8-2018

Fractional Difference Operators and Related Boundary Value Problems

Scott C. Gensler
University of Nebraska-Lincoln, scott.gensler@huskers.unl.edu

FRACTIONAL DIFFERENCE OPERATORS AND RELATED BOUNDARY VALUE PROBLEMS

by

Scott C. Gensler

A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professor Allan Peterson

Lincoln, Nebraska

August, 2018

FRACTIONAL DIFFERENCE OPERATORS AND RELATED BOUNDARY VALUE PROBLEMS

Scott C. Gensler, Ph.D.

University of Nebraska, 2018

Adviser: Allan Peterson

In this dissertation we develop a fractional difference calculus for functions on a discrete domain. We start by showing that the Taylor monomials, which play a role analagous to that of the power functions in ordinary differential calculus, can be expressed in terms of a family of polynomials which I will refer to as the Pochhammer polynomials. These important functions, the Taylor monomials, were previously described by other scholars primarily in terms of the gamma function. With only this description it is challenging to understand their properties. Describing the Taylor monomials in terms of the Pochhammer polynomials has made it easier to understand their behavior, as we demonstrate in this work. We then use the Taylor monomials to define a fractional operator, ∇^ν, which generalizes the standard backward difference operator, ∇. We show that these fractional difference operators have a very simple composition rule and act nicely on the Taylor monomials. We then describe the Riemann-Liouville and Caputo fractional difference operators, ∇_a^ν and ∇_{a*}^ν, in terms of this more general fractional operator and use the properties of the general fractional operator to derive the composition rules for all such operators. Finally, we apply this theory to study a nonlinear boundary value problem described using a Caputo fractional difference and show how to obtain a sequence of approximate solutions which converges quadratically to the unique solution to this problem.

DEDICATION

This dissertation is dedicated to my family. To my parents, Richard and Lynne Gensler, I cannot thank you enough for your encouragement and support over the years. Most parents tell their children they can do anything they set their mind to. Somehow, you made me belive it. To my two beautiful daughters, Ava and Molly, you have been very patient these last six years while I have been studying at UNL and working on this dissertation. Along the way you have offered many welcome study breaks at various and opportune times. I hope one day you will also have the opportunity to study and/or do whatever it is that interests you the most, at least for a season. I will never forget your your unconditional love nor your frequent words of encouragement, for example, “have a good day at school Daddy . . . make sure you learn your letters.” And to my wife, Sarah Gensler, whom I love. Thank you for allowing me to take this "tactical pause" in my career in order to pursue something I have always wanted to. I know this has not been the financially most lucrative path we might have followed. I know this process has not been easy nor without a sacrifice on your part, so thank you.

As a second dedication, I would like to also dedicate this work to two of my former mathematics teachers. To my high school mathematics and physics teacher, Mr. John Radscheid, thank you for first introducing me to the beauty that exists in math and science. That was more than 30 years, but your influence was great. And finally, to COL(R) Joseph Myers thank you for being an inspirational instructor (and later colleague) at USMA. I appreciate all of your encouragement, shared interest in math and science, and all of the (many) letters of recommendation you have written for me over the years.

ACKNOWLEDGMENTS

I would like to thank my adviser, Professor Allan Peterson, as well as Professor Lynn Erbe and the other members of the UNL Fractional Calculus and Dynamical Systems Seminar, Spring 2015-Spring 2018, namely: Julia St. Goar, Kevin Ahrendt, Areeba Ikram, Wei Hu, Ariel Setniker, Xiang Liu, Feifei Du, and Raziye Mert. It was in preparing to speak to you at these Tuesday/Thursday meetings that I was compelled to try to figure out: (i) how to best communicate the thoughts I had (thoughts that eventually became this dissertation) and (ii) how to convince you all that what I was saying was actually true and (hopefully) even helpful. Professor Peterson, thank you for your mathematical inspiration as well as your temperament and patience. I hope to be as interested in math as you still are after 50 years of teaching and advising students. (Of course this means I will need to live to be ... very old.) I also appreciate the space you give us to work on topics and problems that interested us even if they are perhaps only tangentially related to topics in which you yourself are interested. I would also like to thank a few of the professors here at UNL. In particular, Professors Peterson, Pitts, Rammaha, and Radcliff, your lectures were always enlightening, well prepared, and a pleasure. To be guided toward an understanding of some of the deep, general truths/ideas in mathematics was among the things I was hoping gain as I was studying at UNL and working toward my degree. Your lectures never failed to disappoint! Finally, I would like to thank my Omaha to Lincoln carpool buddies: Andrew Becklin, Brent McKain, Nick Kass, and Kody Holmes for making the daily commute, and often even making it productive and/or entertaining. It turns out carpools are a great place to bounce ideas off other mathematicians and where one can be totally honest regarding what they know (or do not know) to be true.

Table of Contents

1 Preliminaries **1**

1.1 Notation and Terminology 1

1.2 Basic Facts 4

1.3 Pochhammer Polynomials 5

1.4 Taylor Monomials 7

1.5 Convolution Product 17

2 Fractional Nabla Operators (∇^{γ}, ∇_a^{α}, ∇_{a*}^{ν}) **33**

2.1 The General Fractional Operator ∇^{γ} 34

2.2 The Action of Nabla Operators on Taylor Monomials 37

2.3 Composition Rules 42

2.4 The Difference between the Riemann-Liouville and the Caputo Fractional Difference Operators 51

3 More about Taylor Monomials and Pochhammer Polynomials **54**

3.1 Additional Properties of Pochhammer Polynomials 54

3.2 Additional Properties of Taylor Monomials 58

4 Boundary Value Problems (BVPs) Involving ∇_{a*}^{ν} **76**

5 Upper and Lower Solutions to BVPs **83**

5.1 Some Definitions and Basic Theorems 84

5.2 Existence and Comparison Results 91

5.3 Main Results 98

5.4 Example 107

A Select Definitions and Results from *Discrete Fractional Calculus* by C. Goodrich and A. C. Peterson **110**

B Summary of Results from Chapters 2 and 3 **113**

Bibliography **118**

Chapter 1

Preliminaries

1.1 Notation and Terminology

For a given real number a we identify the following sets:

$$\mathbb{N}_a := \{a, a+1, a+2, \cdots\}$$
$$\mathbb{Z}_a := \{\cdots, a-2, a-1, a, a+1, a+2, \cdots\}$$

By $\mathbb{R}^{\mathbb{Z}_a} := \{f : \mathbb{Z}_a \to \mathbb{R}\}$ we denote the $\mathbb{R}$-vector space of functions from $\mathbb{Z}_a$ to $\mathbb{R}$ under pointwise addition.

We will make use of the following linear operators that take $\mathbb{R}^{\mathbb{Z}_a} \to \mathbb{R}^{\mathbb{Z}_a}$. For any $f \in \mathbb{R}^{\mathbb{Z}_a}$ we define:

- the **left shift operator**, $L : \mathbb{R}^{\mathbb{Z}_a} \to \mathbb{R}^{\mathbb{Z}_a}$, by $(Lf)(t) := f(t+1)$ for all $t \in \mathbb{Z}_a$
- the **right shift operator**, $R : \mathbb{R}^{\mathbb{Z}_a} \to \mathbb{R}^{\mathbb{Z}_a}$, by $(Rf)(t) := f(t-1)$ for all $t \in \mathbb{Z}_a$
- the **identity operator**, $I : \mathbb{R}^{\mathbb{Z}_a} \to \mathbb{R}^{\mathbb{Z}_a}$, by $(If)(t) := f(t)$ for all $t \in \mathbb{Z}_a$
- the **nabla difference operator** $\nabla : \mathbb{R}^{\mathbb{Z}_a} \to \mathbb{R}^{\mathbb{Z}_a}$ by $\nabla := I - R$, so explicitly, $(\nabla f)(t) := f(t) - f(t-1)$ for all $t \in \mathbb{Z}_a$.

For $k \in \mathbb{N}_0$ we define L^k, R^k, and ∇^k recursively. That is, for $T \in \{L, R, \nabla\}$, $T^0 := I$ and for $k \in \mathbb{N}_1$ we define $\left(T^k f\right)$ by $\left(T^k f\right) := T(T^{k-1} f)$. Explicitly, for $f \in \mathbb{R}^{\mathbb{Z}_a}$ and $k \in \mathbb{N}_1$ we have:

- $L^k = \overbrace{L \circ L \circ \cdots \circ L}^{k}$, hence $(L^k f)(t) = f(t+k)$
- $R^k = \overbrace{R \circ R \circ \cdots \circ R}^{k}$, hence $(R^k f)(t) = f(t-k)$
- $\nabla^k = \overbrace{\nabla \circ \nabla \circ \cdots \circ \nabla}^{k}$, hence $(\nabla^k f)(t) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} f(t-i)$.

This expression for $\nabla^k f$ follows formally from the Binomial Theorem since

$$\nabla^k = \overbrace{(I-R)\cdots(I-R)}^{k} = \sum_{i=0}^{k} \binom{k}{i} I^{k-i}(-R)^i = \sum_{i=0}^{k} (-1)^i \binom{k}{i} R^i.$$

The **regressive function**, $\rho : \mathbb{Z}_a \to \mathbb{Z}_a$ is defined by $\rho(t) := t - 1$.

For $f : \mathbb{Z}_a \to \mathbb{R}$, the **support of** f **on** $\mathbb{Z}_a$ is $\operatorname{spt}(f) := \{t \in \mathbb{Z}_a : f(t) \neq 0\}$.

For $A \subseteq \mathbb{R}$, **the characteristic function on** A is $\mathbb{1}_A(t) = \begin{cases} 1, \text{ if } t \in A \\ 0, \text{ if } t \notin A. \end{cases}$

For $c, d \in \mathbb{Z}_a$ with $c < d$, we use the notation:

$$\mathbb{N}_c^d := [c, d] \cap \mathbb{Z}_a$$

Throughout, we will often use Greek letters to represent variables that may assume any real value and Latin letters to represent variables that assume only integer values. In particular, we will use:

- α and γ to represent *any general* real numbers,
- m and n to represent nonnegative integers, and

- μ and ν to represent nonnegative real numbers (For these, we will often let $M := \lceil \mu \rceil$ and $N := \lceil \nu \rceil$, where $\lceil \cdot \rceil$ is the ceiling function.).

We define the nabla definite integral of $f : \mathbb{Z}_a \to \mathbb{R}$ for $c, d \in \mathbb{Z}_a$ by

$$\int_c^d f(s)\nabla s = \begin{cases} f(c+1) + f(c+2) + \cdots + f(d-1) + f(d), & c < d \\ 0, & c \geq d. \end{cases}$$

Sometimes we will also use the following notation to represent this same quantity:

$$\int_c^d f(s)\nabla s = \int_{(c,d]} f(s)\nabla s = \sum_{s\in(c,d]} f(s) = \sum_{s\in\mathbb{N}_{c+1}^d} f(s) = \sum_{s=c+1}^{d} f(s)$$

Remark Note that if $c \geq d$, then $(c, d] = \{s \in \mathbb{R} : c < s \leq d\} = \emptyset$. Thus,

$$\int_{(c,d]} f(s)\nabla s = \int_{\emptyset} f(s)\nabla s = 0.$$

This is one reason for defining the nabla definite integral to be 0 when $c \geq d$.

Suppose $f \in \mathbb{R}^{\mathbb{Z}_a}$ and $A \subseteq \mathbb{Z}_a$. We say:

- f is **increasing** on A, provided $x, y \in A$ and $x < y \Rightarrow f(x) \leq f(y)$
- f is **strictly increasing** on A, provided $x, y \in A$ and $x < y \Rightarrow f(x) < f(y)$.

Similiarly, we say:

- f is **decreasing** on A, provided $x, y \in A$ and $x < y \Rightarrow f(x) \geq f(y)$
- f is **strictly decreasing** on A, provided $x, y \in A$ and $x < y \Rightarrow f(x) > f(y)$.

1.2 Basic Facts

Gamma Function

In this dissertation, we assume the following five facts regarding the Gamma function:

(1) the Gamma function is analytic (and therefore continuous) on $\mathbb{R}\setminus\{0,-1,-2,\cdots\}$,

(2) the Gamma function has no zeros,

(3) For $n \in \mathbb{N}_0$, $\lim\limits_{t\to(-n)^{\pm}} \Gamma(t) = \pm\infty$ for even n and $\lim\limits_{t\to(-n)^{\pm}} \Gamma(t) = \mp\infty$ for odd n.

(4) $\Gamma(x+1) = x\Gamma(x), \quad x \in \mathbb{R} \setminus \{0,-1,-2,\cdots\}$, and

(5) $\Gamma(n+1) = n!, \quad n \in \mathbb{N}_0$.

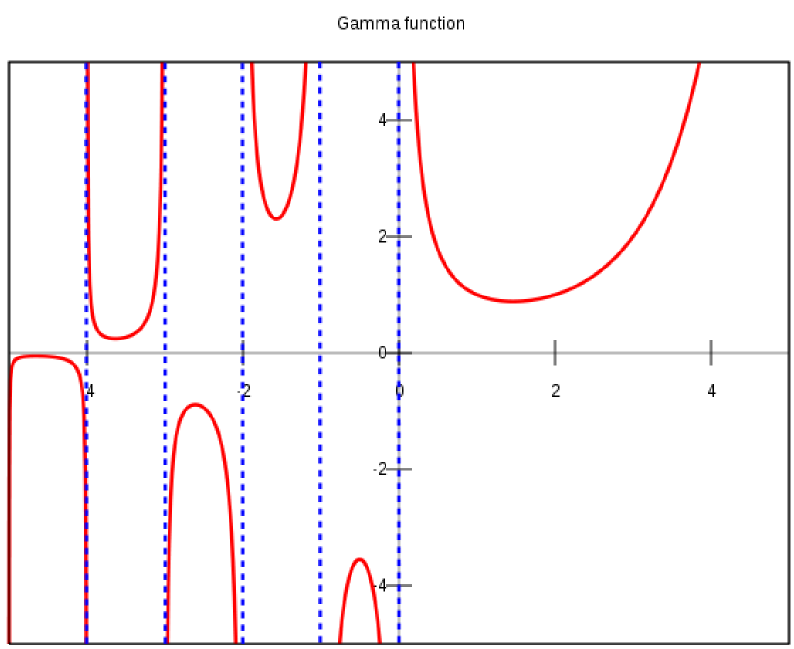


Figure 1.1: Gamma Function

Note: (3) implies that for all $n \in \mathbb{N}_0$, $0 = \lim\limits_{t\to(-n)^+} \frac{1}{\Gamma(t)} = \lim\limits_{t\to(-n)^-} \frac{1}{\Gamma(t)} = \lim\limits_{t\to -n} \frac{1}{\Gamma(t)}$. Thus, (1)-(3) together imply $f(x) := \lim\limits_{t\to x} \frac{1}{\Gamma(t)} \in (-\infty,\infty)$ for all $x \in \mathbb{R}$.

Theorem 1.1. ***(Fundamental Theorem of Nabla Calculus)*** *(See [5, Theorem 3.37]). If $c, d \in \mathbb{Z}_a$ with $c < d$ and $f : \mathbb{N}_c^d \to \mathbb{R}$, then*

$$\int_c^d (\nabla f)(s) \nabla s = f(s)\Big|_c^d = f(d) - f(c). \tag{1.1}$$

Theorem 1.2. *Suppose $c, d \in \mathbb{Z}_a$ with $c < d$.*

If $(\nabla f)(t) \geq 0$ on $[c+1, d] \cap \mathbb{Z}_a$, then f is increasing on $[c, d] \cap \mathbb{Z}_a$.

If $(\nabla f)(t) > 0$ on $[c+1, d] \cap \mathbb{Z}_a$, then f is strictly increasing on $[c, d] \cap \mathbb{Z}_a$.

If $(\nabla f)(t) \leq 0$ on $[c+1, d] \cap \mathbb{Z}_a$, then f is decreasing on $[c, d] \cap \mathbb{Z}_a$.

If $(\nabla f)(t) < 0$ on $[c+1, d] \cap \mathbb{Z}_a$, then f is strictly decreasing on $[c, d] \cap \mathbb{Z}_a$.

1.3 Pochhammer Polynomials

In Section 1.4 we will define and describe the fractional Taylor monomials, which play an important role in the theory of fractional differences. But, before we do that it will be useful to introduce the so-called Pochhammer polynomials and to review some of their basic properties.

Definition 1.3. *For $d \in \mathbb{Z}$ and $x \in \mathbb{R}$, we define* ***the Pochhammer polynomial of degree d*** *to be:*

$$P_d(x) = \binom{x+d}{d} = \begin{cases} \frac{1}{d!}(x+1)(x+2)\cdots(x+d) & , \textit{for } d \in \mathbb{N}_1 \\ 1 & , \textit{for } d = 0 \\ 0 & , \textit{for } d \in \mathbb{Z} \setminus \mathbb{N}_0. \end{cases}$$

So, for $d \geq 1$, $P_d(x)$ is the polynomial with d distinct zeros at $x = -1, -2, \cdots, -d$ normalized so that $P_d(0) = 1$.

Example 1.4.

$$\begin{aligned}
P_3(2) &= \binom{2+3}{3} = \frac{1}{3!}(x+1)(x+2)(x+3)\Big|_{x=\ 2} = 10\\
P_3(-1) &= \binom{-1+3}{3} = \frac{1}{3!}(x+1)(x+2)(x+3)\Big|_{x=-1} = 0\\
P_3(-4) &= \binom{-4+3}{3} = \frac{1}{3!}(x+1)(x+2)(x+3)\Big|_{x=-4} = -1.
\end{aligned}$$

Theorem 1.5. *For $d \in \mathbb{N}_0$ and $k \in \mathbb{N}_0$,*

$$P_d(-k) = (-1)^d \binom{k-1}{d}\ .$$

Proof. To see this equation holds, notice

$$\begin{aligned}
P_d(-k) &= \frac{1}{d!}(-k+1)(-k+2)\cdots(-k+d)\\
&= (-1)^d \cdot \frac{1}{d!}(k-1)(k-2)\cdots(k-d)\\
&= (-1)^d \cdot \frac{1}{d!}(k-d)(k-(d-1))\cdots(k-1)\\
&= (-1)^d \cdot \frac{1}{d!}((k-d-1)+1)((k-d-1)+2)\cdots((k-d-1)+d)\\
&= (-1)^d \cdot \binom{(k-d-1)+d}{d}\\
&= (-1)^d \cdot \binom{k-1}{d}\ .
\end{aligned}$$

□

Theorem 1.6. *For $d \in \mathbb{N}_0$, $P_{d+1}(x) = P_d(x) \cdot \frac{x+d+1}{d+1}$.*

Proof. Clearly the claim is true when $d = 0$. If $d > 0$, then

$$\begin{aligned} P_{d+1}(x) &= \frac{1}{(d+1)!}(x+1)(x+2)\cdots(x+d)(x+(d+1)) \\ &= \frac{(x+1)(x+2)\cdots(x+d)}{d!} \cdot \frac{(x+d+1)}{d+1} \\ &= P_d(x) \cdot \frac{x+d+1}{d+1} \ . \end{aligned}$$

□

1.4 Taylor Monomials

We introduce two kinds of fractional Taylor monomials. Both will be useful. [1, 2]

Definition 1.7. *For all $\gamma \in \mathbb{R}$, and all $s, t \in \mathbb{Z}_a$, we define:*

$$\widetilde{H}_\gamma(t,s) := \lim_{\varepsilon \to 0} \frac{\Gamma(t-s+\gamma+\varepsilon)}{\Gamma(t-s+\varepsilon)\Gamma(\gamma+1+\varepsilon)} \quad \textit{and} \quad H_\gamma(t,s) := \widetilde{H}_\gamma(t,s) \cdot \mathbb{1}_{(s,\infty)}(t).$$

Remark Recall the Gamma function has no zeros, so the denominator in the expression for $\widetilde{H}_\gamma(t,s)$ is never 0. In the next two theorems we show the limit used to define $\widetilde{H}_\gamma(t,s)$ always exists in $\mathbb{R}$ and so $\widetilde{H}_\gamma(t,s)$ is, in fact, well-defined for *all* $\gamma \in \mathbb{R}$, and *all* $s, t \in \mathbb{Z}_a$ as claimed.

[1] In [5] (see [Definition 3.56, page 186]) Goodrich et al define the fractional Taylor monomials, for $\gamma \neq -1, -2, -3, \cdots$, by

$$H_\gamma(t,a) := \frac{(t-a)^{\overline{\gamma}}}{\Gamma(\gamma+1)} = \frac{\Gamma(t-a+\gamma)}{\Gamma(t-a)\Gamma(\gamma+1)}$$

for values of t and γ such that the right-hand side of this equation makes sense. $\widetilde{H}_\gamma(t,s)$ as defined above therefore extends and clarifies the domain of the fractional Taylor monomial as defined in [5]. The $\widetilde{\ }$ is intended to remind the reader of this fact.

[2] The $\widetilde{\ }$ is also intended to remind the reader that $\widetilde{H}_\gamma(t,s)$ is an "extension" of the second kind of Taylor monomial, $H_\gamma(t,s)$, in the sense that $\operatorname{spt}(H_\gamma(\cdot,s)) \subsetneq \operatorname{spt}(\widetilde{H}_\gamma(\cdot,s))$ when $\gamma = n \in \mathbb{N}_0$ (as we will show).

Theorem 1.8. *For $s, t \in \mathbb{Z}_a$ with $t \leq s$ and $\gamma \in \mathbb{R}$,*

$$\widetilde{H}_\gamma(t,s) = \begin{cases} 0, & \text{when } \gamma \in \mathbb{R} \setminus \mathbb{N}_0 \\ (-1)^n \binom{|t-s|}{n}, & \text{when } \gamma = n \in \mathbb{N}_0. \end{cases}$$

Proof. Put $d = d(t,s) := -(t-s) = |t-s| \in \mathbb{N}_0$.

Case 1: $\gamma \in \mathbb{R} \setminus \mathbb{N}_0$.

$$\begin{aligned} \widetilde{H}_\gamma(t,s) :&= \lim_{\varepsilon \to 0} \frac{\Gamma(t-s+\gamma+\varepsilon)}{\Gamma(t-s+\varepsilon)\Gamma(\gamma+1+\varepsilon)} \\ &= \lim_{\varepsilon \to 0} \frac{\Gamma(-d+\gamma+\varepsilon)}{\Gamma(-d+\varepsilon)\Gamma(\gamma+1+\varepsilon)} \\ &= \lim_{\varepsilon \to 0} \frac{1}{\Gamma(-d+\varepsilon)} \lim_{\varepsilon \to 0} \frac{\Gamma(-d+\gamma+\varepsilon)}{\Gamma(\gamma+1+\varepsilon)} \cdot \frac{(-d+\gamma+\varepsilon)\cdots(0+\gamma+\varepsilon)}{(-d+\gamma+\varepsilon)\cdots(0+\gamma+\varepsilon)} \\ &= \lim_{\varepsilon \to 0} \frac{1}{\Gamma(-d+\varepsilon)} \lim_{\varepsilon \to 0} \frac{1}{(-d+\gamma+\varepsilon)\cdots(0+\gamma+\varepsilon)} \\ &= \lim_{\varepsilon \to 0} \frac{1}{\Gamma(-d+\varepsilon)} \cdot \frac{1}{(-d+\gamma)\cdots(0+\gamma)} \\ &= \frac{1}{(-d+\gamma)\cdots(0+\gamma)} \lim_{\varepsilon \to 0} \frac{1}{\Gamma(-d+\varepsilon)} = 0 \\ &= 0. \end{aligned}$$

Case 2a: $\gamma = 0$.

$$\begin{aligned} \widetilde{H}_0(t,s) :&= \lim_{\varepsilon \to 0} \frac{\Gamma(t-s+0+\varepsilon)}{\Gamma(t-s+\varepsilon)\Gamma(0+1+\varepsilon)} \\ &= \lim_{\varepsilon \to 0} \frac{1}{\Gamma(0+1+\varepsilon)} = \frac{1}{0!} = 1 \\ &= (-1)^0 \binom{|t-s|}{0}. \end{aligned}$$

Case 2b: $\gamma = n \in \mathbb{N}_1$.

$$\begin{aligned}
\widetilde{H}_n(t,s) :&= \lim_{\varepsilon\to 0} \frac{\Gamma(t-s+n+\varepsilon)}{\Gamma(t-s+\varepsilon)\Gamma(n+1+\varepsilon)} \\
&= \lim_{\varepsilon\to 0} \frac{\Gamma(-d+n+\varepsilon)}{\Gamma(-d+\varepsilon)\Gamma(n+1+\varepsilon)} \\
&= \lim_{\varepsilon\to 0} \frac{1}{\Gamma(n+1+\varepsilon)} \lim_{\varepsilon\to 0} \frac{\Gamma(-d+n+\varepsilon)}{\Gamma(-d+\varepsilon)} \cdot \frac{(-d+\varepsilon)\cdots(-d+n-1+\varepsilon)}{(-d+\varepsilon)\cdots(-d+n-1+\varepsilon)} \\
&= \frac{1}{\Gamma(n+1)} \lim_{\varepsilon\to 0} \frac{(-d+\varepsilon)\cdots(-d+n-1+\varepsilon)}{1} \\
&= \frac{1}{n!}(-1)^n \lim_{\varepsilon\to 0} \frac{(d-\varepsilon)\cdots(d-n+1-\varepsilon)}{1} \\
&= \frac{(-1)^n}{n!} \cdot (d)\cdots(d-(n-1)) \\
&= (-1)^n \binom{d}{n} \\
&= (-1)^n \binom{|t-s|}{n}.
\end{aligned}$$

□

The reason for introducing the Pochhammer polynomials in the last section is that they are closely related to the Taylor monomials, as the next theorem shows.

Theorem 1.9. *For $s, t \in \mathbb{Z}_a$ with $t > s$ and $\gamma \in \mathbb{R}$,*

$$\widetilde{H}_\gamma(t,s) = P_{t-s-1}(\gamma) \ . \tag{1.2}$$

Proof. Put $d = d(t, s+1) := t - s - 1 \in \mathbb{N}_0$. [3]

$$\begin{aligned}
\widetilde{H}_\gamma(t,s) :&= \lim_{\varepsilon\to 0} \frac{\Gamma(t-s+\gamma+\varepsilon)}{\Gamma(t-s+\varepsilon)\Gamma(\gamma+1+\varepsilon)} \\
&= \lim_{\varepsilon\to 0} \frac{\Gamma(d+1+\gamma+\varepsilon)}{\Gamma(d+1+\varepsilon)\Gamma(\gamma+1+\varepsilon)} \\
&= \lim_{\varepsilon\to 0} \frac{1}{\Gamma(d+1+\varepsilon)} \lim_{\varepsilon\to 0} \frac{\Gamma(d+1+\gamma+\varepsilon)}{\Gamma(\gamma+1+\varepsilon)} \cdot \frac{(\gamma+1+\varepsilon)\cdots(\gamma+d+\varepsilon)}{(\gamma+1+\varepsilon)\cdots(\gamma+d+\varepsilon)} \\
&= \frac{1}{\Gamma(d+1)} \lim_{\varepsilon\to 0} \frac{\Gamma(d+1+\gamma+\varepsilon)}{\Gamma(d+1+\gamma+\varepsilon)} \cdot (\gamma+1+\varepsilon)\cdots(\gamma+d+\varepsilon) \\
&= \frac{1}{d!}(\gamma+1)\cdots(\gamma+d) \\
&= P_d(\gamma) = P_{t-s-1}(\gamma).
\end{aligned}$$

□

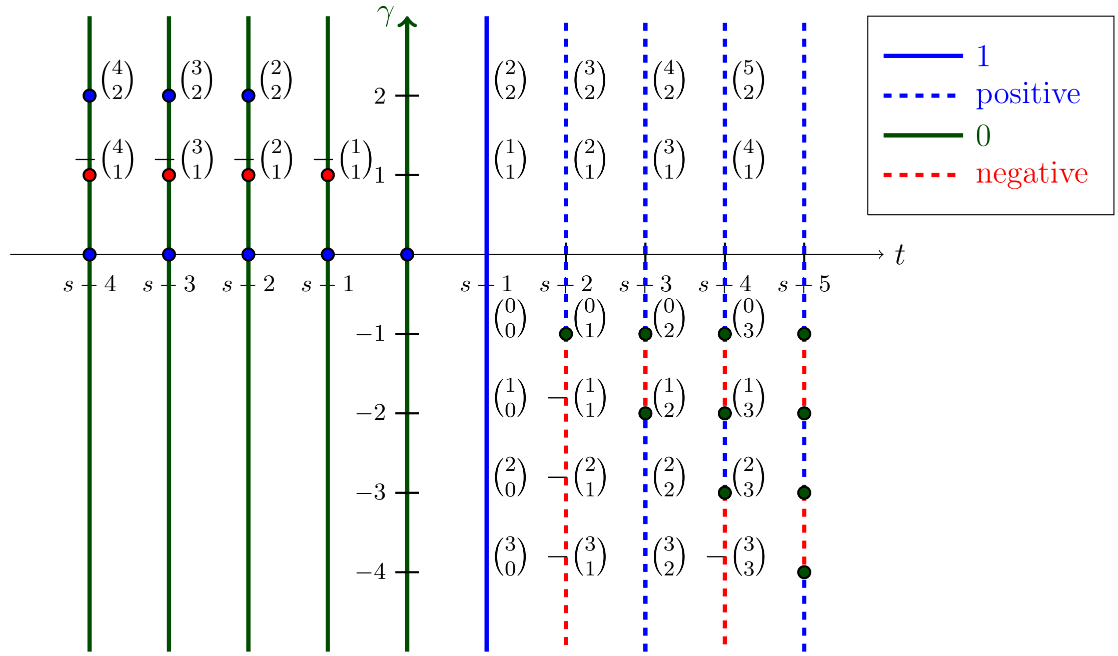
Remark 1.10. *Theorem 1.8 and 1.9 combine to give us the following formula and complete picture of $\widetilde{H}_\gamma(t,s)$ for all $\gamma \in \mathbb{R}$ and all $t, s \in \mathbb{Z}_a$. (See Figure 1.2 below.)*

$$\widetilde{H}_\gamma(t,s) = P_{t-s-1}(\gamma) + (-1)^\gamma \binom{|t-s|}{\gamma} \mathbb{1}_{\mathbb{N}_0}(\gamma)\mathbb{1}_{(-\infty,s]}(t) \ . \tag{1.3}$$

Remark Recall that by Definition 1.3 $P_d(x) \equiv 0$ whenever $d \in \mathbb{Z} \setminus \mathbb{N}_0$, so the first term in equation (1.3) is 0 whenever $t \le s$.

Remark The labelled values in Figure 1.2 can be obtained either: (i) directly from this formula (as in the case of those values that appear in the II quadrant) or (ii) by combining this formula and Definition 1.3 with the results of Theorem 1.5 (as in the case of those values that appear in the I and IV quadrants).

[3] In this proof $d(t, s+1)$ is the integer-valued *distance* between t and $s+1$. It is also the *degree* of the relevant Pochhammer polynomial.



$$\widetilde{H}_\gamma(t,s) := \lim_{\varepsilon\to 0}\frac{\Gamma(t-s+\gamma+\varepsilon)}{\Gamma(t-s+\varepsilon)\Gamma(\gamma+1+\varepsilon)}$$

Figure 1.2: (Extended) Taylor Monomials

A very nice symmetry in Figure 1.2 is reflected in the formula for $\widetilde{H}_\gamma(t,s)$ given in the next corollary.

Corollary 1.11. *For all* $\gamma \in \mathbb{R}$ *and all* $t, s \in \mathbb{Z}_a$

$$\widetilde{H}_\gamma(t,s) = \begin{cases} P_{t-s-1}(\gamma), & \text{for } t > s \text{ and } \gamma \in \mathbb{R} \\ P_\gamma(t-s-1), & \text{for } t \leq s \text{ and } \gamma = n \in \mathbb{N}_0 \\ 0, & \text{for } t \leq s \text{ and } \gamma \in \mathbb{R}\setminus\mathbb{N}_0. \end{cases}$$

Proof. Theorems 1.8 and 1.9 give us formulas for $\widetilde{H}_\gamma(t,s)$ when $t \leq s$ and $t > s$, respectively. Looking back at these, the only thing new here is the formula for $\widetilde{H}_n(t,s)$ when $t \leq s$ and $\gamma = n \in \mathbb{N}_0$. To verify this formula recall the result of Theorem 1.8.

