \sqrt{I_2 + \phi^{-1}[J]} = I_2 \vee \phi^{-1}\,[J].
\end{aligned}$$

This proves that $I_1 \subseteq I_2 \vee \phi^{-1}\,[J]$, thus $\mathrm{RId}(\phi)$ is closed. $\square$

Proposition 2.2.6. *If $\phi : A \to B$ is an onto ring homomorphism, then* $\mathrm{RId}(B) \cong \uparrow\left(\sqrt{\ker\phi}\right)$.

Proof. For brevity, we write Φ for the frame homomorphism $\mathrm{RId}(A) \to \mathrm{RId}(B)$ induced by ϕ. The right adjoin is the mapping $\Phi_*\colon \mathrm{RId}(B) \to \Phi_*[\mathrm{RId}(B)]$. Since ϕ is onto, so is Φ. Recall that $\Phi_*[\mathrm{RId}(B)] = \{\phi^{-1}[J] \mid J \in \mathrm{RId}(B)\}$. We claim that

$$\Phi_*[\mathrm{RId}(B)] = \uparrow(\sqrt{\ker\phi}).$$

Let $a \in \sqrt{\ker\phi}$. Then for some positive integer n we have $\phi(a^n) = \phi(a)^n = 0$, which implies $\phi(a) \in [0]$, so that $a \in \phi^{-1}[0]$, whence $\sqrt{\ker\phi} \subseteq \phi^{-1}[0]$. Since $[0]$ is the zero of $\mathrm{RId}(B)$, it follows that, for any $J \in \mathrm{RId}(B)$, $\sqrt{\ker\phi} \leq \phi^{-1}[J]$, which then shows that $\Phi_*[\mathrm{RId}(B)] \subseteq {\uparrow}(\sqrt{\ker\phi})$. For the reverse inclusion, take any $I \in \mathrm{RId}(A)$ such that $\sqrt{\ker\phi} \subseteq I$. We plan to show that $I = \phi^{-1}[\sqrt{\phi[I]}]$. Since $\phi^{-1}[\sqrt{\phi[I]}] = \Phi_*\Phi(I)$, we immediately have $I \subseteq \phi^{-1}[\sqrt{\phi[I]}]$. On the other hand, let $z \in \phi^{-1}[\sqrt{\phi[I]}]$, so that $\phi(z) \in \sqrt{\phi[I]}$. Pick $m \in \mathbb{N}$ and $u \in I$ such that $\phi(z)^m = \phi(u)$. Then $\phi(z^m - u) = 0$, hence

$$z^m - u \in \ker\phi \subseteq \sqrt{\ker\phi} \subseteq I, \text{ since } I \text{ is radical,}$$

showing that $z^m \in I = \sqrt{I}$. Thus, $z \in I$. We therefore have ${\uparrow}(\sqrt{\ker\phi}) \subseteq \Phi_*[\mathrm{RId}(B)]$, and hence equality. $\square$

Chapter 3

Lattice-ordered rings

In his study of spectra of f-rings via pointfree topology, Banaschewski [6] considers lattices of ℓ-ideals, radical ℓ-ideals, and saturated ℓ-ideals of a given f-ring A. In the previous chapters we considered radical ideals, we also gave a brief required background on frames, whereas in this chapter we shall catalogue all that is known so far regarding lattices of ℓ-ideals, radical ℓ-ideals, and saturated ℓ-ideals of an f-ring. In each case we shall show that the lattice of each of these kinds of ideals is a coherent frame.

3.1 Compact normal frames

To begin with, we shall recall some definitions and facts of pointfree topology that are of particular relevance in our context. These results will be sourced mainly from [3], [4], [5], and [6].

Definition 3.1.1. A frame L is *subfit* if for any $a, b \in L$, $a < b$ implies there exists some $c \in L$ such that $a \vee c < 1 = b \vee c$.

Lemma 3.1.2. *Any normal subfit frame L is regular.*

Proof. We shall prove this by contradiction. Let L be a normal subfit frame and suppose that for some elements $a, b \in L$, $a = \bigvee\{x \in L \mid x \prec b\} \neq b$. Then $a < b$, so there exists some $c \in L$

such that $a \vee c < 1 = b \vee c$. Since L is normal, pick $u, v \in L$ such that $b \vee u = 1 = c \vee v$ and $u \wedge v = 0$. Hence that $v \prec b$, and so $v \leq a$ by definition of a. Since $c \vee v = 1$ we have $a \vee c = 1 < 1$, a contradiction. $\square$

Lemma 3.1.3. *Any codense image of a normal frame is normal.*

Proof. Let L, M be frames, and let $h : L \to M$ be a codense onto mapping with L normal. If $a \vee b = 1$ in M pick $x, y \in L$ such that $h(x) = a$ and $h(y) = b$; then $x \vee y = 1$ since h is codense, and the normality condition for x, y in L then implies this condition for a, b in M, via the homomorphism h. $\square$

Definition 3.1.4. A *nucleus* on a frame L is a closure operator $j : L \to L$ such that for all $a, b \in L$, $j(a \wedge b) = j(a) \wedge j(b)$. The set $\mathrm{Fix}(j) = \{x \in L \mid j(x) = x\}$ is a frame with meet as in L and join given by

$$\bigvee_{\mathrm{Fix}(j)} S = j\left(\bigvee S\right),$$

for every $S \subseteq \mathrm{Fix}(j)$. Further, the map $L \to \mathrm{Fix}(j)$, effected by j, is an onto frame homomorphism. Banaschewski shows in [2, Lemma 2] that if L is compact and j is codense, then $\mathrm{Fix}(j)$ is also compact.

Definition 3.1.5. Let L be a compact frame, then for any $a \in L$, the element $x \in L$ is called *a-small* if $x \vee y = 1$ implies $a \vee y = 1$ for $y \in L$. We then define the *saturation nucleus*, $s_L : L \to L$, by

$$s_L(a) = \bigvee \{x \in L \mid x \text{ is } a\text{-small}\}.$$

We will write SL for the frame $\mathrm{Fix}(s_L)$. When confusion is unlikely, we will drop the subscript on the nucleus s_L.

Lemma 3.1.6. *Let L be a compact frame. Then for every $a \in L$, $s_L(a)$ is the largest a-small element of L.*

Proof. We plan to show that for any $a \in L$, if $s_L(a) \vee b = 1$ then $a \vee b = 1$ for every $b \in L$. If $a, b \in L$ such that $s_L(a) \vee b = 1$, by compactness there exists elements $x_1, \ldots, x_n$ all a-small such that $x_1 \vee \cdots \vee x_n \vee b = 1$. Hence we have

$$\begin{aligned}
x_1\ a\text{-small} &\implies a \vee x_2 \vee x_3 \vee x_4 \vee\ \cdots\ \vee x_n \vee b = 1,\\
x_2\ a\text{-small} &\implies a \vee a \vee x_3 \vee x_4 \vee\ \cdots\ \vee x_n \vee b = 1,\\
x_3\ a\text{-small} &\implies a \vee a \vee a \vee x_4 \vee\ \cdots\ \vee x_n \vee b = 1,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
x_n\ a\text{-small} &\implies a \vee a \vee a \vee a \vee\ \cdots\ \vee a \vee b = 1.
\end{aligned}$$

So $a \vee b = 1$. Therefore $s_L(a)$ itself is a-small, and hence the largest a-small element of L. $\square$

Example 3.1.7. In the lattice $\mathfrak{O}X$ of open sets of a topological space X, the set U is W-small if and only if every closed subset contained in U is contained in W.

