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Curvilinear Integral Theorem for G-Monogenic Mappings in the Algebra of Complex Quaternion

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Abstract. For G-monogenic mappings taking values in the algebra of complex quaternion we prove a curvilinear analogue of the Cauchy integral theorem in the case where a curve of integration lies on the boundary of a domain of G-monogeneity.

Introduction

Let $\mathbb{H}(\mathbb{C})$ be the quaternion algebra over the field of complex numbers $\mathbb{C}$, whose basis consists of the unit 1 of the algebra and of the elements I, J, K satisfying the multiplication rules:

$$I^2 = J^2 = K^2 = -1,$$

$$IJ = -JI = K, \quad JK = -KJ = I, \quad KI = -IK = J.$$

In the algebra $\mathbb{H}(\mathbb{C})$ there exists another basis $\{e_1, e_2, e_3, e_4\}$ such that multiplication table in a new basis can be represented as (see, e. g., [1])

$\cdot$	e_1	e_2	e_3	e_4
e_1	e_1	0	e_3	0
e_2	0	e_2	0	e_4
e_3	0	e_3	0	e_1
e_4	e_4	0	e_2	0

.

The unit of the algebra can be decomposed as $1 = e_1 + e_2$.

Let us consider the vectors

$$i_1 = e_1 + e_2, \quad i_2 = a_1e_1 + a_2e_2, \quad i_3 = b_1e_1 + b_2e_2, \tag{1}$$

$a_k, b_k \in \mathbb{C}$, $k = 1, 2$, which are linearly independent over the field of real numbers $\mathbb{R}$. It means that the equality $\alpha_1 i_1 + \alpha_2 i_2 + \alpha_3 i_3 = 0$ for $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ holds if and only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

In the algebra $\mathbb{H}(\mathbb{C})$ we consider the linear span $E_3 := \{\zeta = xi_1 + yi_2 + zi_3 : x, y, z \in \mathbb{R}\}$ generated by the vectors i_1, i_2, i_3 over the field $\mathbb{R}$. A set $S \subset \mathbb{R}^3$ is associated with the set $S_\zeta := \{\zeta = xi_1 + yi_2 + zi_3 : (x, y, z) \in S\}$ in E_3. We also note that a topological property of a set S_ζ in E_3 understand as the same topological property of the set S in $\mathbb{R}^3$. For example, we will say that a curve $\gamma_\zeta \subset E_3$ is homotopic to a point if $\gamma \subset \mathbb{R}^3$ is homotopic to a point, etc.

Let Ω be a domain in $\mathbb{R}^3$.

We say (see [2]) that a continuous mapping $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ $\big($or $\widehat{\Phi} : \Omega_\zeta \to \mathbb{H}(\mathbb{C})\big)$ is *right-G-monogenic* $\big($or *left-G-monogenic*$\big)$ in a domain $\Omega_\zeta \subset E_3$, if Φ $\big($or $\widehat{\Phi}\big)$ is differentiable in the sense of the Gâteaux at every point of Ω_ζ, i. e. for every $\zeta \in \Omega_\zeta$ there exists an element $\Phi'(\zeta) \in \mathbb{H}(\mathbb{C})$ $\big($or $\widehat{\Phi}'(\zeta) \in \mathbb{H}(\mathbb{C})\big)$ such that

$$\lim_{\varepsilon \to 0+0} \Big(\Phi(\zeta + \varepsilon h) - \Phi(\zeta)\Big)\varepsilon^{-1} = h\Phi'(\zeta) \quad \forall\, h \in E_3$$

$$\left(\text{or} \quad \lim_{\varepsilon\to 0+0}\left(\widehat{\Phi}(\zeta+\varepsilon h)-\widehat{\Phi}(\zeta)\right)\varepsilon^{-1}=\widehat{\Phi}'(\zeta)h \quad \forall\, h\in E_3\right).$$

The Cauchy integral theorems for holomorphic functions of the complex variable are fundamental results of the classical complex analysis. Analogues of these results are also important tools in the quaternionic analysis.

In the paper [3] were established some analogues of classical integral theorems of the theory of analytic functions of the complex variable: the surface and curvilinear Cauchy integral theorems and the Cauchy integral formula. The Morera theorem was proved in the paper [4]. Taylor's and Laurent's expansions of G-monogenic mappings are obtained in [5].

