RANGE OF VALIDITY OF WEAKLY NON-LINEAR THEORY IN RAYLEIGH-BÉNARD CONVECTION

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Abstract

In this paper we examine the equilibrium states of periodic finite amplitude flow in a horizontal channel with differential heating between the two rigid boundaries. The solutions to the Navier-Stokes equations are obtained by means of a perturbation method for evaluating the Landau coefficients and through a Newton-Raphson iterative method that results from the Fourier expansion of the solutions that bifurcate above the linear stability threshold of infinitesimal disturbances. The results obtained from these two different methods of evaluating the convective flow are compared in the neighbourhood of the critical Rayleigh number. We find that for small Prandtl numbers the discrepancy of the two methods is noticeable.

1 Weakly Non-Linear Theory and Amplitude Equations

Weakly non-linear theory for fluid motions was established by Stuart (1960) and Watson (1960) and has been applied to many fundamental basic flows, successfully, during last a half century. The resultant ODE is sometimes referred to as 'Stuart-Landau equation', or simply 'amplitude equation'. For steady onset, Malkus and Veronis (1958) derived a branching equation instead of the amplitude equation. In the development of the theory, the most important contribution was due to Newell and Whitehead (1969), Stewartson and Stuart (1971), and DiPrima, Eckhaus and Segel (1971); they extended the theory to take account of the effect of spatial modulation. Those classical theories are known to give amplitude equations equivalent with those based on the method of centre manifold (See Carr 1981, Cheng and Chang 1990, Fujimura 1991 and 1997, for example), implying that the classical theories are mathematically justified. In practical applications of the weakly non-linear theory, we need to keep in mind that the range of validity of amplitude equations is not always wide enough. Kuo (1961) carried out weakly-non-linear expansion to seventh order and examined the range of validity of the expansion for Rayleigh-Bénard problem under free-free boundary conditions. He showed that a parameter expansion based on $(R - R_c)/R_c$ yields an alternating series which causes a narrow range of validity whereas an expansion in $(R - R_c)/R$ resolves the alternating feature and yields a much wider range of validity. Instead, Herbert (1980) numerically showed that the range of validity is unexpectedly very narrow for plane Poiseuille flow. Recall that the bifurcation of the non-linear state at the linear critical point is subcritical for plane Poiseuille flow. Herbert (1983) showed that the zeroth harmonic resonance arises in the subcritical region. The onset of the resonance corresponds to the fact that the centre manifold has zero divisors in subcritical region (Roberts 1989). In the present paper, we revisit this classical, but practically important, problem on the range of validity of the weakly nonlinear theory for Rayleigh-Bénard convection with perfectly conducting rigid boundaries.

Consider a horizontal layer of Boussinesq fluid with infinite extent. The fluid is confined between boundaries located at $z = 0$ and d and is uniformly heated from below. After an appropriate non-dimensionalization, PDEs governing the disturbance have the form

