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GEOMETRY OF THE WORD PROBLEM FOR 3-MANIFOLD GROUPS

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GEOMETRY OF THE WORD PROBLEM FOR 3-MANIFOLD GROUPS

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Abstract. We provide an algorithm to solve the word problem in all fundamental groups of 3-manifolds that are either closed, or compact with (finitely many) boundary components consisting of incompressible tori, by showing that these groups are autostackable. In particular, this gives a common framework to solve the word problem in these 3-manifold groups using finite state automata.

We also introduce the notion of a group which is autostackable respecting a subgroup, and show that a fundamental group of a graph of groups whose vertex groups are autostackable respecting any edge group is autostackable. A group that is strongly coset automatic over an autostackable subgroup, using a prefix-closed transversal, is also shown to be autostackable respecting that subgroup. Building on work by Antolin and Ciobanu, we show that a finitely generated group that is hyperbolic relative to a collection of abelian subgroups is also strongly coset automatic relative to each subgroup in the collection. Finally, we show that fundamental groups of compact geometric 3-manifolds, with boundary consisting of (finitely many) incompressible torus components, are autostackable respecting any choice of peripheral subgroup.

1. Introduction

One fundamental goal in geometric group theory since its inception has been to find algorithmic and topological characteristics of the Cayley graph satisfied by all closed 3-manifold fundamental groups, to facilitate computations. This was an original motivation for the definition of automatic groups by Epstein, Cannon, Holt, Levy, Paterson, and Thurston [12], and its recent extension to Cayley automatic groups by Kharlampovich, Khoussainov, and Miasnikov [21]. These constructions, as well as finite convergent rewriting systems, provide a solution to the word problem using finite state automata. However, automaticity fails for 3-manifold groups in two of the eight geometries, and Cayley automaticity and finite convergent rewriting systems are unknown for many 3-manifold groups. Autostackable groups, first introduced by the first two authors and Holt in [8], are a natural extension of both automatic groups and groups with finite convergent rewriting systems. In common with these two motivating properties, autostackability also gives a solution to the word problem using finite state automata. In this paper we show that the fundamental group of every compact 3–manifold with incompressible toral boundary, and hence every closed 3-manifold group, is autostackable.

Let G be a group with a finite inverse-closed generating set A. Autostackability is defined using a discrete dynamical system on the Cayley graph $\Gamma := \Gamma_A(G)$ of G over A, as follows. A *flow function* for G with *bound* $K \geq 0$, with respect to a spanning tree T in Γ, is a function Φ mapping the set $\vec{E}$ of directed edges of Γ to the set $\vec{P}$ of directed paths in Γ, such that

- **(F1):** for each $e \in \vec{E}$ the path $\Phi(e)$ has the same initial and terminal vertices as e and length at most K,
- **(F2):** Φ acts as the identity on edges lying in T (ignoring direction), and

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(F3): there is no infinite sequence $e_1, e_2, e_3, ...$ of edges with each $e_i \in \vec{E}$ not in T and each e_{i+1} in the path $\Phi(e_i)$.

These three conditions are motivated by their consequences for the extension $\widehat{\Phi} : \vec{P} \to \vec{P}$ of Φ to directed paths in Γ defined by $\widehat{\Phi}(e_1 \cdots e_n) := \Phi(e_1) \cdots \Phi(e_n)$, where $\cdot$ denotes concatenation of paths. Upon iteratively applying $\widehat{\Phi}$ to a path p, whenever a subpath of $\widehat{\Phi}^n(p)$ lies in T, then that subpath remains unchanged in any further iteration $\widehat{\Phi}^{n+k}(p)$, since conditions (F1-2) show that $\widehat{\Phi}$ fixes any point that lies in the tree T. Condition (F3) ensures that for any path p there is a natural number n_p such that $\widehat{\Phi}^{n_p}(p)$ is a path in the tree T, and hence $\widehat{\Phi}^{n_p+k}(p) = \Phi^{n_p}(p)$ for all $k \geq 0$. The bound K controls the extent to which each application of $\widehat{\Phi}$ can alter a path. Thus when $\widehat{\Phi}$ is iterated, paths in Γ "flow", in bounded steps, toward the tree.

A finitely generated group admitting a bounded flow function over some finite set of generators is called *stackable*. Let $\mathcal{N}_T$ denote the set of words labeling non-backtracking paths in T that start at the vertex labeled by the identity 1 of G (hence $\mathcal{N}_T$ is a prefix-closed set of normal forms for G), and let $\mathsf{label} : \vec{P} \to A^*$ be the function that returns the label of any directed path in Γ. The group G is *autostackable* if there is a finite generating set A with a bounded flow function Φ such that the graph of Φ, written in triples of strings over A as

$$\mathsf{graph}\,(\Phi) := \{(y, a, \mathsf{label}(\Phi(e_{y,a})) \mid y \in \mathcal{N}_T, a \in A, \text{ and } e_{y,a} \in \vec{E} \text{ has initial vertex } y \text{ and label } a\},$$

is recognized by a finite state automaton (that is, $\mathsf{graph}\,(\Phi)$ is a (synchronously) regular language).

To solve the word problem in an autostackable group, given a word w in A^*, by using the finite state automaton recognizing $\mathsf{graph}\,(\Phi)$ to iteratively replace any prefix of the form ya with $y \in \mathcal{N}_T$ and $a \in A$ by $y\mathsf{label}(\Phi(e_{y,a}))$ (when $\mathsf{label}(\Phi(e_{y,a}))$ is not a), and performing free reductions, a word $w' \in \mathcal{N}_T$ is obtained, and $w =_G 1$ if and only if w' is the empty word. Hence autostackability also implies that the group has a finite presentation.

Both autostackability and its motivating property of automaticity also have equivalent definitions as rewriting systems. In particular, a group G is autostackable if and only if G admits a bounded regular convergent prefix-rewriting system [8], and a group G is automatic with prefix-closed normal forms if and only if G admits an interreduced regular convergent prefix-rewriting system [26]. (See Section 2.2 for definitions of rewriting systems and Section 2.3 for a geometric definition of automaticity.)

The class of autostackable groups contains all groups with a finite convergent rewriting system or an asynchronously automatic structure with prefix-closed (unique) normal forms [8]. Beyond these examples, autostackable groups include some groups that do not have homological type FP_3 [9, Corollary 4.2] and some groups whose Dehn function is non-elementary primitive recursive; in particular, Hermiller and Martínez-Pérez show in [15] that the Baumslag-Gersten group is autostackable.

We focus here on the case where G is the fundamental group of a connected, compact 3-manifold M with incompressible toral boundary. In [12] it is shown that if no prime factor of M admits *Nil* or *Sol* geometry, then $\pi_1(M)$ is automatic. However, the fundamental group of any *Nil* or *Sol* manifold does not admit an automatic, or even asynchronously automatic, structure [12, 3]. Replacing the finite state automata by automata with unbounded memory, Bridson and Gilman show in [4] that the group G is asynchronously combable by an indexed language (that is, a set of words recognized by a nested stack automaton), although for some 3-manifolds the language cannot be improved to context-free (and a push-down automaton). Another extension of automaticity, solving the word problem with finite state automata whose alphabets are not based upon a generating set, is given by the more recent concept of Cayley graph automatic groups, introduced by Kharlampovich, Khoussainov and Miasnikov in [21]; however, it is an open question whether all fundamental groups of closed 3-manifolds with *Nil* or *Sol* geometry are Cayley graph automatic. From the rewriting viewpoint, in [16] Hermiller and Shapiro showed that fundamental groups of closed fibered hyperbolic 3-manifolds admit finite convergent rewriting systems, and that all closed geometric 3-manifold groups in the other 7 geometries do as well. However, the question of whether all closed 3-manifold groups admit a finite convergent rewriting system also remains open.

In this paper we show that *every* fundamental group of a connected, compact 3-manifold with incompressible toral boundary is autostackable. The results of [8] above show that the fundamental group of any closed geometric 3-manifold is autostackable; here, we will show that the restriction to geometric manifolds is unnecessary. To do this, we investigate the autostackability of geometric pieces arising in the JSJ decomposition of a 3-manifold, along with closure properties of autostackability under the construction of fundamental groups of graphs of groups, including amalgamated products and HNN extensions.

We begin with background on automata, autostackability, rewriting systems, fundamental groups of graphs of groups, strongly coset automatic groups, relatively hyperbolic groups, and 3-manifolds in Section 2.

Section 3 contains the proof of the autostackability closure property for graphs of groups. We define a group G to be *autostackable respecting* a finitely generated subgroup H if G has an autostackable structure with flow function Φ and spanning tree T_G on a generating set A satisfying:

> *Subgroup closure:* There is a finite inverse-closed generating set B for H contained in A such that T_G contains a spanning tree T_H for the subgraph $\Gamma_B(H)$ of $\Gamma_A(G)$, and for all $h \in H$ and $b \in B$, $\mathsf{label}(\Phi(e_{h,b})) \in B^*$.
>
> *H–translation invariance:* There is a subtree T' of T_G containing the vertex 1 such that the left action of H on $\Gamma_G(A)$ gives $T_G = T_H \cup (\cup_{h \in H} hT')$, and for all $h, \tilde{h} \in H$ the trees $hT', \tilde{h}T'$ are disjoint and the intersection of the trees hT', T_H is the vertex h. Moreover, the group action outside of $\Gamma_B(H)$ preserves the label of the flow function; that is, for all directed edges $e_{g,a}$ of $\Gamma_A(G)$ not in $\Gamma_B(H)$ (with $g \in G$ and $a \in A$) and for all $h \in H$, the flow function satisfies $\mathsf{label}(\Phi(e_{g,a})) = \mathsf{label}(\Phi(e_{hg,a}))$.

(As above, $e_{g,a}$ denotes the directed edge of $\Gamma_A(G)$ with initial vertex g and label a.) The conditions on the tree T_G are equivalent to the requirement that the associated normal form set $\mathcal{N}_G$ satisfy $\mathcal{N}_G = \mathcal{N}_H \mathcal{N}_\mathcal{T}$ for some prefix-closed sets $\mathcal{N}_H \subset B^*$ of normal forms for H and $\mathcal{N}_\mathcal{T} \subset A^*$ of normal forms for the set of right cosets $H \backslash G$. The subgroup closure condition together with Lemma 3.2 imply that the subgroup H is also autostackable. If the requirement that the graph of the flow function is a regular language is removed, we say that G is *stackable respecting* H. We show that autostackability of vertex groups respecting edge groups suffices to preserve autostackability for graphs of groups.

Theorem 3.5 *Let $\mathcal{G}$ be a graph of groups over a finite connected graph Λ with at least one edge. If for each directed edge e of Λ the vertex group G_v corresponding to the terminal vertex $v = t(e)$ of e is autostackable [respectively, stackable] respecting the associated injective homomorphic image of the edge group G_e, then the fundamental group $\pi_1(\mathcal{G})$ is autostackable [respectively, stackable].*

We note that for the two word problem algorithms that motivated autostackability, some closure properties for the graph of groups construction have been found, but with other added restrictions. For automatic groups, closure of amalgamated free products and HNN extensions over finite subgroups is shown in [12, Thms 12.1.4, 12.1.9] and closure for amalgamated products under other restrictive geometric and language theoretic conditions has been shown in [2]. For groups with finite convergent rewriting systems, closure for HNN-extensions in which one of the associated subgroups equals the base group and the other has finite index in the base group is given in [14]. Closure for stackable groups in the special case of an HNN extension under significantly relaxed assumptions (and using left cosets instead of right) are given by the second author and Martínez-Pérez in [15]. They also prove a closure result for HNN extensions of autostackable groups, with a requirement of further technical assumptions.