Namely, in the paper [3] was proved a curvilinear analogue of the Cauchy integral theorem in the case where a curve of integration lies in a domain of G-monogeneity.

In the present paper we prove a curvilinear Cauchy integral theorem for G-monogenic mappings in the case where a curve of integration lies on the boundary of a domain of G-monogeneity.

The main result

Let γ be a Jordan rectifiable curve in $\mathbb{R}^3$. For a continuous mapping $\Psi:\gamma_\zeta\to\mathbb{H}(\mathbb{C})$ of the form

$$\Psi(\zeta)=\sum_{k=1}^{4}\Big(U_k(x,y,z)+iV_k(x,y,z)\Big)e_k, \tag{2}$$

where $(x,y,z)\in\gamma$ and $U_k:\gamma\to\mathbb{R}$, $V_k:\gamma\to\mathbb{R}$, we define integrals along a Jordan rectifiable curve γ_ζ by the equalities

$$\int_{\gamma_\zeta} d\zeta\,\Psi(\zeta):=\sum_{k=1}^{4}e_k\int_\gamma U_k(x,y,z)dx+\sum_{k=1}^{4}i_2e_k\int_\gamma U_k(x,y,z)dy+$$

$$+\sum_{k=1}^{4}i_3e_k\int_\gamma U_k(x,y,z)dz+i\sum_{k=1}^{4}e_k\int_\gamma V_k(x,y,z)dx+$$

$$+i\sum_{k=1}^{4}i_2e_k\int_\gamma V_k(x,y,z)dy+i\sum_{k=1}^{4}i_3e_k\int_\gamma V_k(x,y,z)dz$$

and

$$\int_{\gamma_\zeta} \Psi(\zeta)\,d\zeta:=\sum_{k=1}^{4}e_k\int_\gamma U_k(x,y,z)dx+\sum_{k=1}^{4}e_ki_2\int_\gamma U_k(x,y,z)dy+$$

$$+\sum_{k=1}^{4}e_ki_3\int_\gamma U_k(x,y,z)dz+i\sum_{k=1}^{4}e_k\int_\gamma V_k(x,y,z)dx+$$

$$+i\sum_{k=1}^{4}e_ki_2\int_\gamma V_k(x,y,z)dy+i\sum_{k=1}^{4}e_ki_3\int_\gamma V_k(x,y,z)dz,$$

where $d\zeta:=i_1dx+i_2dy+i_3dz$.

In the paper [3] for right-G-monogenic mappings was obtained the following analogue of the Cauchy integral theorem.

Theorem 1 [3]. *Let $\Phi : \Omega_\zeta \to \mathbb{H}(\mathbb{C})$ be a right-G-monogenic mapping in a domain Ω_ζ. Then for every closed Jordan rectifiable curve γ_ζ homotopic to a point in Ω_ζ, the following equality is true:*

$$\int\limits_{\gamma_\zeta} d\zeta\,\Phi(\zeta) = 0. \tag{3}$$

Below we establish sufficient conditions for the curve γ_ζ lying on the boundary $\partial\Omega_\zeta$ of a domain Ω_ζ such that the equality (3) holds. For this goal we apply a scheme of the proof of Theorem 4 of [6] for G-monogenic mappings.

Let on a boundary $\partial\Omega_\zeta$ of the domain Ω_ζ given closed Jordan rectifiable curve $\gamma_\zeta \equiv \gamma_\zeta(t)$, where $0 \le t \le 1$, homotopic to an interior point $\zeta_0 \in \Omega_\zeta$. It means that there exists the mapping $H(s,t)$ continuous on the square $[0,1]\times[0,1]$, such that $H(0,t) = \gamma_\zeta(t)$, $H(1,t) \equiv \zeta_0$, and all curves $\gamma_\zeta^s \equiv \gamma_\zeta^s(t) := \{\zeta = H(s,t) : 0 \le t \le 1\}$ for $0 < s < 1$ are contained in the domain Ω_ζ.

Consider also the curves $\Gamma_\zeta^t \equiv \Gamma_\zeta^t(s) := \{\zeta = H(s,t) : 0 \le s \le 1\}$. Denote by $\Gamma[\zeta_1,\zeta_2]$ arc of Jordan oriented rectifiable curve, beginning at the point ζ_1 and ending at the point ζ_2, and denote by the mes a linear Lebesgue measure of a rectifiable curve.

