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# Generalization and Improvement of the Levenshtein Lower Bound for Aperiodic Correlation

Fabien Arlery, Uy Hour Tan<sup>ID</sup>, and Olivier Rabaste

**Abstract**—This paper deals with lower bounds on aperiodic correlation of sequences. It intends to solve two open questions. The first one is on the validity of the Levenshtein bound for a set of sequences other than binary sequences or those over the roots of unity. Although this result could be *a priori* extended to polyphase sequences, a formal demonstration is presented here, proving that it does actually hold for these sequences. The second open question is on the possibility to find a bound tighter than Welch's, in the case of a set consisting of two sequences  $M = 2$ . By including the specific structure of correlation sequences, a tighter lower bound is introduced for this case. Besides, this method also provides in the cases  $M = 3$  and  $M = 4$  a tighter bound than the up-to-now tightest bound provided by Liu *et al.*

**Index Terms**—Aperiodic correlation lower bound, Levenshtein bound, Welch bound.

## I. INTRODUCTION

**A**PERIODIC correlation arises as the output of the matched filter in many applications, such as asynchronous Direct-Sequence Spread-Spectrum (DSSS) systems in digital communications, or pulse radars [1]. It is usually of interest to consider sequences with the lowest possible sidelobe level. For instance, in radar applications, a low-sidelobe sequence may avoid a weak target to be buried in the sidelobes generated by a stronger one. Depending on the applications, such as the MIMO radar [1], it may be necessary to consider families of several sequences with low aperiodic auto- and cross- correlations. Lower bounds on the maximum aperiodic correlation sidelobes are thus interesting to determine the best performance a system could achieve.

The computation of lower bounds on a set of sequences is a recurrent topic in [2]–[4]. Two different problematics can be found: some articles consider the problem of computing lower bounds on codebooks, which consists in fact of computing the

lower bound of any inner product between two sequences in the codebook. This optimal codebook is *not* the concern of this paper. Here is considered on the contrary the problem of computing lower bounds on correlation sequences, in other words lower bounds on the inner products between any two sequences among the set with any possible delay shift. This problem is thus completely different from the codebook problem, and in that case no family set is known to meet the existing bounds, which are likely to be not tight enough yet.

Besides, the first lower bound on aperiodic correlation was proposed by Welch [2] for sets of  $M$  sequences of length  $N$  with identical energy. Its proof is based on the computation of the maximum value taken by the inner products between any two pairs of sequences among a given set. A new bound was proposed only almost 25 years later by Levenshtein [3], [5] for the specific case of binary sequences. This bound, based on the introduction of a weight vector, was proved to be tighter than the Welch bound for any  $M \geq 4$  and  $N \geq 2$ . It was shown soon after that the Levenshtein bound also holds for sequences over the roots of unity [6]. Note that both proofs require first to enumerate the number of all possible sequences in the considered set (the set of binary sequences or the set of sequences over the roots of unity) and second to quantify the minimum distance between two different sequences, which is not possible when considering the set of unimodular sequences, as it contains an infinite number of sequences and hence presents a minimum distance equal to zero. Finally, Liu *et al.* [7] recently proposed a new weight vector that provides a tighter Levenshtein bound for  $M = 3$ ,  $N \geq 3$  and any  $M \geq 4$ ,  $N \geq 2$ . However, as stated both in [5] and [7], the Welch bound remains up to now the tightest bound in the case  $M = 2$ , since a better bound cannot be provided by the Levenshtein method in that particular case. Thus two problems remain open: first, the validity of the Levenshtein bound to more general sequence sets than sequences over the roots of unity, and second, the possibility to find a tighter bound than the Welch bound in the case  $M = 2$ .

When studying the proofs of the Welch and the Levenshtein bounds, it can be noticed that the very specific structure of the sets used for the aperiodic correlation is not fully exploited. Based on this observation, this paper provides new results that enable to provide answers to the two open questions above. First, we demonstrate that the Levenshtein bound holds for all unimodular sequences. This result is quite expected, as a natural extension of Boztaş's work [6]. However, in his proof appears the cardinality of the set of sequences over the roots of

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unity for a given  $N$ . In the case of unimodular sequences, this cardinality tends to infinity, so that the extension of Boztaş' proof may not be that straightforward. Thus in this paper is presented a formal proof that does not require this cardinality. Interestingly this proof mixes Welch and Levenshtein methods, removes the usage of the Cauchy-Schwarz inequality, and exploits to some extent the structure provided by the aperiodic correlation. In particular, this proof does not require the enumeration of the set of unimodular sequences and the minimum distance between any two sequences. Second, we show that this specific structure enables to tighten the upper bound over the energy of all inner products used in Welch's proof and thus to get a tighter bound in the case  $M = 2$ . We also show that it provides in the cases  $M = 3$  and  $M = 4$  a tighter bound than the tightest Levenshtein bound provided in [7].

This article is organized as follows. Section II provides a review on the well-known Welch and Levenshtein bounds on the aperiodic correlation. A generalization of the latter for unimodular sequences is presented in Section III. Section IV tightens this Levenshtein bound, introducing a parameter on the number of delays considered. This improvement is illustrated in Section V in several cases according to the number of sequences  $M$ . Finally, some proofs are detailed in the appendices.

*Notation:* In the following, bold letters designate matrices and vectors.  $(\cdot)^*$ ,  $(\cdot)^T$  and  $(\cdot)^H$  denote the conjugate, the transpose and the transpose conjugate operator, respectively.  $\mathbf{0}_{m,n}$  denotes the null matrix (a matrix where all the entries are equal to zero) of size  $m \times n$ .  $\mathbf{I}_n$  is the identity matrix of size  $n$ .  $\|\cdot\|_F$  stands for the Frobenius norm. For an  $m \times n$  matrix  $\mathbf{A}$ , it is defined by  $\|\mathbf{A}\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n |a_{i,j}|^2}$ . Finally, circulant matrices are defined through a map denoted  $\text{circ}$ , and are specified by a vector  $\mathbf{x} = [x_1, \dots, x_n]$  of length  $n$ :

$$\left\{ \begin{array}{l} \mathbb{C}^n \rightarrow \mathbb{C}^{n \times n} \\ \mathbf{x} \mapsto \text{circ}(\mathbf{x}) = \begin{bmatrix} x_1 & x_2 & \dots & x_{n-1} & x_n \\ x_n & x_1 & x_2 & & x_{n-1} \\ \vdots & x_n & x_1 & \ddots & \vdots \\ x_3 & & \ddots & \ddots & x_2 \\ x_2 & x_3 & \dots & x_n & x_1 \end{bmatrix} \end{array} \right.$$

## II. REVIEW ON EXISTING BOUNDS

This paper focuses on the calculation of lower bounds on the maximum sidelobe level of auto- and cross-correlation sidelobes for unimodular sequences. Let  $\{\mathbf{x}^m\}_{m \in \{1, M\}}$  be a set of  $M$  sequences of length  $N$ , such that any  $n$ -th entry of any  $m$ -th sequence satisfies  $|x_n^m|^2 = 1/N$ . Thus the energy of each sequence  $(\mathbf{x}^m)_{m \in \{1, M\}}$ , denoted by  $E_x$ , is constant and equal to 1.