Section 4 contains a discussion of extensions of two autostackability closure results of [9] to autostackability respecting subgroups, namely for extensions of groups and finite index supergroups.

In Section 5 we study the relationship between autostackability of a group G respecting a subgroup H and strong coset automaticity of G with respect to H defined by Redfern [28] and Holt and Hurt [18] (referred to as coset automaticity with the coset fellow-traveler property in the latter paper; see Section 2.3 below for definitions). More precisely, we prove the following.

Theorem 5.1 *Let G be a finitely generated group and H a finitely generated autostackable subgroup of G. If the pair (G,H) is strongly prefix-closed coset automatic, then G is autostackable respecting H.*

Applying this in the case where G is hyperbolic relative to a collection of sufficiently nice subgroups, and building upon work of Antolin and Ciobanu [1], we obtain the following.

Theorem 5.4 *Let G be a group that is hyperbolic relative to a collection of subgroups $\{H_1,...,H_n\}$ and is generated by a finite set A'. Suppose that for every index j, the group H_j is shortlex biautomatic on every finite ordered generating set. Then there is a finite subset $\mathcal{H}' \subseteq \mathcal{H} := \cup_{j=1}^n (H_j \setminus 1)$ such that for every finite generating set A of G with $A' \cup \mathcal{H}' \subseteq A \subseteq A' \cup \mathcal{H}$ and any ordering on A, and for any $1 \leq j \leq n$, the pair (G,H_j) is strongly shortlex coset automatic, and G is autostackable respecting H_j, over A.*

In particular, if G is hyperbolic relative to abelian subgroups, then G is autostackable respecting any peripheral subgroup.

In Section 6 we prove our results on autostackability of 3-manifold groups. We begin by considering compact geometric 3-manifolds with boundary consisting of a finite number of incompressible tori that arise in a JSJ decomposition of a compact, orientable, prime 3-manifold. Considering the Seifert fibered and hyperbolic cases separately, we obtain the following.

Proposition 6.1 and Corollary 5.5 *Let M be a finite volume geometric 3-manifold with incompressible toral boundary. Then for each choice of component T of ∂M, the group $\pi_1(M)$ is autostackable respecting any conjugate of $\pi_1(T)$.*

In comparison, finite convergent rewriting systems have been found for all fundamental groups of Seifert fibered knot complements, namely the torus knot groups, by Dekov [11], and for fundamental groups of alternating knot complements, by Chouraqui [10]. In the case of a finite volume hyperbolic 3-manifold M, the fundamental group $\pi_1(M)$ is hyperbolic relative to the collection of fundamental groups of its torus boundary components by a result of Farb [13], and so by closure of the class of (prefix-closed) biautomatic groups with respect to relative hyperbolicity (shown by Rebbecchi in [27]; see also [1]), the group $\pi_1(M)$ is biautomatic.

Combining this result on fundamental groups of pieces arising from JSJ decompositions with Theorem 3.5, together with other closure properties for autostackability, yields the result on 3-manifold groups.

Theorem 6.2 *Let M be a compact 3-manifold with incompressible toral boundary. Then $\pi_1(M)$ is autostackable. In particular, if M is closed, then $\pi_1(M)$ is autostackable.*

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2. Background: Definitions and notation

In this section we assemble definitions, notation, and theorems that will be used in the rest of the paper, in order to make the paper more self-contained.

Let $G = \langle A \rangle$ be a group. Throughout this paper we will assume that every generating set is finite and inverse-closed, and every generating set for a flow function does not contain a letter representing the identity element of the group. By " $=$ " we mean equality in A^*, while " $=_G$ " denotes equality in the group G. For a word $w \in A^*$, we denote its length by $\ell(w)$. The identity of the group G is written 1 and the empty word in A^* is ε.

Let $\Gamma_A(G)$ be the Cayley graph of G with the generators A. We denote by $\vec{E} = \vec{E}_A(G)$ the set of oriented edges of the Cayley graph, and denote by $\vec{P} = \vec{P}_A(G)$ the set of directed edge paths in $\Gamma_A(G)$. By $e_{g,a}$ we mean the oriented edge with initial vertex g labeled by a.

By a set of *normal forms* we mean the image $\mathcal{N} = \sigma(G) \subset A^*$ of a section $\sigma\colon G \to A^*$ of the natural monoid homomorphism $A^* \to G$. In particular, every element of G has a unique normal form. For $g \in G$, we denote its normal form by $\mathsf{nf}(g)$.

Let $H \leq G$ be a subgroup. By a right coset of H in G we mean a subset of the form Hg for $g \in G$. We denote the set of right cosets by $H\backslash G$. A subset $\mathcal{T} \subseteq G$ is a *right transversal* for H in G if every right coset of H in G has a unique representative in $\mathcal{T}$.

2.1. **Regular languages.**

A comprehensive reference on the contents of this section can be found in [12, 19]; see [8] for a more concise introduction.

Let A be a finite set, called an *alphabet*. The set of all finite strings over A (including the empty word ε) is written A^*. A *language* is a subset $L \subseteq A^*$. Given languages L_1, L_2 the *concatenation* L_1L_2 of L_1 and L_2 is the set of all expressions of the form l_1l_2 with $l_i \in L_i$. Thus A^k is the set of all words of length k over A; similarly, we denote the set of all words of length at most k over A by $A^{\leq k}$. The *Kleene star* of L, denoted L^*, is the union of L^n over all integers $n \geq 0$.

The class of *regular languages* over A is the smallest class of languages that contains all finite languages and is closed under union, intersection, concatenation, complement and Kleene star. (Note that closure under some of these operations is redundant.)

Regular languages are precisely those accepted by finite state automata; that is, by computers with a bounded amount of memory. More precisely, a finite state automaton consists of a finite set of states Q, an initial state $q_0 \in Q$, a set of accept states $P \subseteq Q$, a finite set of letters A, and a transition function $\delta : Q \times A \to Q$. The map δ extends to a function $\delta : Q \times A^* \to Q$; for a word $w = a_1 \cdots a_k$ with each a_i in A, the transition function gives $\delta(q, w) = \delta(\cdots(\delta(\delta(q, a_1), a_2), \cdots, a_k)$. The automaton can also considered as a directed labeled graph whose vertices correspond to the state set Q, with a directed edge from q to $\delta(q, a)$ labeled by a for each $a \in A$ and $q \in Q$. Using this model $\delta(q, w)$ is the terminal vertex of the path starting at q labeled by w. A word w is in the language of this automaton if and only if $\delta(q_0, w) \in P$.

The concept of regularity is extended to subsets of a Cartesian product $(A^*)^n = A^* \times \cdots \times A^*$ of n copies of A^* as follows. Let \$ be a symbol not contained in A. Given any tuple $w = (a_{1,1} \cdots a_{1,m_1}, ..., a_{n,1} \cdots a_{n,m_n}) \in (A^*)^n$ (with each $a_{i,j} \in A$), rewrite w to a *padded word* $\hat{w}$ over the finite alphabet $B := (A \cup \$)^n$ by $\hat{w} := (\hat{a}_{1,1}, ..., \hat{a}_{n,1}) \cdots (\hat{a}_{1,N}, ..., \hat{a}_{n,N})$ where

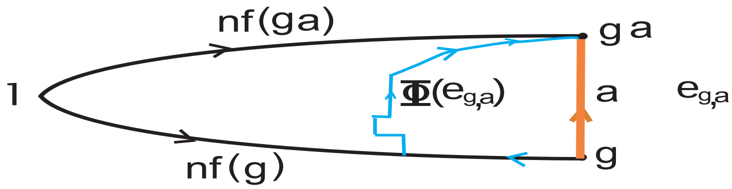

FIGURE 1. The flow function

$N = \max\{m_i\}$ and $\hat{a}_{i,j} = a_{i,j}$ for all $1 \le i \le n$ and $1 \le j \le m_i$ and $\hat{a}_{i,j} = \$$ otherwise. A subset $L \subseteq (A^*)^n$ is called a *regular language* (or, more precisely, *synchronously regular*) if the set $\{\hat{w} \mid w \in L\}$ is a regular subset of B^*.

The following theorem, much of the proof of which can be found in [12, Chapter 1], contains closure properties of regular languages that are used later in this paper.

Theorem 2.1. *Let A, B be finite alphabets, x an element of A^*, L, L_i regular languages over A, K a regular language over B, $\phi\colon A^* \to B^*$ a monoid homomorphism, L' a regular subset of $(A^*)^n$, and $p_i : (A^*)^n \to A^*$ the projection map on the i-th coordinate. Then the following languages are also regular:*

(1) *(Homomorphic image)* $\phi(L)$.
(2) *(Homomorphic preimage)* $\phi^{-1}(K)$.
(3) *(Quotient)* $L_x := \{w \in A^* : wx \in L\}$.
(4) *(Product)* $L_1 \times L_2 \times \cdots \times L_n$.
(5) *(Projection)* $p_i(L)$.

2.2. Autostackability and rewriting systems.

Proofs of the results in this section and more detailed background on autostackability are in [8, 9, 15]. Let $G = \langle A \rangle$ be an autostackable group, with spanning tree T in $\Gamma_A(G)$ and flow function $\Phi : \vec{E} \to \vec{P}$.

As noted in Section 1, the tree T defines a set of prefix-closed normal forms, denoted $\mathcal{N} = \mathcal{N}_T$, for G, namely the words that label non-backtracking paths in T with initial vertex 1; as above, we denote the normal form of $g \in G$ by $\mathsf{nf}(g)$. Since $\mathcal{N} = p_1(\mathsf{graph}\,(\Phi))$, where p_1 denotes projection on the first coordinate, Theorem 2.1 implies that the set $\mathcal{N}$ is a regular language over A.

An illustration of the flow function path associated to the edge $e_{g,a}$ in the Cayley graph $\Gamma_A(G)$ is given in Figure 1. The flow function Φ yields an algorithm to build a van Kampen diagram for any word w over A representing the trivial element of G. Writing $w = a_1 \cdots a_n$, diagrams for each of the words $\mathsf{nf}(a_1 \cdots a_{i-1})a_i\mathsf{nf}(a_1 \cdots a_i)^{-1}$ are recursively constructed, and then glued along the normal forms. (Prefix closure of the normal forms implies that each path labeled by a normal form is a simple path, and hence this gluing preserves planarity of the diagram.) In particular, for $g \in G$ and $a \in A$, the edge $e_{g,a}$ lies in the tree T if and only if either $\mathsf{nf}(g)a = \mathsf{nf}(ga)$ or $\mathsf{nf}(ga)a^{-1} = \mathsf{nf}(g)$, which in turn holds if and only if there is a degenerate van Kampen diagram (i.e., containing no 2-cells) for the word $\mathsf{nf}(g)a\mathsf{nf}(ga)^{-1}$. In the case that $e_{g,a}$ is not in T, the diagram for $\mathsf{nf}(g)a\mathsf{nf}(ga)^{-1}$ is built recursively from the 2-cell bounded by $e_{g,a}$ and $\Phi(e_{g,a})$ together with van Kampen diagrams for the edges in the path $\Phi(e_{g,a})$. See [7, 8] for more details.

Since directed edges in $\Gamma_A(G)$ are in bijection with $\mathcal{N} \times A$ and the set of paths in the Cayley graph based at g is in bijection with the set A^* of their edge labels, the flow function Φ gives the same information as the *stacking function* $\phi\colon \mathcal{N} \times A \to A^{\le K}$ defined by $\phi(\mathsf{nf}(g), a) := \mathsf{label}(\Phi(e_{g,a}))$. Thus the set

$$\mathsf{graph}\,(\Phi) = \{(\mathsf{nf}(g), a, \phi(\mathsf{nf}(g), a)) : g \in G, a \in A\}$$

is the graph of this stacking function. This perspective will be very useful in writing down the constructions of flow functions throughout the paper.

