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# Global Existence and Non-existence Theorems for Nonlinear Wave Equations

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# *Global Existence and Non-existence Theorems for Nonlinear Wave Equations*

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ABSTRACT. In this article we focus on the global well-posedness of an initial-boundary value problem for a nonlinear wave equation in all space dimensions. The nonlinearity in the equation features the damping term $|u|^k |u_t|^m \operatorname{sgn}(u_t)$ and a source term of the form $|u|^{p-1}u$, where $k$, $p \geq 1$ and $0 < m < 1$. In addition, if the space dimension $n \geq 3$, then the parameters $k$, $m$ and $p$ satisfy $p, k/(1-m) \leq n/(n-2)$. We show that whenever $k + m \geq p$, then local weak solutions are global. On the other hand, we prove that whenever $p > k + m$ and the initial energy is negative, then local weak solutions blow-up in finite time, regardless of the size of the initial data.

## 1. INTRODUCTION

In quantum field theory and certain mechanical applications, various examples of the evolution equation

$$u_{tt} - \Delta u + \mathcal{R}(x, t, u, u_t) = \mathcal{F}(x, u), \tag{1.1}$$

satisfying the structural conditions $v\mathcal{R}(x, t, u, v) \geq 0$, $\mathcal{R}(x, t, u, 0) = \mathcal{F}(x, 0) = 0$, and $\mathcal{F}(x, u) \sim |u|^{p-1}u$ for large $|u|$ arise (cf. Jörgens [11] and Segal [26]).

In this paper, we study the long-time behavior of solutions to an initial-boundary value problem for a nonlinear wave equation of the form (1.1). Of central interest is the relationship of the source and damping terms to the behavior of solutions.

Throughout the paper, assume that $\Omega$ is an open, bounded, connected domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega = \Gamma$. Further assume $\Gamma$ is the union of two disjoint, connected $n-1$-dimensional manifolds $\Gamma_0$ and $\Gamma_1$. Our interest in this


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article is focussed on the initial-boundary value problem,

$$(1.2)\qquad \begin{cases} u_{tt} - \Delta u + |u|^k\,|u_t|^m \operatorname{sgn}(u_t) = |u|^{p-1}u, & \text{in } \Omega \times (0,T), \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \Gamma_0 \times (0,T), \\ \dfrac{\partial u}{\partial n}(x,t) = h(x,t), & \text{on } \Gamma_1 \times (0,T), \end{cases}$$

where $0 < m < 1$, $k$, $p \geq 1$, $h \in C^1([0,\infty), L^2(\Gamma_1))$, and $\partial/\partial n$ denotes the outward normal derivative on $\Gamma_1$. Hypotheses on the initial data $u^0$ and $u^1$ are given in (2.2) below.

It is well-known that when the damping term $|u|^k|u_t|^m \operatorname{sgn}(u_t)$ is absent, the source term $|u|^{p-1}u$ drives the solution of (1.2) to blow-up in finite time ([6, 16, 22, 30]). In addition, if the source term $|u|^{p-1}u$ is removed from the equation, then damping terms of various forms are known to yield existence of global solutions, (cf. [2, 3, 10]). However, the interaction between the damping and source terms is often difficult to analyze, as one can see from the work in [5, 17, 19, 24, 28].

The purpose of this paper is to establish sharp results on the long-time behavior of solutions to the initial-boundary value problem (1.2). Our main results are Theorems 4.1, 5.1 and 6.1. First, in Theorem 4.1 we construct a local weak solution to (1.2) by using a standard Galerkin scheme based on the eigenfunctions of the Laplacian. However, there are several technical difficulties in the passage to the limit. One difficulty lies in showing that the sequence of approximate solutions $\{u_N\}$ satisfies

$$|u_N'(t)|^m \operatorname{sgn}(u_N'(t)) \to |u'(t)|^m \operatorname{sgn}(u'(t)) \quad \text{weakly in } L^2(\Omega).$$

Uniqueness of solutions does not follow from the theory of ordinary differential equations, and presents another difficulty. Our next main result, Theorem 5.1, shows that the every local weak solution to (1.2) is global, provided $k + m \geq p$. The proof of Theorem 5.1 relies on obtaining an energy-type estimate for the sequence of the approximate solutions which holds for each bounded time interval $[0, T]$. Finally, in Theorem 6.1 we use an argument similar to the one in [5] to prove that every local weak solution to (1.2) with negative initial energy blows-up in finite time, regardless of the size of the initial data. Moreover, we obtain a precise upper bound for the life span of the solution in terms of the initial data and the other parameters in the equation.

It should be noted here that our approach for establishing the existence of a local weak solution does not directly extend to the case when $m > 1$. Indeed, for the case $m > 1$, the existence of local solutions requires different type of a *priori* estimates, and therefore the case $m > 1$ is not addressed in this paper.

Of particular relevance to our results in this article are those of Georgiev and Todorova [5] and Levine and Serrin [17]. We also note the fundamental work of

Lasiecka and Triggiani [14, 15] and Lions and Strauss [21]. In [5] the authors analyzed the global regularity of solutions to a similar equation, but with the more regular damping term $|u_t|^{m-1}u_t$. Although the blow-up result obtained in [5] is for large data, their proof can be modified to yield the same blow-up result for small initial data. In [17], Levine and Serrin proved several abstract theorems on the global nonexistence of solutions to a large class of nonlinear hyperbolic equations. However, their results are not applicable to the initial-boundary value problem (1.2) due the lack of smoothness in the nonlinearity. For the same reason, a standard fixed-point argument to establish the existence of weak solutions to (1.2) does not apply.

## 2. Preliminaries

In this section we introduce some notation, definitions, and the technical assumptions that are necessary for the remaining sections of the paper. Let $L^2(\Omega)$, $L^2(\Gamma_1)$, etc. denote the standard Lebesgue spaces and $H^s(\Omega)$, $H^s(\Gamma_0)$, $H^s(\Gamma_1), \ldots,$ denote the standard Sobolev spaces. By $H^s(\Gamma)$ we mean the space $H^s(\Gamma_0) \times H^s(\Gamma_1)$, and by $H^s_{0,\Gamma_0}(\Gamma)$, $s \geq 0$, we mean the subspace of $H^s(\Gamma)$ that is given by $H^s_{0,\Gamma_0}(\Gamma) = \{0\} \times H^s(\Gamma_1)$. Also, for $s > \frac{1}{2}$ we set

$$H^s_{0,\Gamma_0}(\Omega) = \{u \in H^s(\Omega) \mid u\,|_{\Gamma_0} = 0\},$$

where the evaluation on $\Gamma_0$ is taken in the sense of traces. For $u \in H^s(\Omega)$ we denote by $\gamma u$ the trace operator (whenever defined) on $\Gamma$, i.e., $\gamma u = u|_\Gamma$. Also, we set $\gamma_1 u = u|_{\Gamma_1}$.

Let $X$ and $Y$ be Banach spaces with $X \subset Y$. We write $X \hookrightarrow Y$ if the injection $i : X \to Y$ is continuous. Also, we denote by $\mathcal{L}(X, Y)$ the space of all continuous linear operators from $X$ to $Y$.

Throughout the paper, we let $A : L^2(\Omega) \to L^2(\Omega)$ be the operator given by: $A = -\Delta$ with its domain

$$\mathcal{D}(A) = \left\{ u \in H^2(\Omega) : u\,\big|_{\Gamma_0} = \frac{\partial u}{\partial n}\bigg|_{\Gamma_1} = 0 \right\}.$$

Also, we let $R : L^2_{0,\Gamma_0}(\Gamma) \to L^2(\Omega)$ be the Dirichlet-Neumann map, which is given by: $Rh = w$ if and only if

$$\Delta w = 0 \quad \text{in } \Omega, \quad w\,\big|_{\Gamma_0} = 0, \quad \frac{\partial w}{\partial n}\bigg|_{\Gamma_1} = h.$$

It is well known that $A$ is positive, self-adjoint, and $A$ is the inverse of a compact operator. Moreover, $A$ has the infinite sequence of positive eigenvalues $\{\lambda_n \mid n = 1, 2, \ldots\}$ and a corresponding sequence of eigenfunctions $\{e_n \mid n = 1, 2, \ldots\}$ that forms an orthonormal basis for $L^2(\Omega)$. Namely, if $u \in L^2(\Omega)$, then $u =$

$\sum_{n=1}^{\infty} u_n e_n$, where the convergence is in $L^2(\Omega)$, with $\|u\|^2_{L^2(\Omega)} = \sum_{n=1}^{\infty} |u_n|^2$ and $u_n = \langle u, e_n \rangle_{L^2(\Omega)}$.

The powers of $A$ are defined as follows: $A^s : \mathcal{D}(A^s) \subseteq L^2(\Omega) \to L^2(\Omega)$, $A^s u = \sum_{n=1}^{\infty} \lambda_n^s u_n e_n$, with the domain of $A^s$ given by

$$\mathcal{D}(A^s) = \Big\{ u \in L^2(\Omega) \mid u = \sum_{n=1}^{\infty} u_n e_n,\ \sum_{n=1}^{\infty} \lambda_n^{2s} |u_n|^2 < \infty \Big\}.$$

We remark here that the results of Grisvard [8] and Seeley [25] give the following characterization for the fractional powers of $A$:

$$\mathcal{D}(A^s) = \begin{cases} H^{2s}(\Omega), & 0 \le s < \frac{1}{4}; \\ H^{2s}_{0,\Gamma_0}(\Omega), & \frac{1}{4} < s < \frac{3}{4}; \\ \Big\{ u \in H^{2s}(\Omega) : u\big|_{\Gamma_0} = 0, \dfrac{\partial u}{\partial n}\Big|_{\Gamma_1} = 0 \Big\}, & \frac{3}{4} < s \le 1. \end{cases} \tag{2.1}$$

Moreover, $\mathcal{D}(A^{1/4}) \hookrightarrow H^{1/2}(\Omega)$, $\mathcal{D}(A^{3/4}) \hookrightarrow H^{3/2}_{0,\Gamma_0}(\Omega)$, and the norm $\|u\|_{H^s(\Omega)}$ is equivalent to $(\sum_{n=1}^{\infty} \lambda_n^s |u_n|^2)^{1/2}$. Therefore, we set

$$||u||^2_{H^s(\Omega)} = \sum_{n=1}^{\infty} \lambda_n^s |u_n|^2.$$

Let $S(t)$ and $C(t)$ be the sine and cosine operators associated with $A$. Specifically, $S(t), C(t) : L^2(\Omega) \to L^2(\Omega)$ are given by $S(t) = A^{-1/2} \sin(A^{1/2} t)$ and $C(t) = \cos(A^{1/2} t)$.

The following assumptions will be valid throughout the paper:

$$u^0 \in H^1_{0,\Gamma_0}(\Omega), \quad u^1 \in L^2(\Omega), \tag{2.2}$$

$$k,\ p \ge 1, \quad 0 < m < 1, \quad \text{and} \quad p,\ \frac{k}{1-m} \le \frac{n}{n-2} \text{ if } n \ge 3. \tag{2.3}$$

Finally, the following Sobolev imbeddings will be used frequently in the paper:

$$\begin{cases} H^1_{0,\Gamma_0}(\Omega) \hookrightarrow L^q(\Omega), & \text{for } 1 \le q \le \dfrac{2n}{n-2},\ n \ge 3, \\ H^1_{0,\Gamma_0}(\Omega) \hookrightarrow L^q(\Omega), & \text{for } 1 \le q < \infty,\ n = 1, 2. \end{cases} \tag{2.4}$$

Also, throughout the paper we set:

$$G(u, u') = |u|^k \, |u'|^m \operatorname{sgn}(u'), \quad g(u') = |u'|^m \operatorname{sgn}(u'), \quad f(u) = |u|^{p-1} u.$$

We shall use the weak formulation of the problem to define what we mean by a solution to the initial-boundary value problem (1.2).