The aperiodic cross-correlation between two sequences  $\mathbf{x}^{m_1}$  and  $\mathbf{x}^{m_2}$  can be defined as:

$$\theta_{\mathbf{x}^{m_1}, \mathbf{x}^{m_2}}(k) = \sum_{n=1}^N x_n^{m_1} (x_{n+k}^{m_2})^*, \quad \text{for } |k| < N, \quad (1)$$

where we set  $x_l^m = 0$  for any  $l \leq 0$  or  $l > N$ . The aperiodic autocorrelation is simply obtained for  $m_1 = m_2$  and will be denoted by  $\theta_{\mathbf{x}^{m_1}}(k)$ .

Several lower bounds have been developed on the maximum sidelobe level of the auto- and cross-correlations. This maximum level will be denoted in this paper by  $\theta_{\max}$  and is provided by:

$$\theta_{\max}^2 = \max \left\{ \max_{\substack{|k| < N \\ m_1 \neq m_2}} |\theta_{\mathbf{x}^{m_1}, \mathbf{x}^{m_2}}(k)|^2, \max_{\substack{k \neq 0 \\ m_1}} |\theta_{\mathbf{x}^{m_1}}(k)|^2 \right\}. \quad (2)$$

The *Peak-to-Sidelobe Level*, denoted by PSL, and used hereafter for comparison purpose, is defined by:

$$\text{PSL} = \frac{\theta_{\max}^2}{E_x^2}, \quad (3)$$

and is simply equal here to  $\text{PSL} = \theta_{\max}^2$  since  $E_x^2 = 1$ .

The most well-known bounds were provided by Welch [2] and Levenshtein [3]. The Welch bound, valid for any family of unit energy sequences, is provided by:

$$\text{PSL} \geq \frac{M-1}{M(2N-1)-1}. \quad (4)$$

More recently, a tighter bound has been established by Levenshtein [3]. It introduces a weight vector  $\mathbf{w}$  of length  $2N-1$  — applied on each correlation sequence — that should satisfy the following *weighting condition*:

$$\left\{ \begin{array}{l} \sum_{i=1}^{2N-1} w_i = 1, \quad \text{with } w_i \geq 0 \text{ if } i \in \{1, 2N-1\}, \\ w_i = 0 \text{ otherwise.} \end{array} \right. \quad (5)$$

Initially determined for binary sequences [3], the Levenshtein bound was shown in [6] to be valid for sequences over the roots of unity. This bound is expressed by:

$$\text{PSL} \geq \frac{1}{N^2} \left( N - \frac{Q_{2N-1} \left( \mathbf{w}, \frac{N(N-1)}{M} \right)}{1 - \frac{1}{M} \sum_{i=1}^{2N-1} w_i^2} \right), \quad (6)$$

where  $\mathbf{w}$  is any weight vector that satisfies the weighting condition (5), and:

$$\left\{ \begin{array}{l} Q_{2N-1}(\mathbf{w}, a) = a \sum_{i=1}^{2N-1} w_i^2 + \sum_{s,t=1}^{2N-1} l_{s,t,N} w_s w_t, \\ l_{s,t,N} = \min(|s-t|, 2N-1-|s-t|). \end{array} \right. \quad (7)$$

If  $w_i = 1/K$  for  $i \leq K$ , and  $w_i = 0$  otherwise, it has been shown that, for all  $K \in \{1, N\}$ :

$$\text{PSL} \geq \frac{1}{N^2} \left( \frac{3NMK - 3N^2 - MK^2 + M}{3(KM-1)} \right). \quad (8)$$

An optimal choice of the parameter  $K$  in the right-hand side of (8) further provides:

$$\text{PSL} \geq \frac{1}{N^2} \left( N - \frac{2N}{\sqrt{3M}} \right) \quad \text{when } M \geq 3. \quad (9)$$

Besides, the previous theorem induces a minimization problem on the quadratic form  $Q_{2N-1}(\mathbf{w}, a)$  under the weighting condition. Levenshtein tackled it in [5], and obtained a tighter bound:

$$\text{PSL} \geq \frac{1}{N^2} \left( N - \left\lceil \frac{\pi N}{\sqrt{8M}} \right\rceil \right) \quad \text{when } 5 \leq M \leq N^2. \quad (10)$$

In [7]–[9], specific weight vectors  $\mathbf{w}$  are used in order to obtain tighter Levenshtein bounds.

### III. GENERALIZATION OF THE LEVENSHTEIN BOUND

In this section, we establish that the Levenshtein bound holds for any set of unimodular sequences. This generalization of Levenshtein's result is performed in two steps: first a calculation of upper and lower bounds on the Frobenius norm of an auto- and cross-correlation matrix, and second the deduction of the lower bound on the PSL.

Let  $\{\mathbf{x}^m\}_{m \in \{1, M\}}$  be a set of  $M$  unimodular sequences of length  $N$  with  $|x_n^m|^2 = 1/N$  for each  $n \in \{1, N\}$ . Consider the matrix  $\mathbf{X}$  of size  $[M(2N-1)] \times [2N-1]$ :

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}^1 \\ \mathbf{X}^2 \\ \vdots \\ \mathbf{X}^M \end{bmatrix} \quad (11)$$

with  $\mathbf{X}^m, m \in \{1, M\}$ , a square matrix of order  $2N-1$  defined by:

$$\mathbf{X}^m = \begin{bmatrix} x_1^m & x_2^m & \cdots & x_{N-1}^m & x_N^m & 0 & 0 & \cdots & 0 \\ 0 & x_1^m & x_2^m & \cdots & x_{N-1}^m & x_N^m & 0 & 0 & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & x_1^m & x_2^m & \cdots & x_{N-1}^m & x_N^m \\ x_N^m & 0 & \cdots & \cdots & 0 & x_1^m & x_2^m & \cdots & x_{N-1}^m \\ x_{N-1}^m & x_N^m & 0 & \cdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & x_1^m & x_2^m & \vdots \\ x_2^m & \cdots & x_{N-1}^m & x_N^m & 0 & \cdots & \cdots & 0 & x_1^m \end{bmatrix}.$$

Remark that  $\mathbf{X}^m$  is circulant and specified by the vector  $[\mathbf{x}^m, \mathbf{0}_{N-1,1}]^T$ .