That is, for $t \leq s$ and $\gamma = n \in \mathbb{N}_0$ recall that

$$\begin{aligned}
\widetilde{H}_\gamma(t,s) = (-1)^n \binom{|t-s|}{n} &= \frac{(-1)^n}{n!}(|t-s|)(|t-s|-1)\cdots(|t-s|-(n-1)) \\
&= \frac{1}{n!}(-|t-s|)(-|t-s|+1)\cdots(-|t-s|+(n-1)) \\
&= \frac{1}{n!}(t-s)(t-s+1)\cdots(t-s+(n-1)) \\
&= \frac{1}{n!}(t-s-1+1)(t-s-1+1+1)\cdots(t-s-1+1+(n-1)) \\
&= \frac{1}{n!}((t-s-1)+1)((t-s-1)+2)\cdots((t-s-1)+n) \\
&= P_n(t-s-1).
\end{aligned}$$

□

Note that the second term in (1.3) is often 0. In fact, it is only nonzero when $\gamma \in \mathbb{N}_0$ *and* $t \leq s$. The next two corollaries follow from this fact.

Corollary 1.12. *For all $s, t \in \mathbb{Z}_a$ and all $\gamma \in \mathbb{R}$,*

$$H_\gamma(t,s) = P_{t-s-1}(\gamma) \ . \tag{1.4}$$

Proof. Recall that, by Definition 1.3, $P_d(x) \equiv 0$ whenever $d \in \mathbb{Z} \setminus \mathbb{N}_0$. Thus, if $t \leq s$, by definition, both the left- and right-hand sides of equation (1.4) are 0. If $t > s$, Theorem 1.9 gives us

$$H_\gamma(t,s) \overset{\text{def}}{=} \widetilde{H}_\gamma(t,s) \cdot \mathbb{1}_{(s,\infty)}(t) = \widetilde{H}_\gamma(t,s) = P_{t-s-1}(\gamma).$$

□

Corollary 1.13. *For all $\gamma \in \mathbb{R} \setminus \mathbb{N}_0$, and all $s, t \in \mathbb{Z}_a$,*

$$\widetilde{H}_\gamma(t,s) = H_\gamma(t,s). \tag{1.5}$$

Proof. By hypothesis $\gamma \notin \mathbb{N}_0$ and so the second term in (1.3) is 0. Thus, by equations (1.3) and (1.4), we have $\widetilde{H}_\gamma(t,s) = P_{t-s-1}(\gamma) + 0 = H_\gamma(t,s)$. □

Thus, for *almost every* value of γ the two kinds of Taylor monomials introduced are the same. It is only when $\gamma = n \in \mathbb{N}_0$ (and $t \leq s$) that they differ.

Remark 1.14. *For all $s, t \in \mathbb{Z}_a$ and all $\gamma \in \mathbb{R}$,*

(i) $\widetilde{H}_\gamma(s+1,s) = H_\gamma(s+1,s) = 1$

(ii) $\widetilde{H}_0(t,s) \equiv 1$

Proof. Both of these facts follow quickly and directly from the definitions.
To see (i), note $H_\gamma(s+1,s) \stackrel{\text{def}}{=} \widetilde{H}_\gamma(s+1,s)\cdot 1 = \lim\limits_{\varepsilon\to 0}\frac{\Gamma(1+\gamma+\varepsilon)}{\Gamma(1+\varepsilon)\Gamma(\gamma+1+\varepsilon)} = \frac{1}{\Gamma(1)} = 1.$
To see (ii), note $\widetilde{H}_0(t,s) = \lim\limits_{\varepsilon\to 0}\frac{\Gamma(t-s+\varepsilon)}{\Gamma(t-s+\varepsilon)\Gamma(1+\varepsilon)} = \frac{1}{\Gamma(1)} = 1.$ □

Theorem 1.15. *(Power Rule [4]) For $p \in \mathbb{R}$ and fixed $s \in \mathbb{Z}_a$,*

$$\nabla \widetilde{H}_p(t,s) = \begin{cases} \widetilde{H}_{p-1}(t,s), & \text{when } 0 \neq p \in \mathbb{R} \\ 0, & \text{when } p = 0. \end{cases}$$

Proof. For $p = 0$, using Remark 1.14 (ii), we have $\nabla \widetilde{H}_0(t,s) = \nabla 1 = 0$.

[4]We refer to this theorem as the Power Rule since it is the analogue of the Power Rule from calculus which describes how the power functions, x^p, behave under differentiation.

For $p \neq 0$, starting with the definition of $\widetilde{H}_p(t,s)$, we obtain

$$\begin{aligned}
\nabla \widetilde{H}_p(t,s) &= \lim_{\varepsilon \to 0} \frac{\Gamma(t-s+p+\varepsilon)}{\Gamma(t-s+\varepsilon)\Gamma(p+\varepsilon+1)} - \lim_{\varepsilon \to 0} \frac{\Gamma(t-1-s+p+\varepsilon)}{\Gamma(t-1-s+\varepsilon)\Gamma(p+\varepsilon+1)} \\
&= \lim_{\varepsilon \to 0} \left[\frac{t-1-s+p+\varepsilon}{t-1-s+\varepsilon} - 1 \right] \frac{\Gamma(t-1-s+p+\varepsilon)}{\Gamma(t-1-s+\varepsilon)\Gamma(p+\varepsilon+1)} \\
&= \lim_{\varepsilon \to 0} \frac{p \cdot \Gamma(t-1-s+p+\varepsilon)}{\Gamma(t-s+\varepsilon)\Gamma(p+1+\varepsilon)} \\
&= \lim_{\varepsilon \to 0} \frac{p \cdot \Gamma(t-1-s+p+\varepsilon)}{(p+\varepsilon)\Gamma(t-s+\varepsilon)\Gamma(p+\varepsilon)} \\
&= \lim_{\varepsilon \to 0} \frac{p}{(p+\varepsilon)} \cdot \lim_{\varepsilon \to 0} \frac{\Gamma(t-s+p-1+\varepsilon)}{\Gamma(t-s+\varepsilon)\Gamma(p+\varepsilon)} \\
&= \widetilde{H}_{p-1}(t,s).
\end{aligned}$$

□

Theorem 1.16. *(Log Rule [5]) For $p \in \mathbb{R}$ and fixed $s \in \mathbb{Z}_a$,*

$$\nabla H_p(t,s) = H_{p-1}(t,s).$$

Proof. If $t \leq s$, then by definition $H_p(t,s) = H_p(t-1,s) = 0$ just as $H_{p-1}(t,s) = 0$ (since $\mathbb{1}_{(s,\infty)}(t) = \mathbb{1}_{(s,\infty)}(t-1) = 0$ when $t \leq s$). Thus,

$$\nabla H_p(t,s) = H_p(t,s) - H_p(t-1,s) = 0 - 0 = 0 = H_{p-1}(t,s).$$

If $t \geq s+2$, then by definition $H_p(t,s) = \widetilde{H}_p(t,s)$ and $H_p(t-1,s) = \widetilde{H}_p(t-1,s)$ just

[5] We refer to this theorem as the Log Rule since one of the (very important) things it says is that $\nabla H_0(t,s) = H_{-1}(t,s) = \widetilde{H}_{-1}(t,s)$. This is the analogue of the fact from calculus that the derivative of $\ln|x|$ is the power function x^{-1}.

as $H_{p-1}(t,s) = \widetilde{H}_{p-1}(t,s)$ (since $\mathbb{1}_{(s,\infty)}(t) = \mathbb{1}_{(s,\infty)}(t-1) = 1$ when $t \geq s+2$). Thus,

$$\begin{aligned}
\nabla H_p(t,s) &= H_p(t,s) - H_p(t-1,s) \\
&= \widetilde{H}_p(t,s) - \widetilde{H}_p(t-1,s) \\
&= \nabla \widetilde{H}_p(t,s) = \widetilde{H}_{p-1}(t,s) = H_{p-1}(t,s). \qquad \surd
\end{aligned}$$

It remains to show that the theorem holds when $t = s+1$. In this case, by the definition and Remark 1.14 (i),

$$\begin{aligned}
\nabla H_p(t,s) &= H_p(t,s) - H_p(t-1,s) \\
&= H_p(s+1,s) - H_p(s,s) \\
&= \widetilde{H}_p(s+1,s) \cdot \mathbb{1}_{(s,\infty)}(s+1) - \widetilde{H}_p(s,s) \cdot \mathbb{1}_{(s,\infty)}(s) \\
&= 1 - 0 = 1 \\
&= H_{p-1}(s+1,s) = H_{p-1}(t,s). \qquad \surd
\end{aligned}$$

□

We conclude this section with a corollary and a theorem that is easy to prove using it. The corollary records the fact that the Taylor monomials only depend on the difference between the values of t and s. Hence the same amount may be added to or subtracted from both arguments without changing the output.

Corollary 1.17. *If $\gamma \in \mathbb{R}$ and $s, t, w, x \in \mathbb{Z}_a$ and $t - s = x - w$,*

$$\widetilde{H}_\gamma(t,s) = \widetilde{H}_\gamma(x,w) \quad \text{and} \quad H_\gamma(t,s) = H_\gamma(x,w).$$

Proof.

$$\widetilde{H}_\gamma(t,s) = \lim_{\varepsilon\to 0} \frac{\Gamma(t-s+\gamma+\varepsilon)}{\Gamma(t-s+\varepsilon)\Gamma(\gamma+1+\varepsilon)} = \lim_{\varepsilon\to 0} \frac{\Gamma(x-w+\gamma+\varepsilon)}{\Gamma(x-w+\varepsilon)\Gamma(\gamma+1+\varepsilon)} = \widetilde{H}_\gamma(x,w)$$

and so

$$H_\gamma(t,s) = \widetilde{H}_\gamma(t,s)\cdot \mathbb{1}_{(s,\infty)}(t) = \widetilde{H}_\gamma(x,w)\cdot \mathbb{1}_{(w,\infty)}(x) = H_\gamma(x,w).$$

□

The next theorem gives us a formula for the nabla difference of a Taylor monomial taken with respect to the second variable. It is quick to prove using the last corollary.

Theorem 1.18. *For $p\in\mathbb{R}$ and fixed $t\in\mathbb{Z}_a$,*

$$\nabla_s H_p(t,s) = -H_{p-1}(t,\rho(s)) \qquad \text{and} \qquad \nabla_s \widetilde{H}_p(t,s) = \begin{cases} \widetilde{H}_{p-1}(t,s), & \text{when } 0\neq p\in\mathbb{R} \\ 0, & \text{when } p=0. \end{cases}$$

Proof.

$$\begin{aligned} \nabla_s H_p(t,s) &= H_p(t,s) - H_p(t,s-1) = -[H_p(t,s-1) - H_p(t,s)] \\ &\overset{\text{Cor 1.17}}{=} -[H_p(t,s-1) - H_p(t-1,s-1)] \\ &= -\nabla H_p(t,\rho(s)) \overset{\text{Thm 1.16}}{=} -H_{p-1}(t,\rho(s)). \end{aligned}$$

The proof of the second equation follows from the same argument with $H_\gamma(t,s)$ replaced by $\widetilde{H}_\gamma(t,s)$. □

1.5 Convolution Product

In this section we define a binary operation on a certain vector subspace of $\mathbb{R}^{\mathbb{Z}_a}$. The operation we define could perhaps be extended to a larger subspace of $\mathbb{R}^{\mathbb{Z}_a}$, but we will only have cause in this dissertation to use it on the subspace V identified below.

Definition 1.19. *For $T \in \mathbb{Z}_a$ we define*

$$\begin{aligned} V_T :&= \{f : \mathbb{Z}_a \to \mathbb{R} : f(t) = 0 \text{ for all } t < T\} \\ &= \{f : \mathbb{Z}_a \to \mathbb{R} : spt(f) \subseteq [T, \infty)\} \end{aligned}$$

and

$$\begin{aligned} V :&= \bigcup_{i \geq 0} V_{a-i} \\ &= \bigcup_{T \in a - \mathbb{N}} V_T \\ &= \{f : \mathbb{Z}_a \to \mathbb{R} : f(t) = 0 \text{ for all } t \leq T \text{ for some } T = T_f \in \mathbb{Z}_a\} \end{aligned}$$

Notice that V is a real vector space.[6]

Definition 1.20. *For $f, g \in V$ we define $f * g : \mathbb{Z}_a \to \mathbb{R}$ by*

$$(f * g)(t) : \ = \int_{-\infty}^{\infty} f(t - s + a) g(s) \nabla s$$

A nice way to visualize this convolution product is to think of the two functions f and g as extending horizontally to the right. f is then rotated 90° counterclockwise about the value $t = a$ and g is rotated 90° clockwise about the value $t = a$. f is then translated down $(t - a)$ units. The values that appear next to each other after these

[6] The convolution operation we define next will make it into an algebra, however we will have no explicit need to use this fact.

two rotations and the translation are then multiplied and the resulting products are summed to yield $(f * g)(t)$.

Considered from this perspective, it is probably clear that $* : V \times V \to V$ is a well-defined binary operation. We prove that it is well defined in Theorem 1.21 below. The proof also yields a formula for the convolution product in terms of a finite sum. Sometimes it is easier to work with this finite sum than with Definition 1.20.

Theorem 1.21. *The convolution operation $* : V \times V \to V$ is well defined.*

Proof. Suppose $f, g \in V$. Then there exist T_f and $T_h \in \mathbb{Z}_a$ such that $f \in V_{T_f}$ and $g \in V_{T_g}$. Put $M = M(f,g) := \max\{a - T_f, a - T_g, 0\} \in \mathbb{N}_0$ and $T := \min\{T_f, T_g, a\} \in \mathbb{Z}_a$. So $T = a - M$ and $f, g \in V_T = V_{a-M}$ and so for all $t < (a - M)$, $f(t) = g(t) = 0$. Thus,

Case 1: $(t \le a - 2M - 1)$

$$
\begin{aligned}
(f * g)(t) : &\overset{\text{Def}}{=} \int_{-\infty}^{\infty} f(t - s + a) g(s) \nabla s \\
&= \left(\int_{-\infty}^{a-M-1} + \int_{a-M-1}^{\infty} \right) f(t - s + a) g(s) \nabla s
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \underbrace{\int_{-\infty}^{a-M-1} f(t-s+a) g(s) \nabla s}_{=0 \text{ (since } g(s)=0 \text{ for all } s<(a-M))} + \underbrace{\int_{a-M-1}^{\infty} f(t-s+a) g(s)) \nabla s}_{=0 \text{ (since } f(\tau)=0 \text{ for all } \tau<(a-M))} \\
\text{Reason} \quad & \left[\left. \begin{cases} t \le a - 2M - 1 \\ -s \le -(a - M) \end{cases} \right\} \Rightarrow \begin{array}{l} t - s \le -M - 1 \\ \underbrace{t - s + a}_{\tau} \le a - M - 1 < (a - M) \end{array} \right] \\
= \quad & 0.
\end{aligned}
$$

Case 2: $(t \geq a-2M)$

$$
\begin{aligned}
(f*g)(t): &\overset{\text{Def}}{=} \int_{-\infty}^{\infty} f(t-s+a)g(s)\nabla s \\
&= \left(\int_{-\infty}^{a-M-1} + \int_{a-M-1}^{t+M} + \int_{t+M}^{\infty}\right) f(t-s+a)g(s)\nabla s \\
&= \underbrace{\int_{-\infty}^{a-M-1} f(t-s+a)g(s)\nabla s}_{=0 \text{ (since } g(s)=0 \text{ for all } s<(a-M))} + \underbrace{\int_{t+M}^{\infty} f(t-s+a)g(s))\nabla s}_{=0 \text{ (since } f(\tau)=0 \text{ for all } \tau<(a-M))} \\
&\qquad + \int_{a-M-1}^{t+M} f(t-s+a)g(s)\nabla s \\
&\left[\text{Reason: } s>t+M \Rightarrow \begin{array}{l} -s<-t-M \\ \underbrace{t-s+a}_{\tau}<(a-M) \end{array}\right] \\
&= \int_{a-M-1}^{t+M} f(t-s+a)g(s)\nabla s \\
&= \sum_{s\in\mathbb{N}_{a-M}^{t+M}} f(t-s+a)g(s).
\end{aligned}
$$

In summary,

$$
(f*g)(t) = \begin{cases} \int_{a-M-1}^{t+M} f(t-s+a)g(s)\nabla s, & t \geq a-2M \\ 0, & t < a-2M \end{cases}
$$

or simply

$$
(f*g)(t) = \int_{a-M-1}^{t+M} f(t-s+a)g(s)\nabla s
$$

if we recall that, by definition, $\int_a^b h(s)\nabla s = 0$ whenever $b \leq a$. □

Remark 1.22. *From the proof of the last theorem we have the useful formula for the* *convolution product:*

$$(f*g)(t) = \int_{a-M-1}^{t+M} f(t-s+a)g(s)\nabla s = \sum_{s\in\mathbb{N}_{a-M}^{t+M}} f(t-s+a)g(s) \tag{1.6}$$

where $M = M(f,g) \in \mathbb{N}_0$ *is sufficiently large that for all* $t < (a-M)$, $f(t) = g(t) = 0$.[7]

Before providing some examples it will be convenient to define a single variable function $H_\gamma(t)$ by fixing $s = \rho(a)$ in the Taylor monomial $H_\gamma(t,s)$ as given in Definition 1.7.

Definition 1.23. *For* $\gamma \in \mathbb{R}$ *we define* $H_\gamma : \mathbb{Z}_a \to \mathbb{R}$ *to be:*

$$H_\gamma(t) := H_\gamma(t, \rho(a)), \text{ for all } t \in \mathbb{Z}_a.$$

Remark 1.24. *Other expressions for the single variable function* $H_\gamma(t)$ *in terms of the Pochhammer polynomials and the standard combinatorial choice function include:*

$$H_\gamma(t) : \overset{Defn}{=} \widetilde{H}_\gamma(t, \rho(a)) \cdot \mathbb{1}_{(\rho(a),\infty)}(t) \tag{1.7}$$

$$\overset{Thm\ 1.12}{=} \begin{cases} P_{\rho(t)-\rho(a)}(\gamma), \text{ for } t \geq a \\ 0, \text{ for } t < a \end{cases} \tag{1.8}$$

$$\tag{1.9}$$

[7]Notice that *if* $f, g \in V_{a+1}$, then $(Lf)(t) := f(t+1) \in V_a$. If we now consider $(Lf*g)(t)$, then here we have $M = M(Lf, g) = 0$ and so

$$(Lf*g)(t) = \sum_{s\in\mathbb{N}_a^t} (Lf)(t-s+a)g(s) = \sum_{s\in\mathbb{N}_a^t} f(t-s+a+1)g(s) = \int_a^t f(t-\rho(s)+a)g(s)\nabla s$$

This last expression is the definition of the convolution product used by Goodrich et al in [5]. (See [5, Definition 3.77].)

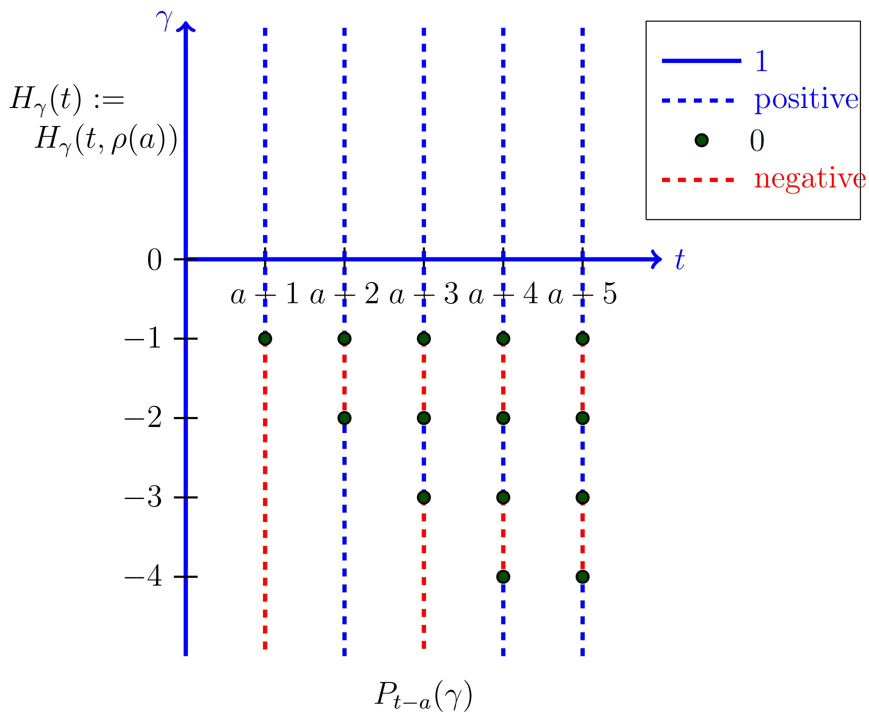


Figure 1.3: Single-Variable Taylor Monomials

$$= P_{t-a}(\gamma) \tag{1.10}$$

$$= \begin{cases} \frac{1}{(t-a)!}(\gamma+1)(\gamma+2)\cdots(\gamma+(t-a)), \textit{ for } t > a \\ 1, \textit{ for } t = a \\ 0, \textit{ for } t < a \end{cases} \tag{1.11}$$

$$= \binom{\gamma + t - a}{t - a}. \tag{1.12}$$

Example 1.25. $H_0(t) = \mathbb{1}_{[a,\infty)}(t)$.

Proof. By Definition 1.7 and Remark 1.14 (ii),

$$H_0(t) = H_0(t, \rho(a)) = \widetilde{H}_0(t, \rho(a)) \cdot \mathbb{1}_{(\rho(a),\infty)}(t) = \mathbb{1}_{[a,\infty)}(t).$$

□

Example 1.26. *If $n \in \mathbb{N}_1$ then by Corollary 1.12,*

$$H_n(t) = H_n(t, \rho(a)) = P_{t-\rho(a)-1}(n) = P_{t-a}(n)$$
$$= \begin{cases} 0, & \text{for } t < a \\ 1, & \text{for } t = a \\ \frac{1}{(t-a)!}(n+1)(n+2)\cdots(n+(t-a)), & \text{for } t > a. \end{cases}$$

The next example is important and will be referenced later. (See Theorem 2.2.)

Example 1.27. *If $n \in \mathbb{N}_1$ then again by Corollary 1.12 and Theorem 1.5,*

$$H_{-n}(t) = H_{-n}(t, \rho(a)) = P_{t-\rho(a)-1}(-n) = P_{t-a}(-n)$$
$$= \begin{cases} 0, & \text{for } t < a \\ 1, & \text{for } t = a \\ \frac{1}{(t-a)!}(-n+1)(-n+2)\cdots(-n+(t-a)), & \text{for } t > a \end{cases}$$
$$= \begin{cases} 0, & \text{for } t < a \\ 1, & \text{for } t = a \\ \frac{(-1)^{(t-a)}}{(t-a)!}(n-1)(n-2)\cdots(n-(t-a)), & \text{for } t > a \end{cases}$$
$$= \begin{cases} 0, & \text{for } t < a \\ 1, & \text{for } t = a \\ (-1)^{(t-a)}\binom{n-1}{t-a}, & \text{for } t > a. \end{cases}$$

Example 1.28. *In particular, for $\gamma = -1, -2, -3,$ and -4, we have:*

$$H_{-1}(t) = \begin{cases} 1, & \text{for } t = a \\ 0, & \text{for } t > a \end{cases}$$

$$H_{-2}(t) = \begin{cases} 1, & \text{for } t = a \\ -1, & \text{for } t = a+1 \\ 0, & \text{otherwise} \end{cases}$$

$$H_{-3}(t) = \begin{cases} 1, & \text{for } t = a \\ -2, & \text{for } t = a+1 \\ 1, & \text{for } t = a+2 \\ 0, & \text{otherwise} \end{cases}$$

$$H_{-4}(t) = \begin{cases} 1, & \text{for } t = a \\ -3, & \text{for } t = a+1 \\ 3, & \text{for } t = a+2 \\ -1, & \text{for } t = a+3 \\ 0, & \text{otherwise.} \end{cases}$$

The next two examples suggest a result we will prove in Chapter 2.

Example 1.29. *Let* $g \in V$*. Then* $(H_{-2} * g)(t) = g(t) - g(t-1) = \nabla g(t)$*.*

Example 1.30. *Let* $g \in V$*. Then* $(H_{-3} * g)(t) = g(t) - 2g(t-1) + g(t-2) = (\nabla^2 g)(t)$*.*

In Theorem 1.31 – Theorem 1.36 we show that V equipped with the operations of $+$ and $*$ is a commutative ring.[8]

Theorem 1.31. V *is commutative with respect to* $*$*.*

Proof. Suppose $f, g \in V$. Then there exist T_f and $T_h \in \mathbb{Z}_a$ such that $f \in V_{T_f}$ and $g \in V_{T_g}$. Put $M = M(f,g) := \max\{a - T_f, a - T_g, 0\} \in \mathbb{N}_0$ and $T := \min\{T_f, T_g, a\} \in \mathbb{Z}_a$. (So $T = a - M$ and $f, g \in V_T = V_{a-M}$.)

Now, fix $t \in \mathbb{Z}_a$. Recall the formula for $f * g$ from Remark 1.22 and consider:

$$\begin{aligned}
(f * g)(t) : &\overset{\text{Rmk 1.22}}{=} \sum_{s \in \mathbb{N}_{a-M}^{t+M}} f(t - s + a) g(s) \\
&\overset{\tau:=\text{t-s+a}}{=} \sum_{\tau \in \mathbb{N}_{a-M}^{t+M}} f(\tau) g(t - \tau + a) \\
&= \sum_{\tau \in \mathbb{N}_{a-M}^{t+M}} g(t - \tau + a) f(\tau) \\
&= (g * f)(t)
\end{aligned}$$

□

Recall that in Section 1.1 we defined the right-shift operator, $R : \mathbb{R}^{\mathbb{Z}_a} \to \mathbb{R}^{\mathbb{Z}_a}$, by $(Rf)(t) := f(t-1)$ for all $t \in \mathbb{Z}_a$ and for $k \in \mathbb{N}_1$, we defined R^k recursively so that $(R^k f)(t) = f(t-k)$. L and L^k were defined similiarly.

Theorem 1.32. *For* $f, g \in V$ *and* $k \in \mathbb{N}_0$*,*

$$(R^k f * g)(t) = R^k (f * g)(t) = (f * g)(t - k).$$

[8] Actually more is true, $(V, +, *)$ is isomorphic to $\mathbb{R}((x))$, the formal Laurent series, and as such is, in fact, a field. (See [8], p.238). However, we shall not need this fact.

Proof. Fix $t \in \mathbb{Z}_a$ and $k \in \mathbb{N}_0$, let $M = M(f,g) := \max\{a - T_f, a - T_g, 0\}$ and consider

$$\begin{aligned}
(R^k f * g)(t) &\overset{\text{Def}}{=} \sum_{s \in \mathbb{N}_{a-M}^{t+M}} (R^k f)(t-s+a)g(s) \\
&= \sum_{s \in \mathbb{N}_{a-M}^{t+M}} f(t-s+a-k)g(s) \\
&= \sum_{s \in \mathbb{N}_{a-M}^{t+M-k} \cup \mathbb{N}_{t+M-k+1}^{t+M}} f(t-k-s+a)g(s) \\
&= \sum_{s \in \mathbb{N}_{a-M}^{t-k+M}} f(t-k-s+a)g(s) + \underbrace{\sum_{s \in \mathbb{N}_{t+M-k+1}^{t+M}} f(t-k-s+a)g(s)}_{=0 \text{ (since } f(\tau)=0 \text{ for all } \tau<(a-M))} \\
&\quad \left[\text{Reason: } s \geq t-k+M+1 \Rightarrow \begin{array}{l} -s \leq -t+k-M-1 \\ \underbrace{t-s-k+a}_{\tau} \leq a-M-1 < (a-M) \end{array} \right] \\
&= \sum_{s \in \mathbb{N}_{a-M}^{(t-k)+M}} f((t-k)-s+a)g(s) \\
&= (f * g)(t-k) = R^k(f * g)(t).
\end{aligned}$$

□

The commutativity of V under $*$ gives us the following corollary.

Corollary 1.33. *For $f, g \in V$ and $m, n \in \mathbb{N}_0$,*

$$(R^m f * R^n g) = R^{m+n}(f * g).$$

Proof. Consider

$$\begin{aligned}
(R^m f * R^n g) &\overset{\text{Thm 1.32}}{=} R^m(f * R^n g) \\
&\overset{\text{Thm 1.31}}{=} R^m(R^n g * f) \\
&= R^m(R^n(g * f)) \\
&\overset{\text{Thm 1.31}}{=} R^{m+n}(f * g).
\end{aligned}$$

□

Lemma 1.34. *V_a is associative with respect to $*$. That is, if $f, g, h \in V_a$, then*

$$f * (g * h) = (f * g) * h.$$

Proof. If $f, g, h \in V_a$ then for all $t < a$, $f(t) = g(t) = h(t) = 0$. Fix $t \in \mathbb{Z}_a$ and consider

$$\begin{aligned}
(f * (g * h))(t) : &\overset{\text{Def}}{=} \sum_{u \in \mathbb{N}_a^t} f(t-u+a)(g * h)(u) \\
&\overset{\text{Def}}{=} \sum_{u \in \mathbb{N}_a^t} f(t-u+a) \sum_{s \in \mathbb{N}_a^u} g(u-s+a)h(s) \\
&= \sum_{u \in \mathbb{N}_a^t} \sum_{s \in \mathbb{N}_a^u} f(t-u+a)g(u-s+a)h(s) \\
&= \sum_{s \in \mathbb{N}_a^t} \sum_{u \in \mathbb{N}_s^t} f(t-u+a)g(u-s+a)h(s) \\
&= \sum_{s \in \mathbb{N}_a^t} h(s) \sum_{u \in \mathbb{N}_s^t} f(t-u+a)g(u-s+a) \\
&\underset{\therefore \text{u=w+s-a}}{\overset{\text{w(u):=u-s+a}}{=}} \sum_{s \in \mathbb{N}_a^t} h(s) \sum_{w \in \mathbb{N}_a^{t-s+a}} f(t-(w+s-a)+a)g(w) \\
&= \sum_{s \in \mathbb{N}_a^t} h(s) \sum_{w \in \mathbb{N}_a^{t-s+a}} f((t-s+a)-w+a)g(w) \\
&= \sum_{s \in \mathbb{N}_a^t} (f * g)(t-s+a)h(s) \\
&= ((f * g) * h)(t).
\end{aligned}$$

□

The next theorem extends the result of Lemma 1.34 to all functions in V.

Theorem 1.35. *V is associative with respect to $*$.*

Proof. Suppose $f, g, h \in V$. There exist $T_f, T_g,$ and $T_h \in \mathbb{Z}_a$ such that $f \in V_{T_f}, g \in V_{T_g}$, and $h \in V_{T_h}$. Put $M = M(f, g, h) := \max\{a - T_f, a - T_g, a - T_h, 0\} \in \mathbb{N}_0$ and $T := \min\{T_f, T_g, T_h, a\} \in \mathbb{Z}_a$. So $T = a - M$ and $f, g, h \in V_T = V_{a-M}$ which implies that for all $t < (a - M)$, $f(t) = g(t) = h(t) = 0$ which, in turn, implies that $(R^M f), (R^M g), (R^M h) \in V_a$. Thus, by Lemma 1.34,

$$R^M f * (R^M g * R^M h) = (R^M f * R^M g) * R^M h.$$

But, by Corollary 1.33,

$$\begin{aligned} R^M f * (R^M g * R^M h) &= R^M f * (R^{2M}(g * h)) \\ &= R^{3M}(f * (g * h)). \end{aligned}$$

On the other hand,

$$\begin{aligned} (R^M f * R^M g) * R^M h &= (R^{2M}(f * g)) * R^M h \\ &= R^{3M}(f * g) * h). \end{aligned}$$

Thus,

$$R^{3M}(f * (g * h)) \quad = \quad R^{3M}(f * g) * h).$$

Applying the injective operator L^{3M} to both sides of this last equation yields,

$$f * (g * h) \quad = \quad (f * g) * h.$$

□

Finally, we can justify our claim that $(V, +, *)$ is a commutative ring.

Theorem 1.36. *$(V, +, *)$ is a commutative ring.*

Proof. The fact that V is commutative and associative with respect to $*$ has already been shown. (See Theorem 1.31 and Theorem 1.35, respectively.) It remains to show distributivity holds.

$$\begin{aligned} f*(g+h) &= \sum_{s\in\mathbb{N}_{a-M}^{t+M}} f(t-s+a)(g+h)(s) \\ &= \sum_{s\in\mathbb{N}_{a-M}^{t+M}} f(t-s+a)[g(s)+h(s)] \\ &= \sum_{s\in\mathbb{N}_{a-M}^{t+M}} f(t-s+a)g(s)+f(t-s+a)h(s) \\ &= \sum_{s\in\mathbb{N}_{a-M}^{t+M}} f(t-s+a)g(s)+\sum_{s\in\mathbb{N}_{a-M}^{t+M}} f(t-s+a)h(s) \\ &= f*g+f*h\ . \end{aligned}$$

□

We conclude this section with a theorem that tells us what the convolution product of H_μ and H_ν is for any $\mu, nu \in \mathbb{R}$, but first we need a lemma.