Lemma 3.1.8. *For any compact frame L, the map $s_L : L \to L$ is a closure operator on L.*

Proof. Following from the definition, a itself is a-small, and so we have $a \leq s_L(a)$. Now if $a \leq b$ then let x be a-small. To show that x is b-small let $x \vee y = 1$. Since x is a-small, it implies $a \vee y = 1$. But $a \leq b$, so $b \vee y = 1$. Hence x is b-small. Therefore $s_L(a) \leq s_L(b)$. In Lemma 3.1.6 we proved that $s_L(a) \vee y = 1$ implies $a \vee y = 1$ using compactness. Hence if x is $s_L(a)$-small, and $x \vee y = 1$, then $s_L(a) \vee y = 1$ implies $a \vee y = 1$ and so x is a-small. Thus,

$$\begin{aligned}
s_L(s_L(a)) &= \bigvee\{x \in L \mid x \text{ is } s_L(a)\text{-small}\}\\
&= \bigvee\{x \in L \mid x \text{ is } a\text{-small}\}\\
&= s_L(a).
\end{aligned}$$

Therefore s_L is a closure operator. $\square$

In the following lemma, we extend our results from Lemma 3.1.8.

Lemma 3.1.9. *For any compact frame L, $s_L(a) \wedge s_L(b)$ is $a \wedge b$-small and, for all $a, b \in L$, $s_L(a) \wedge s_L(b) = s_L(a \wedge b)$.*

Proof. Since s_L preserves order, it is immediate that $s_L(a \wedge b) \leq s_L(a) \wedge s_L(b)$ for every a, b in L. For the reverse inequality, suppose $(s_L(a) \wedge s_L(b)) \vee c = 1$ for some $c \in L$. Then also $s_L(a) \vee c = 1$ and $s_L(b) \vee c = 1$. Hence $a \vee c = 1$ and $b \vee c = 1$ by Lemma 3.1.6, and therefore $(a \vee c) \wedge (b \vee c) = (a \wedge b) \vee c = 1$. Thus $s_L(a) \wedge s_L(b)$ is $a \wedge b$-small. But $s_L(a \wedge b)$ is the largest $a \wedge b$-small element of L and hence $s_L(a) \wedge s_L(b) \leq s_L(a \wedge b)$. □

Remark 3.1.10. By a *point* of a frame L we mean a *prime element*, that is, an element $p < 1$ such that for any $a, b \in L$, $a \wedge b \leq p$ implies $a \leq p$ or $b \leq p$. We denote by $\mathrm{Pr}(L)$ the set of primes of L. For the nucleus s_L on L,

$$\mathrm{Pr}(SL) = \{x \in \mathrm{Pr}(L) \mid s_L(x) = x\},$$

and so the points of SL are exactly the points x of L such that $s_L(x) = x$ [35].

Theorem 3.1.11. *A compact frame L is normal if and only if SL is regular.*

Proof. ($\Rightarrow$) By Lemma 3.1.3, if L is normal then SL is also normal. We know that SL is subfit, and so it is a normal subfit frame. Thus, by Lemma 3.1.2, it follows that SL is regular.

($\Leftarrow$) If $a \vee b = 1$ in L then $s_L(a) \vee s_L(b) = 1$ in SL, hence by compactness and regularity, $x \vee y = 1$ in SL for some $x \prec s_L(a)$ and $y \prec s_L(b)$ in SL. Since s_L is a codense quotient of L, $x \vee y = 1$ in L. Furthermore, since SL is regular there exists u, v in SL such that

$$x \wedge u = 0 \quad \text{and} \quad s_L(a) \vee u = 1,$$
$$y \wedge v = 0 \quad \text{and} \quad s_L(b) \vee v = 1$$

in SL, then also $a \vee u = 1$ and $b \vee v = 1$ in L, while

$$\begin{aligned} u \wedge v &= 1 \wedge (u \wedge v) \\ &= (x \vee y) \wedge (u \wedge v) \\ &= (x \wedge (u \wedge v)) \vee (y \wedge (u \wedge v)) \\ &= ((x \wedge u) \wedge v) \vee ((y \wedge v) \wedge u) \\ &= (0 \wedge v) \vee (0 \wedge u) \\ &= 0, \quad \text{as required.} \end{aligned}$$

□

For any frame L, if L is compact then every maximal element is saturated since s is codense, it follows that L and SL have the same maximal elements [5].

Proposition 3.1.12. *For any compact normal frame L, $s : L \to SL$ has a right inverse $r : SL \to L$ defined by $r(a) = \bigvee\{x \in L \mid x \prec a\}$.*

Proof. We first show that r is a homomorphism. By the definition of rather below relation, r does preserve $0, \wedge$ and e. Concerning join, we show that r preserves updirected and finitary join. Let $a = s(\bigvee D)$ for updirected $D \subseteq SL$ and $x \prec a$, that is, $a \vee x^* = 1$ in L. Then also $(\bigvee D) \vee x^* = 1$ by codensity, hence by compactness $t \vee x^* = 1$ for some $t \in D$. Therefore $x \prec t$, and so $x \leq r(t) \leq \bigvee r[D]$. Thus $r(a) \leq \bigvee r[D]$, while the reverse inequality always holds. Hence $r(\bigvee D) = \bigvee r[D]$. Now for the finitary meet, we show that $r(a \vee b) = r(a) \vee r(b)$. If $x \prec a \vee b$ for $a, b \in SL$, so that $a \vee b \vee x^* = 1$. Then there exists $u, v \in L$ such that $a \vee u = 1 = b \vee x^* \vee v$ and $u \wedge v = 0$. Furthermore, we have $w, z \in L$ such that $w \wedge z = 0$ and $b \vee w = 1 = x^* \vee v \vee z$. Then, $v \prec a$, $z \prec b$ and

$$\begin{aligned}
x &= x \wedge 1 \\
&= x \wedge (x^* \vee v \vee z) \\
&= (x \wedge x^*) \vee (x \wedge v) \vee (x \wedge z) \\
&= 0 \vee (x \wedge v) \vee (x \wedge z) \\
&= (x \wedge v) \vee (x \wedge z).
\end{aligned}$$

But $x \wedge v \prec a$ and $x \wedge z \prec b$, so $(x \wedge v) \vee (x \wedge z) \leq r(a) \vee r(b)$. Thus, $x \leq r(a) \vee r(b)$ showing that $r(a \vee b) \leq r(a) \vee r(b)$, while the reverse inequality always holds.

Lastly we show that $sr(a) = a$ for every $a \in SL$. It is immediate that $sr(a) \leq a$ since $r(a) \leq a$ is trivial. For the reverse inequality, we show that a is $r(a)$-small. If $a \vee b = 1$ for some $b \in L$, then by normality there exists $u, v \in L$ such that $a \vee u = 1 = b \vee v$ and $u \wedge v = 0$. It follows that $v \prec a$, and so $v \leq r(a)$. Therefore $r(a) \vee b = 1$, proving that a is $r(a)$-small. □

Proposition 3.1.13. *For any compact normal frame L, the mapping s_L induces an isomorphism* $\operatorname{Reg} L \to SL$ *with inverse effected by r_L* [6].

Proof. The map s_L is codense and any codense homomorphism h on a regular frame is one-one because $h(a) = h(b)$ and $x \prec a$ implies that $h(b \vee x^*) = h(a \vee x^*) = 1$ so that $b \vee x^* = 1$ and hence $x \leq b$. On the other hand, s_L maps $\operatorname{Reg} L$ onto SL since $s_L r_L = s_L$. Lastly, for $a \in \operatorname{Reg} L$, $r_L s_L(a) = r_L(a) = a$, which shows that r_L induces the inverse of the isomorphism $\operatorname{Reg} L \to SL$ given by s_L. $\square$

Definition 3.1.14. A frame L is *zero-dimensional* if every element of L is a join of complemented elements below it.

Proposition 3.1.15. *A regular, coherent frame is zero-dimensional.*

Proof. Suppose that $a \in L$, then $a = \bigvee\{x \in L \mid x \prec a\}$ since L is regular. Because $x \prec a$ forms an ideal, it implies that $a \prec a$, also expressed as $a \vee a^* = 1$, showing that a is complemented. Furthermore, since every element of L is a join of compact elements by coherence, this shows that L is zero-dimensional. $\square$

Definition 3.1.16. A frame L is *strongly normal* if whenever $a \vee b = 1$ in L there exists $c, d \in L$ such that $c \leq a$, $d \leq b$, $c \vee d = 1$ and $c \wedge d = 0$.

