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Course Portfolio for Math 407 Mathematics for High School Teaching: Refining Conceptual Understanding in a Mathematics Course for Pre-service Teachers

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Abstract

My intention in this portfolio is to present my approach to teaching an upper-level mathematics course for pre-service secondary level mathematics teachers. Several teaching strategies are discussed in the context of designing a coherent approach to this course, which emphasizes the need for conceptual reasoning above all other goals. These strategies are evaluated and assessed in connection to the learning outcomes using samples of student work from the course.

Also presented are samples of course materials that were used to lead students through an organized discussion of the relevant concepts. These materials convey some basic mathematical knowledge and therefore may suited to other courses as well. Additionally, this portfolio includes a survey of students perceptions and attitudes towards conceptual mathematics at the beginning of the course, which can be viewed as baseline information, as well as a sample of student work production and self-reflections at the end of the curse, which establish a certain growth in confidence and abilities.

Keywords: Undergraduate teaching, mathematics, conceptual reasoning, secondary education, pre-service teachers.

Contents

Chapter 1

Introduction and Objectives of Course Portfolio

I enrolled in the Peer Review of Teaching Project with the objective of making my instructional choices more informed and intentional and with a secondary objective of learning from the experience of the other participants.

With respect to my target course, Math 407 – Mathematics for High School Teaching, my first objective is to to examine the design of the course and make purposeful decisions regarding the design in a way that directly influences student learning. In particular, I would like to isolate the core conceptual underpinnings of the high school mathematics curriculum and make sure they are reflected in the course. At the same time, and perhaps as a higher priority, I would like to emphasize conceptual reasoning in the classroom and implement it in all the aspects of the course.

My second objective is to document the major resources, materials, strategies and interventions that I have created for the course, and make them available as a resource for teaching the course in the future. I would also like to document how these strategies and resources influence student learning and make the resulting conclusions available to those who study the education of pre-service teachers.

My third objective is to take a first step towards a more in-depth discussion of how conceptual reasoning and in particular proof writing can be tailored to the needs of future educators, with a view of subsequently redeveloping the prerequisite courses.

The course analysis comprised in this portfolio is performed primarily by means of examining student work. Although this may be surprising in a mathematics course, the analysis is predominantly qualitative as this correlates best with the rigors of conceptual reasoning. This includes a discussion on how systematic feedback on written assignments influences student learning and helps them in preparing for formal evaluations. This is viewed as a cornerstone of refining conceptual understanding.

Chapter 2

Benchmark Memo 1: Reflections on the Course Syllabus

2.1 Description of the Course

UNL's online course catalog presents the following rather cryptic description:

> **UNL Course Bulletin Description:**
> **MATH 407 – Mathematics for High School Teaching I**
>
> Analysis of the connections between college mathematics and high school algebra and precalculus.

Below is my own interpretation of what this should mean, based on reviewing the current algebra and precalculus curriculum for high school and on my knowledge of the topics covered in university courses offered in the mathematics department:

High School Mathematics	**College Mathematics**	**Connections**
Number Systems	Intro to Modern Algebra	Emphasis on the integers
Functions and relations	Calculus I, II, III Intro to Real Analysis	Emphasis on functions Relations not covered
Trigonometry	Calculus I, II, III	Using trig identities and polar coordinates
Analytic Geometry	Calculus I, II, III	Recognizing equations of curves and surfaces

Notice that while all the main components of the high school algebra and precalculus curriculum are *used* in mainstream college-level mathematics courses, such as the Calculus sequence, the only aspects that are *revisited* and *reinforced* at the college level are the number systems, particularly the integers and to a lesser extent the complex numbers, which are treated in college from an abstract algebraic viewpoint.

2.1.1 Purpose of the course

This course serves three major purposes. First: to offer students, who are pre-service teachers, a refresher on some of the main topics of the algebra and precalculus high school curriculum. Second: to strengthen the students' conceptual reasoning skills and emphasize usage of conceptual as opposed to procedural reasoning in the classroom setting. Third: to make explicit connections between the course material and the practice of teaching at the high school level. This is needed as the students are heading into the practicum component of their program in the semester following this course.

2.1.2 Students

The students are all junior and senior level mathematics education majors, although there was one student who was both a mathematics and a mathematics education major. They will have taken the complete Calculus sequence as well as a "proofs-based" course in Modern (Abstract) Algebra, which will give them the intellectual framework to handle the abstract concepts in the course. These are the pre-requisites for the course. An initial survey showed that all students had also completed a course in Differential Equations and all but one student had completed a course in Linear Algebra. During the semester it became apparent that most students were simultaneously enrolled in the Concepts in Geometry course, which is also part of the required curriculum for pre-service mathematics teachers. Normally they were supposed to have had this class the semester before, but the course was not offered that semester.

2.1.3 Position in the Math and Education Curriculum

This course is a required course for education majors seeking an endorsement in secondary mathematics education. It is the first semester in a year-long sequence. The second semester, Math 408 – Mathematics for High School Teaching II, is described as an "analysis of the connections between college mathematics and high school algebra and geometry" and typically focuses more in-depth on topics from geometry.

The course is also cross-listed at the 800-level as Math 807. One student was enrolled in this version of the course while she was pursuing a masters degree in Mathematics Education.

2.1.4 The Course and the Broader Curriculum

The course is not a prerequisite course for any other course. Nevertheless, it is a course which informs the future activities of these students during their practicum and beyond. Furthermore it should be viewed as a course that goes hand in hand with and complements teaching pedagogy courses offered by the education department. For a student view on this symbiosis please consult Appendix 6.5.

2.2 Objectives of the Course

2.2.1 Knowledge

Students should come away from the course knowing the conceptual underpinnings of high school algebra and precalculus.

Concretely, some of the main topics they should be familiar with include:

- Number systems: the integer, rational, real, complex numbers,
- Relations and functions: similarities and differences between these notions, invertible functions and transformations,
- Trigonometric functions derived from the wrapping function of the number line around the unit circle,
- Elements of analytic geometry, with emphasis on conic sections.

2.2.2 Rigorous Communication Abilities

Students should be able to reason using the concepts above in a completely rigorous fashion. This includes making appropriate use of definitions, making mathematical claims and proving them according to the principles of formal and rigorous mathematics communication. At the same time, students should learn how to rephrase their explanations in language accessible to a variety of ages and levels (e.g. middle school and high school students) without compromising the accuracy.

2.2.3 Conceptual Understanding

Student should understand and be able to articulate *why* the standard methods of high school algebra and pre calculus (e.g. the vertical line test) work the way they do starting from the relevant definitions. They should be able to understand this well enough to internalize the need of emphasizing the conceptual reasoning (the *why's*) over the procedural (the *how to's*) in their daily practice.

2.2.4 Perspectives and Attitudes

It is common for many students, even mathematics majors, to have a negative attitude toward writing proofs which they often view either as being overwhelming or as tedious. However, rigorous reasoning and validation of mathematical results via proofs lies at the heart of the discipline. It is often the case that after having experiences which demonstrate the power and beauty of certain proof techniques, some students become drawn to these aspects. Yet other students become at least acclimated to this rigorous environment when the process of proof writing is demystified by being placed into a rigorous framework.

In terms of attitudes toward proof writing, a pre-class survey was administered to Math 407 students asking them to self-rate their abilities on a scale that includes:

1. I understand what proofs are, but I am not confident that I can write proofs.
2. I can write proofs, but am not completely comfortable with some proof methods
3. I can write proofs, but sometimes I get lost in the details.
4. I am completely comfortable with writing proofs and I can do this well.

Figure 2.2.4 presents a survey of students' pre-class self-reported confidence in proof writing, with each column representing an individual student's level of confidence.