Lemma 1.37. *For $x, y \in \mathbb{R}$ and all $k \in \mathbb{N}$,*

$$\sum_{i=0}^{k} \binom{x+k-i}{k-i}\binom{y+i}{i} = \binom{x+y+1+k}{k}.$$

Proof. We first prove the claim for the special case where $x = m$ and $y = n$, where $m, n \in \mathbb{N}$. Fix k, m, and $n \in \mathbb{N}_1$ and consider

$$
\begin{aligned}
F_m(z) &:= \frac{1}{(1-z)^{m+1}} = (1-z)^{-(m+1)} \\
&\Rightarrow F_m'(z) = (m+1)(1-z)^{-m-2} && \Rightarrow F_m'(0) = (m+1) \\
&\qquad\qquad \vdots && \qquad\qquad \vdots \\
&\Rightarrow F_m^{(k)}(z) = (m+1)\ldots(m+k)(1-z)^{-m-2} && \Rightarrow F_m^{(k)}(0) = (m+1)\ldots(m+k) \\
& && = (m+1)^{\overline{k}} \\
& && = (m+k)^{\underline{k}} \cdot \frac{k!}{k!} \\
& && = \binom{m+k}{k} k!.
\end{aligned}
$$

Similiarly,

$$
F_n(z) = \frac{1}{(1-z)^{n+1}} \Rightarrow F_n^{(k)}(0) = \binom{n+k}{k} k!.
$$

Notice,

$$
\begin{aligned}
(F_m \cdot F_n)(z) = F_m(z) \cdot F_n(z) &= \frac{1}{(1-z)^{m+1}} \cdot \frac{1}{(1-z)^{n+1}} \\
&= \frac{1}{(1-z)^{((m+n+1)+1)}} \\
&= F_{(m+n+1)}(z).
\end{aligned}
$$

And so

$$
\frac{d^k}{dz^k}(F_m \cdot F_n)(0) = F_{(m+n+1)}^{(k)}(0) = \binom{(m+n+1)+k}{k} k!.
$$

But, by repeated application of the Product Rule, we get:

$$\begin{aligned}\frac{d^k}{dz^k}(F_m \cdot F_n)(z) &= F_m^{(k)}(z) \cdot F_n(z) + \binom{k}{1} F_m^{(k-1)}(z) \cdot F_n'(z) + \cdots \\ &\qquad + \binom{k}{k} F_m(z) \cdot F_n^{(k)}(z) \\ &= \sum_{i=0}^{k} \binom{k}{i} F_m^{(k-i)}(z) \cdot F_n^{(i)}(z).\end{aligned}$$

And so we also have

$$\begin{aligned}\frac{d^k}{dz^k}(F_m \cdot F_n)(0) &= \sum_{i=0}^{k} \binom{k}{i} F_m^{(k-i)}(0) \cdot F_n^{(i)}(0) \\ &= \sum_{i=0}^{k} \binom{k}{i} \binom{m+(k-i)}{k-i} (k-i)! \cdot \binom{n+i}{i} i! \\ &= \sum_{i=0}^{k} k! \binom{m+k-i}{k-i} \binom{n+i}{i} \\ &= k! \sum_{i=0}^{k} \binom{m+k-i}{k-i} \binom{n+i}{i}.\end{aligned}$$

Since these two expressions for $\frac{d^k}{dz^k}(F_m \cdot F_n)(0)$ must be equal, we have:

$$k! \sum_{i=0}^{k} \binom{m+k-i}{k-i} \binom{n+i}{i} = k! \binom{(m+n+1)+k}{k}$$

or

$$\sum_{i=0}^{k} \binom{m+k-i}{k-i} \binom{n+i}{i} = \binom{m+n+1+k}{k}.$$

Thus, the claim holds whenever $(x, y) = (m, n) \in \mathbb{N} \times \mathbb{N}$.

Now, to see that the claim holds for all $(x, y) \in \mathbb{R} \times \mathbb{R}$, notice the expression on the left-hand side of the original claim is a k-degree polynomial in x and and a k-degree polynomial in y, as is the expression on the right-hand side of the original claim. So their difference is also a k-degree polynomial in both x and y. And, we have shown that this difference is 0 whenever $(x, y) = (m, n) \in \mathbb{N} \times \mathbb{N}$.

Now, a polynomial in two variables that has zeros at all $(m, n) \in \mathbb{N} \times \mathbb{N}$ must be identically 0. That is, it must be the 0 polynomial. Thus the difference polynomial is the 0 polynomial.

Hence we conclude that the two polynomial expressions in the original claim must both represent the same polynomial function in x and y. As such, they must be equal everywhere. And so, the claim must hold for all $(x, y) \in \mathbb{R} \times \mathbb{R}$. □

Theorem 1.38. *For $\mu, \nu \in \mathbb{R}$, $H_\mu * H_\nu = H_{\mu+\nu+1}$.*

Proof. Fix $\mu, \nu \in \mathbb{R}$.

If $t < a$, then

$$\begin{aligned}
(H_\mu * H_\nu)(t) &\overset{\text{Def}}{=} \int_{-\infty}^{\infty} H_\mu(t-s+a)H_\nu(s)\nabla s \\
&= \underbrace{\int_{-\infty}^{\rho(a)} H_\mu(t-s+a)H_\nu(s)\nabla s}_{=0 \text{ (since } H_\nu(s)=0 \text{ for all } s<a)} + \underbrace{\int_{\rho(a)}^{\infty} H_\mu(t-s+a)H_\nu(s)\nabla s}_{=0 \text{ (since } H_\mu(\tau)=0 \text{ for all } \tau<a)} \\
&= 0.
\end{aligned}$$

If $t \geq a$, fix t and put $k := (t-a) \in \mathbb{N}_0$, then consider:

$$\begin{aligned}
(H_\mu * H_\nu)(t) &\overset{\text{Def}}{=} \int_{-\infty}^{\infty} H_\mu(t-s+a)H_\nu(s)\nabla s \\
&= \underbrace{\int_{-\infty}^{\rho(a)} H_\mu(t-s+a)H_\nu(s)\nabla s}_{=0 \text{ (since } H_\mu(\tau)=0 \text{ for all } \tau<a)} + \int_{\rho(a)}^{t} H_\mu(t-s+a)H_\nu(s)\nabla s \\
&\qquad + \underbrace{\int_{t}^{\infty} H_\mu(t-s+a)H_\nu(s)\nabla s}_{=0 \text{ (since } H_\mu(\tau)=0 \text{ for all } \tau<a)} \\
&= \sum_{s\in\mathbb{N}_a^t} H_\mu(t-s+a)H_\nu(s)
\end{aligned}$$

$$
\begin{aligned}
&\overset{\text{Rmk 1.24}}{=} \sum_{s\in\mathbb{N}_a^t} \binom{\mu+(t-s+a)-a}{(t-s+a)-a}\binom{\nu+s-a}{s-a} \\
&\overset{\text{t=a+k}}{=} \sum_{s\in\mathbb{N}_{a+0}^{a+k}} \binom{\mu+(a+k)-s}{(a+k)-s}\binom{\nu+s-a}{s-a} \\
&\overset{\text{i:=s-a}}{=} \sum_{i=0}^{k} \binom{\mu+(a+k)-(a+i)}{(a+k)-(a+i)}\binom{\nu+i}{i} \\
&= \sum_{i=0}^{k} \binom{\mu+k-i}{k-i}\binom{\nu+i}{i} \\
&\overset{\text{Lemma 1.37}}{=} \binom{\mu+\nu+1+k}{k} \overset{\text{k:=t-a}}{=} \binom{(\mu+\nu+1)+t-a}{t-a} \\
&\overset{\text{Rmk 1.24}}{=} H_{\mu+\nu+1}(t).
\end{aligned}
$$

□

Chapter 2

Fractional Nabla Operators (∇^γ, ∇_a^α, ∇_{a*}^ν)

In many treatments of the subject, ∇_a^ν and ∇_{a*}^ν are thought of as operators on functions that take $\mathbb{N}_a$ or $\mathbb{N}_{a-N}$ to $\mathbb{R}$. In this paper we define a more general fractional nabla operator ∇^ν that acts on functions taking $\mathbb{Z}_a$ to $\mathbb{R}$, and show how ∇_a^ν and ∇_{a*}^ν are both manifestations of this more general operator under different circumstances.

One advantage of this perspective is the domain issues one encounters in working with the fractional delta operators and, to a lesser extent, when working with the fractional nabla operators whose domains only extend to the right disappear. All functions under consideration have exactly the same domain, $\mathbb{Z}_a$, both before and after they are operated on by a fractional nabla operator.

A second advantage of this perspective is that many of the known properties of these operators follow very simply and directly from three facts about: convolution, the Taylor monomials, and the fractional nabla operator. Namely:

- convolution is commutative and associative over V (Theorem 1.36)
- $H_\mu * H_\nu = H_{\mu+\nu+1}$ (Theorem 1.38)
- $\nabla^\gamma f = H_{-\gamma-1} * f$ (Theorem 2.2)

A third advantage is that many of the known properties of fractional nabla operators can be extended, so the results we obtain via this approach require fewer caveats than previously established results.

Finally, after realizing that ∇_a^ν and ∇_{a*}^ν are both different manifestations of ∇^ν, it becomes clear that the difference between these two operators is simply the convolution of a Taylor monomial and a short sequence. (See Theorem 2.21.)

2.1 The General Fractional Operator ∇^γ

Recall that in Definition 1.19 we defined the vector space V and in Definition 1.23 we defined the single variable function $H_\gamma(t) := H_\gamma(t, \rho(a)) \in V$ for all $\gamma \in \mathbb{R}$. We now use this function to define the general nabla operator ∇^γ.

Definition 2.1. *For all $f \in V$ and each $\gamma \in \mathbb{R}$ we define the* ***general fractional nabla operator of order*** γ, $\nabla^\gamma : V \to V$, *by* $\nabla^\gamma f := H_{-\gamma-1} * f$.

The first thing we will do is check that we have not made a poor choice of notation. Since there is already a meaning for the operator ∇^γ when $\gamma \in \mathbb{N}$, the first thing we will do is confirm that this new operator is precisely the previously defined (i.e. iterated) nabla difference operator when $\gamma \in \mathbb{N}_0$ and $f \in V$.

Theorem 2.2. *If $N \in \mathbb{N}_0$ and $f \in V$, then* $H_{-N-1} * f = \nabla^N f$.

Proof. Fix $t \in \mathbb{Z}_a$ and consider

$$\begin{aligned}
(H_{-(N+1)} * f)(t) &\overset{\text{Def}}{=} \int_{-\infty}^{\infty} H_{-(N+1)}(t-s+a) f(s) \nabla s \\
&\overset{\text{Example 1.27}}{=} \int_{-\infty}^{\infty} (-1)^{((t-s+a)-a)} \binom{(N+1)-1}{(t-s+a)-a} f(s) \nabla s \\
&= \int_{-\infty}^{\infty} (-1)^{(t-s)} \binom{N}{t-s} f(s) \nabla s \\
&= \sum_{s \in \mathbb{N}_{t-N}^t} (-1)^{(t-s)} \binom{N}{t-s} f(s)
\end{aligned}$$

$$\overset{\text{i := (t-s)}}{=} \sum_{i=0}^{N} (-1)^i \binom{N}{i} f(t-i)$$
$$= (\nabla^N f)(t).$$

□

Thus, when $\gamma \in \mathbb{N}_0$, the general fractional operator corresponds to the standard (iterated) nabla difference operator when $\gamma = n \in \mathbb{N}_0$.

In the next theorem we show the ν^{th} order nabla fractional sum (as defined by Goodrich et al in [5] (see Appendix A, Definition A.3) can be obtained from the general fractional operator of the same order by first replacing every value of f at a and below with 0.

Theorem 2.3. *If $\gamma < 0$ and we put $\nu := -\gamma > 0$, then $\nabla_a^{-\nu} f = \nabla^{-\nu}(\mathbb{1}_{(a,\infty)} f)$.*

Proof. Starting with the right-hand side of the above equation we have:

$$\begin{aligned}
(\nabla^{-\nu}(\mathbb{1}_{(a,\infty)} f))(t) &\overset{\text{Def}}{=} (H_{\nu-1} * \mathbb{1}_{(a,\infty)} f)(t) \\
&= \int_{-\infty}^{\infty} H_{\nu-1}(t-s+a)(\mathbb{1}_{(a,\infty)} f)(s)\nabla s \\
&= \sum_{s=a+1}^{t} H_{\nu-1}(t-s+a, \rho(a)) f(s) \\
&\overset{\text{Cor 1.17}}{=} \sum_{s=a+1}^{t} H_{\nu-1}(t, \rho(s)) f(s) \\
&= \int_a^t H_{\nu-1}(t, \rho(\tau)) f(\tau) \nabla\tau \\
&\overset{\text{Def A.3}}{=} (\nabla_a^{-\nu} f)(t).
\end{aligned}$$

□

At this point we could repeat the above argument for $\gamma = \nu > 0$ to show that $(\nabla^{\nu}(\mathbb{1}_{(a,\infty)} f))(t) = \int_a^t H_{-\nu-1}(t, \rho(\tau)) f(\tau)\nabla\tau \overset{\text{Thm A.6}}{=} (\nabla_a^{\nu} f)(t)$. However, with the results we have already established concerning convolution and the Taylor monomials we can get the desired identity more directly via the method in the proof below.

Theorem 2.4. *If* $\gamma = \nu > 0$ *and* $\gamma = \nu \notin \mathbb{N}$*, then* $\nabla_a^\nu f = \nabla^\nu(\mathbb{1}_{(a,\infty)} f)$.

Proof. Starting with the left-hand side of the above equation we have:

$$\begin{aligned}
\nabla_a^\nu f &\overset{\text{Def A.4}}{=} \nabla^N \nabla_a^{-(N-\nu)} f \text{ where } N := \lceil \nu \rceil \\
&\overset{\text{Thm 2.3}}{=} \nabla^N \nabla^{-(N-\nu)}(\mathbb{1}_{(a,\infty)} f) \\
&\overset{\text{Def 2.1}}{=} H_{-(N+1)} * (H_{N-\nu-1} * (\mathbb{1}_{(a,\infty)} f)) \\
&\overset{\text{Thm 1.35}}{=} (H_{-N-1} * H_{N-\nu-1}) * (\mathbb{1}_{(a,\infty)} f) \\
&\overset{\text{Thm 1.38}}{=} H_{(-N-1)+(N-\nu-1)+1} * (\mathbb{1}_{(a,\infty)} f) \\
&= H_{-(\nu+1)} * (\mathbb{1}_{(a,\infty)} f) \\
&\overset{\text{Def 2.1}}{=} \nabla^\nu(\mathbb{1}_{(a,\infty)} f).
\end{aligned}$$

□

The previous two theorems combine to tell us that for all $\gamma \in \mathbb{R} \setminus \mathbb{N}_0$, $\nabla_a^\gamma f = \nabla^\gamma(\mathbb{1}_{(a,\infty)} f)$. Since we have not previously defined the operator ∇_a^γ for the case when $\gamma = N \in \mathbb{N}_0$, we are free to define $\nabla_a^N f := \nabla^N(\mathbb{1}_{(a,\infty)} f)$, so that the result holds all $\gamma \in \mathbb{R}$. So we make this definition and record the result in the following corollary.

Corollary 2.5. *For all* $\gamma \in \mathbb{R}$ *and all* $f \in V$*,*

$$\nabla_a^\gamma f = \nabla^\gamma(\mathbb{1}_{(a,\infty)} f) \ .$$

Remark 2.6. *Note that if $f \in V_{a+1}$, then $f = \mathbb{1}_{(a,\infty)} f$ and so for $\nu \geq 0$:*

$$\nabla^{-\nu} f = \nabla_a^{-\nu} f \quad \text{and}$$
$$\nabla^{\nu} f = \nabla_a^{\nu} f \ .$$

That is, the general nabla operator is, in fact, a generalization of the nabla fractional sum and the nabla fractional difference as defined by Goodrich et al in [5]. That is, both of these sum and difference operators can be realized as a special case of this more general fractional operator.

2.2 The Action of Nabla Operators on Taylor Monomials

In this section we will show that the operators ∇^{α} and ∇_a^{α} and ∇_{a*}^{ν} act nicely on the Taylor monomials. Recall that the operators ∇^{α} and ∇_a^{α} were introduced in Section 2.1. In [5] Goodrich et al define, for $\nu > 0$, the **Caputo nabla fractional difference** operator, $\nabla_{a*}^{\nu} : \mathbb{R}^{\mathbb{Z}_a} \to \mathbb{R}^{\mathbb{Z}_a}$, by $\nabla_{a*}^{\nu} := \nabla_a^{\nu-N} \nabla^N$ (where $N := \lceil \nu \rceil$). (See [5, Definition 3.117].) [1]

We begin with a few quick facts concerning the single variable function $H_{\beta}(t)$ which was defined in Section 1.5.

Fact 2.7. *If $\alpha, \beta \in \mathbb{R}$, then $\nabla^{\alpha} H_{\beta} = H_{\beta-\alpha}$.*

Proof. This fact follows from Theorem 1.38.

$$\nabla^{\alpha} H_{\beta} \overset{\text{Def 2.1}}{=} H_{-\alpha-1} * H_{\beta} \overset{\text{Thm 1.38}}{=} H_{(-\alpha-1)+\beta+1} = H_{\beta-\alpha} \ .$$

[1] For the convenience of the reader we have recorded several definitions and results from [5] in Appendix A. This definition of the Caputo nabla fractional difference appears in Appendix A, Definition A.5.

□

Fact 2.8. *For all $\beta \in \mathbb{R}$,*

$$RH_\beta(t) = H_\beta(t, a).$$

Proof.

$$RH_\beta(t) = H_\beta(t-1) \overset{\text{Def 1.23}}{=} H_\beta(t-1, \rho(a)) \overset{\text{Cor 1.17}}{=} H_\beta(t, a).$$

□

Theorem 2.9. *For $\alpha, \beta \in \mathbb{R}$ and $\nu > 0$,*

$$\nabla_a^\alpha H_\beta(t, a) = \nabla^\alpha H_\beta(t, a) = H_{\beta-\alpha}(t, a) \quad \textit{and}$$
$$\nabla_{a*}^\nu H_\beta(t, a) = H_{\beta-\nu}(t, a).$$

Proof. This corollary follows from Theorem 1.38 and Corollary 1.33. Since $H_\beta(t, a) \in V_{a+1}$, we have

$$\begin{aligned}
\nabla_a^\alpha H_\beta(t, a) &\overset{\text{Rmk 2.6}}{=} \nabla^\alpha H_\beta(t, a) \\
&\overset{\text{Fact 2.8}}{=} (\nabla^\alpha (RH_\beta))(t) \\
&= (H_{-\alpha-1} * RH_\beta)(t) \\
&\overset{\text{Cor 1.33}}{=} R(H_{-\alpha-1} * H_\beta)(t) \\
&\overset{\text{Thm 1.38}}{=} R(H_{-\alpha-1+\beta+1})(t) \\
&= RH_{\beta-\alpha}(t) \\
&\overset{\text{Fact 2.8}}{=} H_{\beta-\alpha}(t, a). \quad \checkmark
\end{aligned}$$

The second equation follows from the first. To see this put $N := \lceil \nu \rceil$ and note that, by the first equation we have

$$\begin{aligned}\nabla_{a*}^{\nu}H_{\beta}(t,a) &\overset{\text{Def A.5}}{=} \nabla_a^{\nu-N}\nabla^N H_{\beta}(t,a)\\ &= \nabla_a^{\nu-N}H_{\beta-N}(t,a) = H_{\beta-N-(\nu-N)}(t,a)\\ &= H_{\beta-\nu}(t,a). \quad \checkmark\end{aligned}$$

□

Corollary 2.10. *For $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}\setminus\mathbb{N}_0$ and $\nu > 0$,*

$$\nabla_a^{\alpha}\widetilde{H}_{\beta}(t,a) = \nabla^{\alpha}\widetilde{H}_{\beta}(t,a) = H_{\beta-\alpha}(t,a) \quad \textit{and}$$
$$\nabla_{a*}^{\nu}\widetilde{H}_{\beta}(t,a) = H_{\beta-\nu}(t,a).$$

Proof. This follows immediately from Corollary 2.9, since for $\beta \in \mathbb{R}\setminus\mathbb{N}_0$, by Corollary 1.13, $\widetilde{H}_{\beta}(t,a) = H_{\beta}(t,a)$. □

It remains to consider how the various operators act on the extended Taylor monomials, $\widetilde{H}_k(t,a)$, when $k \in \mathbb{N}_0$.

Corollary 2.11. *For $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}_0$,*

$$\nabla_a^{\alpha}\widetilde{H}_k(t,a) = H_{k-\alpha}(t,a).$$

Proof. This follows from Corollary 2.5 and Corollary 2.9, since

$$\nabla_a^{\alpha}\widetilde{H}_k(t,a) \overset{\text{Cor 2.5}}{=} \nabla^{\alpha}(\mathbb{1}_{(a,\infty)}\widetilde{H}_k(t,a)) \overset{\text{Def 1.7}}{=} \nabla^{\alpha}H_k(t,a) \overset{\text{Cor 2.9}}{=} H_{k-\alpha}(t,a).$$

□

When considering how the general operator ∇^α acts on $\widetilde{H}_k(t,a)$ we must exercise a bit of caution since when $k \in \mathbb{N}_0$, $\widetilde{H}_k(t,a) \notin V$. So, in general, it does not make sense to apply ∇^α to $\widetilde{H}_k(t,a)$. However, if $\alpha = n \in \mathbb{N}_0$, it still makes sense to apply the iterated operator ∇^n to $\widetilde{H}_k(t,a)$. In this situation, by repeated application of the Power Rule (Theorem 1.15), we get:

$$\nabla^n \widetilde{H}_k(t,a) = \begin{cases} \widetilde{H}_{k-n}(t,a), & (k-n) \geq 0 \\ 0, & (k-n) < 0. \end{cases}$$

Thus, we have the following corollary.

Corollary 2.12. *For $n, k \in \mathbb{N}_0$ and $\nu > 0$,*

$$\nabla^n \widetilde{H}_k(t,a) = \begin{cases} \widetilde{H}_{k-n}(t,a), & (k-n) \geq 0 \\ 0, & (k-n) < 0 \end{cases}$$

and

$$\nabla^\nu_{a*} \widetilde{H}_k(t,a) = \begin{cases} H_{k-\nu}(t,a), & (k-\nu) \geq 0 \\ 0, & (k-\nu) < 0. \end{cases}$$

Proof. The first equation results from repeated application of the Power Rule (Theorem 1.15). The second equation follows from the first. To see this put $N := \lceil \nu \rceil$ and consider

$$\nabla_{a*}^{\nu}\widetilde{H}_k(t,a) = \nabla_a^{\nu-N}\nabla^N\widetilde{H}_k(t,a) = \begin{cases} \nabla_a^{\nu-N}\widetilde{H}_{k-N}(t,a), & (k-N)\geq 0 \\ \nabla_a^{\nu-N}0, & (k-N)<0 \end{cases}$$

$$= \begin{cases} H_{k-N-(\nu-N)}(t,a), & (k-N)\geq 0 \\ 0, & (k-N)<0 \end{cases}$$

$$= \begin{cases} H_{k-\nu)}(t,a), & (k-\nu)\geq 0 \\ 0, & (k-\nu)<0. \end{cases}$$

Note the last two steps follow from Corollary 2.12/Corollary 2.11 and the fact that $(k-\nu)\geq 0 \iff k\geq\nu \iff k\geq\lceil\nu\rceil \iff (k-N)\geq 0.$ □

We summarize the main results of this section in the following remark.

Remark 2.13. *If $k,n\in\mathbb{N}_0$, $\alpha,\beta,\gamma\in\mathbb{R}$, $\nu>0$, and $N:=\lceil\nu\rceil$, the following describe how the various operators act on the Taylor monomials (for all $t\in\mathbb{Z}_a$).* [2] [3]

$$\nabla_{\cdot}^{\gamma}H_\beta(t,a) = \quad H_{\beta-\gamma}(t,a) \quad \textit{where} \quad \nabla_{\cdot}^{\gamma}\in\{\nabla^\alpha,\nabla_a^\alpha,\nabla_{a*}^\nu\}$$

[2]Recall that if $k\in\mathbb{N}_0$ and $\alpha\in\mathbb{R}\setminus\mathbb{N}_0$, then $\widetilde{H}_k(t,a)\notin V$ and so $\nabla^\alpha\widetilde{H}_k(t,a)$ is not well-defined.

[3]For the convenience of the reader we have recorded the main results from this section, section 2.3, and section 3.2 in Appendix B.

$$\nabla_a^{\alpha} \widetilde{H}_k(t,a) = \quad H_{k-\alpha}(t,a)$$

$$\nabla^n \widetilde{H}_k(t,a) = \begin{cases} \widetilde{H}_{k-n}(t,a), & (k-n) \geq 0 \\ 0, & (k-n) < 0 \end{cases}$$

$$\nabla_{a*}^{\nu} \widetilde{H}_k(t,a) = \begin{cases} H_{k-\nu}(t,a), & (k-\nu) \geq 0 \\ 0, & (k-\nu) < 0. \end{cases}$$

2.3 Composition Rules

We begin this section with a theorem that identifies one of the very nice properties of the general fractional operator ∇^{γ}. The proof follows quickly and easily from the three very important properties mentioned at the beginning of this chapter.

Theorem 2.14. *If $\alpha, \beta \in \mathbb{R}$, then $\nabla^{\alpha}\nabla^{\beta} f = \nabla^{\alpha+\beta} f = \nabla^{\beta}\nabla^{\alpha} f$.*

Proof. For fixed $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned} \nabla^{\alpha}\nabla^{\beta} f &\overset{\text{Definition}}{=} H_{-\alpha-1} * (H_{-\beta-1} * f) \\ &\overset{\text{Thm 1.35}}{=} (H_{-\alpha-1} * H_{-\beta-1}) * f \\ &\overset{\text{Thm 1.38}}{=} H_{(-\alpha-1)+(-\beta-1)+1} * f \\ &= H_{-(\alpha+\beta)-1} * f \\ &\overset{\text{Definition}}{=} \nabla^{\alpha+\beta} f \end{aligned}$$

Relabelling α and β in the above argument yields the other equality in the claim. □

Thus, general fractional operators commute.

Next we state and prove a lemma that will help streamline the next few proofs.

Lemma 2.15. *For $\gamma \in \mathbb{R}$, $f \in V$, $T \in \mathbb{Z}_a$, and $N \in \mathbb{N}_1$:*

(i) $\mathbb{1}_{(T,\infty)} \nabla^{\gamma}(\mathbb{1}_{(T,\infty)} f) = \nabla^{\gamma}(\mathbb{1}_{(T,\infty)} f)$

(ii) $\mathbb{1}_{(-\infty,T+N]} \nabla^{N}(\mathbb{1}_{(-\infty,T]} f) = \nabla^{N}(\mathbb{1}_{(-\infty,T]} f)$

(iii) $\mathbb{1}_{(T,\infty)} \nabla^{N}(\mathbb{1}_{(-\infty,T]} f) = \mathbb{1}_{(T,\infty)} \nabla^{N}(\mathbb{1}_{(T-N,T]} f) = \mathbb{1}_{(T,T+N]} \nabla^{N}(\mathbb{1}_{(T-N,T]} f)$

Proof. Proof of (i). Clearly, when $t > T$, the claim holds. To see that (i) also holds when $t \leq T$, note

$$\begin{aligned}\left(\nabla^{\gamma}(\mathbb{1}_{(T,\infty)} f)\right)(t) &\overset{\text{Def}}{=} \left(H_{-\gamma-1} * \mathbb{1}_{(T,\infty)} f\right)(t) \overset{\text{Def}}{=} \int_{\mathbb{Z}_a} H_{-\gamma-1}(t-s+a)\mathbb{1}_{(T,\infty)}(s) f(s) \nabla s \\ &= \int_{(T,\infty)} \underbrace{H_{-\gamma-1}(t-s+a)}_{0} f(s)\nabla s = 0.\end{aligned}$$

The reason the integrand is 0 here is that

$$\begin{cases} t \leq T \text{ and} \\ s \geq T+1 \end{cases} \Rightarrow \begin{cases} t \leq T \text{ and} \\ -s \leq -T-1 \end{cases} \Rightarrow t-s+a \leq a-1 \ \Rightarrow H_{\gamma-1}(t-s+a) = 0. \quad \checkmark$$

Proof of (ii). In this proof we will require a fact that has not yet been proven.

Fact *: For $k \in \mathbb{N}_1$, $H_{-k}(t) = 0$ for all $t \geq a+k$.

Reason: Fix $k \in \mathbb{N}_1$ and suppose $t \geq a+k > a \Rightarrow t-a \geq k$. Then,

$$\begin{aligned}H_{-k}(t) &= H_{-k}(t,\rho(a)) = P_{t-a}(-k) \\ &= \frac{1}{(t-a)!}(-k+1)(-k+2)\cdots\underbrace{(-k+k)}_{0}\cdots(-k+(t-a)) = 0. \quad \checkmark\end{aligned}$$

Returning to the claim, clearly, when $t \leq T+N$, the claim holds. To see that (ii) also holds when $t > T+N$, fix $t \geq T+N+1$, and consider

$$\left(\nabla^N(\mathbb{1}_{(-\infty,T]}f)\right)(t) = \left(H_{-N-1} * \mathbb{1}_{(-\infty,T]}f\right)(t) \overset{\text{Def}}{=} \int_{\mathbb{Z}_a} H_{-N-1}(t-s+a)\mathbb{1}_{(-\infty,T]}(s)f(s)\nabla s$$
$$= \int_{(-\infty,T]} \underbrace{H_{-(N+1)}(t-s+a)}_{0} f(s)\nabla s = 0.$$

The reason the integrand is 0 here is that

$$\begin{cases} t \geq T+N+1 \text{ and} \\ s \leq T \end{cases} \Rightarrow \begin{cases} t \geq T+N+1 \text{ and} \\ -s \geq -T \end{cases} \Rightarrow t-s+a \geq N+1+a,$$

and so, by the fact just given, we have $H_{-(N+1)}(t-s+a) = 0$. $\surd$

Proof of (iii). Clearly, when $t < T$, the first equality in the claim holds. Suppose $t \geq T+1$, then

$$\begin{aligned}
\left(\mathbb{1}_{(T,\infty)}\nabla^N(\mathbb{1}_{(-\infty,T]}f)\right)(t) &= \left(\nabla^N(\mathbb{1}_{(-\infty,T]}f)\right)(t) = \left(H_{-N-1} * \mathbb{1}_{(-\infty,T]}f\right)(t) \\
&\overset{\text{Def}}{=} \int_{\mathbb{Z}_a} H_{-N-1}(t-s+a)\mathbb{1}_{(-\infty,T]}(s)f(s)\nabla s \\
&= \int_{(-\infty,T]} H_{-(N+1)}(t-s+a)f(s)\nabla s \\
&= \int_{(-\infty,T-N]} \underbrace{H_{-(N+1)}(t-s+a)}_{0} f(s)\nabla s \qquad (*) \\
&\quad + \int_{(T-N,T]} H_{-(N+1)}(t-s+a)f(s)\nabla s
\end{aligned}$$

The reason the integrand in $(*)$ is 0 is that

$$\begin{cases} t \geq T+1 \text{ and} \\ s \leq T-N \end{cases} \Rightarrow \begin{cases} t \geq T+1 \text{ and} \\ -s \geq -T+N \end{cases} \Rightarrow t-s+a \geq N+1+a,$$

and so, by Fact * again, we have $H_{-(N+1)}(t-s+a) = 0$.

Thus,

$$\begin{aligned} \left(\mathbb{1}_{(T,\infty)} \nabla^N (\mathbb{1}_{(-\infty,T]} f)\right)(t) &= \int_{(T-N,T]} H_{-(N+1)}(t-s+a) f(s) \nabla s \\ &= \int_{\mathbb{Z}_a} H_{-N-1}(t-s+a) \mathbb{1}_{(T-N,T]}(s) f(s) \nabla s \\ &= \left(H_{-N-1} * \mathbb{1}_{(-\infty,T]} f\right)(t) \\ &= \nabla^N (\mathbb{1}_{(T-N,T]} f)(t). \quad \checkmark \end{aligned}$$

So the first equality in (iii) holds. The second equality in (iii) follows from

$$\begin{aligned} \mathbb{1}_{(T,T+N]} \nabla^N (\mathbb{1}_{(T-N,T]} f) &= \mathbb{1}_{(T,\infty)} \underbrace{\mathbb{1}_{(-\infty,T+N]} \nabla^N (\mathbb{1}_{(-\infty,T]}} \mathbb{1}_{(T-N,\infty)} f) \\ &\overset{(ii)}{=} \mathbb{1}_{(T,\infty)} \nabla^N (\mathbb{1}_{(-\infty,T]} \mathbb{1}_{(T-N,\infty)} f) \\ &= \mathbb{1}_{(T,\infty)} \nabla^N (\mathbb{1}_{(T-N,T]} f). \quad \checkmark \end{aligned}$$

This completes the proof. □

Corollary 2.16. *If* $\alpha, \beta \in \mathbb{R}$, *then* $\nabla_a^\alpha \nabla_a^\beta f = \nabla_a^{\alpha+\beta} f = \nabla_a^\beta \nabla_a^\alpha f$.