***Definition 2.1.*** Let $u^0 \in H^1_{0,\Gamma_0}(\Omega)$, $u^1 \in L^2(\Omega)$. We say that $u$ is a weak solution to the initial-boundary value problem (1.2) on $[0,T]$ if

$$u \in L^2(0,T,H^1_{0,\Gamma_0}(\Omega)), \quad u' \in L^2(0,T,L^2(\Omega))$$

and $u$ satisfies:

$$\begin{aligned}(2.5)\quad \langle u'(t),\varphi\rangle_{L^2(\Omega)} - \langle u^1,\varphi\rangle_{L^2(\Omega)} + \int_0^t \langle A^{1/2}u(s), A^{1/2}\varphi\rangle_{L^2(\Omega)}\, ds \\ + \int_0^t \Big[\langle G(u(s),u'(s)),\varphi\rangle_{L^2(\Omega)} - \langle f(u(s)),\varphi\rangle_{L^2(\Omega)}\Big]\, ds \\ - \int_0^t \langle h(s),\gamma_1\varphi\rangle_{L^2(\Gamma_1)}\, ds = 0,\end{aligned}$$

for all $\varphi \in H^1_{0,\Gamma_0}(\Omega)$ and almost every $t \in [0,T]$.

In order for us to obtain certain estimates, we now derive the integral equations that must be satisfied by a weak solution to the initial-boundary value problem (1.2). Let $v(t) = u(t) - w(t)$, where $w(t) = Rh(t)$. Then, $v$ formally satisfies the abstract initial value problem:

$$(2.6)\qquad v'' + Av = -w'' + f(v+w) - G(v+w, v'+w') \quad \text{on } (0,T),$$

$$(2.7)\qquad v(0) = u^0 - w(0), \quad v'(0) = u^1 - w'(0).$$

Thus, by the variation of parameters formula, we have

$$\begin{aligned}(2.8)\quad v(t) &= C(t)(u^0 - w(0)) + S(t)(u^1 - w'(0)) \\ &\quad - \int_0^t S(t-\tau)[w''(\tau) - f(u(\tau)) + G(u(\tau),u'(\tau))]\, d\tau.\end{aligned}$$

Formal integration by parts yields

$$\begin{aligned}(2.9)\quad u(t) &= C(t)(u^0 - Rh(0)) + S(t)u^1 + Rh(t) - \int_0^t C(t-\tau)Rh'(\tau)\, d\tau \\ &\quad + \int_0^t S(t-\tau)[f(u(\tau)) - G(u(\tau),u'(\tau))]\, d\tau.\end{aligned}$$

By differentiating (2.9), one has

$$\begin{aligned}(2.10)\quad u'(t) &= C(t)u^1 - AS(t)(u^0 - Rh(0)) + \int_0^t AS(t-\tau)Rh'(\tau)\, d\tau \\ &\quad + \int_0^t C(t-\tau)[f(u(\tau)) - G(u(\tau),u'(\tau))]\, d\tau.\end{aligned}$$

At this end we let

$$U_0(t) = C(t)(u^0 - Rh(0)) + S(t)u^1 + Rh(t) - \int_0^t C(t-\tau)Rh'(\tau)\,d\tau, \tag{2.11}$$

$$V_0(t) = C(t)u^1 - AS(t)(u^0 - Rh(0)) + \int_0^t AS(t-\tau)Rh'(\tau)\,d\tau. \tag{2.12}$$

The following regularity results are well-known (for example, see [14, 15, 20]), and thus their proofs are omitted.

***Lemma 2.2.*** *For $s \geq 0$, we have*

(i) $C(\cdot) \in \mathcal{L}(\mathcal{D}(A^s), C([0,T], \mathcal{D}(A^s)))$,
(ii) $S(\cdot) \in \mathcal{L}(\mathcal{D}(A^s), C([0,T], \mathcal{D}(A^{s+1/2})))$,
(iii) $R \in \mathcal{L}(L^2_{0,\Gamma_0}(\Gamma), H^{3/2}_{0,\Gamma_0}(\Omega))$.

***Remark 2.3.*** In view of Lemma 2.2 and the fact that

$$h \in C^1([0,\infty), L^2(\Gamma_1)),$$

it is easy to see that $U_0 \in C([0,\infty), H^1_{0,\Gamma_0}(\Omega))$, and $V_0 \in C([0,\infty), L^2(\Omega))$. Moreover, it is not too difficult to show that if $u \in L^2(0,T,H^1_{0,\Gamma_0}(\Omega))$, $u' \in L^2(0,T,L^2(\Omega))$, and $u$ satisfies the integral equations:

$$u(t) = U_0(t) + \int_0^t S(t-\tau)[f(u(\tau)) - G(u(\tau),u'(\tau))]\,d\tau, \tag{2.13}$$

$$u'(t) = V_0(t) + \int_0^t C(t-\tau)[f(u(\tau)) - G(u(\tau),u'(\tau))]\,d\tau, \tag{2.14}$$

then $u$ is a weak solution to (1.2) in the sense of Definition 2.1. Moreover, the converse is also valid. The proof of this fact is similar to that of Remark 2.1 in [2] and thus it is omitted.

## 3. APPROXIMATE SOLUTIONS

Our first step is to construct the sequence of approximate solutions to the initial boundary value problem (1.2) and obtain the necessary estimates for the passage to the limit, without further restriction on the damping or source terms. Let $\{e_k\}_{k=1}^\infty$ be the orthonormal basis for $L^2(\Omega)$, as described in Section 2. Let $\mathcal{P}_N$ be the orthogonal projection of $L^2(\Omega)$ onto the linear span of $\{e_1,\dots,e_N\}$. Let $u_N(t) = \sum_{k=1}^N u_{N,k}(t)e_k$ be a weak solution to the Galerkin system associated with the initial-boundary value problem (1.2), i.e., $u_N(t)$ satisfies the initial value problem:

$$\begin{aligned}(3.1)\quad &\frac{d}{dt}\langle u_N'(t), e_k\rangle_{L^2(\Omega)} + \langle A^{1/2}u_N(t), A^{1/2}e_k\rangle_{L^2(\Omega)} - \langle h(t), \gamma_1 e_k\rangle_{L^2(\Gamma_1)}\\ &\quad + \langle \mathcal{P}_N G(u_N(t), u_N'(t)), e_k\rangle_{L^2(\Omega)} - \langle \mathcal{P}_N f(u_N(t)), e_k\rangle_{L^2(\Omega)} = 0\end{aligned}$$

$$(3.2)\qquad u_{N,k}(0) = u_k^0, \quad u_{N,k}'(0) = u_k^1,$$

for $k = 1, 2\ldots, N$, where $u_k^0 = \langle u^0, e_k\rangle_{L^2(\Omega)}$ and $u_k^1 = \langle u^1, e_k\rangle_{L^2(\Omega)}$. Now, (3.1) and (3.2) are equivalent to:

$$\begin{aligned}(3.3)\qquad &u_{N,k}''(t) + \lambda_k u_{N,k}(t) = \langle h(t), \gamma_1 e_k\rangle_{L^2(\Gamma_1)}\\ &\quad - \langle \mathcal{P}_N G(u_N(t), u_N'(t)), e_k\rangle_{L^2(\Omega)} + \langle \mathcal{P}_N f(u_N(t)), e_k\rangle_{L^2(\Omega)},\end{aligned}$$

$$(3.4)\qquad u_{N,k}(0) = u_k^0, \quad u_{N,k}'(0) = u_k^1,$$

for $k = 1, 2\ldots, N$.

Since (3.3), (3.4) is an initial value problem for a second order $N \times N$ system of ordinary differential equations with continuous nonlinearities in the unknown functions $u_{N,k}$ and their derivatives $u_{N,k}'$, it follows from the Cauchy-Peano Theorem that for every $N \geq 1$, (3.3), (3.4) has at least one solution $u_{N,k}$ defined on $[0, T_N]$, for some $T_N > 0$. Moreover, for $1 \leq k \leq N$, $u_{N,k} \in C^2[0, T_N]$, and they satisfy the following integral equation on $[0, T_N]$:

$$\begin{aligned}(3.5)\quad u_{N,k}(t) &= u_k^0 \cos\lambda_k^{1/2}t + u_k^1\lambda_k^{-1/2}\sin\lambda_k^{1/2}t\\ &\quad + \int_0^t \lambda_k^{-1/2}\sin\lambda_k^{1/2}(t-\tau)H_{N,k}(\tau)\,d\tau,\end{aligned}$$

for $k = 1, 2\ldots, N$, and where

$$\begin{aligned}H_{N,k}(\tau) &= \langle h(\tau), \gamma_1 e_k\rangle_{L^2(\Gamma_1)} - \langle \mathcal{P}_N G(u_N(\tau), u_N'(\tau)), e_k\rangle_{L^2(\Omega)}\\ &\quad + \langle \mathcal{P}_N f(u_N(\tau)), e_k\rangle_{L^2(\Omega)}.\end{aligned}$$

Now by noting the definition of the Dirichlet-Neumann map $Rh(t) = w(t)$, we have

$$(3.6)\qquad \langle h(\tau), \gamma_1 e_k\rangle_{L^2(\Gamma_1)} = \langle A^{1/2}w(\tau), A^{1/2}e_k\rangle_{L^2(\Omega)} = \lambda_k\langle Rh(\tau), e_k\rangle_{L^2(\Omega)}.$$

Therefore, by using (3.6) and integration by parts, the first term in the convolution in (3.5) is given by

$$
\begin{aligned}
(3.7)\qquad & \int_0^t \lambda_k^{-1/2} \sin \lambda_k^{1/2}(t-\tau)\langle h(\tau), \gamma_1 e_k\rangle_{L^2(\Gamma_1)}\, d\tau \\
&= \int_0^t \lambda_k^{1/2} \sin \lambda_k^{1/2}(t-\tau)\langle Rh(\tau), e_k\rangle_{L^2(\Omega)}\, d\tau \\
&= \langle Rh(t), e_k\rangle_{L^2(\Omega)} - \cos \lambda_k^{1/2} t\langle Rh(0), e_k\rangle_{L^2(\Omega)} \\
&\quad - \int_0^t \cos \lambda_k^{1/2}(t-\tau)\langle Rh'(\tau), e_k\rangle_{L^2(\Omega)}\, d\tau.
\end{aligned}
$$

It follows easily from (3.5), (3.7), and the definitions of the sine and cosine operators that $u_N$ satisfies the following integral equations on $[0, T_N]$:

$$
(3.8)\quad u_N(t) = U_{N,0}(t) + \int_0^t S(t-\tau)\mathcal{P}_N[f(u_N(\tau)) - G(u_N(\tau), u_N'(\tau))]\, d\tau,
$$

$$
(3.9)\quad u_N'(t) = V_{N,0}(t) + \int_0^t C(t-\tau)\mathcal{P}_N[f(u_N(\tau)) - G(u_N(\tau), u_N'(\tau))]\, d\tau,
$$

where

$$
\begin{aligned}
U_{N,0}(t) &= C(t)\mathcal{P}_N(u^0 - Rh(0)) + S(t)\mathcal{P}_N u^1 + \mathcal{P}_N Rh(t) \\
&\quad - \int_0^t C(t-\tau)\mathcal{P}_N Rh'(\tau)\, d\tau, \\
V_{N,0}(t) &= C(t)\mathcal{P}_N u^1 - AS(t)\mathcal{P}_N(u^0 - Rh(0)) \\
&\quad + \int_0^t AS(t-\tau)\mathcal{P}_N Rh'(\tau)\, d\tau.
\end{aligned}
$$

***A priori estimates.*** Here, we shall show that $T_N$ can be replaced by some $T > 0$, for all $N \geq 1$. In the remaining parts of the paper, we shall refer to the following Hilbert spaces repeatedly:

$$
X_T = L^2(0, T, L^2(\Omega)) \quad \text{and} \quad Y_T = L^2(0, T, H^1_{0,\Gamma_0}(\Omega)).
$$

***Lemma 3.1.*** *There exists a constant $T > 0$ such that the sequence of approximate solutions $\{u_N\}$ satisfies the following:*

(i) $\{u_N\}$ *is bounded in* $Y_T$*;*
(ii) $\{u_N'\}$ *is bounded in* $X_T$*;*
(iii) $\mathcal{P}_N|u_N'|^m \operatorname{sgn}(u_N')$ *is bounded in* $L^2(0, T, L^{2/m}(\Omega))$.