Denote by  $X_i^m, i \in \{1, 2N-1\}$  the  $i$ -th row of  $\mathbf{X}^m$ , and by  $X_{i,q}^m$  its  $q$ -th element. It appears that every auto- and cross-correlation value between two sequences of  $\{\mathbf{x}^m\}$  is reduced to a scalar product of some vectors  $X_i^m$ . These values are contained in the matrix  $\mathbf{R} = \mathbf{X}\mathbf{X}^H$  of order  $M(2N-1)$ .

If necessary, a weighted version of the row vectors may be considered, by defining the following matrix:

$$\tilde{\mathbf{X}}_i^m := X_i^m \sqrt{w_i} \quad i \in \{1, 2N-1\} \text{ and } m \in \{1, M\} \quad (12)$$

where the weights  $w_i$  satisfies the weighting condition (5).

The associated matrices  $\tilde{\mathbf{R}}$  and  $\tilde{\mathbf{X}} \in \mathbb{C}^{[M(2N-1)] \times [2N-1]}$  are therefore expressed by:

$$\tilde{\mathbf{R}} = \tilde{\mathbf{X}}\tilde{\mathbf{X}}^H \quad \text{with } \tilde{\mathbf{X}} = \begin{bmatrix} \tilde{\mathbf{X}}^1 \\ \tilde{\mathbf{X}}^2 \\ \vdots \\ \tilde{\mathbf{X}}^M \end{bmatrix}. \quad (13)$$

In the particular case where the weights  $w_i$  are non zero only for the first  $K$  values, the matrix  $\tilde{\mathbf{R}}$  then contains only all auto- and cross-correlation values of the set up to the  $K$ -th lag. This matrix  $\tilde{\mathbf{R}}$  satisfies the following lemma:

*Lemma 1 (Upper Bound):* Under the above-mentioned hypothesis, the Frobenius norm of the matrix  $\tilde{\mathbf{R}}$  can be upper-bounded by:

$$\|\tilde{\mathbf{R}}\|_F^2 \leq M^2 \theta_{\max}^2 + M \left(1 - \theta_{\max}^2\right) \sum_{i=1}^{2N-1} w_i^2 \quad (14)$$

*Proof:* This result is similar to [5, Lemma 1], and is also obtained by a similar proof.  $\square$

A lower bound on the squared Frobenius norm of  $\tilde{\mathbf{R}}$  can also be computed.

*Lemma 2 (Lower Bound):* Under the above-mentioned hypothesis, the Frobenius norm of the matrix  $\tilde{\mathbf{R}}$  can be lower-bounded by:

$$\|\tilde{\mathbf{R}}\|_F^2 \geq \frac{M^2}{N^2} \left( N - \sum_{s,t=1}^{2N-1} l_{s,t,N} w_s w_t \right) \quad (15)$$

with  $l_{s,t,N} = \min(|t-s|, 2N-1-|t-s|)$

*Proof:* See Appendix A. This proof mixes methods used by Welch and Levenshtein but, instead of using the Cauchy-Schwarz inequality, it exploits to some extent the particular structure provided by the aperiodic correlation and the constant modulus constraint. Besides, it removes the requirement of sequences over the roots of unity.  $\square$

From these lemmas can then easily be deduced a lower bound on the PSL, akin to Levenshtein's proof:

*Theorem 1:* For any set of  $M$  unimodular sequences of length  $N$ , and with any weight vector  $\mathbf{w}$  that satisfies the weighting condition (5), a lower bound on the Peak-to-Sidelobe Level is given by:

$$\theta_{\max}^2 \geq \frac{1}{N^2} \left[ N - \frac{Q_{2N-1} \left( \mathbf{w}, \frac{N(N-1)}{M} \right)}{1 - \frac{1}{M} \sum_{i=1}^{2N-1} w_i^2} \right] \quad (16)$$

with  $Q_{2N-1}(\mathbf{w}, a) = a \sum_{i=1}^{2N-1} w_i^2 + \sum_{s,t=1}^{2N-1} l_{s,t,N} w_s w_t$

$$l_{s,t,N} = \min(|s-t|, 2N-1-|s-t|).$$

*Proof:* Combining Lemma 1 and Lemma 2 yields the desired result (16).  $\square$

This expression is identical to the Levenshtein bound, but it is proved here that it is still valid for any set of unimodular polyphase sequences — the Levenshtein bound

was originally meant for binary sequences and those over the roots of unity. This suggests that any bound obtained from Levenshtein expression using a specific weight vector also holds for unimodular sequences. In particular, optimal weight vectors considered by Levenshtein [5] and Liu *et al.* [7] do.

#### IV. IMPROVEMENT OVER THE EXISTING BOUNDS

Theorem 1 states that the Levenshtein bound is valid for any set of unimodular sequences. However, this lower bound does not take into account additional information that can be extracted from the specific structure of aperiodic auto- and cross-correlations for a unimodular sequence, *e.g.*, the last delay satisfies:

$$|\theta_{x^l, x^m}(k)|^2 = 1/N^2, \text{ for } |k| = N - 1, \quad \forall(l, m), \quad (17)$$

and, using the Cauchy-Schwarz inequality, the  $d$ -th last delay satisfies (for  $d \in \{1, N\}$  and  $d \neq N$  if  $l = m$ ):

$$|\theta_{x^l, x^m}(k)|^2 \leq (d/N)^2 \text{ for } |k| = N - d, \quad \forall(l, m). \quad (18)$$

These properties have already been exploited in [10], but in another context (an estimation of a gap between an aperiodic lower bound and a periodic one). Here, they enable to provide a new upper bound, as stated in the following lemma:

*Lemma 3 (Upper Bound Considering the  $D$  Last Delays):*

$$\begin{aligned} \|\tilde{\mathbf{R}}\|_F^2 &\leq M^2 \theta_{\max}^2 + M \left(1 - \theta_{\max}^2\right) \sum_{i=1}^{2N-1} w_i^2 \\ &\quad - \sum_{d=1}^D \left( M^2 \left( \theta_{\max}^2 - \frac{d^2}{N^2} \right) \sum_{\substack{i,j=1 \\ i,j,N=N-d}}^{2N-1} w_i w_j \right). \end{aligned} \quad (19)$$

*Proof:* See appendix C.  $\square$

Clearly for  $D = 0$ , this upper bound is equal to the Levenshtein upper bound provided in Lemma 1. But it can be proved to be tighter if it satisfies the following property:

$$\exists (w_i, w_j) \text{ s.t. } \sum_{\substack{i,j=1 \\ i,j,N=N-d}}^{2N-1} w_i w_j \neq 0 \quad \text{with } \theta_{\max}^2 \geq \frac{d^2}{N^2}. \quad (20)$$

Minimizing the right-hand-side of (19) with respect to  $D$  — denote the optimum by  $D_{\text{opt}}$  — insures to provide an upper bound at least equal or tighter than the Levenshtein bound.