In [7] the first two authors show that if G is a stackable group whose flow function bound is K, then G is finitely presented with relators given by the labels of loops in $\Gamma_A(G)$ of length at most $K + 1$ (namely the relations $\phi(\mathsf{nf}(g), a) =_G a$). Although it is unknown if autostackability is invariant under changes in finite generating sets, we note that it is straightforward to show that if G is autostackable with generating set A and $A \subseteq B$, then G is autostackable with the generators B using the same set of normal forms (see [15, Proposition 4.3] for complete details).

Not every stackable group has decidable word problem; Hermiller and Martínez-Pérez [15] show that there exist groups with a bounded flow function but unsolvable word problem. A group with a bounded flow function Φ whose graph is a recursive (i.e., decidable) language has a word problem solution using the automaton that recognizes $\mathsf{graph}\,(\Phi)$ [8]. Thus autostackable groups have word problem solutions using finite state automata.

Autostackability also has an interpretation in terms of prefix-rewriting systems. A *convergent prefix-rewriting system* for a group G consists of a finite set A and a subset $R \subseteq A^* \times A^*$ such that G is presented as a monoid by $\langle A \mid \{uv^{-1} : (u, v) \in R\}\rangle$ and the rewriting operations of the form $uz \to vz$ for all $(u, v) \in R$ and $z \in A^*$ satisfy:

Termination. There is no infinite sequence of rewritings $x \to x_1 \to x_2 \to \dots$

Normal Forms. Each $g \in G$ is represented by a unique irreducible word (i.e., one that cannot be rewritten) over A.

A prefix-rewriting system is called *regular* if R is a regular subset of $A^* \times A^*$, and R is called *interreduced* if for each $(u, v) \in R$ the word v is irreducible over R and the word u is irreducible over $R \setminus \{(u, v)\}$. A prefix-rewriting system R is called *bounded* if there is a constant $K > 0$ such that for each $(u, v) \in R$ there are words $s, t, w \in A^*$ such that $u = ws$, $v = wt$ and $\ell(s), \ell(t) \leq K$.

The following is a recharacterization of autostackability which is useful in interpreting the results and proofs later in this paper.

Theorem 2.2 (Brittenham, Hermiller, Holt [8])**.** *Let G be a finitely generated group.*

(1) *The group G is autostackable if and only if G admits a regular bounded convergent prefix-rewriting system.*
(2) *The group G is stackable if and only if G admits a bounded convergent prefix-rewriting system.*

A *finite convergent rewriting system* for G consists of a finite set A and a finite subset $R' \subset A^* \times A^*$ presenting G as a monoid such that the regular bounded prefix-rewriting system $R := \{(yu, yv) \mid y \in A^*, (u, v) \in R'\}$ is convergent. Thus, Theorem 2.2 shows that autostackability is a natural extension of finite convergent rewriting systems, in which the choice of direction of rewritings of bounded length subwords depends upon the prefix appearing before the subword to be rewritten.

2.3. **Automatic groups and coset automaticity.**

In [28], Redfern introduced the notion of coset automatic group, as well as the geometric condition of strong coset automaticity (using different terminology), studied in more detail by Holt and Hurt in [18]. We consider strong coset automaticity in this paper.

Let G be a group generated by a finite inverse-closed set A and let $\Gamma = \Gamma_A(G)$ be the corresponding Cayley graph. Let d_Γ denote the path metric distance in Γ. For any word v

in A^* and integer $i \geq 0$, let $v(i)$ denote the prefix of v of length i; that is, if $v = a_1 \cdots a_m$ with each $a_j \in A$, then $v(i) := a_1 \cdots a_i$ if $i \leq m$ and $v(i) = v$ if $i \geq m$.

Definition 2.3. [18] Let G be a group, let H be a subgroup of G, and let $K \geq 1$ be a constant. The pair (G, H) satisfies the *H–coset K–fellow traveler property* if there exists a finite inverse-closed generating set A of G and a language $L \subset A^*$ containing a representative for each right coset Hg of H in G, together with a constant $K \geq 0$, satisfying the property that for any two words $v, w \in L$ with $d_{\Gamma_A(G)}(v, hw) \leq 1$ for some $h \in H$, we have $d_{\Gamma_A(G)}(v(i), hw(i)) \leq K$ for all $i \geq 0$. The pair (G, H) is *strongly coset automatic* if there exists a finite inverse-closed generating set A of G and a regular language containing a representative for each right coset of H in G satisfying the H–coset K–fellow traveler property for some $K \geq 0$.

The pair (G, H) is *strongly prefix-closed coset automatic* if the language L is also prefix-closed and contains exactly one representative of each right coset. Given a total ordering on A, the pair (G, H) is *strongly shortlex coset automatic* if in addition L contains only the shortlex least representative (using the shortlex ordering induced by the ordering on A) of each coset.

A group G is *automatic* if the pair $(G, \{1\})$ is strongly coset automatic. Prefix-closed automaticity and shortlex automaticity are obtained similarly.

We also consider the 2-sided notions of fellow traveling and automaticity in Sections 2.5 and 5. The group G is *biautomatic* if there is a regular language $L \subset A^*$ containing a representative of each element of G and a constant $K \geq 0$ such that for any $v, w \in L$ and $a \in A \cup \{\varepsilon\}$ with $d_{\Gamma_A(G)}(av, w) \leq 1$, we have $d_{\Gamma_A(G)}(\tilde{v}(i), w(i)) \leq K$ for all $i \geq 0$, where $\tilde{v} := av$. The adjective *shortlex* is added if L is the set of shortlex least representatives of the elements of G.

Holt and Hurt show that in the case that the language L is the shortlex transversal (i.e., the shortlex least representatives for the cosets), the coset fellow-traveler property yields regularity of the language.

Theorem 2.4. [18, Theorem 2.1] *Let H be a subgroup of G, and let A be a finite inverse-closed totally ordered generating set for G. If (G, H) satisfies the H–coset K–fellow traveler property over A using the shortlex transversal for the right cosets of H in G, then the pair (G, H) is strongly shortlex coset automatic.*

In their work on strong coset automaticity, Holt and Hurt also describe finite state automata that perform multiplication by a generator in strongly shortlex coset automatic groups. As we note in the next Proposition, their construction works without the shortlex ordering as well. We provide some of the details of their construction (with a slight modification), in order to use them later in the proof of Theorem 5.1. As above, $d_{\Gamma_A(G)}$ denotes the path metric distance in the Cayley graph $\Gamma_A(G)$. For any radius $r \geq 0$, let

$$B_{\Gamma_A(G)}(r) := \{g \in G \mid d_{\Gamma_A(G)}(1, g) \leq r\},$$

be the set of vertices in the closed ball of radius r in the Cayley graph.

Proposition 2.5. [18] *Let H be a subgroup of a group G, and suppose that (G, H) is strongly coset automatic over a generating set A of G with H–coset K–fellow traveling regular language L of representatives of the right cosets of H in G. Then for each $h \in H \cap B_{\Gamma_A(G)}(K)$ and $a \in A$, there is a finite state automaton $M_{h,a}$ accepting the set of padded words corresponding to the set of word pairs*

$$L_{h,a} := \{(x, y) \mid x, y \in L \text{ and } xa =_G hy\}.$$

Proof. Regularity of the set L together with closure of regular languages under product (Theorem 2.1) implies that the language $L \times L \subset A^* \times A^*$ is also regular. Hence the set of padded words corresponding to the pairs of words in $L \times L$ is accepted by a finite state automaton M, with state set Q, initial state q_0, accept states P, alphabet $(A \cup \$)^2$, and transition function $\delta : Q \times A \to Q$.

Note that the H–coset K–fellow traveler property implies that for all (x, y) in $L_{h,a}$, we have $x(i)^{-1}hy(i) \in B_{\Gamma_A(G)}(K)$ for all $i \geq 0$, and so we can also write

$$L_{h,a} = \{(x, y) \mid x, y \in A^*, x(i)^{-1}hy(i) \in B_{\Gamma_A(G)}(K) \text{ for all } i \geq 0, \text{ and } xa =_G hy\}.$$

We construct a finite state automaton $M_{h,a}$ as follows. The set of states of $M_{h,a}$ is $\widetilde{Q} := (Q \times B_{\Gamma_A(G)}(K)) \cup \{F\}$, the initial state is $\widetilde{q}_0 := (q_0, h)$, the set of accept states is $\widetilde{P} := P \times \{a\}$, and the alphabet is $(A \cup \$)^2$. The transition function $\widetilde{\delta} : \widetilde{Q} \times A \to \widetilde{Q}$ is defined by $\widetilde{\delta}((q, g), (a, b)) := (\delta(q, a), a^{-1}gb)$ if $a^{-1}gb \in B_{\Gamma_A(G)}(K)$ and $\widetilde{\delta}((q, g), (a, b)) := F$ otherwise, and $\widetilde{\delta}(F, (a, b)) := F$ (here if either a or b is \$, it is treated as the group identity in the expression $a^{-1}gb$). The language of $M_{h,a}$ is $L_{h,a}$. □

2.4. **Graphs of groups.**

A general reference to the material in this section, with an algebraic approach together with proofs of basic facts about graphs of groups (e.g., invariance under change of spanning tree, injectivity of the natural inclusion of G_v and G_e, existence of the Bass-Serre tree, etc.), can be found in [31]. A more topological viewpoint on this topic is given in [30].

Let Λ be a connected graph with vertex set V, and directed edge set $\vec{E}_\Lambda$. Each undirected edge is considered to underlie two directed edges with opposite orientations. For an edge $e \in \vec{E}_\Lambda$, the symbol $\overline{e}$ denotes the directed edge associated with the same undirected edge as e but with opposite orientation. The initial vertex of e will be called $i(e)$ and the terminal vertex $t(e)$.

Definition 2.6. A *graph of groups* is a quadruple $\mathcal{G} = (\Lambda, \{G_v\}, \{G_e\}, \{h_e\})$, where Λ is a graph, $\{G_v\}$ is a collection of groups indexed by V, $\{G_e\}$ is a collection of groups indexed by $\vec{E}_\Lambda$ subject to the condition that for all $e \in \vec{E}_\Lambda$, $G_e = G_{\overline{e}}$, and $\{h_e\}$ is a collection of injective homomorphisms $h_e \colon G_e \hookrightarrow G_{t(e)}$.

Definition 2.7. Let $\mathcal{G} = (\Lambda, \{G_v\}, \{G_e\}, \{h_e\})$ be a graph of groups and let T_Λ be a spanning tree of Λ. The *fundamental group* of $\mathcal{G}$ at T_Λ, denoted $\pi_1(\mathcal{G}) = \pi_1(\mathcal{G}, T_\Lambda)$, is the group generated by the union of all of the groups G_v and the set $\vec{E}_{\Lambda\setminus T}$ of edges in $\vec{E}_\Lambda$ whose underlying undirected edge is not in T_Λ, with three types of relations:

(1) $\overline{e} = e^{-1}$ for all $e \in \vec{E}_{\Lambda\setminus T}$,
(2) $h_e(g) = h_{\overline{e}}(g)$ for all e in $\vec{E}_\Lambda \setminus \vec{E}_{\Lambda\setminus T}$ and $g \in G_e$, and
(3) $eh_e(g)e^{-1} = h_{\overline{e}}(g)$ for all $e \in \vec{E}_{\Lambda\setminus T}$ and $g \in G_e$.