*Proof.* Fix $T_0 > 0$. First note that by using Lemma 2.2, it is easy to check that $U_{N,0} \in C([0, T_0], H^1_{0,\Gamma_0}(\Omega))$ and $V_{N,0} \in C([0, T_0], L^2(\Omega))$. Also, we note that $\mathcal{P}_N u^0 \to u^0$ in $H^1_{0,\Gamma_0}(\Omega)$, and $\mathcal{P}_N u^1 \to u^1$ in $L^2(\Omega)$. Therefore, it follows from Lemma 2.2 that $U_{N,0} \to U_0$ in $C([0,T_0], H^1_{0,\Gamma_0}(\Omega))$ and $V_{N,0} \to V_0$ in $C([0,T_0], L^2(\Omega))$ as $N \to \infty$. So, there exists a constant $M > 0$ such that for all $N$,

$$\sup_{t\in[0,T_0]} \|U_{N,0}(t)\|_{H^1_{0,\Gamma_0}(\Omega)} \le \sup_{t\in[0,T_0]} \|U_0(t)\|_{H^1_{0,\Gamma_0}(\Omega)} + M \equiv \alpha, \tag{3.10}$$

$$\sup_{t\in[0,T_0]} \|V_{N,0}(t)\|_{L^2(\Omega)} \le \sup_{t\in[0,T_0]} \|V_0(t)\|_{L^2(\Omega)} + M \equiv \beta. \tag{3.11}$$

It should be noted here that in (3.10) and (3.11), the constants $\alpha$ and $\beta$ depend on $T_0$. It follows from (3.8) and Lemma 2.2 that, for all $t \in [0,T_0]$ and all $N \ge 1$,

$$\begin{aligned} \|u_N(t)\|_{H^1_{0,\Gamma_0}(\Omega)} &\le \|U_{N,0}(t)\|_{H^1_{0,\Gamma_0}(\Omega)} \\ &\quad + \int_0^t \|S(t-\tau)\mathcal{P}_N[f(u_N(\tau)) - G(u_N(\tau), u_N'(\tau))]\|_{H^1_{0,\Gamma_0}(\Omega)}\, d\tau \\ &\le \alpha + \int_0^t \|f(u_N(\tau)) - G(u_N(\tau), u_N'(\tau))\|_{L^2(\Omega)}\, d\tau. \end{aligned} \tag{3.12}$$

However, by using Hölder's inequality, (2.3), and (2.4), one has

$$\begin{aligned} &\|f(u_N(\tau)) - G(u_N(\tau), u_N'(\tau))\|_{L^2(\Omega)} \\ &\quad \le \|\, |u_N(\tau)|^{p-1} u_N(\tau)\|_{L^2(\Omega)} + \|\, |u_N(\tau)|^k\, |u_N'(\tau)|^m\|_{L^2(\Omega)} \\ &\quad \le ||u_N(\tau)||_{2p}^p + ||u_N'(\tau)||_2^m ||u_N(\tau)||_{2k/(1-m)}^k \\ &\quad \le C[||u_N(\tau)||^p_{H^1_{0,\Gamma_0}(\Omega)} + ||u_N'(\tau)||_2^m ||u_N(\tau)||^k_{H^1_{0,\Gamma_0}(\Omega)}], \end{aligned} \tag{3.13}$$

for some positive constant $C = C(m,k,\Omega)$. It follows from (3.12) and (3.13) that

$$\begin{aligned} \|u_N(t)\|_{H^1_{0,\Gamma_0}(\Omega)} \le \alpha + C\int_0^t [||u_N(\tau)||^p_{H^1_{0,\Gamma_0}(\Omega)} \\ + ||u_N'(\tau)||_2^m\, ||u_N(\tau)||^k_{H^1_{0,\Gamma_0}(\Omega)}]\, d\tau. \end{aligned} \tag{3.14}$$

Similarly, we have

$$\begin{aligned} \|u_N'(t)\|_{L^2(\Omega)} &\le \|V_{N,0}(t)\|_{L^2(\Omega)} \\ &\quad + \int_0^t \|C(t-\tau)\mathcal{P}_N[f(u_N(\tau)) - G(u_N(\tau), u_N'(\tau))]\|_{L^2(\Omega)}\, d\tau \\ &\le \beta + \int_0^t \|f(u_N(\tau)) - G(u_N(\tau), u_N'(\tau))\|_{L^2(\Omega)}\, d\tau \quad \le \end{aligned} \tag{3.15}$$

$$\leq \beta + C \int_0^t [\|u_N(\tau)\|^p_{H^1_{0,\Gamma_0}(\Omega)} + \|u'_N(\tau)\|_2^m \|u_N(\tau)\|^k_{H^1_{0,\Gamma_0}(\Omega)}]\, d\tau.$$

Therefore, it follows from (3.14) and (3.15) that

$$\begin{aligned} (3.16)\quad & \|u_N(t)\|_{H^1_{0,\Gamma_0}(\Omega)} + \|u'_N(t)\|_{L^2(\Omega)} \\ & \leq \alpha + \beta + C \int_0^t [\|u_N(\tau)\|^p_{H^1_{0,\Gamma_0}(\Omega)} + \|u'_N(\tau)\|_2^m \|u_N(\tau)\|^k_{H^1_{0,\Gamma_0}(\Omega)}]\, d\tau. \end{aligned}$$

Let $y_N(t) := 1 + \|u_N(t)\|_{H^1_{0,\Gamma_0}(\Omega)} + \|u'_N(t)\|_{L^2(\Omega)}$ and $\sigma = \max\{p, k+m\}$. Then, for $t \in [0, T_0]$ we have

$$y_N(t) \leq 1 + \alpha + \beta + C \int_0^t y_N(\tau)^\sigma\, d\tau. \tag{3.17}$$

Hence, by using a standard comparison theorem, (3.17) yields that $y_N(t) \leq z(t)$, where $z(t) = [(1+\alpha+\beta)^{1-\sigma} - C(\sigma-1)t]^{-1/(\sigma-1)}$ is the solution to the Volterra integral equation

$$z(t) = 1 + \alpha + \beta + C \int_0^t z(\tau)^\sigma\, d\tau. \tag{3.18}$$

Although $z(t)$ blows-up in finite time (since $\sigma > 1$), there exists a time $T > 0$, $T < T_0$ such that $y_N(t) \leq z(t) \leq C$ for all $t \in [0, T]$ and all $N \geq 1$. This shows that $\{u_N\}$ is bounded in $Y_T$ and $\{u'_N\}$ is bounded in $X_T$. Consequently, $\mathcal{P}_N |u'_N|^m \operatorname{sgn}(u'_N)$ is also bounded in $X_T$ and in $L^2(0, T, L^{2/m}(\Omega))$, by Hölder's inequality. □

The following compactness theorem is a special case of Aubin's compactness theorem which can be found in [27], for example.

***Compactness Theorem.*** *Let $X_T$ and $Y_T$ be the Hilbert spaces as described above. Let $\mathcal{Y}$ be the space of functions $\mathcal{Y} = \{u \in Y_T,\ u' \in X_T\}$ endowed with the natural norm $\|u\|^2_{\mathcal{Y}} = \|u\|^2_{Y_T} + \|u'\|^2_{X_T}$. Then the imbedding $\mathcal{Y} \overset{i}{\hookrightarrow} X_T$ is compact.*

Now, by using Lemma 3.1 and the compactness theorem above, we can extract a subsequence of $\{u_N\}$ (still denoted by $\{u_N\}$) and find functions $u$ and $\eta$ with

$u \in Y_T$, $u' \in X_T$ and $\eta \in L^2(0,T,L^{2/m}(\Omega)) \subset X_T$ such that:

$$(3.19)\qquad \begin{cases} u_N \to u \text{ strongly in } X_T, \\ u_N \to u \text{ weakly in } Y_T, \\ u'_N \to u' \text{ weakly in } X_T, \\ \mathcal{P}_N |u'_N|^m \operatorname{sgn}(u'_N) \to \eta \text{ weakly in } X_T \\ u_N(t) \to u(t) \text{ strongly in } L^2(\Omega) \text{ for almost all } t \in [0,T], \\ u_N(t) \to u(t) \text{weakly in } H^1_{0,\Gamma_0}(\Omega) \text{ for almost all } t \in [0,T], \\ u'_N(t) \to u'(t) \text{ weakly in } L^2(\Omega) \text{ for almost all } t \in [0,T], \\ u_N(.,t) \to u(.,t) \text{ point-wise a.e. } \Omega, \text{ for almost all } t \in [0,T]. \end{cases}$$

Before passing to the limit we shall need some auxiliary lemmas. First, let us introduce the following temporary notation:

$$F(s) = |s|^k,$$

and

$$F_j(s) = \begin{cases} |s|^k; & |s| < j, \\ j^k; & |s| \geq j. \end{cases}$$

Also, we let $\Omega_{j,v} := \{x \in \Omega : |v(x)| \geq j\}$.

***Lemma 3.2.*** *With $k$ satisfying* (2.3)*, then $F_j(\cdot) \to F(\cdot)$ in $L^2(\Omega)$, uniformly on bounded sets in $H^1_{0,\Gamma_0}(\Omega)$. More specifically, given $\varepsilon > 0$ and $L > 0$, then there exists $j_0 \geq 1$, such that for all $v \in H^1_{0,\Gamma_0}(\Omega)$ with $\|v\|_{H^1_{0,\Gamma_0}(\Omega)} \leq L$, we have*

$$(3.20)\qquad ||F_j(v) - F(v)||^2_{L^2(\Omega)} = \int_{\Omega_{j,v}} |j^k - |v(x)|^k|^2\, dx < \varepsilon^2,$$

*for all $j \geq j_0$.*

*Proof.* Let $v \in H^1_{0,\Gamma_0}(\Omega)$ with $\|v\|_{H^1_{0,\Gamma_0}(\Omega)} \leq L$. Then,

$$(3.21)\qquad \begin{aligned} ||F_j(v) - F(v)||^2_{L^2(\Omega)} &= \int_{\Omega_{j,v}} |j^k - |v(x)|^k|^2\, dx \\ &\leq 2\int_{\Omega_{j,v}} (j^{2k} + |v(x)|^{2k})\, dx. \end{aligned}$$

With the assumptions on $k$, we can choose $q$ such that $2k < q < \infty$, when $n = 1$, 2; and $2k < q \leq 2n/(n-2)$, when $n \geq 3$. Then,

$$\int_{\Omega_{j,v}} j^q \, dx = j^q |\Omega_{j,v}| \leq \int_{\Omega_{j,v}} |v(x)|^q \, dx \leq C \|v\|^q_{H^1_{0,\Gamma_0}(\Omega)} \leq CL^q, \tag{3.22}$$

where $|\Omega_{j,v}|$ denotes the Lebesgue measure of $\Omega_{j,v}$. In particular, we have

$$|\Omega_{j,v}| \leq CL^q j^{-q}, \quad j = 1, 2 \dots. \tag{3.23}$$

Therefore,

$$\int_{\Omega_{j,v}} j^{2k} \, dx \leq CL^q j^{-q+2k}, \quad j = 1, 2 \dots. \tag{3.24}$$