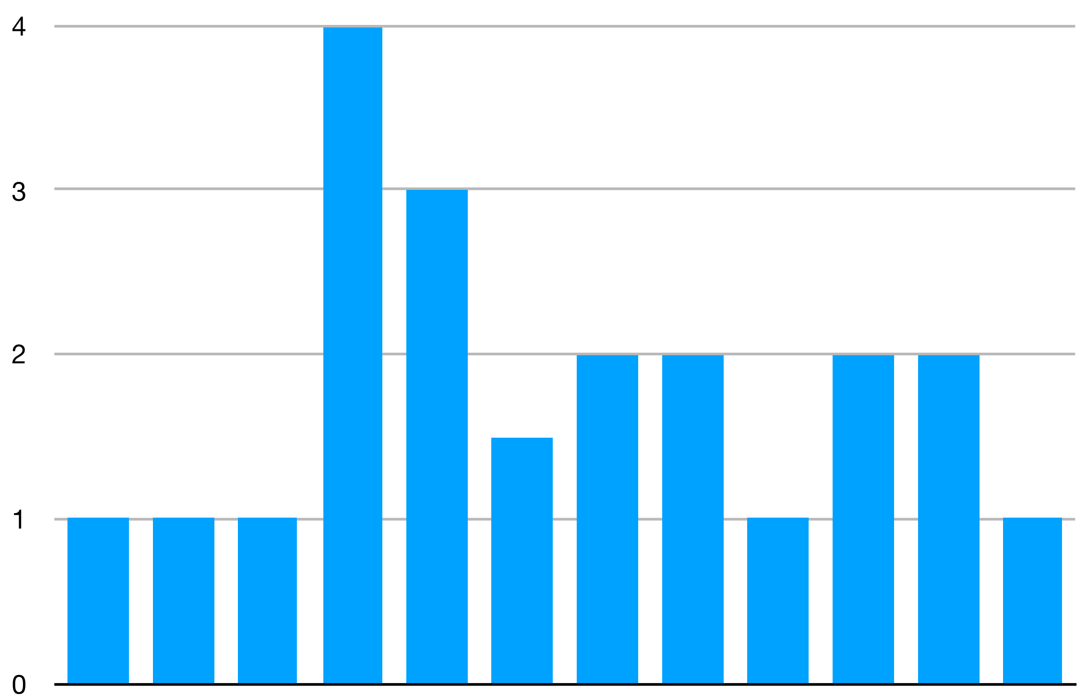


Figure 2.1: Pre class survey on student's confidence in their proof writing skills

One of my key objectives in this course was to improve both the skills and the self-confidence level of the students with respect to proof writing.

2.2.5 Impact on the Larger Society

As the focus of this course is on future educators, this is a major opportunity to make and impact on many more individuals that just the students in this course, since the current students will in turn use their knowledge to educate their students. It would be interesting (and I believe it is an ongoing effort in the mathematics education community) to track how teacher education is reflected in the later outcomes of their instructional practices.

2.2.6 Justification for the Goals

We want the student to come away from the course with an ability to make in impact in the lives of others through the teaching of mathematics at the high school level. Therefore, we want to give them the knowledge necessary for this purpose, as well as the rigorous thinking required to present the mathematics accurately and to involve their future students in a process of discovery and validation as they mature as mathematical thinkers. We also want to develop some practical skills, especially the practice of interacting with peers and students in regards to the mathematical content of the course.

It is necessary for students to achieve these goals if they want to understand more deeply and systematically the material involved in their future profession. Without conceptual mathematical training, they run the risk of passing on a mentality that mathematics is a collection of rote procedures and rules to their future students.

Chapter 3

Benchmark Memo 2: Description of Course Activities

3.1 Teaching Methods and Classroom Time

The classroom time scheduled for this course is atypical, namely the course meets once day per week from 6-8:30 pm. Inquiring with the students enrolled in the Fall 2018 class, it appears that the main reason for this choice of course organization, which is that some of the students enrolled would be participating in practicum activities, is in fact not an applicable reason. In fact, students reported that they would start their practicum in the *following* academic year.

Therefore, one of the changes I would suggest is to switch the format to the class meeting for 75 minutes twice a week during regular classroom hours (before 5 pm). This would allow sufficient time to maintain the active learning format described below, while also ensuring that the students' attention remains within optimal levels throughout the duration of the class meeting.

Given the objectives of the course as well as the late meeting time, it is important that the course be structured in a manner that helps keep the students focused and maximizes student learning. The main method employed to ensure these results consists of designing the course as an active learning environment, where students are presented with guiding questions and problems which they explore, in groups, during the class meetings.

In order to provide the structure and model good mathematics communication, summary notes are typed up to record the classroom discussion and disseminate the main concepts and results.

3.1.1 How Are The Methods Used in Class?

A typical class is structured around an activity worksheet. Every week, the class discussion opens with an opening mathematical inquiry. For example, for the section on complex numbers the opening inquiry consists of an activity meant to get students to think about transformations that are obtained by composing a given transformation with itself. One can easily see that the transformation given by $x \mapsto x + 4$ can be decomposed as a sequence $x \mapsto x + 2 \mapsto (x + 2) + 2$ of two transformations which are each defined by adding the number 2 (or, geometrically as translation on the number line by 2).

A natural question arises as to what other transformations can be decomposed in a similar manner. In particular, for the transformation $x \mapsto -x$ (which can be vied as a rotation of the number line by $180°$) to be written as a composition of two identical transformations, one needs to decompose it into two rotations by $90°$, which correspond to $x \mapsto ix$. This is a good jumping off point into the need for and meaning of *complex numbers*, see the sample course activity worksheet 6.2.

In the rest of the time, students engage with each other in groups to solve various problems aimed at either clarifying certain mathematical notions or discovering new properties of these notions. At various points during the class (say 2-5 in the course of a 2.5 hour meeting), the students are asked to come together and summarize their findings to the class. They are often asked to report on the findings of their working group from their previous discussion. A formal summary of the findings is recorded by the entire class to solidify their ideas.

During the working sessions, I observe the progress of the groups, asking questions that alert them to any issues that arise or making suggestions aimed at resolving individual difficulties.

3.1.2 How Do The Methods Facilitate Course Goals?

1. **Knowledge:** Active discussion is activity that allows the student to engage deeply with the material. It also gives the instructor a chance to informally assess how students are doing, judge whether there are any misconceptions or clarifications that need to be addressed and proceed to do so in a timely manner. I found this instant feedback very beneficial and the direct interaction between the students and the material quite effective in boosting student confidence in manipulating the various mathematical concepts.

2. **Rigorous Communication Abilities:** Since much of the in-class communication between students is oral, maintaining high standards for rigorous mathematics communication presents a particular challenge. During the class discussions it became apparent that students would more naturally be able to come up with

the ideas involved in solving a particular task than be able to communicate their solution in completely rigorous terms. This was partially addressed in class by setting up rigorous proof frameworks whenever such a method of justification was appropriate. Models for rigorous proofs were discussed demonstrated on a few different occasions and more example were included in the course summary that was distributed to students weekly. This is intended to facilitate the students in learning further on their own, as they see different methods or further examples for certain proofs.

However, due to the active learning design of the class fewer fully worked out examples of mathematical proofs were presented in class than would have been presented in a lecture-based course. This sometimes lead to insufficient models for students to base their homework-solving efforts on.

3. **Conceptual Understanding:** Throughout the class discussions a recurrent theme was to emphasize the conceptual explanations over the procedural language. This was often stated explicitly in the assigned tasks, especially when the tasks involved the practice of explaining mathematics to a hypothetical high school student audience. To a large extend and to the extent to which the students developed their communication abilities referred to above this motto eventually resonated with the students, as stated in a few of their end-of-semester reflections.

3.2 Course Activities Outside of Class

Outside of class, student had weekly problem sets and a term project to work on. The problem sets are meant to reinforce and expand upon the notions discussed in class. The projects involve a combination of several techniques learned in class: discovering new mathematics (making conjectures and proving the) and explaining it using a rigorous language that is appropriate language for the target audience.

3.2.1 Problem sets

The goal of the homework problems is to get students to work though the details of problems, focusing more thoroughly on the rigorous communication skills than the classroom discussion affords. The optimal distribution of the problems would include one or two problems that were discussed in class or closely resemble the classroom discussion and one or two problems that are designed to push the students out of their comfort level and pinpoint their understanding of the conceptual underpinnings of the material in a novel context. This distribution matches the two main goals of the problem sets: reinforcement and consolidation of the knowledge acquired and building upon this knowledge. The represent two successive steps toward mastery of the material.

Whenever possible, teaching scenarios are built into the assigned problems. Tasks related to these scenarios may include clarifying misconceptions, reformulating explanations in language appropriate for a specific level of student development or writing an organized explanation of a definition or mathematical test as if it was being presented to a class.

3.2.2 Projects

Here is the full description of the term project assignment:

> Your project, to be produced by teams of 3–4 students, will include a typed paper as well as a presentation to the class. The projects will consist of a guided discovery of a new topic, together with notes on how to teach this to middle/high school students in inquiry-based form.
>
> Here are the main parts of the project:
>
> 1. A brief introduction to the problem you are addressing in your project.
>   - Explain the importance of your topic and where it would fit within the mathematics curriculum for middle or high school.
>   - Either in the introduction or in a separate background section include all the mathematics notions and facts that are required to understand your project. Typically this will consist of several definitions and other useful preliminary facts.
> 2. An account of your solution to the problem. This will include the following:
>   - Clarifications on any assumptions you are making. Note that the class presentations should have alerted you to what clarifications others may need to fully understand your topic.
>   - Revised versions of propositions/theorems and examples as per feedback on the first draft.
>   - Proofs for all propositions in paper, revised per the feedback on the first draft and on presentations.
> 3. A plan to lead the class through your findings in an inquiry-based manner. For this, please come up with two sets of questions for the class:

- A set of conjectural inquiry questions, meant to lead the class towards guessing (conjecturing) at least one of your propositions.
- A set of proof inquiry questions, meant to lead the class through discovering the proof for at least one of your propositions.

