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Lynn Erbe
University of Nebraska-Lincoln, lerbe@unl.edu

Christopher S. Goodrich
Creighton Preparatory School

Baoguo Jia
Sun Yat-Sen University

Allan C. Peterson
University of Nebraska-Lincoln, apeterson1@unl.edu

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Survey of the qualitative properties of fractional difference operators: monotonicity, convexity, and asymptotic behavior of solutions

Lynn Erbe[1], Christopher S Goodrich[2], Baoguo Jia[3] and Allan Peterson[1*]

*Correspondence:
apeterson1@math.unl.edu
[1]Department of Mathematics, University of Nebraska-Lincoln, Lincoln, NE 68588-0130, USA
Full list of author information is available at the end of the article

Abstract
In this article we discuss some of the qualitative properties of fractional difference operators. We especially focus on the connections between the fractional difference operator and the monotonicity and convexity of functions. In the integer-order setting, these connections are elementary and well known. However, in the fractional-order setting the connections are very complicated and muddled. We survey some of the known results and suggest avenues for future research. In addition, we discuss the asymptotic behavior of solutions to fractional difference equations and how the nonlocal structure of the fractional difference can be used to deduce these asymptotic properties.

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1 Introduction and preliminaries

1.1 The integer-order calculus

Consider a map $f : \mathbb{N}_a \to \mathbb{R}$, where we put $\mathbb{N}_a := \{a, a+1, a+2, \ldots\}$. A well-known operation on f is the forward difference operator Δ, which is defined by

$$\Delta f(t) := f(t+1) - f(t), \quad t \in \mathbb{N}_a. \tag{1.1}$$

In some sense this can be thought of as a discrete form of the ordinary derivative of a function. In particular, (1.1) computes the amount of change in f as we move the time point from t to $t+1$. An important feature of the forward difference operator is its local structure. By this we mean that in (1.1) only the time points t and $t+1$ are considered. Thus, the behavior of f at earlier points or later points is ignored and plays no role whatsoever in the computation of $\Delta f(t)$.

This, of course, is similar to the ordinary derivative of a map $f : X \subseteq \mathbb{R} \to \mathbb{R}$, with X open. Given $c \in X$ at which f' exists, we have

$$f'(c) := \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

So, again, $f'(c)$ really only considers the local behavior of f near the point $x = c$.

Although perhaps taken for granted, one very important consequence of this local nature of the preceding operators is that the operators possess a strong connection to the monotone behavior of f. In particular, as every first semester calculus student learns, given a differentiable function $f : X \subset \mathbb{R} \to \mathbb{R}$, it is geometrically obvious that if $f'(x) > 0$ for all x in some open set $U \Subset X$, then it follows that f is increasing on $\overline{U}$. More precisely, the mean value theorem establishes this connection rigorously. The situation in the discrete case is even more transparent, for if $\Delta f(t) \geq 0$, say for $t \in \mathbb{N}_a$, then we immediately obtain $f(t+1) \geq f(t)$, for each $t \in \mathbb{N}_a$, whence f is increasing on $\mathbb{N}_a$. Hence, here we do not even need a deep result from analysis such as the mean value theorem.

Similarly, as one also learns in first semester calculus, there is a deep connection between whether a function may be classified as convex or concave and the associated behavior of f''. Thus, for example, we know that if $f''(x) > 0$ for all x in the domain of f, then f is a convex mapping, whereas if $f''(x) < 0$ on the domain of f, then f is a concave mapping. Convexity and concavity are very important ideas; it is well known that, for example, convex maps behave much more nicely than a 'typical' function. Moreover, convex and concave maps obey important inequalities such as, for example, Jensen's inequality. As yet another example, if one has a map $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$ with $\Omega \subseteq \mathbb{R}^n$ open and bounded and one wishes to minimize the map

$$J[u; \Omega] := \int_\Omega f(x, u, Du)\, dx$$

among all maps u in the Sobolev space $W^{1,p}(\Omega)$, then a typical assumption (among others) to ensure the existence of a minimizer is the convexity of the partial map $\xi \mapsto f(\cdot, \cdot, \xi)$. All in all, then, the connection between f'' and the convexity or concavity of f is an indispensable one in both pure and applied mathematics.

In the discrete case, as with monotonicity, the relationship is ever more transparent. Indeed, if we have $\Delta^2 y(t) > 0$, then obviously

$$\Delta^2 y(t) = \Delta y(t+1) - \Delta y(t) > 0,$$

which means that the map $t \mapsto \Delta y(t)$ is increasing. As such, by definition we recover immediately that y is a convex map. All in all, we thus recover a connection between convexity and the sign of the second-order difference map $t \mapsto \Delta^2 y(t)$.

1.2 The fractional-order calculus: the delta difference case

Recently there has been much interest in a 'fractionalized' version of (1.1). In this setting we allow for the *order* of the difference to be a real number that is *not necessarily* an integer. While our interest in this survey will mostly be confined to the pure mathematical interest in this generalization, suffice it to say there are applicative reasons to consider a fractional difference operator - for example, the reader may consult the article by Atici and Şengül [1], which details some possible applications of discrete fractional differences to tumor modeling, wherein the authors use the order of the fractional difference to modulate their growth model so as to better align with collected data.

While there are many possible definitions of the discrete fractional difference in use, one of the more common ones is the so-called Riemann-Liouville forward fractional difference. It is defined by first defining a fractional sum. To accomplish this, we give the following definition.

Definition 1.1 We define

$$t^{\underline{r}} := \frac{\Gamma(t+1)}{\Gamma(t+1-r)}$$

for all values of t and r for which the right-hand side above is well defined. Moreover, in the case where $t+1-r$ is a pole of the gamma function but $t+1$ is not, we declare $t^{\underline{r}} := 0$.

Example 1.2 Note that

$$\left(\frac{1}{2}\right)^{\underline{\frac{3}{4}}} = \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{4})} = \frac{1}{2\sqrt{2\pi}}\Gamma\left(\frac{1}{4}\right).$$

Also note that by convention we have

$$\nu^{\underline{\nu+1}} := 0,$$

for any $\nu \in \mathbb{R} \setminus \{\dots,-2,-1\}$. Another useful identity, which is easy to establish, is

$$\nu^{\underline{\nu}} = \Gamma(\nu+1).$$

With Definition 1.1 in hand, we then can define the fractional sum and difference. First we define the map h_ν by

$$h_\nu(t,s) := \frac{(t-s)^{\underline{\nu}}}{\Gamma(\nu+1)}.$$

We call this map the νth order fractional Taylor monomial based at s. The νth order fractional sum and difference are then defined as follows.

Definition 1.3 Assume that $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$. Then we define the νth order fractional sum (based at a) of f by

$$\Delta_a^{-\nu} f(t) := \int_a^{t-\nu+1} h_{\nu-1}(t,\tau+1)f(\tau)\Delta\tau = \sum_{\tau=a}^{t-\nu} h_{\nu-1}(t,\tau+1)f(\tau), \tag{1.2}$$

for each $t \in \mathbb{N}_{a+\nu}$.

Definition 1.4 Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$. Define $N \in \mathbb{N}$ to be the unique positive integer satisfying $N - 1 < \nu < N$. Then the νth fractional difference is defined by

$$\Delta_a^\nu f(t) := \Delta^N \Delta_a^{-(N-\nu)} f(t), \tag{1.3}$$

for each $t \in \mathbb{N}_{a+N-\nu}$.

A very important point regarding Definition 1.4 is that the domain of the map $t \mapsto \Delta_a^\nu f(t)$ is ***different*** from the domain of the map $t \mapsto f(t)$. This is (to those familiar with the discrete fractional calculus) the well-known domain shifting peculiarity of the forward fractional difference. To emphasize this point going forward, let us make the following remark.

Remark 1.5 As Definition 1.4 demonstrates, by definition, the fractional forward difference shifts the domain of f from $\mathbb{N}_a$ to $\mathbb{N}_{a+N-\nu}$. On the one hand, this is really just a minor irritation and causes little more than bookkeeping difficulties when analyzing fractional differences. On the other hand, as we shall see later the nabla (*i.e.*, backward) fractional difference does not possess this peculiarity, and, as such, it is suggestive of some dissimilarities between the forward and backward fractional differences and their associated operational properties. We shall see this more fully later.

The key fact that we wish to emphasize at this juncture regarding (1.2) is the ***nonlocal*** nature of the fractional difference. Indeed, observe that since

$$\begin{aligned}\Delta_a^{-\nu} f(t) &= \sum_{\tau=a}^{t-\nu} h_{\nu-1}(t, \tau + 1) f(\tau) \\ &= h_{\nu-1}(t, a+1) f(a) + h_{\nu-1}(t, a+2) f(a+1) + \cdots + h_{\nu-1}(t, t-\nu+1) f(t-\nu),\end{aligned}$$

it follows that the value of the map $t \mapsto \Delta_a^{-\nu} f(t)$ at each $t \in \mathbb{N}_{a+\nu}$ is, in fact, a linear combination of the collection $\{f(a), f(a+1), \ldots, f(t-\nu)\}$. In particular, the νth order difference of f at time t depends on the value of f at all previous times. This observation is why we often remark that the fractional difference has a 'memory' property, for it, in some mathematical sense, recalls and weights the values of f at all previous time points when 'determining' the value of the fractional difference at a fixed time.

We cannot emphasize enough in what bold relief this stands against the classical (*i.e.*, integer-order) difference defined in (1.1). Indeed, this implicit nonlocal structure is responsible not only for the mathematical interest of the discrete fractional difference but also its tremendous complexity. In particular, because we take rather for granted the implication that $\Delta y(t) \geq 0$ implies that y is increasing, it is all too easy to forget what a strong role the ***local*** nature of the integer-order operator plays in facilitating the proof of that result (or the analogous result regarding f', for that matter). By introducing the nonlocal structure, it turns out that things are no longer so simple and straightforward. Indeed, as we shall quickly see in Section 2, the connection, for example, between the sign of $\Delta_a^\nu f(t)$, in the case $1 < \nu < 2$, and the monotone behavior of f is quite muddled, complex, and even, at times, unexpected. All of this is due to the nonlocal nature of the fractional difference.

1.3 The fractional-order calculus: the nabla difference case

Having provided some of the basics of the fractional delta difference in the preceding subsection, in this subsection we provide some basic details regarding the nabla or, if one prefers, backward fractional difference. While constructed in an evidently similar way with respect to the fractional delta difference, in the end we shall see some key, and, indeed, perhaps unexpected, dissimilarities between the two operators.

Therefore, to begin let us first recall the integer-order nabla difference. For a function $f : \mathbb{N}_a \to \mathbb{R}$ we define the nabla (or backward) difference map of f, denoted $\nabla f(t)$, by

$$\nabla f(t) := f(t) - f(t-1), \quad t \in \mathbb{N}_{a+1}.$$

Thus, in the integer-order setting we see that ostensibly there is little difference between the nabla and delta differences. As we shall see later in this survey, in the fractional-order setting things are not so simple.