This definition was introduced by Banaschewski in his study of zero-dimensionality in point-free topology [8].

Lemma 3.1.17. *For any compact frame L, if L is normal and SL is zero-dimensional, then L is strongly normal.*

Proof. Suppose that $a \vee b = 1$ in L, then also $r_L(a) \vee r_L(b) = 1$ in $\operatorname{Reg} L \cong SL$. Hence there exists $c \leq r_L(a)$ and $d \leq r_L(b)$ such that $c \vee d = 1$ and $c \wedge d = 0$. Since $\operatorname{Reg} L$ is a subframe of L and $r_L(x) \leq x$ for all $x \in L$ this proves the claim [6]. $\square$

To that end, we close this section by investigating for which rings is $\mathrm{RId}(A)$ zero-dimensional and strongly normal. The particular rings to be considered here are defined below, and we also mention some examples of these rings from Banaschewski's paper [6]. Recall that all our rings are commutative with identity.

Definition 3.1.18. A ring A is called a *semiprime ring* if it has no nonzero nilpotent elements. Such rings are also called *reduced* .

Definition 3.1.19. A ring A is called a *Gelfand ring* if for every $a \in A$, there exists an idempotent $u \in A$ such that $a + u$ is invertible.

Definition 3.1.20. A ring A is called an *exchange ring* if for some $r, s \in A$, $a + b = 1$ in A implies $(1 + ar)(1 + bs) = 0$.

The following are examples of Gelfand rings.

Example 3.1.21. (1) Exchange rings. If $a + b = 1$ and u is an idempotent such that $u - a$ is invertible, then

$$\begin{aligned}\left(1+\frac{a}{u-a}\right)\left(1-\frac{b}{u-a}\right) &= \frac{1}{(u-a)^2}(u)(u-a-b)\\ &= \frac{1}{(u-a)^2}(u)(u-(1-b)-b)\\ &= \frac{1}{(u-a)^2}(u)(u-1)\\ &= 0, \text{ as required.}\end{aligned}$$

(2) f-rings with bounded inversion. If $a + b = 1$ then $a \vee b$ is invertible, and so

$$\begin{aligned}\left(1-\frac{a}{a\vee b}\right)\left(1-\frac{b}{a\vee b}\right) &= \frac{1}{(a\vee b)^2}(0\vee(b-a))(0\vee(a-b))\\ &= \frac{1}{(a\vee b)^2}(0)\\ &= 0, \text{ as required.}\end{aligned}$$

The following are examples of exchange rings.

Example 3.1.22. (1) Boolean rings. A Boolean ring has only one invertible element, namely 1, and every element is idempotent so that $(1 - a)a = 1$.

(2) Regular rings. For any element a, if $a = a^2b$ by regularity then $1 - ab$ is idempotent.

Thus,

$$\begin{aligned}
(a+1-ab)\,(1+ab(b-1)) &= (a+1-ab)\left(1+ab^2-ab\right)\\
&= a+a^2b^2-a^2b+1+ab^2-ab-ab-a^2b^3+a^2b^2\\
&= a+2a^2b^2-a+1+ab^2-2ab-a^2b^3\\
&= 2a^2b^2+1+ab^2-2ab-ab^2\\
&= 1-2ab\,(1-ab)\\
&= 1
\end{aligned}$$

since $a^2b = a$ and $(1-ab)ab = 0$.

In the following proofs we will refer to Banaschewski's results [3, Lemma 1], which states:

For any ring A, the map $\operatorname{Idp} A \to B(\operatorname{RId}(A))$ taking u to $[u]$ is an isomorphism.

Theorem 3.1.23. *A semiprime ring A is regular if and only if $\operatorname{RId}(A)$ is zero-dimensional.*

Proof. ($\Rightarrow$) If $a = a^2b$ and $u = ab$ is an idempotent, then $[a] = [b]$. By [3, Lemma 1], $[u]$ is complemented, and since $\operatorname{RId}(A)$ is generated by $[a]$ then the results hold.

($\Leftarrow$) For any $a \in A$, $[a]$ is compact in $\operatorname{RId}(A)$ and hence complemented by zero-dimensionality so that $[a] = [u]$ for some idempotent u. Consequently, $u = ab$ and $a^n = uc$ for some $b, c \in A$ and exponent n. So $(a(1-u))^n = 0$. Since A is semiprime, then $a = au = a^2b$. $\square$

Theorem 3.1.24. *For any ring A, if A is an exchange ring then $\operatorname{RId}(A)$ is strongly normal.*

Proof. If $I \vee J = [1]$ in $\operatorname{RId}(A)$, pick $a \in I$ and $b \in J$ such that $a + b = 1$. Furthermore, pick an idempotent u such that $u - b$ is invertible by c. Then

$$\begin{aligned}
auc &= (1-b)uc\\
&= (u-b)uc\\
&= (u-b)(u-b)^{-1}u\\
&= u, \text{ showing that } u \in I.
\end{aligned}$$

And,

$$
\begin{aligned}
b(u-1)c &= (b-u)(u-1)c \\
&= (u-b)(1-u)(u-b)^{-1} \\
&= 1-u, \text{ showing that } 1-u \in J.
\end{aligned}
$$

Therefore $[u] \subseteq I$ and $[1-u] \subseteq J$, as required. □

3.2 The frame of ℓ-ideals

In this section we introduce the three spectra's that form the principal subject of this study, and establish some of their basic properties. These results will be sourced mainly from [3] and [6].

Definition 3.2.1. An *ℓ-ring* A is a ring together with a partial order such that for every $a, b, c \in A$ we have $a, b \geq 0$ implies $ab \geq 0$, and $a \leq b$ implies $a + c \leq b + c$.

Definition 3.2.2. For any ℓ-ring A, we define an ideal I of A an *ℓ-ideal* if whenever $a \in I$ and $|x| \leq |a|$ implies $x \in I$. We denote the lattice of these ideals by $\mathfrak{L}(A)$.

Lemma 3.2.3. *The sum of any two ℓ-ideals is an ℓ-ideal.*

Proof. Let L and J be ℓ-ideals with $a \in I$, $b \in J$ and $x \in A$. We must show that $I + J$ is an ℓ-ideal such that $|x| \leq |a + b|$ where $a + b \in I + J$ implies $x \in I + J$. If $|x| \leq |a + b|$, then also $|x| \leq |a| + |b|$ since $|a + b| \leq |a| + |b|$. Thus,

$$
\begin{aligned}
|x| &= |x| + |a| \wedge |x| - |a| \wedge |x| \\
&= |a| \wedge |x| + |x| + (-|a| \vee -|x|) \\
&= |a| \wedge |x| + (|x| - |a|) \vee (|x| - |x|) \\
&= |a| \wedge |x| + (|x| - |a|) \vee 0.
\end{aligned}
$$

But $|a| \wedge |x| \in I$, and since $|x| - |a| \leq |b| \in J$ then $(|x| - |a|) \vee 0 \in J$. Hence, $|x| \in I + J$. But $|x| = x^+ + x^-$, so $x^+, x^- \in I + J$. Also, $x = x^+ + x^- = x^+ - (|x| - x^+) = 2x^+ - |x| \in I + J$. Thus, $x \in I + J$, as required. □

Theorem 3.2.4. *For any ℓ-ring A, $\mathfrak{L}(A)$ is a compact algebraic frame.*

Proof. $\mathfrak{L}(A)$ is a complete lattice with intersection as meet, and directed join as union, making binary meet distribute over arbitrary joins. So, to see that $\mathfrak{L}(A)$ is finitary distributive, we let $x \in I \cap (J \vee H) = I \cap (J + H)$ for any $I, J, H \in \mathfrak{L}(A)$ so that $x = a + b$ for some $a \in J$ and $b \in H$. Consequently, $|x| \leq |a + b|$, which implies that

$$|x| \leq |a| + |b| = (|a| + (|a| \wedge |b|)) \vee (|b| + (|a| \wedge |b|)),$$

and so $|x| \in (I \cap J) + (I \cap H)$. Thus, $x \in (I \cap J) + (I \cap H)$, as required by Lemma 3.2.3.