$$Pr^{-1}\frac{D\boldsymbol{v}}{Dt} = -\nabla\pi + R\theta\boldsymbol{e}_z + \Delta\boldsymbol{v}, \ \ \nabla\cdot\boldsymbol{v} = 0, \ \ \frac{D\theta}{Dt} + (-1)w = \Delta\theta, \tag{1}$$
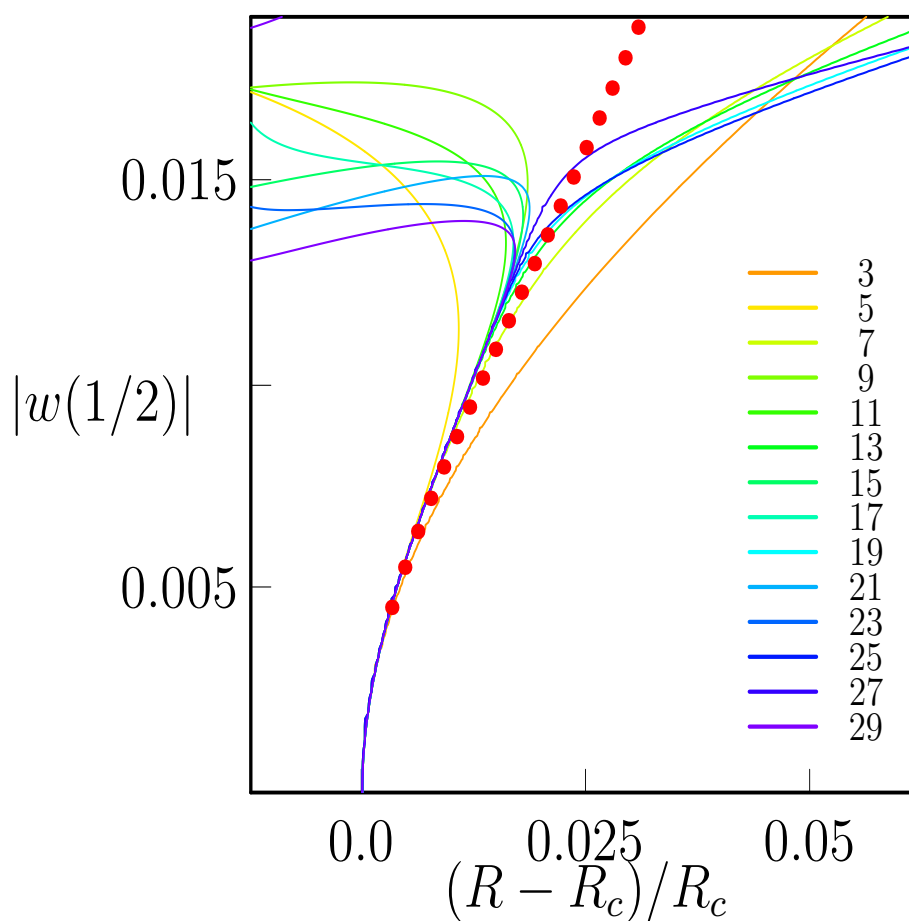
Figure 1: *Results of the calculations for the non-linear flow for $Pr = 0.001$. The streamwise harmonics use $k_c = 3.11632355$, and R_c is the critical Rayleigh number for Rayleigh-Bnard convection: $R_c = 1707.76178$. In order to measure the agreement between the two methods studied in this work we use the amplitude of the z-component of the velocity field evaluated at midplane: $|w(1/2)|$. The truncation levels of the Stuart-Landau equation (2) are also provided.*

where $Pr = \dfrac{\nu_0}{\kappa_0}$, $R = \dfrac{\alpha_0 g \delta T d^3}{\nu_0 \kappa_0}$, d being the depth of the fluid layer, and δT being a temperature difference between the top and bottom boundaries.

Under the assumption that the solution of (1) is $2\pi/k_c$-periodic, (1) is invariant under $x \to x + l \pmod{2\pi/k_c}$ and $x \to -x$ together with $u \to -u$, where k_c is the critical wavenumber. Let the solution of (1) be written as $(\boldsymbol{v}, \pi, \theta)^T = A(t)(\boldsymbol{\Psi}(z), \Pi(z), \Theta(z))^T \, \mathrm{e}^{ik_c x} + c.c. + \cdots$. Orthogonal group O(2) acts on $A \in \mathbb{R}^2$ as $A \to A \, \mathrm{e}^{i\varphi}$ for $0 \le \varphi < 2\pi$ and $A \to \overline{A}$. The equation for $A(t)$ generated by O(2)-equivariant vector field is written as $\dot{A} = A f(|A|^2; \mu)$, where $\mu = (R - R_c)/R_c \in \mathbb{R}$ and $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Taylor expansion of the invariant function f about the origin yields the Stuart-Landau equation

$$\dot{A} = A \left(f_\mu(0)\mu + f_{|A|^2}(0)|A|^2 + \cdots \right) = \sigma A + \sum_{n=1}^{\infty} \lambda_{2n+1} |A|^{2n} A. \tag{2}$$