The fractional-order nabla difference and sum are defined in a manner analogous with the delta fractional difference. In particular, we first introduce a suitable Taylor monomial, and then use this to construct the associated nabla difference and sum. In particular, let us make the following definition, which is analogous with Definition 1.1.

Definition 1.6 The (generalized) rising function is defined by

$$t^{\overline{r}} := \frac{\Gamma(t+r)}{\Gamma(t)}$$

for those values of t and r so that the right-hand side of this equation is sensible. Also, we use the convention that if t is a nonpositive integer, but $t + r$ is not a nonpositive integer, then $t^{\overline{r}} := 0$.

We then can define the nabla fractional Taylor monomial functions as follows.

Definition 1.7 The *nabla Taylor monomials* based at a, denoted $H_r(t,a)$, $r \neq -1, -2, -3, \ldots$, are defined by

$$H_r(t,a) := \frac{(t-a)^{\overline{r}}}{\Gamma(r+1)},$$

whenever the right-hand side of this equation makes sense.

Finally, using Definitions 1.6-1.7 we can define a fractional nabla difference and sum.

Definition 1.8 Let $f : \mathbb{N}_{a+1} \to \mathbb{R}$ be given and assume that $\mu > 0$. Then we define the μth order nabla fractional sum, denoted $\nabla_a^{-\mu} f(t)$, by

$$\nabla_a^{-\mu} f(t) := \int_a^t H_{\mu-1}(t, s-1) f(s) \nabla s,$$

for $t \in \mathbb{N}_a$. By convention we put

$$\nabla_a^{-\mu} f(a) := 0.$$

Definition 1.9 Let $f : \mathbb{N}_{a+1} \to \mathbb{R}$ be given and assume that $\nu \in \mathbb{R}^+$. Select N to be the unique number such that $N - 1 < \nu \leq N$. Then we define the νth order nabla fractional difference, denoted $\nabla_a^\nu f(t)$, by

$$\nabla_a^\nu f(t) := \nabla^N \nabla_a^{-(N-\nu)} f(t),$$

for $t \in \mathbb{N}_{a+N}$.

We would like to note, as this will be important later in Section 2.2, that Erbe *et al.* have noticed (see [2]) that the form of ∇_a^ν in Definition 1.9 can be written in a slightly different and sometimes more useful way. We state this result next as Lemma 1.10.

Lemma 1.10 *Assume that $f : \mathbb{N}_a \to \mathbb{R}$ and let $\nu \in (0, +\infty) \setminus \mathbb{N}_1$ be given. Choose $N \in \mathbb{N}_1$ such that $N - 1 < \nu < N$. Then*

$$\nabla_a^\nu f(t) = \int_a^t H_{-\nu-1}(t, \tau - 1) f(\tau) \nabla \tau,$$

for $t \in \mathbb{N}_{a+1}$.

Once again, a cursory examination of Definitions 1.8-1.9 reveals that much like the fractional delta difference and sum, the fractional nabla difference and sum are nonlocal operators. In particular, we note that

$$\nabla_a^{-\mu} f(t) = \int_a^t H_{\mu-1}(t, s-1) f(s) \nabla s = \sum_{\tau=a+1}^{t} H_{\mu-1}(t, s-1) f(s). \tag{1.4}$$

Thus, (1.4) demonstrates that the nabla fractional sum $\nabla_a^{-\mu} f(t)$ involves the values of f from the collection $\{f(a+1), f(a+2), \dots, f(t)\}$. Consequently, we again have a memory property since the operator in (1.4) weights all of the previous values of f and, in particular, weights these by means of the nabla Taylor monomials. As we shall see in Sections 2, 3, and 4, this nonlocal structure then greatly affects and complicates the analysis of these operators.

There is a second important point worth making regarding Definitions 1.8-1.9 as compared to Definitions 1.3-1.4. Notice that the fractional nabla sum and difference do ***not*** induce a domain shift in the way that the fractional delta sum and difference induce. Practically this means that if f is defined on $\mathbb{N}_a$, then so, too, is the map $t \mapsto \nabla_a^{-\mu} f(t)$. Thus, at least in this one sense, nabla fractional operators are somewhat easier to work with than their delta counterparts. As we go through the remainder of this survey we shall see some additional dissimilarities between these two operators, particularly as concerns their relationship to monotonicity and convexity.

1.4 Overview of the article

As suggested in the previous subsections, our goal in this article is to survey some recent results in the theory of the fractional difference calculus. Especially we are concerned with a couple of aspects of this theory - namely,

- the relationship between the sign of the fractional difference and monotonicity and convexity; and
- asymptotic behavior of solutions to fractional difference equations.

In particular, one of the chief goals of this survey is to illustrate how the nonlocal structure present in (1.2)-(1.3) seriously complicates and confounds the relationship between the sign of $\Delta_a^\nu f(t)$ and the monotonicity (in the case $1 < \nu < 2$) and convexity (in the case $2 < \nu < 3$) of f. As we shall see and as we have suggested to the reader earlier in this section, these relationships in the fractional setting are rather complicated and frankly nontrivial. And properties and relationships that we rather take for granted in the integer-order setting may fail to hold in the fractional-order setting.

All in all, then, the outline of the remainder of this survey is as follows. In Section 2 we provide an up-to-date treatment of the relationship between $\Delta_a^\nu f(t)$ and $\nabla_a^\nu f(t)$, in the case where $1 < \nu < 2$, and the associated monotone behavior of the map $t \mapsto f(t)$. We mention both the results in the delta case and the nabla case. In Section 3 we then conduct the same sort of study, but here we focus on the relationship between these operators and the convexity or concavity of the map f; thus, here we shall focus on the setting where $2 < \nu < 3$. Once again, we treat both the delta and the nabla cases. Finally, in Section 4 we demonstrate how these operators induce certain asymptotic properties in the solutions of fractional initial value problems.

For the most part in our discussion we eschew the proofs of the relevant results since we wish here to give the reader a broad overview of the current frontier of this area of research. Nonetheless, we do, at times, provide some of the proofs in order to illustrate, broadly speaking, the techniques that are utilized to establish the results; we hope this will be of interest to readers wishing to contribute to this area. In any case, for more details on the proofs of these results, the interested reader can consult either the relevant papers, which we cite, or Chapter 7 of the textbook by Goodrich and Peterson [3].

1.5 Suggestions for further reading

Finally, we wish to conclude our introduction by highlighting some relevant articles for the interested reader, who wishes to go beyond the introduction provided by this survey. There are a great many articles nowadays on discrete fractional calculus; here we briefly recall a few of these, and we then direct the reader to the relevant references in these articles for additional reading.

First of all, for a general overview of the discrete fractional calculus, together with substantial background on the integer-order difference calculus, we direct the interested reader to the textbook by Goodrich and Peterson [3]. This book contains not only a treatment of the integer- and fractional-order discrete calculus on the time scale $\mathbb{Z}$ with both delta and nabla differences, but also contains a treatment of the well-known q-calculus as well as the so-called mixed time scales. Of particular relevance, a greatly expanded discussion of fractional Taylor monomials is contained in the book.

Second of all, regarding research articles, depending upon one's interest there are a great many relevant articles that have appeared in the past five to ten years. After the initial work of Atici and Eloe [4–7], which served to pique mathematicians' interest in discrete fractional calculus, a number of subsequent works have appeared. For example, if one wishes to delve further into the operational properties of the discrete fractional calculus, one can consult the papers by Anastassiou [8–16], Atici and Acar [17], Atici and Eloe [4–6, 18, 19], Atici and Uyanik [20], Baoguo *et al.* [21, 22], Jia *et al.* [2, 23, 24], Čermák and Nechvátal [25], Čermák *et al.* [26], Dahal and Goodrich [27, 28], Ferreira [29], Goodrich [30, 31], and Holm [32]. Ahrendt *et al.* [33] and Holm [34] have considered the Laplace transform and its application in various discrete fractional problems. On the other hand, for those interested in the development and analysis of boundary and initial value problems with fractional differences, one may consult, for example, the works by Agarwal *et al.* [35], Atici and Eloe [7, 36], Awasthi [37, 38], Aswathi *et al.* [39], Baleanu *et al.* [40], Dahal *et al.* [41], Ferreira [42], Ferreira and Goodrich [43, 44], Goodrich [45–54], He *et al.* [55], Holm [56], Lv *et al.* [57], and Sitthiwirattham *et al.* [58], and Sitthiwirattham [59]. There are also some papers detailing extensions of the fractional calculus on the time scale $\mathbb{Z}$ to other time scales, and one may consult Bastos *et al.* [60], Ferreira [61, 62], Ferreira and Torres [63], and Graef and Kong [64]. Consideration of various inequalities in the discrete fractional calculus (*e.g.*, Grüss- and Gronwall-type inequalities) have been considered by Akin *et al.* [65], Güvenilir *et al.* [66], and Xu and Zhang [67]. Very recently Jia *et al.* have investigated asymptotic behavior of solutions to initial value problems in discrete fractional calculus [68–70]. Finally, there has recently been some interesting attempts to investigate the chaotic behavior of fractional discrete dynamical systems, and the paper by Wu and Baleanu [71] may be consulted to see the directions that this research has taken.

All in all, then, there is a wide and growing body of literature on discrete fractional calculus. In particular, the nonlocal structure of the fractional operators induce substantial difficulties in their analysis and significant dissimilarities in comparison with their integer-order counterparts. Therefore, we believe that this will continue to be a wellspring of interesting mathematics in the foreseeable future. We hope that this paper serves as an invitation for additional researchers to join us in investigating this surprisingly complex and subtle area of analysis.

2 Monotonicity

2.1 Results for the delta fractional difference

As mentioned in Section 1, the following is a well-known fact in the difference calculus, and indeed requires almost no effort to prove.

Proposition 2.1 *Let* $f : \mathbb{N}_a \to \mathbb{R}$. *Then* $\Delta y(t) \geq 0$ *for each* $t \in \mathbb{N}_a$ *if and only if* y *is increasing on* $\mathbb{N}_a$.

In light of this, a natural question is whether such a result holds in the discrete fractional setting. In particular, we might wonder whether the following statement is true: 'If $1 < \nu < 2$ and $\Delta_a^\nu y(t) \geq 0$ for each $t \in \mathbb{N}_{a+N-\nu}$, then y is increasing on $\mathbb{N}_a$.' As it turns out, the answer to this seemingly innocuous conjecture is rather complicated and subtle.

The first researchers to consider this question were Dahal and Goodrich. They obtained the following result, which can be seen as a partial affirmative answer to the preceding question. In order to illustrate to the reader the method of proof utilized by Dahal and Goodrich, we provide the proof in full. This will also illustrate the seriously complicating effect of the nonlocal structure of the fractional difference. We also point out that the statement and proof of this result may also be found in [27], Theorem 2.2, and [3], Theorem 7.2. Prior to stating and proving the monotonicity result, we need to recall a preliminary lemma, which is due to Holm [32] and is of independent interest.