Let us consider the algebra $\widetilde{\mathbb{H}}(\mathbb{R})$ with the basis $\{e_k, ie_k\}_{k=1}^4$ over the field $\mathbb{R}$ which is isomorphic to the algebra $\mathbb{H}(\mathbb{C})$ over the field $\mathbb{C}$. In the algebra $\widetilde{\mathbb{H}}(\mathbb{R})$ there exist another basis $\{i_k\}_{k=1}^8$, where the vectors i_1, i_2, i_3 are the same as in the equalities (1).

For the element $a := \sum\limits_{k=1}^{8} a_k i_k,\ a_k \in \mathbb{R}$, we define the Euclidian norm

$$\|a\| := \sqrt{\sum_{k=1}^{8} a_k^2}\,.$$

Accordingly, $\|\zeta\| = \sqrt{x^2+y^2+z^2}$ and $\|i_1\| = \|i_2\| = \|i_3\| = 1$.

Using Theorem of equivalents of norms, for the element $b := \sum\limits_{k=1}^{4}(b_{1k} + ib_{2k})e_k$, $b_{1k}, b_{2k} \in \mathbb{R}$, we have the following inequalities:

$$|b_{1k} + ib_{2k}| \le \sqrt{\sum_{k=1}^{4}\left(b_{1k}^2 + b_{2k}^2\right)} \le c\|b\|,$$

where c is a positive constant does not dependent on b.

Theorem 2. *Suppose that $\Phi : \overline{\Omega_\zeta} \to \mathbb{H}(\mathbb{C})$ is a continuous mapping in the closure $\overline{\Omega_\zeta}$ of a domain Ω_ζ and Φ is right-G-monogenic in Ω_ζ. Suppose also that $\gamma_\zeta \subset \partial\Omega_\zeta$ is any closed Jordan rectifiable curve homotopic to a point $\zeta_0 \in \Omega_\zeta$ such that the curves of the family $\{\Gamma_\zeta^t : 0 \le t \le 1\}$ are rectifiable and the set $\{\text{mes}\,\gamma_\zeta^s : 0 \le s \le 1\}$ is bounded. Then the equality* (3) is true.

Proof. Let $\varepsilon > 0$. We fix the number $\rho \in \left(0, \frac{1}{2}\,\text{mes}\,\gamma_\zeta\right)$ such that for arbitrary $\zeta_1, \zeta_2 \in \overline{\Omega_\zeta}$ from the condition $||\zeta_1 - \zeta_2|| < 2\rho$ follows the inequality

$$||\Phi(\zeta_1) - \Phi(\zeta_2)|| < \varepsilon. \tag{4}$$

Since the mapping H is uniformly continuous on the square $[0,1]\times[0,1]$, then there exists $\delta > 0$ such that for all $s \in (0,\delta)$ and $t, t' \in [0,1] : |t - t'| < \delta$ the inequality $|H(0,t) - H(s,t')| < \rho$ is true.

Let numbers $0 = t_0 < t_1 < \ldots < t_n < 1$ such that for corresponding points $\zeta_{0,k} := H(0,t_k)$ of the curve γ_ζ the following relations are fulfilled

$$\text{mes}\,\gamma_\zeta[\zeta_{0,k}, \zeta_{0,k+1}] = \rho \quad \text{for} \quad k = \overline{0, n-1},$$

$$\text{mes}\,\gamma_\zeta[\zeta_{0,n},\zeta_{0,0}]\le\rho.$$

It is obvious that $2\le n\le\left[\frac{\text{mes}\,\gamma_\zeta}{\rho}\right]+1$.

Let us consider the points $\zeta_{s,k}:=H(s,t_k)$ of the curve γ_ζ^s and the curves

$$\Upsilon_{[k]}^s:=\gamma_\zeta[\zeta_{0,k},\zeta_{0,k+1}]\cup\Gamma_\zeta^{t_{k+1}}[\zeta_{0,k+1},\zeta_{s,k+1}]\cup\gamma_\zeta^s[\zeta_{s,k+1},\zeta_{s,k}]\cup\Gamma_\zeta^{t_k}[\zeta_{s,k},\zeta_{0,k}]$$

for $k=\overline{0,n}$, where $\zeta_{s,n+1}:=\zeta_{s,0}$ for $0\le s\le1$, setting that the orientation of curves $\Upsilon_{[k]}^s$ is induced by orientation of the curve γ_ζ.