To derive equation (2) from (1), we apply the refined version of the amplitude expansion scheme (See Herbert 1983 or Fujimura 1989, for example). We first rewrite (1) into a simplified form

$$\left[\frac{\partial}{\partial t} \mathcal{S} + \mathcal{L}(R) \right] \boldsymbol{\psi} = \mathcal{N}(\boldsymbol{\psi}, \boldsymbol{\psi}) \quad \text{for} \quad \boldsymbol{\psi} = (\boldsymbol{v}, \pi, \theta)^T. \tag{3}$$

The $\boldsymbol{\psi}$ is subject to homogeneous boundary conditions

$$\mathcal{H}\boldsymbol{\psi} = 0 \text{ at } z = 0, 1, \quad \text{i.e.,} \quad \boldsymbol{v} = 0 \text{ and } \theta = 0 \text{ at } z = 0, 1. \tag{4}$$

Let us assume the solution $\boldsymbol{\psi}$ to be two-dimensional and seek $\boldsymbol{\psi}$ in the form of Fourier series

$$\boldsymbol{\psi}(\boldsymbol{x}, t) = \sum_{j=-\infty}^{\infty} \Psi_j(z, t) \, \mathrm{e}^{ijk_c x} \quad \text{where} \quad \Psi_{-j} = \overline{\Psi_j}. \tag{5}$$

Each Fourier component Ψ_j is governed by

$$\left[\frac{\partial}{\partial t} S_j + L_j(R) \right] \Psi_j = \sum_{m=-\infty}^{\infty} N(\Psi_{j-m}, \Psi_m), \ S_j = \mathcal{S}|_{\partial_x \to ijk_c, \partial_z \to d/dz}, \ L_j = \mathcal{L}|_{\partial_x \to ijk_c, \partial_z \to d/dz}. \tag{6}$$

Following (2), we expand Ψ_j in $|A|^2$ such that

$$\Psi_j(z, t) = \sum_{n=1}^{\infty} \Phi_{j,j+2(n-1)}(z)|A|^{2(n-1)} A^j \quad \text{and} \quad \Psi_0(z, t) = \sum_{n=1}^{\infty} \Phi_{0,2n}(z)|A|^{2n}. \tag{7}$$

At $O(|A|)$, we have

$$[\sigma S_1 + L_1(R)] \Phi_{11} = 0 \quad \text{subject to} \quad H_1 \Phi_{11} = 0 \text{ at } z = 0, 1 \tag{8}$$

where $H_j = \mathcal{H}|_{\partial_x \to ijk_c, \partial_z \to d/dz}$.[1] At $O(|A|^2)$, the mean-flow-distortion and the second harmonic are governed by

$$[(\sigma + \bar{\sigma})S_0 + L_0(R)] \Phi_{02} = N_{02} \quad \text{and} \quad [2\sigma S_2 + L_2(R)] \Phi_{22} = N_{22}, \tag{9}$$

where $N_{02} = N(\Phi_{11}, \overline{\Phi_{11}}) + N(\overline{\Phi_{11}}, \Phi_{11})$ and $N_{22} = N(\Phi_{11}, \Phi_{11})$. At $O(|A|^3)$, the deformation of the fundamental is governed by

$$[(2\sigma + \bar{\sigma})S_1 + L_1(R)] \Phi_{13} = N_{13} - \lambda_3 S_1 \Phi_{11}, \tag{10}$$

where $N_{13} = N(\overline{\Phi_{11}}, \Phi_{22}) + N(\Phi_{22}, \overline{\Phi_{11}}) + N(\Phi_{11}, \Phi_{02}) + N(\Phi_{02}, \Phi_{11})$.