4. A conclusion summarizing your findings and presenting any advice you might give to an instructor who might use your notes to present the topic to their class.
5. Finally, include citations for your references at the very end of the paper

3.3 Course Materials

There is no required textbook for this class, however weekly typed up course summary notes were made available to students in addition to the weekly activity worksheets. The course material was developed based on the lecture notes for **MODULE(S2): Algebra for Secondary Mathematics Teaching** developed by Prof. Yvonne Lai at UNL with collaborators from other universities. MODULE(S2) means Mathematics Of Doing, Understand, Learning, and Educating Secondary Schools and is a project partially supported by funding from a collaborative grant of the National Science Foundation.

3.4 Rationale for Choice of Teaching Methods

Discussing mathematics is not a skill that is often practiced in the college classroom because of the time pressures of the curriculum. Changing the main classroom activity to discussion challenges the students to get engaged with the material in deeper ways. Homework is used to solidify the knowledge, practice rigorous mathematics communication, and expand upon the knowledge acquired in the classroom.

3.4.1 Effect On Students' Future Courses And Career

The course develops the fundamentals of mathematics encountered in the high school curriculum. It also affords pre-service teachers the opportunity to practice explaining these notions as illustrated in section 3.2.1. This has direct connections to the curriculum in the education department, in particular to the teaching methods courses, where students learn lesson planning and teaching methods applied to the high school mathematics curriculum; see Appendix 6.5 for a student testimonial.

Chapter 4

Benchmark Memo 3: Document and Analyze Student Learning

In this chapter I describe the relationship between the activities documented in Chapter 3 and their effect on promoting and enhancing student understanding.

4.1 The Nature of Student Understanding

There are several layers involved in understanding and interacting with the material in any mathematics class. While present in any mathematics course through paired/ group activities where students share their knowledge with each other, the third level of knowledge listed below deserves a particular emphasis in a course for pre-service teachers:

1. **Working/procedural knowledge:** This comprises an intuitive understanding of the mathematical concepts as well as a knowledge of the procedures involved in accomplishing some standard tasks such as solving an equation, producing the formula for the inverse of a function, applying a given transformation to a function.

2. **Conceptual knowledge:** This comprises conceptual understanding of the mathematical notions, illustrated by proficiency in using the formal *definitions* of the notions involved as well as the ability to *justify* and *validate* the use and applicability of any procedures described above. A paramount

3. **Transferable knowledge:** This refers to being able to *explain* the concepts and procedures above in a correct and coherent manner, usually for the benefit of others.

4.2 Student Understanding in Work Samples

It is clear that some students struggled with the concepts, and some concepts, for example de formal definition for the graph of a function, were never fully grasped even by the time of the final exam for a few students. On the other hand, some students demonstrated solid understanding, and were commended in the end-of-semester reflections by their peers for sharing their insight.

We examine three example drawn from student's work on the first midterm exam. The question they address is:

> *Teach the definitions of invertible function and inverse function. Write this as if you were presenting a 5-10 minute lesson to a high school class.*

The first sample, Appendix 6.3.1, shows work from a student whose understanding is mostly at the working/procedural level. She shows an intuitive understanding of the notion of inverse function as reversing the roles of the input and output of a function and the ability to recognize invertible functions based on their property of sending distinct inputs to distinct outputs. At the conceptual level, however, the sample displays some imprecision in stating the appropriate definitions. In fact, the first definition is stated in a manner that makes it vacuously true *An invertible function is invertible if*... and the second definition is stated in an incoherent manner. However, the student *does* use the correct definition (f^{-1} is a function) to verify that the function in her example is invertible. This is surprising evidence that the content she is able to employ for knowledge transfer may exceed the the limitation of this student's personal understanding.

The second sample, Appendix 6.3.2, demonstrates some level of conceptual level understanding manifested through stating the appropriate definitions and relating the notions described to the graph representation for a function. However, by contrast to the previous case study, in this case there is a strong reliance on procedural language and techniques (the application of the horizontal line test for determining whether a function is invertible and the method of finding the rule for the inverse function) for the purpose of explaining the respective concepts.

The third sample, Appendix 6.3.3, shows work from a student who possesses strong conceptual understanding and has some experience in a teaching assistant capacity. A few points to note are that this mock lesson uses language that is appropriate for beginning learners, while also demonstrating knowledge of and conveying the formal definitions for the concepts under analysis. The explanations draw from previous knowledge of the concept of a function, which is recalled and integrated seamlessly into the explanation. Although only one representation for functions is used in the examples, this sample demonstrates superior knowledge transfer techniques.

Notice that none of the student's work is truly perfect. As these are samples obtained from work early in the semester, the need for further fine-tuning is to be expected.

4.3 The Formative Assessment-Feedback-Evaluation Cycle

Because of the importance of writing in higher level mathematics, students were given feedback on all the major writing assignments, including the weekly problem sets and the drafts of their final project. One of the goals of this portfolio is to analyze to what extent the feedback obtained on their work on weekly problem sets effected the performance of the students on subsequent assessments (examinations).

In order for this connection to be explored independently of any confounding factors, each midterm and the final exam contained one problem which was closely related to a homework problem. We include a case study of an an example of a pre-test (student work on the formative problem set assessment shown in Appendix 6.3.4) and post-test (student work on the midterm evaluation shown in Appendix 6.3.5) on two paired problems. One can remark the following responses to the feedback received:

- Better use of notation: on the formative assessment functions are always represented by their rules. On the midterm evaluation the notation $f(x)$ is used, although functions are still occasionally confounded with their rules.
- Clearer, well motivated algebra: Algebraic manipulations are made more clearly to explain why $16^{ax} = 2^{4ax}$. Furthermore the use of closure under multiplication to explain why $4a$ is an integer is made clear.

Note that the feedback received (see the margins of the formative assessment in Appendix 6.3.4) refers directly to the two items that exhibit improvement as shown above.

It is not clear whether improvements on formal evaluations can be related directly to feedback, even for paired problems. The following table summarizes the relationship between the average score on a problem on each of the three major evaluations: midterm 1, midterm 2 and the final exam and a paired homework problem.

Paired probl. 1	Midterm 1	Paired probl. 2	Midterm 2	Paired probl. 3	Final
87.8%	80.9%	69.4%	76.0%	86.2%	86.8%

A possible explanation for the greater improvement shown on the second midterm is that the students learned to take the feedbacks more seriously and implement the corresponding changes after having encountered the paired problem on the first midterm. A possible explanation for the equalization on the final paired problem is that by the end of the semester students had learned to produce reliable work which did not necessitate much adjustment any more.

4.4 Evidence of Students Meeting Learning Goals

4.4.1 Perceived Outcomes

In terms of the final grades, 4 of the 12 students were in the "A" range, and 8 were in the "B" range, which seems to be a typical distribution for an upper-level course in mathematics for teaching.

A major goal of the course was to emphasize conceptual understanding and rigorous mathematical reasoning and writing (proofs). Students self-reported increased confidence in their proof writing abilities. This may be due to a large extent to the introduction of "proof frameworks" which gave the students an entry point into the structure of the proof, which both helps them organize their reasoning and overcome the difficulty of staring at an empty page without knowing where to start. This shows a marked improvement in attitudes, compared to the data in section 2.2.4.

In terms of content knowledge for teaching, all students made a genuine effort to master the material. It can be stated with certainty that each of them gained at least a working knowledge of the material.

In terms of conceptual understanding, students displayed a larger variability with respect to their levels of achievement. By the end of the course only a select number of students had internalized the formal definition for the graph of a function (although they all possess an intuitive, working knowledge of this notion). This is a matter that requires further attention and will be focused on in the future, see Chapter 5.

4.4.2 What Students Are Learning Beyond the Curriculum

Beyond the course material and good mathematics communication skills, students in this course reported learning other important skills. Perhaps the most prominent one is the ability to interact with their classmates in regards to the material. The same skill development is often reported by advanced mathematics students (for example, graduate students) as an adaptation response to the demands of higher-level mathematics. Therefore, in many respects, this course has contributed to the formation of a strong cohort.