Lemma 2.2 *Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$. Assume that N is the unique positive integer such that $N-1 < \nu < N$. Then*

$$\Delta_a^\nu f(t) = \int_a^{t+\nu+1} h_{-\nu-1}(t, \tau+1) f(\tau)\Delta\tau$$

for each $t \in \mathbb{N}_{a+N-\nu}$.

Theorem 2.3 *Let $y : \mathbb{N}_0 \to \mathbb{R}$ be a nonnegative function satisfying $y(0) = 0$. Fix $\nu \in (1,2)$ and suppose that $\Delta_0^\nu y(t) \geq 0$ for each $t \in \mathbb{N}_{2-\nu}$. Then y is increasing on $\mathbb{N}_0$.*

Proof The manner in which Dahal and Goodrich proved this result was by the principle of strong induction. So, we follow the same method here.

First of all, it is easy to observe that the base case holds somewhat trivially since we calculate

$$y(1) - y(0) = y(1) \geq 0,$$

due to the fact that $y(0) = 0$, by assumption, and the fact that $y(1) \geq 0$, also by assumption. Therefore, it remains to complete the induction step.

To this end, fix $k \in \mathbb{N}$ and suppose that

$$y(i) - y(i-1) \geq 0$$

for each $i \in \mathbb{N}_1^{k-1}$. By assumption we have $\Delta_0^\nu y(t) \geq 0$ for each $t \in \mathbb{N}_{2-\nu}$. Combining this with Lemma 2.2 we obtain, for fixed $k \in \mathbb{N}$, the following collection of estimates.

$$\begin{aligned}
-\Delta_0^\nu y(2-\nu) &= \nu y(1) - y(2) \leq 0, \\
-\Delta_0^\nu y(3-\nu) &= \frac{1}{2}\nu(1-\nu)y(1) + \nu y(2) - y(3) \leq 0, \\
-\Delta_0^\nu y(4-\nu) &= \frac{1}{6}\nu(1-\nu)(2-\nu)y(1) + \frac{1}{2}\nu(1-\nu)y(2) + \nu y(3) - y(4) \leq 0, \\
&\vdots \\
-\Delta_0^\nu y(k-\nu) &= \frac{1}{(k-1)!}\nu(1-\nu)\cdots(k-2-\nu)y(1) + \cdots + \nu y(k-1) - y(k) \leq 0.
\end{aligned} \tag{2.1}$$

Note that in (2.1) we have used the assumption that $y(0) = 0$ to rewrite $\Delta_0^\nu y(j-\nu)$ for each $j \in \mathbb{N}_2^k$. All in all, we see that (2.1) implies the inequality

$$y(k) \geq \frac{1}{(k-1)!}\nu(1-\nu)\cdots(k-2-\nu)y(1) + \cdots + \nu y(k-1), \tag{2.2}$$

for each fixed $k \in \mathbb{N}$. Inequality (2.2) shall be used repeatedly in the sequel.

We claim that for the value of k fixed at the beginning of the preceding paragraph

$$\Delta y(k-1) = y(k) - y(k-1) \geq 0, \tag{2.3}$$

which evidently will complete the induction step. To prove (2.3) we first calculate

$$\begin{aligned}
&y(k) - y(k-1) \\
&\quad = (\nu-1)y(k-1) \\
&\qquad + \left[\left(\frac{1}{2}\nu(1-\nu)y(k-2) - \frac{1}{2}\nu(1-\nu)y(k-1)\right) + \frac{1}{2}\nu(1-\nu)y(k-1)\right] \\
&\qquad + \left[\left(\frac{1}{6}\nu(1-\nu)(2-\nu)y(k-3) - \frac{1}{6}\nu(1-\nu)(2-\nu)y(k-1)\right)\right. \\
&\qquad \left. + \frac{1}{6}\nu(1-\nu)(2-\nu)y(k-1)\right] \\
&\qquad \vdots \\
&\qquad + \left[\left(\frac{1}{(k-1)!}\nu(1-\nu)(2-\nu)\cdots(k-2-\nu)y(1)\right.\right. \\
&\qquad \left. - \frac{1}{(k-1)!}\nu(1-\nu)(2-\nu)\cdots(k-2-\nu)y(k-1)\right) \\
&\qquad \left. + \frac{1}{(k-1)!}\nu(1-\nu)(2-\nu)\cdots(k-2-\nu)y(k-1)\right].
\end{aligned} \tag{2.4}$$

Here we have used the fact that inequality (2.2) holds. On the other hand, invoking the induction hypothesis yields the following $k-2$ estimates:

$$\begin{aligned}
&\underbrace{\frac{1}{2}\nu(1-\nu)}_{<0}\underbrace{\bigl(-y(k-1)+y(k-2)\bigr)}_{\leq 0} \geq 0, \\
&\underbrace{\frac{1}{6}\nu(1-\nu)(2-\nu)}_{<0}\underbrace{\bigl(-y(k-1)+y(k-3)\bigr)}_{\leq 0} \geq 0, \\
&\vdots \\
&\underbrace{\frac{1}{(k-1)!}\nu(1-\nu)(2-\nu)\cdots(k-2-\nu)}_{<0}\underbrace{\bigl(-y(k-1)+y(1)\bigr)}_{\leq 0} \geq 0.
\end{aligned} \tag{2.5}$$

Notice that in (2.5) we utilize the observation that since $y(k-1) \geq y(k-2)$ it follows that $y(k-1) - y(k-3) \geq y(k-2) - y(k-3) \geq 0$, so that, in general,

$$y(k-1) - y(k-j) \geq y\bigl(k-(j-1)\bigr) - y(k-j) \geq 0,$$

for each $j \in \mathbb{N}_2^{k-1}$. In any case, putting the inequalities in (2.5) into estimate (2.4) yields

$$\begin{aligned}
&y(k) - y(k-1) \\
&\quad \geq \left[(\nu-1) + \frac{1}{2}\nu(1-\nu) + \frac{1}{6}\nu(1-\nu)(2-\nu) + \cdots\right. \\
&\qquad \left. + \frac{1}{(k-1)!}\nu(1-\nu)(2-\nu)\cdots(k-2-\nu)\right]y(k-1).
\end{aligned}$$

Recalling that $y(k-1) \geq 0$ by assumption, to complete the proof it suffices to show that

$$(\nu-1)+\frac{1}{2}\nu(1-\nu)+\frac{1}{6}\nu(1-\nu)(2-\nu)+\cdots+\frac{1}{(k-1)!}\nu(1-\nu)(2-\nu)\cdots(k-2-\nu)\geq 0,$$

for each $1 < \nu < 2$.

To this end, let us define the $(k-1)$th degree polynomial function $P_{k-1} : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned}P_{k-1}(\nu) := {} & (\nu-1)+\frac{1}{2}\nu(1-\nu)+\frac{1}{6}\nu(1-\nu)(2-\nu)+\cdots \\ & +\frac{1}{(k-1)!}\nu(1-\nu)(2-\nu)\cdots(k-2-\nu).\end{aligned}$$

It can be shown (see [27]) that

$$P_{k-1}(\nu)=\frac{-1}{(k-1)!}(1-\nu)(2-\nu)\cdots(k-2-\nu)(k-1-\nu),$$

from which it follows that

$$\begin{aligned}P_{k-1}(\nu) &= -\frac{1}{(k-1)!}(1-\nu)(2-\nu)\cdots(k-2-\nu)(k-1-\nu) \\ &= \frac{1}{(k-1)!}(-1)^2(\nu-1)(2-\nu)\cdots(k-2-\nu)(k-1-\nu) \\ &= \frac{1}{(k-1)!}(-1)^3(\nu-1)(\nu-2)(3-\nu)\cdots(k-2-\nu)(k-1-\nu) \\ &\ \ \vdots \\ &= \frac{1}{(k-1)!}(-1)^k(\nu-1)(\nu-2)\cdots(\nu-k+2)(\nu-k+1). \qquad (2.6)\end{aligned}$$

The factorization of P_{k-1} given by (2.6) implies that P_{k-1} has $k-1$ distinct zeros and these zeros are, in particular, $\nu = 1, 2, \ldots, k-1$. Carefully observing the distribution of these zeros implies that for each $k \in \mathbb{N}$ it follows that $P_{k-1}(\nu) > 0$ whenever $\nu \in (1, 2)$ is true. And this implies that (2.3) holds. By the arbitrariness of k, it follows that the proof is complete. □

One may make a couple of observations regarding Theorem 2.3 and its proof. First of all, we notice that the proof is vastly more complicated than that of Proposition 2.1, and this is due precisely to the presence of the nonlocal elements in the definition of the fractional difference. Second of all, however, one may notice that the hypotheses of Theorem 2.3 contain some unusual restrictions - namely, that $y(0) = 0$ and that y be nonnegative. This is slightly surprising since such restrictions evidently are ***not*** required in the integer-order setting. Indeed, it makes no difference whether y is negative or positive in Proposition 2.1, nor is it necessary that $y(0) = 0$.

As such, in light of the statement of Theorem 2.3 one may reasonably ask the following question: Is it possible to eliminate the condition $\Delta y(0) \geq 0$, which implicitly occurs in the statement of this theorem? Note that this is because we assume that $y(0) = 0$ and that y is also nonnegative. In some sense, the implicit requirement that $\Delta y(0) \geq 0$ hold is odd, for it implies that $\Delta_a^\nu f(t) \geq 0$ is, in isolation, insufficient to ensure the monotonicity of f, that, rather, we need a sort of 'initial increasingness' as embodied by the condition $\Delta y(0) \geq 0$.

Certainly, this is ***not*** required in the integer-order setting - *i.e.*, if $\Delta y(t) \geq 0$ for $t \in \mathbb{N}_1$, then f is increasing on $\mathbb{N}_1$ by Proposition 2.1; we do ***not*** need $\Delta f(0) \geq 0$.

Perhaps surprisingly, the answer to the preceding query is that the condition $\Delta y(0) \geq 0$ ***cannot*** be eliminated without replacing it with other suitable auxiliary condition. That is to say, the condition that the fractional difference be nonnegative is insufficient *in isolation* to guarantee the monotonicity of f. And it was Jia *et al.* who discovered this. To illustrate explicitly their discovery, we present next the original example that they provided - see [2, 3] for this example.

Example 2.4 Define the map $f : \mathbb{N}_0 \to \mathbb{R}$ by $f(t) = 2^{-t}$, and assume that

$$\frac{2+\sqrt{2}}{2} < \nu < 2.$$

We claim that the following collection of statements are true.

- $\Delta_0^\nu f(t) \geq 0$ for $t \in \mathbb{N}_{2-\nu}$,
- $f(t) \geq 0$ for $t \in \mathbb{N}_0$,
- f is nonincreasing on $\mathbb{N}_1$.