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[1] We formally retained the general form of homogeneous boundary conditions although the conditions here are very simple. In what follows, Φ_{jn} is subject to the boundary conditions $H_j \Phi_{jn} = 0$ at $z = 0, 1$.
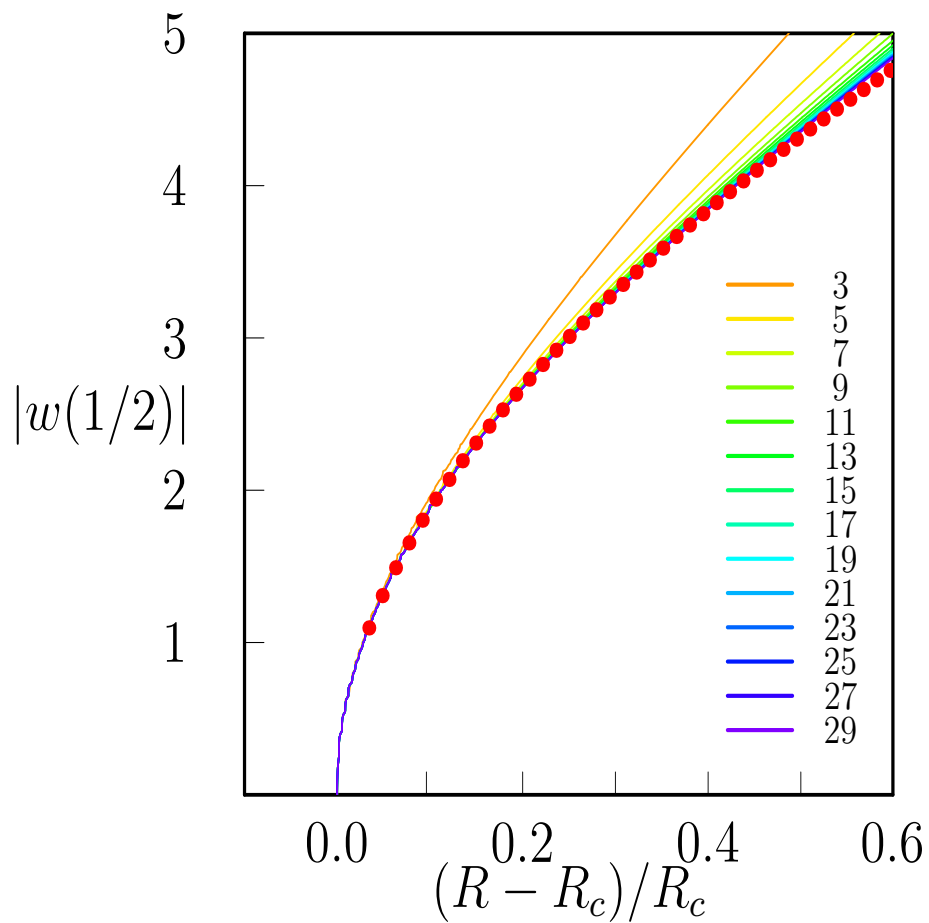
Figure 2: *Results of the calculations for the non-linear flow for $Pr = 7.0$. The streamwise harmonics use $k_c = 3.11632355$, and R_c is the critical Rayleigh number for Rayleigh-Bnard convection: $R_c = 1707.76178$. In order to measure the agreement between the two methods studied in this work we use the amplitude of the z-component of the velocity field evaluated at midplane: $|w(1/2)|$. The truncation levels of the Stuart-Landau equation (2) are also provided.*

For $\operatorname{Re}\sigma = 0$, the first Landau constant λ_3 is determined from the solvability condition for (10). For non-vanishing $\operatorname{Re}\sigma$, on the other hand, we follow Watson (1960) and set $\Phi_{13} = \chi_{13}^{(1)} - \lambda_3 \chi_{13}^{(2)}$. Equation (10) is then divided into

$$[(2\sigma + \bar{\sigma})S_1 + L_1(R)]\chi_{13}^{(1)} = N_{13} \quad \text{and} \quad [(2\sigma + \bar{\sigma})S_1 + L_1(R)]\chi_{13}^{(2)} = S_1\Phi_{11}. \tag{11}$$