Since the directed join in $\mathfrak{L}(A)$ is union, and the compact ideals $J \in \mathfrak{L}(A)$ are the finitely generated ones, which makes the unit element of $\mathfrak{L}(A)$ compact, and so $\mathfrak{L}(A)$ is algebraic. Thus, $\mathfrak{L}(A)$ is a compact algebraic frame. $\square$

Remark 3.2.5. An ℓ-ring A is called an f-ring if $(a \wedge b)\, c = ac \wedge bc$ for every $a, b \in A$ and $c \in A^+ = \{x \in A \mid x \geq 0\}$.

Definition 3.2.6. For any ℓ-ring A and $a \in A$, we define the *principal ℓ-ideal* of A generated by a as $[a] = \{x \in A \mid |x| \leq |a|b \text{ for some } b \geq 0\}$. Furthermore, $[a] + [b] = [|a| \vee |b|]$ and the finitely generated ℓ-ideals are the principal ℓ-ideals.

Following from Definition 3.2.6, now one might wonder when is $\mathfrak{L}(A)$ coherent, which is the same as asking "when are all $[a] \cap [b]$ principal?" This would occur if $[a] \cap [b] = [a \wedge b]$ for every $a, b \in A^+$. In short, $\mathfrak{L}(A)$ is coherent whenever A is an f-ring, which is shown in the following theorem.

Theorem 3.2.7. *Let A be an ℓ-ring. Then for every $a, b \in A^+$, $[a] \cap [b] = [a \wedge b]$ if and only if $(a \wedge b)c = ac \wedge bc$ for every $a, b \in A$ and $c \in A^+$.*

Proof. ($\Rightarrow$) The hypothesis implies that $ac \wedge bc = 0$ whenever $a \wedge b = 0$. So now by the $-$ and $\wedge$ rules we have, for arbitrary $a, b \in A$,

$$(a - (a \wedge b)) \wedge (b - (a \wedge b)) \;=\; (a \wedge b) - (a \wedge b) \;=\; 0.$$

Hence, for arbitrary $c \in A^+$ we have

$$\begin{aligned} 0 &= (\,(a-(a \wedge b))\,c\,) \wedge (\,(b-(a \wedge b))\,c\,) \\ &= (ac-(a \wedge b)\,c) \wedge (bc-(a \wedge b)\,c) \\ &= (ac \wedge bc)-(a \wedge b)\,c, \end{aligned}$$

showing that $ac \wedge bc = 0$ whenever $a \wedge b = 0$.

($\Leftarrow$) Let $x \in [a] \cap [b]$. We must show that $x \in [a \wedge b]$. Since $x \in [a] \cap [b]$ we have $|x| \leq as, bt$ for some $s, t \in A^+$, which implies $|x| \leq ac \wedge bc$ where $c = s + t \in A$. But $ac \wedge bc = (a \wedge b)\,c$, so $|x| \leq (a \wedge b)\,c$. Thus, $x \in [a \wedge b]$, as required. □

Definition 3.2.8. A frame L is *coherently normal* if it is coherent and, for every compact $a \in L$, $\downarrow a$ is normal.

Theorem 3.2.9. *For any f-ring A, $\mathfrak{L}(A)$ is coherently normal.*

Proof. In Theorem 3.2.7 we have shown that $\mathfrak{L}(A)$ is coherent, so now we need only show that $\mathfrak{L}(A)$ is normal. Let $I + J = [a]$ where $I, J \in \mathfrak{L}(A)$ and $a \in A^+$. Then $a = b + c$ for some $b \in I$ and $c \in J$. We claim that $I + [u] = [a] = J + [v]$ for $u = |c| - |b| \wedge |c|$ and $v = |b| - |b| \wedge |c|$. But $|b| \wedge |c| \leq |c|$ and $c \in J$, so $u \in J$. Thus, $I + [u] \subseteq [a]$. On the other hand, observe that

$$a = \; |a| \leq \; |b| + |c| = \; |b| + (u - |b| \wedge |c|) = \; (|b| - |b| \wedge |c|) + u,$$

showing that $[a] \subseteq I + [u]$, and hence proving the first identity. Similarly, $[a] = J + [v]$ by symmetry. Furthermore, since A is a f-ring we then have $[u] \cap [v] = [u \wedge v] = [0]$. □

Definition 3.2.10. For any ℓ-ring A, an ℓ-ideal I of A is called a *saturated ℓ-ideal* if $s_L(J) = J$ for the saturation operator s on $\mathfrak{L}(A)$. We denote the lattice of all saturated ℓ-ideals by $S\mathfrak{L}(A)$.

Another class of ideals of an f-ring that we will investigate consist of those ideals which are simultaneously radical ideals and ℓ-ideals. We also prove that these ideals are coherently normal.

Definition 3.2.11. For any ℓ-ring A, an ℓ-ideal I of A is called a *radical ℓ-ideal* if whenever $a^n \in I$ implies $a \in I$ for every $a \in A$ and $n \in \mathbb{N}$. We denote the lattice of all radical ℓ-ideals by $R\mathfrak{L}(A)$.

Lemma 3.2.12. *For any f-ring A, we have an operator r on $\mathfrak{L}(A)$ such that*

$$r(I) = \{a \in A \mid a^n \in I \text{ for some } n \in \mathbb{N}\} \text{ is a radical } \ell\text{-ideal.}$$

Proof. For any $I \in \mathfrak{L}(A)$, by the ring ideal property we have $a^n, a^m \in I$ implies $(a+b)^{n+m} \in I$, while by the f-ring property we have $|ab| = |a||b|$, and hence $|a^n| = |a|^n$. Now to show that r is a nucleus $\mathfrak{L}(A)$: for any $a \in r(I) \cap r(J)$. If $a^n \in I$ and $a^m \in J$ then $a^{n+m} \in I \cap J$ which implies $a \in r(I \cap J)$, proving that $r(I) \cap r(J) \subseteq r(I \cap J)$. On the other hand, if $a \in r(I \cap J)$, then $a^n \in I \cap J \subseteq r(I) \cap r(J)$. Thus $a \in r(I) \cap r(J)$. Furthermore, $R\mathfrak{L}(A) = \mathrm{Fix}(r)$. □

Theorem 3.2.13. *For any f-ring A, $R\mathfrak{L}(A)$ is coherently normal.*

Proof. Since $R\mathfrak{L}(A)$ is closed under directed unions, the compact elements of $R\mathfrak{L}(A)$ are exactly the $r[a]$ with $a \in A^+$. For coherence, we need to show that $r\,[a] \cap r\,[b] = r([a] \cap [b]) = r\,[a \wedge b]$ for any $a, b \in A^+$. Let $x \in r\,[a] \cap r\,[b]$. Then there exists $s, t \in A^+$ such that $x^n = sa$ and $x^m = tb$ for some $n, m \in \mathbb{N}$. So $x^{n+m} = (sa)(tb) \in [a] \cap [b]$, which implies that $x \in r[a \wedge b]$. On the other hand, let $x \in r\,[a \wedge b]$ so that $x^n = sab$ for some $s \in A^+$ and $n \in \mathbb{N}$. Then $x^n = (sa)\,b \in [b]$, and also $x^n = (sb)\,a \in [a]$. Hence, $x \in r[a] \cap r[b]$.