This in turn enables us to determine a more general bound on the aperiodic correlation that takes into account the additional information on the  $D$  last delays of the auto- and cross-correlation. This bound is provided in the following theorem, with the help of matrices  $\mathbf{A}_d$  and  $\mathbf{L}$  such that:

$$\begin{cases} \sum_{\substack{i,j=1 \\ i,j,N=N-d}}^{2N-1} w_i w_j = \mathbf{w}^T \mathbf{A}_d \mathbf{w}, \\ \sum_{s,t=1}^{2N-1} l_{s,t,N} w_s w_t = \mathbf{w}^T \mathbf{L} \mathbf{w}. \end{cases} \quad (21)$$

*Clarification Example :* Set  $N = 3$ . The matrix  $\mathbf{L}$  is of size  $5 \times 5$  ( $2N - 1 \times 2N - 1$  in fact) and is defined as follows :

$$\mathbf{L} = \begin{bmatrix} 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 \end{bmatrix} = \text{circ}([0, 1, 2, 2, 1]^T). \quad (22)$$

As  $\mathbf{A}_d$  matrices are related :

$$(\mathbf{A}_d)_{i,j} = \begin{cases} 1 & \text{if } L_{i,j} = N - d, \\ 0 & \text{otherwise,} \end{cases} \quad (23)$$

it comes that :

$$\begin{aligned} \mathbf{A}_1 &= \text{circ}([0, 0, 1, 1, 0]^T), \\ \mathbf{A}_2 &= \text{circ}([0, 1, 0, 0, 1]^T). \end{aligned} \quad (24)$$

*Theorem 2:* For any set of  $M$  unimodular sequences of length  $N$ , and any weight vector  $\mathbf{w}$  that satisfies the weighting condition (5), a lower bound on the Peak-to-Sidelobe Level is given by:

$$\begin{aligned} \theta_{\max}^2 &\geq \max_D \frac{1}{N^2} \left[ N - \frac{\tilde{Q} \left( \mathbf{w}, \frac{N(N-1)}{M}, \sum_{d=1}^D (d^2 - N) \mathbf{A}_d \right)}{1 - \mathbf{w}^T \left( \frac{1}{M} \mathbf{I} + \sum_{d=1}^D \mathbf{A}_d \right) \mathbf{w}} \right] \\ &\quad \text{with } \tilde{Q}(\mathbf{w}, a, \mathbf{B}) = \mathbf{w}^T (a \mathbf{I} + \mathbf{B} + \mathbf{L}) \mathbf{w}. \end{aligned}$$

*Proof:* This expression is directly obtained by combining the upper bound (19) and the lower bound provided by Lemma 2.  $\square$

Since this new lower bound on aperiodic correlation has been obtained with a single change — on the upper bound of  $\|\tilde{\mathbf{R}}\|_F^2$ , while keeping the lower bound — it implies that if there exists a value  $D$  for which the upper bound (19) is tighter than the one provided by Lemma 1 — for a given weight vector  $\mathbf{w}$  — that satisfies (20), then the resulting lower bound on aperiodic correlation provided by Theorem 2 is also tighter than the Levenshtein bound for the same weight vector.

*Corollary 1:* Let the Levenshtein bound be denoted by  $B_{\text{Lev}}$  for given  $M$  and  $N$ , and a given weight vector  $\mathbf{w}$ . If there exists  $d$  such that  $\mathbf{w}$  satisfies (20) and  $B_{\text{Lev}} \geq d^2/N^2$ , then the bound provided by Theorem 2 is tighter than the Levenshtein bound for the same  $\mathbf{w}$ .

Corollary 1 presents a sufficient condition on the value  $d$  in order to get a tighter bound than Levenshtein's. If such a value exists, and if the aforementioned conditions are verified, it implies the following :

$$B_{\text{Lev}} \geq \frac{d^2}{N^2} \implies d \in [0, \lfloor N \sqrt{B_{\text{Lev}}} \rfloor], \quad (25)$$

where  $\lfloor \cdot \rfloor$  denotes the floor function. In other words, each value that lies in this interval assures a tighter bound. One of these values, denoted  $D_{\text{max}}$ , provides the tightest: a rough approximation of it is the maximum value  $d$  for which  $B_{\text{Lev}} \geq d^2/N^2$ , *i.e.*  $D_{\text{max}} \approx \lfloor N \sqrt{B_{\text{Lev}}} \rfloor$ . However, note that the previous cited interval does not include all the values of interest (see Figure 1 for instance).

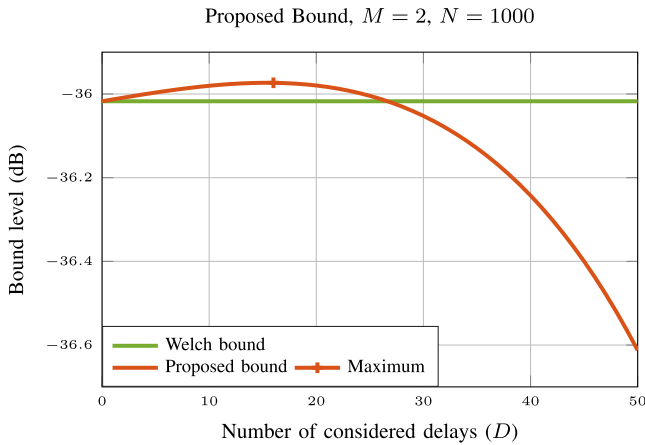


Fig. 1. Proposed bound versus last delays  $D$ .

As explained in [11], an open question remains on the search for set of sequences that achieve the Levenshtein bound. At this point, a first partially negative answer can be given, thanks to Corollary 1. If the latter can indeed be applied, the Levenshtein bound is proved not to be the tightest, meaning that it cannot logically be reached by any set. We will see thereafter that it is the case for instance for  $M = 2, 3$  or  $4$ .