To prove the preceding claims, let us note that for $t = 2 - \nu + k$, with $k \in \mathbb{N}_0$, we have

$$\Delta_0^\nu f(t) = \int_0^{3+k} h_{-\nu-1}(2-\nu+k, \tau+1) f(\tau)\Delta\tau = \sum_{i=0}^{k+2} h_{-\nu-1}(2-\nu+k, i+1)2^{-i}.$$

Now, for $0 \leq i \leq k+2$, with $1 < \nu < 2$, we observe that

$$\begin{aligned} h_{-\nu-1}(2-\nu+k, i+1) &= \frac{(1-\nu+k-i)^{\underline{-\nu-1}}}{\Gamma(-\nu)} \\ &= \frac{\Gamma(2-\nu+k-i)}{\Gamma(3+k-i)\Gamma(-\nu)} \\ &= \frac{(-\nu+1+k-i)\cdots(-\nu+1)(-\nu)}{(2+k-i)!}. \end{aligned} \tag{2.7}$$

From (2.7) we see that if $k - i \geq 1$, then it follows that $h_{-\nu-1}(2-\nu+k, i+1) > 0$. And when $i = k, k+1, k+2$, we compute

$$\begin{aligned} h_{-\nu-1}(2-\nu+k, k+1) &= \frac{\Gamma(2-\nu)}{2!\Gamma(-\nu)} = \frac{(-\nu+1)(-\nu)}{2}, \\ h_{-\nu-1}(2-\nu+k, k+2) &= \frac{\Gamma(1-\nu)}{\Gamma(-\nu)} = -\nu, \quad \text{and} \\ h_{-\nu-1}(2-\nu+k, k+3) &= \frac{\Gamma(-\nu)}{\Gamma(-\nu)} = 1, \end{aligned} \tag{2.8}$$

respectively. So from (2.8), together with the fact that $\nu \in (\frac{2+\sqrt{2}}{2}, 2)$, we estimate

$$\begin{aligned} \Delta_0^\nu f(t) &\geq \sum_{i=k}^{k+2} h_{-\nu-1}(2-\nu+k, i+1)2^{-i} \\ &= \frac{(-\nu+1)(-\nu)}{2} \cdot \frac{1}{2^k} - \frac{\nu}{2^{k+1}} + \frac{1}{2^{k+2}} \end{aligned}$$

$$= \frac{2\nu^2 - 4\nu + 1}{2^{k+2}}$$
$$> 0.$$

Consequently, we conclude that $\Delta_0^\nu f(t) \geq 0$, for each $t \in \mathbb{N}_{2-\nu}$. But since f is obviously nonincreasing, it follows that the some additional condition is necessary above and beyond the positivity of the fractional difference. Consequently, we are left with the following surprising conclusion: If $1 < \nu < 2$ and $\Delta_a^\nu f(t) \geq 0$ for each $t \in \mathbb{N}_{2-\nu}$, one does not need to have f increasing. In particular, it is ***necessary*** that ***some*** additional condition be imposed beyond simply the nonnegativity of the fractional delta difference on $\mathbb{N}_{2-\nu}$.

Thus, Example 2.4 definitively establishes that, in general, Proposition 2.1 does ***not*** carry over to the discrete fractional setting with delta difference. On the one hand, this is surprising, for one might expect, at a minimum, the fractional difference to preserve this well-known property of the integer-order operator. However, upon more careful thought, due to the nonlocal structure of the fractional operator, this is perhaps less surprising.

It turns out that this is not quite the end of the story, however, for it is possible to generalize Theorem 2.3 by introducing a suitable hypothesis. This generalization was accomplished by Jia *et al.* (see [2]), and their work represented the first significant extension and, more importantly, refinement of the original work by Dahal and Goodrich.

Corollary 2.5 *Let $y : \mathbb{N}_0 \to \mathbb{R}$ be a nonnegative function. Fix $\nu \in (1, 2)$ and suppose that $\Delta_0^\nu y(t) \geq 0$, for each $t \in \mathbb{N}_{2-\nu}$. If $\Delta y(0) \geq 0$, then y is increasing on $\mathbb{N}_0$.*

Thus, we see that the addition of the condition $\Delta y(0) \geq 0$ allows the result to hold. Obviously, the statement of Theorem 2.3 ensures that this is so since if y is nonnegative and satisfies $y(0) = 0$, then $\Delta y(0) \geq 0$.

It turns out that Corollary 2.5 is not even the last word on the subject. In fact, the condition $\Delta y(0) \geq 0$ while sufficient is not necessary insofar as it can be replaced by a somewhat weaker condition. And this particular refinement was recently produced jointly by Baoguo *et al.* (see [22]). Before stating this most up-to-date monotonicity theorem, which is Theorem 2.7, we first state a preliminary lemma, namely Lemma 2.6. This lemma was originally proved by Jia *et al.* [2]. In fact, this was the key discovery necessary to upgrade the result of Theorem 2.3 to the result of Corollary 2.5. Moreover, the lemma plays a key role in establishing Theorem 2.7.

Lemma 2.6 *Assume that $\Delta_a^\nu f(t) \geq 0$, for each $t \in \mathbb{N}_{a+2-\nu}$, with $1 < \nu < 2$. Then*

$$\Delta f(a + k + 1) \geq -h_{-\nu}(a + k + 2 - \nu, a)f(a) - \sum_{\tau=a}^{a+k} h_{-\nu}(a + k + 2 - \nu, \tau + 1)\Delta f(\tau)$$

for each $k \in \mathbb{N}_0$, where

$$h_{-\nu}(t, \tau + 1) = \frac{(t - \tau)^{\underline{-\nu}}}{(t - \nu - \tau)!(a + k + 2 - \tau)!} < 0,$$

for $t \in \mathbb{N}_{a+2-\nu}$, $a \leq \tau + 1 \leq t + \nu - 1$.

As can be seen, Lemma 2.6 establishes a lower bound on $\Delta f(a+k+1)$ that must be observed whenever $\Delta_a^\nu f(t) \geq 0$ holds. This lower control over the first-order difference of f is a truly important observation. Indeed, it facilitates all of the refined monotonicity results (*e.g.*, Corollary 2.5 and Theorem 2.7). It seems that without this lemma in hand, one must prove associated monotonicity results in the more technical manner that, say, Theorem 2.3 is established. In any case, we now state and prove our next monotonicity result.

Theorem 2.7 *Assume that $f : \mathbb{N}_a \to \mathbb{R}$ and that $\Delta_a^\nu f(t) \geq 0$, for each $t \in \mathbb{N}_{a+2-\nu}$, with $1 < \nu < 2$. If*

$$f(a+1) \geq \frac{\nu}{k+2} f(a)$$

for each $k \in \mathbb{N}_0$, then $\Delta f(t) \geq 0$, for $t \in \mathbb{N}_{a+1}$.

Proof We argue that $\Delta f(a+k+1) \geq 0$, for each $k \geq 0$, by means of the principle of strong induction, much as was the case with the proof of Theorem 2.3. To this end, from Lemma 2.6, in the case $k = 0$, together with the hypothesis $f(a+1) \geq \frac{\nu}{2} f(a)$, we are able to establish the following inequality:

$$\begin{aligned}
\Delta f(a+1) &\geq -h_{-\nu}(a+2-\nu, a) f(a) - h_{-\nu}(a+2-\nu, a+1)\Delta f(a) \\
&= -\left[\frac{\Gamma(3-\nu)}{\Gamma(3)\Gamma(-\nu+1)} f(a) + \frac{\Gamma(2-\nu)}{\Gamma(2)\Gamma(-\nu+1)} \Delta f(a)\right] \\
&= -\frac{\Gamma(2-\nu)}{\Gamma(-\nu+1)} \left[\frac{1}{2}(2-\nu) f(a) + \Delta f(a)\right] \\
&\geq -\frac{\Gamma(2-\nu)}{\Gamma(-\nu+1)} \underbrace{\left[\frac{1}{2}(2-\nu) + \frac{\nu}{2} - 1\right]}_{=0} f(a) \\
&= 0.
\end{aligned}$$

Suppose next that $k \geq 1$ and $\Delta f(a+i) \geq 0$, for $i \in \mathbb{N}_1^k$. From Lemma 2.6 together with the hypothesis $f(a+1) \geq \frac{\nu}{k+2} f(a)$ for each $k \in \mathbb{N}_0$, we can also establish the inequality

$$\begin{aligned}
\Delta f(a+k+1) &\geq -f(a) h_{-\nu}(a+k+2-\nu, a) - \sum_{\tau=a}^{a+k} h_{-\nu}\big(a+k+2-\nu, \sigma(\tau)\big)\Delta f(\tau) \\
&\geq -f(a) h_{-\nu}(a+k+2-\nu, a) - h_{-\nu}(a+k+2-\nu, a+1)\Delta f(a) \\
&= -\frac{\Gamma(k+3-\nu)}{\Gamma(k+3)\Gamma(-\nu+1)} f(a) - \frac{\Gamma(k+2-\nu)}{\Gamma(k+2)\Gamma(-\nu+1)} \Delta f(a) \\
&= \underbrace{-\frac{\Gamma(k+2-\nu)}{\Gamma(k+2)\Gamma(-\nu+1)}}_{>0} \left[\frac{k+2-\nu}{k+2} f(a) + \Delta f(a)\right] \\
&\geq -\frac{\Gamma(k+2-\nu)}{\Gamma(k+2)\Gamma(-\nu+1)} \underbrace{\left[\frac{k+2-\nu}{k+2} + \frac{\nu}{k+2} - 1\right]}_{=0} f(a) \\
&= 0.
\end{aligned}$$

But this inequality implies that f is monotone increasing. And so the proof is complete. $\square$

Example 2.8 Throughout this example, we shall assume that $f : \mathbb{N}_a \to [0, +\infty)$ - *i.e.*, f is a nonnegative map. With this in mind, to illustrate the application of Theorem 2.7 we first construct the following table, which illustrates the inequality in the statement of the theorem for some different choices of $1 < \nu < 2$ for $k = 1$.

ν	1.2	1.4	1.6	1.8
Condition:	$f(a+1) \geq \frac{3}{5}f(a)$	$f(a+1) \geq \frac{7}{10}f(a)$	$f(a+1) \geq \frac{4}{5}f(a)$	$f(a+1) \geq \frac{9}{10}f(a)$

Note that since $k \mapsto \frac{\nu}{k+2}$ is a decreasing map in k, for each fixed ν, it follows that if the conditions in the above table hold, then for each $k \in \mathbb{N}_0$ the inequality $f(a+1) \geq \frac{\nu}{k+2}f(a)$ is satisfied; here we use the nonnegativity assumption on f.