Among the students at the very top of the class, some reported their intellectual curiosity being stimulated by various aspects of the course material. Furthermore, the students were challenged to learn professional typesetting for their projects using the typesetting environment LaTeX, which is the standard choice for professional mathematicians.

Chapter 5

Planned Changes and Assessment of the Portfolio Process

5.1 Possible Changes, Redesigns, and Future Plans

If I were to teach this course in the future, I would implement the following changes:

1. Set aside designated class time to emphasize good mathematics writing *during every class*. This would involve designing relatively short and easy writing tasks that match the topic of each lesson and can be turned into more complex problems fit for homework assignments.

2. Continue to refine the formative assessment–feedback–evaluation cycle described in section 4.3. A first step would be to make this process more transparent to students, i.e., make them aware of the value of incorporating feedback into their intellectual products. Another idea would be to incentivize students to incorporate this feedback by offering an opportunity to write homework correction for partial or extra credit.

3. Make the connection between in-class work, problem set assignments and evaluations more evident by starting some of the homework assignment problems in class and continuing the paired problem strategy for exams.

4. It is clear that those students who had the added benefit of attending office hours on a regular basis had a far richer experience and fared better in the course. I would like to find a way to incentivize those students with a weaker background to make use of this opportunity and find a study group early in the semester.

5. Ponder some of the choices made for introducing mathematical concepts (definitions) in the course material to see if they can be improved. For example, the

current definition for invertible function corresponds to the notion of *injective* function, which is not standard in mathematics.

6. Make the connections between the course material and *college* mathematics more evident. This was originally planned for this course but this objective was removed in favor of focusing in depth on the other core objectives of the course.

7. Clarify the place of the course in the curriculum for mathematics education majors. In particular, clarify what elements of geometry can be assumed known.

5.2 Summary and Overall Assessment of the Peer Review Process

The review process influenced me in the design of the following elements of the course:

1. I designed a pre-class survey in hopes of measuring student attitudes and background. I felt that this helped me to know my students better and tailor my material appropriately to their proficiency level.

2. I examined the overarching goals of the class and decided to keep only those that would best serve the students in their capacity as pre-service teachers. This countered of my first predisposition to overcrowd the curriculum and I ended up adding a single topic to the the syllabus from the previous year, namely complex numbers. I feel strongly that this topic is underemphasized in high school mathematics and also in college mathematics. Therefore, I believe this was a needed addition.

3. I determined the value of active learning activities as an organizational cornerstone for the course. It is clear that this afforded the students to interact with the material more directly than a traditional lecture format would have challenged them to do. However, sometimes this method leads to insufficient rigor and it seems that more dedicated time for sample worked-out examples and short in-class proofs needs to be incorporated in the future.

4. I devised a way to evaluate students on their class participation based on an end of semester self-reflection that asked them to summarize the highlights of their in-class activities. This is a direct result of the peer review program, as this idea was suggested to me by a fellow participant during one of the PRTP small group meetings.

I am grateful to the Peer Review of Teaching Project for giving me the opportunity and motivation to look at teaching from an informed perspective and to record my experience with this course.

Chapter 6

Appendix

6.1 Syllabus

Below is the syllabus for the course.

Instructor: Dr. Alexandra Seceleanu

Office: 338 Avery Hall. Office hours to be determined during the first week of class.

Email: aseceleanu@unl.edu

Class Times and Location: W 6:00 –8:30 pm, Avery Hall – Room 109.

Prerequisites: Math 208 and Math 310

Text: None required. Lecture summaries and handouts will be distributed in class.

Web Page: All course material, including your grades, will be posted on Canvas.

Course Objectives: There are two primary goals in this course:

- Understand the typical mathematical notions, constructions, problems and routines of high school algebra and precalculus – about sets, number systems, functions, and equations – from a rigorous, advanced viewpoint;
- Gain fluency at communicating mathematics notions, justifications and proofs in precise yet accessible ways, identifying the destination of a proof and how this influences both the reasoning toward a solution and the writing up of the solution.

Material covered: The topics we will address in this course are:

- The formal language of mathematics, proof logic, and sets
- Number systems and their applications
- Relations, functions, and transformations of functions
- Topics in geometry and trigonometry

Grades: Grades for the course will be computed as follows:

Weekly homework	33% (10–11 problem sets)
Class participation	10%
Group project	12% (includes paper & presentation)
Midterms	30% (2 midterms ×15% each)
Final	15%

Guaranteed cutoffs for letter grades are: 97-100% A+, 92-96.99% A, 90-91.99% A- etc. These cutoffs may be adjusted at the end of the semester, but only in favor of the students.

Homework: Homework will be assigned weekly on Thursday through Canvas. Homework is to be submitted individually, and is due in class on Wednesday of the week following the week it was assigned. Students are allowed to collaborate on solving the homework problems, but the writing of the solutions must be done individually. Late homework will not be accepted.

Participation: This class will build on your experiences as learners, observers, and doers of mathematics and mathematics teaching. Moreover, an integral part of the work of teaching is talking through and listening to mathematical ideas. The portion of your grade intended for participation will come from a combination of:

- Collaborating on in-class explorations with a group of peers and actively participating in class discussions.
- Explaining a piece of mathematics in response to a question or prompt, either from a student or from the instructor. Often this will involve coming up to the board.
- Your collaboration with your final project team members.

Project: Your project, to be produced by teams of 3–4 students, will include a typed paper as well as a presentation to the class. Potential topics will be announced by the third week of class. The projects will consist of a guided discovery of a new topic, together with notes on how to teach this to middle/high school students in inquiry-based form.

Oral Presentation: Each presentation will be 20-25 minutes in length. All team members will address the entire class at some point during the presentation. Presentations will take place either on **Wednesday, April 18** or on **Wednesday, April 25**.

The structure for your presentation will include:

Conjectural inquiry (2-5 mins) + Proof inquiry (∼15 mins) + Wrap up (2-5 mins).

Written Report: The final project paper is due on **Friday, April 27**. Prior to the final paper deadline, there are the following deadlines for successive drafts.

Outline – Due end of 7th week The outline will include at least the following.

- A clear statement of each of the propositions you will prove in your final paper.
- Examples illustrating what the propositions mean, and why the hypotheses (conditions) of the propositions are important.
- Summary of who is working on which proofs of the propositions.
- Conjectural inquiry questions for the propositions.

First Draft of paper – Due end of 12th week. The first final draft of your paper will include at least the following.

- Revised versions of propositions and examples as per feedback on the outline.
- Proofs for all propositions in paper.
- Conjectural inquiry questions, as revised per feedback on outline.
- Proof inquiry questions for the proofs of propositions.

Final Draft of paper – Due Friday, April 27. The final draft of your paper will include all the elements of the first draft above and additionally

- A "Wrapping things up" section
- A "Notes to teacher and curricular alignment" section.

Midterm Exams: There will be two in-class midterm exams on

Wednesday, February 14 and **Wednesday, March 28**

Final Exam: The final exam is scheduled on **Wednesday, May 2nd, 8:15 to 10:15 p.m.**. Mutually agreeable alternatives will be discussed in class.

Department Grading Policy: Students who believe their academic evaluation has been prejudiced or capricious have recourse for appeals to (in order) the instructor, the department chair, the departmental appeals committee, and the college appeals committee.

Students with Disabilities: Students with disabilities are encouraged to contact me for a confidential discussion of their individual needs for academic accommodation. It is the policy of UNL to provide flexible and individualized accommodation to students with documented disabilities that may affect their ability to fully participate in course activities or to meet course requirements. To receive accommodation services, students must be registered with the Services for Students with Disabilities (SSD) office, 132 Canfield Administration, 402-472-3787 voice or TTY.

Academic Honesty and Integrity Academic honesty is the foundation of intellectual inquiry and academic pursuit. Every student must adhere to UNL's policy for academic integrity, set forth in the Student Code of Conduct in the UNL Undergraduate Bulletin. Students who plagiarize or academically dishonest are at risk of receiving a failing grade on an assignment or for an entire course and may be reported to the Student Judicial Review Board. Academic dishonesty includes:

- Handing in work that is in whole or in part the work of another person;
- Submitting work from a class taken previously for work in this class;
- Using notes or other study aids or otherwise obtaining unauthorized answers for a quiz, assessment or examination; and/or
- Using the words, opinions, or ideas of another person without giving full credit and accurate citation information.
- Committing plagiarism by using phrases, sentences, paragraphs, and ideas from any source, without attribution. Sources include, but are not limited to books, magazines, newspapers, television or radio reports, websites and papers of other students. Of course, you may refer to textbooks or other sources for mathematical facts or definitions if you need a refresher, but you may not copy a proof of a homework problem from a textbook or any other source.