Proof. Consider

$$\begin{aligned} \nabla_a^\alpha \nabla_a^\beta f &\overset{2.5}{=} \nabla^\alpha(\mathbb{1}_{(a,\infty)}\nabla^\beta(\mathbb{1}_{(a,\infty)}f)) \\ &\overset{\text{Lem 2.15(i)}}{=} \nabla^\alpha(\nabla^\beta(\mathbb{1}_{(a,\infty)}f)) \\ &\overset{\text{Thm 2.14}}{=} \nabla^{\alpha+\beta}(\mathbb{1}_{(a,\infty)}f) \\ &\overset{2.5}{=} \nabla_a^{\alpha+\beta} f \end{aligned}$$

Relabelling α and β in the above argument yields the other equality in the claim. □

Corollary 2.17. *The following composition rules hold for all $\alpha, \gamma \in \mathbb{R}$, $\nu \in [0, \infty)$, $N := \lceil \nu \rceil$, and $k \in \mathbb{Z}$:*

(i) (a) $\nabla^\gamma \nabla_a^\alpha = \nabla_a^{\gamma+\alpha}$

(b) $\nabla_a^\gamma \nabla_a^\alpha = \nabla_a^{\gamma+\alpha}$

(c) $\nabla_{a*}^\nu \nabla_a^\alpha = \nabla_a^{\nu+\alpha}$

(ii) $\nabla_{a*}^\nu \nabla^k = \nabla_{a*}^{\nu+k}$ *(whenever $\nu+k \geq 0$)*

Proof. To see (i)(a) note:

$$\begin{aligned} \nabla^\gamma \nabla_a^\alpha f &\overset{2.5}{=} \nabla^\gamma \nabla^\alpha(\mathbb{1}_{(a,\infty)}f) \\ &\overset{\text{Thm 2.14}}{=} \nabla^{\gamma+\alpha}(\mathbb{1}_{(a,\infty)}f) \\ &\overset{2.5}{=} \nabla_a^{\gamma+\alpha} f \qquad \checkmark \end{aligned}$$

The proof of (i)(b) was given above in Corollary 2.16. ✓

To see (i)(c) note:

$$\begin{aligned}
\nabla_{a*}^{\nu}\nabla_{a}^{\alpha}f &\overset{\text{Defn}}{=} \nabla_{a}^{-(N-\nu)}\nabla^{N}\nabla_{a}^{\alpha}f \\
&\overset{(i)(a)}{=} \nabla_{a}^{-(N-\nu)}\nabla_{a}^{N+\alpha}f \\
&\overset{(i)(b)}{=} \nabla_{a}^{\nu+\alpha} \qquad \surd
\end{aligned}$$

To see (ii) note that, by assumption, we have $0 \leq (\nu+k) \leq \lceil \nu + k \rceil = \lceil \nu \rceil + k = N+k$. So,

$$\begin{aligned}
\nabla_{a*}^{\nu}\nabla^{k}f &\overset{\text{Defn}}{=} \nabla_{a}^{-(N-\nu)}\nabla^{N}\nabla^{k}f \\
&\overset{\text{Thm 2.14}}{=} \nabla_{a}^{-(N-\nu)}\nabla^{N+k}f \\
&= \nabla_{a}^{-(N+k-\nu-k)}\nabla^{N+k}f \\
&= \nabla_{a}^{-(N+k-(\nu+k))}\nabla^{N+k}f \\
&= \nabla_{a}^{-(\lceil \nu+k \rceil-(\nu+k))}\nabla^{N+k}f \\
&\overset{\text{Defn}}{=} \nabla_{a*}^{\nu+k}f \qquad \surd
\end{aligned}$$

□

Part (ii) of the last corollary gives a nice formula for what happens when we first apply an iterated nabla diference and then apply a Caputo fractional difference. Next we look at what happens when we apply an iterated nabla difference and then apply a Reiman-Liousville fractional nabla operator.

Theorem 2.18. *For $f \in \mathbb{R}^{\mathbb{Z}_a}$ and for all $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}_1$:*

$$\nabla_{a}^{\alpha}\nabla^{n}f = \nabla_{a}^{\alpha+n}\left[f - \sum_{k=0}^{n-1}[(\nabla^{k}f)(a)]H_{k}(\cdot,a)\right]. \tag{2.1}$$

Proof. The proof is by induction on n.

Base Case: $(n = 1)$

In this case we have:

$$\begin{aligned}
(\nabla_a^\alpha \nabla^1 f)(t) &= \nabla_a^\alpha \left(\nabla\left(\left[\mathbb{1}_{(a,\infty)} + \mathbb{1}_{(-\infty,a]}\right] f\right)\right) \\
&= \nabla_a^\alpha \nabla^1 \mathbb{1}_{(a,\infty)} f + \nabla_a^\alpha \nabla^1 \mathbb{1}_{(-\infty,a]} f \\
&\overset{\text{Cor 2.5}}{=} \nabla^\alpha (\mathbb{1}_{(a,\infty)} \nabla^1 \mathbb{1}_{(a,\infty)} f) + \nabla^\alpha (\mathbb{1}_{(a,\infty)} \nabla^1 \mathbb{1}_{(-\infty,a]} f) \\
&\overset{\text{Lem 2.15}}{=} \nabla^\alpha (\nabla^1 \mathbb{1}_{(a,\infty)} f) + \nabla^\alpha (\mathbb{1}_{(a,a+1]} \nabla^1 \mathbb{1}_{(a-1,a]} f) \\
&\overset{\text{Thm 2.14}}{=} (\nabla^{\alpha+1} \mathbb{1}_{(a,\infty)} f) + \nabla^\alpha \left(\mathbb{1}_{\{a+1\}}[0 - f(a)]\right) \\
&\overset{\text{Cor 2.5}}{=} (\nabla_a^{\alpha+1} f) + \nabla^\alpha \left([-f(a)] H_{-1}(\cdot, a)\right) \\
&= (\nabla_a^{\alpha+1} f) + \nabla^\alpha \left([-f(a)] \nabla H_0(\cdot, a)\right) \\
&= (\nabla_a^{\alpha+1} f) + \nabla^\alpha \left(\nabla [-f(a)] H_0(\cdot, a)\right) \\
&= (\nabla_a^{\alpha+1} f) + \nabla^{\alpha+1} \left([-f(a)] \mathbb{1}_{(a,\infty)} H_0(\cdot, a)\right) \\
&= \nabla_a^{\gamma+1} \left[f - [f(a)] H_0(\cdot, a)\right] \quad \checkmark
\end{aligned}$$

With the base case established, we proceed by induction.

Induction Step:

Assume the claim holds for some $n_0 \in \mathbb{N}$ and consider:

$$\begin{aligned}
(\nabla_a^\alpha \nabla^{n_0+1} f)(t) &= (\nabla_a^\alpha \nabla^{n_0} \nabla^1 f)(t) = (\nabla_a^\alpha \nabla^{n_0} (\nabla f))(t) \\
&= (\nabla_a^{\alpha+n_0} (\nabla f))(t) - \nabla^\alpha \sum_{k=0}^{n_0-1} [(\nabla^k (\nabla f))(a)] \nabla_a^{n_0-k} H_0(t, a) \\
&= (\nabla_a^{\alpha+n_0+1} f)(t) - \nabla^{\alpha+n_0} [f(a)] \nabla_a^1 H_0(t, a) \\
&\qquad - \nabla^\alpha \sum_{k=0}^{n_0-1} [(\nabla^k (\nabla f))(a)] \nabla_a^{n_0-k} H_0(t, a)
\end{aligned}$$

$$= (\nabla_a^{\alpha+n_0+1} f)(t) - \nabla^\alpha [f(a)] \nabla_a^{n_0+1} H_0(t,a)$$
$$- \nabla^\alpha \sum_{k'=1}^{n_0} [(\nabla^{k'} f)(a)] \nabla_a^{n_0-k'+1} H_0(t,a)$$
$$= (\nabla_a^{\alpha+(n_0+1)} f)(t) - \nabla^\alpha \sum_{k'=0}^{(n_0+1)-1} [(\nabla^{k'} f)(a)] \nabla_a^{(n_0+1)-k'} H_0(t,a) \quad \surd$$

Thus, the claim holds for all $n \in \mathbb{N}$. □

Theorem 2.19. *Suppose $f : \mathbb{Z}_a \to \mathbb{R}$. Then, for all $\alpha, \gamma \in \mathbb{R}$, $\mu, \nu \in [0, \infty)$, $M := \lceil \mu \rceil$, $N := \lceil \nu \rceil$, and $m, n \in \mathbb{N}_0$ we have the following* [4]*:*

(i) *(a)* $\nabla^\gamma \nabla_a^\alpha = \nabla_a^{\gamma+\alpha}$

(b) $\nabla_a^\gamma \nabla_a^\alpha = \nabla_a^{\gamma+\alpha}$

(c) $\nabla_{a*}^\mu \nabla_a^\alpha = \nabla_a^{\mu+\alpha}$

(ii) *(a)* $\nabla^\gamma \nabla^n = \nabla^{\gamma+n}$

(b) $\nabla_a^\gamma \nabla^n f = \nabla_a^{\gamma+n} \left[f - \sum_{k=0}^{n-1} [(\nabla^k f)(a)] H_k(\cdot, a) \right]$

(c) $\nabla_{a*}^\mu \nabla^n = \nabla_{a*}^{\mu+n}$

(iii) *(a)* $\nabla^\gamma \nabla_{a*}^\nu f = \nabla_a^{\gamma+\nu} \left[f - \sum_{k=0}^{N-1} [(\nabla^k f)(a)] H_k(\cdot, a) \right]$

(b) $\nabla_a^\gamma \nabla_{a*}^\nu f = \nabla^\gamma \nabla_{a*}^\nu f$

(c) $\nabla_{a*}^\mu \nabla_{a*}^\nu f = \nabla^\mu \nabla_{a*}^\nu f$

Proof. Equations (i)(a)-(c) are just a restatement of the facts proven in Corollary 2.17 (i)(a)-(c), (ii)(a) is a special case of Theorem 2.14, and (ii)(b) is the result of theorem 2.18, and (ii)(c) is a restatement of the fact given in Corollary 2.17 (ii).

[4]Setting $\gamma = 0$ in (iii)(a) gives us a formula for the Caputo fractional difference in terms of the Riemann-Liousville fractional difference.

To see (iii)(a) note:

$$\begin{aligned}\nabla^{\gamma}\nabla^{\nu}_{a*}f &= \nabla^{\gamma}\nabla^{\nu-N}_{a}\nabla^{N}f\\ &\overset{(ii)(b)}{=} \nabla^{\gamma}\nabla^{\nu}_{a}\left[f-\sum_{k=0}^{N-1}[(\nabla^{k}f)(a)]H_{k}(\cdot,a)\right]\\ &\overset{(i)(a)}{=} \nabla^{\gamma+\nu}_{a}\left[f-\sum_{k=0}^{N-1}[(\nabla^{k}f)(a)]H_{k}(\cdot,a)\right]. \quad \checkmark\end{aligned}$$

Note that if we set $\gamma = 0$ in (iii)(a) we get:

$$\nabla^{\nu}_{a*}f = \nabla^{\nu}_{a}\left[f-\sum_{k=0}^{N-1}[(\nabla^{k}f)(a)]H_{k}(\cdot,a)\right] \tag{2.2}$$

We can obtain (iii)(b) by applying ∇^{α}_{a} to both sides of equation (2.2) and then using (i)(b) and (iii)(a). That is,

$$\begin{aligned}\nabla^{\gamma}_{a}\nabla^{\nu}_{a*}f &= \nabla^{\gamma}_{a}\nabla^{\nu}_{a}\left[f-\sum_{k=0}^{N-1}[(\nabla^{k}f)(a)]H_{k}(\cdot,a)\right]\\ &\overset{(i)(b)}{=} \nabla^{\gamma+\nu}_{a}\left[f-\sum_{k=0}^{N-1}[(\nabla^{k}f)(a)]H_{k}(\cdot,a)\right] \overset{(iii)(a)}{=} \nabla^{\gamma}\nabla^{\nu}_{a*}f.\end{aligned}$$

Similiarly, we can obtain (iii)(c) by applying ∇^{α}_{a*} to both sides of equation 2.2 and then using (i)(c) and (iii)(a). That is,

$$\begin{aligned}\nabla^{\gamma}_{a*}\nabla^{\nu}_{a*}f &= \nabla^{\gamma}_{a*}\nabla^{\nu}_{a}\left[f-\sum_{k=0}^{N-1}[(\nabla^{k}f)(a)]H_{k}(\cdot,a)\right]\\ &\overset{(i)(c)}{=} \nabla^{\gamma+\nu}_{a}\left[f-\sum_{k=0}^{N-1}[(\nabla^{k}f)(a)]H_{k}(\cdot,a)\right] \overset{(iii)(a)}{=} \nabla^{\gamma}\nabla^{\nu}_{a*}f.\end{aligned}$$

$\square$

2.4 The Difference between the Riemann-Liouville and the Caputo Fractional Difference Operators

In this section we take a closer look at how the Caputo fractional difference and the Riemann-Liouville fractional difference differ. The next definition makes it easier to state the main theorem of this section (Theorem 2.21).

Definition 2.20. *For all $f \in V$ and all $N \in \mathbb{N}_1$, we define $\widehat{N} : V \to V$ by*

$$\begin{aligned}\widehat{N}(f) &:= \mathbb{1}_{(a,a+N]}(\nabla^N(f\mathbb{1}_{(a-N,a]}))\\ &= \begin{cases} 0, & \textit{for } t \leq a \\ (\nabla^N f\mathbb{1}_{(a-N,a]})(t), & \textit{for } t \in \mathbb{N}_{a+1}^{a+N} \\ 0, & \textit{for } t \geq a+N+1 \end{cases}\end{aligned}$$

and refer to $\widehat{N}(f)$ as the ***N sequence derived from*** *f. Note that $\widehat{N}f$ has, at most, N nonzero values.*

Example 2.1 For $N = 3$ we have the 3 sequence derived from f,

$$\begin{aligned}\widehat{3}(f) &:= \mathbb{1}_{(a,a+3]}(\nabla^3(f\mathbb{1}_{(a-3,a]}))\\ &= \begin{cases} 0, & \text{for } t \leq a \\ (\nabla^3 f\mathbb{1}_{(a-3,a]})(a+1), & \text{for } t = a+1 \\ (\nabla^3 f\mathbb{1}_{(a-3,a]})(a+2), & \text{for } t = a+2 \\ (\nabla^3 f\mathbb{1}_{(a-3,a]})(a+3), & \text{for } t = a+3 \\ 0, & \text{for } t \geq a+4 \end{cases}\end{aligned}$$

$$= \begin{cases} 0, & \text{for } t \le a \\ -3f(a) + 3f(a-1) - f(a-2), & \text{for } t = a+1 \\ 3f(a) - f(a-1), & \text{for } t = a+2 \\ -f(a), & \text{for } t = a+3 \\ 0, & \text{for } t \ge a+4. \end{cases}$$

Theorem 2.21. *For all $\nu > 0$ and $f \in V$,*

$$\nabla_{a*}^{v} f = \nabla_a^{v} f + H_{\epsilon-1} * \widehat{N} f$$

where $N := \lceil \nu \rceil$ and $\epsilon := N - \nu$ and $\widehat{N}$ is as defined above.

Proof. Consider

$$\begin{aligned}
\nabla_{a*}^{\nu} f &\overset{\text{Definition}}{=} \nabla_a^{-(N-\nu)} \nabla^N f \\
&= \nabla^{-\epsilon} \mathbb{1}_{(a,\infty)} \nabla^N (\mathbb{1}_{(-\infty,a]} f + \mathbb{1}_{(a,\infty)} f) \\
&= \nabla^{-\epsilon} \mathbb{1}_{(a,\infty)} \nabla^N (\mathbb{1}_{(-\infty,a]} f) + \nabla^{-\epsilon} \mathbb{1}_{(a,\infty)} \nabla^N (\mathbb{1}_{(a,\infty)} f) \\
&= \nabla^{-\epsilon} \mathbb{1}_{(a,\infty)} \nabla^N (\mathbb{1}_{(-\infty,a]} f) + \nabla^{-\epsilon} \nabla^N (\mathbb{1}_{(a,\infty)} f) \\
&= \nabla^{-\epsilon} \mathbb{1}_{(a,\infty)} \nabla^N (\mathbb{1}_{(-\infty,a]} f) + \nabla^{\nu} (\mathbb{1}_{(a,\infty)} f) \\
&= \nabla^{-\epsilon} \mathbb{1}_{(a,a+N]} \nabla^N (\mathbb{1}_{(-\infty,a]} f) + \nabla_a^{\nu} f \\
&= \nabla^{-\epsilon} \mathbb{1}_{(a,a+N]} \nabla^N (\mathbb{1}_{(a-N,a]} f) + \nabla_a^{\nu} f \\
&\overset{\text{Definitions}}{=} H_{\epsilon-1} * \widehat{N} f + \nabla_a^{\nu} f.
\end{aligned}$$

□

Corollary 2.22. *For all* $\nu > 0$ *and* $f \in V$ *,*

$$(\nabla^{\nu}_{a*} f - \nabla^{\nu}_{a} f)(t) \to 0, \text{ as } t \to \infty \ .$$

Proof. This corollary follows from the fact that $\widehat{N}f(t) = 0$ for all but at most N values of t and the fact that $H_{\epsilon-1}(t) \to 0$ as $t \to \infty$. (We have deferred the proof of this second fact until chapter 3 (Thm 3.5).) To see that the claim holds put $M := \max\limits_{t \in [a+1, a+N]} |(\widehat{N}f)(t)|$. Then, from Theorem 2.21, we have:

$$|(\nabla^{v}_{a*} f - \nabla^{v}_{a} f)(t)| = |(H_{\epsilon-1} * \widehat{N}f)(t)| \underset{(\text{for } t \geq a+N)}{\leq} N \cdot M \cdot |H_{\epsilon-1}(t-N)| \underset{(\text{as } t \to \infty)}{\to} 0.$$

□

The following corollaries are immediate consequences of Theorem 2.21.

Corollary 2.23. *If* $\widehat{N}f \equiv 0$ *(or equivalently, if* $f(t) = 0$ *for all* $t \in (a - N, a]$*), then*

$$\nabla^{\nu}_{a*} f = \nabla^{\nu}_{a} f \ .$$

Corollary 2.24. *If* $f \equiv c$ *for some* $c \in \mathbb{R}$*, then*

$$\nabla^{\nu}_{a} f = -H_{\epsilon-1} * \widehat{N}f \ .$$

Corollary 2.25. *If* $\epsilon = 0$ *(or equivalently, if* $\nu = N$*), then*

$$\nabla^{\nu}_{a*} f = \nabla^{\nu}_{a} f + \widehat{N}f \ .$$

Chapter 3

More about Taylor Monomials and Pochhammer Polynomials

3.1 Additional Properties of Pochhammer Polynomials

In this brief section we provide an example that deomonstrates how it is sometimes advantageous to work with the Pochhammer polynomials instead of working directly with the Taylor monomials. Compare the statement and proof of the following theorem with that given in Theorem 3.96 from [5].

Theorem 3.1. *For all $m \in \mathbb{N}_0$, $d \in \mathbb{Z}$, and $\tau \in \mathbb{R}$, $\nabla^m P_d(\tau) = P_{d-m}(\tau)$.*

Proof. **Step 1:** The claim holds when $m = 1$ for all $d \in \mathbb{Z}$ and all $\tau \in \mathbb{R}$.
First, note that if $d \in \mathbb{N}_1$, then $\nabla P_d(\tau) = P_{d-1}(\tau)$, for all $\tau \in \mathbb{R}$ since

$$\begin{aligned}
\nabla P_d(\tau) &= P_d(\tau) - P_d(\tau - 1) \\
&= \frac{(\tau+1)(\tau+2)\ldots(\tau+d)}{d!} - \frac{(\tau-1+1)(\tau-1+2)\ldots(\tau-1+d)}{d!} \\
&= \frac{(\tau+1)(\tau+2)\ldots(\tau+d-1)}{(d-1)!\cdot d}\left[\frac{(\tau+d)-(\tau)}{1}\right] \\
&= \frac{(\tau+1)(\tau+2)\ldots(\tau+d-1)}{(d-1)!}\cdot\frac{d}{d} \\
&= P_{d-1}(\tau).
\end{aligned}$$

Second, note that if $d = 0$, the claim holds, since in this case we have $P_0 \equiv 1$ which implies $\nabla P_0(\tau) \equiv 0 \equiv P_{-1}(\tau)$.

Third, note that if $d = -k \in -\mathbb{N}_1$, the claim also holds, since in this case we have $P_{-k} \equiv 0$ which implies $\nabla P_{-k}(\tau) \equiv 0 \equiv P_{-k-1}(\tau)$.

Conclusion: The formula $\nabla P_d(\tau) = P_{d-1}(\tau)$ holds for all $d \in \mathbb{Z}$ and $\tau \in \mathbb{R}$.

Step 2: Fix $d_0 \in \mathbb{Z}$ and proceed by induction on m. For $m = 0$, the claim is true (since ∇^0 is the identity operator). Also, we have shown in Step 1 that the claim holds for d_0 when $m = 1$. Now, suppose $M \in \mathbb{N}_0$ for which the claim holds when $d = d_0$. That is,

$$\nabla^M P_{d_0}(\tau) \quad = \quad P_{d_0-M}(\tau), \text{ for all } \tau \in \mathbb{R} \; .$$

Taking the nabla difference of both sides of this equation yields

$$\begin{aligned} \nabla\nabla^M P_{d_0}(\tau) \quad &= \quad \nabla P_{d_0-M}(\tau) \\ \nabla^{M+1} P_{d_0}(\tau) \quad &\underset{\text{since } (d_0-M)\in\mathbb{Z}}{\overset{\text{(by the above)}}{=}} \quad P_{(d_0-M)-1}(\tau) \\ &= \quad P_{d_0-(M+1)}(\tau) \end{aligned}$$

Thus, by induction, the claim holds for all $m \in \mathbb{N}_0$ (when $d = d_0$). But since d_0 was arbitrary in $\mathbb{Z}$, we have that the claim holds for all $d \in \mathbb{Z}$, $m \in \mathbb{N}$, and $\tau \in \mathbb{R}$. $\square$

By setting $\tau = \nu - N$ and $d = t - a - 1$, the following calculation shows Theorem 3.1 implies and is a bit more general than Theorem 3.96 in [5] as there are no restrictions on ν or N.

Corollary 3.2. *For $\nu, N \in \mathbb{R}$, $m \in \mathbb{N}$, and $t \in \mathbb{Z}_a$*

$$H_{\nu-N}(t, a+m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} H_{\nu-N-k}(t, a) \ .$$

Proof. Put $d = d(t, a+1) = t - a - 1 \in \mathbb{Z}$. From Theorem 3.1and Corollary 1.12,

$$\begin{aligned}
P_{d-m}(\nu - N) &= \nabla^m P_d(\nu - N) \\
P_{d-m}(\nu - N) &= \sum_{k=0}^{m} (-1)^k \binom{m}{k} P_d(\nu - N - k) \\
H_{\nu-N}(d - m + 1 + a, a) &= \sum_{k=0}^{m} (-1)^k \binom{m}{k} H_{\nu-N-k}(d + 1 + a, a) \\
H_{\nu-N}(\underbrace{t - a - 1}_{d} - m + 1 + a, a) &= \sum_{k=0}^{m} (-1)^k \binom{m}{k} H_{\nu-N-k}(\underbrace{t - a - 1}_{d} + 1 + a, a) \\
H_{\nu-N}(t, a+m) &= \sum_{k=0}^{m} (-1)^k \binom{m}{k} H_{\nu-N-k}(t, a).
\end{aligned}$$

□

Remark Recall $H_\gamma(t, s) = \widetilde{H}_\gamma(t, s)$ if $\gamma \notin \mathbb{N}_0$ or if $t > s$. (See Corollary 1.13 and/or Definition 1.7.) Thus, if $\nu - N \notin \mathbb{N}_0$ (which implies $\nu - N - k \notin \mathbb{N}_0$ for all $k \in \mathbb{N}_0^m$) or if $d \geq m$ (which is equivalent to having $t \geq m + a + 1$), we also have

$$\widetilde{H}_{\nu-N}(t, a+m) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \widetilde{H}_{\nu-N-k}(t, a),$$

The following fact is often used to simplify sums of the given form.

Corollary 3.3. *For $n \in \mathbb{N}_0$ and $s, t \in \mathbb{Z}_a$ with $t > s$,*

$$\sum_{k=0}^{n} H_k(t, s) = H_n(t+1, s).$$

Proof. The proof follows from Corollary 1.12, the Fundamental Theorem of Nabla

Calculus, and Theorem 3.1.

$$\begin{aligned}\sum_{k=0}^{n} H_k(t,s) = \sum_{k=0}^{n} P_{t-s-1}(k) &= \int_{(-1,n]} P_{t-s-1}(k)\nabla k \\ &= \int_{(-1,n]} (\nabla P_{t-s})(k)\nabla k \\ &= P_{t-s}(n) - \underbrace{P_{t-s}(-1)}_{=0 \text{ when } t\neq s} \\ &= H_n(t+1,s).\end{aligned}$$

□

This result extends to sums of extended Taylor monomials of this form.

Corollary 3.4. *For $n \in \mathbb{N}_0$ and $s, t \in \mathbb{Z}_a$,*

$$\sum_{k=0}^{n} \widetilde{H}_k(t,s) = \widetilde{H}_n(t+1,s).$$

Proof. If $t > s$, $\widetilde{H}_k(t,s) = H_k(t,s)$ for all k and the result follows from Corollary 3.3. If $t = s$,

$$\sum_{k=0}^{n} \widetilde{H}_k(t,s) = \underbrace{\widetilde{H}_0(s,s)}_{=1} + \sum_{k=1}^{n} \underbrace{\widetilde{H}_k(s,s)}_{=0} = \widetilde{H}_n(s+1,s).$$

If $t < s$, then $t < t+1 \leq s$ and so by Corollary 1.11 we have $\widetilde{H}_k(t,s) = P_k(t-s-1)$ and $\widetilde{H}_n(t+1,s) = P_n(t+1-s-1) = P_n(t-s)$. Thus, by the Log Rule and the

Fundamental Theorem of Nabla Calculus,

$$\begin{aligned}
\sum_{k=0}^{n} \widetilde{H}_k(t,s) &= \sum_{k=0}^{n} P_k(t-s-1) \overset{\text{Cor 1.12}}{=} \sum_{k=0}^{n} H_{t-s-1}(a+k+1,a) \\
&= \int_{(a,a+n+1]} H_{t-s-1}(\tau,a)\nabla\tau \\
&= H_{t-s}(\tau,a)\Big|_{a}^{a+n+1} = H_{t-s}(a+n+1,a) - \underbrace{H_{t-s}(a,a)}_{=0} \\
&= P_n(t-s) \\
&= \widetilde{H}_n(t+1,s).
\end{aligned}$$

□

3.2 Additional Properties of Taylor Monomials

Theorem 3.5. *For fixed* $\nu > 0$*,* $H_{-\nu}(t,a) \to 0$ *as* $t \to \infty$*.*

Proof. By a theorem (see Rudin [6], Theorem 15.5, page 322), whenever $0 \leq u_n < 1$,

$$\sum_{n=1}^{\infty} u_n < \infty \quad \text{if and only if} \quad \prod_{n=1}^{\infty} (1-u_n) > 0.$$

So, by the contrapositive of this theorem, we have that

$$\sum_{n=1}^{\infty} u_n = \infty \quad \text{if and only if} \quad \prod_{n=1}^{\infty} (1-u_n) = 0.$$

Let $N := \lceil \nu \rceil$ and put $0 \leq u_n := \frac{\nu}{N+n} < 1$ for all $n \in \mathbb{N}_1$. Thus,

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{\nu}{N+n} = \sum_{i=N+1}^{\infty} \frac{\nu}{i} = \nu \left(\sum_{i=N+1}^{\infty} \frac{1}{i} \right) = \infty$$

which therefore implies

$$0 = \prod_{n=1}^{\infty} (1 - u_n) = \prod_{n=1}^{\infty} \left(\frac{N+n}{N+n} - \frac{\nu}{N+n} \right) = \prod_{n=1}^{\infty} \frac{N+n-\nu}{N+n} = \prod_{j=N+1}^{\infty} \frac{j-\nu}{j}.$$

Hence,

$$\prod_{j=1}^{\infty} \frac{j-\nu}{j} = \left(\prod_{j=1}^{N} \frac{j-\nu}{j} \right) \underbrace{\left(\prod_{j=N+1}^{\infty} \frac{j-\nu}{j} \right)}_{0} = 0.$$

And so,

$$\begin{aligned} 0 &= \prod_{j=1}^{\infty} \frac{j-\nu}{j} = \lim_{k\to\infty} \prod_{j=1}^{k} \frac{j-\nu}{j} \\ &= \lim_{k\to\infty} \left(\frac{1-\nu}{1} \right) \left(\frac{2-\nu}{2} \right) \cdots \left(\frac{k-\nu}{k} \right) \\ &\overset{\text{Def 1.3}}{=} \lim_{k\to\infty} P_k(-\nu) \\ &\overset{\text{Cor 1.12}}{=} \lim_{k\to\infty} H_{-\nu}(a+1+k, a) \overset{\text{t:=(a+1+k)}}{=} \lim_{t\to\infty} H_{-\nu}(t, a). \end{aligned}$$

That is, $\lim_{t\to\infty} H_{-\nu}(t, a) = 0$. □

Theorem 3.6. *For fixed $\nu > 0$, $H_\nu(t, a) \to \infty$ as $t \to \infty$.*

Proof. By a well known result (see Apostol [7], Theorem 8.52, page 208), whenever $a_n > 0$,

$$\prod_{n=1}^{\infty} (1 + a_n) < \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty} a_n < \infty.$$

Put $t := a + k$ where $k \in \mathbb{N}_1$ and notice

$$\begin{aligned} H_\nu(t, a) = H_\nu(a+k, a) = P_{k-1}(\nu) &= \left(\frac{\nu+1}{1} \right) \left(\frac{\nu+2}{2} \right) \cdots \left(\frac{\nu+k-1}{k-1} \right) \\ &= \left(1 + \frac{\nu}{1} \right) \left(1 + \frac{\nu}{2} \right) \cdots \left(1 + \frac{\nu}{k-1} \right). \end{aligned}$$

So,

$$\begin{aligned}\lim_{t\to\infty} H_\nu(t,a) &= \lim_{k\to\infty} H_\nu(a+k,a)\\ &= \lim_{k\to\infty}\left(1+\frac{\nu}{1}\right)\left(1+\frac{\nu}{2}\right)\cdots\left(1+\frac{\nu}{k-1}\right)\\ &= \lim_{k\to\infty}\prod_{n=1}^{k-1}\left(1+\frac{\nu}{n}\right)\\ &= \prod_{n=1}^{\infty}\left(1+\frac{\nu}{n}\right).\end{aligned}$$

Thus,

$$\lim_{t\to\infty} H_\nu(t,a) = \prod_{n=1}^{\infty}\left(1+\frac{\nu}{n}\right) < \infty \quad \text{if and only if} \quad \sum_{n=1}^{\infty}\left(\frac{\nu}{n}\right) < \infty.$$

But, $\sum_{n=1}^{\infty}\left(\frac{\nu}{n}\right) = \nu\left(\sum_{n=1}^{\infty}\frac{1}{n}\right) = \infty$, and so $\lim_{t\to\infty} H_\nu(t,a) = \prod_{n=1}^{\infty}\left(1+\frac{\nu}{n}\right) = \infty$, as well.