Since $\operatorname{Re}\sigma \neq 0$, the first equation of (11) is solvable whereas the second equation has a solution of the form of $\chi_{13}^{(2)} = \Phi_{11}/(\sigma + \bar{\sigma})$ so that $\Phi_{13} = \chi_{13}^{(1)} - \lambda_3 \Phi_{11}/(\sigma + \bar{\sigma})$. At this stage, following Herbert (1983), we define the amplitude of the fundamental mode at a reference point $z_0 \in (0,1)$ by requiring that

$$\Phi_{11}(z_0) = 1, \quad \Phi_{1n}(z_0) = 0 \text{ for } n \geq 3. \tag{12}$$

Equation (12) then yields

$$\lambda_3 = (\sigma + \bar{\sigma})\chi_{13}^{(1)}(z_0). \tag{13}$$

Formal analysis to the higher-order approximation is straightforward. The n-th Landau constant λ_{2n+1} is given by

$$\lambda_{2n+1} = n(\sigma + \bar{\sigma})\chi_{1,2n+1}^{(1)}(z_0) \tag{14}$$

where $\chi_{1,2n+1}^{(1)}$ is a solution of

$$\left[\{(n+1)\sigma + n\bar{\sigma}\}S_1 + L_1(R)\right]\chi_{1,2n+1}^{(1)} = N_{1,2n+1} - \sum_{j=2}^{n}\{j\lambda_{2(n-j)+3} + (j-1)\overline{\lambda_{2(n-j)+3}}\}S_1\Phi_{1,2j-1}$$

and $N_{1,2n+1}$ is a summation of nonlinear terms arising at $O(|A|^{2n+1})$.

2 Fully Non-Linear Solution of the PDEs

Since there is no preferred direction in the horizontal plane, the most dangerous modes have wavevectors on the critical circle with radius k_c. Let us restrict ourselves on the situation similar to that in section 1 and consider the roll-structure which is $2\pi/k_c$-periodic in the x-direction and uniform in the y-direction. Two-dimensional equilibrium solutions were obtained numerically by using the collocation method combined with the Newton-Raphson iterative method for some high enough truncation numbers M, N. Details of the numerical method to obtain non-linear solutions were presented recently in Nagata & Generalis (2002) and Generalis & Nagata (2003, 2004). Although the non-linear solutions bifurcate from the conduction state along the neutral curve of the linearized theory it is important to obtain converged solutions that represent the flow globally. The latter is important as the series

$$w = \sum_{n=0}^{N}\sum_{m=-M}^{M} a_{nm}(1 - \zeta^2)^2 T_n(\zeta)e^{imk_c x}, \tag{15}$$

$$\theta = \sum_{n=0}^{N}\sum_{m=-M}^{M} b_{nm}(1 - \zeta^2)T_n(\zeta)e^{imk_c x} \tag{16}$$