Let $s\in(0,\delta)$. Since for all $\zeta\in\Upsilon_{[k]}^s$ the inequality $||\zeta-\zeta_{0,k}||\le2\rho$ is true, then by Theorem 2 [3], Lemma 4.1 [4] and the inequality (4), we have

$$\left\|\int\limits_{\gamma_\zeta}d\zeta\Phi(\zeta)\right\|=\left\|\sum_{k=0}^{n}\int\limits_{\Upsilon_{[k]}^s}d\zeta(\Phi(\zeta)-\Phi(\zeta_{0,k}))\right\|\le$$

$$\le c\sum_{k=0}^{n}\int\limits_{\Upsilon_{[k]}^s}||d\zeta||\,||\Phi(\zeta)-\Phi(\zeta_{0,k})||\le c\varepsilon\sum_{k=0}^{n}\text{mes}\,\Upsilon_{[k]}^s\le$$

$$\le c\varepsilon\Big(\text{mes}\,\gamma_\zeta+\text{mes}\,\gamma_\zeta^s+2(n+1)\max_{k=\overline{0,n}}\text{mes}\,\Gamma_\zeta^{t_k}[\zeta_{s,k},\zeta_{0,k}]\Big)\le$$

$$\le M\varepsilon\left(1+\frac{1}{\rho}\max_{k=\overline{0,n}}\text{mes}\,\Gamma_\zeta^{t_k}[\zeta_{s,k},\zeta_{0,k}]\right)\tag{5}$$

and a constant M does not depend on ε and ρ.

Passing to the limit in the inequality (5) as $s\to0$, we have the inequality

$$\left\|\int\limits_{\gamma_\zeta}d\zeta\Phi(\zeta)\right\|\le M\varepsilon.$$

Now passing to the limit in the last inequality as $\varepsilon\to0$, we obtain the equality (3). The Theorem is proved.

The similar statement is true for the left-G-monogenic mappings.

Theorem 3. *Suppose that $\widehat{\Phi}:\overline{\Omega}_\zeta\to\mathbb{H}(\mathbb{C})$ is a continuous mapping in the closure $\overline{\Omega}_\zeta$ of a domain Ω_ζ and $\widehat{\Phi}$ is left-G-monogenic in Ω_ζ. Suppose also that $\gamma_\zeta\subset\partial\Omega_\zeta$ is any closed Jordan rectifiable curve homotopic to a point $\zeta_0\in\Omega_\zeta$ such that the curves of the family $\{\Gamma_\zeta^t:0\le t\le1\}$ are rectifiable and the set $\{\text{mes}\,\gamma_\zeta^s:0\le s\le1\}$ is bounded. Then the following equality is true:*

$$\int\limits_{\gamma_\zeta}\widehat{\Phi}(\zeta)d\zeta=0.$$

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References

[1] E. Cartan, Les groupes bilinéares et les systèmes de nombres complexes, Annales de la faculté des sciences de Toulouse. **12**:1 (1898) 1–64.

[2] V. S. Shpakivskyi, T. S. Kuzmenko, On one class of quaternionic mappings. Ukr. Math. J. **68**:1 (2016) 117–130.

[3] V. S. Shpakivskyi, T. S. Kuzmenko, Integral theorems for the quaternionic G-monogenic mappings. An. Şt. Univ. Ovidius Constanţa. **24**:2 (2016).

[4] V. S. Shpakivskyi, T. S. Kuzmenko, On monogenic mappings of the quaternionic variable. Ukr. Math. Bull. (submitted).

[5] T. S. Kuzmenko, Power and Laurent series in the algebra of complex quaternion. Zb. Pr. Inst. Mat. NAN Ukr. **12**:3 (2015) 164–174 (Ukrainian).

[6] S. A. Plaksa, V. S. Shpakivskyi, Integral theorems for differentiable functions in a three-dimensional harmonic algebra. Reports of the NAS of Ukraine. **5** (2010) 23–30 (Russian).