The fundamental group of a graph of groups can be obtained by iterated HNN extensions (corresponding to the edges in $\vec{E}_{\Lambda\setminus T}$) and amalgamated free products (corresponding to edges in $\vec{E}_\Lambda \setminus \vec{E}_{\Lambda\setminus T}$). It is also the fundamental group of the corresponding graph of spaces formed from the disjoint union of Eilenberg-MacLane spaces $\{K(G_v, 1)\}_{v\in V}$ by adding tubes corresponding to $\{K(G_e, 1)\times I\}_{e\in\vec{E}_\Lambda}$ and gluing the tubes using identifications corresponding to the maps h_e and $h_{\overline{e}}$.

2.5. **Relatively hyperbolic groups.**

Background and details on relatively hyperbolic groups used in this paper can be found in [25, 13, 20, 1].

For a group G with a finite inverse-closed generating set A, let $\vec{P}_{\Gamma_A(G)}$ denote the set of directed paths in the associated Cayley graph. Given $p, q \in \vec{P}_{\Gamma_A(G)}$, write $i(p)$ for the group element labeling the initial vertex of p, and $t(p)$ for the terminal vertex. Given $\lambda \geq 1$ and $\epsilon \geq 0$, the path p is a *(λ, ϵ)-quasigeodesic* if for every subpath r of p the inequality $\ell(r) \leq \lambda d_{\Gamma_A(G)}(i(r), t(r)) + \epsilon$ holds, where $d_{\Gamma_A(G)}$ is the path metric distance in $\Gamma_A(G)$.

Definition 2.8. Let G be a group with a finite inverse-closed generating set A and let $\{H_1, ..., H_n\}$ be a collection of proper subgroups of G. Let $\Gamma_A(G)$ be the Cayley graph of G with respect to A. For each index j let $\widetilde{H_j}$ be a set in bijection with H_j, and let $\mathcal{H} := \coprod_{j=1}^{n}(\widetilde{H}_j \setminus \{1\})$.

- The *relative* Cayley graph of G relative to $\mathcal{H}$, denoted $\Gamma_{A\cup\mathcal{H}}(G)$ is the Cayley graph of G with generating set $A \cup \mathcal{H}$ (with the natural map from $(A \cup \mathcal{H})^* \to G$).
- A path p in $\Gamma_{A\cup\mathcal{H}}(G)$ *penetrates* a left coset gH_j if p contains an edge labeled by a letter in $\widetilde{H_j}$ connecting two vertices in gH_j.
- An H_j*–component* of a path p in $\Gamma_{A\cup\mathcal{H}}(G)$ is a nonempty subpath s of p labeled by a word in $\widetilde{H_j}^*$ that is not properly contained in a longer subpath of p with label in $\widetilde{H_j}^*$.
- A path $p \in \vec{P}_{\Gamma_{A\cup\mathcal{H}}(G)}$ is *without backtracking* if whenever $1 \leq j \leq n$ and the path p is a concatenation of subpaths $p = p'srs'p''$ with two H_j–components s, s', then the initial vertices $i(s)$, $i(s')$ lie in different left cosets of H_j (intuitively, p penetrates every left coset at most once).

Geometrically, the relative Cayley graph collapses each left coset of a subgroup in $\{H_1, ..., H_n\}$ to a diameter 1 subset.

The following definition is a slight modification of the definition originally due to Farb [13], which is shown by Osin in [25, Appendix] to be equivalent to both Farb's and Osin's definitions of relative hyperbolicity for finitely generated groups. Many other equivalent definitions can also be found in the literature (see for example [20, Section 3] for a list of many of these).

Definition 2.9. Let G be a group with a finite generating set A and let $\{H_1, ..., H_n\}$ be a collection of proper subgroups. G is *hyperbolic relative to* $\{H_1, ..., H_n\}$ if:

(1) $\Gamma_{A\cup\mathcal{H}}(G)$ is Gromov hyperbolic, and
(2) given any $\lambda \geq 1$ there exists a constant $B(\lambda)$ such that for any $1 \leq j \leq n$ and any two $(\lambda, 0)$–quasigeodesics $p, q \in \vec{P}_{\Gamma_{A\cup\mathcal{H}}(G)}$ without backtracking that satisfy $i(p) = i(q)$ and $d_{\Gamma_A(G)}(t(p), t(q)) \leq 1$, the following hold:
   (a) If s is an H_j–component of p and the path q does not penetrate the coset $i(s)H_j$, then $d_{\Gamma_A(G)}(i(s), t(s)) < B(\lambda)$.
   (b) If s is an H_j–component of p and s' is an H_j–component of q satisfying $i(s)H_j = i(s')H_j$, then $d_{\Gamma_A(G)}(i(s), i(s')) < B(\lambda)$ and $d_{\Gamma_A(G)}(t(s), t(s')) < B(\lambda)$.

Property (1) in Definition 2.9 is sometimes called *weak relative hyperbolicity* and the property (2) is called *bounded coset penetration*. The collection $\{H_1, ..., H_n\}$ is called the set of *peripheral* or *parabolic* subgroups. A form of bounded coset penetration for (λ, ϵ)–quasigeodesics is given by Osin in [25, Theorem 3.23]; we record that here for use in Section 5.

Proposition 2.10. [25, Theorem 3.23] *Let G be a group with a finite generating set A that is hyperbolic relative to $\{H_1, ..., H_n\}$. Given any $\lambda \geq 1$ and $\epsilon \geq 0$ there exists a constant $B = B(\lambda, \epsilon)$ such that the statement of Definition* 2.9*(2) holds for any two (λ, ϵ)–quasigeodesics p, q.*

Remark 2.11. We note that if a subgroup H of $G = \langle A \rangle$ is a peripheral subgroup in a relatively hyperbolic structure for G (that is, G is hyperbolic relative to $\{H_1, ..., H_n\}$

and $H = H_i$ for some i), and if g is any element of G, then the conjugate subgroup gHg^{-1} is also a peripheral subgroup in a relatively hyperbolic structure for G (namely $\{gH_1g^{-1}, ..., gH_ng^{-1}\}$). In particular, the isometry $\Gamma_{A\cup\mathcal{H}}(G) \to \Gamma_{gAg^{-1}\cup(\cup(\widetilde{gH_jg^{-1}}\setminus\{1\}))}(G)$ preserves both hyperbolicity and bounded coset penetration.

When a finitely generated group G is hyperbolic relative to $\{H_1, ..., H_n\}$, each of the subgroups H_j is also finitely generated [25, Proposition 2.29]. Since relative hyperbolicity of the pair $(G, \{H_1, ..., H_n\})$ is independent of the finite generating set for G [25, Theorem 2.34], for the remainder of the paper we assume that for any relatively hyperbolic group that $A\cap H_j$ generates H_j for all $1 \le j \le n$.

Definition 2.12. [1, Construction 4.1] Let p be a path in $\Gamma_A(G)$ with label $w \in A^*$, and write $w = w_0u_1w_1 \cdots u_nw_n$ with each $w_k \in (A \setminus (\cup_{j=1}^n(A \cap H_j)))^*$ and each $u_k \in (A \cap H_{j_k})^*$ for some index j_k, such that whenever $w_i = \varepsilon$ and x is the first letter of u_{i+1}, then u_ix does not lie in $(A \cap H_j)^*$ for any j. The path p has *no parabolic shortenings* if for each k the subpath labeled by the subword u_k is a geodesic in the subgraph $\Gamma_{A\cap H_{j_k}}(H_{j_k})$. If the path p has no parabolic shortenings, let $\hat{p}$ be the path in $\Gamma_{A\cup\mathcal{H}}(G)$ with label $w_0h_1w_1 \cdots h_nw_n$ where for each k the symbol h_k denotes the letter in $\mathcal{H}$ representing the element u_k of H_{j_k}; the path $\hat{p}$ is the path *derived from* p.

Antolin and Ciobanu studied geodesics and language theoretic properties of relatively hyperbolic groups in [1]. For example, they show that every finitely generated relatively hyperbolic group has a finite generating set, A, such that each $\Gamma_{A\cap H_j}(H_j)$ isometrically embeds in $\Gamma_A(G)$ [1, Lemma 5.3] (in fact every finite generating set can be extended to one with this property). We apply the following results from their paper in Section 5.

Definition 2.13. Let G be a finitely generated group hyperbolic relative to $\{H_1, ..., H_n\}$ and suppose that $\lambda \ge 1$ and $\epsilon \ge 0$. A finite inverse-closed generating set A for G is called (λ, ϵ)*–nice* if

(1) every path derived from a geodesic in $\Gamma_A(G)$ is a (λ, ϵ)–quasigeodesic in $\Gamma_{A\cup\mathcal{H}}(G)$ without backtracking,
(2) $H_j = \langle A \cap H_j\rangle$ for all j, and
(3) for every total ordering on A satisfying the property that H_j is shortlex biautomatic on $A \cap H_j$ (with the restriction of the ordering from A) for all j, the group G is shortlex biautomatic over A with respect to that ordering.

Theorem 2.14. [1, Lemma 5.3, Theorem 7.7] *Let G be a group with finite generating set A' that is hyperbolic relative to $\{H_1, ..., H_n\}$. Then there are constants $\lambda \ge 1$ and $\epsilon \ge 0$ and a finite subset $\mathcal{H}'$ of $\mathcal{H}$ such that every finite generating set A of G satisfying $A' \cup \mathcal{H}' \subseteq A \subseteq A' \cup \mathcal{H}$ is a (λ, ϵ)–nice generating set.*

Remark 2.15. In [12, Theorem 4.3.1], Holt shows that every finitely generated abelian group is shortlex automatic over every finite generating set with respect to every ordering of that set; moreover, the structure is also biautomatic. Combining this with Theorem 2.14 implies that any finitely generated group hyperbolic relative to abelian subgroups is shortlex biautomatic (on a (λ, ϵ)–nice generating set), and hence autostackable.

2.6. **3-manifolds.**

We review some important facts about 3-manifolds that will be used later in the paper. For background, an interested reader can consult [24, 29, 32]. Let $M = M^3$ be a connected, compact, orientable, three-dimensional manifold with *incompressible toral boundary*; that is, the boundary of M consists of a finite number of incompressible (i.e., π_1–injective) tori.

Definition 2.16. A 3-manifold $M = M^3$ is called *prime* if whenever M is a connected sum $M \cong M_1 \# M_2$, then one of M_1 or M_2 is homeomorphic to S^3.

Decomposing the compact 3-manifold M along a disjoint collection of S^2's via the connected sum operation, there is a decomposition $M = M_1 \# \cdots \# M_k$, where each of the M_i are prime, which is unique up to reordering. This gives a decomposition of $\pi_1(M)$ as a free product of the fundamental groups of its prime factors.

If M is prime then, using Thurston's Geometrization Conjecture (proved by Perelman; see, e.g., [23]), either M admits a geometric structure based on one of $S^3, S^2 \times \mathbb{R}, \mathbb{E}^3, \mathbb{H}^2 \times \mathbb{R}, \widetilde{PSL_2}, Nil, Sol$ or $\mathbb{H}^3$, in which case M is called *geometric*, or else M contains an incompressible torus.