So, a cursory examination of the table reveals that as $\nu \to 1^+$ the inequality in the statement of the theorem becomes a weaker restriction, whereas as $\nu \to 2^-$ the inequality approaches the 'initial monotonicity' condition $\Delta f(a) \geq 0$, which was required in the original monotonicity theorem, namely Theorem 2.3. Nonetheless, for each $\nu \in (1, 2)$, we observe that the hypotheses of Theorem 2.7 do ***not*** require any 'initial monotonicity'. For example, if we fix $\nu = 1.2$, then we see from the table that the only condition required in Theorem 2.7, other than that $\Delta_0^{\frac{6}{5}} y(t) \geq 0$, is that

$$f(a+1) \geq \frac{3}{5}f(a).$$

Thus, it is possible that $f(a+1) < f(a)$ and, thus, that $\Delta f(a) \ngeq 0$.

Remark 2.9 The statement of Theorem 2.7 is the most general known result for monotonicity theorems when using the delta fractional difference. Obviously, the condition given in the statement of this theorem is sufficient. Whether it is also necessary, however, is, to the best of the authors' knowledge still an open question. It would be interesting to establish whether this is the case, and, if it is not necessary, whether one can derive a sharp result.

Remark 2.10 Note also that Theorem 2.7 does ***not*** require that f be a nonnegative map. This represents an improvement over Theorem 2.3. However, if f happens to be nonnegative, as we assumed in Example 2.8, for instance, then one can obtain more refined information from Theorem 2.7. For refinements in this direction, we encourage the interested reader to consult the forthcoming paper [22].

2.2 Results for the nabla fractional difference

In this section we consider some additional monotonicity-type results but, crucially, when the nabla difference is utilized instead. One might reasonably suppose that the results would parallel the delta case. However, it turns out that this is not quite the case. Instead, as we shall see momentarily, there is a rather stronger connection between monotonicity and the sign of $\nabla_a^\nu f(t)$.

To begin, as we did in the previous subsection, let us recall the following, obvious result from the integer-order difference calculus.

Proposition 2.11 *The map* $y : \mathbb{N}_a \to \mathbb{R}$ *is increasing on* $\mathbb{N}_a$ *if and only if* $\nabla y(t) \geq 0$ *for* $t \in \mathbb{N}_{a+1}$.

As with the delta case, we see that Proposition 2.11 is really a triviality. Passing to the fractional-order case, however, something very interesting happens. Before stating the monotonicity result we obtain in the nabla setting, we require a preliminary lemma. This particular lemma is the nabla analog of Lemma 2.6, and it was similarly established by Jia *et al.* - see [2], Section 2; we omit its proof.

Lemma 2.12 *Assume that* $f : \mathbb{N}_a \to \mathbb{R}$ *and that* $\nabla_a^\nu f(t) \geq 0$, *for each* $t \in \mathbb{N}_{a+1}$, *where* $1 < \nu < 2$. *Then we have*

$$\nabla f(t) \geq -f(a+1)\big[H_{-\nu-1}(t,a) + H_{-\nu}(t,a+1)\big] - \sum_{\tau=a+2}^{t-1} H_{-\nu}(t,\tau-1)\nabla f(\tau).$$

As with Lemma 2.6 we see that Lemma 2.12 establishes a lower bound on the nabla difference, and it does so in terms of the Taylor monomials. This is the key observation needed to establish a monotonicity result in the nabla setting. Indeed, with Lemma 2.12 in hand, we can then easily obtain the following result, Theorem 2.13, which was originally discovered by Jia *et al.* [2].

Theorem 2.13 *Suppose that* $f : \mathbb{N}_a \to \mathbb{R}$. *If* $\nabla_a^\nu f(t) \geq 0$ *for each* $t \in \mathbb{N}_{a+1}$, *where* $1 < \nu < 2$, *then* $\nabla f(t) \geq 0$ *for each* $t \in \mathbb{N}_{a+2}$.

Proof We argue that $\nabla f(a+k) \geq 0$ for each $k \in \mathbb{N}_a$, and we do so by means of an induction argument. To this end, from Lemma 1.10 we deduce that

$$\begin{aligned}
\nabla_a^\nu f(a+1) &= \int_a^{a+1} H_{-\nu-1}(t,\tau-1)f(\tau)\nabla\tau \\
&= \sum_{\tau=a+1}^{a+1} H_{-\nu-1}(t,\tau-1)f(\tau) \\
&= f(a+1) \\
&\geq 0, \qquad (2.9)
\end{aligned}$$

where the inequality follows by the hypotheses assumed in the statement of the theorem. Moreover, we have

$$\begin{aligned}
\nabla_a^\nu f(a+2) &= \int_a^{a+2} H_{-\nu-1}(a+2,\tau-1)f(\tau)\nabla\tau \\
&= \sum_{\tau=a+1}^{a+2} H_{-\nu-1}(a+2,\tau-1)f(\tau) \\
&= f(a+2) - \nu f(a+1) \\
&= \nabla f(a+2) - (\nu-1)f(a+1). \qquad (2.10)
\end{aligned}$$

Recall that we are assuming that $\nabla_a f(a+2) \geq 0$. Thus, combining estimates (2.9) and (2.10) we obtain

$$\nabla f(a+2) \geq (\nu-1)f(a+1) = (\nu-1)\nabla_a^\nu f(a+1) \geq 0,$$

which establishes the basis for induction.

Now we complete the actual induction argument. Therefore, assume that $\nabla f(t) \geq 0$ for each $t \in \mathbb{N}_{a+2}^{a+k}$ for some $k \in \mathbb{N}_2$ fixed. Then using Lemma 2.12 we obtain

$$\begin{aligned}\nabla f(a+k+1) &\geq -f(a+1)\underbrace{\left[H_{-\nu-1}(a+k+1,a) + H_{-\nu}(a+k+1,a+1)\right]}_{<0 \text{ for each } k\in\mathbb{N}_2} \\ &\quad - \sum_{\tau=a+2}^{t-1} \underbrace{H_{-\nu}(a+k+1,\tau-1)}_{<0 \text{ for each } k\in\mathbb{N}_2} \nabla f(\tau) \\ &> 0. \end{aligned} \tag{2.11}$$

Note that in (2.11) we are using the fact that $f(a+1) \geq 0$, which follows from the equality in inequality (2.9). Since this completes the induction step, the proof is complete, and we conclude that f is monotone increasing on its domain, as claimed. □

As a comparison of Theorem 2.3 to Theorem 2.13 reveals, we observe immediately that there are substantial differences between the delta and nabla cases. For example, we notice that in Theorem 2.3 we require the 'initial monotonicity' of the map f - *i.e.*, we have $f(a+1) \geq 0$ and $f(a) = 0$. Even in the more refined results, namely Corollary 2.5 and Theorem 2.7, we see that *some* sort of additional hypothesis is required above and beyond the nonnegativity of the fractional difference.

In great contrast, we see in Theorem 2.13 that in the nabla setting the imposition of the condition $\nabla_a^\nu f(t) \geq 0$, for $t \in \mathbb{N}_{a+1}$, is sufficient that f is increasing on $\mathbb{N}_{a+2}$. Thus, the nabla operator behaves differently in this case. And, in fact, the nabla operator behaves more naturally insofar Theorem 2.13 is a more natural analog of Proposition 2.1 than are the results in the delta difference setting.

3 Convexity

3.1 Results for the delta fractional difference

We now discuss the known connections between the delta fractional difference and the convexity of the map f. As with monotonicity the connections are not as straightforward as one might hope and certainly are more complicated than in the integer-order setting. To emphasize this fact going forward, let us first recall the following basic result.

Proposition 3.1 *Let $f : \mathbb{N}_a \to \mathbb{R}$. Then $\Delta^2 y(t) > 0$ for each $t \in \mathbb{N}_a$ if and only if y is a convex map on $\mathbb{N}_a$. Similarly, $\Delta^2 y(t) < 0$ for each $t \in \mathbb{N}_a$ if and only if y is a concave map on $\mathbb{N}_a$.*

As was mentioned in Section 1 the proof of this result is essentially trivial. If we pass to the fractional-order setting, however, then the relationship is much more complicated. Goodrich [31] was the first to investigate this relationship, and his original result in this area was the following - see also Dahal and Goodrich [28]. (It should be noted that the original statement, *i.e.*, in [31], omitted the hypothesis $\Delta^{N-1}f(0) \geq 0$. This was later pointed out by Jia *et al.* [23]. Thus, as a technical matter, the statement of Theorem 3.2 provided here really is that given in [23].)

Theorem 3.2 *Assume that $\Delta_a^\nu f(t) \geq 0$ for each $t \in \mathbb{N}_{N+a-\nu}$ where $N \in \mathbb{N}_3$ is selected such that $N-1 < \nu < N$. In addition, suppose that*

(1) $(-1)^{N-i}\Delta^i f(a) \geq 0$, *for each $i \in \mathbb{N}_0^{N-2}$; and*
(2) $\Delta^{N-1}f(a) \geq 0$.

Then $\Delta^{N-1}f(t) \geq 0$, for each $t \in \mathbb{N}_a$.

We omit the proof, which can be found in [31], but, essentially, the result is proved by applying Theorem 2.3 to the map $w : \mathbb{N}_0 \to \mathbb{R}$ defined by $w(t) := \Delta^{N-2}y(t)$ to establish that w is an increasing map. An interesting corollary of this result is the following.

Corollary 3.3 *Fix $\mu \in (N-1, N)$ with $N \in \mathbb{N}_3$. Let $f : \mathbb{N}_0 \to \mathbb{R}$ be a function satisfying $\Delta_0^\mu y(t) \geq 0$ for each $t \in \mathbb{N}_{N-\mu}$. In case N is odd, assume that*

$$\begin{cases} \Delta^j y(0) < 0, & j = 0, 2, \ldots, N-3, \\ \Delta^j y(0) > 0, & j = 1, 3, \ldots, N-4, \end{cases}$$

whereas in the case N is even, assume that

$$\begin{cases} \Delta^j y(0) > 0, & j = 0, 2, \ldots, N-4, \\ \Delta^j y(0) < 0, & j = 1, 3, \ldots, N-3. \end{cases}$$

If in addition $\Delta^{N-2}y(0) \geq 0$ and $\Delta^{N-1}y(0) \geq 0$, then $\Delta^{N-1}y(t) \geq 0$, for each $t \in \mathbb{N}_0$.

For the proof of this result, one may consult either Goodrich [31] or Goodrich and Peterson [3]. Here we wish to focus on some interesting examples that follow from Corollary 3.3.