must be truncated at a certain level of the harmonics that are taken into account. Here $\zeta = 2z - 1$. This pre-determined truncation level is also present in the weakly non-linear theory of the previous section and in the case of Poiseuille flow Herbert (1980) showed the deviation from agreement between the two methods for small amplitudes, i.e. in the vicinity of the neutral curve. This was attributed by Herbert (1980) to the limited scope of the weakly non-linear theory as the latter approximates the mean flow rather poorly already at small amplitudes and therefore the use of the weakly non-linear theory should only be employed at regions that are very close to the neutral curve.
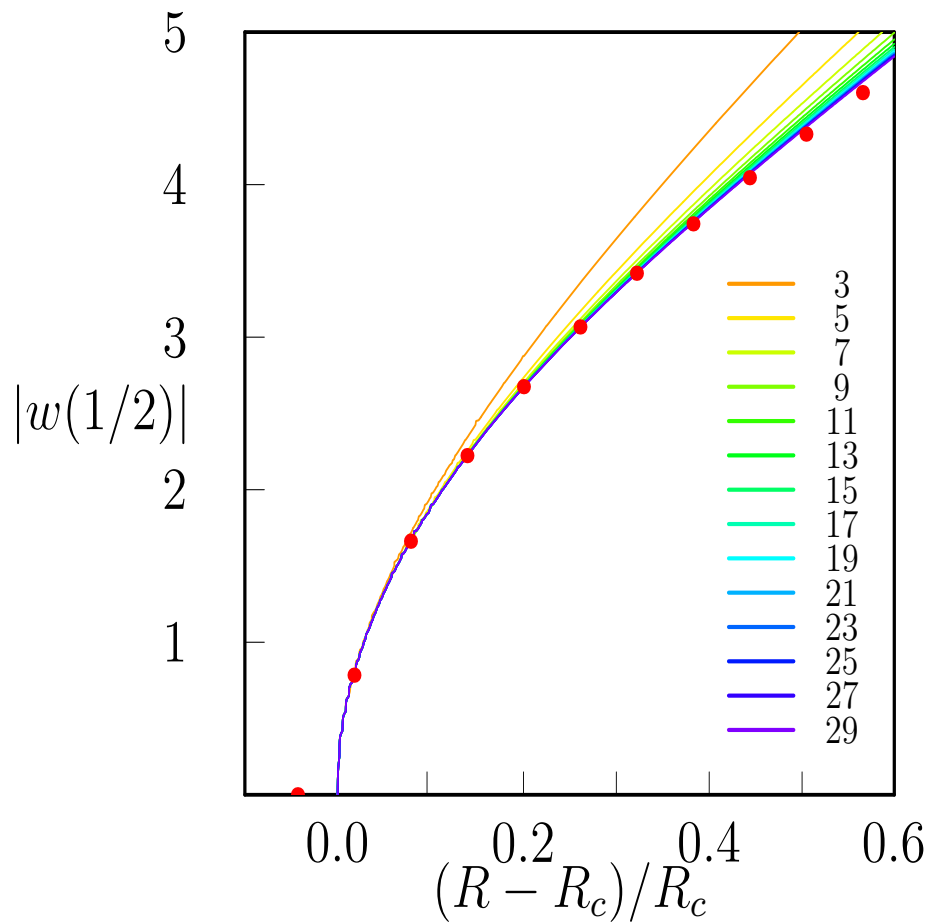
Figure 3: *Results of the calculations for the non-linear flow for $Pr = 1000$. The streamwise harmonics use $k_c = 3.11632355$, and R_c is the critical Rayleigh number for Rayleigh-Bnard convection: $R_c = 1707.76178$. In order to measure the agreement between the two methods studied in this work we use the amplitude of the z-component of the velocity field evaluated at midplane: $|w(1/2)|$. The truncation levels of the Stuart-Landau equation (2) are also provided.*

3 Numerical Results and Concluding Remarks

In order to examine the convergence of the results between the two methods, we evaluated non-linear solutions for various values of the Prandtl number. Non-linear amplitudes were derived from equations of the form (Nagata & Generalis 2002, Generalis & Nagata 2003)

$$A\boldsymbol{x} + \boldsymbol{x}^T B \boldsymbol{x} = 0, \tag{17}$$

where by x_i we denote collectively the unknown two-dimensional amplitudes that we evaluate at the collocation points $z_i = \cos((i+1)\pi/(N+2))$, $i = 0, \cdots, N$. The rank of the square matrices A, B is $(N+1)(2M+1)$. The Newton-Raphson iterative method is employed in order to obtain solutions to the resulting finite system of equations.

As can be seen from Figures 1-3 the agreement between the two methods in the vicinity of the critical Rayleigh number is very good provided that $Pr \geq O(1)$. If the latter condition is not satisfied then even a large number of non-linear terms in the amplitude expansion or a large number of harmonics in the Newton-Raphson iterative method[2] does not make the two methods converge to the same values of the quantity used here ($|w(1/2)|$). For $Pr < 0.25$, λ_5, λ_9, λ_{11}, and λ_{15} are positive whereas λ_3, λ_7, λ_{13}, λ_{17}, and λ_{19} are negative. Therefore the RHS of (2) is almost alternating. This situation is similar to the one in plane Poiseuille flow. All the positive $\lambda's$ for small Pr change their sign and become negative almost simultaneously at around $Pr = 0.25$. For $Pr > 0.25$, the λ_{2n+1} on the RHS of (2) are all negative. The agreement between the two methods for moderate and large values of the Prandtl number testifies to the fact that the range of validity of weakly non-linear expansion is fairly extensive. For $Pr > 0.25$ the range of validity of the Stuart-Landau equation (2) is $0 < (R - R_c)/R_c < 1$ or even wider, whereas it is within $0 < (R - R_c)/R_c < 0.02$ for $Pr < 0.25$. The apparent narrow range of agreement for small values of Pr even when higher order non-linear terms are taken into account in the amplitude equation and a large number of harmonics is taken into account in the Newton-Raphson iterative method is a matter of on-going investigation.