Academic dishonesty will not be tolerated in this course. Students who choose to violate the UNL Student Code of Conduct will be, at a minimum, reported to the Student Judicial Board and receive zero credit for associated assignments or projects. If you have any questions regarding this policy, please speak to the instructor. To review the Student Code of Conduct policy visit `http://stuafs.unl.edu/ja/code/three.shtml`.

Technology in the Classroom Policy Technology has become a necessary part of working in any field. However, communications technology, such as smart phones, tablets, and laptop computers, presents challenges in the classroom, especially when close attention to and involvement in course discussion are necessary for success. Therefore, the course policy regarding technology is as follows:

- Technology use in the classroom is to be for the exclusive purpose of class-related activities, such as note-taking or typesetting mathematics.
- Students who choose to violate this policy will be first asked to discontinue the non-class-related activity.
- Persistent problems with the inappropriate use of technology during class will result in a reduction of the student's participation grade.

Attendance Policy Class attendance is essential for success in this course. Because the course meets once a week, unexcused absences will not be tolerated and you will lose participation points for every unexcused absence. Excused absences are limited to the following, all of which must be documented in writing: medical issues and university-related extra curricular activities. Students unable to attend class for any reason are responsible for the material covered in class and for turning in homework.

6.2 Sample In-class Activity Worksheet

OPENING INQUIRY: TWO-STEP TRANSFORMATIONS

Let's think about the following transformations (functions) on the real number line:

- $\mathrm{shift}_2 : \mathbb{R} \to \mathbb{R}, x \mapsto x + 2$
- $\mathrm{scale}_4 : \mathbb{R} \to \mathbb{R}, x \mapsto 4x$
- $\mathrm{scale}_7 : \mathbb{R} \to \mathbb{R}, x \mapsto 7x$
- $\mathrm{scale}_{-1} : \mathbb{R} \to \mathbb{R}, x \mapsto -x$

For each of these, find a new transformation T_i such that the *i*-th transformation above can be obtained by applying T_i twice.

Original Transformation	Two-step transformation		
shift_2	$\mathrm{shift}_2 = T_1 \circ T_1$	for	$T_1(x) =$
scale_4	$\mathrm{scale}_4 = T_2 \circ T_2$	for	$T_2(x) =$
scale_7	$\mathrm{scale}_7 = T_3 \circ T_3$	for	$T_3(x) =$
scale_{-1}	$\mathrm{scale}_{-1} = T_4 \circ T_4$	for	$T_4(x) =$

Checking that

- $\mathrm{shift}_2 = T_1 \circ T_1$

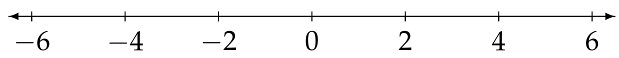
- $\mathrm{scale}_4 = T_2 \circ T_2$

- $\mathrm{scale}_7 = T_3 \circ T_3$

- $\mathrm{scale}_{-1} = T_4 \circ T_4$

The Complex Plane

The standard picture of the real numbers is a line:



Every real number is associated to a point on the line and every point is associated to a real number.

Definition. A **complex number** is a number of the form $z = a + bi$, where $a, b \in \mathbb{R}$.

The set of all complex numbers is denoted by $\mathbb{C}$:

$$\mathbb{C} = \{z \mid z = a + bi, \text{ and } a, b \in \mathbb{R}\}.$$

Definition. For a complex number $z = a + bi$ we denote by $\mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = b$ the real and the imaginary coordinate for z respectively and we say that the **standard form** for z is

$$z = \mathrm{Re}(z) + \mathrm{Im}(z)i.$$

Can we devise a representation for complex numbers similar to the real number line?

What would be associated to the complex number $z = a + bi$ in this representation?

Draw the set of points in the complex plane corresponding to complex numbers z such that

1. $\mathrm{Re}(z) = 1$
2. $\mathrm{Im}(z) + \mathrm{Re}(z) = 1$
3. $\mathrm{Im}(z) < 1$

Polar form for complex numbers

We have used rectangular coordinates $(\mathrm{Re}(z), \mathrm{Im}(z))$ to locate points in the complex plane, but these are not the only coordinates we can use.

Definition. The **polar form** for a complex number z is an expression of the form

$$z = r(\cos\theta + i\sin\theta) = r\mathrm{cis}(\theta),$$

where $r \geq 0$ is a real number and θ is an angle (in radians).

1. What do r and θ represent for the point associated to z in the complex plane?
2. Express $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ in terms of r and θ.
3. Express r and θ in terms of $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$.
4. Find the polar form for all powers of i, namely i^n, where $n \in \mathbb{N}$.

Definition. The **complex absolute value** (or *modulus* or *magnitude*) of a complex number $z = a + bi$ is defined to be $|z| = \sqrt{a^2 + b^2}$.

Definition. The **argument** of a complex number z, denoted $\mathrm{Arg}(z)$, is the unique angle $\mathrm{Arg}(z) \in [-\pi, \pi]$ such that

$$z = |z|(\cos \mathrm{Arg}(z) + i \sin \mathrm{Arg}(z)).$$

The real numbers $|z|$ and $\mathrm{Arg}(z)$ are the **polar coordinates** of the complex number z.

Definition. The **conjugate** of a complex number $z = a + bi$ is the complex number $\bar{z} = a - bi$.

How do $z, \bar{z}$ and $|z|$ relate?

How does the point in the complex plane corresponding to z relate to the point corresponding to $\bar{z}$?

Operations with complex numbers

Which form is better suited to describe each of the following operations?

Operation	Description in rectangular form	Description in polar form
Addition	$(a+bi)+(c+di)=$	
Subtraction	$(a+bi)-(c+di)=$	
Multiplication by $s \in \mathbb{R}$	$s(a+bi)=$	$s(r\text{cis}(\theta))=$
Multiplication in $\mathbb{C}$	$(a+bi)(c+di)=$	$r_1\text{cis}(\theta_1)\cdot r_2\text{cis}(\theta_2)=$

Notes on proofs for the above descriptions:

Geometric descriptions of the operations:

Draw the set of points in the complex plane corresponding to complex numbers z such that

1. $|z|=2$,
2. $|z-(2-i)|=1$
3. $|z-(2-i)|=|z-(1+2i)|$

De Moivre's Theorem

Compute the first 10 powers of $z = 2(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}))$ in polar form:

$z^1 =$

$z^2 =$

$z^3 =$

$z^4 =$

$z^5 =$

$z^6 =$

$z^7 =$

$z^8 =$

$z^9 =$

$z^{10} =$

Make a conjecture regarding a formula for the positive powers of a complex numbers with polar form $z = r(\cos(\theta) + i\sin(\theta))$. Can you prove your conjecture?

If $z = r(\cos(\theta) + i\sin(\theta))$ is a complex number and $n \in \mathbb{N}$ then

$$z^n =$$

Notes on proof:

What does your conjecture say about the powers of the complex number $\text{cis}(\theta) = \cos(\theta) + i\sin(\theta)$?

$$\text{cis}(\theta)^n =$$

List all the functions f you know which have the property $f(t)^n = f(nt)$ for any $n \in \mathbb{N}$.

EULER'S FORMULA AND EXPONENTIAL FORM

Theorem. (Euler's Formula) For all $t \in \mathbb{R}$ we have

$$e^{it} = \text{cis}(t) = \cos(t) + i\sin(t).$$

Corollary. (Euler's Identity)

$$e^{i\pi} + 1 = 0$$

Euler's Identity provides a connection between five of the most important constants in mathematics.

Write the complex number $e^{\frac{3}{2}+i\frac{\pi}{4}}$ in standard form (rectangular coordinates).

Definition. We can rewrite the polar form of a complex number using Euler's Formula as

$$z = re^{i\theta} = |z|e^{i\text{Arg}(z)}.$$

This is called the **exponential form** for z.

Write the complex number -1 in exponential form.
What are all the values of the exponents that you could use to write -1 in exponential form?

6.3 Samples of Student Work

6.3.1 Sample 1: Predominantly intuitive understanding

4. Teach the definitions of invertible function and inverse function. [30pts]
   Write this as if you were presenting a 5–10 minute lesson to a high school class.
   The rubric for this problem is included on the next page.

An invertible function is invertible if its inverse relation f^{-1} is a function.

Lesson:

This definition is incorrect

Definition: An invertible function is a function if the outputs of the function $f(x)$ map to the inputs of the function, also known as the inverse function.