Moreover, by using Hölder's inequality, (2.4) and (3.23), we have

$$\begin{aligned} \int_{\Omega_{j,v}} |v(x)|^{2k} \, dx &\leq \|v\|^{2k}_{2k/(1-m)} |\Omega_{j,v}|^m \leq C \|v\|^{2k}_{H^1_{0,\Gamma_0}(\Omega)} |\Omega_{j,v}|^m \\ &\leq C^{1+m} L^{2k+mq} j^{-mq}. \end{aligned} \tag{3.25}$$

It follows from (3.24), (3.25) and (3.21) that there exists $j_0 \geq 1$ such that

$$\|F_j(v) - F(v)\|^2_{L^2(\Omega)} \leq 2(CL^q j^{-q+2k} + C^{1+m} L^{2k+mq} j^{-mq}) \leq \varepsilon^2, \tag{3.26}$$

for all $j \geq j_0$, and the proof is complete. □

***Lemma 3.3.*** *Let* $\{u_N\}$ *be the sequence of approximate solutions satisfying* (3.19). *Then, for any* $k \geq 1$ *satisfying* (2.3), *we have*

$$|u_N(t)|^k \to |u(t)|^k \quad \text{strongly in } L^2(\Omega) \text{ for almost all } t \in [0, T]. \tag{3.27}$$

*Proof.* Let $k \geq 1$ and satisfying (2.3). Let $L > 0$ be such that, for all $t \in [0, T]$ and all $N \geq 1$,

$$\|u_N(t)\|_{H^1_{0,\Gamma_0}(\Omega)} \quad \text{and} \quad \|u(t)\|_{H^1_{0,\Gamma_0}(\Omega)} \leq L.$$

We choose $q$ as in the proof of Lemma 3.2 and $j_0$ large enough so that

$$\|F_{j_0}(v) - F(v)\|^2_{L^2(\Omega)} = \int_{\Omega_{j_0,v}} |j_0^k - |v(x)|^k|^2 \, dx < \frac{\varepsilon^2}{16}, \tag{3.28}$$

and

$$8 \int_{\Omega_{j_0,v}} j_0^{2k} \, dx \leq 8CL^q j_0^{-q+2k} < \frac{\varepsilon^2}{16}, \tag{3.29}$$

for all $v \in H^1_{0,\Gamma_0}(\Omega)$ with $\|v\|_{H^1_{0,\Gamma_0}(\Omega)} \le L$. Now, for all $N \ge 1$, we have

$$\begin{aligned}\| \, |u_N(t)|^k - |u(t)|^k \, \|_{L^2(\Omega)} &= \|F(u_N(t)) - F(u(t))\|_{L^2(\Omega)} \\ &\le \|F(u_N(t)) - F_{j_0}(u_N(t))\|_{L^2(\Omega)} \\ &\quad + \|F_{j_0}(u_N(t)) - F_{j_0}(u(t))\|_{L^2(\Omega)} \\ &\quad + \|F_{j_0}(u(t)) - F(u(t))\|_{L^2(\Omega)} \\ &< \frac{\varepsilon}{2} + \|F_{j_0}(u_N(t)) - F_{j_0}(u(t))\|_{L^2(\Omega)}.\end{aligned} \tag{3.30}$$

For fixed $t \in [0,T]$, let

$$\begin{aligned}\Omega_{1,j_0,N} &= \{x \colon |u_N(t)|, \ |u(t)| < j_0\}, \\ \Omega_{2,j_0,N} &= \{x \colon |u_N(t)| < j_0, \ |u(t)| \ge j_0\}, \\ \Omega_{3,j_0,N} &= \{x \colon |u_N(t)| \ge j_0, \ |u(t)| < j_0\}, \\ \Omega_{4,j_0,N} &= \{x \colon |u_N(t)|, \ |u(t)| \ge j_0\}.\end{aligned}$$

Now, by using Lemma 3.2, one has

$$\begin{aligned}&||F_{j_0}(u_N(t)) - F_{j_0}(u(t))||^2_{L^2(\Omega)} \\ &\quad = ||F_{j_0}(u_N(t)) - F_{j_0}(u(t))||^2_{L^2(\Omega_{1,j_0,N})} \\ &\qquad + ||F_{j_0}(u_N(t)) - F_{j_0}(u(t))||^2_{L^2(\Omega_{2,j_0,N})} \\ &\qquad\quad + ||F_{j_0}(u_N(t)) - F_{j_0}(u(t))||^2_{L^2(\Omega_{3,j_0,N})}.\end{aligned} \tag{3.31}$$

However, for all $N \ge 1$

$$\begin{aligned}||F_{j_0}(u_N(t)) - F_{j_0}(u(t))||^2_{L^2(\Omega_{2,j_0,N})} &= \int_{\Omega_{2,j_0,N}} | \, |u_N(t)|^k - j_0^k|^2 \, dx \\ &\le 4 \int_{\Omega_{2,j_0,N}} j_0^{2k} \, dx \\ &\le 4 \int_{\Omega_{j_0,u(t)}} j_0^{2k} \, dx < \frac{\varepsilon^2}{32},\end{aligned} \tag{3.32}$$

where we have used (3.29). Similarly, one has

$$||F_{j_0}(u_N(t)) - F_{j_0}(u(t))||^2_{L^2(\Omega_{3,j_0,N})} < \frac{\varepsilon^2}{32}. \tag{3.33}$$

Since $u_N(.,t) \to u(.,t)$ point-wise a.e. $\Omega$ as $N \to \infty$, for almost all $t \in [0,T]$ and $\chi_{\Omega_{1,j_0,N}}(x)| \, |u_N(t)|^k - |u(t)^k| \, |^2 < 4j_0^{2k}$ (where $\chi_{\Omega_{1,j_0,N}}$ denotes the characteristic function on $\Omega_{1,j_0,N}$), then by the dominated convergence theorem, there exists an $N_0 \geq 1$ such that, for all $N \geq N_0$, we have

$$\begin{aligned}(3.34) \qquad &\|F_{j_0}(u_N(t)) - F_{j_0}(u(t))\|^2_{L^2(\Omega_{1,j_0,N})} \\ &= \int_\Omega \chi_{\Omega_{1,j_0,N}}(x) |\,|u_N(t)|^k - |u(t)^k|\,|^2\,dx < \frac{\varepsilon^2}{16}.\end{aligned}$$

Finally, it follows from (3.30), (3.31) and (3.32)-(3.34) that

$$(3.35) \qquad \|\,|u_N(t)|^k - |u(t)|^k\,\|_{L^2(\Omega)} < \varepsilon.$$

For all $N \geq N_0$ and almost all $t \in [0,T]$, which completes the proof. □

The following Lemma is an immediate consequence of Lemma 3.3.

***Lemma 3.4.*** *The sequence of approximate solutions* $\{u_N\}$ *satisfies:*

$$(3.36) \qquad \mathcal{P}_N |u_N|^k\,|u_N'|^m \operatorname{sgn}(u_N') \to |u|^k \eta \quad \text{weakly in } X_T.$$

*Proof.* The fact that $|u(t)|^k\eta(t) \in L^2(\Omega)$ for almost all $t \in [0,T]$ follows easily from Hölder's inequality. The convergence in (3.36) follows easily by using Lemma 3.3 and a density argument. □

***Lemma 3.5.*** *Let* $p \geq 1$ *satisfying* (2.3). *Then, there exists a subsequence of* $\{u_N\}$, *still denoted by* $\{u_N\}$, *such that, for almost all* $t \in [0,T]$

$$(3.37) \qquad |u_N(t)|^{p-1}u_N(t) \to |u(t)|^{p-1}u(t) \quad \text{strongly in } L^2(\Omega),$$

*for* $p \geq 1$, $n = 1, 2$; *and for* $1 \leq p < n/(n-2)$, $n \geq 3$. *Moreover, the mode of convergence in* (3.37) *is replaced by weakly in* $L^2(\Omega)$, *if* $p = n/(n-2)$, $n \geq 3$.

*Proof.* Here, the proof is similar to the proof of Lemma 3.3, and thus will not be repeated. However, from the proof of Lemma 3.3, we only need the restriction that $1 \leq p < n/(n-2)$ when $n \geq 3$ to deduce (3.37). Moreover, if $p = n/(n-2)$, then obviously $p - 1 < n/(n-2)$, and one has $|u_N(t)|^{p-1} \to |u(t)|^{p-1}$ strongly in $L^2(\Omega)$. The fact that $|u_N(t)|^{p-1}u_N(t)$ is bounded in $L^2(\Omega)$ yields the second statement of the lemma. □

Now, the sequence of approximate solutions in (3.19) satisfies

$$\begin{aligned}(3.38) \quad &\langle u_N'(t), e_k\rangle_{L^2(\Omega)} - \langle \mathcal{P}_N u^1, e_k\rangle_{L^2(\Omega)} \\ &+ \int_0^t \langle A^{1/2}u_N(s), A^{1/2}e_k\rangle_{L^2(\Omega)}\,ds \\ &+ \int_0^t [\langle \mathcal{P}_N G(u_N(s), u_N'(s)), e_k\rangle_{L^2(\Omega)} - \langle \mathcal{P}_N f(u_N(s)), e_k\rangle_{L^2(\Omega)}\,ds] \\ &- \int_0^t \langle h(s), \gamma_1 e_k\rangle_{L^2(\Gamma_1)}\,ds = 0,\end{aligned}$$

for all $k = 1, 2\ldots, N$, and almost every $t \in [0, T]$.

By letting $N \to \infty$, and by using (3.19) and Lemmas 3.4, 3.5, one finds that the limit function $u$ satisfies

$$\begin{aligned}(3.39)\qquad &\langle u'(t), \varphi\rangle_{L^2(\Omega)} - \langle u^1, \varphi\rangle_{L^2(\Omega)} + \int_0^t \langle A^{1/2}u(s), A^{1/2}\varphi\rangle_{L^2(\Omega)}\, ds\\ &+ \int_0^t [\langle |u(s)|^k\eta(s), \varphi\rangle_{L^2(\Omega)} - \langle f(u(s)), \varphi\rangle_{L^2(\Omega)}]\, ds\\ &- \int_0^t \langle h(s), \gamma_1\varphi\rangle_{L^2(\Gamma_1)}\, ds = 0,\end{aligned}$$

for all $\varphi \in H^1_{0,\Gamma_0}(\Omega)$ and almost every $t \in [0, T]$. In view of Remark 2.3, then $u$ satisfies the integral equations

$$(3.40)\qquad u(t) = U_0(t) + \int_0^t S(t-\tau)[f(u(\tau)) - |u(\tau)|^k\eta(\tau)]\, d\tau,$$

$$(3.41)\qquad u'(t) = V_0(t) + \int_0^t C(t-\tau)[f(u(\tau)) - |u(\tau)|^k\eta(\tau)]\, d\tau,$$

where $U_0$ and $V_0$ are given in (2.11) and (2.12). Therefore, we deduce that $u \in C([0, T], H^1_{0,\Gamma_0}(\Omega))$ and $u' \in C([0, T], L^2(\Omega))$. Moreover, $u$ satisfies the initial-boundary value problem,

$$(3.42)\qquad \begin{cases} u_{tt} - \Delta u + |u|^k\eta = |u|^{p-1}u, & \text{in } \Omega \times (0, T),\\ u(x, 0) = u^0(x),\ u_t(x, 0) = u^1(x), & \text{in } \Omega,\\ u(x, t) = 0, & \text{on } \Gamma_0 \times (0, T),\\ \dfrac{\partial u}{\partial n}(x, t) = h(x, t), & \text{on } \Gamma_1 \times (0, T).\end{cases}$$

In addition, we have the following lemma.

***Lemma 3.6.*** *Let* $u^0 \in H^1_{0,\Gamma_0}(\Omega)$, $u^1 \in L^2(\Omega)$, $\eta \in L^2(0, T, L^{2/m}(\Omega))$ *and* $u$ *be a weak solution to the initial-boundary value problem* (3.42). *Then,* $u'' \in L^2(0, T, (H^1_{0,\Gamma_0}(\Omega))')$.