In the nongeometric case, geometrization says that M can be split along a collection $\{T_i\}$ of non-isotopic, incompressible, two-sided tori in M in such a way that every connected component of $M \setminus \cup N(T_i)$ (where each $N(T_i)$ is an open regular neighborhood of T_i), known as a *piece*, has interior admitting a geometric structure with finite volume; this is commonly referred to as a JSJ decomposition. Moreover each piece in $M \setminus \cup N(T_i)$ is either Seifert fibered or atoroidal, and again by geometrization, the atoroidal pieces have interior that is hyperbolic. In this nongeometric case the fundamental group $\pi_1(M)$ decomposes as the fundamental group of a graph of groups whose vertex groups are the fundamental groups of the pieces and whose edge groups are all $\mathbb{Z}^2$ subgroups corresponding to fundamental groups of the tori T_i in the decomposition.

Now suppose that N is a piece from the JSJ decomposition in the nongeometric case, or else that $N = M$ in the geometric case. Then N is also a compact 3-manifold with incompressible toral boundary. A collection of $\mathbb{Z}^2$ subgroups arising from this boundary, one for each free homotopy class of boundary component of N, or equivalently one for each conjugacy class of $\mathbb{Z}^2$ subgroup, is a collection of *peripheral subgroups* of $\pi_1(N)$.

If N is a Seifert fibered 3-manifold with boundary, then N is a circle bundle over a two-dimensional orbifold with boundary. Consequently, $\pi_1(N)$ is an extension of the orbifold fundamental group of that 2-dimensional orbifold by $\mathbb{Z}$ (see [29, Lemma 3.2] for more details). On the other hand, if the interior of N is a finite volume hyperbolic 3-manifold, then $\pi_1(N)$ is hyperbolic relative to a collection of peripheral ($\mathbb{Z}^2$) subgroups (see [13, Theorem 5.1]).

3. Autostackability for graphs of groups

In this section we prove Theorem 3.5, the closure of autostackability under the construction of fundamental groups of graphs of groups, in the case that the vertex groups are autostackable respecting their respective edge groups.

We begin by noting that a small extension of the proof that autostackability is invariant under increasing the generating set [15, Proposition 4.3] yields the following Lemma, which will be useful in our closure proof.

Lemma 3.1. *Let G be autostackable over a generating set A respecting a subgroup H, and let $A' \supseteq A$ be another finite inverse-closed generating set for G. Then G is also autostackable over A' respecting H.*

Proof. The set of normal forms over A' is taken to be the same as the normal form set over A, and the flow function on edges with labels in A is also unchanged. The flow function maps edges labeled by any letter $a \in A' \setminus A$ to paths labeled by $\mathsf{nf}(a)$. □

Lemma 3.2. *Let G be autostackable over a generating set A respecting a subgroup H with generating set $B \subseteq A$. Suppose that the set of normal forms is $\mathcal{N}_G = \mathcal{N}_H \mathcal{N}_{\mathcal{T}}$, where $\mathcal{N}_H$ is* *a set of normal forms for H over B and $\mathcal{N}_{\mathcal{T}}$ is a set of normal forms over A for a right transversal of H in G. Suppose further that $\mathcal{N}_G$ is regular and prefix-closed. Then*

$$\mathcal{N}_H = \mathcal{N}_G \cap B^* \qquad \text{and} \qquad \mathcal{N}_{\mathcal{T}} = \{\varepsilon\} \cup \big(\mathcal{N}_G \cap [(A \setminus B) \cdot A^*]\big),$$

and both $\mathcal{N}_H$ and $\mathcal{N}_{\mathcal{T}}$ are also regular prefix-closed languages.

Proof. Note that since $\mathcal{N}_G$ is prefix-closed, the empty word ε is an element of $\mathcal{N}_G$, and so we also have $\varepsilon \in \mathcal{N}_H$ and $\varepsilon \in \mathcal{N}_{\mathcal{T}}$.

Let w be any word in $\mathcal{N}_{\mathcal{T}}$ and write $w = yz$ where y is the maximal prefix of w representing an element of H. Let $y' \in \mathcal{N}_H \subset B^*$ be the normal form for the inverse of y. Then the word $y'w = y'yz$ is in $\mathcal{N}_G$, and so by prefix-closure of this set, $y'y \in \mathcal{N}_G$ as well. Since $y'y =_G 1$, and normal forms are unique, then $y = y' = \varepsilon$. Hence no word in $\mathcal{N}_{\mathcal{T}}$ has a nonempty prefix representing an element in H. The rest of the result follows from the closure properties of regular languages. □

Next we establish some notation used throughout the rest of this section.

Let $\mathcal{G} = (\Lambda, \{G_v\}, \{G_e\}, \{h_e\})$ be a graph of groups on a finite connected graph Λ with vertex set V, basepoint $v_0 \in V$, directed edge set $\vec{E}_\Lambda$, and spanning tree T_Λ. Let $\vec{E}_{\Lambda\setminus T} \subseteq \vec{E}_\Lambda$ denote the subset of $\vec{E}_\Lambda$ of directed edges whose underlying undirected edges do not lie in T_Λ. For each $v \in V$ let A_v be a finite inverse-closed generating set for G_v. Let $\widetilde{A} := (\cup_{v\in V} A_v) \cup \vec{E}_\Lambda$ and $A := (\cup_{v\in V} A_v) \cup \vec{E}_{\Lambda\setminus T}$.

The fundamental group $\pi_1(\mathcal{G})$ of this graph of groups is a quotient of the free product $(*_{v\in V} G_v) * F(\vec{E}_{\Lambda\setminus T})$, where $F(\vec{E}_{\Lambda\setminus T})$ is the free group on the set $\vec{E}_{\Lambda\setminus T}$, satisfying $\overline{e} = e^{-1}$ for each $e \in \vec{E}_{\Lambda\setminus T}$; thus, the set A is a finite inverse-closed generating set for $\pi_1(\mathcal{G})$. With each of the elements of $\vec{E}_\Lambda \setminus \vec{E}_{\Lambda\setminus T}$ as representatives of the identity of this group, the set $\widetilde{A}$ also is a finite inverse-closed generating set for $\pi_1(\mathcal{G})$.

Define two functions $i, t : \widetilde{A}^* \to V$ as follows. Let $i(\varepsilon) = t(\varepsilon) := v_0$. For each letter $a \in \vec{E}_\Lambda$, define $i(a)$ to be the initial vertex of a and $t(a)$ to be the terminal vertex of a, and for each vertex $u \in V$ and letter $a \in A_u$, define $i(a) = t(a) := u$. Finally, for an arbitrary nonempty word $w \in A^*$, define $i(w)$ to be $i(a)$ where a is the first letter of w, and define $t(w)$ to be $t(b)$ where b is the last letter of w.

Let $\rho : \widetilde{A}^* \to \vec{E}_\Lambda^*$ be the monoid homomorphism determined by $\rho(a) := a$ for all $a \in \vec{E}_\Lambda$ and $\rho(a) := \varepsilon$ for all $e \in \cup_{v\in V} A_v$. Given any two vertices $u, v \in V$, let $\mathsf{path}_{T_\Lambda}(u, v)$ denote the word in $(\vec{E}_\Lambda \setminus \vec{E}_{\Lambda\setminus T})^*$ that is the unique path without backtracking in the tree T_Λ from u to v. Note that if $u = v$ then $\mathsf{path}_{T_\Lambda}(u, v) = \varepsilon$.

We construct an "inflation" function $\mathsf{infl} : A^* \to \widetilde{A}^*$ by defining infl on any word $w = a_1 \cdots a_k$, with each $a_i \in A$, as

$$\mathsf{infl}(w) := \mathsf{path}_{T_\Lambda}(v_0, i(a_1))a_1\mathsf{path}_{T_\Lambda}(t(a_1), i(a_2))a_2 \cdots \mathsf{path}_{T_\Lambda}(t(a_{n-1}), i(a_n))a_n.$$

Note that for any word w over A, the word $\rho(\mathsf{infl}(w))$ is a directed edge path in Λ; moreover, it is the shortest directed edge path in Λ starting at the basepoint v_0 and ending at the vertex $t(w)$ that traverses the vertices $t(a_i)$ of the letters a_i that lie in $\cup_{v\in V} A_v$ and the edges corresponding to the letters a_i lying in $\vec{E}_{\Lambda\setminus T}$, in the order that they appear in w, and otherwise remains in the tree T_Λ.

Define a "deflation" (monoid) homomorphism $\mathsf{defl} : \widetilde{A}^* \to A^*$ by $\mathsf{defl}(a) := a$ for all $a \in A$ and $\mathsf{defl}(e) := \varepsilon$ for all $e \in \vec{E}_\Lambda \setminus \vec{E}_{\Lambda\setminus T}$. Note that for any word $w \in A^*$ we have $\mathsf{defl}(\mathsf{infl}(w)) = w$.

Finally, define a "pruning" function $\mathsf{prune} : \widetilde{A}^* \to \widetilde{A}^*$ by $\mathsf{prune}(w) :=$ the maximal prefix of w whose last letter lies in A. Since e represents the identity of $\pi_1(\mathcal{G})$ for all $e \in \vec{E}_\Lambda \setminus \vec{E}_{\Lambda\setminus T}$, all three maps infl, defl, and prune preserve the group element represented by the word.

Each element g of the fundamental group $\pi_1(\mathcal{G})$ is represented by a loop at v_0 in the graph Λ together with vertex group elements at each vertex group traversed by the loop; that is, $g =_{\pi_1(\mathcal{G})} g_0 e_1 g_1 \cdots e_k g_k$ where $e_1 \cdots e_k$ is a loop in Λ based at v_0, $g_0 \in G_{v_0}$, and $g_i \in t(e_i)$ for each i. This format for elements is realized by the inflation function; given any word over A representing an element of $\pi_1(\mathcal{G})$, the inflation function inserts letters that are edges in the tree in order to produce a word representing a path starting at the basepoint in the graph Λ, interspersed with letters from vertex groups at vertices along that path; however the inflation function does not add a final path through the tree to make the path end at the basepoint. The deflation map reverses this process, removing all letters corresponding to edges in the tree.

With this notation, we use a construction of normal forms for fundamental groups given by Higgins in [17] to obtain a set of normal forms for elements of $\pi_1(\mathcal{G})$.

Proposition 3.3. *Let $\mathcal{G} = (\Lambda, \{G_v\}, \{G_e\}, \{h_e\})$ be a graph of groups on a finite connected graph Λ with vertex set V, basepoint $v_0 \in V$, directed edge set $\vec{E}_\Lambda$, spanning tree T_Λ, and subset $\vec{E}_{\Lambda\setminus T} \subseteq \vec{E}_\Lambda$ consisting of edges not lying in T_Λ. For each $v \in V$ let A_v be a finite inverse-closed generating set for G_v. Let $\widetilde{A} := (\cup_{v\in V} A_v) \cup \vec{E}_\Lambda$ and $A := (\cup_{v\in V} A_v) \cup \vec{E}_{\Lambda\setminus T}$, and let $\mathsf{defl}\colon \widetilde{A}^* \to A^*$ be the deflation map. Suppose that $\mathcal{N}_0$ is a regular prefix-closed set of normal forms for G_{v_0} over A_{v_0}, and for each $e \in \vec{E}_\Lambda$ suppose that $\mathcal{N}_{\mathcal{T},e}$ is a regular prefix-closed set of normal forms for a right transversal over $A_{t(e)}$ of $h_e(G_e)$ in $G_{t(e)}$. Define*

$$\widetilde{\mathcal{N}} := \{w_0 e_1 t_1 e_2 t_2 \cdots e_k t_k \mid k \geq 0, i(e_1) = v_0, w_0 \in \mathcal{N}_0, \forall\ i,\ e_i \in \vec{E}_\Lambda,\ t(e_i) = i(e_{i+1}), t_i \in \mathcal{N}_{\mathcal{T},e_i},\ \textit{and } t_i \neq \varepsilon \textit{ if } e_{i+1} = \overline{e_i}\ \textit{and either } t_k \neq \varepsilon,\ e_k \in \vec{E}_{\Lambda\setminus T},\ \textit{or } k = 0\},$$

and let $\mathcal{N} := \mathsf{defl}\left(\widetilde{\mathcal{N}}\right)$. Then $\widetilde{\mathcal{N}}$ is a regular set of normal forms over $\widetilde{A}$, and $\mathcal{N}$ is a regular prefix-closed set of normal forms over A, for the fundamental group $\pi_1(\mathcal{G})$.