## V. COMPARISON TO THE WELCH AND THE LEVENSHTEIN BOUNDS

The bound provided by Theorem 2 depends on several parameters, such as the number of sequences  $M$  and their length  $N$ , but also on the choice of the weight vector  $\mathbf{w}$  and the number of considered delays  $D$ . Several results have been given in the literature according to the number of sequences  $M$  [5], [7], [8]. This section studies the behavior of this new bound, according to this criterion. In particular, we will show here that the proposed bound is tighter than the Welch bound in the case  $M = 2$  — an achievement that is not possible with the Levenshtein bound — and tighter than the (up-to-now) tightest Levenshtein bound in the case  $M = 3$  and  $M = 4$ .

### A. $M = 2$ Case

It has been proved that the tightest Levenshtein bound is obtained with constant weight vectors  $w_i = 1/(2N - 1)$  for  $i = 1, \dots, 2N - 1$  [5]. In this case, the Levenshtein bound is equivalent to the Welch bound, and:

$$B_{\text{Welch}} = \frac{1}{4N - 3}. \quad (26)$$

However, this weight vector clearly satisfies Condition (20). It means that there exists  $d \geq 1$  (for instance,  $d = 1$  works) such that  $B_{\text{Welch}} \geq d^2/N^2$  for  $N \geq 3$ . Applying Corollary 1 thus proves that the proposed bound is tighter than the Welch bound for  $M = 2$ , an improvement that cannot be achieved by the Levenshtein bound. An explicit expression of this bound is:

$$\theta_{\text{max}}^2 \geq \frac{1}{3N^2} \frac{3MN^2 - 3N^2 - MD(D+1)(2D+1)}{M(2N - 2D - 1) - 1}, \quad (27)$$

and the corresponding proof is given in Appendix D.

TABLE I  
LOWER BOUND COMPARISON FOR SEVERAL VALUES OF  $N$ , CONSTANT WEIGHT,  $M = 2$

$N$	$D_{\text{max}}$	Proposed (dB)	Welch (dB)
10	1	-15.36	-15.68
50	3	-22.77	-22.94
100	5	-25.86	-25.99
500	11	-32.94	-33.00
1000	15	-35.97	-36.02
10000	50	-46.01	-46.02

Fig. 1 shows the behavior of the proposed bound with  $M = 2$  and  $N = 1000$  according to  $D$ . A better bound, compared to the Welch bound, is observed considering 1 to 25 delays, while the optimal bound is achieved with  $D_{\text{max}} = 15$ . As above-mentioned, this value can be approximately computed:

$$\frac{15^2}{N^2} \approx 2, 25 \cdot 10^{-4} < B_{\text{Welch}} \approx 2, 5 \cdot 10^{-4} < \frac{16^2}{N^2} \approx 2, 56 \cdot 10^{-4}.$$

Table I compares the Welch bound and the proposed bound for several sequence lengths.

### B. $M = 3$ and $M = 4$ Cases

As shown by Liu *et al.* [7], the “Positive–Cycle–of–a–Sine–Wave” weight vector leads to the *up-to-now* tightest Levenshtein bound for  $M \geq 3, N \geq 3$  and  $M \geq 4, N \geq 2$  (which is in particular also tighter than the Welch bound). These weights are defined by (with  $K \in \{2, 2N - 1\}$ ):

$$w_i = \begin{cases} \tan\left(\frac{\pi}{2K}\right) \sin\left(\frac{\pi(i-1)}{K}\right) & \text{if } i \in \{1, K\}, \\ 0 & \text{otherwise.} \end{cases} \quad (28)$$

Using these weights, the associated bound, defined in [7], reaches its maximum for a certain value of  $K$ , denoted  $K_{\text{opt}}$ . Actually, it is possible to show that our bound is even tighter, using again Corollary 1 with the “Positive–Cycle–of–a–Sine–Wave” weights. In turn, to prove that Corollary 1 is satisfied in that case, it is sufficient to check that  $K_{\text{opt}}$  is greater than  $N$ , which directly implies Condition (20).

As defined in [7], the Levenshtein bound using Liu’s weight is given by:

$$B(K) = \frac{1}{N^2} \left[ N - \frac{((N-1)N - \frac{M}{2})K \tan^2(\frac{\pi}{2K}) + \frac{MK}{2}}{2M - K \tan^2(\frac{\pi}{2K})} \right] \quad (29)$$

for  $K \in \{2, N\}$  (its expression is different for  $K > N$  and is given in [7]). It can be shown that this bound is an increasing function of  $K$  on its definition interval, for  $M = 3$  and  $M = 4$ . Therefore,  $K_{\text{opt}}$  is necessarily greater or equal to  $N$  so that Condition (20) is satisfied and Corollary 1 can be applied. However, further analysis should be performed to draw a conclusion for  $M \geq 5$ , as the optimum value  $K_{\text{opt}}$  may be smaller than  $N$ .

Fig. 2 compares the Levenshtein bound, the proposed bound —both with Liu’s weight [7] — and the Welch bound,

TABLE II  
LOWER BOUND COMPARISON FOR SEVERAL VALUES  
OF  $N$ , LIU'S WEIGHT [7],  $M = 3$

$N$	$D_{\max}$	Proposed (dB)	Levenshtein (dB)
10	1	-14.15	-14.28
50	4	-21.36	-21.40
100	6	-24.40	-24.43
500	13	-31.42	-31.43
1000	18	-34.43	-34.44
10000	58	-44.442	-44.445

as functions of  $K$ , in the case  $M = 3$ . The number of delays to consider in the computation of our proposed bound was estimated with the maximum value of  $D$  for which  $B_{\text{Liu}} \geq D^2/N^2$ . In this figure is found again that optimal value  $K_{\text{opt}}$  is indeed greater than the length of the sequences. Table II gives some values of these bounds, according to several sequence lengths  $N$ . The case  $M = 4$  is quite similar.

According to the application, it may be needed to consider every delay, *i.e.*,  $K = 2N - 1$ . In that case, Liu *et al.* [9] have also developed another weight vector:

$$w_i = \frac{1}{2N-1} \left( 1 + \frac{\cos\left(\frac{2\pi(i+q)}{2N-1}\right)}{\cos\left(\frac{\pi}{2N-1}\right)} \right), \quad i \in \{1, 2N-1\}, \quad (30)$$

for any integer  $q$ . Fig. 2 also compares the Levenshtein bound and the proposed one with these weights. The ‘‘Positive-Cycle-of-a-Sine-Wave’’ still provides a better global bound with a wise selection of the parameter  $K$  but, if all delays are considered, tide is turned. That is a *not-so-surprising* result, as that weight has been precisely defined for that case. That said, the proposed bound remains tighter than Levenshtein's, whatever weights.

## VI. CONCLUSION

Two contributions have been given in this paper.