□

Corollary 3.7. *For $\nu > 0$, $H_\nu(\cdot,a)$ is strictly increasing and without bound on $\mathbb{N}_a$.*

Proof. Note that for $d \in \mathbb{N}_0$, $P_d(x) = \frac{1}{d!}(x+1)(x+2)\cdots(x+d) > 0$ whenever $x > -1$. Thus, by the Log Rule (Theorem 1.16) and Corollary 1.12,

$$\nabla H_\nu(t,a) = H_{\nu-1}(t,a) = P_{t-a-1}(\nu-1) > 0 \quad \text{whenever } \nu > 0 \text{ and } t \in \mathbb{N}_{a+1}.$$

So, by Theorem 1.2, $H_\nu(\cdot,a)$ is strictly increasing on $[a,\infty)\cap\mathbb{Z}_a = \mathbb{N}_a$. The previous theorem shows that this increase is without bound. □

Theorem 3.8. *(P-Integrals)*

$$\int_a^\infty H_{-p}(t,a)\nabla t = \begin{cases} 0 \ , \ \textit{whenever } p > 1 \\ 1 \ , \ \textit{when } p = 1 \\ \infty \ , \ \textit{whenever } p < 1. \end{cases}$$

Proof. The proof follows from Theorems 3.5 and 3.6 and the Fundamental Theorem of Nabla Calculus.

$$\begin{aligned} \int_a^\infty H_{-p}(t,a)\nabla t &= \lim_{b\to\infty} \int_a^b H_{-p}(t,a)\nabla t \\ &= \lim_{b\to\infty} \int_a^b \nabla H_{-p+1}(t,a)\nabla t \\ &= \lim_{b\to\infty} \left(H_{-p+1}(b,a) - \underbrace{H_{-p+1}(a,a)}_{=0} \right) \\ &= \begin{cases} 0 \ , \ \text{whenever } \ -p+1 < 0 \\ 1 \ , \ \text{when } \ -p+1 = 0 \\ \infty \ , \ \text{whenever } \ -p+1 > 0. \end{cases} \end{aligned}$$

□

Theorem 3.9. *For $0 < \nu \notin \mathbb{N}_1$ and $t \in \mathbb{Z}_a$,*

$$\left(\nabla |H_{-\nu}(\cdot,a)|\right)(t) \begin{cases} = 0 \ \textit{when } t \leq a \\ > 0 \ \textit{when } a < t \leq a + \lfloor \frac{\nu}{2} \rfloor + 1 \\ < 0 \ \textit{when } a + \lfloor \frac{\nu}{2} \rfloor + 1 < t. \end{cases} \tag{3.1}$$

For $\nu = n \in \mathbb{N}_1$ *and* $t \in \mathbb{Z}_a$,

$$(\nabla |H_{-\nu}(\cdot, a)|)(t) \begin{cases} = 0 \ \textit{when } t \leq a \\ > 0 \ \textit{when } a < t < a + \frac{n}{2} + 1 \\ = 0 \ \textit{when } t = a + \frac{n}{2} + 1 \\ < 0 \ \textit{when } a + \frac{n}{2} + 1 < t \leq a + n + 1 \\ = 0 \ \textit{when } a + n + 1 < t. \end{cases} \tag{3.2}$$

Proof. Recall that

$$\begin{aligned}(\nabla |H_{-\nu}(\cdot, a)|)(t) &= |H_{-\nu}(t, a)| - |H_{-\nu}(t-1, a)| \\ &= |P_{t-a-1}(-\nu)| - |P_{t-1-a-1}(-\nu)|.\end{aligned}$$

Clearly the claim is true when $t \leq a$, since by Theorem 1.8, $|H_{-\nu}(t, a)| = 0$ whenver $t \leq a$. Also, if $t = a + 1$, the claim holds since in this case we have

$$a < a + 1 = t < a + \frac{\nu}{2} + 1 \quad \text{and}$$

$$(\nabla |H_{-\nu}(\cdot, a)|)(a+1) = |H_{-\nu}(a+1, a)| - |H_{-\nu}(a, a)| = 1 - 0 = 1 > 0. \quad \checkmark$$

Next, we consider the case $t = a+2$. In order to show the claim holds in this situation we must show:

$$(\nabla|H_{-\nu}(\cdot,a)|)(a+2)\begin{cases}<0 \text{ when } \nu<2\\ =0 \text{ when } \nu=2\\ >0 \text{ when } \nu>2.\end{cases}$$

Doing the computation we get:

$$\begin{aligned}(\nabla|H_{-\nu}(\cdot,a)|)(a+2) &= |H_{-\nu}(a+2,a)|-|H_{-\nu}(a+1,a)|\\ &= |P_1(-\nu)|-|P_0(-\nu)|\\ &= |-\nu+1|-1=|1-\nu|-1\\ &\begin{cases}<0 \text{ when } 0<\nu<2\\ =0 \text{ when } \nu=2\\ <0 \text{ when } 2<\nu, \qquad\qquad \text{as desired.}\end{cases}\end{aligned}$$

Finally, for $t\geq a+3$,

$$\begin{aligned}(\nabla|H_{-\nu}(\cdot,a)|)(t) &= |H_{-\nu}(t,a)|-|H_{-\nu}(t-1,a)|\\ &= |P_{t-a-1}(-\nu)|-|P_{t-1-a-1}(-\nu)|\\ &= \left.\frac{|x+1|\cdot|x+2|\cdots|x+(t-a-2)|\cdot|x+(t-a-1)|}{(t-a-1)!}\right|_{x=-\nu}\\ &\quad -\left.\frac{|x+1|\cdot|x+2|\cdots|x+(t-a-2)|}{(t-a-2)!}\cdot\frac{(t-a-1)}{(t-a-1)}\right|_{x=-\nu}\\ &= \underbrace{\left[\frac{|-\nu+1|\cdots|-\nu+(t-a-2)|}{(t-a-1)!}\right]}_{I}\cdot\underbrace{\left[\Big|-\nu+(t-a-1)\Big|-(t-a-1)\right]}_{II}.\end{aligned}$$

Note that the first factor, I, is always greater than or equal to 0, while the second factor , II, is $\begin{cases} > 0 \text{ when } t-a-1 < \frac{\nu}{2} \\ = 0 \text{ when } t-a-1 = \frac{\nu}{2} \\ < 0 \text{ when } t-a-1 > \frac{\nu}{2}. \end{cases}$

Reason: To see this quickly consider, for $\nu > 0$, the graphs of the real-valued functions $f(x) = |x-\nu|$ (the absolute value function shifted ν units to the right) and $g(x) = x$. It is clear that $f(x) - g(x) \begin{cases} > 0 \text{ when } x < \frac{\nu}{2} \\ = 0 \text{ when } x = \frac{\nu}{2} \\ < 0 \text{ when } x > \frac{\nu}{2}. \end{cases}$ So, setting $x = (t-a-1)$ gives us that II is positive, 0, or negative when $(t-a-1)$ is less than, equal to, or greater than $\frac{\nu}{2}$, respectively. Thus, the second term, II, is $\begin{cases} > 0 \text{ when } t < a + \frac{\nu}{2} + 1 \\ = 0 \text{ when } t = a + \frac{\nu}{2} + 1 \\ < 0 \text{ when } t > a + \frac{\nu}{2} + 1. \end{cases}$

And so, in this case (i.e. for $a + 3 \leq t$) we have

$$(\nabla |H_{-\nu}(\cdot, a)|)(t) \begin{cases} \geq 0 \text{ when } t < a + \frac{\nu}{2} + 1 \\ = 0 \text{ when } t = a + \frac{\nu}{2} + 1 \\ \leq 0 \text{ when } a + \frac{\nu}{2} + 1 < t. \end{cases}$$

Summarizing, so far we have shown that for $0 < \nu$ and $t \in \mathbb{Z}_a$,

$$(\nabla |H_{-\nu}(\cdot, a)|)\,(t) \begin{cases} = 0 \text{ when } t \leq a \\ \geq 0 \text{ when } a < t < a + \frac{\nu}{2} + 1 \\ = 0 \text{ when } t = a + \frac{\nu}{2} + 1 \\ \leq 0 \text{ when } a + \frac{\nu}{2} + 1 < t. \end{cases} \tag{3.3}$$

Note that for $0 < \nu \notin \mathbb{N}_1$, since $t \in \mathbb{Z}_a$, $t \neq a + \frac{\nu}{2} + 1$. Furthermore, the inequalities in equation (3.3) are strict in this case since $\nu \notin \mathbb{N}_1$ implies I is strictly positive and also that II is nonzero (as $t \in \mathbb{Z}_a$ implies $(t - a - 1) \in \mathbb{Z}$ and so $t - a - 1 \neq \frac{\nu}{2}$). The statement in (3.1) follows.

If $\nu = n \in \mathbb{N}_1$, I is 0 for all $t \geq n + a + 2$ and strictly positive when $t \leq n + a + 1$. Also, II is only 0 when $t = \frac{n}{2} + a + 1 < n + a + 1$. Thus, the statement in (3.2) follows. □

Definition 3.10. *For $\nu > 0$, we put $N := \lceil \nu \rceil$ and define*

$$t_\nu := \sup\{t \in \mathbb{Z}_a : t \leq a + N \text{ and } (\nabla |H_{-\nu}(\cdot, a)|)(t) \geq 0\}.$$

Corollary 3.11. *For all $\nu > 0$,*

$$t_\nu = a + \lfloor \frac{\nu}{2} \rfloor + 1.$$

$|H_{-\nu}(\cdot, a)|$ is increasing on $(-\infty, t_\nu]$ and decreasing on $[t_\nu, \infty)$. Therefore, [1]

$$\|H_{-\nu}(\cdot, a)\|_\infty = |H_{-\nu}(t_\nu, a)| = \left| P_{\lfloor \nu/2 \rfloor}(-\nu) \right| = \begin{cases} 1, & \text{if} \quad 0 < \nu < 2 \\ \left| \frac{(\nu-1)\cdots(\nu - \lfloor v/2 \rfloor)}{\lfloor v/2 \rfloor !} \right|, & \text{if} \quad 2 \leq \nu. \end{cases}$$

Furthermore, if:

- $\nu \in (0, \infty) \setminus \mathbb{N}_1$, *then $|H_{-\nu}(\cdot, a)|$ is strictly increasing on $[a, t_\nu]$ and strictly decreasing on $[t_\nu, \infty)$,*
- $\nu = n \in \mathbb{N}_1 \setminus (2\mathbb{N}_1)$, *then $|H_{-n}(\cdot, a)|$ is strictly increasing on $[a, t_\nu]$ and strictly decreasing on $[t_\nu, a + n + 1]$,*

[1] Interesting fact: The function $f(\nu) := \|H_{-\nu}(\cdot, a)\|_\infty$ is continuous and increasing on $[0, \infty)$.

- *$\nu = n \in 2\mathbb{N}_1$, then $|H_{-n}(\cdot, a)|$ is strictly increasing on $[a, t_\nu - 1]$ and strictly decreasing on $[t_\nu, a + n + 1]$.*

Thus, for any $0 < \nu$, $|H_{-\nu}(\cdot, a)|$ is always strictly increasing on $[a, t_\nu - 1]$ and always strictly decreasing on $[t_\nu, a + N + 1]$.

Proof. Recall that if $(\nabla f)(t) \geq 0$ on $[c + 1, d]$, then f is increasing on $[c, d]$ and that similar statements hold to give us intervals on which f is strictly increasing, decreasing, and strictly decreasing. (See Theorem 1.2.) Thus, the formula for t_ν and the statements concerning the monotonicity/strict monotonicity of $|H_{-\nu}(\cdot, a)|$ are an immediate consequence of the previous theorem. The formula for $\|H_{-\nu}(\cdot, a)\|_\infty$ follows from

$$|H_{-\nu}(t_\nu, a)| = \left|H_{-\nu}(a + \lfloor \frac{\nu}{2} \rfloor + 1, a)\right| = \left|P_{\lfloor \nu/2 \rfloor}(-\nu)\right|.$$

□

A second important time associated with the function $|H_{-\nu}(\cdot, a)|$ is the latest time for which $|H_{-\nu}(t, a)| \geq 1$.

Definition 3.12. *For $\nu > 0$, we define*

$$T_\nu := \sup\{t \in \mathbb{Z}_a : |H_{-\nu}(t, a)| \geq 1\}.$$

Note that by Corollary 3.11, $|H_{-\nu}(\cdot, a)|$ is increasing on $(-\infty, t_\nu]$ and $a + 1 \leq t_\nu$, thus we have $1 = |H_{-\nu}(a + 1, a)| \leq |H_{-\nu}(t_\nu, a)|$. Also, by Corollary 3.11 and Theorem 3.5, $|H_{-\nu}(\cdot, a)|$ decreases to 0 on $[t_\nu, \infty)$. Thus we are guaranteed such a time, T_ν, exists and that $T_\nu \geq t_\nu$. The next few theorems lead to Corollary 3.16, which gives a formula for T_ν in terms of ν.

Theorem 3.13. *For $\nu = n \in \mathbb{N}_1$ and $t \in \mathbb{Z}_a$,*

$$|H_{-\nu}(t,a)| = |H_{-n}(t,a)| \begin{cases} = 0 \text{ when } t \leq a \\ = 1 \text{ when } t = a+1 \\ > 1 \text{ when } a+1 < t < a+n \\ = 1 \text{ when } t = a+n \\ = 0 \text{ when } a+n < t. \end{cases}$$

Proof. Fix $n \in \mathbb{N}_1$. The fact that $H_{-n}(t,a) = 0$ when $t \leq a$ and $H_{-n}(a+1,a) = 1$ has already been shown. (See Theorem 1.8 and Remark 1.14.) Note that by Theorem 1.5 when $t = a+n$ we have

$$\left|H_{-n}(a+n,a)\right| = \left|P_{n-1}(-n)\right| = \left|(-1)^{n-1}\binom{n-1}{n-1}\right| = \left|(-1)^{n-1}\right| = 1.$$

By Corollary 3.11, $|H_{-n}(\cdot,a)|$ is increasing on $(-\infty, t_n]$ and decreasing on $[t_n, \infty)$, where $t_n = a + \lfloor\frac{n}{2}\rfloor + 1 \in [a+1, a+n]$ (since $a+1 \leq \underbrace{a + \lfloor\frac{n}{2}\rfloor + 1}_{t_n} \leq a+n$). Therefore, $|H_{-n}(t,a)| \geq 1$ on $[a+1, a+n]$. Finally, when $t = a+n+1$ we have

$$H_{-n}(a+n+1,a) = P_n(-n) = \frac{1}{n!}(x+1)\cdots(x+n)\Big|_{x=-n} = 0.$$

So the fact that $|H_{-n}(\cdot,a)|$ is decreasing on $[t_n, \infty)$ implies $|H_{-n}(\cdot,a)|$ is decreasing on $[a+n+1, \infty)$ which, in turn, implies $|H_{-n}(t,a)| = 0$ for all $t \geq a+n+1$. □

Theorem 3.14. *For $\nu \in (0,2) \setminus \mathbb{N}_1$ and $t \in \mathbb{Z}_a$,*

$$|H_{-\nu}(t,a)| \begin{cases} = 0 \text{ when } t \leq a \\ = 1 \text{ when } t = a+1 \\ < 1 \text{ when } a+1 < t. \end{cases}$$

Proof. The fact that $H_{-\nu}(t,a) = 0$ when $t \leq a$ and $H_{-\nu}(a+1,a) = 1$ has already been shown. (See Theorem 1.8 and Remark 1.14.) By hypothesis, $0 < \nu < 2$ and so $t_\nu = a+1$. Also, $\nu \notin \mathbb{N}_1$ so, by the first bullet of Corollary 3.11 above, $|H_{-\nu}(\cdot,a)|$ is *strictly* decreasing on $[t_\nu, \infty)$. Thus, when $t > t_\nu = a+1$, $|H_{-\nu}(t,a)| < |H_{-\nu}(t_\nu,a)| = 1$. □

Theorem 3.15. *For $\nu \in (2,\infty) \setminus \mathbb{N}_1$ and $t \in \mathbb{Z}_a$,*

$$|H_{-\nu}(t,a)| \begin{cases} = 0 \text{ when } t \leq a \\ = 1 \text{ when } t = a+1 \\ > 1 \text{ when } a+1 < t \leq a + \lfloor \nu \rfloor \\ < 1 \text{ when } a + \lfloor \nu \rfloor < t. \end{cases}$$

Proof. That $H_{-\nu}(t,a) = 0$ when $t \leq a$ and $H_{-\nu}(a+1,a) = 1$ have already been shown. (See Theorem 1.8 and Remark 1.14.) So we just need to verify the theorem for $t > a+1$. Toward that end, put $N = \lceil \nu \rceil \geq 3$. Note that in this case, since

$$a+1 < \underbrace{a + \lfloor \frac{n}{2} \rfloor + 1}_{t_n} < a + \lfloor \nu \rfloor + 1 = a + \lceil \nu \rceil = a + N$$

we have $a+1 < t_\nu \leq a + N - 1 = a + \lfloor \nu \rfloor$. By the first bullet of Corollary

3.11, $|H_{-\nu}(\cdot, a)|$ is *strictly* increasing on $[a+1, t_\nu]$ and *strictly* decreasing on $[t_\nu, \infty)$. Therefore, to complete the proof it suffices to show that

$$\left|H_{-\nu}(a+N-1, a)\right| > 1 \tag{3.4}$$

and

$$\left|H_{-\nu}(a+N, a)\right| < 1. \tag{3.5}$$

To see (3.4) put $\epsilon := \nu - (N-1) \in (0,1)$ and note

$$\begin{aligned}
\left|H_{-\nu}(a+N-1, a)\right| = \left|P_{N-2}(-\nu)\right| &= \left| \left(\frac{-\nu+1}{1}\right)\left(\frac{-\nu+2}{2}\right)\cdots\left(\frac{-\nu+N-2}{N-2}\right) \right| \\
&= \left|\frac{-\nu+1}{N-2}\right|\left|\frac{-\nu+2}{N-3}\right|\cdots\left|\frac{-\nu+(N-2)}{1}\right| \\
&= \left|\frac{\nu-1}{N-2}\right|\left|\frac{\nu-2}{N-3}\right|\cdots\left|\frac{\nu-(N-2)}{1}\right| \\
&= \underbrace{\left|\frac{N-2+\epsilon}{N-2}\right|}_{>1}\underbrace{\left|\frac{N-3+\epsilon}{N-3}\right|}_{>1}\cdots\underbrace{\left|\frac{1+\epsilon}{1}\right|}_{>1} > 1.
\end{aligned}$$

To see (3.5) note

$$\begin{aligned}
\left|H_{-\nu}(a+N, a)\right| = \left|P_{N-1}(-\nu)\right| &= \left| \left(\frac{-\nu+1}{1}\right)\left(\frac{-\nu+2}{2}\right)\cdots\left(\frac{-\nu+N-1}{N-1}\right) \right| \\
&= \left|\frac{-\nu+1}{N-1}\right|\left|\frac{-\nu+2}{N-2}\right|\cdots\left|\frac{-\nu+(N-1)}{1}\right| \\
&= \underbrace{\left|\frac{\nu-1}{N-1}\right|}_{<1}\underbrace{\left|\frac{\nu-2}{N-2}\right|}_{<1}\cdots\underbrace{\left|\frac{\nu-(N-1)}{1}\right|}_{<1} < 1.
\end{aligned}$$

□

Theorems 3.13, 3.14, and 3.15 give us the following formula for T_ν.

Corollary 3.16. *For $\nu > 0$ and T_ν as defined above,*

$$T_\nu = a + \max\{\lfloor \nu \rfloor, 1\} = \begin{cases} a+1, & 0 < \nu < 1 \\ a + \lfloor \nu \rfloor, & 1 \leq v. \end{cases}$$

Proof. The statements of Theorems 3.13, 3.14, and 3.15 together give us the largest value of t for which $|H_{-\nu}(t,a)| \geq 1$ for any $\nu > 0$. This value is summarized by the given formula. □

Theorem 3.17. *For $\nu \in (0,1)$ and $t \in \mathbb{Z}_a$,*

$$\text{sgn} H_{-\nu}(t,a) = \begin{cases} 0, & t \leq a \\ 1, & a+1 \leq t \leq T_\nu = a+1 \\ 1, & T_\nu + 1 \leq t. \end{cases} \tag{3.6}$$

For $\nu \in (1,\infty) \setminus \mathbb{N}_1$ and $t \in \mathbb{Z}_a$,

$$\text{sgn} H_{-\nu}(t,a) = \begin{cases} 0, & t \leq a \\ (-1)^{t-(a+1)}, & a+1 \leq t \leq T_\nu \\ (-1)^{(T_\nu+1)-(a+1)} & T_\nu + 1 \leq t. \end{cases} \tag{3.7}$$

For $\nu = n \in \mathbb{N}_1$ and $t \in \mathbb{Z}_a$,

$$\text{sgn} H_{-\nu}(t,a) = \begin{cases} 0, & t \leq a \\ (-1)^{t-(a+1)}, & a+1 \leq t \leq T_\nu \\ 0 & T_\nu + 1 \leq t. \end{cases} \tag{3.8}$$

Thus $sgnH_{-\nu}(\cdot,a)$ *alternates on* $[a+1,T_\nu]$ *and is constant on* $[T_\nu+1,\infty)$. *That is,*

$$sgnH_{-\nu}(t,a)=c \text{ for all } t\in\mathbb{N}_{T_\nu+1}, \text{ where } c=\begin{cases}1, & \text{when } \nu\in(0,1)\\ (-1)^{T_\nu-a}, & \text{when } \nu\in(1,\infty)\setminus\mathbb{N}_1\\ 0, & \text{when } \nu\in\mathbb{N}_1.\end{cases}$$

Proof. When $t\le a$, $H_{-\nu}(t,a)=0$ and when $t=a+1$, $H_{-\nu}(t,a)=H_{-\nu}(a+1,a)=1$. So the content of this theorem is that equations (3.6), (3.7), and (3.8) hold for $t>a+1$. Recall that for $t>a+1$,

$$H_{-\nu}(t,a)=P_{t-a-1}(-\nu)=\left(\frac{-\nu+1}{1}\right)\left(\frac{-\nu+2}{2}\right)\cdots\left(\frac{-\nu+(t-a-1)}{(t-a-1)}\right)$$

and so

$$\begin{aligned}\text{sgn}H_{-\nu}(t,a)&=\text{sgn}\left(\frac{-\nu+1}{1}\right)\text{sgn}\left(\frac{-\nu+2}{2}\right)\cdots\text{sgn}\left(\frac{-\nu+(t-a-1)}{(t-a-1)}\right)\\&=\text{sgn}\,(-\nu+1)\,\text{sgn}\,(-\nu+2)\cdots\text{sgn}\,(-\nu+(t-a-1))\\&=\prod_{i=1}^{t-a-1}\text{sgn}(-\nu+i).\end{aligned}\tag{3.9}$$

Case 1: First, we consider the case where $\nu\ge 2$. (So $(\lfloor\nu\rfloor-1)\ge 1$.) (The situation when $0<\nu<2$ will be considered in three special cases at the end.)

From equation (3.9) when $t-a-1\ge\lfloor\nu\rfloor+1$ (that is, for $t\ge a+\lfloor\nu\rfloor+2=T_\nu+2$),

$$\text{sgn}H_{-\nu}(t,a)=\left[\prod_{i=1}^{\lfloor\nu\rfloor-1}\underbrace{\text{sgn}(-\nu+i)}_{=-1}\right]\left[\underbrace{\text{sgn}(-\nu+\lfloor\nu\rfloor)}_{=-1\text{ or }0}\right]\left[\prod_{i=\lfloor\nu\rfloor+1}^{t-a-1}\underbrace{\text{sgn}(-\nu+i)}_{=+1}\right]\tag{3.10}$$

The first factor of equation (3.10) contributes $(-1)^{\lfloor\nu\rfloor-1}=(-1)^{T_\nu-a-1}=(-1)^{T_\nu-(a+1)}$. The second factor is 0 or -1, depending on whether $\nu\in\mathbb{N}_1$ or not. The third factor is 1. Thus, for $t\ge T_\nu+2$, we have $\text{sgn}H_{-\nu}(t,a)=\begin{cases}(-1)^{T_\nu-(a+1)+1} & \text{when } \nu\notin\mathbb{N}_1\\ 0 & \text{when } \nu\in\mathbb{N}_1.\end{cases}$

If $t = T_\nu + 1$ we just lose the third factor from equation (3.10) which does not change the output. Hence, for $t \geq T_\nu + 1$, we have

$$\text{sgn} H_{-\nu}(t,a) = \begin{cases} (-1)^{(T_\nu+1)-(a+1)} & \text{when } \nu \notin \mathbb{N}_1 \\ 0 & \text{when } \nu \in \mathbb{N}_1. \end{cases}$$

These are the third conditions in equations (3.7) and (3.8), respectively.

The middle conditions of equations (3.7) and (3.8) follow from the fact that when $t - a - 1 \leq \lfloor \nu \rfloor - 1$. (That is, for $a + 1 < t \leq a + \lfloor \nu \rfloor = T_\nu$), the second and third factors of equation (3.10) never appear. And so, in this case equation (3.9) is simply:

$$\text{sgn} H_{-\nu}(t,a) = \left[\prod_{i=1}^{t-a-1} \underbrace{\text{sgn}(-\nu + i)}_{=-1} \right] \tag{3.11}$$

Thus, for $t \geq T_\nu + 2$, we have $\text{sgn} H_{-\nu}(t,a) = (-1)^{t-a-1} = (-1)^{t-(a+1)}$, as desired.

Case 2a: If $\nu \in (0,1)$, then $\text{sgn}(-\nu + i) = 1$ for all $i \geq 1$. So, equation (3.9) gives $\text{sgn} H_{-\nu}(t,a) = 1$ for all $t \geq a + 2$. Thus, equation (3.6) holds.

Case 2b: If $\nu = 1$, equation (3.9) gives $\text{sgn} H_{-\nu}(t,a) = 0$ for all $t \geq a + 2$. Thus, equation (3.8) holds when $\nu = n = 1$.

Case 2c: If $\nu \in (1,2)$, then $\text{sgn}(-\nu + 1) = -1$ and $\text{sgn}(-\nu + i) = 1$ for all $i \geq 2$. So, equation (3.9) gives

$$\text{sgn} H_{-\nu}(t,a) = \text{sgn}(-\nu + 1) = -1, \qquad \text{when } t = a + 2$$

and

$$\text{sgn} H_{-\nu}(t,a) = \underbrace{\text{sgn}(-\nu + 1)}_{=-1} \cdot \prod_{i=2}^{t-a-1} \underbrace{\text{sgn}(-\nu + i)}_{=+1} = -1, \text{ when } t \geq a + 3.$$

Thus, equation (3.7) holds when $\nu \in (1,2)$. This completes the proof. □

The next few results show that iterated and fractional differences can be taken term-by-term.

Theorem 3.18. *Suppose that for a sequence of functions* $\{f_n : \mathbb{Z}_a \to \mathbb{R}\}$, $\sum\limits_{n=0}^{\infty} f_n(t)$ *converges for each* $t \in \mathbb{Z}_a$ *and* $f(t) = \sum\limits_{n=0}^{\infty} f_n(t)$. *Then, for each* $k \in \mathbb{N}_0$,

$$\left(\nabla^k f\right)(t) = \sum_{n=0}^{\infty} \left(\nabla^k f_n\right)(t).$$

Proof. The proof follows from the fact that convergent sums can be multiplied by a constant and added term-by-term, hence linear combinations of convergent sums can be taken term-by-term (which justifies (*) in the calculation below). Fix $k \in \mathbb{N}_1$ and $t \in \mathbb{Z}_a$ and consider,

$$\begin{aligned}
\left(\nabla^k f\right)(t) &= \sum_{i=0}^{k} (-1)^i \binom{k}{i} f(t-i) \\
&= \sum_{i=0}^{k} (-1)^i \binom{k}{i} \sum_{n=0}^{\infty} f_n(t-i) \\
&= (-1)^0 \binom{k}{0} \sum_{n=0}^{\infty} f_n(t-0) + (-1)^1 \binom{k}{1} \sum_{n=0}^{\infty} f_n(t-1) + \\
&\qquad\qquad \cdots + (-1)^k \binom{k}{k} \sum_{n=0}^{\infty} f_n(t-k) \\
&\overset{(*)}{=} \sum_{n=0}^{\infty} \left[\begin{array}{r} (-1)^0 \binom{k}{0} f_n(t-0) + (-1)^1 \binom{k}{1} f_n(t-1) + \\ \cdots + (-1)^k \binom{k}{k} f_n(t-k) \end{array} \right] \\
&= \sum_{n=0}^{\infty} \left[\sum_{i=0}^{k} (-1)^i \binom{k}{i} f_n(t-i) \right] \\
&= \sum_{n=0}^{\infty} \left(\nabla^k f_n\right)(t).
\end{aligned}$$

□

The next theorem shows that, provided we have a sequence of functions all of which are in V_T for some $T \in \mathbb{Z}_a$, then a general fractional difference can be taken term-by-term as well.

Theorem 3.19. *Suppose that a sequence of functions $\{f_n\} \subseteq V_T$ for some $T \in \mathbb{Z}_a$ and that for each $t \in \mathbb{Z}_a$, $\sum_{n=0}^{\infty} f_n(t)$ converges. Then, if $f(t) = \sum_{n=0}^{\infty} f_n(t)$, $f \in V_T$ and*

$$\left(\nabla^\gamma f\right)(t) = \sum_{n=0}^{\infty} \left(\nabla^\gamma f_n\right)(t).$$

Proof. This proof also follows from the fact that convergent sums can be multiplied by a constant and added term-by-term, hence linear combinations of convergent sums can be taken term-by-term (which justifies (*) in the calculation below). Put $M = \max\{a - T, 0\}$ so for all $t < (a - M)$, $f(t) = f_n(t) = 0$. Fix $t \in \mathbb{Z}_a$ and consider,

$$\begin{aligned}
\left(\nabla^\gamma f\right)(t) &= \left(H_{-\gamma-1} * f\right)(t) \\
&\overset{\text{Rem 1.22}}{=} \sum_{s=a-M}^{t+M} H_{-\gamma-1}(t-s+a) f(s) \\
&= \sum_{s=a-M}^{t+M} H_{-\gamma-1}(t-s+a) \sum_{n=0}^{\infty} f_n(s) \\
&\overset{(*)}{=} \sum_{n=0}^{\infty} \left[\sum_{s=a-M}^{t+M} H_{-\gamma-1}(t-s+a) f_n(s) \right] \\
&\overset{\text{Rem 1.22}}{=} \sum_{n=0}^{\infty} \left(H_{-\gamma-1} * f_n\right)(t) \\
&= \sum_{n=0}^{\infty} (\nabla^\gamma f_n)(t).
\end{aligned}$$

□

Corollary 3.20. *Suppose that, $\{f_n\} \subseteq \mathbb{R}^{\mathbb{Z}_a}$, $\sum_{n=0}^{\infty} f_n(t)$ converges for each $t \in \mathbb{Z}_a$ and*

$f(t) = \sum_{n=0}^{\infty} f_n(t)$. Then, for all $\alpha \in \mathbb{R}$ and $\nu \in [0, \infty)$,

$$\left(\nabla_a^\alpha f\right)(t) = \sum_{n=0}^{\infty} \left(\nabla_a^\alpha f_n\right)(t) \quad \textit{and} \quad \left(\nabla_{a*}^\nu f\right)(t) = \sum_{n=0}^{\infty} \left(\nabla_{a*}^\nu f_n\right)(t).$$

Proof. To see the first equation holds,

$$\begin{aligned}
\left(\nabla_a^\alpha f\right)(t) &= \nabla^\alpha \left(\mathbb{1}_{(a,\infty)} f\right)(t) = \nabla^\alpha \left(\mathbb{1}_{(a,\infty)} \sum_{n=0}^{\infty} f_n\right)(t) \\
&= \nabla^\alpha \left(\sum_{n=0}^{\infty} \underbrace{\mathbb{1}_{(a,\infty)} f_n}_{\in V_{a+1}}\right)(t) \overset{\text{Thm 3.19}}{=} \sum_{n=0}^{\infty} \left(\nabla^\alpha \mathbb{1}_{(a,\infty)} f_n\right)(t) \\
&= \sum_{n=0}^{\infty} \left(\nabla_a^\alpha f_n\right)(t).
\end{aligned}$$

The second equation follows from the first and Theorem 3.18, since if we put $N := \lceil \nu \rceil$,

$$\begin{aligned}
\left(\nabla_{a*}^\nu f\right)(t) &= \left(\nabla_a^{\nu-N} \nabla^N f\right)(t) = \left(\nabla_a^{\nu-N} \nabla^N \sum_{n=0}^{\infty} f_n\right)(t) \\
&\overset{\text{Thm 3.18}}{=} \left(\nabla_a^{\nu-N} \sum_{n=0}^{\infty} \nabla^N f_n\right)(t) \overset{\text{1st eqn}}{=} \left(\sum_{n=0}^{\infty} \nabla_a^{\nu-N} \nabla^N f_n\right)(t) \\
&= \sum_{n=0}^{\infty} \left(\nabla_{a*}^\nu f_n\right)(t).
\end{aligned}$$

□

Corollary 3.21. *If $\gamma \in \mathbb{R}$ and $\alpha \in \mathbb{R}$ and (a_n) is a sequence of real numbers such that $f(t) = \sum_{n=0}^{\infty} a_n H_{\gamma+n}(t, a) \in \mathbb{R}$ for all $t \in \mathbb{N}_{a+1}^b$, then $(\nabla_a^\alpha f)(t) = \sum_{n=0}^{\infty} a_n H_{\gamma+n-\alpha}(t, a)$.*

Proof. Set $f_n(t) = a_n H_{\gamma+n}(t, a)$, apply Corollary 3.20, and recall that $\nabla_a^\alpha H_{\gamma+n}(t, a) = H_{\gamma+n-\alpha}(t, a)$. □

Chapter 4

Boundary Value Problems (BVPs) Involving ∇_{a*}^{ν}

Theorem 4.1. *When $1 < \nu$ and $N := \lceil \nu \rceil$ and $a < b \in \mathbb{Z}_a$, the Green's function for the boundary value problem:*

$$(*)\begin{cases} -(\nabla_{a*}^{\nu} x)(t) = 0, & \text{for } t \in \mathbb{N}_{a+1}^{b} \\ x(a-i) = 0, & \text{for } 1 \leq i \leq N-1 \\ x(b) = 0 \end{cases}$$

is given by

$$G(t,s) = \left[\frac{H_{N-1}(t,\rho(a))}{H_{N-1}(b,\rho(a))}\right] H_{\nu-1}(b,\rho(s)) - H_{\nu-1}(t,\rho(s)) \tag{4.1}$$

for all $(t,s) \in \mathbb{N}_{a-N+1}^{b} \times \mathbb{N}_{a+1}^{b}$.