*Proof.* Let $\langle \cdot, \cdot\rangle$ denote the standard pairing of $(H^1_{0,\Gamma_0}(\Omega))'$ and $H^1_{0,\Gamma_0}(\Omega)$. Then, it follows from (3.39) that

$$\begin{aligned}|\langle u''(t), \varphi\rangle| &= \left|\frac{d}{dt}\langle u'(t), \varphi\rangle\right| = \left|\frac{d}{dt}\langle u'(t), \varphi\rangle_{L^2(\Omega)}\right|\\ &\le |\langle A^{1/2}u(t), A^{1/2}\varphi\rangle_{L^2(\Omega)}| + |\langle |u(t)|^k\eta(t), \varphi\rangle_{L^2(\Omega)}|\\ &\qquad + |\langle f(u(t)), \varphi\rangle_{L^2(\Omega)}| + |\langle h(t), \gamma_1\varphi\rangle_{L^2(\Gamma_1)}|\end{aligned}$$

for all $\varphi \in H^1_{0,\Gamma_0}(\Omega)$ and for almost all $t \in [0,T]$.

However, by the Cauchy-Schwarz inequality and Hölder's inequality, we have

$$\begin{aligned}
&|\langle u''(t), \varphi\rangle| \\
&\quad \le C\|\varphi\|_{H^1_{0,\Gamma_0}(\Omega)}(\|A^{1/2}u(t)\|_{L^2(\Omega)} + \|\,|u(t)|^k\eta(t)\|_{L^2(\Omega)}) \\
&\qquad + C\|\varphi\|_{H^1_{0,\Gamma_0}(\Omega)}\,||u(t)||^p_{L^{2p}(\Omega)} + \|h(t)\|_{L^2(\Gamma_1)}\,\|\gamma_1\varphi\|_{L^2(\Gamma_1)} \\
&\quad \le C\|\varphi\|_{H^1_{0,\Gamma_0}(\Omega)}\big(\|u(t)\|_{H^1_{0,\Gamma_0}(\Omega)} + \|\eta(t)\|_{L^{2/m}(\Omega)}\,||u(t)||^k_{L^{2k/(1-m)}(\Omega)} \\
&\qquad + ||u(t)||^p_{H^1_{0,\Gamma_0}(\Omega)} + \|h(t)\|_{L^2(\Gamma_1)}\big)
\end{aligned}$$

for all $\varphi \in H^1_{0,\Gamma_0}(\Omega)$ and for almost all $t \in [0,T]$. Therefore, for almost every $t \in [0,T]$

$$\begin{aligned}
(3.43)\quad &||u''(t)||^2_{(H^1_{0,\Gamma_0}(\Omega))'} \\
&\quad \le C(||u(t)||^2_{H^1_{0,\Gamma_0}(\Omega)} + ||\eta(t)||^2_{L^{2/m}(\Omega)}\,||u(t)||^{2k}_{H^1_{0,\Gamma_0}(\Omega)} \\
&\qquad\qquad + ||u(t)||^{2p}_{H^1_{0,\Gamma_0}(\Omega)} + ||h(t)||^2_{L^2(\Gamma_1)}),
\end{aligned}$$

for some constant $C$. Hence, the lemma follows from (3.43). □

## 4. LOCAL SOLUTIONS

In this section, we shall show that the initial-boundary value problem (1.2) has a unique local solution under the general restrictions on the damping and source terms. We accomplish this by first showing that $\eta = g(u')$ and then prove uniqueness. Specifically, we have the following Theorem.

***Theorem 4.1.*** *Let $u^0 \in H^1_{0,\Gamma_0}(\Omega)$, $u^1 \in L^2(\Omega)$, and $h \in C^1([0,\infty), L^2(\Gamma_1))$. Then there exists a constant $T > 0$ such that the initial-boundary value problem* (1.2) *has a unique weak solution on* $[0,T]$ *with*

$$u \in C([0,T]; H^1_{0,\Gamma_0}(\Omega)) \quad \text{and} \quad u' \in C([0,T]; L^2(\Omega)).$$

*Proof.* The proof follows immediately from the previous section and from Lemma 4.4 and Lemma 4.5 below. □

We start by deriving an energy identity for the approximate solutions $\{u_N\}$. By multiplying equation (3.3) by $u'_{N,k}(t)$, summing from 1 to $N$, and integrating from 0 to $t$, we obtain

$$
\begin{aligned}
(4.1)\qquad E_N(t) := {} & \frac{1}{2}(\|u_N'(t)\|^2_{L^2(\Omega)} + \|A^{1/2}u_N(t)\|^2_{L^2(\Omega)}) \\
& - \frac{1}{p+1}\|u_N(t)\|^{p+1}_{L^{p+1}(\Omega)} - \langle h(t), \gamma_1 u_N(t)\rangle_{L^2(\Gamma_1)} \\
& + \int_0^t \langle h'(\tau), \gamma_1 u_N(\tau)\rangle_{L^2(\Gamma_1)}\, d\tau \\
& + \int_0^t \langle |u_N(\tau)|^k g(u_N'(\tau)), u_N'(\tau)\rangle_{L^2(\Omega)}\, d\tau = E_N(0).
\end{aligned}
$$

Due to the fact that $u$, the solution of the initial-boundary value problem (3.42), is not sufficiently regular, obtaining the energy identity in Lemma 4.2 is not straightforward. However, by modifying the proof of Lemma 8.3 of Lions and Magenes [20], Lemma 4.2 follows. Thus, its proof is omitted.

***Lemma 4.2.*** *Let $u \in C([0,T], H^1_{0,\Gamma_0}(\Omega))$ and $u' \in C([0,T], L^2(\Omega))$ such that $u$ is a weak solution to the initial-boundary value problem* (3.42). *Then $u$ satisfies*

$$
\begin{aligned}
(4.2)\qquad E(t) := {} & \frac{1}{2}(\|u'(t)\|^2_{L^2(\Omega)} + \|A^{1/2}u(t)\|^2_{L^2(\Omega)}) - \frac{1}{p+1}\|u(t)\|^{p+1}_{L^{p+1}(\Omega)} \\
& - \langle h(t), \gamma_1 u(t)\rangle_{L^2(\Gamma_1)} + \int_0^t \langle h'(\tau), \gamma_1 u(\tau)\rangle_{L^2(\Gamma_1)}\, d\tau \\
& + \int_0^t \langle |u(\tau)|^k \eta(\tau), u'(\tau)\rangle_{L^2(\Omega)}\, d\tau = E(0).
\end{aligned}
$$

***Lemma 4.3.*** *Let $v \in H^1_{0,\Gamma_0}(\Omega)$ be fixed, and $k \geq 1$ satisfying* (2.3). *Let $g(\varphi) = |\varphi|^m \operatorname{sgn}(\varphi)$. Then the mapping $\varphi \to |v|^k g(\varphi)$ generates a monotone operator from $L^2(\Omega)$ into $L^2(\Omega)$. More precisely,*

$$
\langle |v|^k [g(\varphi) - g(\psi)], \varphi - \psi\rangle_{L^2(\Omega)} \geq 0,
$$

*for all $\varphi$, $\psi \in L^2(\Omega)$, and every fixed $v \in H^1_{0,\Gamma_0}(\Omega)$. Moreover, the mapping $\lambda \mapsto \langle |v|^k g(\varphi + \lambda\psi), \mu\rangle_{L^2(\Omega)}$ is continuous from $\mathbb{R}$ to $\mathbb{R}$ for every fixed $\varphi$, $\psi$, $\mu \in L^2(\Omega)$, and $v \in H^1_{0,\Gamma_0}(\Omega)$.*

*Proof.* The first statement is trivial and follows easily from the monotonicity of the function $g(\varphi) = |\varphi|^m \operatorname{sgn}(\varphi)$. The second statement of the lemma also follows easily from the fact that $g$ is Hölder continuous with $|g(\varphi) - g(\psi)| \leq 2|\varphi - \psi|^m$, for all $\varphi$, $\psi \in \mathbb{R}$. ◻

***Lemma 4.4.*** *Let $u \in C([0,T], H^1_{0,\Gamma_0}(\Omega))$ and $u' \in C([0,T], L^2(\Omega))$ such that $u$ is a weak solution to the initial-boundary value problem* (3.42). *Then, $\eta = g(u')$.*

*Proof.* Since $\mathcal{P}_N u^0 \to u^0$ in $H^1_{0,\Gamma_0}(\Omega)$ and $\mathcal{P}_N u^1 \to u^1$ in $L^2(\Omega)$, it follows from (2.3) and (2.4) that $\|\mathcal{P}_N u^0\|_{L^{p+1}(\Omega)} \to \|u^0\|_{L^{p+1}(\Omega)}$. Moreover, by the continuity of the mapping $H^1_{0,\Gamma_0}(\Omega) \xrightarrow{\gamma_1} L^2(\Gamma_1)$, we have $\gamma_1 \mathcal{P}_N u^0 \to \gamma_1 u^0$ in $L^2(\Gamma_1)$. Now, since

$$E_N(0) = \frac{1}{2}(\|\mathcal{P}_N u^1\|^2_{L^2(\Omega)} + \|A^{1/2}\mathcal{P}_N u^0\|^2_{L^2(\Omega)}) - \langle h(0), \gamma_1 \mathcal{P}_N u^0\rangle_{L^2(\Gamma_1)} - \frac{1}{p+1}\|\mathcal{P}_N u^0\|^{p+1}_{L^{p+1}(\Omega)},$$

then,

$$\lim_{N\to\infty} E_N(0) = E(0) = \frac{1}{2}(\|u^1\|^2_{L^2(\Omega)} + \|A^{1/2}u^0\|^2_{L^2(\Omega)}) - \langle h(0), \gamma_1 u^0\rangle_{L^2(\Gamma_1)} - \frac{1}{p+1}\|u^0\|^{p+1}_{L^{p+1}(\Omega)}. \tag{4.3}$$

Therefore, for almost all $t \in [0, T]$, we have

$$\liminf_{N\to\infty} E_N(t) = \lim_{N\to\infty} E_N(0) = E(0) = E(t). \tag{4.4}$$

We remark here that by virtue of the proof of Lemma 3.3, one can conclude that $\|u_N(t)\|_{L^{p+1}(\Omega)} \to \|u(t)\|_{L^{p+1}(\Omega)}$, for almost all $t \in [0, T]$. Therefore, it follows from (3.19) and (4.1) that

$$\begin{aligned}
E(t) &= \liminf_{N\to\infty} E_N(t) \\
&\geq \frac{1}{2}\liminf_{N\to\infty}\|u_N'(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\liminf_{N\to\infty}\|A^{1/2}u_N(t)\|^2_{L^2(\Omega)} \\
&\quad - \frac{1}{p+1}\|u(t)\|^{p+1}_{L^{p+1}(\Omega)} - \langle h(t), \gamma_1 u(t)\rangle_{L^2(\Gamma_1)} \\
&\quad + \int_0^t \langle h'(\tau), \gamma_1 u(\tau)\rangle_{L^2(\Gamma_1)}\, d\tau \\
&\quad + \liminf_{N\to\infty}\int_0^t \langle |u_N(\tau)|^k g(u_N'(\tau)), u_N'(\tau)\rangle_{L^2(\Omega)}\, d\tau \\
&\geq \frac{1}{2}\|u'(t)\|^2_{L^2(\Omega)} + \frac{1}{2}\|A^{1/2}u(t)\|^2_{L^2(\Omega)} \\
&\quad - \langle h(t), \gamma_1 u(t)\rangle_{L^2(\Gamma_1)} + \int_0^t \langle h'(\tau), \gamma_1 u(\tau)\rangle_{L^2(\Gamma_1)}\, d\tau \\
&\quad + \liminf_{N\to\infty}\int_0^t \langle |u_N(\tau)|^k g(u_N'(\tau)), u_N'(\tau)\rangle_{L^2(\Omega)}\, d\tau.
\end{aligned} \tag{4.5}$$