To prove for the normality condition, we let $r(I+J) = r[a]$ where $I, J \in \mathfrak{L}(A)$ and $a \in A^+$. Then $a^n = b + c$ for some $b \in I$ and $c \in J$. We claim that $I + r[u] = r[a] = J + r[v]$ for $u = |c| - |b| \wedge |c|$ and $v = |b| - |b| \wedge |c|$. But $|b| \wedge |c| \leq |c|$ and $c \in J$, so $u \in J$. Thus, $I + r[u] \subseteq r[a]$. On the other hand, observe that

$$a^n = |a|^n \leq |b| + |c| = |b| + (u - |b| \wedge |c|) = (|b| - |b| \wedge |c|) + u,$$

showing that $r[a] \subseteq I + r[u]$, and hence proving the first identity. Similarly, $r[a] = J + r[v]$ by symmetry. Furthermore, since A is a f-ring we then have $r[u] \cap r[v] = r[u \wedge v] = [0]$. □

The basic relation between the three pointfree spectra mentioned above is given by the commutative diagram

$$\begin{tikzcd}
\mathfrak{L}(A) \arrow[r, "r_L"] \arrow[dr, "s_L"'] & R\mathfrak{L}(A) \arrow[d] \\
& S\mathfrak{L}(A)
\end{tikzcd}$$

where s_L factors through r_L because r_L is codense, and has a right inverse by Proposition 3.1.12 [6].

Chapter 4

The d-nucleus on an algebraic frame

In this last chapter we revisit the d-elements of Martínez and Zenk [30], and characterise them analogously to d-ideals in commutative rings.

4.1 Characterisations of d-ideals

To start, we recall the definition of d-ideals in commutative rings, and some characterisations of these ideals. The references for this section are [1] and [33]. Throughout, A stands for a reduced commutative ring with identity. We should emphasise that the definitions we recall are not restricted to reduced rings only.

Definition 4.1.1. An ideal I in a ring A is a *minimal ideal* if $I \neq [0]$ and there is no ideal J with $[0] \subsetneq J \subsetneq I$. The set of all minimal prime ideals of A is denoted by $\min(A)$.

Definition 4.1.2. The *annihilator* of a set $S \subseteq A$ is the ideal

$$\mathrm{Ann}(S) = \{a \in A \,|\, as = 0 \text{ for every } s \in S\}.$$

In the case of a singleton $\{a\}$, the annihilator is denoted by $\mathrm{Ann}(a)$ or $a^{\perp}$; whereas double annihilators will be denoted by $\mathrm{Ann}^2(S)$ or $S^{\perp\perp}$, and $\mathrm{Ann}^2(a)$ or $a^{\perp\perp}$.

Definition 4.1.3. An ideal I of A is a *d-ideal* if $\mathrm{Ann}^2(a) \subseteq I$ for every $a \in I$.

This is the definition given by Mason in [33, page 944]. It is equivalent to that stated in [1] because, as shown in [33, Lemma 1.2],

$$\mathrm{Ann}^2(a) = \bigcap \{P \in \min(A) \mid a \in P\}.$$

The characterisations given in Theorem 4.1.1 below appear as [33, Theorem 2.3]. Let us recall some notation. For any $a \in A$, the set $\mathscr{V}(a)$ is defined by

$$\mathscr{V}(a) = \{P \in \min(A) \mid a \in P\}.$$

Theorem 4.1.1. *The following are equivalent for an ideal I of a ring A.*

1. *I is a d-ideal.*
2. *$\mathrm{Ann}^2(b) \supseteq \mathrm{Ann}^2(y)$ and $b \in I$ implies $y \in I$.*
3. *$\mathrm{Ann}^2(b) = \mathrm{Ann}^2(y)$ and $b \in I$ implies $y \in I$.*
4. *$\mathscr{V}(y) = \mathscr{V}(b)$ and $b \in I$ implies $y \in I$.*
5. *$\mathscr{V}(y) \supseteq \mathscr{V}(b)$ and $b \in I$ implies $y \in I$.*

Definition 4.1.4. An algebraic frame A has the *finite intersection property* (FIP) if for every $a, b \in \mathfrak{k}(A)$, $a \wedge b \in \mathfrak{k}(A)$.

We are going to extend these characterisations to algebraic frames with FIP. We should point out that, in the context of algebraic frames with FIP, our results are new. Let us start by laying the requisite background.

4.2 Revisiting d-elements in algebraic frames

The *d-nucleus* on an algebraic frame with FIP was introduced by Martínez and Zenk [30] as a way of abstracting the algebraic concept of d-ideals in commutative rings. From here to the end of the chapter, L denotes an algebraic frame with FIP. The d-nucleus is defined by

$$d(a) = \bigvee \{c^{\perp\perp} \mid c \in \mathfrak{k}(L) \text{ and } c \leq a\}.$$

The resulting quotient frame (which is, of course a sublocale of L) is denoted by dL, and its elements are called *d-elements*. If follows immediately from the definition that, for any $a \in L$,

$$a \text{ is a } d\text{-element} \quad \Leftrightarrow \quad c \leq a \text{ and } c \text{ compact implies } c^{\perp\perp} \leq a.$$

We shall use this characterisation frequently. As a frame homomorphism, the map $d\colon L \to dL$ is a dense onto frame homomorphism. Its right adjoint (as for the case of all nuclei) is the inclusion map $dL \to L$.

There are several properties of the d-nucleus and of minimal prime elements (which we will define shortly) that are not proved in [30] and are left to the reader. Here we will prove some of those.

Definition 4.2.1. An element $p \in L$ is a *minimal prime* if it is prime, and for any prime $q \leq p$, we have $q = p$. We write $\min(L)$ for the set of all minimal prime elements of L.

When dealing with algebraic frames, to test if an element is prime it suffices to restrict to compact elements. This is observed (without proof) in [30]. We give the proof.

Lemma 4.2.2. *If L is an algebraic frame, then for any $p \in L$, we have that p is prime if and only if for any $c, d \in \mathfrak{k}(L)$, $c \wedge d \leq p$ implies $c \leq p$ or $d \leq p$.*

Proof. One implication is trivial. Conversely, suppose the stated condition holds. Let $x \wedge y \leq p$, where $x, y \in L$. We must show that $x \leq p$ or $y \leq p$. Suppose not; then since L is algebraic, there exists compact elements c and d such that $c \leq x$, $d \leq y$, but $c \nleq p$ and $d \nleq p$. Then $c \wedge d \leq x \wedge y \leq p$, which by the stated condition implies $c \leq p$ or $d \leq p$. We thus have a contradiction. Therefore $x \leq p$ or $y \leq p$, and hence p is prime. $\square$

The following useful result was proved by Martínez in [27].

Proposition 4.2.3. *Let L be an algebraic frame with* FIP*. If $p \in \min(L)$, then for any compact c, either $c \leq p$ or $c^{\perp} \leq p$, but not both.*

In the lemma that follows we collect some easy, but useful, facts regarding d-elements and the frame dL. In one of the proofs we will apply Zorn's Lemma "downwards". Let us explain.

If P is a poset and every chain in P has a lower bound, then P has a minimal element. This, of course, follows from applying Zorn's Lemma to the poset P^{op}.

Lemma 4.2.4. *The following facts hold in any algebraic frame L with* FIP*.*

1. *For any $a \in L$, $a^{\perp}$ is a d-element.*

2. *$\mathfrak{k}(dL) = \{d(c) \mid c \in \mathfrak{k}(L)\}$.*

3. *If p is prime in L, then there is a $q \in \min(L)$ such that $q \leq p$.*

4. *Every minimal prime in L is a d-element.*

Proof. (1) Suppose $c \in \mathfrak{k}(L)$ and $c \leq a^{\perp}$. Then $c^{\perp\perp} \leq a^{\perp\perp\perp} = a^{\perp}$, and so $a^{\perp}$ is a d-element.

(2) Let $c \in \mathfrak{k}(L)$. Since $d : L \to dL$ is a coherent map, $d(c)$ is compact in dL. But $d(c) = c^{\perp\perp}$, so $\{d(c) \mid c \in \mathfrak{k}(L)\} \subseteq \mathfrak{k}(dL)$. On the other hand, let $x \in dL$ be compact, which implies

$$\begin{aligned}
x &= d(x) \\
&= d\left(\bigvee_{L}\{c^{\perp\perp} \mid c \in \mathfrak{k}(L) \text{ and } c \leq x\}\right) \qquad \text{since the supremum is upward directed} \\
&= \bigvee_{dL}\{d(c^{\perp\perp}) \mid c \in \mathfrak{k}(L) \text{ and } c \leq x\} \\
&= \bigvee_{dL}\{c^{\perp\perp} \mid c \in \mathfrak{k}(L) \text{ and } c \leq x\}.
\end{aligned}$$

Therefore, since x is compact and the set $\{c^{\perp\perp} \mid c \in \mathfrak{k}(L) \text{ and } c \leq x\}$ is directed, we have $x = a^{\perp\perp}$ for some $a \in \mathfrak{k}(L)$, and so $a \leq x$. Thus, $\mathfrak{k}(dL) \subseteq \{d(c) \mid c \in \mathfrak{k}(L)\}$. Therefore we have equality.