Finally we note that Kuo (1961) reported that, other than $(R - R_c)/R_c$, the $(R - R_c)/R$ drastically improves the range of validity of the parameter expansion. As Pr decreases, however, the alternating feature recovers and the convergence of the expansion gets worse. For $Pr = 0.001$, the range of validity is almost comparable with our result. Therefore, an introduction of the parameter expansion in terms of $(R - R_c)/R$ is not expected to improve the range of validity for small Pr fluids under rigid-rigid boundary conditions.

References

[1] Carr, J. 1981, *Applications of centre manifold theory*, (Springer Verlag).

[2] Cheng, M. & Chang, H.-C. 1990, A generalized sideband stability theory via center manifold projection, *Phys. Fluids*, A**2**, 1364–1379.

[3] R.C. DiPrima, W. Eckhaus & L.A. Segel, 1971, Non-linear wave-number interaction in near-critical two-dimensional flows, *J. Fluid Mech.*, **49**, 705–744.

[4] Fujimura, K. 1989, The equivalence between two perturbation methods in weakly non-linear stability theory for parallel shear flows, *Proc. R. Soc. A*, **424**, 373–392.

[5] Fujimura, K. 1991, Methods of centre manifold and multiple scales in the theory of weakly non-linear stability for fluid motions, *Proc. R. Soc. A*, **434**, 719–733.

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[2]For $Pr < 1$ the value $M = 91$ was used in order to establish that the two methods agreed.

[6] Fujimura, K. 1997, Centre manifold reduction and the Stuart-Landau equation for fluid motions, *Proc. R. Soc. A*, **453**, 181–203.

[7] Generalis, S. & Nagata, M. 2003, Transition in Homegeneously Heated Inclined Plane-Parallel Shear Flows, *ASME Journal of Heat Transfer*, **125**, 795-803.

[8] Generalis, S. & Nagata, M. 2004, Transition in Plane-Parallel Shear Flows Heated Internally, *Comptes R. Mech.* , **332**, 9-16.

[9] Herbert, T. 1980, Non-linear stability of parallel flows by high-order amplitude expansions, *AIAA J.*, **18**, 243–248.

[10] Herbert, T. 1983, On perturbation methods in non-linear theory, *J. Fluid Mech.*, **126**, 167–186.

[11] Kuo, H.L. 1961, Solution of the non-linear equations of cellular convection and heat transport, *J. Fluid Mech.*, **10**, 611–634.

[12] Malkus, W.V.R. & Veronis, G. 1958, Finite amplitude cellular convection, *J. Fluid Mech.*, **4**, 225–260.

[13] Nagata, M. & Generalis, S. 2002, Transition in Convective Flows Heated Internally, *ASME Journal of Heat Transfer*, **124**, 635-642.

[14] Newell, A.C. & Whitehead, J.A. 1969, Finite bandwidth, finite amplitude convection, *J. Fluid Mech.*, **38**, 279–303.

[15] Roberts, A.J. 1989, The utility of an invariant manifold description of the evolution of a dynamical system, *SIAM J. Math Analysis*, **20**, 1447–1458.

[16] Stewartson, K. & Stuart, J.T. 1971, A non-linear instability theory for a wave system in plane Poiseuille flow, *J. Fluid Mech.*, **48**, 529–545.

[17] Stuart, J.T. 1960, On the non-linear mechanics of wave disturbances in stable and unstable parallel flows, Part I. The basic behaviour in plane Poiseuille flow, *J. Fluid Mech.*, **9**, 353–370.

[18] Watson, A. 1960, On the non-linear mechanics of wave disturbances in stable and unstable parallel flows. Part 2. The development of a solution for plane Poiseuille and for plane Couette flow, *J. Fluid Mech.*, **9**, 371–389.