We can find the inverse function by using any function $f(x)$ and switching the x and y variables. For example, if we have $f(x)=3x+5$, also written as $y=3x+5$, to find the inverse we switch x and y to get $x=3y+5$. Next we solve for y. ① $x-5=3y$ ② $\frac{x}{3}-5=y$ We now have the equation $\frac{1}{3}x-5=y$,

this does not follow from ①, should be $y=\frac{x}{3}-\frac{5}{3}$

which can also be written as $f^{-1}(x)=\frac{1}{3}x-5$ and is called the inverse function. Now, lets look at some inputs & outputs of the function and its inverse (see charts) notice that for $f(x)$, our inputs each map to a different output. Now if we look at our chart for $f(x)$, notice that our x values from $f(x)$ are now in the y column of $f^{-1}(x)$ and our y values from $f(x)$ are now in our x column of $f^{-1}(x)$. In other words, our inputs from $f(x)$ are now our outputs for the inverse of $f(x)$, and our outputs from $f(x)$ are now our inputs for the inverse of $f(x)$. Now, if

xy chart for f(x)

X	Y
0	5
1	8
2	11
3	14

xy chart for $f^{-1}(x)$

X	Y
5	0
8	1
11	2
14	3

we look closer at our xychart for $f(x)$, we can note that every input has no more than one output. This means that $f(x)$ is an invertible function. Now, if we look at $f(x)=x^2$ and ~~solve for the~~ actually the inverse of the function, we get $f^{-1}(x)=\sqrt{x}$. Then if we the inverse relation is $x \mapsto \pm\sqrt{x}$ make xycharts, we see that $f(x)$ is a function because each input has no more than one output. However, if we look at the xychart for $f^{-1}(x)$, we notice that the input of value 1 has two outputs, 1, and -1. This means that $f^{-1}(x)$ is not a function. Therefore, $f(x)=x^2$ is not an invertible function.

$f(x)$

x	y
0	0
1	1
-1	1
2	4

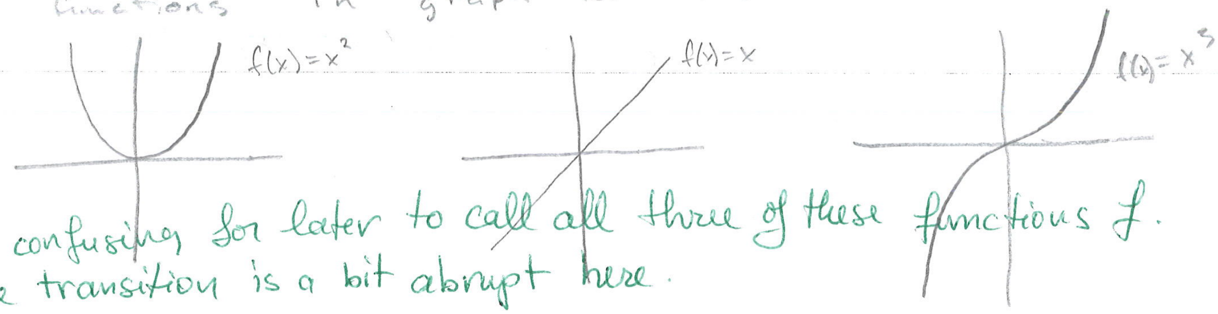
$f^{-1}(x)$

x	y
0	0
1	1
1	-1
4	2

6.3.2 Sample 2: Mixed conceptual and procedural understanding

4. Teach the definitions of invertible function and inverse function. [30pts]
Write this as if you were presenting a 5–10 minute lesson to a high school class.
The rubric for this problem is included on the next page.

Let us first look at some examples of functions in graph form and their function form.



It's confusing for later to call all three of these functions f. The transition is a bit abrupt here.

A function $f: D \to R$ is an invertible function if the inverse relation of the function $f: R \to D$ is also a function. If f is invertible then $f^{-1}: R \to D$ is called the inverse function of f.

~~So~~ there are two ways to look at finding if the function is invertible. The first way is to flip the graph over the $y=x$ line (because it flips (a,b) to (b,a) for every coordinate) and then check the vertical line test. The second way is to use a horizontal line test which uses the same criteria as a vertical line test only going horizontal on the original function.

Why does this way work?

So the x^2 ~~graph~~ function is the only one that is not invertible. In order to find the inverse function of the function is to switch the y and x in the function and then solve for y.

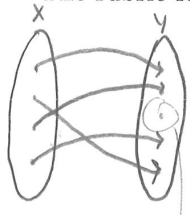
So $y=x \Rightarrow x=y \Rightarrow y=x$ so $f^{-1}(x)=x$

and $y=x^3 \Rightarrow y=y^3 \Rightarrow y=\sqrt[3]{x}$ so $f^{-1}(x)=\sqrt[3]{x}$

So after finding out if a function is an invertible function you can find out what the inverse function is by switching the y and x.

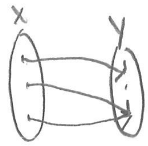
6.3.3 Sample 3: Mastery of conceptual understanding and knowledge transfer

4. Teach the definitions of invertible function and inverse function. **[30pts]**
Write this as if you were presenting a 5–10 minute lesson to a high school class.
The rubric for this problem is included on the next page.



As we have previously learned, you see a function here. We know this is a function b/c each x maps to only a single y. Now, hypothetically, what if the functions were going the other way? Would this still be a function? (yes.) Why? (Each pt on the right only goes to one point on the left.)
at most

this point goes to no point



Okay, is this a function? (yes.) Good. Now tell me, what if we flipped the arrows on this graph? Would this be a function? (No.) Right, and that's because one of the points on the right would point to two different points on the left.

The first picture is an example of an invertible function. This is b/c we can "invert" it and it stays a function. ✓ Similarly, the second example is not an invertible function

Formally, an invertible function is a function whose inverse relation is also a function This inverse relation is called an inverse function ✓

Let's make some more sense of this using some more examples.

As we know, the relation $F(x) = x^2$ is a function b/c each x-value gives us only <u>one</u> y-value. Is this an invertible function? In order for it to be invertible, there cannot be more than one x going to the same y, as previously demonstrated. But, $x=2$ and $x=-2$ both map to $y=4$. Thus, <u>the inverse relation gives us points $(4, 2)$ and $(4,-2)$</u>, making it <u>not</u> a function. So, $F(x) = x^2$ is not invertible.

You mean $(4,2)$ and $(4,-2)$ are point on the graph of the inverse relation or simply the inverse relation maps $4 \mapsto 2$ and $4 \mapsto -2$

x	y
-3	4
-2	7
-1	-2
0	6
1	4
2	0
3	9

In the table on the ~~right~~ left, we can clearly see that the relation described is a function. Each x-value ~~has~~ maps to only one y-value. But, looking the other direction to build our inverse relation, we find that $y=4$ is mapped to by $x=-3$ and $x=1$, so this is <u>also</u> not invertible.

x	y
-3	4
-2	6
-1	-7
0	2
1	0
2	-21
3	1

This relation is also clearly a function. Furthermore, the graph of the inverse relation contains the points $(4,-3), (6,-2), (-7,-1), (2,0), (0,1), (-21,2), (1,3)$. <u>None of these x-values map to more than one function.</u> Thus, the inverse is a function & the function is invertible!

I'm puzzled this

6.3.4 Sample 4: Formative Assesment with Feedback

2) If $A = \{f : \mathbb{R} \to \mathbb{R}, x \mapsto 16^{ax} : a \in \mathbb{Q}\}$ and $B = \{f : \mathbb{R} \to \mathbb{R}, x \mapsto 2^{ax} : a \in \mathbb{Q}\}$, then $A = B$.

Proof: Let $A = \{f : \mathbb{R} \to \mathbb{R}, x \mapsto 16^{ax} : a \in \mathbb{Q}\}$ and $B = \{f : \mathbb{R} \to \mathbb{R}, x \mapsto 2^{ax} : a \in \mathbb{Q}\}$

In order for $A = B$ two subclaims need to be proved.

1) $A \subseteq B$

2) $B \subseteq A$

Subclaim 1): $A \subseteq B$

(This means that every element of A is an element of B) elements of A are functions

Let $f \in A$ be given. You mean $x \mapsto 16^{ax}$

f maps $x = 16^{ax}$ for some $a \in \mathbb{Q}$ by the definition of A

$x = (2^{ax})(2^{ax})(2^{ax})(2^{ax}) = (2^{ax})^4$ by closure under multiplication

this is not relevant to what you did

$\cancel{4x}$ $(2^{ax})^4 = 2^{4ax} = 2^{dx}$ where $d = 4a$

Thus $x \in B$, since $x \mapsto 2^{dx}$, $d \in \mathbb{Q}$ so x is a member of B. here you need to use closure under multiplication

Therefore we have shown every member of A is a member of B. Hence $A \subseteq B$ ✓

it's not clear how you transformed 16^{ax} to get $(2^{ax})^4$. I would be more clear if you wrote $16^{ax} = (2^4)^{ax} = 2^{4ax}$

Subclaim 2) $A \supseteq B$

Let $f \in B$ be given then f maps $x \mapsto 2^{ax}$

$x \mapsto 2^{ax}$ for some $a \in \mathbb{Q}$ by definition of B

$a = 4n$ for some $n \in \mathbb{Q}$ such that $n = \frac{a}{4} \in \mathbb{Q}$

why?