Therefore, (4.5) and (4.2) yield

$$\int_0^t \langle |u(\tau)|^k \eta(\tau), u'(\tau)\rangle_{L^2(\Omega)}\, d\tau \geq \liminf_{N\to\infty} \int_0^t \langle |u_N(\tau)|^k g(u_N'(\tau)), u_N'(\tau)\rangle_{L^2(\Omega)}\, d\tau \geq 0. \tag{4.6}$$

Now, let $\varphi \in C_0(\Omega \times [0,T])$ be arbitrary. Since

$$\lim_{N\to\infty} \langle |u(\tau)|^k g(\varphi(\tau)), u_N'(\tau)\rangle = \lim_{N\to\infty} \langle |u_N(\tau)|^k g(\varphi(\tau)), u_N'(\tau)\rangle,$$

then (4.6) implies

$$\int_0^t \langle |u(\tau)|^k [\eta(\tau) - g(\varphi(\tau))], u'(\tau) - \varphi(\tau)\rangle_{L^2(\Omega)}\, d\tau \geq \liminf_{N\to\infty} \int_0^t \langle |u_N(\tau)|^k [g(u_N'(\tau)) - g(\varphi(\tau))], u_N'(\tau) - \varphi(\tau))\rangle_{L^2(\Omega)}\, d\tau. \tag{4.7}$$

Therefore, Lemma (4.3) and (4.7) yield

$$\int_0^t \langle |u(\tau)|^k [\eta(\tau) - g(\varphi(\tau))], u'(\tau) - \varphi(\tau)\rangle_{L^2(\Omega)}\, d\tau \geq 0, \tag{4.8}$$

for all $\varphi \in C_0(\Omega \times [0,T])$. By density, (4.8) holds for all $\varphi \in L^2(0,T,L^2(\Omega))$. By choosing $\varphi(t) = u'(t) - \lambda\psi(t)$, where $\lambda \geq 0$, then (4.8) yields

$$\int_0^t \langle |u(\tau)|^k [\eta(\tau) - g(u'(\tau) - \lambda\psi(\tau))], \psi(\tau)\rangle_{L^2(\Omega)}\, d\tau \geq 0, \tag{4.9}$$

for all $\lambda \geq 0$ and for all $\psi \in L^2(0,T,L^2(\Omega))$. By letting $\lambda \to 0^+$ and using Lemma 4.3, one has

$$\int_0^t \langle |u(\tau)|^k [\eta(\tau) - g(u'(\tau))], \psi(\tau)\rangle_{L^2(\Omega)}\, d\tau \geq 0, \tag{4.10}$$

for all $\psi \in L^2(0,T,L^2(\Omega))$. In particular, by letting $\psi(t) = g(u'(t)) - \eta(t)$, then (4.10) shows that $g(u'(t)) - \eta(t) = 0$ almost everywhere in $\Omega$, for almost all $t \in [0,T]$. $\square$

***Lemma 4.5.*** *Let $v_1$, $v_2 \in C([0,T], H^1_{0,\Gamma_0}(\Omega))$ and $v_1'$, $v_2' \in C([0,T], L^2(\Omega))$ such that $v_1$ and $v_2$ are weak solutions to the initial-boundary value problem* (1.2). *Then, $v_1 = v_2$.*

*Proof.* Let $v = v_1 - v_2$. Let $L > 0$ be such that $\|v_j\|_{C([0,T],H^1_{0,\Gamma_0}(\Omega))} \le L$ and $\|v_j'\|_{C([0,T],L^2(\Omega))} \le L$, for $j = 1, 2$. First, we note that $v \in C([0,T], H^1_{0,\Gamma_0}(\Omega))$, $v' \in C([0,T], L^2(\Omega))$ and $v$ satisfies the following initial-boundary value problem

$$
(4.11)\qquad \begin{cases} v_{tt} - \Delta v + G(v_1, v_1') - G(v_2, v_2') = f(v_1) - f(v_2), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } \Omega \times (0,T), \\ v(x,0) = 0,\ v_t(x,0) = 0, & \text{in } \Omega, \\ v(x,t) = 0 & \text{on } \Gamma_0 \times (0,T), \\ \dfrac{\partial v}{\partial n}(x,t) = 0, & \text{on } \Gamma_1 \times (0,T). \end{cases}
$$

By an argument similar to the proof of Lemma 4.2, we have

$$
\begin{aligned}
(4.12)\quad \frac{1}{2}(||v'(t)||^2_{L^2(\Omega)} &+ ||A^{1/2}v(t)||^2_{L^2(\Omega)}) \\
&= -\int_0^t \langle G(v_1(\tau), v_1'(\tau)) - G(v_2(\tau), v_2'(\tau)), v'(\tau)\rangle_{L^2(\Omega)}\, d\tau \\
&\qquad + \int_0^t \langle f(v_1(\tau)) - f(v_2(\tau)), v'(\tau)\rangle_{L^2(\Omega)}\, d\tau.
\end{aligned}
$$

However, by Lemma 4.3, and the use of the elementary inequality

$$
|\,|a|^k - |b|^k| \le C|a-b|(|a|^{k-1} + |b|^{k-1}),
$$

for some constant $C > 0$, all $k \ge 1$, and all $a, b \in \mathbb{R}$, we have

$$
\begin{aligned}
(4.13)\qquad & -\langle G(v_1, v_1') - G(v_2, v_2'), v'\rangle_{L^2(\Omega)} \\
&= -\langle |v_1|^k[g(v_1') - g(v_2')] + g(v_2')[|v_1|^k - |v_2|^k], v'\rangle_{L^2(\Omega)} \\
&\le -\langle g(v_2')[|v_1|^k - |v_2|^k], v'\rangle_{L^2(\Omega)} \\
&\le C\langle |v_2'|^m\, |v_1 - v_2|\, [|v_1|^{k-1} + |v_2|^{k-1}], |v'|\rangle_{L^2(\Omega)}.
\end{aligned}
$$

Now, for space dimensions $n \ge 3$, we choose

$$
\alpha = \frac{2}{m}, \quad \beta = \frac{2n}{n-2}, \quad \gamma = \frac{2n}{(n-2)(k-1)}, \quad \delta = \frac{2n}{2(n+k) - n(m+k)}.
$$

By recalling (2.3), it easy to see that $\alpha$, $\beta$, $\gamma$, $\delta$ are Hölder's conjugate exponents, and in particular $1 < \delta \le 2$. Therefore, by using the generalized Hölder inequality and (2.4), we have

$$\begin{aligned}
(4.14)\qquad &\langle |v_2(\tau)'|^m\, |v_1(\tau) - v_2(\tau)|\, |v_j(\tau)|^{k-1}, |v'(\tau)| \rangle_{L^2(\Omega)} \\
&\le \|v_2'(\tau)\|_2^m\, \|v(\tau)\|_{2n/(n-2)}\, \|v_j(\tau)\|_{2n/(n-2)}^{k-1}\, \|v'(\tau)\|_\delta \\
&\le C\|v_2'(\tau)\|_2^m\, \|v(\tau)\|_{H^1_{0,\Gamma_0}(\Omega)}\, \|v_j(\tau)\|_{H^1_{0,\Gamma_0}(\Omega)}^{k-1}\, \|v'(\tau)\|_2 \\
&\le CL^{m+k-1}\|v'(\tau)\|_2\, \|v(\tau)\|_{H^1_{0,\Gamma_0}(\Omega)} \\
&\le C[\|v'(\tau)\|_2^2 + \|v(\tau)\|_{H^1_{0,\Gamma_0}(\Omega)}^2]
\end{aligned}$$

for all $\tau \in [0, T]$ and $j = 1, 2$.

Similarly, by noting that $p \le n/(n-2)$ is equivalent to $n(p-1) \le 2n/(n-2)$, then we have

$$\begin{aligned}
(4.15)\qquad &\langle f(v_1(\tau)) - f(v_2(\tau)), v'(\tau) \rangle_{L^2(\Omega)} \\
&= \langle |v_1(\tau)|^{p-1} v_1(\tau) - |v_2(\tau)|^{p-1} v_2(\tau), v'(\tau) \rangle_{L^2(\Omega)} \\
&\le C\langle |v_1(\tau) - v_2(\tau)| [|v_1(\tau)|^{p-1} + |v_2(\tau)|^{p-1}], |v'(\tau)| \rangle_{L^2(\Omega)} \\
&\le C\|v(\tau)\|_{2n/(n-2)}\, \|v'(\tau)\|_2 [\|v_1(\tau)\|_{n(p-1)}^{p-1} + \|v_2(\tau)\|_{n(p-1)}^{p-1}] \\
&\le C\|v(\tau)\|_{H^1_{0,\Gamma_0}(\Omega)}\, \|v'(\tau)\|_2 [\|v_1(\tau)\|_{H^1_{0,\Gamma_0}(\Omega)}^{p-1} + \|v_2(\tau)\|_{H^1_{0,\Gamma_0}(\Omega)}^{p-1}] \\
&\le C[\|v'(\tau)\|_2^2 + \|v(\tau)\|_{H^1_{0,\Gamma_0}(\Omega)}^2].
\end{aligned}$$

We remark here that the estimates in (4.14)-(4.15) are also valid for the space dimensions $n = 1, 2$, by a similar argument. Therefore, it follows from (4.12)-(4.15) that

$$(4.16)\qquad \|v'(t)\|_2^2 + \|v(t)\|_{H^1_{0,\Gamma_0}(\Omega)}^2 \le C \int_0^t [\|v'(\tau)\|_2^2 + \|v(\tau)\|_{H^1_{0,\Gamma_0}(\Omega)}^2]\, d\tau,$$

for $t \in [0, T]$. Hence, by Gronwall's inequality

$$\|v'(t)\|_2^2 + \|v(t)\|_{H^1_{0,\Gamma_0}(\Omega)}^2 = 0, \quad \text{for } t \in [0, T]. \qquad \square$$

## 5. Global solutions

In this section we shall show that every local weak solution to the initial-boundary value problem (1.2) is a global solution provided $k + m \ge p$. More specifically, we have the following theorem.