- A generalization of the Levenshtein bound. While Levenshtein [5] and Boztas [6] have proved its validity to a set of binary sequences and over the roots of unity respectively, this article showed that it also holds for a set of unimodular sequences.
- An improvement of the Levenshtein bound. This improvement has been obtained by taking into account additional informations that can be extracted from the specific structure of the aperiodic auto- and cross-correlation sequences, and more precisely by refining the upper bound using the  $D$  last delays. It allows to tighten the existing Levenshtein bound for  $M = 2$ ,  $M = 3$  and  $M = 4$ .
- Some work remains on the case  $M \geq 5$ . Actually, the Levenshtein bound has been tightened, but in a negligible and an unnoticeable way. However, any (yet to be found) weight that will improve the Levenshtein bound in that case will also improve the present one.

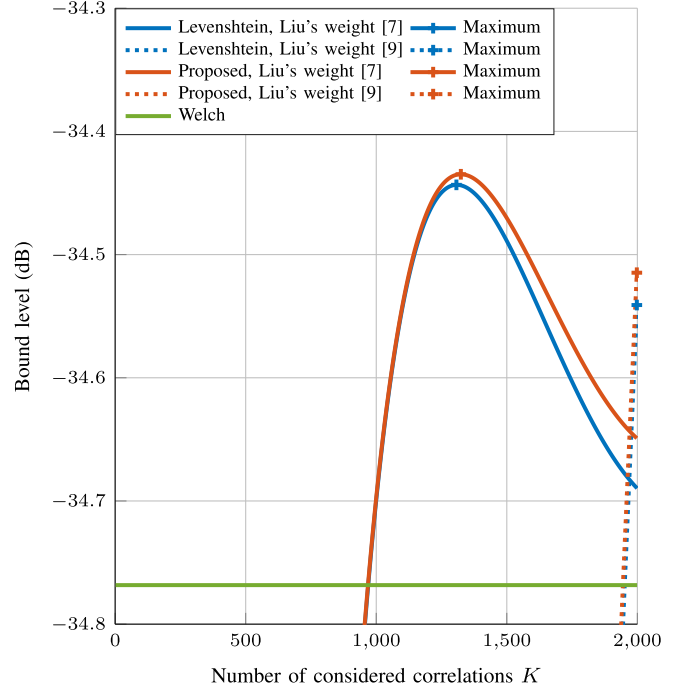


Fig. 2. Comparison of the Levenshtein bound and the proposed bound, both using Liu's weights [7], [9],  $M = 3$ .

- In any case, it is worth insisting on the fact that any weight vector that satisfies Corollary 1 gives a Levenshtein bound that can be tightened by Theorem 2.

In practical cases, this bound can easily be extended considering some constraints on the spectrum, the mainlobe width, *etc.*

## APPENDIX A

### PROOF OF THE LOWER BOUND OF $\|\tilde{\mathbf{R}}\|_F^2$

$\tilde{\mathbf{R}}$  can be developed and lower-bounded as:

$$\begin{aligned} \|\tilde{\mathbf{R}}\|_F^2 &= \|\tilde{\mathbf{X}}\tilde{\mathbf{X}}^H\|_F^2 = \|\tilde{\mathbf{X}}^H\tilde{\mathbf{X}}\|_F^2 \\ &= \sum_{k,k'=1}^{2N-1} \left| \sum_{m=1}^M \sum_{i=1}^{2N-1} X_{i,k}^m (X_{i,k'}^m)^* w_i \right|^2 \\ &\geq \sum_{k=1}^{2N-1} \left( \sum_{m=1}^M \sum_{i=1}^{2N-1} |X_{i,k}^m|^2 w_i \right)^2 \end{aligned} \quad (31)$$

where the last inequality is obtained by removing all terms  $k \neq k'$  in the first summation.

By construction, the structure of the different  $X^m$  are similar. Thus, using the constant modulus property, it comes that  $|X_{i,k}^m|^2 = |X_{i,k}^l|^2$  for any  $l, m, i, k$  ( $|X_{i,k}^m|^2$  can only be equal to 0 or  $1/N$ ). Inserting this into previous inequality, we obtain:

$$\|\tilde{\mathbf{R}}\|_F^2 \geq M^2 \sum_{k=1}^{2N-1} \left( \sum_{i=1}^{2N-1} |X_{i,k}^m|^2 w_i \right)^2. \quad (32)$$

At that step, both Welch's and Levenshtein's proofs use the Cauchy-Schwarz inequality. We do not resort to this inequality

here but rather exploit the specific structure of the matrix  $\mathbf{X}^m$  in the case of aperiodic correlations. Indeed, it appears, for each column  $k$ , that there are exactly  $N$  entries  $X_{i,k}^m$  that are non zero. Exploiting this structure and the fact that the square modulus of the non zero entries is equal to  $1/N$ , it directly comes the following.

$$\sum_{i=1}^{2N-1} |X_{i,k}^m|^2 w_i = \begin{cases} \sum_{i=1}^k w_i/N + \sum_{i=N+k}^{2N-1} w_i/N & \text{if } k \leq N-1 \\ \sum_{i=k-N+1}^k w_i/N & \text{if } k \geq N \end{cases} \quad (33)$$

Including these expressions in (32) gives:

$$\|\tilde{\mathbf{R}}\|_F^2 \geq \frac{M^2}{N^2} \left[ \sum_{k=1}^{N-1} \left( \sum_{i=1}^k w_i + \sum_{i=N+k}^{2N-1} w_i \right)^2 + \sum_{k=N}^{2N-1} \left( \sum_{i=k-N+1}^k w_i \right)^2 \right]. \quad (34)$$

It is possible to show that the right hand side is equal to (cf. Appendix B):

$$\sum_{k=1}^{N-1} \left( \sum_{i=1}^k w_i + \sum_{i=N+k}^{2N-1} w_i \right)^2 + \sum_{k=N}^{2N-1} \left( \sum_{i=k-N+1}^k w_i \right)^2 = \sum_{s,t=1}^{2N-1} (N - l_{s,t,N}) w_s w_t \quad (35)$$

with  $l_{s,t,N} = \min(|t-s|, 2N-1-|t-s|)$ .