Proof. In [17, Corollary 1], Higgins proved that the set

$$\widehat{\mathcal{N}} := \{w_0 e_1 t_1 e_2 t_2 \cdots e_\ell t_\ell \mid \ell \geq 0, e_i \in \vec{E}_\Lambda, i(e_1) = t(e_\ell) = v_0, t(e_i) = i(e_{i+1}), w_0 \in \mathcal{N}_0, t_i \in \mathcal{N}_{\mathcal{T},e_i}, t_i \neq \varepsilon \text{ if } e_{i+1} = \overline{e_i}\}$$

is a set of normal forms for the fundamental group $\pi_1(\mathcal{G})$. Note that for each word $w \in \widetilde{\mathcal{N}}$, the last letter of w cannot lie in $\vec{E}_\Lambda \setminus \vec{E}_{\Lambda\setminus T}$. Hence the concatenated word $w\mathsf{path}_{T_\Lambda}(t(w), v_0)$ lies in $\widehat{\mathcal{N}}$, and moreover, for each word $x \in \widehat{\mathcal{N}}$, the pruned word $\mathsf{prune}(x)$ lies in $\widetilde{\mathcal{N}}$. Thus the maps $\widetilde{\mathcal{N}} \to \widehat{\mathcal{N}}$, defined by $w \mapsto w\mathsf{path}_{T_\Lambda}(t(w), v_0)$, and $\widehat{\mathcal{N}} \to \widetilde{\mathcal{N}}$, defined by $x \mapsto \mathsf{prune}(x)$, are inverses. These maps preserve the group element being represented, and so $\widetilde{\mathcal{N}}$ is also a set of normal forms over $\widetilde{A}$ for $\pi_1(\mathcal{G})$. Thus to prove that $\mathcal{N}$ is also a set of normal forms for $\pi_1(\mathcal{G})$, it suffices to prove that the restriction of the deflation map defl to $\widetilde{\mathcal{N}}$ is a bijection.

By definition, $\mathcal{N} = \mathsf{defl}(\widetilde{\mathcal{N}})$, and thus $\mathsf{defl}\colon \widetilde{\mathcal{N}} \to \mathcal{N}$ is surjective. Thus, we need only show that it is injective. Suppose that $w, w' \in \widetilde{\mathcal{N}}$ and $\mathsf{defl}(w) = \mathsf{defl}(w')$. As deflation only eliminates edges in T_Λ, we have that $w =_{\pi_1(\mathcal{G})} \mathsf{defl}(w)$ and $w' =_{\pi_1(\mathcal{G})} \mathsf{defl}(w')$. Now, this implies that $w =_{\pi_1(\mathcal{G})} w'$; however, since $\widetilde{\mathcal{N}}$ is a set of normal forms for $\pi_1(\mathcal{G})$, we must have that $w = w'$. Thus, defl is injective when restricted to $\widetilde{\mathcal{N}}$. We also note that the restriction of the inflation map to $\mathcal{N}$ gives $\mathsf{infl}: \mathcal{N} \to \widetilde{\mathcal{N}}$, which is the inverse of defl. Thus $\mathcal{N}$ is also a set of normal forms for $\pi_1(\mathcal{G})$.

Since $\mathsf{defl}\colon \widetilde{A}^* \to A^*$ is a monoid homomorphism and the image of a regular language is regular (Theorem 2.1), in order to show that $\mathcal{N}$ is regular it suffices to show that $\widetilde{\mathcal{N}}$ is regular. For all $w \in \widetilde{\mathcal{N}}$, the word w is a concatenation $w = w_0 w'$, where $w_0 \in \mathcal{N}_0$ and the suffix w' alternates between edges and transversal normal forms from the transversals corresponding to the edges; that is, $w \in \mathcal{N}_0 \cdot (\cup_{e\in\vec{E}_\Lambda} e\mathcal{N}_{\mathcal{T},e})^*$. On the other hand, a word $w \in \mathcal{N}_0 \cdot (\cup_{e\in\vec{E}_\Lambda} e\mathcal{N}_{\mathcal{T},e})^*$ will fail to be in the language $\widetilde{\mathcal{N}}$ if and only if either w contains an edge and its reverse consecutively, the terminal vertex of one edge is not the initial vertex of the next, the first edge in w' does not have initial vertex v_0, or the last letter of w lies in $\vec{E}_\Lambda \setminus \vec{E}_{\Lambda\setminus T}$. That is,

$$\widetilde{\mathcal{N}} = \mathcal{N}_0 \cdot \Big(\cup_{e\in\vec{E}_\Lambda} e\mathcal{N}_{\mathcal{T},e}\Big)^* \setminus \Big[\Big(\cup_{e\in\vec{E}_\Lambda} \widetilde{A}^* e\overline{e}\widetilde{A}^*\Big) \cup \Big(\cup_{t(e)\neq i(e')} \widetilde{A}^* e\mathcal{N}_{\mathcal{T},e}e'\widetilde{A}^*\Big) \cup \Big(\cup_{i(e)\neq v_0} \mathcal{N}_0 e\widetilde{A}^*\Big) \cup \Big(\widetilde{A}^*(\vec{E}_\Lambda \setminus \vec{E}_{\Lambda\setminus T}\Big) \Big].$$

Hence $\widetilde{\mathcal{N}}$ is formed from regular languages by concatenation, union, Kleene star, and complementation. Thus, by Theorem 2.1, $\widetilde{\mathcal{N}}$ is regular, and so $\mathcal{N}$ is also regular.

Finally, suppose that w' is a prefix of a word $w \in \mathcal{N}$. Then $w = w'w''$ for some $w'' \in A^*$, and so the "lift" $\mathsf{infl}(w)$ of w to $\widetilde{\mathcal{N}}$ satisfies $\mathsf{infl}(w) = \mathsf{infl}(w')x''$ for some word $x'' \in \widetilde{A}^*$. Now prefix-closure of the sets $\mathcal{N}_0$ and of $\mathcal{N}_{\mathcal{T},e}$ for all $e \in \vec{E}_\Lambda$ implies that the word $\mathsf{infl}(w')$ lies in the set

$$\mathcal{N}_0 \cdot \Big(\cup_{e\in\vec{E}_\Lambda} e\mathcal{N}_{\mathcal{T},e}\Big)^* \setminus \Big[\Big(\cup_{e\in\vec{E}_\Lambda} \widetilde{A}^* e\overline{e}\widetilde{A}^*\Big) \cup \Big(\cup_{t(e)\neq i(e')} \widetilde{A}^* e\mathcal{N}_{\mathcal{T},e}e'\widetilde{A}^*\Big) \cup \Big(\cup_{i(e)\neq v_0} \mathcal{N}_0 e\widetilde{A}^*\Big)\Big].$$

Moreover, the last letter of $\mathsf{infl}(w')$ is the last letter of w', since the inflation procedure does not insert any letters at the end of a word over A. Hence $\mathsf{infl}(w')$ lies in $\widetilde{\mathcal{N}}$, and so $w' = \mathsf{defl}(\mathsf{infl}(w'))$ lies in $\mathcal{N}$. Therefore the set $\mathcal{N}$ is also prefix-closed. □

In the following lemma we illustrate the effect on normal forms of multiplication by a word in a vertex group. This information will be integral to our proof in Theorem 3.5 that the function Φ satisfies the property (F3), showing that iteration of Φ on any path eventually stabilizes at a path in the tree.

Lemma 3.4. *Let $\mathcal{G}$, Λ, v, v_0, $\vec{E}_\Lambda$, T_Λ, $\vec{E}_{\Lambda\setminus T}$, A_v, G_v, $\widetilde{A}$, $\mathcal{N}_0$, $\mathcal{N}_{\mathcal{T},e}$, and $\widetilde{\mathcal{N}}$ be as in the statement of Proposition* 3.3. *Let $\tilde{y} = w_0e_1t_1e_2t_2\cdots e_kt_k$ be an element of $\widetilde{\mathcal{N}}$ with $k \geq 0$, $i(e_1) = v_0$, $w_0 \in \mathcal{N}_0$, and each $e_i \in \vec{E}_\Lambda$, $t(e_i) = i(e_{i+1})$, and $t_i \in \mathcal{N}_{\mathcal{T},e_i}$. Moreover, let $w \in A_u^*$ for some $u \in V$ and write $\mathsf{path}_{T_\Lambda}(t(\tilde{y}), i(w)) = f_1 \cdots f_m$ with $m \geq 0$ and each $f_j \in \vec{E}_\Lambda \setminus \vec{E}_{\Lambda\setminus T}$. Then the normal form of $\tilde{y}w$ in $\widetilde{\mathcal{N}}$ is*

$$\mathsf{nf}_{\widetilde{\mathcal{N}}}(\tilde{y}w) = \mathsf{prune}(w_0'e_1x_1e_2\cdots e_kx_kf_1w_1f_2\cdots f_mw_m)$$

for some words $w_0' \in \mathcal{N}_0$, $x_i \in \mathcal{N}_{\mathcal{T},e_i}$, and $w_j \in \mathcal{N}_{\mathcal{T},f_j}$.

Proof. Note that since $\tilde{y} \in \widetilde{\mathcal{N}}$, then $\tilde{y}$ is freely reduced and $\tilde{y}$ does not end with a letter in $\vec{E}_\Lambda \setminus \vec{E}_{\Lambda\setminus T}$. Hence the word $\tilde{y}\mathsf{path}_{T_\Lambda}(t(\tilde{y}), i(w))$ is also freely reduced.

We can uniquely factor the element of $G_{t(f_m)}$ represented by w as $w =_{G_{t(f_m)}} g_mw_m$ where $g_m \in h_{f_m}(G_{f_m})$ and $w_m \in \mathcal{N}_{\mathcal{T},f_m}$. Let $\widetilde{g}_m$ denote a representative of g_m in $G_{i(f_m)}$; that is, $\widetilde{g}_m = h_{\overline{f_m}}(h_{f_m}^{-1}(g_m))$. Repeating this process, for each $1 \leq i \leq m-1$, write the element $\widetilde{g}_{i+1}$ in $G_{t(f_i)}$ as $\widetilde{g}_{i+1} =_{G_{t(f_i)}} g_iw_i$ where $g_i \in h_{f_i}(G_{f_i})$ and $w_i \in \mathcal{N}_{\mathcal{T},f_i}$, and let $\widetilde{g}_i$ denote a representative of g_i in $G_{i(f_i)}$. Then $\tilde{y}w =_{\pi_1(\mathcal{G})} \tilde{y}\mathsf{path}_{T_\Lambda}(t(y), i(w))w =_{\pi_1(\mathcal{G})} \tilde{y}\widetilde{g}_1f_1w_1f_2\cdots f_mw_m$.