Proof. First, note that the bounary conditions at a imply

$$x(a) = (\nabla x)(a) = \cdots = (\nabla^{N-1})(a).$$

Then, note that by setting $\gamma = 0$ in (iii)(a) of Theorem 2.19 we have

$$(\nabla_{a*}^{\nu} x)(t) = \nabla_a^{\nu}\left[x(t) - \sum_{k=0}^{N-1}[(\nabla^k x)(a)]H_k(t,a)\right]$$
$$= \nabla_a^{\nu}\left[x(t) - \sum_{k=0}^{N-1}[(\nabla^k x)(a)]\widetilde{H}_k(t,a)\right]$$

since $\widetilde{H}_k(t,a) = H_k(t,a)$ for all $t \in \mathbb{N}_{a+1}$. And so if x is a solution to the boundary value problem

$$(**)\begin{cases} -(\nabla_{a*}^{\nu} x)(t) = f(t), & \text{for } t \in \mathbb{N}_{a+1}^{b} \\ x(a-i) = 0, & \text{for } 1 \le i \le N-1 \\ x(b) = 0 \end{cases}$$

then

$$-(\nabla_{a*}^{\nu} x)(t) = -\nabla_a^{\nu}\left[x(t) - \sum_{k=0}^{N-1}[(\nabla^k x)(a)]\widetilde{H}_k(t,a)\right] = f(t), \quad \text{for } t \in \mathbb{N}_{a+1}^{b}$$

which implies

$$\nabla_a^{\nu}\left[x(t) - \sum_{k=0}^{N-1}[(\nabla^k x)(a)]\widetilde{H}_k(t,a)\right] = -f(t), \quad \text{for } t \in \mathbb{N}_{a+1}^{b}.$$

Applying $\nabla_a^{-\nu}$ to this last equation and isolating $x(t)$ yields

$$x(t) = \sum_{k=0}^{N-1}[(\nabla^k x)(a)]\widetilde{H}_k(t,a) - (\nabla_a^{-\nu} f)(t). \tag{4.2}$$

The boundary conditions at a tell us that all of the coefficients in the sum are equal to $x(a)$ and so we get

$$x(t) = [x(a)]\sum_{k=0}^{N-1}\widetilde{H}_k(t,a) - (\nabla_a^{-\nu} f)(t).$$

By Corollary 3.4, the sum simplifies to $\widetilde{H}_{N-1}(t+1,a)$ which gives us

$$x(t) = [x(a)]\widetilde{H}_{N-1}(t+1,a) - (\nabla_a^{-\nu} f)(t). \tag{4.3}$$

Applying the boundary condition at b yields

$$0 = x(b) = [x(a)]\underbrace{\widetilde{H}_{N-1}(b+1,a)}_{=H_{N-1}(b,\rho(a))\geq 1} - (\nabla_a^{-\nu} f)(b).$$

Thus, $x(a) = \frac{(\nabla_a^{-\nu} f)(b)}{H_{N-1}(b,\rho(a))}$. Substituting back into equation (4.3) yields

$$x(t) = \left[\frac{(\nabla_a^{-\nu} f)(b)}{H_{N-1}(b,\rho(a))}\right]\widetilde{H}_{N-1}(t+1,a) - (\nabla_a^{-\nu} f)(t) \quad \text{for } t \in \mathbb{N}_{a-N+1}^{b}.$$

Note that by Theorem 1.8, $\widetilde{H}_{N-1}(t+1,a) = 0$ when $t \in \mathbb{N}_{a-N+1}^{a-1}$. As such, for $t \in \mathbb{N}_{a-N+1}$ we have $\widetilde{H}_{N-1}(t+1,a) = H_{N-1}(t+1,a)$, so we may drop the $\widetilde{}$ in the last equation if desired to obtain

$$x(t) = \left[\frac{(\nabla_a^{-\nu} f)(b)}{H_{N-1}(b,\rho(a))}\right]H_{N-1}(t+1,a) - (\nabla_a^{-\nu} f)(t) \quad \text{for } t \in \mathbb{N}_{a-N+1}^{b}.$$

Rearranging slightly (and applying Theorem 1.17) we get that the solution to BVP (**) is

$$x(t) = \left[\frac{H_{N-1}(t,\rho(a))}{H_{N-1}(b,\rho(a))}\right](\nabla_a^{-\nu} f)(b) - (\nabla_a^{-\nu} f)(t) \quad \text{for } t \in \mathbb{N}_{a-N+1}^{b}. \tag{4.4}$$

Using the definition of the operator $\nabla_a^{-\nu}$ and recalling that $H_\gamma(t,\rho(s)) = 0$ when $s > t$

we can rewrite this solution in terms of an integral involving $f(t)$ as follows:

$$\begin{aligned}
x(t) &= \left[\frac{H_{N-1}(t,\rho(a))}{H_{N-1}(b,\rho(a))}\right](\nabla_a^{-\nu} f)(b) - (\nabla_a^{-\nu} f)(t) \\
&= \left[\frac{H_{N-1}(t,\rho(a))}{H_{N-1}(b,\rho(a))}\right]\int_a^b H_{\nu-1}(b,\rho(s))f(s)\nabla s - \int_a^t H_{\nu-1}(t,\rho(s))f(s)\nabla s \\
&= \left[\frac{H_{N-1}(t,\rho(a))}{H_{N-1}(b,\rho(a))}\right]\int_a^b H_{\nu-1}(b,\rho(s))f(s)\nabla s - \int_a^b H_{\nu-1}(t,\rho(s))f(s)\nabla s \\
&= \int_a^b \left[\left[\frac{H_{N-1}(t,\rho(a))}{H_{N-1}(b,\rho(a))}\right] H_{\nu-1}(b,\rho(s)) - H_{\nu-1}(t,\rho(s))\right] f(s)\nabla s \\
&= \int_a^b G(t,s)f(s)\nabla s
\end{aligned}$$

where $G(t,s)$ is as in the statement of the theorem, equation (4.1). □

Theorem 4.2. *For $1 < \nu$ and $N := \lceil \nu \rceil$ and $a < b \in \mathbb{Z}_a$, the Green's function for the boundary value problem (*) is the difference of two nonnegative functions. As such, for $(t,s) \in D := \mathbb{N}_{a-N+1}^b \times \mathbb{N}_{a+1}^b$,*

$$\|G\|_\infty := \max_{(t,s)\in D} |G(t,s)| < H_{\nu-1}(b,a) =: M.$$

Proof. Recall from Theorem 4.1, the Green's function for (*) is

$$G(t,s) = \left[\frac{H_{N-1}(t,\rho(a))}{H_{N-1}(b,\rho(a))}\right] H_{\nu-1}(b,\rho(s)) - H_{\nu-1}(t,\rho(s)), \quad \text{for all } (t,s) \in D.$$

So, $G(t,s) = G_1(t,s) - G_2(t,s)$ where

$$\begin{aligned}
G_1(t,s) &:= \left[\frac{H_{N-1}(t,\rho(a))}{H_{N-1}(b,\rho(a))}\right] H_{\nu-1}(b,\rho(s)) \quad \text{and} \\
G_2(t,s) &:= H_{\nu-1}(t,\rho(s)).
\end{aligned}$$

First, we will show that G_1 and G_2 are both nonnegative. To see this recall that, by definition, $H_\gamma(t,s) = 0$ whenever $t \le s$ and note that for $d \in \mathbb{N}_0$, $P_d(x) = \frac{1}{d!}(x+1)(x+2)\cdots(x+d) > 0$ whenever $x > -1$. So by Corollary 1.12,

$$H_{\nu-1}(t,\rho(s)) = P_{t-s}(\nu-1) \ge 0 \quad \text{whenever } \nu > 0. \tag{4.5}$$

Furthermore, $1 < \nu \Rightarrow 1 < N \Rightarrow 0 < N-1$. Thus, $H_{N-1}(\cdot,\rho(a))$ is (strictly) increasing on $\mathbb{N}_a$ (by Corollary 3.7) and so

$$0 \le H_{N-1}(t,\rho(a)) \overset{\text{Cor 3.7}}{\le} H_{N-1}(b,\rho(a)) \text{ for all } t \in \mathbb{N}_{a-N+1}^{b} \tag{4.6}$$

and (since $\rho(a) < a < b$)

$$1 = H_{N-1}(a,\rho(a)) \le H_{N-1}(b,\rho(a)). \tag{4.7}$$

Hence, by equations (4.6) and (4.7) we have $\left[\frac{H_{N-1}(t,\rho(a))}{H_{N-1}(b,\rho(a))}\right] \in [0,1]$. Combining this with equation (4.5) gives us that all of the Taylor monomials that occur in the formula for the Green's function (4.1) are nonnegative and the denominator in the quotient is nonzero for all t and s. As such $G_1(t,s)$ and $G_2(t,s)$ are well-defined, nonnegative functions for all t and s and so the Green's function is the difference of (these) two nonnegative functions, as claimed.

Next, recall that since $1 < \nu \le N$, we have $0 < \nu - 1 \le N - 1$ which implies that $H_{\nu-1}(\cdot,a)$ and $H_{N-1}(\cdot,a)$ are strictly increasing on $[a,\infty)$ (by Corollary 3.7). Also, note that when $t = b$,

$$G(b,s) = 1 \cdot H_{\nu-1}(b,\rho(s)) - H_{\nu-1}(b,\rho(s)) = 0.$$

Thus, for all $(t,s) \in D$,

$$\begin{aligned} G(t,s) &= G(t,s) \cdot \mathbb{1}_{D_-}(t,s) \\ &= \big[G_1(t,s) - G_2(t,s)\big] \cdot \mathbb{1}_{D_-}(t,s) \\ &= G_1(t,s) \cdot \mathbb{1}_{D_-}(t,s) - G_2(t,s) \cdot \mathbb{1}_{D_-}(t,s) \end{aligned}$$

where $D_- := \mathbb{N}_{a-N+1}^{b-1} \times \mathbb{N}_{a+1}^{b} \subset D$. That is, we may also consider G to be the difference of the two nonnegative functions $G_1 \cdot \mathbb{1}_{D_-}$ and $G_2 \cdot \mathbb{1}_{D_-}$. If we can show that each of these nonnegative functions is *strictly* bounded above by M on D we will be done, since then G, as the difference of these two nonnegative functions, must satisfy $\|G\|_\infty < M$.

To see $\|G_1 \cdot \mathbb{1}_{D_-}\|_\infty < M$ note that

$$\begin{aligned} \|G_1 \cdot \mathbb{1}_{D_-}\|_\infty &= \max_{(t,s)\in D} G_1(t,s) \cdot \mathbb{1}_{D_-}(t,s) \\ &= \max_{(t,s)\in D} \underbrace{\left[\frac{H_{N-1}(t,\rho(a))}{H_{N-1}(b,\rho(a))}\right] \cdot \mathbb{1}_{D_-}(t,s)}_{\in [0,1)} \cdot H_{\nu-1}(b,\rho(s)) \\ &< \max_{(t,s)\in D} H_{\nu-1}(b,\rho(s)) \overset{\text{Cor 1.17}}{=} \max_{(t,s)\in D} H_{\nu-1}(b-s+1+a,a) \\ &= H_{\nu-1}(b-(a+1)+a+1,a) \\ &= H_{\nu-1}(b,a) = M. \end{aligned}$$

To see $\|G_2 \cdot \mathbb{1}_{D_-}\|_\infty < M$ note that

$$\begin{aligned} \|G_2 \cdot \mathbb{1}_{D_-}(t,s)\|_\infty &= \max_{(t,s)\in D} G_2(t,s) \cdot \mathbb{1}_{D_-}(t,s) \\ &= \max_{(t,s)\in D} H_{\nu-1}(t,\rho(s)) \cdot \mathbb{1}_{D_-}(t,s) \end{aligned}$$

$$= \max_{(t,s)\in D} H_{\nu-1}(t-s+1+a,a)\cdot \mathbb{1}_{D_-}(t,s)$$

$$= H_{\nu-1}((b-1)-(a+1)+a+1,a)$$

$$= H_{\nu-1}(b-1,a) < H_{\nu-1}(b,a) = M.$$

This completes the proof. □

Green's functions for boundary value problems involving ordinary differential equations often have a single sign. We conclude this section with an example that shows this is not necessarily the case for boundary value problems involving fractional difference equations.

Corollary 4.3. *The Green's function for the boundary value problem (*) does not necessarily have a single sign. (That is, it may assume both positive and negative values.)*

Example Suppose $\nu = \frac{3}{2} \Rightarrow N = 2$ and $b = a+9$, and $G(t,s)$ is the Green's function for the boundary value problem (*).

If $(t,s) = (a+2, a+1)$, then $G(t,s) = -\frac{32667}{65536} < 0$.

If $(t,s) \in \{b\} \times \mathbb{N}_{a+1}^{b}$, then $G(t,s) = G(b,s) = 0$.

If $(t,s) = (a,b)$, then $G(t,s) = G(a,b) = G_1(a,b) = \frac{1}{H_{\nu-1}(b,\rho(a))} > 0$.

Chapter 5

Upper and Lower Solutions to BVPs

It is well-known that fractional difference equations can be used to model many problems, such as population models, tumor growth models, and so on. In many situations, fractional difference equations have proved to be better than their counterparts with integer differences. Therefore, research in the theory of fractional difference equations has become very important. Previous studies have mainly focused on the theory of integer-order difference equations, classical results have been established, and we can refer to the monographs [1, 2] etc. Recently, there has been a great deal of interest in fractional difference equations. The basic theory of the linear and nonlinear fractional difference equations can be found in [3, 4, 5, 9, 10, 11]. However, we note that the qualitative theory of nonlinear fractional difference equations is not complete and the convergence of approximate solutions plays an important role in the development of the qualitative theory.

In this section we explore a generalized quasi-linearization method that can be used to construct approximate solutions for nonlinear problems. There are many applications of this method. In [13] - [22] the authors have used this method to obtain a convergent sequence of approximate solutions to various kinds of differential equations. Further, in [23, 24], the authors have accelerated the convergence of these sequences by the Gauss-Seidel method. However, there are few applicable results of the above methods to nonlinear fractional difference equations. In [10, 25], the authors only discuss the existence and convergence of solutions for nonlinear fractional difference equations with initial conditions.

In this section, we study the convergence of solutions for a nonlinear Caputo nabla fractional difference equation with boundary conditions. We obtain quadratic convergence by a generalized quasi-linearization method when the forcing function is the sum of a convex and a concave function. Furthermore, we show that the convergence of the sequences obtained is potentially improved by the Gauss-Seidel method. Finally, a numerical example is given to illustrate our [1] resutls.

5.1 Some Definitions and Basic Theorems

In this chapter we use $\mathcal{D}_{a-N+1} := \{x : \mathbb{N}_{a-N+1} \to \mathbb{R}\}$ to denote the vector space of functions from $\mathbb{N}_{a-N+1}$ to $\mathbb{R}$. We define the operator $L_a : \mathcal{D}_{a-N+1} \to \mathcal{D}_{a+1}$ by $(L_a x)(t) := \nabla[(\nabla^{\nu}_{a^*} x)(t+1)]$, $t \in \mathbb{N}_{a+1}$, for $x \in \mathcal{D}_{a-N+1}$.

Lemma 5.1 *Assume $\nu > 0$ and N is the positive integer such that $N - 1 < \nu \leq N$. Then a general solution of the fractional difference equation $(L_a x)(t) = 0$, $t \in \mathbb{N}_{a+1}$ is given by*

$$x(t) = c_0 \widetilde{H}_0(t,a) + c_1 \widetilde{H}_1(t,a) + c_2 \widetilde{H}_2(t,a) + \cdots + c_{N-1} \widetilde{H}_{N-1}(t,a) + c H_{\nu}(t,a)$$

for $t \in \mathbb{N}_{a-N+1}$.

Proof. Put $x_k(t) := \widetilde{H}_k(t,a)$ for $k \in \mathbb{N}_0^{N-1}$. Then, by Corollary 2.12 we have

[1] This chapter is the result of joint work with Xiang Liu, a visiting graduate student studying under Professor Baoguo Jia from the School of Mathematics, Sun Yat-Sen University, Guangzhou, China, while she was visiting University of Nebraska-Lincoln in the Spring of 2018. These results were recently submitted for publication to the *Electronic Journal of Differential Equations* in April 2018.

$(\nabla_{a^*}^{\nu} x_k)(t) = \nabla_{a*}^{\nu} \widetilde{H}_k(t,a) \equiv 0$. So $(L_a x_k)(t) = \nabla[(\nabla_{a^*}^{\nu} x_k)(t+1)] = \nabla 0 = 0$ for $k \in \mathbb{N}_0^{N-1}$.

Also, if we put $\bar{x}(t) := H_\nu(t,a)$. Then, by Theorem 2.9 we have $(\nabla_{a^*}^{\nu}\bar{x})(t) = \nabla_{a*}^{\nu} H_\nu(t,a) = H_0(t,a)$. And so, $(L_a\bar{x})(t) = \nabla[(\nabla_{a^*}^{\nu}\bar{x})(t+1)] = \nabla[H_0(t+1,a)] = \nabla[\mathbb{1}_{(a,\infty)}(t+1)] = \nabla[\mathbb{1}_{[a,\infty)}(t)] = 0$ for $t \in \mathbb{N}_{a+1}$.

Thus, $\widetilde{H}_0(t,a), \widetilde{H}_1(t,a), \cdots, \widetilde{H}_{N-1}(t,a)$, and $H_\nu(t,a)$ are all solutions to $(L_a x)(t) = 0$ for all $t \in \mathbb{N}_{a+1}$.

Next, we show these solutions are linearly independent. We want to show that if

$$c_0\widetilde{H}_0(t,a) + c_1\widetilde{H}_1(t,a) + c_2\widetilde{H}_2(t,a) + \cdots + c_{N-1}\widetilde{H}_{N-1}(t,a) + cH_\nu(t,a) = 0 \qquad (5.1)$$

for all $t \in \mathbb{N}_{a-N+1}$, then it must be the case that $c_0 = c_1 = \cdots = c_{N-1} = c = 0$. Taking $t = a,\ a-1,\ \cdots,\ a-(N-1)$, and $a+1$ in (5.1), we obtain

$$\left[\begin{array}{ccc|c} \widetilde{H}_0(a,a) & \cdots & \widetilde{H}_{N-1}(a,a) & H_\nu(a,a) \\ \widetilde{H}_0(a-1,a) & \cdots & \widetilde{H}_{N-1}(a-1,a) & H_\nu(a-1,a) \\ \vdots & \ddots & \vdots & \vdots \\ \widetilde{H}_0(a-(N-1),a) & \cdots & \widetilde{H}_{N-1}(a-(N-1),a) & H_\nu(a-(N-1),a) \\ \hline \widetilde{H}_0(a+1,a) & \cdots & \widetilde{H}_{N-1}(a+1,a) & H_\nu(a+1,a) \end{array}\right] \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

That is,

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ * & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & (-1)^{N-1} & 0 \\ * & * & \cdots & * & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

So, we arrive at $c_0 = c_1 = \cdots = c_{N-1} = c = 0$. Therefore, we conclude that these solutions are linearly independent. The proof is complete. □

Lemma 5.2 *(See [5, Theorem 3.175]). Assume* $0 < \nu \leq 1$*,* a*,* $b \in \mathbb{R}$*, and* $b - a \in \mathbb{N}_2$*. Then the Green's function for the BVP*

$$\begin{cases} (L_a x)(t) = 0, \quad t \in \mathbb{N}_{a+1}^{b-1}, \\ x(a) = 0, \quad x(b) = 0 \end{cases} \tag{5.2}$$

is given by

$$G(t,s) = \begin{cases} u(t,s), \quad (t,s) \in \mathbb{N}_a^s \times \mathbb{N}_t^b, \\ v(t,s), \quad (t,s) \in \mathbb{N}_s^b \times \mathbb{N}_a^t, \end{cases}$$

where

$$u(t,s) = -\frac{(b-s)^{\overline{\nu}}(t-a)^{\overline{\nu}}}{\Gamma(\nu+1)(b-a)^{\overline{\nu}}},$$

and

$$v(t,s) = u(t,s) + \frac{(t-s)^{\overline{\nu}}}{\Gamma(\nu+1)} = u(t,s) + x(t,s).$$

Lemma 5.3 *(See [5, Theorem 3.177]). Assume* $0 < \nu \leq 1$*,* a*,* $b \in \mathbb{R}$*, and* $b - a \in \mathbb{N}_2$*. Then the Green's function for the BVP*

$$\begin{cases} (L_a x)(t) = 0, \quad t \in \mathbb{N}_{a+1}^{b-1}, \\ x(a) = 0, \quad x(b) = 0 \end{cases}$$

satisfies the inequalities

(i) $G(t,s) \leq 0$*,*

(ii) $G(t,s) \geq -\left(\frac{b-a}{4}\right)\left(\frac{\Gamma(b-a+1)}{\Gamma(\nu+1)\Gamma(b-a+\nu)}\right)$*,*

(iii) $\int_a^b |G(t,s)| \nabla s \leq \frac{(b-a)^2}{4\Gamma(\nu+2)}$*, for* $t \in \mathbb{N}_a^b$*, and*

(iv) $\int_a^b |\nabla_t G(t,s)| \nabla s \leq \frac{b-a}{\nu+1}$*, for* $t \in \mathbb{N}_{a+1}^b$*.*

The next corollary is an immediate consequence of [5, Theorem 3.173]. It relates to the nonhomogeneous BVP with homogeneous boundary conditions:

$$\begin{cases} (L_a y)(t) = h(t), & t \in \mathbb{N}_{a+1}^{b-1}, \\ y(a) = 0, & y(b) = 0. \end{cases} \tag{5.3}$$

Corollary 5.4 *(See [5, Theorem 3.173]). Assume* $0 < \nu \leq 1$, a, $b \in \mathbb{R}$, *and* $b - a \in \mathbb{N}_2$. *The unique solution of the BVP* (5.3) *is given by*

$$y(t) = \int_a^b G(t,s)h(s)\nabla s = \sum_{s=a+1}^{b} G(t,s)h(s), \quad t \in \mathbb{N}_a^b,$$

where $G(t,s)$ *is the Green's function of the BVP* (5.2).

Proof. This is a special case of [5, Theorem 3.173], where $\alpha = \gamma = 1$ and $\beta = \delta = 0$ in equation (3.115) of that theorem. Note that, by [5, Theorem 3.170], $\alpha = \gamma = 1$ and $\beta = \delta = 0$ ensure the hypotheses of [5, Theorem 3.173] are satisfied. □

Lemma 5.5 *Assume* $0 < \nu \leq 1$. *Then the solution of the BVP*

$$\begin{cases} (L_a z)(t) = 0, & t \in \mathbb{N}_{a+1}^{b-1}, \\ z(a) = A, & z(b) = B \end{cases} \tag{5.4}$$

is given by

$$z(t) = A + (B - A)\frac{H_\nu(t,a)}{H_\nu(b,a)}, \quad t \in \mathbb{N}_a^b.$$

Proof. Let z be a solution of the fractional difference equation $(L_a z)(t) = 0$. It follows from Lemma 5.1 that

$$z(t) = c_0 \widetilde{H}_0(t,a) + cH_\nu(t,a) = c_0 + cH_\nu(t,a).$$

Using the first boundary condition $z(a) = A$, we obtain

$$A = c_0 + cH_\nu(a, a) = c_0,$$

which implies $c_0 = A$. Using the second boundary condition $z(b) = B$, we get

$$B = A + cH_\nu(b, a).$$

Solving for c, we get

$$c = \frac{(B - A)}{H_\nu(b, a)}.$$

Thus, we have

$$\begin{aligned} z(t) &= c_0 + cH_\nu(t, a) \\ &= A + (B - A)\frac{H_\nu(t, a)}{H_\nu(b, a)}. \end{aligned}$$

The proof is complete. □

Lemma 5.6 *Assume* $0 < \nu \leq 1$, *and* $h : \mathbb{N}_{a+1}^{b-1} \to \mathbb{R}$. *Then the solution of the nonhomogeneous BVP*

$$\begin{cases} (L_a x)(t) = h(t), \quad t \in \mathbb{N}_{a+1}^{b-1}, \\ x(a) = A, \quad x(b) = B \end{cases} \tag{5.5}$$

is given by

$$x(t) = z(t) + \sum_{s=a+1}^{b} G(t, s)h(s), \quad t \in \mathbb{N}_a^b,$$

where $G(t, s)$ *is the Green's function of the BVP* (5.2) *and* $z(t)$ *is the unique solution of the BVP* (5.4).

Proof. Let

$$y(t) = \sum_{s=a+1}^{b} G(t,s)h(s), \quad t \in \mathbb{N}_a^b.$$

By Corollary 5.4, $y(t)$ is the solution of the BVP (5.3) on $\mathbb{N}_a^b$. Let $z(t)$ be as in the statement of this theorem. Then, we have

$$x(a) = z(a) + y(a) = A + 0 = A,$$

and

$$x(b) = z(b) + y(b) = B + 0 = B.$$

Finally,

$$(L_a x)(t) = (L_a z)(t) + (L_a y)(t) = 0 + h(t) = h(t)$$

for $t \in \mathbb{N}_{a+1}^{b-1}$. The proof is complete. □

Lemma 5.7 *Assume* $0 < \nu \leq 1$, $x : \mathbb{N}_a^b \to \mathbb{R}$, *and put* $M := \max\{x(t) : t \in \mathbb{N}_a^b\}$. *If* $x(t_0) = M$ *for some* $t_0 \in \mathbb{N}_{a+1}^{b-1}$, *then* $(L_a x)(t_0) \leq 0$.

Proof. To see why this is true, note that for $\nu = 1$ we have

$$\begin{aligned}(L_a x)(t_0) &= (\nabla^2 x)(t)|_{t=t_0+1} \\ &= \underbrace{x(t_0+1)}_{\leq M} - \underbrace{2x(t_0)}_{=2M} + \underbrace{x(t_0-1)}_{\leq M} \\ &\leq M - 2M + M = 0\end{aligned}$$

In general, when $0 < \nu \leq 1$, we will use the fact that for any t,

$$(L_a x)(t) = x(t+1) - H_{-\nu-1}(t+1,a)x(a) + \sum_{s=a+1}^{t} H_{-\nu-2}(t+1,\rho(s))x(s)\ . \qquad (5.6)$$

To see this note:

$$
\begin{aligned}
(L_a x)(t) &= \nabla[(\nabla^{\nu}_{a*}x)(t+1)] = (\nabla\nabla^{\nu}_{a*}x)(t+1) \\
&\qquad [\text{Reason: } \nabla[(\nabla^{\nu}_{a*}x)(t+1)] = (\nabla L\nabla^{\nu}_{a*}x)(t) = (L\nabla\nabla^{\nu}_{a*}x)(t).] \\
&\overset{\text{Def}}{=} (\nabla\nabla^{\nu-1}_{a}\nabla x)(t+1) \\
&\overset{\text{Thm 2.19}(i)(a)}{=} (\nabla^{\nu}_{a}\nabla x)(t+1) \\
&\overset{\text{Thm 2.19}(ii)(b)}{=} \nabla^{\nu+1}_{a}\Big[x(\tau) + [x(a)]H_0(\tau,a)\Big]\Big|_{\tau=t+1} \\
&= (\nabla^{\nu+1}_{a}x)(t+1) + [x(a)]H_{-\nu-1}(t+1,a) \\
&\overset{\text{Thm A.6}}{=} \int_{(a,t+1]} H_{-\nu-2}(t+1,\rho(s))x(s)\nabla s + x(a)H_{-\nu-1}(t+1,a) \\
&= \int_{(a,t]} H_{-\nu-2}(t+1,\rho(s))x(s)\nabla s + x(t+1) + x(a)H_{-\nu-1}(t+1,a) \\
&= x(t+1) + \sum_{s=a+1}^{t} H_{-\nu-2}(t+1,\rho(s))x(s) - H_{-\nu-1}(t+1,a)x(a). \qquad \checkmark
\end{aligned}
$$

Note that for $0 < \nu \leq 1$, $-H_{-\nu-1}(t_0+1,a)$ is positive whenever $t_0 \in \mathbb{N}_{a+1}$ and $H_{-\nu-2}(t_0+1,\rho(s))$ is positive whenever $s \in \mathbb{N}_a^{t_0-1}$. Thus, we obtain

$$
\begin{aligned}
(L_a x)(t_0) &= x(t_0+1) - H_{-\nu-1}(t_0+1,a)x(a) + \sum_{s=a+1}^{t_0} H_{-\nu-2}(t_0+1,\rho(s))x(s) \\
&= \underbrace{x(t_0+1)}_{\leq M} - H_{-\nu-1}(t_0+1,a)\underbrace{x(a)}_{\leq M} + \sum_{s=a+1}^{t_0-1} H_{-\nu-2}(t_0+1,\rho(s))\underbrace{x(s)}_{\leq M} \\
&\qquad\qquad + H_{-\nu-2}(t_0+1,\rho(t_0))\underbrace{x(t_0)}_{=M} \\
&\leq M\left[1 - H_{-\nu-1}(t_0+1,a) + \sum_{s=a+1}^{t_0-1} H_{-\nu-2}(t_0+1,\rho(s)) + H_{-\nu-2}(t_0+1,\rho(t_0))\right] \\
&= M\left[1 - H_{-\nu-1}(t_0+1,a) + \sum_{s=a+1}^{t_0} H_{-\nu-2}(t_0+1,\rho(s))\right]
\end{aligned}
$$

$$= M\left[1 - H_{-\nu-1}(t_0+1,a) + \int_{(a,t_0]} H_{-\nu-2}(t_0+1,\rho(s))\nabla s\right]$$
$$\overset{\text{Thm 1.18}}{=} M\left[1 - H_{-\nu-1}(t_0+1,a) + [(-H_{-\nu-1}(t_0+1,t_0) + H_{-\nu-1}(t_0+1,a)]\right]$$
$$= 0.$$

The proof is complete. □

5.2 Existence and Comparison Results

Consider the following BVP for a nonlinear Caputo nabla fractional difference equation

$$\begin{cases} (L_a x)(t) = f(t,x(t)), \quad t \in \mathbb{N}_{a+1}^{b-1}, \\ x(a) = A, \quad x(b) = B, \end{cases} \tag{5.7}$$

where $f : \mathbb{N}_{a+1}^{b-1} \times \mathbb{R} \to \mathbb{R}$ is continuous with respect to x, $x : \mathbb{N}_a^b \to \mathbb{R}$, and $0 < \nu \le 1$.

In this paper, we define the norm of x on $\mathbb{N}_a^b$ by $\|x\| = \max_{s\in\mathbb{N}_a^b} |x(s)|$. Throughout this paper, we use the notation $f^{(k)}(t,x) := \frac{\partial^k f(t,x)}{\partial^k x}$ $(k = 0,1,2\ldots)$. For convenience, when $\alpha_0(t)$ and $\beta_0(t)$ are two functions such that $\alpha_0(t) \le \beta_0(t)$ on $\mathbb{N}_a^b$, we use the following sets:

$$\Omega = \Omega(\alpha_0,\beta_0) := \{(t,x) \in \mathbb{Z}_a \times \mathbb{R} : \alpha_0(t) \le x \le \beta_0(t), \text{ for all } t \in \mathbb{N}_{a+1}^{b-1}\}, \text{ and}$$

$$\mathcal{S} = \mathcal{S}(\alpha_0,\beta_0) := \{x : \mathbb{N}_a^b \to \mathbb{R} | \alpha_0(t) \le x(t) \le \beta_0(t), \text{ for all } t \in \mathbb{N}_a^b\}.$$

We will simply refer to the sets Ω and $\mathcal{S}$ when it is clear from context what α_0 and β_0 are.