Accordingly, the lower bound of  $\|\tilde{\mathbf{R}}\|_F^2$  is given by:

$$\|\tilde{\mathbf{R}}\|_F^2 \geq \frac{M^2}{N^2} \left[ N - \sum_{s,t=1}^{2N-1} l_{s,t,N} w_s w_t \right]. \quad (36)$$

## APPENDIX B

### PROOF OF EQUALITY (35)

This appendix details the proof of Equality (35). Its left hand side can be developed as:

$$\begin{aligned} & \sum_{k=1}^{N-1} \left( \sum_{i=1}^k w_i + \sum_{i=N+k}^{2N-1} w_i \right)^2 + \sum_{k=N}^{2N-1} \left( \sum_{i=k-N+1}^k w_i \right)^2 \\ &= 2 \sum_{k=1}^{N-1} \sum_{s=1}^k \sum_{t=N+k}^{2N-1} w_s w_t + \sum_{k=1}^{N-1} \left( \sum_{i=N+k}^{2N-1} w_i \right)^2 \\ & \quad + \sum_{k=1}^{2N-1} \left( \sum_{i=\max(1,k-N+1)}^k w_i \right)^2 \end{aligned} \quad (37)$$

where this last expression is obtained by observing that:

$$\sum_{i=1}^k w_i = \sum_{i=\kappa}^k w_i \quad \text{for } k \in \{1, N-1\}, \quad (38)$$

with  $\kappa = \max(1, k-N+1)$ .

Let us set  $\mathbf{w} = \{w_i\}_{i=1}^{2N-1}$  and  $\mathbf{M}_{s_1, t_1}^{s_2, t_2}$  the  $(2N-1) \times (2N-1)$  matrix such that the submatrix of row index  $s_1 \leq i \leq s_2$  and of column index  $t_1 \leq j \leq t_2$  is a matrix of ones while the other entries are null:

$$\mathbf{M}_{s_1, t_1}^{s_2, t_2} = \begin{bmatrix} 0 & \dots & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 & 1 & \dots & 1 & 0 & \dots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & \dots & 0 & 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \vdots & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & \dots & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix} \begin{array}{l} \leftarrow s_1 \\ \\ \\ \leftarrow s_2 \\ \\ \\ \end{array} \quad (39)$$

$\uparrow \quad \uparrow$   
 $t_1 \quad t_2$

With such a matrix, we have:

$$\sum_{s=s_1}^{s_2} \sum_{t=t_1}^{t_2} w_s w_t = \mathbf{w}^T \mathbf{M}_{s_1, t_1}^{s_2, t_2} \mathbf{w}, \quad (40)$$

so that each term of (37) may be written with some matrices  $\mathbf{M}_{s_1, s_2}^{t_1, t_2}$ :

$$\begin{aligned} \sum_{k=1}^{N-1} \left( \sum_{i=\kappa}^k w_i \right)^2 &= \mathbf{w}^T \left[ \sum_{k=1}^{2N-1} \mathbf{M}_{\kappa, \kappa}^{k, k} \right] \mathbf{w}, \\ \sum_{k=1}^{N-1} \left( \sum_{i=N+k}^{2N-1} w_i \right)^2 &= \mathbf{w}^T \left[ \sum_{k=1}^{N-1} \mathbf{M}_{N+k, N+k}^{2N-1, 2N-1} \right] \mathbf{w}, \\ 2 \sum_{k=1}^{N-1} \sum_{s=1}^k \sum_{t=N+k}^{2N-1} w_s w_t &= \mathbf{w}^T \left[ \sum_{k=1}^{N-1} \left( \mathbf{M}_{1, N+k}^{k, 2N-1} + \mathbf{M}_{N+k, 1}^{2N-1, k} \right) \right] \mathbf{w}. \end{aligned} \quad (41)$$

It can be observed first that:

$$\sum_{k=1}^{2N-1} \mathbf{M}_{\kappa, \kappa}^{k, k} = \begin{bmatrix} \mathbf{M}_1 & \mathbf{M}_2^T \\ \mathbf{M}_2 & \mathbf{M}_3 \end{bmatrix}, \quad (42)$$

where  $\mathbf{M}_1$  is an  $N \times N$  Toeplitz matrix with generating vector  $[N, N-1, \dots, \dots, 1]$ ,  $\mathbf{M}_2$  is a  $(N-1) \times N$  matrix given by:

$$\mathbf{M}_2 = \begin{bmatrix} 0 & 1 & 2 & \dots & N-1 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 2 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix}, \quad (43)$$

and  $\mathbf{M}_3$  is a  $(N-1) \times (N-1)$  matrix given by:

$$\mathbf{M}_3 = \begin{bmatrix} N-1 & N-2 & \dots & 2 & 1 \\ N-2 & N-2 & & \vdots & \vdots \\ \vdots & & & \ddots & \vdots \\ 2 & \dots & \dots & 2 & \vdots \\ 1 & \dots & \dots & \dots & 1 \end{bmatrix}, \quad (44)$$



second that:

$$\sum_{k=1}^{N-1} \mathbf{M}_{N+k, N+k}^{2N-1, 2N-1} = \begin{bmatrix} \mathbf{0}_{N, N} & \mathbf{0}_{N, N-1} \\ \mathbf{0}_{N-1, N} & \mathbf{M}_4 \end{bmatrix}, \quad (45)$$

where  $\mathbf{M}_4$  is a  $(N-1) \times (N-1)$  matrix given by:

$$\mathbf{M}_4 = \begin{bmatrix} 1 & \cdots & \cdots & \cdots & 1 \\ \vdots & 2 & \cdots & \cdots & 2 \\ \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & & N-2 & N-2 \\ 1 & 2 & \cdots & N-2 & N-1 \end{bmatrix}, \quad (46)$$

and third that:

$$\sum_{k=1}^{N-1} \left( \mathbf{M}_{1, N+k}^{k, 2N-1} + \mathbf{M}_{N+k, 1}^{2N-1, k} \right) = \begin{bmatrix} \mathbf{0}_{N-1, N-1} & \mathbf{M}_2 \\ \mathbf{M}_2^T & \mathbf{0}_{N, N} \end{bmatrix}. \quad (47)$$

The summation of these different matrices then easily provides a simple  $(2N-1) \times (2N-1)$  Toeplitz matrix with the generating vector  $[N, N-1, \dots, \dots, 2, 1, 1, 2, \dots, N-1]$ , and whose entry  $(s, t)$  is thus equal to  $N - l_{s, t, N}$  with:

$$l_{s, t, N} = \min(|t-s|, 2N-1-|t-s|). \quad (48)$$

This then provides the wanted expression:

$$\begin{aligned} \sum_{k=1}^{N-1} \left( \sum_{i=1}^k w_i + \sum_{i=N+k}^{2N-1} w_i \right)^2 + \sum_{k=N}^{2N-1} \left( \sum_{i=k-N+1}^k w_i \right)^2 \\ = \sum_{s, t=1}^{2N-1} (N - l_{s, t, N}) w_s w_t. \end{aligned} \quad (49)$$