So $x \mapsto 2^{4nx} = 16^{nx}$

Thus $x \in A$, since $x \mapsto 16^{nx}$, $n \in \mathbb{Q}$ so the function $x \mapsto 16^{nx}$ is a member of A.

We have shown that every member of B is an element of A. Hence $A \supseteq B$. ✓

By definition of set equality we have shown $A = B$ by proving $A \subseteq B$ and $B \subseteq A$. ✓ $\square$

6.3.5 Sample 5: Paired Evaluation

(d) **Prove** the true statements among the ones in part (c). [10pts]

Prove $A \supseteq B$

Proof: We want to show that every element of B is an element of A to satisfy the definition of a superset. We are given that the ~~function~~ elements of set A ~~is~~ are functio[ns] $f(x) = 2^{ax}$; $a \in \mathbb{Z}$ and the ~~function~~ elements of set B ~~is~~ " $f(x) = 16^{bx}$: $b \in \mathbb{Z}$. So we want to show that every 16^{bx} is ~~an element of 2^{ax}~~. $16^{bx} = (2^4)^{bx}$ because $2^4 = 16$. So $(2^4)^{bx} = 2^{4bx}$, $4b \in \mathbb{Z}$ because it is two integers multiplied together. Therefore it is in the set A ~~for every x~~ for every $b \in \mathbb{Z}$. Thus every element of B is an element of A so $A \supseteq B$. □

this is not a set so you can't speak of it having elements

Good!

what? the function $f(x) = 16^{bx}$

Prove $A \supsetneq B$

Proof: In order to show $A \supsetneq B$ we need to show two things: $A \supseteq B$ and an element of set A is not in set B ($A \not\subseteq B$). We have shown $A \supseteq B$ in the proof above. So let us try and find an element of set A that is not in set B.

If $x = 1$ ~~and~~ $f(x) = 2^{ax}$ ~~then~~ $f(1) = 2^a$ for the function of set A? and $f(1) = 16^b$ for the function of set B. In order for 16^b to be equal to 2^a b has to be $\frac{1}{4}$ of a. Since $a \in \mathbb{Z}$ it can be 1 and $\frac{1}{4}$ of 1 is $\frac{1}{4} \notin \mathbb{Z}$ and $b \in \mathbb{Z}$ so thus this is an element that is not in set B. Since $A \supseteq B$ and an element of set A is not in set B, $A \supsetneq B$. □

↑ better call this g, because it's not the same as f above

Notice that you are re-writing the proof in part (b). You could have just referenced it here.

Prove $A \neq B$

Proof: In proof of $A \supsetneq B$ we showed that there was an element of set A that was not in set B. This showed that A is not a subset of B. By definition in order to have $A = B$ we need to have $A \supseteq B$ and $A \subseteq B$. Since $A \subseteq B$ is not true it must be that $A \neq B$. □

−2

6.4 Sample Project

Graphs and Rotations

Little Sparkly Donuts
Taylor Turner, Elizabeth Tyler, Kelly Glaser, Jenny Kortus

April 29, 2018

1 Introduction

In this paper, we will be investigating, if possible, under what conditions you can rotate the graph of a function about the origin, and still have the resulting graph be the graph of a function? We will first recall the Vertical Line Test and how the Vertical Line Test helps us determine if the graph of a relation is a function. We will then look at three different examples, such as parabolas, lines, and exponential graphs, using the Vertical Line Test. Let θ be the angle between the line $L = mx+b$ and the $y - axis$.

2 Definitions and Examples

- A relation P from a set D to a set R a set of assignments from elements of D, called inputs, to elements of R, called outputs.[1]
- The graph of a relation $r : D \to R$ is defined as the set of points $(a, b) \in R^2$ such that r assigns $a \to b$.[1]
- A function f from D to R is a relation from D to R where each input in D is assigned to no more than one output in R. [1]
- The graph of a function is defined as the point (a, b) is on the graph of f if and only if $f(a) = b$.[1]
- Vertical Line Test: A relation $r : R \to R$ is a function iff any vertical line in the coordinate plane intersects the graph of r in at most one point. [1]

Question 1. Given a function below, can you find a point of rotation and angle of rotation such that the function is no longer a function?

$y = e^x$ $\qquad$ $y = x^2$ $\qquad$ $y = mx + b$

While exploring each of the above examples, we noticed that if you pick two points on the function and draw a line through them, this line can be rotated by some angle, θ, such that the line becomes vertical and fails the Vertical Line Test. This led us to conjecture that given an arbitrary function, there exists a line that intersects the graph of a function in at least two points. This also led us to conjecture that given the graph of an arbitrary function f, $G(f)$, there exists an angle, θ, such that $G(f)$ rotated by θ is not the graph of a function.

3 Proofs

3.1 Prove that a line exists such that it intersects a given arbitrary function twice.

Claim 1. *Given an arbitrary function, there exists a line that intersects the graph of a function in at least two points.*

Proof. Given a function, f, we can represent the function as a graph such that $f : D \to R$ maps coordinates (x, y) from $x \to y$. Because x can be assigned values across its domain, that means many values of y will result, too. Thus, the graph of f will have at least two coordinate points: say (x, y) and (a, b). Connect (x, y) to (a, b) and, by definition, we have a line. This shows that there exists a line that intersects the graph of f in at least two point. □

Now that we have determined how to find a line that intersects an arbitrary function twice, lets figure out how to choose a point of rotation.

3.2 How to choose a point of rotation.

To choose the best point of rotation we simply need to choose a point that intersects with function f and line L such that L intersects the graph of f in at least two places.

Now that we have determined how to choose a point of rotation, let's figure out how to find the angle of rotation when rotating about this point.

3.3 How to find the angle of rotation when rotating about a point that lies on the intersection of $G(f)$ and the line $L = mx + b$.

Claim 2. *Given the graph of f, $G(f)$, there exists an angle, θ, such that $G(f)$ rotated by θ around point P is not the graph of a function.*

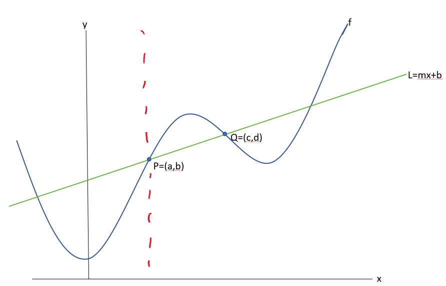


Figure 1: G(f) rotating about point P.

Proof. Let the center of rotation be the point P that lies on the intersection of $G(f)$ and the line $L = mx + b$.
Let $P = (a, b)$ and $Q = (c, d)$ be two points on the line L.
Then $m = \frac{d-b}{c-a}$.
Geometrically, $\tan(\theta_1) = m$ where θ_1 is the angle between L and the x-axis.
Solve for θ_1:

$$\begin{aligned} \arctan(\tan(\theta_1)) &= \arctan(m) \\ \theta_1 &= \arctan(m) \end{aligned}$$

Therefore, $\theta_1 = \arctan(m)$ where θ_1 is the angle between L and x-axis.
We need to solve for the angel θ_2 between the y-axis and L.
To find θ_2, the angle between the y-axis and L, we take $\pi/2$ - θ_1.
Hence, $\theta_2 = \pi/2 - \arctan(m)$.
Therefore, Given the graph of f, $G(f)$, there exists an angle, $\theta = \pi/2 - \arctan(m)$, such that $G(f)$ rotated by θ around point P is not the graph of a function. □

Now that we have determined the angle of rotation when rotating about a point that lies on the intersection of $G(f)$ and the line L, let's see what happens when we rotate about the origin.

3.4 How to find the angle of rotation when centered at the origin.

Claim 3. *Given the graph of of f, G(f), there exists an angle, θ, such that G(f) rotated by θ around the origin is not the graph of a function.*

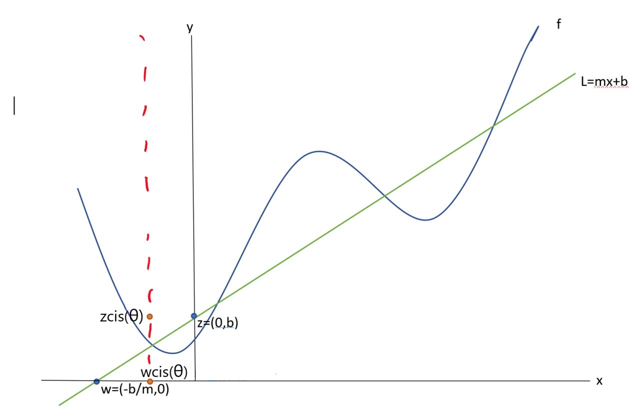


Figure 2: G(f) rotating about the origin.