Definition 5.8 The function $\alpha_0(t)$ is said to be a lower solution of BVP (3.1), if

$$\begin{cases} (L_a\alpha_0)(t) \geq f(t, \alpha_0(t)), & t \in \mathbb{N}_{a+1}^{b-1}, \\ \alpha_0(a) \leq A, \quad \alpha_0(b) \leq B. \end{cases} \tag{5.8}$$

If all three inequalities in (5.8) are reversed, we have an upper solution. Now we present an existence result relative to BVP (5.7), which we will use in our main results. Since the proof is a standard application of Schauder's fixed point theorem we will omit the proof of this lemma.

Lemma 5.9 *Assume that*

$(H_{3.1})$ *the function* $f : \mathbb{N}_{a+1}^{b-1} \times \mathbb{R} \to \mathbb{R}$ *is continuous with respect to* x*, and for* $M \geq 0$*, define* $C = C(M) := \max\{|f(t,x)| : t \in \mathbb{N}_{a+1}^{b-1}, |x| \leq 2M\}$*.*

Then the nonlinear BVP (5.7) *has a solution provided there is some* $M \geq \|z\| = \max\{|A|, |B|\}$*, where* z *is the unique solution of BVP* (5.4)*, such that* $C(M) > 0$ *and*

$$(b-a)^2 \leq \frac{4M\Gamma(\nu+2)}{C(M)}.$$

In particular, if $f \not\equiv 0$ and bounded, then BVP (5.7) has a solution.

Lemma 5.10 *Assume that*

$(H_{3.2})$ *the function* $f : \mathbb{N}_{a+1}^{b-1} \times \mathbb{R} \to \mathbb{R}$ *is nondecreasing with respect to* x *for each fixed* $t \in \mathbb{N}_{a+1}^{b-1}$*.*

$(H_{3.3})$ *the functions* α_0*,* $\beta_0 : \mathbb{N}_a^b \to \mathbb{R}$ *are lower and upper solutions respectively of BVP* (5.7)*.*

Then $\alpha_0(t) \leq \beta_0(t)$ *on* $\mathbb{N}_a^b$*.*

Proof. Let us first prove the lemma for strict inequality. That is, suppose

$$\begin{cases} (L_a\alpha_0)(t) > f(t,\alpha_0(t)), & t \in \mathbb{N}_{a+1}^{b-1}, \\ \alpha_0(a) \leq A, \ \alpha_0(b) \leq B. \end{cases}$$

and

$$\begin{cases} (L_a\beta_0)(t) \leq f(t,\beta_0(t)), & t \in \mathbb{N}_{a+1}^{b-1}, \\ \beta_0(a) \geq A, \ \beta_0(b) \geq B. \end{cases}$$

The boundary conditions immediately give us $\alpha_0(a) \leq A \leq \beta_0(a)$ and $\alpha_0(b) \leq B \leq \beta_0(b)$. Next, we will show that $\alpha_0(t) < \beta_0(t)$ for $t \in \mathbb{N}_{a+1}^{b-1}$. If it is not true, then there exists $t_0 \in \mathbb{N}_{a+1}^{b-1}$ such that $x(t) := \alpha_0(t) - \beta_0(t)$ has a nonnegative maximum at t_0. That is, for some $t_0 \in \mathbb{N}_{a+1}^{b-1}$,

$$x(t_0) = \alpha_0(t_0) - \beta_0(t_0) = \max\{\alpha_0(t) - \beta_0(t), \ t \in \mathbb{N}_{a+1}^{b-1}\} \geq 0,$$

But then, $x(t_0) = \max\limits_{t\in\mathbb{N}_a^b} x(t)$, since $x(a) = \alpha_0(a) - \beta_0(a) \leq 0$ and $x(b) = \alpha_0(b) - \beta_0(b) \leq 0$. Therefore, by Lemma 5.7, we obtain

$$(L_a x)(t_0) = (L_a\alpha_0)(t_0) - (L_a\beta_0)(t_0) \leq 0.$$

So, we have

$$f(t_0,\alpha_0(t_0)) < (L_a\alpha_0)(t_0) \leq (L_a\beta_0)(t_0) \leq f(t_0,\beta_0(t_0)),$$

On the other hand, our assumption that $x(t_0) = \alpha_0(t_0) - \beta_0(t_0) \geq 0 \Rightarrow \beta_0(t_0) \leq \alpha_0(t_0)$. By $(H_{3.2})$, $f(t,x)$ is nondecreasing with respect to x for each t, so this, in turn, implies

$$f(t_0,\beta_0(t_0)) \leq f(t_0,\alpha_0(t_0))$$

which is a contradiction. Hence, we conclude that $\alpha_0(t) < \beta_0(t)$ on $\mathbb{N}_{a+1}^{b-1}$. $\surd$

Now, we define $\tilde{\alpha}_0(t) = \alpha_0(t) + \varepsilon(H_{\nu+1}(t,a) - H_{\nu+1}(b,a))$, where $\varepsilon > 0$. Then $\tilde{\alpha}_0(t) < \alpha_0(t)$ for $t \in \mathbb{N}_{a+1}^{b-1}$. Using the condition $(H_{3.2})$, we get

$$\begin{aligned}(L_a\tilde{\alpha}_0)(t) &= (L_a\alpha_0)(t) + (L_a\varepsilon(H_{\nu+1}(\cdot,a) - H_{\nu+1}(b,a)))(t) \\ &= (L_a\alpha_0)(t) + \varepsilon \\ &\geq f(t,\alpha_0(t)) + \varepsilon \\ &\geq f(t,\tilde{\alpha}_0(t)) + \varepsilon \\ &> f(t,\tilde{\alpha}_0(t)).\end{aligned}$$

Thus $\tilde{\alpha}_0(t)$ is a lower solution for which strict inequality holds. It therefore follows from the previous argument that $\tilde{\alpha}_0(t) < \beta_0(t)$ for all $t \in \mathbb{N}_{a+1}^{b-1}$. Note this holds for all $\varepsilon > 0$. Letting $\varepsilon \to 0$, we get $\alpha_0(t) \leq \beta_0(t)$ for all $t \in \mathbb{N}_{a+1}^{b-1}$. Thus, we have $\alpha_0(t) \leq \beta_0(t)$ on $\mathbb{N}_a^b$. The proof is complete. □

Corollary 5.11 *Assume that* $(H_{3.2})$ *holds. If BVP* (5.7) *has a solution, then it is unique.*

Proof. Suppose $x(t)$ and $\bar{x}(t)$ are two solutions of BVP (5.7). Since any solution is both a lower and an upper solution, by Lemma 5.10, we have

$$x(t) \leq \bar{x}(t) \leq x(t) \Rightarrow x(t) = \bar{x}(t) \text{ on } \mathbb{N}_a^b.$$

That is, the solution of BVP (5.7), if one exists, is unique. By hypothesis a solution exists, so $x \equiv \bar{x}$. The proof is complete. □

Next, we consider BVP (5.7) in the special case where $f(t,x) = C(t)x$ and

$A = B = 0$. That is, we consider the BVP

$$(L_a x)(t) = C(t)x(t) \text{ for } \ t \in \mathbb{N}_{a+1}^{b-1}, \ \ x(a) = 0, \ x(b) = 0. \tag{5.9}$$

Corollary 5.12 *Assume that*

$(H_{3.4})$ *the function* $C(t) \geq 0$ *for* $t \in \mathbb{N}_{a+1}^{b-1}$.

If $x(t)$ *is a lower solution to BVP* (5.9), *then* $x(t) \leq 0$ *on* $\mathbb{N}_a^b$. *If* $y(t)$ *is an upper solution, then* $y(t) \geq 0$ *on* $\mathbb{N}_a^b$.

Proof. By hypothesis, $x(t)$ is a lower solution and, by inspection, $z(t) \equiv 0$ is a (upper) solution to BVP (5.9). So Lemma 5.10 guarantees $x(t) \leq 0$. Similiarly, $y(t)$ is an upper solution and $z(t) \equiv 0$ is a (lower) solution to BVP (5.9). So Lemma 5.10 guarantees $y(t) \geq 0$. The proof is complete. □

Since $x(t)$ is a lower solution to BVP (5.9), it satisfies the inequalities

$$(L_a x)(t) \geq C(t)x(t) \text{ for } \ t \in \mathbb{N}_{a+1}^{b-1}, \ \ x(a) \leq 0, \ x(b) \leq 0. \tag{5.10}$$

Lemma 5.13 *Assume that*

$(H_{3.5})$ *the functions* α_0, $\beta_0 : \mathbb{N}_a^b \to \mathbb{R}$ *are lower and upper solutions respectively of BVP* (5.7) *such that* $\alpha_0(t) \leq \beta_0(t)$ *on* $\mathbb{N}_a^b$

$(H_{3.6})$ *the function* $f : \Omega \to \mathbb{R}$ *is continuous in its second variable and* $f \not\equiv 0$ *on* Ω.

Then there exists a solution $x(t)$ *of BVP* (5.7) *satisfying* $\alpha_0(t) \leq x(t) \leq \beta_0(t)$ *on* $\mathbb{N}_a^b$.

Proof. Let $P : \mathbb{N}_{a+1}^{b-1} \times \mathbb{R} \to \mathbb{R}$ be defined by $P(t, x) = \max\left\{\alpha_0(t), \min\{x, \beta_0(t)\}\right\}$. Then $f(t, P(t, x))$ defines an extension of f to $\mathbb{N}_{a+1}^{b-1} \times \mathbb{R}$, which is continuous in its second variable, $f \not\equiv 0$ and bounded. Therefore, by Lemma 5.9, the BVP

$$(L_a x)(t) = \bar{f}(t,x) := f(t, P(t,x)) \text{ for } t \in \mathbb{N}_{a+1}^{b-1}, \quad x(a) = A, \ x(b) = B. \tag{5.11}$$

has a solution $x(t)$ on $\mathbb{N}_a^b$.

To complete the proof, we just need to show that $\alpha_0(t) \leq x(t) \leq \beta_0(t)$ on $\mathbb{N}_a^b$. Doing so will mean that $\bar{f}(t, x(t)) = f(t, x(t))$, which will imply $x(t)$ is not only a solution to BVP (5.11) but also actually a solution to BVP (5.7). Toward this end, we first show that $\alpha_0(t) \leq x(t)$ on $\mathbb{N}_a^b$. The boundary conditions immediately give us that $\alpha_0(a) \leq x(a)$ and $\alpha_0(b) \leq x(b)$. To see that $\alpha_0(t) \leq x(t)$ on $\mathbb{N}_{a+1}^{b-1}$, as in the proof of Lemma 5.10, for $\varepsilon > 0$ define $\tilde{\alpha}_\varepsilon(t) := \alpha_0(t) + \varepsilon[H_{\nu+1}(t,a) - H_{\nu+1}(b,a)]$. Then $\tilde{\alpha}_\varepsilon(t) \leq \alpha_0(t)$ for $t \in \mathbb{N}_a^b$ and, in particular, $\tilde{\alpha}_\varepsilon(a) < \alpha_0(a) \leq x(a)$, and $\tilde{\alpha}_\varepsilon(b) = \alpha_0(b) \leq x(b)$. If we can show $\tilde{\alpha}_\varepsilon(t) < x(t)$ on $\mathbb{N}_{a+1}^{b-1}$, then letting $\varepsilon \to 0$, we get $\lim\limits_{\varepsilon \to 0} \tilde{\alpha}_\varepsilon(t) = \alpha_0(t) \leq x(t)$ for $t \in \mathbb{N}_{a+1}^{b-1}$ and we will be done. So, toward contradiction, assume that for some fixed $\varepsilon > 0$ this is not true. That is, assume that for some fixed $\varepsilon > 0$ there exists $t_1 \in \mathbb{N}_{a+1}^{b-1}$ such that $d(t) := \tilde{\alpha}_\varepsilon(t) - x(t)$ has a nonnegative maximum at t_1, i.e. for this $t_1 \in \mathbb{N}_{a+1}^{b-1}$,

$$d(t_1) = \tilde{\alpha}_\varepsilon(t_1) - x(t_1) = \max\{\tilde{\alpha}_\varepsilon(t) - x(t), \ t \in \mathbb{N}_{a+1}^{b-1}\} \geq 0.$$

But then, $d(t_1) = \max\limits_{t \in \mathbb{N}_a^b} d(t)$, since $d(a) = \tilde{\alpha}_\varepsilon(a) - x(a) \leq 0$ and $d(b) = \tilde{\alpha}_\varepsilon(b) - x(b) \leq 0$. And so, by Lemma 5.7, we must have

$$(L_a d)(t_1) \leq 0. \tag{5.12}$$

On the other hand, our assumption that $\tilde{\alpha}_\varepsilon(t_1) - x(t_1) \geq 0 \Rightarrow x(t_1) \leq \tilde{\alpha}_\varepsilon(t_1) \leq \alpha_0(t_1)$

and so we have $P(t_1, x(t_1)) = \alpha_0(t_1)$. Thus,

$$(L_a\alpha_0)(t_1) \geq f(t_1, \alpha_0(t_1)) = f(t_1, P(t_1, x(t_1))) = (L_a x)(t_1)$$

which implies

$$(L_a\alpha_0)(t_1) - (L_a x)(t_1) \geq 0$$

which, in turn, implies,

$$\begin{aligned}
(L_a d)(t_1) &= (L_a\tilde{\alpha}_\varepsilon)(t_1) - (L_a x)(t_1) \\
&= (L_a\alpha_0)(t_1) + \varepsilon(L_a[H_{\nu+1}(\cdot, a) - H_{\nu+1}(b, a)](t_1)) - (L_a x)(t_1) \\
&= (L_a\alpha_0)(t_1) + \varepsilon(1 - 0) - (L_a x)(t_1) \\
&= (L_a\alpha_0)(t_1) - (L_a x)(t_1) + \varepsilon \\
&> 0.
\end{aligned}$$

which is in contradiction to (5.12). Thus, our assumption must be incorrect and it must be the case that for all $\varepsilon > 0$, $\tilde{\alpha}_\varepsilon(t) < x(t)$ on $\mathbb{N}_{a+1}^{b-1}$. And so, as argued above, letting $\varepsilon \to 0$ we obtain our result that $\alpha_0(t) \leq x(t)$ on $\mathbb{N}_a^b$. Similarly, we can show that $x(t) \leq \beta_0(t)$ on $\mathbb{N}_a^b$. It follows that $x(t)$ is a solution of BVP (5.7) which lies between α_0 and β_0. The proof is complete. □

Remark 5.14 The proof of the preceding theorem shows that if a function x is a solution to BVP (5.11), then $x \in \mathcal{S}$ and x is a solution to BVP (5.7). The converse is also true, since if x is a solution to BVP (5.7) and $x \in \mathcal{S}$ we have

$$(L_a x)(t) = f(t, x(t)) = f(t, P(t, x(t))) = \bar{f}(t, x(t)) \text{ for all } t \in \mathbb{N}_{a+1}^{b-1}.$$

We will make use of this equivalence at the beginning of the proof to Theorem 4.1.

Lemma 5.15 *Assume that* $(H_{3.2})$*,* $(H_{3.5})$ *and* $(H_{3.6})$ *hold. Then BVP* (5.7) *has a unique solution,* $x(t)$*. Furthermore,* $\alpha_0(t) \leq x(t) \leq \beta_0(t)$ *on* $\mathbb{N}_a^b$.

Proof. By Lemma 5.13, there exists a solution to BVP (5.7) which lies in $\mathcal{S}$. By Corollary 5.11 this solution is unique. The proof is complete. □

5.3 Main Results

In this section, we consider BVP (5.7) in the special case where $f(t,x)$ is increasing with respect to x and $f(t,x) = f_1(t,x) + f_2(t,x)$, where $f_1(t,x)$ is concave up with respect to x and $f_2(t,x)$ is concave down with respect to x. We then obtain sequences of successive approximations by applying the generalized quasi-linearization method to our nonlinear Caputo nabla fractional difference equation and show that the sequences so obtained converge quadratically to the solution. Furthermore, we use the Gauss-Seidel method to improve the rate of convergence.

Theorem 5.16 *Assume that the condition* $(H_{3.5})$ *holds, and*

$(A_{4.1})$ $f \not\equiv 0$ *and* $f = f_1 + f_2$ *where the functions* $f_1,\ f_2 : \Omega \to \mathbb{R}$ *are such that:*

(i) $f_1^{(i)}(t,x)$, $f_2^{(i)}(t,x)$ $(i = 0,1,2)$ *exist and are continuous in the second variable,*

(ii) $f_1^{(1)}(t,x) + f_2^{(1)}(t,y) \geq 0$ *for all* $(t,x),(t,y) \in \Omega$*, and*

(iii) $f_1^{(2)}(t,x) \geq 0$ *and* $f_2^{(2)}(t,x) \leq 0$ *on* Ω.

Then there exist two sequences $\{\alpha_n(t)\}$ *and* $\{\beta_n(t)\}$*, which converge monotonically to the one function* x *in* $\mathcal{S}(\alpha_0,\beta_0)$ *which is a solution to BVP* (5.7)*. Furthermore, the convergence is quadratic.*[2]

[2] As in R. P. Agarwal et al (see [22, 3. Main Results, Theorem 1]), we define the quadratic convergence of two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ which both converge to a common x, by the condition

Proof. The proof proceeds as outlined below.

Step 1: Show there exists exactly one function $x \in \mathcal{S}(\alpha_0, \beta_0)$ that is a solution to BVP (5.7).

Step 2: Show how to obtain the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ and that for all $n \geq 0$, the functions α_n and β_n are lower and upper solutions respectively to BVP (5.7).

Step 3: Show the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are monotonic and converge pointwise to x.

Step 4: Show the convergence is quadratic.

Step 1: Let $P(t,x)$ and $\bar{f}(t,x)$ be as in the proof of Lemma 5.13. Since, by condition $(A_{4.1})$(ii), we have that $f^{(1)}(t,x) = f_1^{(1)}(t,x) + f_2^{(1)}(t,x) \geq 0$, the function $f : \Omega \to \mathbb{R}$ is nondecreasing with respect to x for each t which, in turn, implies $\bar{f} : \mathbb{N}_{a+1}^{b-1} \times \mathbb{R} \to \mathbb{R}$ is nondecreasing with respect to x for each t. Therefore, by Lemma 5.15, BVP (5.11) has a unique solution, x. By Remark 5.14, this function x sits in $\mathcal{S}$ and is also a solution to BVP (5.7). Furthermore, by the converse mentioned in Remark 5.14, if x, $\bar{x} \in \mathcal{S}$ are two (possibly different) solutions to BVP (5.7), then they must both also be solutions to BVP (5.11). However, as the solution to BVP (5.11) is unique, this means $x \equiv \bar{x}$. Thus, there exists exactly one function $x \in \mathcal{S}(\alpha_0, \beta_0)$ that is a solution to BVP (5.7).

Step 2: Next, we develop two sequences of successive approximations $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ that both converge pointwise to the function x that we identified in Step 1.

---

that there exist constants C_1, C_2, C_3, and C_4 such that for all $n \geq 0$,

$$\|x - \alpha_{n+1}\| \leq C_1 \|x - \alpha_n\|^2 + C_2 \|\beta_n - x\|^2$$

and

$$\|\beta_{n+1} - x\| \leq C_3 \|\beta_n - x\|^2 + C_4 \|x - \alpha_n\|^2.$$

First, from the condition $(A_{4.1})$ (iii), we obtain, for all (t,x), $(t,y) \in \Omega$,

$$f_1(t,y) \geq f_1(t,x) + f_1^{(1)}(t,x)(y-x), \tag{5.13}$$

and

$$f_2(t,y) \geq f_2(t,x) + f_2^{(1)}(t,y)(y-x). \tag{5.14}$$

Next, consider the following two BVPs where $\alpha, \beta : \mathbb{N}_a^b \to \mathbb{R}$ are fixed functions

$$\begin{cases} (L_a y)(t) = f_1(t,\alpha) + f_1^{(1)}(t,\beta)(y-\alpha) + f_2(t,\alpha) + f_2^{(1)}(t,\alpha)(y-\alpha) \\ \qquad\qquad = f(t,\alpha) + [f_1^{(1)}(t,\beta) + f_2^{(1)}(t,\alpha)](y-\alpha) \\ \qquad\qquad \equiv F(t,\alpha,\beta;y), \quad t \in \mathbb{N}_{a+1}^{b-1}, \\ \quad y(a) = A, \ y(b) = B, \end{cases} \tag{5.15}$$

and

$$\begin{cases} (L_a z)(t) = f_1(t,\beta) + f_1^{(1)}(t,\beta)(z-\beta) + f_2(t,\beta) + f_2^{(1)}(t,\alpha)(z-\beta) \\ \qquad\qquad = f(t,\beta) + [f_1^{(1)}(t,\beta) + f_2^{(1)}(t,\alpha)](z-\beta) \\ \qquad\qquad \equiv G(t,\alpha,\beta;z), \quad t \in \mathbb{N}_{a+1}^{b-1}, \\ \quad z(a) = A, \ z(b) = B. \end{cases} \tag{5.16}$$

Letting $\alpha = \alpha_0$, $\beta = \beta_0$ in BVPs (5.15), (5.16). We first prove that $\alpha_0(t)$ and $\beta_0(t)$ are lower and upper solutions of BVP (5.15), respectively. In fact, from the condition

$(H_{3.5})$, we have

$$
\begin{aligned}
(L_a\alpha_0)(t) &\geq f_1(t,\alpha_0) + f_2(t,\alpha_0) = F(t,\alpha_0,\beta_0;\alpha_0), \quad t \in \mathbb{N}_{a+1}^{b-1}, \\
\alpha_0(a) &\leq A, \;\; \alpha_0(b) \leq B,
\end{aligned}
$$

and by using the inequalities (5.13), (5.14), it follows that

$$
\begin{aligned}
(L_a\beta_0)(t) &\leq f_1(t,\beta_0) + f_2(t,\beta_0) \\
&\leq f_1(t,\alpha_0) + f_1^{(1)}(t,\beta_0)(\beta_0-\alpha_0) + f_2(t,\alpha_0) + f_2^{(1)}(t,\alpha_0)(\beta_0-\alpha_0) \\
&= F(t,\alpha_0,\beta_0;\beta_0), \quad t \in \mathbb{N}_{a+1}^{b-1}, \\
\beta_0(a) &\geq A, \;\; \beta_0(b) \geq B.
\end{aligned}
$$

These show that $\alpha_0(t)$ and $\beta_0(t)$ are lower and upper solutions of BVP (5.15). Furthermore, note that $F : \mathbb{N}_{a+1}^{b-1} \times \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing with respect to y. Thus, by Lemma 5.15, it follows that there exists a unique solution $\alpha_1(t)$ of BVP (5.15) such that $\alpha_0(t) \leq \alpha_1(t) \leq \beta_0(t)$ on $\mathbb{N}_a^b$.

Similarly, using the condition $(H_{3.5})$, and the inequalities (5.13), (5.14), we obtain

$$
\begin{aligned}
(L_a\alpha_0)(t) &\geq f_1(t,\alpha_0) + f_2(t,\alpha_0) \\
&\geq f_1(t,\beta_0) + f_1^{(1)}(t,\beta_0)(\alpha_0-\beta_0) + f_2(t,\beta_0) + f_2^{(1)}(t,\alpha_0)(\alpha_0-\beta_0) \\
&= G(t,\alpha_0,\beta_0;\alpha_0), \quad t \in \mathbb{N}_{a+1}^{b-1}, \\
\alpha_0(a) &\leq A, \;\; \alpha_0(b) \leq B,
\end{aligned}
$$

and

$$
\begin{aligned}
(L_a\beta_0)(t) &\leq f_1(t,\beta_0) + f_2(t,\beta_0) = G(t,\alpha_0,\beta_0;\beta_0), \quad t \in \mathbb{N}_{a+1}^{b-1}, \\
\beta_0(a) &\geq A, \;\; \beta_0(b) \geq B.
\end{aligned}
$$

These show that $\alpha_0(t)$ and $\beta_0(t)$ are lower and upper solutions of BVP (5.16). Furthermore, note that $G : \mathbb{N}_{a+1}^{b-1} \times \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing with respect to z. Thus, by Lemma 5.15, it follows that there exists a unique solution $\beta_1(t)$ of BVP (5.16) such that $\alpha_0(t) \leq \beta_1(t) \leq \beta_0(t)$ on $\mathbb{N}_a^b$.

Next, we show that $\alpha_1(t)$ and $\beta_1(t)$ are lower and upper solutions of the original BVP (5.7). Toward this end, using the fact that $\alpha_1(t)$ is the unique solution of BVP (5.15), the condition $(A_{4.1})$ (iii), and the inequalities (5.13), (5.14), we have

$$\begin{aligned}(L_a\alpha_1)(t) &= f_1(t,\alpha_0) + f_1^{(1)}(t,\beta_0)(\alpha_1 - \alpha_0) + f_2(t,\alpha_0) + f_2^{(1)}(t,\alpha_0)(\alpha_1 - \alpha_0)\\ &\geq f_1(t,\alpha_0) + f_1^{(1)}(t,\alpha_1)(\alpha_1 - \alpha_0) + f_2(t,\alpha_0) + f_2^{(1)}(t,\alpha_0)(\alpha_1 - \alpha_0)\\ &\geq f_1(t,\alpha_1) + f_2(t,\alpha_1), \quad t \in \mathbb{N}_{a+1}^{b-1},\\ \alpha_1(a) &= A, \;\; \alpha_1(b) = B,\end{aligned}$$

which proves $\alpha_1(t)$ is a lower solution of BVP (5.7). Similar arguments show that

$$\begin{aligned}(L_a\beta_1)(t) &\leq f_1(t,\beta_1) + f_2(t,\beta_1), \quad t \in \mathbb{N}_{a+1}^{b-1},\\ \beta_1(a) &= A, \;\; \beta_1(b) = B,\end{aligned}$$

which shows that $\beta_1(t)$ is an upper solution of BVP (5.7).

Finally, to see that $\alpha_1(t) \leq \beta_1(t)$ recall that, as we have shown above, for each fixed $t \in \mathbb{N}_a^b$, $\alpha_0(t) \leq \alpha_1(t), \beta_1(t) \leq \beta_0(t)$. That is, $\alpha_1, \beta_1 \in \mathcal{S}(\alpha_0, \beta_0)$. As such, $\bar{f}(t,\alpha_1(t)) = f(t,\alpha_1(t))$ and $\bar{f}(t,\beta_1(t)) = f(t,\beta_1(t))$ and so α_1 and β_1 are also lower and upper solutions to BVP (5.11). Since $\bar{f} : \mathbb{N}_{a+1}^{b-1} \times \mathbb{R} \to \mathbb{R}$ is nondecreasing with respect to x and, from Step 1, we know that x is the unique solution to BVP (5.11), we can invoke Lemma 5.10, to conclude $\alpha_1(t) \leq x(t) \leq \beta_1(t)$ on $\mathbb{N}_a^b$. And so, we obtain

$$\alpha_0(t) \le \alpha_1(t) \le x(t) \le \beta_1(t) \le \beta_0(t) \quad \text{on } \mathbb{N}_a^b.$$

Thus, by iteration, we get

$$\alpha_0(t) \le \alpha_1(t) \le \cdots \le \alpha_n(t) \le x(t) \le \beta_n(t) \le \cdots \le \beta_1(t) \le \beta_0(t) \quad \text{on } \mathbb{N}_a^b$$

where we obtain the functions α_{n+1} and β_{n+1} by the iterative schemes:

$$\begin{cases} (L_a\alpha_{n+1})(t) = F(t, \alpha_n, \beta_n; \alpha_{n+1}), \quad t \in \mathbb{N}_{a+1}^{b-1}, \\ \qquad\qquad = f(t, \alpha_n) + [f_1^{(1)}(t, \beta_n) + f_2^{(1)}(t, \alpha_n)](\alpha_{n+1} - \alpha_n) \\ \alpha_{n+1}(a) = A, \; \alpha_{n+1}(b) = B, \end{cases} \tag{5.17}$$

and

$$\begin{cases} (L_a\beta_{n+1})(t) = G(t, \alpha_n, \beta_n; \beta_{n+1}), \quad t \in \mathbb{N}_{a+1}^{b-1}, \\ \qquad\qquad = f(t, \beta_n) + [f_1^{(1)}(t, \beta_n) + f_2^{(1)}(t, \alpha_n)](\beta_{n+1} - \beta_n) \\ \beta_{n+1}(a) = A, \; \beta_{n+1}(b) = B. \end{cases} \tag{5.18}$$

Step 3: For any fixed $t \in \mathbb{N}_a^b$, the monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ are bounded above/below by $x(t)$. As such, they converge pointwise to some limit functions, ρ and r. That is, the functions ρ, $r : \mathbb{N}_a^b \to \mathbb{R}$ and satisfy

$$\lim_{n\to\infty} \alpha_n(t) = \rho(t) \le x(t) \le r(t) = \lim_{n\to\infty} \beta_n(t).$$

By taking the limit as $n \to \infty$ of the difference equation in BVPs (5.17) and (5.18) we can show that $\rho(t)$ and $r(t)$ are solutions of BVP (5.7). Since $\rho(t)$ and $r(t)$ also lie in $\mathcal{S}(\alpha_0, \beta_0)$, it must be the case that $\rho(t) = x(t) = r(t)$ on $\mathbb{N}_a^b$. Hence $\alpha_n(t)$ and

$\beta_n(t)$ both converge monotonically to $x(t)$.

Step 4: Finally, we show that the convergence of the sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ is quadratic. For this purpose, set

$$p_{n+1}(t) = x(t) - \alpha_{n+1}(t) \geq 0, \text{ and } q_{n+1}(t) = \beta_{n+1}(t) - x(t) \geq 0, \quad t \in \mathbb{N}_a^b.$$

By Corollary 5.4, the condition $(A_{4.1})$ and the Mean Value Theorem, Lemma 5.3, and Cauchy's Inequality (respectively), we obtain

$$\begin{aligned}
p_{n+1}(t) &= \int_a^b G(t,s)f(s,x(s))\nabla s - \int_a^b G(t,s)F(s,\alpha_n(s),\beta_n(s);\alpha_{n+1}(s))\nabla s \\
&= \int_a^b G(t,s)\big[f_1(s,x) + f_2(s,x)\big]\nabla s \\
&\qquad - \int_a^b G(t,s)\begin{bmatrix} f_1(s,\alpha_n) + f_1^{(1)}(s,\beta_n)(\alpha_{n+1}-\alpha_n) + \\ f_2(s,\alpha_n) + f_2^{(1)}(s,\alpha_n)(\alpha_{n+1}-\alpha_n) \end{bmatrix}\nabla s \\
&= \int_a^b G(t,s)\begin{bmatrix} \big[f_1^{(1)}(s,\beta_n) + f_2^{(1)}(s,\alpha_n)\big]p_{n+1} + \\ \big[f_1^{(1)}(s,\xi_3) - f_1^{(1)}(s,\beta_n)\big]p_n + \\ \big[f_2^{(1)}(s,\xi_4) - f_2^{(1)}(s,\alpha_n)\big]p_n \end{bmatrix}\nabla s \\
&\leq \int_a^b G(t,s)\big[f_1^{(2)}(s,\eta_1)(\xi_3-\beta_n)p_n + f_2^{(2)}(s,\eta_2)(\xi_4-\alpha_n)p_n\big]\nabla s \\
&\leq \int_a^b |G(t,s)|\big[A_1\|p_n+q_n\|\|p_n\| + B_1\|p_n\|\|p_n\|\big]\nabla s \\
&\leq MA_1\|p_n\|(\|p_n\|+\|q_n\|) + MB_1\|p_n\|^2 \\
&= \Big(\frac{3}{2}MA_1 + MB_1\Big)\|p_n\|^2 + \frac{1}{2}MA_1\|q_n\|^2
\end{aligned}$$

where $M := \max_{t\in\mathbb{N}_a^b}\sum_{s=a+1}^b |G(t,s)| = \frac{(b-a)^2}{4\Gamma(\nu+2)}$, $\alpha_n(t) \leq \xi_3(t), \xi_4(t) \leq x(t)$, $\xi_3(t) \leq \eta_1(t) \leq \beta_n(t)$, $\alpha_n(t) \leq \eta_2(t) \leq \xi_4(t)$, $|f_1^{(2)}(t,x)| \leq A_1$, $|f_2^{(2)}(t,x)| \leq B_1$ for $t \in \mathbb{N}_a^b$.

So, we have

$$\|p_{n+1}\| \leq \Big(\frac{3}{2}MA_1 + MB_1\Big)\|p_n\|^2 + \frac{1}{2}MA_1\|q_n\|^2.$$

Similarly, we may obtain

$$\|q_{n+1}\| \leq \Big(MA_1 + \frac{3}{2}MB_1\Big)\|q_n\|^2 + \frac{1}{2}MB_1\|p_n\|^2.$$

The proof is complete. □

Next, we will apply the Gauss-Seidel method to possibly improve upon the convergence rate of the iterative scheme described in Theorem 5.16.