Example 3.4 Suppose that $N = 3$. Then Corollary 3.3 demonstrates that $\Delta^2 y(t) \geq 0$, for example, provided that

$$\begin{aligned} &y(0) < 0, \\ &\Delta y(0) > 0, \\ &\Delta^2 y(0) > 0, \\ &\Delta_0^\mu y(t) > 0 \quad \text{for some } \mu \in (2,3), t \in \mathbb{N}_{3-\nu}. \end{aligned}$$

On the other hand, suppose now that $N = 4$. Then Corollary 3.3 implies that $\Delta^3 y(t) \geq 0$ if we have

$$\begin{aligned} &y(0) > 0, \\ &\Delta y(0) < 0, \end{aligned}$$

$$\Delta^2 y(0) > 0,$$
$$\Delta^3 y(0) > 0,$$
$$\Delta_0^{\mu} y(t) > 0 \quad \text{for some } \mu \in (3,4), t \in \mathbb{N}_{4-\nu}.$$

Let us focus on the case $N = 3$ for a moment. Observe from the above application of Corollary 3.3 that we have a sort of unusual collection of conditions - namely, that y must be 'initially' negative, increasing, and convex. If all this is so ***and*** also $\Delta_0^{\mu} y(t)$ is nonnegative, then we may deduce that $\Delta^2 y(t) \geq 0$; in fact, if $\Delta_0^{\mu} y(t)$ is positive, then we can actually deduce that y is a convex map - *i.e.*, that $\Delta^2 y(t) > 0$.

So, one perhaps unexpected aspect of this is that we require a certain 'initial convexity', roughly speaking, in order to obtain the result. We also have to require the auxiliary conditions on $y(0)$ and $\Delta y(0)$. All in all, this stands in bold relief to the integer-order case as embodied by Proposition 3.1.

In light of Example 3.4 we might wonder whether we can, for example, eliminate the condition $\Delta^2 y(0) \geq 0$. The following surprising example, constructed by Jia *et al.* [23], shows that this is not quite so. In other words, the collection of conditions

- $f(a) \leq 0$,
- $\Delta f(a) \geq 0$,
- $\Delta_a^{\mu} f(t) \geq 0$

is ***not*** sufficient to deduce that $\Delta^2 y(t) \geq 0$ holds for $t \in \mathbb{N}_{a+1}$. We note that this is the same example as is presented in [23].

Example 3.5 Define the map $f : \mathbb{N}_0 \to \mathbb{R}$ by

$$f(t) := \left(t - \frac{1}{2}\right)^{\underline{0.5}}.$$

We compute the following:

$$f(0) = 0,$$
$$f(1) = \frac{1}{2}\sqrt{\pi},$$
$$f(2) = \frac{3}{4}\sqrt{\pi}.$$

One can show both that $\Delta f(t) > 0$ and $\Delta^2 f(t) < 0$, for each $t \in \mathbb{N}_0$. At the same time we have

$$\Delta_0^{2.6} f(t) = \int_0^{t+\nu+1} h_{-\nu-1}(t, \tau + 1) f(\tau) \Delta\tau > 0,$$

for each $t \in \mathbb{N}_{3-\nu}$. In other words, we conclude that

- $f(0) \leq 0$,
- $\Delta f(0) \geq 0$,
- $\Delta_0^{2.6} f(t) \geq 0$, $t \in \mathbb{N}_{3+a-\nu}$,

but yet $\Delta^2 f(t) < 0$ for each $t \in \mathbb{N}_0$. Hence, we conclude that the preceding collection of hypotheses is insufficient to guarantee the convexity of the map f.

Thus, in light of Example 3.5 a natural question is in what way, if any way at all, can Theorem 3.2 be ameliorated. As with the connection between delta fractional differences and monotonicity this answer is frankly complicated. And to the best of the authors' knowledge, sharp results are not known at present. Nonetheless, we can improve Theorem 3.2. The following theorem, which we state without proof, was recently proved by Goodrich in [72]. Associated to it are several corollaries, which we omit - see [72]. But we do provide an example here to illustrate the specific improvement it affords over Theorem 3.2.

Theorem 3.6 *Fix $\nu \in (2,3)$ and suppose that $\Delta_a^\nu f(t) \geq 0$ for each $t \in \mathbb{N}_{3+a-\nu}$. If for each $k \in \mathbb{N}_{-1}$ we have*

$$\frac{1}{-\nu+1} f(a+2) + \frac{\nu+2+k}{(\nu-1)(3+k)} f(a+1) - \frac{\nu}{(3+k)(4+k)} f(a) \leq 0, \tag{3.1}$$

then $\Delta^2 f(t) \geq 0$ for each $t \in \mathbb{N}_{a+1}$.

Example 3.7 Suppose that we put $f(a) := 0$, $f(a+1) := 1$, and $f(a+2) := 1.9$. Let us also fix $\nu = \frac{5}{2}$. Then it is easy to show that inequality (3.1) holds. In spite of this, one finds that $\Delta^2 f(a) = -\frac{1}{10} < 0$. Thus, we see that Theorem 3.6 does ***not*** require any 'initial convexity', and in this sense, then, Theorem 3.6 can be seen as both a refinement and an improvement of Theorem 3.2.

We end this subsection with the following remark.

Remark 3.8 As previously mentioned, it is not known if the results presented here are sharp. As with the monotonicity results of Section 2 for the delta fractional difference, it would be interesting to determine precisely the optimal convexity-type result one can obtain.

3.2 Results for the nabla fractional difference

We now look at some related convexity results in the nabla fractional setting. As with the monotonicity results, we shall see quickly that the results in the nabla setting are simpler, cleaner, and more natural than those in the delta setting. Also as with the delta setting the nabla results follow from the following inequality, which is of independent interest and was first discovered by Erbe *et al.* [23].

Lemma 3.9 *Assume that $\nabla_a^\nu f(t) \geq 0$, for each $t \in \mathbb{N}_{a+1}$, where $2 < \nu < 3$. Then for $t = a+k$, with $k \in \mathbb{N}_4$, we have*

$$\nabla^2 f(t) \geq -\sum_{i=a+2}^{a+k-2} H_{-\nu+1}(a+k,i)\nabla^2 f(a+i+1) + \frac{(a+k-2)\Gamma(-\nu+k)}{(k-1)!\Gamma(-\nu+1)} f(a+1).$$

With Lemma 3.9 in hand, it is a simple matter to obtain the following convexity-type result for the nabla fractional difference.

Theorem 3.10 *Assume that* $f : \mathbb{N}_{a+1} \to \mathbb{R}$ *satisfies* $\nabla_a^\nu f(t) \geq 0$, *for each* $t \in \mathbb{N}_{a+1}$, *where* $2 < \nu < 3$. *Then* $\nabla^2 f(t) \geq 0$, *for each* $t \in \mathbb{N}_{a+3}$.

Proof As with the other results we have established, we shall provide an induction proof of this result. In particular, by means of Lemma 1.10 we obtain

$$\begin{aligned}\nabla_a^\nu f(a+3) &= \int_a^{a+3} H_{-\nu-1}(a+3, \tau-1) f(\tau) \nabla\tau \\ &= \nabla^2 f(a+3) - (\nu-2) f(a+2) - f(a+1) + \frac{\nu(\nu-1)}{2} f(a+1),\end{aligned}$$

from which it follows that

$$\nabla^2 f(a+3) \geq (\nu-2) f(a+2) + f(a+1) - \frac{\nu(\nu-1)}{2} f(a+1) \geq \frac{(\nu-2)(\nu-1)}{2} f(a+1) \geq 0,$$

where we use the fact that the inequality $f(a+2) \geq \nu f(a+1)$ can be shown. Thus, the base case is established. Finally, by using Lemma 3.9 the induction step can be established in a manner similar to, for example, the proof of Theorem 2.13. □

As with the monotonicity results presented in Section 2, a comparison of Theorem 3.10 to either Theorem 3.2 or 3.6 demonstrates that in the nabla setting we are able to obtain a convexity-type result that is a much more natural analog of Proposition 3.1. In particular, we do not need to impose a number of extra conditions, which are not necessary in the integer-order setting, and so, Theorem 3.10 is cleaner in this way. So, all in all, we see yet another dissimilarity between the delta and nabla fractional differences.

3.3 Further results

In this subsection we collect some further results on the relationship between fractional differences of a map f and associated properties of f itself. We state each of these results without proof. Our first collection of results provides a relationship between the nabla fractional difference and the sign of $\nabla^k f(t)$ for various choices of $k \in \mathbb{N}$. These results were proved by Baoguo *et al.* [21].

Theorem 3.11 *Assume that* $f : \mathbb{N}_a \to \mathbb{R}$ *satisfies* $\nabla_a^\nu f(t) \geq 0$ *for each* $t \in \mathbb{N}_{a+1}$, *where* $3 < \nu < 4$. *Then* $\nabla^3 f(t) \geq 0$, *for each* $t \in \mathbb{N}_{a+4}$.

Theorem 3.12 *Assume that* $f : \mathbb{N}_a \to \mathbb{R}$ *satisfies* $\nabla_a^\nu f(t) \geq 0$ *for each* $t \in \mathbb{N}_{a+1}$, *where* $4 < \nu < 5$. *Then* $\nabla^4 f(t) \geq 0$, *for each* $t \in \mathbb{N}_{a+5}$.

Theorem 3.13 *Assume that* $f : \mathbb{N}_a \to \mathbb{R}$ *satisfies* $\nabla_a^\nu f(t) \geq 0$ *for each* $t \in \mathbb{N}_{a+1}$, *where* $5 < \nu < 6$. *Then* $\nabla^5 f(t) \geq 0$, *for each* $t \in \mathbb{N}_{a+6}$.

Notice that in each of Theorems 3.11, 3.12, and 3.13 we obtain the conclusion *without* the imposition of additional hypotheses. As before, this is typical when considering the nabla fractional difference.

Remark 3.14 It is possible to extend the preceding theorems to the case where $\nu > 6$. The results are analogous to those stated above, and we leave it to the reader to state and prove such results - see the discussion in [21] for additional details.

To illustrate results of a different flavor, we can study the reversal of some of the relationships we have previously deduced. That is to say, we have thus far focused on what properties of f the sign of $\Delta_a^\nu f(t)$ or $\nabla_a^\nu f(t)$ imply. But this question can be reversed, and we can ask what, if anything, the sign of the integer-order differences imply about the fractional-order difference. As a particular case of this sort of result, we consider the following theorem, which was also recently proved by Baoguo *et al.* [22].

Theorem 3.15 *Assume that* $f : \mathbb{N}_a \to \mathbb{R}$ *and that each of the following conditions holds*:

(1) $\Delta^N f(t) \geq 0$, *for each* $t \in \mathbb{N}_a$,

(2) $(-1)^{N-i}\Delta^i f(a) \leq 0$, *for each* $i \in \mathbb{N}_0^{N-1}$.

Then $\Delta_a^\nu f(t) \geq 0$, *for each* $t \in \mathbb{N}_{a+N-\nu}$, *where* $N - 1 < \nu < N$.

By specializing $N \in \mathbb{N}_1$ to various particular values, we can obtain a suite of corollaries that follow from Theorem 3.15, one of which was earlier discovered independently as a special case. We identify certain of these special cases in the following examples.