(3) We apply Zorn's Lemma "downwards". Let $S = \{q \in L \mid q \leq p\}$. Consider any chain $C \subseteq S$. We show that C has a lower bound in S. Put $\ell = \bigwedge C$. Then ℓ is a lower bound for C in L. We show that $\ell \in S$; and for this we need only show that ℓ is prime. Suppose it is not. Then there exists $x, y \in L$ such that $x \wedge y \leq \ell$ but $x \not\leq \ell$ and $y \not\leq \ell$. Since $x \not\leq \ell$, it follows that x is not a lower bound for C, and so there exists some $q_1 \in C$ such that $x \not\leq q_1$. Similarly, there exists $q_2 \in C$ such that $y \not\leq q_2$. Since C is a chain, we may assume, without loss of generality,

that $q_1 \leq q_2$. Since q_1 is prime and $x \wedge y \leq \ell \leq q_1$, we must have $x \leq q_1$ or $y \leq q_1$, neither of which is possible. This contradiction shows that $\bigwedge C$ is a lower bound for the chain C which lies in S. So S has a minimal element, that is, there is a minimal prime q with $q \leq p$.

(4) Let q be a minimal prime element. If c is compact and $c \leq q$, then, by Proposition 4.2.3, $c^\perp \nleq q$; hence $c^{\perp\perp} \leq q$ since q is prime. Therefore q is a d-element. □

We shall extend the characterisations of d-ideals stated above to d-elements. We start with a lemma.

Lemma 4.2.5. *Let L be an algebraic frame.*

1. *For any $a \in L$, $a^\perp = \bigwedge\{q \in \min(L) \mid a \nleq q\}$.*
2. *For any $c \in \mathfrak{k}(L)$, $c^{\perp\perp} = \bigwedge\{q \in \min(L) \mid c \leq q\}$.*

Proof. (1) Let q be a minimal prime with $a \nleq q$. Then $a^\perp \leq q$ since q is prime. This shows that $a^\perp \leq \bigwedge\{q \in \min(L) \mid a \nleq q\}$. To see the other inequality, set $b = \bigwedge\{q \in \min(L) \mid a \nleq q\}$. We claim that $b \wedge a = 0$. If not, then, by spatiality of L, there is a prime element p in L such that $a \wedge b \nleq p$. Let q be a minimal prime with $q \leq p$. Then $a \wedge b \nleq q$, and so $a \nleq q$. This implies $b \leq q$ (since b is the infimum of the set of minimal primes not above a), and so $a \wedge b \leq q \leq p$, and we have a contradiction. Therefore $b \wedge a = 0$, and hence

$$\bigwedge\{q \in \min(L) \mid a \nleq q\} = b \leq a^\perp,$$

and we have the claimed equality.

(2) We apply the first part and Proposition 4.2.3. We have

$$\begin{aligned} c^{\perp\perp} = (c^\perp)^\perp &= \bigwedge\{q \in \min(L) \mid c^\perp \nleq q\} && \text{by the first part} \\ &= \bigwedge\{q \in \min(L) \mid c \leq q\} && \text{by Proposition 4.2.3,} \end{aligned}$$

which proves the result. □

We now introduce the following notation. For any $c \in \mathfrak{k}(L)$,

$$V(c) = \{q \in \min(L) \mid c \leq q\}.$$

Thus, by the foregoing lemma, $c^{\perp\perp} = \bigwedge V(c)$.

Theorem 4.2.6. *The following are equivalent for an element a of an algebraic frame L with* FIP.

1. *a is a d-element.*
2. *For any $b, c \in \mathfrak{k}(L)$, $b^{\perp\perp} \geq c^{\perp\perp}$ and $b \leq a$ implies $c \leq a$.*
3. *For any $b, c \in \mathfrak{k}(L)$, $b^{\perp\perp} = c^{\perp\perp}$ and $b \leq a$ implies $c \leq a$.*
4. *For any $b, c \in \mathfrak{k}(L)$, $V(c) = V(b)$ and $b \leq a$ implies $c \leq a$.*
5. *For any $b, c \in \mathfrak{k}(L)$, $V(c) \supseteq V(b)$ and $b \leq a$ implies $c \leq a$.*

Proof. (1) $\Rightarrow$ (2): If $b \leq a$, then $b^{\perp\perp} \leq a$ since a is a d-element. Now, since $c \leq c^{\perp\perp} \leq b^{\perp\perp}$, it follows that $c \leq a$.

(2) $\Rightarrow$ (3): This is trivial.

(3) $\Rightarrow$ (4): Since $b^{\perp\perp} = \bigwedge V(b)$ and $c^{\perp\perp} = V(c)$, by Lemma 4.2.5, it is clear that (4) is implied by (3).

(4) $\Rightarrow$ (5): Assume that (4) holds, and that $V(c) \supseteq V(b)$ and $b \leq a$, for some compact b and c. We must show that $c \leq a$. We first show that $V(c) = V(b \wedge c)$. It is immediate that $V(c) \subseteq V(b \wedge c)$ because if $c \leq q$, for some minimal prime q, then $b \wedge c \leq q$. On the other hand, let q be a minimal prime element in $V(b \wedge c)$. Then $b \wedge c \leq q$, so that $b \leq q$ or $c \leq q$. In the first case, $q \in V(b) \subseteq V(c)$, and in the latter case $q \in V(c)$. Thus, in either case, $q \in V(c)$, whence $V(b \wedge c) \subseteq V(c)$, and hence $V(b \wedge c) = V(c)$. Now since L has FIP, $b \wedge c$ is compact and $b \wedge c \leq a$, so, by (4), we have $c \leq a$.

(5) $\Rightarrow$ (1): To show that a is a d-element, assuming (5) holds, let b be a compact element with $b \leq a$. We must show that $b^{\perp\perp} \leq a$. Consider any compact element $c \leq b^{\perp\perp}$. Then $c^{\perp\perp} \leq b^{\perp\perp}$. We show that $V(b) \subseteq V(c)$. Indeed, if $q \in V(b)$, then $b \leq q$, and hence $b^{\perp\perp} \leq q$, as q is a minimal prime element. Thus, $c \leq q$, establishing that $V(b) \subseteq V(c)$. Since $b \leq a$, condition (5) implies that $c \leq a$. But $b^{\perp\perp}$ is a join of compact elements below it, so we deduce that $b^{\perp\perp} \leq a$, proving that a is a d-element. $\square$

A combination of this theorem with Theorem 4.1.1 enables us to deduce the following results which is mentioned in passing in [15].

Corollary 4.2.7. *An ideal of a reduced ring A is a d-ideal if and only if it is a d-element of the frame* $\mathrm{RId}(A)$.

As another application, we characterise coherent maps $h\colon L \to M$ whose right adjoints send d-elements to d-elements. From this characterisation, we will be able to deduce the result of Azarpanah, Karamzadeh and Rezai Aliabad [1] that states that a ring homomorphism contracts d-ideals to d-ideals if and only if it contracts minimal prime ideals to d-ideals.

Let us first give an example of coherent map whose right adjoint sends d-elements to d-elements. Recall that a frame homomorphism $h\colon L \to M$ is called *skeletal* if it maps dense elements to dense elements. One characterisation of skeletal maps is that whenever $x^{\perp} = y^{\perp}$, then $h(x)^{\perp} = h(y)^{\perp}$. It is well known that every dense onto frame homomorphism is skeletal.