Continuing this process further, let $\widetilde{g}'_{k+1} := \widetilde{g}_1$, and for all $1 \le i \le k$, write the element $t_i\widetilde{g}'_{i+1}$ in $G_{t(f_i)}$ as $t_i\widetilde{g}'_{i+1} =_{G_{t(f_i)}} g'_i x_i$ where $g'_i \in h_{f_i}(G_{f_i})$ and $x_i \in \mathcal{N}_{\mathcal{T},f_i}$, and let $\widetilde{g}'_i$ denote a representative of g'_i in $G_{i(f_i)}$. Also let w'_0 be the normal form in $\mathcal{N}_0$ of the element represented by $w_0\widetilde{g}'_1$. Then

$$\tilde{y}w =_{\pi_1(\mathcal{G})} yw =_{\pi_1(\mathcal{G})} w'_0 e_1 x_1 e_2 \cdots e_k x_k f_1 w_1 f_2 \cdots f_m w_m.$$

If there is an index i such that $x_i = \varepsilon$, then $t_i =_{\pi_1(\mathcal{G})} g'_i(\widetilde{g}'_{i+1})^{-1}$ and since $g'_i, \widetilde{g}'_{i+1}$ are in $h_{f_i}(G_{f_i})$, then t_i is in $h_{f_i}(G_{f_i})$ as well. Since t_i is an element of $\mathcal{N}_{\mathcal{T},f_i}$, then $t_i = \varepsilon$ in this case. Note that if $x_k = \varepsilon$, then $t_k = \varepsilon$, and since $\tilde{y}$ is pruned, this implies that $e_k \in \vec{E}_{\Lambda\setminus T}$. Thus, the word $\tilde{y}\mathsf{path}_{T_\Lambda}(t(\tilde{y}), i(w))$ does not contain any consecutive inverse pair of letters, and the process in this proof cannot create any subword that is not freely reduced. Thus the word produced by this process satisfies all of the properties of words in $\widetilde{\mathcal{N}}$, except for the possibility that it has a suffix of letters in $(\vec{E}_\Lambda \setminus \vec{E}_{\Lambda\setminus T})^*$. Therefore $\mathsf{nf}_{\widetilde{\mathcal{N}}}(\tilde{y}w) = \mathsf{prune}(w'_0 e_1 x_1 e_2 \cdots e_k x_k f_1 w_1 f_2 \cdots f_m w_m)$. □

We are now ready to show autostackability of fundamental groups of graphs of autostackable groups in which each vertex group has an autostackable structure respecting each incident edge group. En route we also prove this closure property for stackable groups.

Theorem 3.5. *Let $\mathcal{G}$ be a graph of groups over a finite connected graph Λ with at least one edge. If for each directed edge e of Λ the vertex group G_v corresponding to the terminal vertex $v = t(e)$ of e is autostackable [respectively, stackable] respecting the associated injective homomorphic image of the edge group G_e, then the fundamental group $\pi_1(\mathcal{G})$ is autostackable [respectively, stackable].*

Proof. Let $\mathcal{G} = (\Lambda, \{G_v\}, \{G_e\}, \{h_e\})$ be a graph of groups with vertex set V, basepoint $v_0 \in V$, directed edge set $\vec{E}_\Lambda$, spanning tree T_Λ, and subset $\vec{E}_{\Lambda\setminus T} \subseteq \vec{E}_\Lambda$ of edges not lying in T_Λ, of the finite connected graph Λ.

Since Λ is connected and has at least one edge, every vertex of Λ is the terminus of an edge in Λ, including v_0, and so every vertex group is autostackable. Suppose that G_{v_0} has an autostackable structure over an alphabet A_0 and normal form set $\mathcal{N}_0$. Also for each $e \in \vec{E}_\Lambda$, and $v \in V$ with $t(e) = v$, suppose that G_v is autostackable respecting $h_e(G_e)$ over a finite inverse-closed generating set $A_{v,e}$, with normal form set of the form $\mathcal{N}_{G_v,e} = \mathcal{N}_{h_e(G_e)}\mathcal{N}_{\mathcal{T},e}$ where $\mathcal{N}_{h_e(G_e)}$ is a set of normal forms for $h_e(G_e)$ over a finite inverse-closed generating set B_e of $h_e(G_e)$ contained in $A_{v,e}$, and $\mathcal{N}_{\mathcal{T},e}$ is a set of normal forms over $A_{v,e}$ for a right transversal of $h_e(G_e)$ in G_v.

For each $v \in V \setminus \{v_0\}$, let $A_v := \cup_{e\in\vec{E}_\Lambda, t(e)=v} A_{v,e}$, and let $A_{v_0} := A_0 \cup (\cup_{e\in\vec{E}_\Lambda, t(e)=v_0} A_{v_0,e})$. By Lemma 3.1, the group G_{v_0} is also autostackable over A_{v_0} with normal form set $\mathcal{N}_0$, and for each vertex v and edge e with $t(e) = v$ the group G_v is autostackable respecting $h_e(G_e)$ over A_v with the same normal form set $\mathcal{N}_{G_v,e}$. Let Φ_0 be the associated flow function for G_0, with bound K_0, and let Φ_e denote the flow function for the autostackable structure on G_v respecting $h_e(G_e)$, with bound K_e.

Let $A := (\cup_{v\in V} A_v) \bigcup \vec{E}_{\Lambda\setminus T}$. Let Γ be the Cayley graph of $\pi_1(\mathcal{G})$ over A and as usual let $\vec{E}, \vec{P}$ be the sets of directed edges and paths in Γ. Having satisfied all of the hypotheses of Proposition 3.3, let $\mathcal{N}$ be the regular prefix-closed set of normal forms for $\pi_1(\mathcal{G})$ over A from that Proposition.

For every $g \in \pi_1(\mathcal{G})$, let $\mathsf{nf}(g) \in \mathcal{N}$ denote the normal form of g. Let T be the spanning tree in Γ determined by the normal form set $\mathcal{N}$.

The flow function and stackability:

Next we define a (stacking) function $\phi : \mathcal{N} \times A \to A^*$ and show that the associated function $\Phi : \vec{E} \to \vec{P}$ defined by $\Phi(e_{g,a}) :=$ the path in Γ starting at g labeled by $\phi(\mathsf{nf}(g), a)$ satisfies the properties of a bounded flow function.

Let $\phi_0 : \mathcal{N}_0 \times A_{v_0} \to A^*_{v_0}$ be the stacking function associated to Φ_0, and let $\phi_e : \mathcal{N}_{G_{t(e),e}} \times A_{t(e)} \to A^*_{t(e)}$ be the stacking functions associated to the flow functions Φ_e for each $e \in \vec{E}_\Lambda$, respectively.

Before giving the full details of the definition of the stacking function ϕ, we include an informal description. Given a normal form word $y \in \mathcal{N}$ and a letter $a \in A$, there are three possible options. The first is that $a \in \vec{E}_{\Lambda\setminus T}$, in which case we define $\phi(y, a) = a$ so that the flow function fixes the edge $e_{y,a}$. Otherwise $a \in A_v$ for some vertex v of Λ. Let p be the path in Λ associated to the inflation of the word ya. The second option is that the path p ends with an edge e such that a is a generator of the edge group $h(G_e)$ and y does not end with a letter in A_v; then we use relation (2) or (3) of Definition 2.7 (depending on whether or not e is in T_Λ) to define $\phi(y, a)$ so that the path in Λ associated to the inflation of the normal form for $y\phi(y, a)$ is shorter. Finally, in the third option (that is, in any other case) we apply the stacking function ϕ_e (or ϕ_0 if p is the empty path). To make this more precise, we again define some notation.

For any word $w \in A^*$ and letter $a \in \cup_{v\in V} A_v$, the word $\rho(\mathsf{infl}(wa))$ is a directed edge path in Λ from v_0 via $t(w)$, to $i(a)$, and in particular this path satisfies $\rho(\mathsf{infl}(wa)) = \rho(\mathsf{infl}(w))\mathsf{path}_{T_\Lambda}(t(w), i(a))$. Let $\mathsf{last}(w, a) := 0$ if $\rho(\mathsf{infl}(wa)) = \varepsilon$ (equivalently, if $w \in A^*_{v_0}$ and $a \in A_{v_0}$), and let $\mathsf{last}(w, a)$ be the last letter of $\rho(\mathsf{infl}(wa))$ otherwise. That is, $\mathsf{last}(w, a)$ is the last edge encountered on the path $\rho(\mathsf{infl}(wa))$. For each word w in A^* (or $\widetilde{A}^*$) and vertex $u \in V$, let $\mathsf{suf}_u(w)$ denote the maximal suffix of w contained in A^*_u.

For any edge $e \in \vec{E}_\Lambda$ and letter $a \in B_e$, let $\widehat{a}_e$ denote the normal form over $B_{\overline{e}}$ in the autostackable structure of $G_{i(e)}$ respecting $h_{\overline{e}}(G_{\overline{e}})$ of the element $h_{\overline{e}}(h_e^{-1}(a))$; that is, if $e \in \vec{E}_\Lambda \setminus \vec{E}_{\Lambda\setminus T}$ then $a =_G \widehat{a}_e$ and if $e \in \vec{E}_{\Lambda\setminus T}$ then $a =_G e^{-1}\widehat{a}_e e$.

Let $y \in \mathcal{N}$ and let $a \in A$. The function $\phi : \mathcal{N} \times A \to A^*$ evaluated at (y, a) is given by

$$\phi(y,a) := \begin{cases} a & \text{if } a \in \vec{E}_{\Lambda\setminus T} \\ \mathsf{defl}(\mathsf{last}(y,a)^{-1}\ \widehat{a}_{\mathsf{last}(y,a)}\ \mathsf{last}(y,a)) & \text{if } \mathsf{last}(y,a) \in \vec{E}_\Lambda, a \in B_{\mathsf{last}(y,a)}, \\ & \quad \text{and } \mathsf{suf}_{i(a)}(y) = \varepsilon \\ \phi_{\mathsf{last}(y,a)}(\mathsf{suf}_{i(a)}(y), a) & \text{otherwise.} \end{cases}$$

Let $\Phi : \vec{E} \to \vec{P}$ be defined by $\Phi(e_{g,a}) :=$ the path in Γ starting at g labeled by $\phi(\mathsf{nf}(g), a)$, for all $g \in G$ and $a \in A$.

Property (F1): It follows immediately from this definition that for each directed edge $e \in \vec{E}$ in the Cayley graph Γ, the path $\Phi(e)$ has the same initial and terminal vertices as e. Also the length of the path $\Phi(e)$ is at most $2 + \max(\{\ell(\widehat{a}_e) \mid e \in \vec{E}_\Lambda, a \in B_e\} \cup \{K_f \mid f \in \{0\} \cup \vec{E}_\Lambda\})$. This is a maximum over a finite set, hence (F1) holds for Φ.

Property (F2): To check that Φ fixes directed edges lying in the spanning tree T in Γ, suppose that $e_{g,a}$ is a directed edge whose underlying undirected edge lies in T, and let $y := \mathsf{nf}(g)$. Then either $\mathsf{nf}(ga) = ya$, or else y ends with the letter a^{-1}.

If $a \in \vec{E}_{\Lambda\setminus T}$, then $\Phi(e_{g,a}) = e_{g,a}$ from the definition above, as required.

On the other hand, suppose that $a \in A_u$ for some $u \in \vec{E}_\Lambda$. Now let $\tilde{y} = \mathsf{infl}(y)$, so that $y = \mathsf{defl}(\tilde{y})$. We can write $\tilde{y} = w_0 e_1 t_1 e_2 t_2 \cdots e_k t_k \in \widetilde{\mathcal{N}}$ such that $e_1 \cdots e_k$ is an edge path in Λ from v_0 to $t(y)$, $w_0 \in \mathcal{N}_0$, each $t_i \in \mathcal{N}_{\mathcal{T},e_i}$, no inverse pair $e\overline{e}$ appears as a subword, and the last letter of $\tilde{y}$ lies in A (or $\tilde{y} = \varepsilon$). Now $\mathsf{infl}(ya) = \mathsf{infl}(y)\mathsf{path}_{T_\Lambda}(t(y), i(a))a$ and $\rho(\mathsf{infl}(ya)) = e_1 \cdots e_k \mathsf{path}_{T_\Lambda}(t(y), i(a))$.