Proof. Let the point of rotation be the origin.
Let θ be the positive angle between the $y - axis$ and $L = mx + b$.
In polar form, the imaginary axis is the $y - axis$ and the real axis is the $x - axis$. Therefore, since we want both of the x-coordinates to be equal, we set the real coordinates in polar form equal to each other.
Then, $Re(bi\,\mathrm{cis}(\theta)) = Re(\frac{-b}{m}\,\mathrm{cis}(\theta))$.
We can write the above as:

$$\begin{aligned}
Re((0+bi)(\cos(\theta))) + i\sin(\theta))) &= Re((\frac{-b}{m} + 0i)(\cos(\theta) + i\sin(\theta)) \\
0(\cos(\theta) + bi(i\sin(\theta)) &= \frac{-b}{m}\cos(\theta) + 0i(i\sin(\theta) \\
0 + (bi^2)\sin(\theta) &= \frac{-b}{m}\cos(\theta) \\
0 - b\sin(\theta) &= \frac{-b}{m}\cos(\theta) \\
-b\sin(\theta) &= \frac{-b}{m}\cos(\theta) \\
\frac{-m}{b}(-b\sin(\theta)) &= \cos(\theta) \\
m\sin(\theta) &= \cos(\theta) \\
\tan(\theta) &= \frac{1}{m} \\
\arctan(\tan(\theta)) &= \arctan(\frac{1}{m}) \\
\theta &= \arctan(\frac{1}{m})
\end{aligned}$$

Hence, $\theta = \arctan(\frac{1}{m})$. Therefore, given the graph of of f, G(f), there exists an angle, θ, such that G(f) rotated by θ around the origin is not the graph of a function. □

In part 3.3 we concluded that when rotating around a point on the intersection of $G(f)$ and line L, the angle, $\theta = \pi/2 - \arctan(m)$. In part 3.4 we concluded that when rotating around the origin, the angle, $\theta = \arctan(\frac{1}{m})$. By trig identity, we know that $\theta = \pi/2 - \arctan(m) = \arctan(\frac{1}{m})$. Therefore, the point of rotation does not effect the angle of rotation.

Now that we know how to choose an angle based on an arbitrary function, we will find how we can choose a function based on an arbitrary function.

3.5 Finding an $f(x)$ given an arbitrary θ.

Claim 4. *Given an arbitrary angle, θ, such that θ is not a multiple of π, there exists a function $f(x)$ such that when rotated about the origin θ degrees, $f(x)$ is no longer a function.*

Proof. Given an angle θ, such that θ is not a multiple of π, we want to find a function such that when the function is rotated counterclockwise θ degrees, it is no longer a function.

In a geometric sense, this means that a rotation of θ degrees would result in the vertical line test failing, meaning that there exists a line such that it intersects the graph more than once. Thus, we can choose $f(x)$ to be a line since $f(x)$ would then intersect $f(x)$ infinitely many times. So, we just need to find $f(x)$ so that a rotation by θ degrees would make it vertical, and thus no longer a function.

A graph is located below to show where θ is found on the graph.

Therefore, this line has the form $f(x) = mx+b$ where $b = 0$ since it goes through the origin. m can be found by calculating our change in y divided by our change in x. Since θ is the angle between the graph and the y-axis, we must use $\frac{\pi}{2} - \theta$ to calculate the slope, m. Thus, the change in y would be $\sin(\frac{\pi}{2} - \theta)$ and the change in x would be $\cos(\frac{\pi}{2} - \theta)$. Therefore, $m = \tan(\frac{\pi}{2} - \theta)$. So, $f(x) = \tan(\frac{\pi}{2} - \theta)x$ for any given θ. Rotating $f(x)$ by θ results in a vertical line based on our construction of $f(x)$, so $f(x)$ is no longer a function when rotated counterclockwise by θ. $\square$

Claim 5. *Given an arbitrary angle, θ, such that θ is a multiple of π, there does not exist a function $f(x)$ such that when rotated about the origin θ degrees, $f(x)$ is no longer a function.*

Proof. There is the unique case of if $\theta = k\pi$, where k is an integer. In this case, our slope would be $\tan(\frac{\pi}{2} - k\pi) = \tan(\frac{j\pi}{2})$ where j is an integer. Since $\cos(\frac{j\pi}{2}) = 0$, then m is undefined for this case. Therefore, the chosen line does not even start as a function. Since a rotation of π radians is equivalent to changing $f(x)$ to $-f(-x)$, and we know that $-f(-x)$ is still a function, there is no function that becomes a non-function after rotation of π radians. $\square$

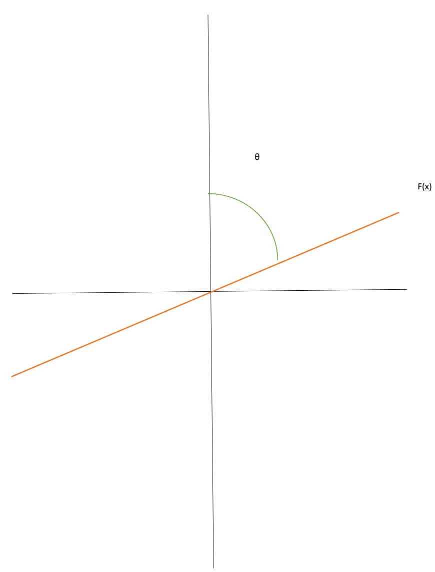


Figure 3: Location of angle θ

4 Conclusion

In conclusion, we first used the Vertical Line Test to conjecture whether or not an arbitrary function could be rotated about the origin and have the resulting relation still be a function. Next we explored how to find a line that intersects a given arbitrary function twice. Then we explored the best strategy for choosing a point of rotation which led us to determining the angle of rotation when rotating about a point that lies on the intersection of $G(f)$ and the line L. This then led us to determining the angle of rotation when rotating about the origin where we concluded that the point of rotation has no effect on the angle of rotation. Lastly, we explored given an arbitrary angle, θ, that there exists a function $f(x)$ such that when rotated about the origin θ degrees, $f(x)$ is no longer a function. Through proving our conjectures in each of these explorations, we have come to the conclusion that given the graph of an arbitrary function f, $G(f)$, there exists an angle, θ, such that $G(f)$ rotated by θ around any point is not the graph of a function.

We believe that this topic fits into a High School Trigonometry course. First introduce students to relations, functions, basic trig functions, and rotations. After these topics have been covered, assign students this project to help them build connections between different topics.

References

[1] Alexandra Seceleanu, *Course Summary*, Math 407 – Math for High School Teaching, Nebraska, Spring 2018.

6.5 Sample Reflection

When I think over this semester in Math 407, I think first of a specific moment. It was the moment I saw my grade on the first homework and looked at the class' average and high score. These numbers were unbelievably low. As a class, we were thrown into a bit of a panic, and I was a bit annoyed with how "picky" the grading had been. However, as the semester went on, I really began to appreciate Alexandra's attention to detail, and I started becoming just as "picky" when it came to my own work and the work of my classmates. These experiences, paired with my experiences in Math 325, gave me much better communication skills.

I started to see how these skills would truly affect my teaching through various assignments and activities we did in TEAC 451P, a methods course. For one group project, we were supposed to evaluate different curriculums based on the Common Core State Standards as if we were looking for one to implement in our school. One thing that my group noted when going through the materials is that some textbooks used the terms equation and function interchangeability without mention of a difference. We claimed that this, among other similar examples, would be misleading for students. Our instructor for that class, insinuated in the comments of our work that this was not a valid claim and that using these terms interchangeably didn't really make a difference. We however, disagreed and had the means to back up our claim, thanks to Math 407. This is one of many examples from TEAC 451P.

I also saw the things I was learning in Math 407 directly affect my effectiveness as a Learning Assistant for Math 103. I was able to better help students because I had thoroughly thought through some of the same material for Math 407 homework assignments. Because I had gone over many of the details for myself, I was much more confident in helping students in multiple ways, wherever they were stuck in the problem.

Overall, because of my new perspective on precision in communication of math, I will be a better mathematician and teacher. To incoming Math 407 students, I would say, don't be discouraged with a poor grade, but instead, be willing to learn a new viewpoint.