Example 4.2.8. The right adjoint of any skeletal coherent map between algebraic frames with FIP sends d-elements to d-elements. To see this, let $h\colon L \to M$ be such a map. Let z be a d-element in M, and consider any two compact elements b and c in L such that $b^{\perp\perp} = c^{\perp\perp}$ and $b \leq h_*(z)$. Then $b^{\perp} = c^{\perp}$, and hence $h(b)^{\perp\perp} = h(c)^{\perp\perp}$, and also $h(b) \leq z$. Since $h(b)$ and $h(c)$ are compact elements and z is a d-element, it follows from Theorem 4.2.6 that $h(c) \leq z$, whence $c \leq h_*(z)$. Therefore $h_*(z)$ is a d-element.

Theorem 4.2.9. *Let $h\colon L \to M$ be a coherent map between algebraic frames with* FIP. *Then h_* sends d-elements to d-elements if and only if it sends minimal prime elements to d-elements.*

Proof. Since minimal primes are d-elements, the left-to-right implication is obvious. Conversely, suppose $h_*(q)$ is a d-element in L for every minimal prime q in M. Let a be a d-element in M. We must show that $h_*(a)$ is a d-element in L. Consider any two compact elements b and c in L such that $V(b) = V(c)$ and $b \leq h_*(a)$. We must show that $c \leq h_*(a)$. Since h is a coherent map, $h(b)$ and $h(c)$ are compact elements in M. We claim that $V(h(b)) = V(h(c))$. To prove this claim, let $q \in V(h(b))$. Then q is a minimal prime element in M with $h(b) \leq q$. Then $b \leq h_*(q)$, and since $h_*(q)$ is a d-element, by hypothesis, $b^{\perp\perp} \leq h_*(q)$. But $b^{\perp\perp} = c^{\perp\perp}$

since $V(b) = V(c)$, so $c \leq c^{\perp\perp} \leq h_*(q)$, which implies $h(c) \leq hh_*(q) \leq q$, so that $q \in V(h(c))$, showing that $V(h(b)) \subseteq V(h(c))$. The other inclusion is shown similarly; so $V(h(b)) = V(h(c))$. Since a is a d-element and $h(b) \leq hh_*(a) \leq a$, we have $h(c) \leq a$, by Theorem 4.2.6. Therefore $c \leq h_*(a)$, and hence $h_*(a)$ is a d-element. □

Recall that if $\phi\colon A \to B$ is a ring homomorphism, and I is an ideal in B, then the ideal $\phi^{-1}[I]$ of A is called the *contraction* of I by ϕ.

Corollary 4.2.10. *Let $\phi\colon A \to B$ be a ring homomorphism between reduced rings. Then ϕ contracts d-ideals to d-ideals if and only if it contracts minimal prime ideals to d-ideals.*

Proof. This follows from the foregoing theorem because the right adjoint of the frame homomorphism $\mathrm{RId}(\phi)\colon \mathrm{RId}(A) \to \mathrm{RId}(B)$ is the mapping $I \mapsto \phi^{-1}[I]$. □

Now, in analogy with ideals in rings, we formulate the following definition.

Definition 4.2.11. Let L be an algebraic frame, and let $a \in L$. We say an element p of L is *minimal prime over* a if p is a prime element, $a \leq p$, and whenever $a \leq q \leq p$ with q prime, then $q = p$.

Since $\uparrow a$ is a sublocale of L, the prime elements of $\uparrow a$ are precisely the prime elements of L that belong to $\uparrow a$. Consequently,

> *p is minimal prime over a if and only if $p \in \min(\uparrow a)$, that is, if and only if p is a minimal prime element in the frame $\uparrow a$.*

We aim to extend the ring-theoretic result that says a prime ideal that is minimal over a d-ideal is itself a d-ideal (see [33, Theorem 2.5] and [1, Theorem 1.16]). We first need to record some results. Some of these may be known, but we have not found them recorded anywhere in the literature.

Lemma 4.2.12. *If L is an algebraic frame, then, for any $a \in L$, $\uparrow a$ is an algebraic frame. Furthermore, the frame homomorphism $\kappa\colon L \to \uparrow a$ given by $\kappa(x) = a \vee x$ is a coherent map.*

Proof. First, we show that for any $c \in \mathfrak{k}(L)$, $a \vee c$ is compact in $\uparrow a$. Indeed, suppose $a \vee c \leq \bigvee T$ for some $T \subseteq \uparrow a$. Then $c \leq \bigvee T$, and so, by compactness, there is a finite $S \subseteq T$ such that $c \leq \bigvee S$. Since $a \leq \bigvee S$, it follows that $a \vee c \leq \bigvee S$, showing that $a \vee c$ is compact.

Next, we show that the compact elements in $\uparrow a$ generate this frame. So let $z \in \uparrow a$. Since L is an algebraic frame, there is a set $\{c_i \mid i \in I\} \subseteq \mathfrak{k}(L)$ such that $z = \bigvee c_i$. Consequently,

$$z = a \vee z = a \vee \bigvee c_i = \bigvee (a \vee c_i),$$

so that, by what we showed above, z is a join of compact elements in $\uparrow a$. Therefore $\uparrow a$ is an algebraic frame. That κ is a coherent map follows from what we have shown in the first part of the proof. $\square$

We show next that if L has FIP, then so does $\uparrow a$. But before that we recall the following result which appears as [16, Lemma 3.8].

Lemma 4.2.13. *If $h \colon L \to M$ is a surjective coherent map between algebraic frames, then for every $b \in \mathfrak{k}(M)$, there is an $a \in \mathfrak{k}(L)$ such that $h(a) = b$.*

An immediate consequence of this result is that a homomorphic image under coherent map of an algebraic frame with FIP is an algebraic frame with FIP. In particular, if L is an algebraic frame with FIP, then for any $a \in L$, $\uparrow a$ is an algebraic frame with FIP.

For use in the proof of the following result, note that if q is a minimal prime element in an algebraic frame M, then for any $b \in \mathfrak{k}(M)$ with $b \leq q$, there is a $c \in \mathfrak{k}(M)$ such that $b \wedge c = 0$ and $c \not\leq q$. To see this, recall that b compact and $b \leq p$ imply $b^{\perp} \not\leq q$. Now, since M is algebraic, there is a compact element $c \leq b^{\perp}$ such that $c \not\leq q$. Thus $b \wedge c = 0$, and $c \not\leq q$.

In the proof that follows, we shall use the fact that, in any frame, $(x \wedge y)^{\perp\perp} = x^{\perp\perp} \wedge y^{\perp\perp}$.

Theorem 4.2.14. *Let L be an algebraic frame with* FIP*. If z is a d-element in L, and p is minimal prime over z, then p is a d-element.*

Proof. Consider any two compact elements a and b in L with $a^{\perp\perp} = b^{\perp\perp}$ and $a \leq p$. We must show that $b \leq p$. As remarked earlier, p is a minimal prime in the frame $\uparrow z$. Since $z \vee a$ is a compact element in $\uparrow z$ with $z \vee a \leq p$, the minimality of p implies that there is a compact element $u \in \mathfrak{k}(\uparrow z)$ such that $u \wedge (z \vee a) = 0_{\uparrow z} = z$ and $u \nleq p$. Pick $c \in \mathfrak{k}(L)$ such that $u = z \vee c$. Thus,

$$(z \vee c) \wedge (z \vee a) = z \qquad \text{and} \qquad z \vee c \nleq p. \tag{$\dagger$}$$

Now, $a \wedge c \leq z \vee a$ and $b \wedge c \leq z \vee c$, so from the equality in (†) we have $(a \wedge c) \wedge (b \wedge c) \leq z$. Since

$$(a \wedge c)^{\perp\perp} = a^{\perp\perp} \wedge c^{\perp\perp} = b^{\perp\perp} \wedge c^{\perp\perp} = (b \wedge c)^{\perp\perp},$$

and $a \wedge c$ and $b \wedge c$ are compact elements in L, and z is a d-element with

$$z = (z \vee c) \wedge (z \vee a) = z \vee (a \wedge c),$$

from (†), we have that $a \wedge c \leq z$, we deduce from Theorem 4.2.6 that $b \wedge c \leq z$, and hence $b \wedge c \leq p$. Since p is prime, $b \leq p$ or $c \leq p$. The latter is not possible, for if it were, then we would have $z \vee c \leq p$ (as $z \leq p$), contradicting the fact that $z \vee c \nleq p$. Consequently, $b \leq p$, and it follows that p is a d-element in L. □

Corollary 4.2.15. *In any reduced ring, a prime ideal that is minimal over a d-ideal is itself a d-ideal.*

Observe that if p is a prime element in an algebraic frame L and $z \in L$ is such that $z \leq p$, then p is a prime element in the frame $\uparrow z$, and hence, by Lemma 4.2.4(3), there is a $q \in \min(\uparrow z)$ with $q \leq p$. Thus, there is a prime element $q \leq p$ of L such that q is minimal prime over z. Since the meet of d-elements is a d-element, the following corollary follows from Theorem 4.2.14.