Example 3.16 Suppose that we fix $N = 1$ in Theorem 3.15. We deduce that if

- $\Delta f(t) \geq 0$, for each $t \in \mathbb{N}_a$,
- $f(a) \geq 0$

then $\Delta_a^\nu f(t) \geq 0$, for each $t \in \mathbb{N}_{a+1-\nu}$. Thus, if f is increasing and initially nonnegative, then its fractional difference must be nonnegative. We note that this particular result was originally discovered by Atici and Uyanik [20].

Example 3.17 Suppose that f satisfies the following collection of hypotheses.

- $\Delta^2 f(t) > 0$, for each $t \in \mathbb{N}_a$,
- $\Delta f(a) > 0$,
- $f(a) \leq 0$.

Then by applying Theorem 3.15 in the case $N = 2$ we deduce that $\Delta_a^\nu f(t) \geq 0$, for each $t \in \mathbb{N}_{a+2-\nu}$. Thus, we conclude that if f is initially nonpositive and, in addition, both 'initially increasing' and 'initially convex', then it follows that $\Delta_a^\nu f(t)$ is nonnegative whenever $t \in \mathbb{N}_{a+2-\nu}$.

Example 3.18 Suppose that f satisfies the following collection of hypotheses.

- $\Delta^3 f(t) \geq 0$, for each $t \in \mathbb{N}_a$,
- $\Delta^2 f(t) \geq 0$,
- $\Delta f(a) \leq 0$,
- $f(a) \geq 0$.

Then by applying Theorem 3.15 in the case $N = 3$ we deduce that $\Delta_a^\nu f(t) \geq 0$, for each $t \in \mathbb{N}_{a+3-\nu}$.

As our final collection of examples in this subsection, we consider briefly the relationship between the so-called Caputo nabla fractional difference of a map $t \mapsto f(t)$ and the monotonicity or convexity of f. These sorts of results were recently investigated by Erbe *et al.* [24]; for a more thorough introduction to Caputo fractional differences than what we provide here, the reader is encouraged to consult the textbook by Goodrich and Peterson [3].

First of all, the definition of the Caputo nabla fractional difference is as follows.

Definition 3.19 Assume that $f : \mathbb{N}_{a-N+1} \to \mathbb{R}$ and $\mu > 0$. Then the μth order Caputo nabla fractional difference of f, denoted $\nabla_{a*}^{\mu} f$, is defined by

$$\nabla_{a*}^{\mu} f(t) := \nabla_a^{-(N-\mu)} \nabla^N f(t), \tag{3.2}$$

for $t \in \mathbb{N}_{a+1}$, and where $N - 1 < \mu < N$.

At first glance, it may be difficult to see the difference between Definition 1.9 and Definition 3.19. But a careful inspection reveals that it lies in the *order* in which the integer-order nabla difference ∇^N is applied. In the Caputo-type difference it is the right-hand factor in the composition, whereas in the Riemann-Liouville-type difference (*i.e.*, Definition 1.9) it is the left-hand factor in the composition. This switch, while seemingly minor, does induce some interesting changes in the properties of the Caputo-type difference. One prominent alteration is that

$$\nabla_{a*}^{\mu} [C](t) \equiv 0,$$

for C a constant. Thus, one advantage of the Caputo-type difference is that it preserves the well-known property that differences of constant polynomials are the zero polynomials.

Remark 3.20 As Definition 3.19 demonstrates, the Caputo difference utilizes the 'original' nabla sum. As such, when working with Caputo nabla differences we utilize Definition 1.8 for our notion of a fractional sum. In other words, there is not, as such, a specialized 'Caputo-type fractional nabla sum'.

Our interest in Definition 3.19 in this paper is primarily that we can obtain the following inequality, which then leads, as a consequence, to an interesting monotonicity-type result. We omit the proofs of these results - see [3, 24].

Lemma 3.21 *Assume that* $N - 1 < \nu < N$ *and* $f : \mathbb{N}_{a-N+1} \to \mathbb{R}$. *If we have*
(1) $\nabla_{a*}^{\nu} f(t) \geq 0$, *for each* $t \in \mathbb{N}_{a+1}$; *and*
(2) $\nabla^{N-1} f(a) \geq 0$;
then $\nabla^{N-1} f(t) \geq 0$, *for each* $t \in \mathbb{N}_a$.

Theorem 3.22 *Assume that* $1 < \nu < 2$ *and that* $f : \mathbb{N}_{a-1} \to \mathbb{R}$. *If we have*
(1) $\nabla_{a*}^{\nu} f(t) \geq 0$, *for each* $t \in \mathbb{N}_{a+1}$; *and*
(2) $f(a) \geq f(a-1)$;
then f *is increasing on* $\mathbb{N}_{a-1}$.

While it turns out that the converse of Theorem 3.22 is not necessarily true, we can obtain a partial converse. This is the following theorem.

Theorem 3.23 *Assume that* $0 < \nu < 1$ *and that* $f : \mathbb{N}_{a-1} \to \mathbb{R}$. *If* f *is increasing on* $\mathbb{N}_a$, *then* $\nabla_{a*}^{\nu} f(t) \geq 0$, *for each* $t \in \mathbb{N}_{a+1}$.

We conclude with an example to demonstrate that the converse of Theorem 3.22 is not true, in general. This example can also be found in the textbook by Goodrich and Peterson - see [3], Example 3.135.

Example 3.24 Define the map $f:\mathbb{N}_1\to\mathbb{R}$ by $f(t):=-\sqrt{t}$; put $a:=2$. Observe that $f''(t)\geq 0$ on $[1,+\infty)$. It can be argued by means of Taylor's theorem that if f is of class $\mathcal{C}^2([a,+\infty))$ and $f''(t)\geq 0$ on $[a,+\infty)$, then it follows that $\nabla^{\nu}_{a*}f(t)\geq 0$ for each $t\in\mathbb{N}_{a+1}$, provided that $1<\nu<2$. An application of this result implies that $\nabla^{\nu}_{a*}f(t)\geq 0$. However, f is actually a decreasing map, as is easily seen. Thus, we conclude that the converse of Theorem 3.22 is, in general, not true.

3.4 Results for the q-fractional difference

Most of the results in this section appear in the paper by Baoguo *et al.* [73]. First we introduce some notation used in the quantum calculus (q-calculus) (see [25]). For $f:q^{\mathbb{N}_0}\to\mathbb{R}$ the nabla q-difference operator is defined by

$$\nabla_q f(t):=\frac{f(t)-f(q^{-1}t)}{(1-q^{-1})t},\quad t\in q^{\mathbb{N}_1}.$$

For any real number α and $q>0$, $q\neq 1$, we set $[\alpha]_q:=\frac{q^\alpha-1}{q-1}$. Then we have the q-analogy of $n!$ in the form $[n]_q!:=[n]_q[n-1]_q\cdots[1]_q$ for $n=1,2,\ldots$, whereas for $n=0$ we put $[0]_q!:=1$. If $q=1$, then $[\alpha]_1:=\alpha$ and $[n]_1!$ becomes the standard factorial. Further, the q-binomial coefficients are defined by

$$\begin{bmatrix}\alpha\\0\end{bmatrix}_q:=1,$$
$$\begin{bmatrix}\alpha\\n\end{bmatrix}_q:=\frac{[\alpha]_q[\alpha-1]_q\cdots[\alpha-n+1]_q}{[n]_q!},$$

where $\alpha\in\mathbb{R}$ and $n\in\mathbb{N}_0$. The extension of the q-binomial coefficient to non-integer values, n, is in terms of the Γ_q function defined for $0<q<1$ as

$$\Gamma_q(t):=\frac{(q,q)_\infty(1-q)^{1-t}}{(q^t,q)_\infty},\tag{3.3}$$

where $(a,q)_\infty=\prod_{j=0}^{\infty}(1-aq^j)$ and $t\in\mathbb{R}\setminus\{0,-1,-2,\ldots\}$. It is easy to check that Γ_q satisfies the functional relation $\Gamma_q(t+1)=[t]_q\Gamma_q(t)$. The q-analog of the power function is introduced as

$$(t-s)_q^{(\alpha)}:=t^\alpha\frac{(\frac{s}{t},q)_\infty}{(q^\alpha\frac{s}{t},q)_\infty},\quad t\neq 0, 0<q<1, \alpha\in\mathbb{R}.\tag{3.4}$$

For $\alpha=n$, a positive integer, this expression reduces to

$$(t-s)_q^{(n)}=t^n\prod_{j=0}^{n-1}\left(1-q^j\frac{s}{t}\right).$$

The following two definitions appear in [25].

Definition 3.25 (Nabla fractional sum) Let $f : q^{\mathbb{N}_0} \to \mathbb{R}$ be given and $q > 1$, $\nu > 0$. Then

$$\nabla_{q,\rho(1)}^{-\nu} f(t) := \frac{1}{\Gamma_{q^{-1}}(\nu)} \int_{\rho(1)}^{t} \left(t - q^{-1}\tau\right)_{q^{-1}}^{(\nu-1)} f(\tau) \nabla_q \tau, \tag{3.5}$$

for $t \in q^{\mathbb{N}_0}$, where $\rho(1) = q^{-1}$ and by convention $\nabla_{q,\rho(1)}^{-\nu} f(\rho(1)) = 0$.

Definition 3.26 Let $\nu \in \mathbb{R}^+$, $f : q^{\mathbb{N}_0} \to \mathbb{R}$, and let $t \in q^{\mathbb{N}_0}$. Then we define the nabla q-fractional difference of f at t by

$$\left(\nabla_{q,\rho(1)}^{\nu} f\right)(t) := \left(\nabla_q^m \nabla_{q,\rho(1)}^{-(m-\nu)} f\right)(t),$$

where $m \in \mathbb{N}_1$ satisfies $m - 1 < \nu < m$.

The power rules in the following lemma are very useful.

Lemma 3.27 *For* $q > 1$,

(1) *the nabla q-difference of the q-factorial function* $(t-s)_{q^{-1}}^{(\alpha)}$ *with respect to t is*

$$_t\nabla_q (t-s)_{q^{-1}}^{(\alpha)} = \frac{1-q^{-\alpha}}{1-q^{-1}} (t-s)_{q^{-1}}^{(\alpha-1)},$$

(2) *the nabla q-difference of the q-factorial function* $(t-s)_{q^{-1}}^{(\alpha)}$ *with respect to s is*

$$_s\nabla_q (t-s)_{q^{-1}}^{(\alpha)} = -\frac{1-q^{-\alpha}}{1-q^{-1}} \left(t - q^{-1}s\right)_{q^{-1}}^{(\alpha-1)},$$

where $\alpha \in \mathbb{R}$.

The following lemma appears in [74].

Lemma 3.28 (Leibniz rule) *Assume* $f : q^{\mathbb{N}_1} \times q^{\mathbb{N}_1} \to \mathbb{R}$. *Then*

$$_t\nabla_q \left[\int_1^t f(t,s) \nabla_q s\right] = \int_1^t {_t\nabla_q} f(t,s) \nabla_q s + f\left(q^{-1}t, t\right)$$

for $t \in q^{\mathbb{N}_1}$.