#### APPENDIX C PROOF OF LEMMA IV

Let us develop the Frobenius norm of  $\tilde{\mathbf{R}}$ :

$$\begin{aligned} \|\tilde{\mathbf{R}}\|_F^2 &= \sum_{l, m=1}^M \sum_{i, j=1}^{2N-1} \left| \tilde{\mathbf{X}}_i^l \left( \tilde{\mathbf{X}}_j^m \right)^H \right|^2 \\ &= \sum_{l, m=1}^M \sum_{i, j=1}^{2N-1} |\theta_{x^l, x^m}(l, i, j, N)|^2 w_i w_j. \end{aligned} \quad (50)$$

$\tilde{\mathbf{R}}$  is a matrix of size  $M(2N-1) \times M(2N-1)$ . The energy constraint provides its diagonal coefficients:

$$|\theta_{x^m, x^m}(0)|^2 = 1, \quad \forall m \in \{1, 2N-1\}, \quad (51)$$

while the aperiodic correlation satisfies:

$$|\theta_{x^l, x^m}(i-j)|^2 \leq \frac{d^2}{N^2} \quad \text{when } |i-j| = N-d \quad \forall (l, m). \quad (52)$$

Using these properties, an upper bound of  $\|\tilde{\mathbf{R}}\|_F^2$  may be obtained. Each term of  $\tilde{\mathbf{R}}$  may be upper bounded by  $\theta_{\max}^2$ ,

except the ones that refer to the autocorrelations mainlobes (51) and the  $D$ -last delays (52). Hence,

$$\begin{aligned} \|\tilde{\mathbf{R}}\|_F^2 &= \sum_{l, m=1}^M \sum_{i, j=1}^{2N-1} |\theta_{x^l, x^m}(l, i, j, N)|^2 w_i w_j \\ &\leq M^2 \theta_{\max}^2 \sum_{i, j=1}^{2N-1} w_i w_j + M \left(1 - \theta_{\max}^2\right) \sum_{i=1}^{2N-1} w_i^2 \\ &\quad - M^2 \left( \theta_{\max}^2 - \frac{1}{N^2} \right) \sum_{\substack{i, j=1 \\ l_{i, j, N} = N-1}}^{2N-1} w_i w_j - \dots \\ &\quad - M^2 \left( \theta_{\max}^2 - \frac{D^2}{N^2} \right) \sum_{\substack{i, j=1 \\ l_{i, j, N} = N-D}}^{2N-1} w_i w_j. \end{aligned} \quad (53)$$

This proof can be concluded using the weighting condition (5):

$$\begin{aligned} \|\tilde{\mathbf{R}}\|_F^2 &\leq M^2 \theta_{\max}^2 + M \left(1 - \theta_{\max}^2\right) \sum_{i=1}^{2N-1} w_i^2 \\ &\quad - \sum_{d=1}^D M^2 \left( \theta_{\max}^2 - \frac{d^2}{N^2} \right) \sum_{\substack{i, j=1 \\ l_{i, j, N} = N-d}}^{2N-1} w_i w_j. \end{aligned} \quad (54)$$

#### APPENDIX D APPLICATIONS WITH A CONSTANT WEIGHT VECTOR

In this appendix are developed the calculations enabling to obtain equation (27) for our proposed bound stated in Theorem 2, in the particular case of a constant weight vector  $w_i := 1/(2N-1)$  for  $i \in \{1, 2N-1\}$ . Remind that for a generic weight vector  $\mathbf{w}$ , the obtained PSL bound is:

$$\theta_{\max}^2 \geq \frac{1}{N^2} \left[ N - \frac{\tilde{Q} \left( \mathbf{w}, \frac{N(N-1)}{M}, \sum_{d=1}^D (d^2 - N) A_d \right)}{1 - \mathbf{w}^T \left( \frac{1}{M} \mathbf{I} + \sum_{d=1}^D A_d \right) \mathbf{w}} \right] \quad (55)$$

with  $\tilde{Q}(\mathbf{w}, a, \mathbf{B}) = \mathbf{w}^T (a\mathbf{I} + \mathbf{B} + \mathbf{L}) \mathbf{w}$ . We will detail the computation of the different terms involved in that expression.

Let us first consider the term  $\mathbf{w}^T \mathbf{L} \mathbf{w}$  given by:

$$\mathbf{w}^T \mathbf{L} \mathbf{w} = \sum_{s, t=1}^{2N-1} l_{s, t, N} w_s w_t, \quad (56)$$

with  $l_{s, t, N} = \min(|t-s|, 2N-1-|t-s|)$ . As already mentioned, the matrix  $\mathbf{L}$  is circulant :

$$\mathbf{L} = \text{circ}([0, 1, \dots, N-1, N-1, \dots, 1]^T). \quad (57)$$

With constant weights, it can be seen that:

$$\mathbf{w}^T \mathbf{L} \mathbf{w} = \frac{1}{(2N-1)^2} \sum_{s, t=1}^{2N-1} L_{s, t}. \quad (58)$$

The particular structure of the matrix  $L$  gives us the following development:

$$\begin{aligned} \mathbf{w}^T L \mathbf{w} &= \frac{1}{(2N-1)^2} \sum_{s=1}^{2N-1} \left( \sum_{k=0}^{N-1} k + \sum_{k=1}^{N-1} k \right) \\ &= \frac{N(N-1)}{2N-1}. \end{aligned} \quad (59)$$

Consider now  $\mathbf{w}^T A_d \mathbf{w}$ :

$$\begin{aligned} \mathbf{w}^T A_d \mathbf{w} &= \sum_{\substack{i,j=1 \\ l_i,j,N=N-d}} w_i w_j \\ &= \frac{1}{(2N-1)^2} \sum_{\substack{i,j=1 \\ |i-j|=N-d \\ \text{or } |i-j|=N+d-1}}^{2N-1} 1 \\ &= \frac{2}{(2N-1)^2} \left[ \sum_{s=1}^{N+d-1} 1 + \sum_{s=1}^{N-d} 1 \right] \\ &= \frac{2}{2N-1}. \end{aligned} \quad (60)$$

From this calculation, it also comes straightforwardly that:

$$\mathbf{w}^T \left( \sum_{d=1}^D A_d \right) \mathbf{w} = \frac{2D}{2N-1}, \quad (61)$$

and that:

$$\mathbf{w}^T \left( \sum_{d=1}^D d^2 A_d \right) \mathbf{w} = \frac{2}{2N-1} \frac{D(D+1)(2D+1)}{6}. \quad (62)$$

Putting all these results together gives rise to the following bound:

$$\theta_{\max}^2 \geq \frac{1}{3N^2} \frac{3MN^2 - 3N^2 - MD(D+1)(2D+1)}{M(2N-2D-1) - 1}. \quad (63)$$

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