***Theorem 5.1.*** *Let $k$, $m$ and $p$ satisfy* (2.3). *Assume that $k + m \geq p$, $u^0 \in H^1_{0,\Gamma_0}(\Omega)$, $u^1 \in L^2(\Omega)$, and $h \in C^1([0,\infty), L^2(\Gamma_1))$. Then for any $T > 0$, the initial-boundary value problem* (1.2) *has a unique weak solution on $[0, T]$ with*

$$u \in C([0,T]; H^1_{0,\Gamma_0}(\Omega)) \quad \text{and} \quad u' \in C([0,T]; L^2(\Omega)).$$

*Proof.* The theorem will follow immediately from Lemma 5.2 below. The a priori bounds for the approximate solutions in Lemma 5.2, followed by the application of the results in Sections 3 and 4 allow us to pass to the limit and obtain a weak solution to (1.2) on any time interval $[0, T]$. □

***Lemma 5.2.*** *Let $k$, $m$ and $p$ satisfy* (2.3). *If $k + m \geq p$ then on any bounded time interval $[0, T]$, the sequence of the approximate solutions $\{u_N\}$ is bounded in $Y_T$ and $\{u'_N\}$ is bounded in $X_T$.*

*Proof.* Let

$$e_N(t) = \frac{1}{2}(\|u'_N(t)\|^2_{L^2(\Omega)} + \|A^{1/2}u_N(t)\|^2_{L^2(\Omega)}), \tag{5.1}$$

$$F_N(t) = e_N(t) + \frac{1}{p+1}\|u_N(t)\|^{p+1}_{L^{p+1}(\Omega)}, \tag{5.2}$$

$$\sigma_N(t) = \langle h(t), \gamma_1 u_N(t)\rangle_{L^2(\Gamma_1)} - \int_0^t \langle h'(\tau), \gamma_1 u_N(\tau)\rangle_{L^2(\Gamma_1)}\, d\tau. \tag{5.3}$$

We shall show that $F_N(t)$ remains bounded for bounded time. First, we note that (4.1) yields

$$\begin{aligned} e'_N(t) = &\int_\Omega |u_N(t)|^{p-1}u_N(t)u'_N(t)\, dx \\ &- \int_\Omega |u_N(t)|^k\, |u'_N(t)|^{m+1}\, dx + \sigma'_N(t). \end{aligned} \tag{5.4}$$

Therefore,

$$\begin{aligned} F'_N(t) = 2&\int_\Omega |u_N(t)|^{p-1}u_N(t)u'_N(t)\, dx \\ &- \int_\Omega |u_N(t)|^k\, |u'_N(t)|^{m+1}\, dx + \sigma'_N(t). \end{aligned} \tag{5.5}$$

By Hölder's and Young's inequalities, we have

$$\begin{aligned}
(5.6)\qquad & \left| \int_\Omega |u_N(t)|^{p-1} u_N(t) u_N'(t)\, dx \right| \\
&\le \int_\Omega |u_N(t)|^{p-(p-m)/(m+1)} |u_N(t)|^{(p-m)/(m+1)}\, |u_N'(t)|\, dx \\
&\le \left( \int_\Omega |u_N(t)|^{p-m}\, |u_N'(t)|^{m+1}\, dx \right)^{1/(m+1)} \\
&\quad \times \left( \int_\Omega |u_N(t)|^{p+1}\, dx \right)^{m/(m+1)} \\
&\le \varepsilon \int_\Omega |u_N(t)|^{p-m}\, |u_N'(t)|^{m+1}\, dx + \frac{1}{2} C_0 \int_\Omega |u_N(t)|^{p+1}\, dx,
\end{aligned}$$

for some constant $C_0 > 0$ that depends on $\varepsilon$ and $m$. By taking $\varepsilon = \frac{1}{2}$, then we have

$$\begin{aligned}
(5.7)\qquad F_N'(t) &\le C_0 \int_\Omega |u_N(t)|^{p+1}\, dx + \int_\Omega |u_N(t)|^{p-m}\, |u_N'(t)|^{m+1}\, dx \\
&\quad - \int_\Omega |u_N(t)|^k\, |u_N'(t)|^{m+1}\, dx + \sigma_N'(t) \\
&= \int_\Omega |u_N'(t)|^{m+1} (|u_N(t)|^{p-m} - |u_N(t)|^k)\, dx \\
&\quad + C_0 \int_\Omega |u_N(t)|^{p+1}\, dx + \sigma_N'(t) \\
&= I_N^1(t) + I_N^2(t) + C_0 \int_\Omega |u_N(t)|^{p+1}\, dx + \sigma_N'(t),
\end{aligned}$$

where

$$I_N^1(t) = \int_{\{x\in\Omega:\ |u_N(t)|>1\}} |u_N'(t)|^{m+1} (|u_N(t)|^{p-m} - |u_N(t)|^k)\, dx,$$

and

$$I_N^2(t) = \int_{\{x\in\Omega:\ |u_N(t)|\le 1\}} |u_N'(t)|^{m+1} (|u_N(t)|^{p-m} - |u_N(t)|^k)\, dx.$$

Since $k \ge p - m$, $I_N^1(t) \le 0$. By Hölder's and Young's inequalities

$$\begin{aligned}
(5.8)\qquad I_N^2(t) &\le 2 \int_\Omega |u_N'(t)|^{m+1}\, dx \le C_1 ||u_N'(t)||_{L^2(\Omega)}^{m+1} \\
&\le C_1 ||u_N'(t)||_{L^2(\Omega)}^2 + C_2,
\end{aligned}$$

for some constants $C_1, C_2 > 0$. Therefore, it follows from (5.7) that

$$\begin{aligned}
(5.9)\qquad F_N'(t) &\le C_1 ||u_N'(t)||_{L^2(\Omega)}^2 + C_0 ||u_N(t)||_{p+1}^{p+1} + C_2 + \sigma_N'(t) \\
&\le b F_N(t) + C_2 + \sigma_N'(t),
\end{aligned}$$

for some constant $b > 0$.

By multiplying (5.9) by $e^{-bt}$ and integrating from 0 to $t$, we obtain

$$e^{-bt}F_N(t) \le \left(F_N(0) - \sigma_N(0) + \frac{C_2}{b}\right) + e^{-bt}\sigma_N(t) + b\int_0^t e^{-b\tau}\sigma_N(\tau)\,d\tau. \tag{5.10}$$

However,

$$\begin{aligned} &F_N(0) + |\sigma_N(0)| \\ &\quad = \frac{1}{2}(||\mathcal{P}_N u^1||^2_{L^2(\Omega)} + ||A^{1/2}\mathcal{P}_N u^0||^2_{L^2(\Omega)}) \\ &\qquad + \frac{1}{p+1}||\mathcal{P}_N u^0||^{p+1}_{L^{p+1}(\Omega)} + |\langle h(0), \gamma_1 \mathcal{P}_N u^0\rangle_{L^2(\Gamma_1)}| \\ &\quad \le ||u^1||^2_{L^2(\Omega)} + ||u^0||^2_{H^1_{0,\Gamma_0}(\Omega)} \\ &\qquad + C(||u^0||^{p+1}_{H^1_{0,\Gamma_0}(\Omega)} + \|h(0)\|_{L^2(\Gamma_1)}\|u^0\|_{H^1_{0,\Gamma_0}(\Omega)}) \\ &\quad := C_3. \end{aligned} \tag{5.11}$$

Now, we estimate $\sigma_N(t)$ as follows:

$$\begin{aligned} |\sigma_N(t)| &\le \|h(t)\|_{L^2(\Gamma_1)}\|\gamma_1 u_N(t)\|_{L^2(\Gamma_1)} \\ &\quad + \int_0^t \|h'(\tau)\|_{L^2(\Gamma_1)}\|\gamma_1 u_N(\tau)\|_{L^2(\Gamma_1)}\,d\tau \\ &\le C\left(\frac{1}{4\varepsilon}||h(t)||^2_{L^2(\Gamma_1)} + \varepsilon||u_N(t)||^2_{H^1_{0,\Gamma_0}(\Omega)}\right) \\ &\quad + C\int_0^t \left(\frac{1}{4\varepsilon}||h'(\tau)||^2_{L^2(\Gamma_1)} + \varepsilon||u_N(\tau)||^2_{H^1_{0,\Gamma_0}(\Omega)}\right)d\tau \\ &\le q(t) + \frac{1}{2}F_N(t) + \frac{1}{2}\int_0^t F_N(\tau)\,d\tau, \end{aligned} \tag{5.12}$$

where $q(t) = C^2(\,||h(t)||^2_{L^2(\Gamma_1)} + \int_0^t ||h'(\tau)||^2_{L^2(\Gamma_1)}\,d\tau)$ and $\varepsilon = 1/(4C)$.

Now, it follows from (5.10)-(5.12) that

$$\begin{aligned} &\frac{1}{2}e^{-bt}F_N(t) \\ &\quad \le \frac{C_2}{b} + C_3 + e^{-bt}q(t) + \frac{1}{2}e^{-bt}\int_0^t F_N(\tau)\,d\tau \\ &\qquad + b\int_0^t e^{-b\tau}\left(q(\tau) + \frac{1}{2}\left[F_N(\tau) + \int_0^\tau F_N(s)\,ds\right]\right)d\tau \quad \le \end{aligned} \tag{5.13}$$

$$\le C_4 + e^{-bt}q(t) + b\int_0^t e^{-b\tau}q(\tau)\,d\tau + \frac{2+b}{2}\int_0^t e^{-b\tau}F_N(\tau)\,d\tau$$
$$:= Q(t) + \frac{2+b}{2}\int_0^t e^{-b\tau}F_N(\tau)\,d\tau,$$

where $Q(t) = C_4 + e^{-bt}q(t) + b\int_0^t e^{-b\tau}q(\tau)\,d\tau$ and $C_4 = \frac{C_2}{b} + C_3$. By Gronwall's inequality,

$$\frac{1}{2}e^{-bt}F_N(t) \le Q(t) + (2+b)e^{(2+b)t}\int_0^t Q(\tau)\,d\tau. \tag{5.14}$$

Thus, (5.14) shows that $F_N(t)$ remains bounded for every $t > 0$, which completes the proof. □

## 6. Blow-up of solutions

Throughout this section, we assume that $p > k+m$, and for simplicity, we assume $h(t) \equiv 0$. In particular, the energy identity (4.2) in Lemma 4.2 becomes

$$E(t) := \frac{1}{2}(||u'(t)||^2_{L^2(\Omega)} + ||A^{1/2}u(t)||^2_{L^2(\Omega)}) - \frac{1}{p+1}||u(t)||^{p+1}_{L^{p+1}(\Omega)} + \int_0^t\int_\Omega |u(\tau)|^k\,|u'(\tau)|^{m+1}\,dx\,d\tau = E(0). \tag{6.1}$$

Let

$$F(t) = ||u(t)||^2_{L^2(\Omega)}, \tag{6.2}$$
$$H(t) = -\frac{1}{2}(||u'(t)||^2_{L^2(\Omega)} + ||A^{1/2}u(t)||^2_{L^2(\Omega)}) + \frac{1}{p+1}||u(t)||^{p+1}_{L^{p+1}(\Omega)}. \tag{6.3}$$

Our main result in this section is the following theorem.

***Theorem 6.1.*** *Let $u \in C([0,T), H^1_{0,\Gamma_0}(\Omega))$ and $u' \in C([0,T), L^2(\Omega))$ such that $u$ is a weak solution to the initial-boundary value problem* (1.2) *in the sense of Definition 2.1. If $p > k + m$ and $H(0) > 0$, then $T$ is necessarily finite, i.e., $u$ cannot be continued for all $t > 0$.*

*Proof.* First, (6.1) yields that

$$H'(t) = \int_\Omega |u(t)|^k\,|u'(t)|^{m+1}\,dx \ge 0.$$

Therefore,

$$0 < H(0) \le H(t) \le \frac{1}{p+1}||u(t)||^{p+1}_{L^{p+1}(\Omega)}, \tag{6.4}$$

for $0 \le t < T$.

Let $\alpha = \min\{(p-(k+m))/(m(p+1)), (p-1)/(2(p+1))\}$. In particular, $0 < \alpha < \frac{1}{2}$. Let $K$ and $L$ be the constants given by

$$K = 2|\Omega|^{(p-k-m)/((m+1)(p+1))} \quad \text{and}$$
$$L = \frac{(p+1)^{(k+m+1)/(m(p+1))}}{(p-1)^{1/m}} H(0)^{\alpha-(p-k-m)/(m(p+1))} \tag{6.5}$$

Let $0 < \varepsilon < 1$ be small enough so that

$$1 - \alpha - \varepsilon K^{1+1/m} L \ge 0. \tag{6.6}$$

Later, we may need to adjust $\varepsilon$ again.

In the remainder of the proof, most generic constants will be denoted by $C$, $C_0, \ldots$; they may depend on various parameters, but they are totally independent from $\varepsilon$ and the initial data, and they may change from line to line.

First, we note that (6.6) implies

$$H(0) \ge C\varepsilon^{\theta}, \tag{6.7}$$

where $\theta = m(p+1)/(p-(k+m)-\alpha m(p+1)) > 0$.