Theorem 5.17 *Let all the hypotheses of Theorem 5.16 hold. Consider the iterative schemes given by*

$$\begin{cases} (L_a\alpha^*_{n+1})(t) = f_1(t,\alpha^*_n) + f_1^{(1)}(t,\beta^*_n)(\alpha^*_{n+1} - \alpha^*_n) \\ \qquad\qquad + f_2(t,\alpha^*_n) + f_2^{(1)}(t,\alpha^*_n)(\alpha^*_{n+1} - \alpha^*_n) \\ \qquad \equiv F(t,\alpha^*_n,\beta^*_n;\alpha^*_{n+1}), \quad t \in \mathbb{N}_{a+1}^{b-1}, \\ \alpha^*_{n+1}(a) = A, \ \alpha^*_{n+1}(b) = B, \end{cases} \tag{5.19}$$

and

$$\begin{cases} (L_a\beta^*_{n+1})(t) = f_1(t,\beta^*_n) + f_1^{(1)}(t,\beta^*_n)(\beta^*_{n+1} - \beta^*_n) \\ \qquad\qquad + f_2(t,\beta^*_n) + f_2^{(1)}(t,\alpha^*_{n+1})(\beta^*_{n+1} - \beta^*_n) \\ \qquad \equiv G(t,\alpha^*_{n+1},\beta^*_n;\beta^*_{n+1}), \quad t \in \mathbb{N}_{a+1}^{b-1}, \\ \beta^*_{n+1}(a) = A, \ \beta^*_{n+1}(b) = B. \end{cases} \tag{5.20}$$

starting with $\alpha^*_0 = \alpha_0$, $\beta^*_0 = \beta_0$ *on* $\mathbb{N}_a^b$. *The two sequences obtained via this iterative scheme*$\{\alpha^*_n(t)\}$ *and* $\{\beta^*_n(t)\}$, $n \geq 0$ *converge monotonically to the* $x(t)$, *the one solution of the BVP* (5.7) *which lies between* α_0 *and* β_0, *and the convergence is faster* *than (or equal to) quadratic.*

Proof. Initially, we compute $\alpha_1^*(t)$ using the following BVPs

$$\begin{cases} (L_a\alpha_1^*)(t) = f_1(t,\alpha_0^*) + f_1^{(1)}(t,\beta_0^*)(\alpha_1^* - \alpha_0^*) + f_2(t,\alpha_0^*) + f_2^{(1)}(t,\alpha_0^*)(\alpha_1^* - \alpha_0^*) \\ \qquad\qquad \equiv F(t,\alpha_0^*,\beta_0^*;\alpha_1^*), \quad t \in \mathbb{N}_{a+1}^{b-1}, \\ \quad \alpha_1^*(a) = A, \ \alpha_1^*(b) = B. \end{cases} \tag{5.21}$$

Now, we compute $\beta_1^*(t)$ using $\beta_0^*(t)$ and $\alpha_1^*(t)$, that is, $\beta_1^*(t)$ is a solution of

$$\begin{cases} (L_a\beta_1^*)(t) = f_1(t,\beta_0^*) + f_1^{(1)}(t,\beta_0^*)(\beta_1^* - \beta_0^*) + f_2(t,\beta_0^*) + f_2^{(1)}(t,\alpha_1^*)(\beta_1^* - \beta_0^*) \\ \qquad\qquad \equiv G(t,\alpha_1^*,\beta_0^*;\beta_1^*), \quad t \in \mathbb{N}_{a+1}^{b-1}, \\ \quad \beta_1^*(a) = A, \ \beta_1^*(b) = B. \end{cases} \tag{5.22}$$

It is clear that $\alpha_0(t) = \alpha_0^*(t) \le \alpha_1^*(t)$ and $\beta_1^*(t) \le \beta_0^*(t) = \beta_0(t)$ on $\mathbb{N}_a^b$.
Put $p(t) := \beta_1^*(t) - \beta_1(t)$. Then $p(a) = p(b) = 0$. Also, we have

$$\begin{aligned} (L_ap)(t) =& \,[f_1(t,\beta_0^*) + f_1^{(1)}(t,\beta_0^*)(\beta_1^* - \beta_0^*) + f_2(t,\beta_0^*) + f_2^{(1)}(t,\alpha_1^*)(\beta_1^* - \beta_0^*)] \\ & - [f_1(t,\beta_0) + f_1^{(1)}(t,\beta_0)(\beta_1 - \beta_0) + f_2(t,\beta_0) + f_2^{(1)}(t,\alpha_0)(\beta_1 - \beta_0)] \\ \ge& \, f_1^{(1)}(t,\beta_0)(\beta_0^* - \beta_1) + f_2^{(1)}(t,\alpha_1^*)(\beta_0^* - \beta_1) \\ =& \,[f_1^{(1)}(t,\beta_0) + f_2^{(1)}(t,\alpha_1^*)]p. \end{aligned}$$

So p is a lower solution to a BVP of the form BVP (5.9). Thus, by Corollary 5.12, we know $p(t) \le 0$ on $\mathbb{N}_a^b$. That is, $\beta_1^*(t) \le \beta_1(t)$ on $\mathbb{N}_a^b$. Using similar arguments we are iteratively able to show that for all $n \ge 0$, $\alpha_n(t) \le \alpha_n^*(t)$ and $\beta_n^*(t) \le \beta_n(t)$ for all $t \in \mathbb{N}_a^b$. Hence the sequences $\{\alpha_n^*(t)\}$ and $\{\beta_n^*(t)\}$ must converge at least as fast as the sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$ that were computed using the iterative scheme described in Theorem 5.16. The proof is complete. □

Remark 5.18 When the function $f(t,x)$ is the sum of $(n-1)$-hyperconvex and $(n-1)$-hyperconcave functions (i.e. $f(t,x) = f_1(t,x) + f_2(t,x)$, where $f_1^{(n)}(t,x) \geq 0$, and $f_2^{(n)}(t,x) \leq 0$), we can obtain two monotone sequences $\{\alpha_n(t)\}$ and $\{\beta_n(t)\}$, whose convergence is of order n $(n \geq 2)$. The proof is similar to that of Theorem 5.16, so we omit the details.

5.4 Example

Now, we give an example to illustrate the results established in the previous section.

Example 5.19 Consider the following BVP

$$\begin{cases} (L_a x)(t) = -\dfrac{1}{3}x^3(t) + \dfrac{1}{2}x^2(t) + x(t), & t \in \mathbb{N}_1^4, \\ x(0) = 0, \ x(5) = 1, \end{cases} \tag{5.23}$$

where $a = 0$, $\nu = \frac{1}{2}$.

Taking $\alpha_0(t) \equiv 0$, $\beta_0(t) \equiv 1$, it is quick to verify that $\alpha_0(t)$, $\beta_0(t)$ are lower and upper solutions of BVP (5.23), respectively. Let $f(t,x) = f(x) = -\frac{1}{3}x^3 + \frac{1}{2}x^2 + x$. Then $f(x) = f_1(x) + f_2(x)$ where $f_1(x) = \frac{1}{2}x^2 + x$ and $f_2(x) = -\frac{1}{3}x^3$. Thus,

$$f_1^{(1)}(t,x) = x + 1 > 0, \ f_1^{(2)}(t,x) = 1 > 0 \ \text{ on } \Omega = \mathbb{N}_1^4 \times [0,1],$$

$$f_2^{(1)}(t,x) = -x^2 \leq 0, \ f_2^{(2)}(t,x) = -2x(t) \leq 0 \ \text{ on } \Omega.$$

Table 5.1: Table of three α, β-iterates of (5.23).

t	$\alpha_0(t)$	$\alpha_1(t)$	$\alpha_2(t)$	$\alpha_3(t)$	$\beta_3(t)$	$\beta_2(t)$	$\beta_1(t)$	$\beta_0(t)$
0	0	0	0	0	0	0	0	1
1	0	0.007355	0.021545	0.027694	0.028628	0.067598	0.354038	1
2	0	0.025743	0.058914	0.069876	0.071137	0.115480	0.405802	1
3	0	0.087341	0.152969	0.167201	0.168294	0.205791	0.454208	1
4	0	0.295581	0.391591	0.403170	0.403700	0.424171	0.582092	1
5	0	1	1	1	1	1	1	1

Table 5.2: Table of three α^*, β^*-iterates of (5.23).

t	$\alpha_0^*(t)$	$\alpha_1^*(t)$	$\alpha_2^*(t)$	$\alpha_3^*(t)$	$\beta_3^*(t)$	$\beta_2^*(t)$	$\beta_1^*(t)$	$\beta_0^*(t)$
0	0	0	0	0	0	0	0	1
1	0	0.007355	0.021572	0.028417	0.028532	0.066598	0.353610	1
2	0	0.025743	0.058982	0.070027	0.070946	0.112921	0.404336	1
3	0	0.087341	0.153110	0.167290	0.168026	0.200221	0.449635	1
4	0	0.295581	0.391971	0.403220	0.403523	0.418531	0.570861	1
5	0	1	1	1	1	1	1	1

Now, if we apply the iteration scheme of Theorem 5.16, after three iterations we find the α, β-iterates given in Table 5.1. The graph in Figure 5.1 shows the α-iterates (with broken line) and the β-iterates (with unbroken line).

If we apply the iteration scheme of Theorem 5.17, after three iterations we find the α^*, β^*-iterates given in Table 5.2. The graph in Figure 5.2 shows the α^*-iterates (with broken line) and the β^*-iterates (with unbroken line).

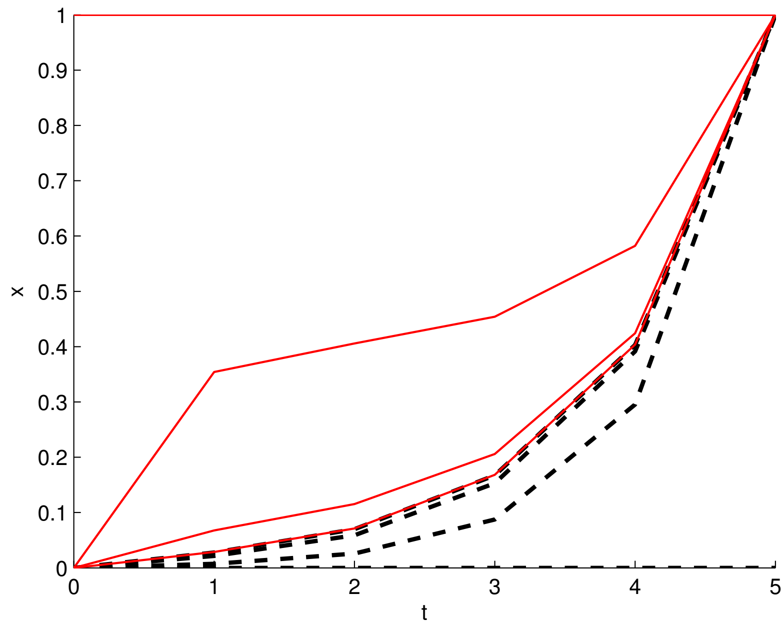


Figure 5.1: α, β-iterates

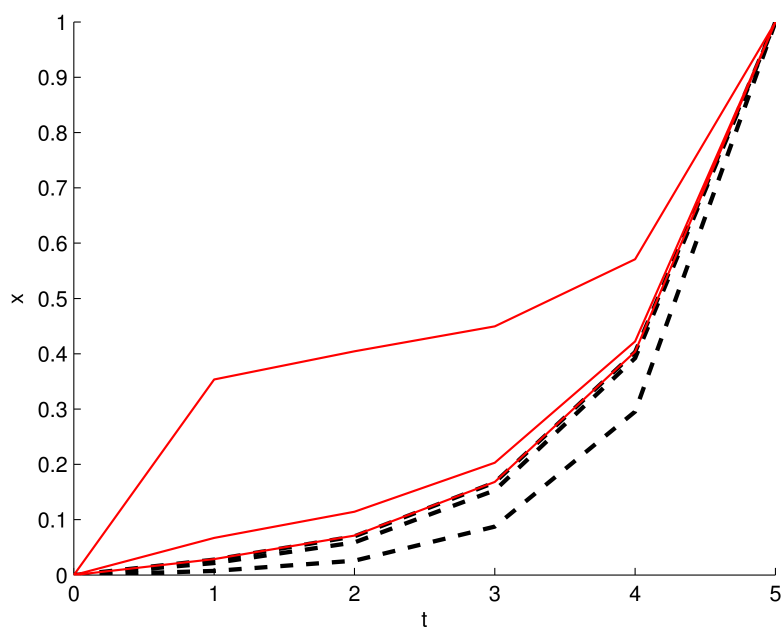


Figure 5.2: α^*, β^*-iterates

Appendix A

Select Definitions and Results from *Discrete Fractional Calculus* by C. Goodrich and A. C. Peterson

Definition A.1. *(See [5, Definition 3.4]). The (generalized) rising function is defined in terms of the Gamma function by*

$$t^{\overline{r}} := \frac{\Gamma(t+r)}{\Gamma(t)} \tag{A.1}$$

for those values of t and r so that the right-hand side of (A.1) *is sensible.*

Definition A.2. *(See [5, Definition 3.56]). Let $\nu \neq -1, -2, \cdots$. Then we define the ν-th order nabla fractional Taylor monomial $H_\nu(t,a)$, by*

$$H_\nu(t,a) := \frac{(t-a)^{\overline{\nu}}}{\Gamma(\nu+1)} = \frac{\Gamma(t-a+\nu)}{\Gamma(t-a)\Gamma(\nu+1)} \tag{A.2}$$

whenever the right-hand side of this equation is sensible. [1]

---

[1] In this dissertation, we extend the ν-th order Taylor monomials by defining

$$\widetilde{H}_\nu(t,a) := \lim_{\varepsilon\to 0} \frac{\Gamma(t-a+\nu+\varepsilon)}{\Gamma(t-a+\varepsilon)\Gamma(\nu+1+\varepsilon)}$$

for *all* $\nu \in \mathbb{R}$, and *all* $t \in \mathbb{Z}_a$. Since the Gamma function is continuous on $(0,\infty)$ and has no zeros, it is clear that when $t > a$ and $\nu > -1$, $\widetilde{H}_\nu(t,a) = H_\nu(t,a)$.

Definition A.3. *(Nabla Fractional Sum [5, Definition 3.58]). Let* $f : \mathbb{N}_a \to \mathbb{R}$, $\nu > 0$ *be given. Then*

$$(\nabla_a^{-\nu} f)(t) = \int_a^t H_{\nu-1}(t, \rho(s)) f(s) \nabla s, \quad t \in \mathbb{N}_a. \tag{A.3}$$

Definition A.4. *(Nabla Fractional Difference [5, Definition 3.61]). Let* $f : \mathbb{N}_a \to \mathbb{R}$, $\nu > 0$ *be given, and let* $N := \lceil \nu \rceil$, *where* $\lceil \cdot \rceil$ *is the ceiling function. Then we define the* ν*th-order nabla fractional difference* $\nabla_a^{\nu} f(t)$ *by*

$$(\nabla_a^{\nu} f)(t) = (\nabla^N (\nabla_a^{-(N-\nu)} f))(t), \quad t \in \mathbb{N}_{a+N}. \tag{A.4}$$

Definition A.5. *(Caputo Nabla Fractional Difference [5, Definition 3.117]). Let* $f : \mathbb{N}_{a-N+1} \to \mathbb{R}$, $\nu > 0$ *be given, and let* $N := \lceil \nu \rceil$. *Then we define the* ν*th-order Caputo nabla fractional difference* $\nabla_{a^*}^{\nu} f(t)$ *by*

$$(\nabla_{a^*}^{\nu} f)(t) = (\nabla_a^{-(N-\nu)} (\nabla^N f))(t), \quad t \in \mathbb{N}_{a+1}. \tag{A.5}$$

Theorem A.6. *(See [5, Definition 3.61 and Theorem 3.62]). Assume* $f : \mathbb{N}_a \to \mathbb{R}$, $\nu > 0$, $\nu \notin \mathbb{N}_1$, *and choose* $N \in \mathbb{N}_1$ *such that* $N - 1 < \nu < N$. *Then*

$$(\nabla_a^{\nu} f)(t) = \int_a^t H_{-\nu-1}(t, \rho(s)) f(s) \nabla s, \quad t \in \mathbb{N}_{a+N}, \tag{A.6}$$

Theorem A.7. *(Nabla Leibniz Formula [5, Theorem 3.41]). Assume* $f : \mathbb{N}_a \times \mathbb{N}_{a+1} \to \mathbb{R}$*. Then for* $t \in \mathbb{N}_{a+1}$*,*

$$\nabla\Big(\int_a^t f(t,\tau)\nabla\tau\Big) = \int_a^t \nabla_t f(t,\tau)\nabla\tau + f(\rho(t),t). \tag{A.7}$$

Also,

$$\nabla\Big(\int_a^t f(t,\tau)\nabla\tau\Big) = \int_a^{t-1} \nabla_t f(t,\tau)\nabla\tau + f(t,t). \tag{A.8}$$

Corollary A.8. *(See [5, Corollary 3.122]). For* $\nu > 0$*,* $N = \lceil \nu \rceil$*, and* $h : \mathbb{N}_{a+1} \to \mathbb{R}$*, we have that*

$$(\nabla_a^{-(N-\nu)}(\nabla_a^{N-\nu}h))(t) = h(t), \quad t \in \mathbb{N}_{a+1}.$$

Appendix B

Summary of Results from Chapters 2 and 3

Definition B.1. *For all* $\gamma \in \mathbb{R}$*, and all* $s, t \in \mathbb{Z}_a$*, we define:*

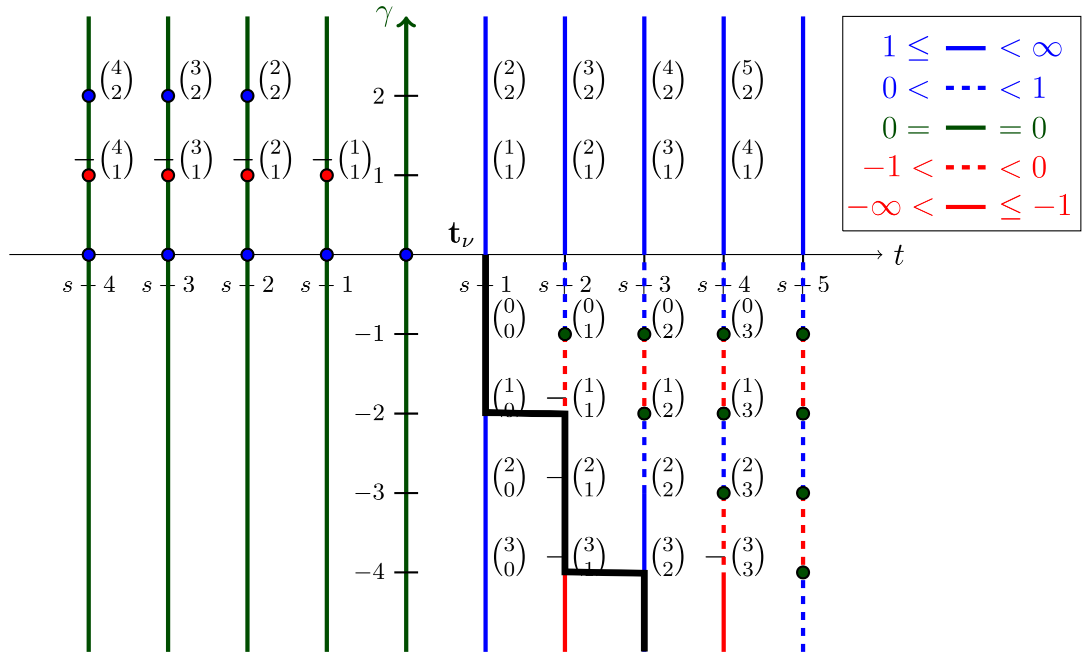
$$\widetilde{H}_\gamma(t,s) := \lim_{\varepsilon \to 0} \frac{\Gamma(t-s+\gamma+\varepsilon)}{\Gamma(t-s+\varepsilon)\Gamma(\gamma+1+\varepsilon)} \quad \text{and} \quad H_\gamma(t,s) := \begin{cases} \widetilde{H}_\gamma(t,s), & t > s, \\ 0 & , \ t \leq s. \end{cases}$$

Corollary B.2. *For all* $\gamma \in \mathbb{R}$ *and all* $t, s \in \mathbb{Z}_a$

$$\widetilde{H}_\gamma(t,s) = \begin{cases} P_{t-s-1}(\gamma), & \text{for } t > s \text{ and } \gamma \in \mathbb{R} \\ P_\gamma(t-s-1), & \text{for } t \leq s \text{ and } \gamma = n \in \mathbb{N}_0 \\ 0, & \text{for } t \leq s \text{ and } \gamma \in \mathbb{R} \setminus \mathbb{N}_0. \end{cases}$$

Remark B.3. *Theorem 1.8 and 1.9 and Corollary 3.11 combine to give us the following formula and complete picture of* $\widetilde{H}_\gamma(t,s)$ *for all* $\gamma \in \mathbb{R}$ *and all* $t, s \in \mathbb{Z}_a$*.*

$$\widetilde{H}_\gamma(t,s) = P_{\rho(t)-s}(\gamma) + (-1)^\gamma \binom{|t-s|}{\gamma} \mathbb{1}_{\mathbb{N}_0}(\gamma) \mathbb{1}_{(-\infty,s]}(t) \ .$$



$$\widetilde{H}_\gamma(t,s) := \lim_{\epsilon\to 0}\frac{\Gamma(t-s+\gamma+\epsilon)}{\Gamma(t-s+e)\Gamma(\gamma+1+e)}$$

Figure B.1 - $\widetilde{\mathbf{H}}_\gamma(\mathbf{t},\mathbf{s})$ with $\mathbf{t}_\nu$ Annotated

Theorem B.4. *For fixed $\nu > 0$, $H_\nu(t,a) \to \infty$ as $t \to \infty$.*

Corollary B.5. *For $\nu > 0$, $H_\nu(\cdot,a)$ is strictly increasing and without bound on $\mathbb{N}_a$.*

Theorem B.6. *For fixed $\nu > 0$, $H_{-\nu}(t,a) \to 0$ as $t \to \infty$.*

Theorem B.7. *(P-Integrals)*

$$\int_a^\infty H_{-p}(t,a)\nabla t = \begin{cases} 0 \ , \ \textit{whenever } p > 1 \\ 1 \ , \ \textit{when } p = 1 \\ \infty \ , \ \textit{whenever } p < 1. \end{cases}$$

Definition B.8. *For $\nu > 0$, we put $N := \lceil \nu \rceil$ and define*

$$t_\nu := \sup\{t \in \mathbb{Z}_a : t \le a + N \ \textit{and} \ \left(\nabla|H_{-\nu}(\cdot,a)|\right)(t) \ge 0\}.$$

Corollary B.9. *($\|H_{-\nu}\|_\infty$ and Monotonicity) For all $\nu > 0$,*

$$t_\nu = a + \lfloor \frac{\nu}{2} \rfloor + 1.$$

$|H_{-\nu}(\cdot, a)|$ is increasing on $(-\infty, t_\nu]$ and decreasing on $[t_\nu, \infty)$. Therefore, [1]

$$\|H_{-\nu}(\cdot, a)\|_\infty = |H_{-\nu}(t_\nu, a)| = \left| P_{\lfloor \nu/2 \rfloor}(-\nu) \right|.$$

Definition B.10. *For $\nu > 0$, we define*

$$T_\nu := \sup\{t \in \mathbb{Z}_a : |H_{-\nu}(t, a)| \geq 1\}.$$

That is, T_ν is the latest time for which $|H_{-\nu}(t, a)| \geq 1$.

Corollary B.11. *For $\nu > 0$ and T_ν as defined above,*

$$T_\nu = a + \max\{\lfloor \nu \rfloor, 1\}.$$

Theorem B.12. *For $\nu > 0$, $sgnH_{-\nu}(\cdot, a)$ alternates on $[a + 1, T_\nu]$ and is constant on $[T_\nu + 1, \infty)$. That is, $sgnH_{-\nu}(t, a) = c$ for all $t \in \mathbb{N}_{T_\nu + 1}$, where*

$$c = \begin{cases} 1, & \text{when } \nu \in (0, 1) \\ (-1)^{T_\nu - a}, & \text{when } \nu \in (1, \infty) \setminus \mathbb{N}_1 \\ 0, & \text{when } \nu \in \mathbb{N}_1 \end{cases} = \begin{cases} (-1)^{\lfloor \nu \rfloor}, & \text{when } \nu \in (0, \infty) \setminus \mathbb{N}_1 \\ 0, & \text{when } \nu \in \mathbb{N}_1. \end{cases}$$

[1] Interesting fact: The function $f(\nu) := \|H_{-\nu}(\cdot, a)\|_\infty$ is continuous and increasing on $[0, \infty)$.

Remark B.13. *Recall that for all $\gamma \in \mathbb{R}$, $\nu > 0$, $N := \lceil \nu \rceil$, and for all $t \in \mathbb{Z}_a$:*

$$(\nabla_a^\gamma f)(t) = \int_a^t H_{-\gamma-1}(t, \rho(s)) f(s) \nabla s$$
$$(\nabla^\gamma f)(t) = \int_{-\infty}^t H_{-\gamma-1}(t, \rho(s)) f(s) \nabla s = \int_{-\infty}^\infty H_{-\gamma-1}(t, \rho(s)) f(s) \nabla s = (H_{-\gamma-1} * f)(t)$$
$$(\nabla_{a*}^\nu f(t) := (\nabla_a^{\nu-N} \nabla^N f)(t).$$

Theorem B.14. *For $k, n \in \mathbb{N}_0$, $\alpha, \beta \in \mathbb{R}$, $\nu > 0$, and $N := \lceil \nu \rceil$ the following equations show how the various nabla operators act on the Taylor monomials. For all $t \in \mathbb{Z}_a$,*

$$\nabla_\cdot^\gamma H_\beta(t,a) = H_{\beta-\gamma}(t,a) \quad \textit{where} \quad \nabla_\cdot^\gamma \in \{\nabla^\alpha, \nabla_a^\alpha, \nabla_{a*}^\nu\}$$
$$\nabla_a^\alpha \widetilde{H}_k(t,a) = H_{k-\alpha}(t,a)$$
$$\nabla^n \widetilde{H}_k(t,a) = \begin{cases} \widetilde{H}_{k-n}(t,a), & (k-n) \geq 0 \\ 0, & (k-n) < 0 \end{cases}$$
$$\nabla_{a*}^\nu \widetilde{H}_k(t,a) = \begin{cases} H_{k-\nu}(t,a), & (k-\nu) \geq 0 \\ 0, & (k-\nu) < 0. \end{cases}$$

Remark B.15. *The following operator composition rules hold for all $f : \mathbb{Z}_a \to \mathbb{R}$. For all $\alpha, \gamma \in \mathbb{R}$, $\mu, \nu \in [0, \infty)$, $M := \lceil \mu \rceil$, $N := \lceil \nu \rceil$, and $m, n \in \mathbb{N}_0$:* [2]*:*

(i) (a) $\nabla^\gamma \nabla_a^\alpha = \nabla_a^{\gamma+\alpha}$

(b) $\nabla_a^\gamma \nabla_a^\alpha = \nabla_a^{\gamma+\alpha}$

(c) $\nabla_{a*}^\mu \nabla_a^\alpha = \nabla_a^{\mu+\alpha}$

[2] Setting $\gamma = 0$ in (iii)(a) gives us a formula for the Caputo fractional difference in terms of the Riemann-Liousville fractional difference.

(ii) *(a)* $\nabla^{\gamma}\nabla^{n} = \nabla^{\gamma+n}$

(b) $\nabla_a^{\gamma}\nabla^{n} f = \nabla_a^{\gamma+n}\left[f - \sum_{k=0}^{n-1}[(\nabla^k f)(a)]H_k(\cdot, a)\right]$

(c) $\nabla_{a*}^{\mu}\nabla^{n} = \nabla_{a*}^{\mu+n}$

(iii) *(a)* $\nabla^{\gamma}\nabla_{a*}^{\nu} f = \nabla_a^{\gamma+\nu}\left[f - \sum_{k=0}^{N-1}[(\nabla^k f)(a)]H_k(\cdot, a)\right]$

(b) $\nabla_a^{\gamma}\nabla_{a*}^{\nu} f = \nabla^{\gamma}\nabla_{a*}^{\nu} f$

(c) $\nabla_{a*}^{\mu}\nabla_{a*}^{\nu} f = \nabla^{\mu}\nabla_{a*}^{\nu} f.$

Bibliography

[1] W. G. Kelley, A. C. Peterson, *Difference Equations: An Introduction with Applications.* 2nd ed., Academic Press, New York, 2001.

[2] R. P. Agarwal, *Difference Equations and Inequalities.* Marcel Dekker, New York, NY, USA, 1992.

[3] C. S. Goodrich, Some new existence results for fractional difference equations. Int. *J. Dyn. Syst. Differ. Equ.* 3 (2014), 145-162.

[4] C. S. Goodrich, Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions. Comput. Math. Appl. 61 (2009), 191-202.

[5] G. S. Goodrich, A. C. Peterson, *Discrete Fractional Calculus.* Springer, New York, 2015.

[6] Rudin, Walter, *Real and Complex Analysis.* 2nd ed., New York: McGraw-Hill, 1974.

[7] Apostol, Tom M., *Mathematical Analysis* 2nd ed., Massachusetts: Addison-Wesley Publishing Company, 1974.

[8] D. Dummit, R. Foote, *Abstract Algebra.* 3rd ed., John Wiley and Sons, United States, 2004.

[9] J. Jonnalagadda, Solutions of perturbed linear nabla fractional difference equations. Differ. Equ. Dyn. Syst. 22 (3) (2014), 281-292.

[10] F. M. Atici, F. Wu, Existence of solutions for nonlinear fractional difference equations with initial conditions. Dynam. Systems Appl. 23 (2-3) (2014), 265-276.

[11] F. M. Atici, P. W. Eloe, Two-point boundary value problems for finite fractional difference equations. J. Difference Equ. Appl. 17 (4) (2011), 445-456.

[12] T. Abdeljawad, On Riemann and Caputo fractional differences. Comput. Math. Appl. 62 (2011), 1602-1611.

[13] R. Bellman, R. Kalaba, *Quasi-linearization and Nonlinear Boundary Value Problems.* Elsevier, New York, 1965.

[14] V. Lakshmikantham, A. S. Vatsala, *Generalized Quasi-linearization for Nonlinear Problems.* Kluwer Academic Publishers, Dordrecht, 1998.

[15] B. Ahmad, R. A. Khan, S. Sivasundaram, Generalized quasi-linearization method for nonlinear functional differential equations. J. Appl. Math. Stoch. Anal. 16 (2003), 33-43.

[16] B. Ahmad, A. Alsaedi, An extended method of quasi-linearization for nonlinear impulsive differential equations with a nonlinear three-point boundary condition. Electron. J. Qual. Theory Differ. Equ. 1 (2007), 1-19.

[17] P. Amster, P. De Napoli, A quasi-linearization method for elliptic problems with a nonlinear boundary condition. Nonlinear Anal. 66 (10) (2007), 2255-2263.

[18] Z. Drici, F. A. McRae, J. V. Devi, Quasi-linearization for functional differential equations with retardation and anticipation. Nonlinear Anal. 70 (2009), 1763-1775.

[19] M. El-Gebeily, D. O'Regan, Existence and generalized quasi-linearization methods for singular nonlinear differential equations. Dynam. Systems Appl. 21 (2008), 67-81.

[20] A. Yakar, Initial time difference quasi-linearization for Caputo fractional differential equations. Adv. Difference Equ. 2012, 2012: 92, 9pages.

[21] P. G. Wang, H. X. Wu, Y. H. Wu, Higher even-order convergence and coupled solutions for second-order boundary value problems on time scales. Compu. Math. Appl. 55 (2008), 1693-1705.

[22] R. P. Agarwal, S. Hristova, Quasilinearization for initial value problems involving differential equations with “maxima". Math. Comput. Modelling 55 (2012), 2096-2105.

[23] V. Anderson, C. Bettis, S. Brown, et al, Superlinear convergence via mixed generalized quasi-linearization method and generalized monotone method. Involve. A Journal of Mathematics 7 (5) (2014), 699-712.

[24] M. Sowmya, A. S. Vatsala, Generalized iterative methods for Caputo fractional differential equations via coupled lower and upper solutions with superlinear convergence. Nonlinear Dyn. Syst. Theory 15 (2) (2015), 198-208.

[25] X. Liu, B. G. Jia, L. Erbe, A. C. Peterson, Existence and rapid convergence results for nonlinear Caputo nabla fractional difference equations. Electron. J. Qual. Theory Differ. Equ. 51 (2017), 1-16.