Corollary 4.2.16. *An element of an algebraic frame with* FIP *is a d-element if and only if it is a meet of prime d-elements.*

In any frame, maximal elements are prime. This of course does not mean that an element which is maximal with a certain property is necessarily prime. In light of the previous corollary, we have the following result.

Corollary 4.2.17. *In any algebraic frame with* FIP*, every maximal d-element is prime.*

We have yet another characterisation of d-elements which is an analogue of some characterisation of d-ideals in rings. Let us expatiate. In [13], Contessa calls d-ideals "Baer ideals". She also defines a ring homomorphism $\phi\colon R \to S$ to be *R-compatible* if for any $a, b \in R$ with $\mathrm{Ann}(a) = \mathrm{Ann}(b)$ in R, $\mathrm{Ann}(\phi(a)) = \mathrm{Ann}(\phi(b))$ in S. She then proves in [13, Proposition 2.8] that an ideal of a reduced ring R is a Baer ideal if and only if it is the kernel of an R-compatible ring homomorphism having R as a source. In extending this to frames, we first introduce the following definition.

Definition 4.2.18. A coherent map $h\colon L \to M$ between algebraic frames is *nearly skeletal* if for any $a, b \in \mathfrak{k}(L)$ with $a^\perp = b^\perp$, $h(a)^\perp = h(b)^\perp$.

Equalities are frequently harder to show than inequalities. We thus prove a lemma (which we shall use below) which characterises nearly skeletal maps in terms of inequalities.

Lemma 4.2.19. *A coherent map $h\colon L \to M$ is nearly skeletal if and only if whenever $a^\perp \leq b^\perp$, with $a, b \in \mathfrak{k}(L)$, then $h(a)^\perp \leq h(b)^\perp$.*

Proof. If h satisfies the condition stated in terms of inequalities, then it is nearly skeletal, quite easily. Conversely, suppose h is nearly skeletal. Consider any $a, b \in \mathfrak{k}(L)$ with $a^\perp \leq b^\perp$. Then $a^\perp = a^\perp \wedge b^\perp = (a \vee b)^\perp$. Since $a \vee b$ is compact and h is nearly skeletal,

$$h(a)^\perp = (h(a \vee b))^\perp = (h(a) \vee h(b))^\perp = h(a)^\perp \wedge h(b)^\perp,$$

showing that $h(a)^\perp \leq h(b)^\perp$, and thus proving the claim. □

Now here is the characterisation.

Theorem 4.2.20. *An element z of an algebraic frame L with* FIP *is a d-element if and only if $z = h_*(0)$ for some nearly skeletal map $h\colon L \to M$.*

Proof. Assume first that z is a d-element. Consider the map $\kappa\colon L \to {\uparrow}z$ given by $x \mapsto z \vee x$. We observed above that κ is a coherent map. Let us show that it is nearly skeletal. Let $a, b \in \mathfrak{k}(L)$ be such that $a^\perp \leq b^\perp$. To avoid ambiguity, we denote the pseudocomplement of any $w \in {\uparrow}z$, taken in this frame, by w^*. So we must show that $\kappa(a)^* \leq \kappa(b)^*$, that is, $(z \vee a)^* \leq (z \vee b)^*$.

Let w be a compact element in $\uparrow z$ with $w \leq (z \vee a)^*$. Then $w \wedge (z \vee a) = 0_{\uparrow z}$. As observed above, there is a compact element $c \in \mathfrak{k}(L)$ such that $w = z \vee c$. Thus, $(z \vee c) \wedge (z \vee a) = z$, thus $z \vee (c \wedge a) = z$, which implies $c \wedge a \leq z$. At this stage we aim to show that $c \wedge b$ is also below z. Since $a^\perp \leq b^\perp$, we have $b^{\perp\perp} \leq a^{\perp\perp}$, and hence

$$(c \wedge b)^{\perp\perp} = c^{\perp\perp} \wedge b^{\perp\perp} \leq c^{\perp\perp} \wedge a^{\perp\perp} = (c \wedge a)^{\perp\perp}.$$

Now, since $c \wedge a$ and $c \wedge b$ are compact elements in L, and z is a d-element in L, and $c \wedge a \leq z$, it follows from Theorem 4.2.6 that $c \wedge b \leq z$. Thus, in turn, implies

$$w \wedge (z \wedge b) = (z \vee c) \wedge (z \vee b) = z \vee (c \wedge b) = z,$$

showing that, in the frame $\uparrow z$, w misses $z \wedge b$, and hence $w \leq \kappa(b)$, whence $\kappa(a)^* \leq \kappa(b)^*$. Therefore κ is nearly skeletal. But $\kappa_*(0_{\uparrow z}) = z$, since κ_* is the inclusion map $\uparrow z \to L$, so the left-to-right implication is proved.

Conversely, suppose $h\colon L \to M$ is a nearly skeletal map. We must show that $h_*(0)$ is a d-element. Let a and b be compact elements of L with $a^\perp = b^\perp$ and $a \leq h_*(0)$. Then $h(a) = 0$, which implies $1 = h(a)^\perp = h(b)^\perp$, since h is nearly skeletal. Thus $h(b) = 0$, and hence $b \leq h_*(0)$. Therefore $h_*(0)$ is a d-element. $\square$

We end with an example that shows that sending minimal primes to minimal primes with the right adjoint is strictly stronger than sending minimal primes to d-ideals. The example was suggested to the author by her dissertation supervisor.

First, we need some bit of background. Recall that a ring A is *von Neumann regular* if, for every $a \in A$ there is an $x \in A$ such that $a^2x = a$. A completely regular Hausdorff space X is called a *P-space* if the ring $C(X)$ is von Neumann regular. On the other hand, a completely regular space X is called an *almost P-space* [25] if every G_δ-set in X has a dense interior. Every P-space is an almost P-space. An ideal I of $C(X)$ is called a *z-ideal* [19] if for any two functions $f, g \in C(X)$,

$$f^{-1}(0) = g^{-1}(0) \text{ and } f \in I \quad \Longrightarrow \quad g \in I.$$

In [28], Martínez and Zenk prove that X is an almost P-space if and only if every z-ideal in $C(X)$ is a d-ideal.

Example 4.2.21. Let X be an almost P-space which is not a P-space. Since a reduced ring is von Neumann regular if and only if every maximal ideal in it is minimal prime, $C(X)$ has a maximal ideal which is not minimal prime. Therefore if we let $L = \mathrm{RId}(C(X))$, then there is a maximal element m in L which is not minimal prime. Consider the frame homomorphism $\kappa\colon L \to {\uparrow}m$, given by $\kappa(x) = m \vee x$. Its right adjoint is, as is well known, the inclusion map ${\uparrow}m \hookrightarrow L$. In ${\uparrow}m$ (which is, in fact, the two-element frame), there are only two d elements, m and the top element. Since m (when viewed as a radical ideal in $C(X)$) is a maximal ideal, it is a z-ideal, and hence a d-ideal as X is an almost P-space. Thus, m is a d-element in L. Therefore κ_* sends d-elements to d-elements. However, since m is not minimal prime, κ_* does not send minimal primes to minimal primes.

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