As in [5], we let

$$y(t) = H(t)^{1-\alpha} + \varepsilon F'(t). \tag{6.8}$$

Due to Lemma 3.6 and the work thereafter, one has $u'' \in C([0,T), (H^1_{0,\Gamma_0}(\Omega))')$. Consequently, $F''(t)$ exists for $t \in [0,T)$ and

$$\begin{aligned} F''(t) = {} & 2(\|u'(t)\|^2_{L^2(\Omega)} - \|A^{1/2}u(t)\|^2_{L^2(\Omega)} + \|u(t)\|^{p+1}_{L^{p+1}(\Omega)}) \\ & - 2\int_\Omega |u(t)|^k u(t) g(u'(t))\,dx. \end{aligned} \tag{6.9}$$

It follows from (6.8)-(6.9) that

$$\begin{aligned} y'(t) = {} & (1-\alpha)H(t)^{-\alpha}H'(t) + 4\varepsilon\|u'(t)\|^2_{L^2(\Omega)} + 4\varepsilon H(t) \\ & + 2\varepsilon\frac{p-1}{p+1}\|u(t)\|^{p+1}_{L^{p+1}(\Omega)} - 2\varepsilon\int_\Omega |u(t)|^k u(t) g(u'(t))\,dx. \end{aligned} \tag{6.10}$$

Since $p > k + m$, then by Hölder's inequality

$$
\begin{aligned}
(6.11)\qquad & \left| \int_\Omega |u(t)|^k u(t) g(u'(t))\, dx \right| \\
&\le \int_\Omega |u(t)|^{k+1-km/(m+1)} |u(t)|^{km/(m+1)} |u'(t)|^m\, dx \\
&\le \left( \int_\Omega |u(t)|^k\, |u'(t)|^{m+1}\, dx \right)^{m/(m+1)} \\
&\quad \times \left( \int_\Omega |u(t)|^{k+m+1}\, dx \right)^{1/(m+1)} \\
&\le \frac{1}{2} K H'(t)^{m/(m+1)} ||u(t)||_{L^{p+1}(\Omega)}^{(k+m+1)/(m+1)},
\end{aligned}
$$

where $K$ is as given in (6.5). However, Young's inequality and (6.11) yield

$$
(6.12)\qquad \left| \int_\Omega |u(t)|^k u(t) g(u'(t))\, dx \right| \le \frac{1}{2} K \left[ \frac{1}{\delta} H'(t) + \delta^m ||u(t)||_{L^{p+1}(\Omega)}^{k+m+1} \right],
$$

where $\delta > 0$ is to be chosen later. Therefore, it follows from (6.10) and (6.12) that

$$
\begin{aligned}
(6.13)\qquad y'(t) \ge & \left[ (1-\alpha) H(t)^{-\alpha} - K \frac{\varepsilon}{\delta} \right] H'(t) + 4\varepsilon ||u'(t)||_{L^2(\Omega)}^2 \\
& + 4\varepsilon H(t) + 2\varepsilon \frac{p-1}{p+1} ||u(t)||_{L^{p+1}(\Omega)}^{p+1} - K\varepsilon\delta^m ||u(t)||_{L^{p+1}(\Omega)}^{k+m+1}.
\end{aligned}
$$

By choosing $\delta = [(p-1)/((p+1)K) \|u(t)\|_{L^{p+1}(\Omega)}^{p-k-m}]^{1/m}$, then

$$
\varepsilon \frac{p-1}{p+1} ||u(t)||_{L^{p+1}(\Omega)}^{p+1} - K\varepsilon\delta^m ||u(t)||_{L^{p+1}(\Omega)}^{k+m+1} = 0.
$$

Therefore, we have

$$
\begin{aligned}
(6.14)\qquad y'(t) \ge & \left[ (1-\alpha) H(t)^{-\alpha} - K \frac{\varepsilon}{\delta} \right] H'(t) + 4\varepsilon ||u'(t)||_{L^2(\Omega)}^2 \\
& + 4\varepsilon H(t) + \varepsilon \frac{p-1}{p+1} ||u(t)||_{L^{p+1}(\Omega)}^{p+1}.
\end{aligned}
$$

Since $H(t) \le 1/(p+1) \|u(t)\|_{L^{p+1}(\Omega)}^{p+1}$, then

$$
\begin{aligned}
(6.15)\qquad & (1-\alpha) H(t)^{-\alpha} - K \frac{\varepsilon}{\delta} \\
&= H^{-\alpha}(t) \left[ 1 - \alpha - K \frac{\varepsilon}{\delta} H(t)^\alpha \right] \\
&\ge H(t)^{-\alpha} \left[ 1 - \alpha - \varepsilon K^{1+1/m} \frac{(p+1)^{1/m-\alpha}}{(p-1)^{1/m}} ||u(t)||_{L^{p+1}(\Omega)}^{(k+m-p+\alpha m(p+1))/m} \right].
\end{aligned}
$$

Furthermore, since $\|u(t)\|_{L^{p+1}(\Omega)} \geq [(p+1)H(0)]^{1/(p+1)} > 0$ and $\alpha$ was chosen so that $k + m - p + \alpha m(p+1) \leq 0$, then it follows from (6.15) that

$$(6.16)\quad \begin{aligned} &(1-\alpha)H(t)^{-\alpha} - K\frac{\varepsilon}{\delta} \\ &\geq H(t)^{-\alpha} \\ &\quad\times\left[1-\alpha-\varepsilon K^{1+1/m}\frac{(p+1)^{(k+m+1)/(m(p+1))}}{(p-1)^{1/m}}H(0)^{\alpha-(p-k-m)/(m(p+1))}\right] \\ &\equiv H(t)^{-\alpha}[1-\alpha-\varepsilon K^{1+1/m}L] \geq 0, \end{aligned}$$

by our choice of $\varepsilon$ in (6.6). Therefore, (6.14) and (6.16) yield

$$(6.17)\qquad y'(t) \geq \varepsilon C[H(t) + ||u'(t)||^2_{L^2(\Omega)} + ||u(t)||^{p+1}_{L^{p+1}(\Omega)}],$$

for $t \in [0,T)$ and where $C > 0$ is a constant that does not depend on $\varepsilon$. In particular (6.17) shows that $y(t)$ is increasing on $[0,T)$, with

$$(6.18)\qquad y(t) = H(t)^{1-\alpha} + \varepsilon F'(t) \geq H(0)^{1-\alpha} + \varepsilon F'(0).$$

If $F'(0) \geq 0$, then no further condition on $\varepsilon$ is needed. However, if $F'(0) < 0$, then we further adjust $\varepsilon$ so that $0 < \varepsilon \leq -H(0)^{1-\alpha}/(2F'(0))$. In any case, one has $y(t) > 0$, for $t \in [0,T)$.

Finally, we show that $y(t)$ satisfies the differential inequality

$$(6.19)\qquad y'(t) \geq \varepsilon^{1+\sigma}C_0 y(t)^{1/(1-\alpha)}, \quad 0 \leq t < T,$$

where $C_0$ is some positive constant and $\sigma = \theta(1 - 2/((1-2\alpha)(p+1))) \geq 0$.

If $F'(t) \leq 0$ for some $t \in [0,T)$, then for such values of $t$ we have

$$(6.20)\qquad y(t)^{1/(1-\alpha)} = [H(t)^{1-\alpha} + \varepsilon F'(t)]^{1/(1-\alpha)} \leq H(t).$$

Thus, (6.20) and (6.17) show that (6.19) is valid for all $t \in [0,T)$ for which $F'(t) \leq 0$. If $t \in [0,T)$ is such that $F'(t) > 0$, then (6.19) will be valid, if for such values of $t \in [0,T)$ the following inequality holds

$$(6.21)\qquad \begin{aligned} H(t) + ||u(t)||^{p+1}_{L^{p+1}(\Omega)} + ||u'(t)||^2_{L^2(\Omega)} \\ \geq \varepsilon^\sigma C[H(t)^{1-\alpha} + \varepsilon F'(t)]^{1/(1-\alpha)}. \end{aligned}$$

So, assume that $F'(t) > 0$, and let $\beta = 1/(1-\alpha)$. Since $1 < \beta < 2$ and $0 < \varepsilon < 1$, then by convexity

$$(6.22)\qquad [H(t)^{1-\alpha} + \varepsilon F'(t)]^\beta \leq 2^{\beta-1}[H(t) + F'(t)^\beta].$$

However, since $p \geq 1$, we have

$$F'(t)^\beta = \left(2\int_\Omega u(t)u'(t)\,dx\right)^\beta \leq C(\|u(t)\|_{L^2(\Omega)}\,\|u'(t)\|_{L^2(\Omega)})^\beta \\ \leq C||u(t)||^\beta_{L^{p+1}(\Omega)}||u'(t)||^\beta_{L^2(\Omega)}. \tag{6.23}$$

Since $2/\beta = 2(1-\alpha) > 1$, by using Young's inequality, we obtain

$$F'(t)^\beta \leq C(||u'(t)||^2_{L^2(\Omega)} + ||u(t)||^{2\beta/(2-\beta)}_{L^{p+1}(\Omega)}). \tag{6.24}$$

Now, by recalling $\|u(t)\|^{p+1}_{L^{p+1}(\Omega)} > (p+1)H(0) > 0$ and by noting that $\alpha \leq (p-1)/(2(p+1))$ is equivalent to $2\beta/(2-\beta) := 2/(1-2\alpha) \leq p+1$, then there exists a constant $C_1 > 0$ that is independent of $\varepsilon$ and the initial data, such that

$$||u(t)||^{2\beta/(2-\beta)}_{L^{p+1}(\Omega)} \leq C_1 H(0)^{2/((1-2\alpha)(p+1))-1}||u(t)||^{p+1}_{L^{p+1}(\Omega)}. \tag{6.25}$$

Since $2/((1-2\alpha)(p+1)) - 1 \leq 0$, then it follows from (6.7) that

$$||u(t)||^{2\beta/(2-\beta)}_{L^{p+1}(\Omega)} \leq C\varepsilon^{-\sigma}||u(t)||^{p+1}_{L^{p+1}(\Omega)}, \tag{6.26}$$

where

$$\sigma = \theta\left(1 - \frac{2}{(1-2\alpha)(p+1)}\right) \\ = \frac{m}{p-(k+m)-\alpha m(p+1)}\left(p+1-\frac{2}{(1-2\alpha)}\right) \geq 0.$$

Thus, it follows from (6.26) and (6.24) that

$$F'(t)^{1/(1-\alpha)} \leq C\varepsilon^{-\sigma}(||u'(t)||^2_{L^2(\Omega)} + ||u(t)||^{p+1}_{L^{p+1}(\Omega)}). \tag{6.27}$$

By combining (6.22) and (6.27), then (6.21) follows. Consequently (6.19) holds, and therefore, $y(t) = H(t)^{1-\alpha} + \varepsilon F'(t)$ blows up in finite-time $T$, where

$$T < C\varepsilon^{-1-\sigma}y(0)^{-\alpha/(1-\alpha)}. \tag{6.28}$$

□

***Remark 6.2.*** If $F'(0) \geq 0$, then (6.28) yields the following upper bound for the life span of the solution

$$T < C\varepsilon^{-1-\sigma}[H(0)^{1-\alpha} + \varepsilon F'(0)]^{-\alpha/(1-\alpha)} \leq C\varepsilon^{-1-\sigma}H(0)^{-\alpha}. \tag{6.29}$$

However, if $F'(0) < 0$, then (6.29) is still valid, since we have chosen $\varepsilon$ in the proof of Theorem 6.1 so that $0 < \varepsilon \le -H(0)^{1-\alpha}/(2F'(0))$. Now, if the initial data is sufficiently small, then in view of (6.7), $\varepsilon \sim H(0)^{1/\theta}$, and therefore we have

$$\begin{aligned} T &< CH(0)^{-\alpha-1/\theta-\sigma/\theta} \\ &= CH(0)^{-(p-(k+m))/(m(p+1))-(1-2/((1-2\alpha)(p+1))}. \end{aligned} \tag{6.30}$$

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