With the preceding preliminary results in hand, it is then possible to state monotonicity- and convexity-type results associated to the q-fractional difference. Here we state a couple of representative results.

In particular, in [73] the following monotonicity result is given.

Theorem 3.29 *Assume* $f : q^{\mathbb{N}_0} \to \mathbb{R}$, $\nabla_q^{\nu} f(t) \geq 0$ *for each* $t \in q^{\mathbb{N}_0}$, *with* $1 < \nu < 2$, *then* $\nabla_q f(t) \geq 0$ *for* $t \in q^{\mathbb{N}_1}$.

Also in [73] the important convexity result is proved.

Theorem 3.30 *Assume* $f : q^{\mathbb{N}_0} \to \mathbb{R}$, $\nabla_q^{\nu} f(t) \geq 0$ *for each* $t \in q^{\mathbb{N}_1}$, *with* $2 < \nu < 3$, *then* $\nabla_q^2 f(t) \geq 0$ *for* $t \in q^{\mathbb{N}_2}$.

4 Qualitative behavior of solutions to fractional difference equations

4.1 Results for the delta fractional difference

Many of the results in this subsection can be found in the paper by Baoguo *et al.* [75] and in the references given there. First we give a comparison theorem for certain delta fractional equations of order ν, $0 < \nu < 1$.

Theorem 4.1 *Assume $c_1(t) \geq c_2(t) \geq -\nu$, $0 < \nu < 1$, and $x(t)$, $y(t)$ are the solutions of the equations*

$$\Delta^{\nu}_{a+\nu-1}x(t) = c_1(t)x(t+\nu-1) \tag{4.1}$$

and

$$\Delta^{\nu}_{a+\nu-1}y(t) = c_2(t)y(t+\nu-1), \tag{4.2}$$

respectively, for $t \in \mathbb{N}_a$ satisfying $x(a+\nu-1) \geq y(a+\nu-1) > 0$. Then

$$x(t) \geq y(t),$$

for $t \in \mathbb{N}_{a+\nu-1}$.

Theorem 4.2 *Assume $0 < \nu < 1$, b is a constant and $a_0 \in \mathbb{R}$. Then the solution of the IVP*

$$\Delta^{\nu}_{a+\nu-1}y(t) = by(t+\nu-1), \quad t \in \mathbb{N}_a, \tag{4.3}$$

$$y(a+\nu-1) = \Delta^{\nu-1}_{a+\nu-1}y(t)|_{t=a} = a_0, \tag{4.4}$$

is given by

$$y(t) = a_0 \sum_{i=0}^{t-a-\nu+1} b^i h_{i\nu+\nu-1}\big(t, a - i(\nu-1)\big), \quad t \in \mathbb{N}_{a+\nu-1}. \tag{4.5}$$

Theorem 4.3 *Assume $0 < b \leq c(t)$, $0 < \nu < 1$, and $x(t)$ is the solution of the fractional equation*

$$\Delta^{\nu}_{a+\nu-1}x(t) = c(t)x(t+\nu-1), \quad t \in \mathbb{N}_a, \tag{4.6}$$

satisfying $x(a+\nu-1) > 0$. Then

$$\lim_{t\to\infty} x(t) = \infty.$$

Theorem 4.4 *Assume $-\nu < c(t) \leq 0$ and $0 < \nu < 1$. Then for all solutions $x(t)$ of the fractional equation*

$$\Delta^{\nu}_{a+\nu-1}y(t) = c(t)y(t+\nu-1), \quad t \in \mathbb{N}_a, \tag{4.7}$$

satisfying $y(a+\nu-1) > 0$, we have

$$\lim_{t\to\infty} y(t) = 0.$$

Theorems 4.3 and 4.4 give the following two results.

Theorem 4.5 *Assume* $0 < \nu < 1$ *and there exists a constant* b *such that* $c(t) \geq b > 0$. *Then the solutions of the equation* (4.6) *satisfying* $x(a + \nu - 1) < 0$, *satisfy*

$$\lim_{t\to\infty} x(t) = -\infty.$$

Theorem 4.6 *Assume* $0 < \nu < 1$ *and* $-\nu \leq c(t) < 0$. *Then the solutions of equation* (4.6), *satisfying* $x(a + \nu - 1) < 0$, *satisfy*

$$\lim_{t\to\infty} x(t) = 0.$$

4.2 Results for the nabla fractional difference

Many of the results in this subsection can be found in the paper by Baoguo *et al.* [75] and in the references given therein. First we define the nabla Mittag-Leffler function.

Definition 4.7 For $|p| < 1$, $0 < \alpha < 1$, we define the discrete Mittag-Leffler function by

$$E_{p,\alpha,\alpha-1}\bigl(t, \rho(a)\bigr) := \sum_{k=0}^{\infty} p^k H_{\alpha k+\alpha-1}\bigl(t, \rho(a)\bigr), \quad t \in \mathbb{N}_a.$$

The next two results give us some properties of the Mittag-Leffler function.

Theorem 4.8 *Assume that* $0 < \nu < 1$, $|b| < 1$. *Then*

$$\nabla^{\nu}_{\rho(a)} E_{b,\nu,\nu-1}\bigl(t, \rho(a)\bigr) = E_{b,\nu,-1}\bigl(t, \rho(a)\bigr)$$

for $t \in \mathbb{N}_a$.

Theorem 4.9 *Assume that* $0 < \nu < 1$, $|b| < 1$. *Then* $E_{b,\nu,\nu-1}(t, \rho(a))$ *is the unique solution of the initial value problem*

$$\begin{aligned} &\nabla^{\nu}_{\rho(a)} x(t) = bx(t), \quad t \in \mathbb{N}_{a+1}, \\ &x(a) = \frac{1}{1-b} > 0. \end{aligned} \tag{4.8}$$

The following result is an important comparison theorem for nabla fractional equations.

Theorem 4.10 *Assume* $c_2(t) \leq c_1(t) < 1$, $0 < \nu < 1$. *If* $x(t)$, $y(t)$ *are the solutions of the equations*

$$\nabla^{\nu}_{\rho(a)} x(t) = c_1(t)x(t) \tag{4.9}$$

and

$$\nabla^{\nu}_{\rho(a)} y(t) = c_2(t)y(t), \tag{4.10}$$

respectively, for $t \in \mathbb{N}_{a+1}$*, satisfying* $x(a) \geq y(a) > 0$*, then*

$$x(t) \geq y(t),$$

for $t \in \mathbb{N}_a$*.*

Theorem 4.10 gives us the following result.

Theorem 4.11 *Assume* $c_2(t) \leq c_1(t) < 1$, $0 < \nu < 1$*. If* $x(t)$, $y(t)$ *are the solutions of the equations*

$$\nabla^{\nu}_{\rho(a)} x(t) = c_1(t)x(t) \tag{4.11}$$

and

$$\nabla^{\nu}_{\rho(a)} y(t) = c_2(t)y(t), \tag{4.12}$$

respectively, for $t \in \mathbb{N}_{a+1}$*, satisfying* $x(a) \geq y(a) > 0$*, then*

$$x(t) \geq y(t),$$

for $t \in \mathbb{N}_a$*.*

The following result follows from the comparison theorem (Theorem 4.11).

Theorem 4.12 *Assume* $0 < b \leq c(t) < 1$, $0 < \nu < 1$*. Then for any solution* $x(t)$ *of*

$$\nabla^{\nu}_{\rho(a)} x(t) = c(t)x(t), \quad t \in \mathbb{N}_{a+1}, \tag{4.13}$$

satisfying $x(a) > 0$ *we have*

$$x(t) \geq \frac{(1-b)x(a)}{2} E_{b,\nu,\nu-1}\bigl(t, \rho(a)\bigr), \quad t \in \mathbb{N}_a.$$

Using the result that for $\Re(z) > 0$, we have

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\cdots(z+n)},$$

one can then prove the following result giving interesting properties of certain nabla fractional Taylor monomials.

Theorem 4.13 *Assume that* $0 < \nu < 1$*. Then we have*

$$\begin{aligned} &\lim_{t\to\infty} H_{\nu k+\nu-1}\bigl(t, \rho(a)\bigr) = \infty, \quad \textit{for } k > \frac{1-\nu}{\nu}, \\ &\lim_{t\to\infty} H_{\nu k+\nu-1}\bigl(t, \rho(a)\bigr) = \frac{1}{\nu k+\nu}, \quad \textit{for } k = \frac{1-\nu}{\nu}, \\ &\lim_{t\to\infty} H_{\nu k+\nu-1}\bigl(t, \rho(a)\bigr) = 0, \quad \textit{for } k < \frac{1-\nu}{\nu}. \end{aligned} \tag{4.14}$$

Theorem 4.14 *For* $0 < b < 1$, *we have*

$$\lim_{t\to\infty} E_{b,\nu,\nu-1}\bigl(t,\rho(a)\bigr) = +\infty.$$

From Theorem 4.12 and Theorem 4.14, we get the following result, which can be regarded as an extension of a result which appears in Atici and Eloe [36].

Theorem 4.15 *Assume* $0 < \nu < 1$ *and there exists a constant* b *such that* $0 < b \leq c(t) < 1$. *Then the solutions of the equation* (4.13) *satisfy*

$$\lim_{t\to\infty} x(t) = \infty.$$

In the following theorem we get conditions under which the solutions of a nabla fractional equation tend to zero.

Theorem 4.16 *Assume* $c(t) \leq 0$, $0 < \nu < 1$. *Then for all solutions* $x(t)$ *of the fractional equation*

$$\nabla^{\nu}_{\rho(a)} y(t) = c(t)y(t), \quad t \in \mathbb{N}_{a+1}, \tag{4.15}$$

satisfying $y(a) > 0$ *we have*

$$\lim_{t\to\infty} y(t) = 0.$$

In particular

$$\lim_{t\to\infty} E_{-b,\nu,\nu-1}\bigl(t,\rho(a)\bigr) = 0.$$

Theorems 4.15 and 4.16 give us the following result.

Theorem 4.17 *Assume* $0 < \nu < 1$. *If there exists a constant* b *such that* $0 < b \leq c(t) < 1$, *then the solutions of the equation* (4.13) *satisfy*

$$\lim_{t\to\infty} x(t) = -\infty.$$

If $c(t) \leq 0$, *then the solutions of equation* (4.15) *satisfy*

$$\lim_{t\to\infty} x(t) = 0.$$

For results similar to the results in these last two subsections for the q-calculus we refer the reader to the paper by Jia *et al.* [68].

Competing interests
The authors declare to have no competing interests.

Authors' contributions
All four authors contributed equally in writing this survey paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, University of Nebraska-Lincoln, Lincoln, NE 68588-0130, USA. [2]Department of Mathematics, Creighton Preparatory School, Omaha, NE 68114, USA. [3]School of Mathematics and Computational Science, Sun Yat-Sen University, Guangzhou, 510275, China.

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