Suppose further that $ya \in \mathcal{N}$; then $\mathsf{infl}(ya) \in \widetilde{\mathcal{N}}$. If $\mathsf{last}(y,a) = 0$, then $\mathsf{suf}_{i(a)}(y) = y \in \mathcal{N}_0$, $a \in A_{v_0}$, and $ya \in \mathcal{N}_0$, and so $\mathsf{label}(\Phi(e_{g,a})) = \phi(y,a) = \phi_0(y,a) = \mathsf{label}(\Phi_0(e_{y,a})) = a$ since the flow function Φ_0 fixes edges in the associated spanning tree for G_{v_0}. If instead $\mathsf{last}(y,a) \neq 0$, then $\mathsf{last}(y,a) \in \vec{E}_\Lambda$. In the case that $t(y) = i(a)$ (i.e., $a \in A_{t(y)}$), we have $\mathsf{last}(y,a) = e_k$ and $\mathsf{suf}_{i(a)}(y) = t_k$. Applying the properties of elements of $\widetilde{\mathcal{N}}$ to $\mathsf{infl}(ya)$ shows that $t_k a \in \mathcal{N}_{\mathcal{T},e_k}$; that is, $\mathsf{suf}_{i(a)}(y)a \in \mathcal{N}_{\mathcal{T},\mathsf{last}(y,a)}$. If $\mathsf{suf}_{i(a)}(y) = t_k = \varepsilon$, then (recalling that $\mathcal{N}_{\mathcal{T},e_k} = \{\varepsilon\} \cup (\mathcal{N}_{G_{t(e_k)},e_k} \cap (A_{t(e_k)} \setminus B_{e_k}) \cdot A^*_{t(e_k)}))$ we have $a \notin B_{\mathsf{last}(y,a)}$. Hence $\mathsf{label}(\Phi(e_{g,a})) = \phi_{\mathsf{last}(y,a)}(\mathsf{suf}_{i(a)}(y), a) = a$, using Property (F2) for the flow function $\Phi_{\mathsf{last}(y,a)}$. In the case that $t(y) \neq i(a)$, the edge $\mathsf{last}(y,a)$ is the last edge of $\mathsf{path}_{T_\Lambda}(t(y), i(a))$ and $\mathsf{suf}_{i(a)} = \varepsilon$. Again using the fact that $\mathsf{infl}(ya) \in \widetilde{\mathcal{N}}$, the edge $\mathsf{last}(y,a)$ must be followed in $\mathsf{infl}(ya)$ by a suffix in the normal form set $\mathcal{N}_{\mathcal{T},\mathsf{last}(y,a)}$; this subword is the single letter a, and so $a \in \mathcal{N}_{\mathcal{T},\mathsf{last}(y,a)}$ and again $a \notin B_{\mathsf{last}(y,a)}$. Therefore $\mathsf{label}(\Phi(e_{g,a})) = \phi_{\mathsf{last}(y,a)}(\mathsf{suf}_{i(a)}(y), a) = a$, as needed.

Suppose instead that y ends with the letter $a^{-1} \in A_{i(a)}$. Then $t(y) = i(a)$ and $\mathsf{suf}_{i(a)}(y)$ is a nonempty word in $A^*_{i(a)}$ ending with the letter a^{-1}. Therefore $\mathsf{label}(\Phi(e_{g,a})) = \phi_{\mathsf{last}(y,a)}(\mathsf{suf}_{i(a)}(y), a) = a$. This concludes the last case, showing that Φ fixes every directed edge lying in the spanning tree T in Γ, and so (F2) is satisfied.

Property (F3): Finally we consider sequences $e_1, e_2, ...$ of directed edges $e_i \in \vec{E}$ in Γ that do not lie in T, satisfying the property that e_{i+1} lies in the path $\Phi(e_i)$ for each i. To show that no infinite sequence of this form can exist, we define a function $\alpha : \vec{E} \to \mathbb{N}^2$ and show that $\alpha(e) > \alpha(e')$ (using the lexicographic order on $\mathbb{N}^2$) whenever $e, e' \in \vec{E}$ do not lie in the tree T, and e' is on the path $\Phi(e)$, as follows.

For each edge $e^0_{g,a}$ with $g \in G_{v_0}$ and $a \in A_{v_0}$ (in the Cayley graph Γ_{v_0} of G_{v_0} over A_{v_0}), the *descending chain length* of $e^0_{g,a}$, denoted $\mathsf{dcl}_0(e^0_{g,a})$, is defined to be the maximum possible number of edges of Γ_{v_0} in a sequence $e^0_{g,a} = e^0_1, e^0_2, ...$ such that each e^0_i is an edge of Γ_0 lying outside of the spanning tree associated to the flow function Φ_0 and e^0_{i+1} lies on $\Phi_0(e^0_i)$ for all indices i. Because Φ is a flow function, this maximum is finite.

For each letter $f \in \vec{E}_\Lambda$, let $\vec{E}_{t(f)}$ be the set of directed edges $e^f_{g,a}$ (with $g \in G_{t(f)}$ and $a \in A_{t(f)}$) in the Cayley graph $\Gamma_{t(f)}$ of $G_{t(f)}$ over $A_{t(f)}$. Also let $\vec{E}_{sub,f}$ be the subset of $\vec{E}_{t(f)}$ of edges $e^f_{h,b}$ satisfying that $h \in h_f(G_f)$ and $b \in B_f$; that is, $e^f_{h,b}$ is in the subgraph of $\Gamma_{t(f)}$ that is the Cayley graph for $h_f(G_f)$ over its generating set B_f. For each edge $e^f_{g,a}$ in $\vec{E}_{t(f)}$, its f*–descending chain length*, denoted $\mathsf{dcl}_f(e^f_{g,a})$, is defined to be the maximum possible number of edges in a sequence $e^f_{g,a} = e^f_1, e^f_2, ...$ such that each e^f_i is an edge in $\vec{E}_{t(f)} \setminus \vec{E}_{sub,f}$ lying outside of the spanning tree associated to the flow function Φ_f and e^f_{i+1} lies on $\Phi_f(e^f_i)$ for all indices i. Note that for all $h \in h_f(G_f)$ the edge $e^f_{hg,a}$ also lies in $\vec{E}_{t(f)}$; the *invariant* f*–descending chain length* of $e^f_{g,a}$ is

$$\mathsf{idcl}_f(e^f_{g,a}) := \min\{\mathsf{dcl}_f(e^f_{hg,a}) \mid h \in h_f(G_f)\}.$$

Note that for all edges $e^f_{h,b} \in \vec{E}_{sub,f}$, we have $\mathsf{dcl}_f(e^f_{h,b}) = \mathsf{idcl}_f(e^f_{h,b}) = 0$, and for all edges $e^f_{g,a} \in \vec{E}_{t(f)} \setminus \vec{E}_{sub,f}$, we have $\mathsf{dcl}_f(e^f_{g,a}) \geq \mathsf{idcl}_f(e^f_{g,a}) \geq 1$.

Recall that for any word w over A^*, the symbol $\ell(w)$ denotes the length of the word w. We now define $\alpha : \vec{E} \to \mathbb{N}^2$ by

$$\begin{array}{rcl} \alpha(e_{g,a}) & := & (\alpha_1(e_{g,a}), \alpha_2(e_{g,a})) \text{ where} \\ \alpha_1(e_{g,a}) & := & \ell(\rho(\mathsf{infl}(\mathsf{nf}(g)a))), \text{ and} \\ \alpha_2(e_{g,a}) & := & \begin{cases} \mathsf{dcl}_0(e^0_{\mathsf{nf}(g),a}) & \text{if } \mathsf{last}(\mathsf{nf}(g),a) = 0 \\ \mathsf{idcl}_{\mathsf{last}(\mathsf{nf}(g),a)}(e^{\mathsf{last}(\mathsf{nf}(g),a)}_{\mathsf{suf}_{i(a)}(\mathsf{nf}(g)),a}) & \text{if } \mathsf{last}(\mathsf{nf}(g),a) \in \vec{E}_\Lambda. \end{cases} \end{array}$$

Let $e = e_{g,a}$ be any element of $\vec{E}$ that does not lie in the spanning tree T, and let $e' = e_{g',a'}$ be any edge on $\Phi(e)$ that also is not in T. Let $y := \mathsf{nf}(g)$ and $y' := \mathsf{nf}(g')$, and let $f := \mathsf{last}(y, a)$.

We consider the three cases of the definition of $\phi(y, a)$ in turn.

Case 1. Suppose that $a \in \vec{E}_{\Lambda\setminus T}$. Since the word y is in $\mathcal{N}$, then the inflation $\mathsf{infl}(y)$ is an element of $\widetilde{\mathcal{N}}$. Thus either the word $\mathsf{infl}(y)\mathsf{path}_{T_\Lambda}(t(y), i(a))a$ is also in $\widetilde{\mathcal{N}}$, in which case $ya \in \mathcal{N}$, or else $\mathsf{infl}(y)$, and hence also the word y, ends with $\overline{a} = a^{-1}$. Then the edge $e = e_{g,a}$ lies in the spanning tree T in Γ, giving a contradiction. So this case cannot occur.

Case 2. Suppose that $f := \mathsf{last}(y, a) \in \vec{E}_\Lambda$, $a \in B_f$, and $\mathsf{suf}_{i(a)}(y) = \varepsilon$. Then $\phi(y, a) = \mathsf{defl}(f^{-1}\, \widehat{a}_f\, f)$ with $\widehat{a}_f \in A^*_{i(f)}$. If $f \in \vec{E}_{\Lambda\setminus T}$, then the edges $e_{g,f^{-1}}$ and $e_{gf^{-1}\widehat{a}_f,f}$ lying in the path $\Phi(e)$ are labeled by elements of $\vec{E}_{\Lambda\setminus T}$, and hence (by Case 1) lie in the spanning tree T of the Cayley graph. Consequently the edge e' cannot be one of those two edges. Thus there is a factorization $\phi(y, a) = \mathsf{defl}(f^{-1})wa'w'\mathsf{defl}(f)$ for some words $w, w' \in A^*_{i(f)}$ such that $e' = e_{g',a'}$ and $g' =_G gf^{-1}w$.

Write (as usual) $\mathsf{infl}(y) = w_0e_1t_1e_2t_2 \cdots e_kt_k \in \widetilde{\mathcal{N}}$ such that $e_1 \cdots e_k$ is an edge path in Λ from v_0 to $t(y)$, $w_0 \in \mathcal{N}_0$, each $t_i \in \mathcal{N}_{\mathcal{T},e_i}$, no inverse pair $e\overline{e}$ appears as a subword, and the last letter of $\mathsf{infl}(y)$ (which is also the last letter of y) lies in A. Write $\mathsf{path}_{T_\Lambda}(t(y), i(a)) = f_1 \cdots f_m$ with $m \geq 0$ and each $f_j \in \vec{E}_\Lambda \setminus \vec{E}_{\Lambda\setminus T}$. Then $\mathsf{infl}(ya) = \mathsf{infl}(y)\mathsf{path}_{T_\Lambda}(t(y), i(a))a$ and so $\alpha_1(e) = \ell(\rho(\mathsf{infl}(ya))) = k + m$.

Note that either $m > 0$, in which case $f = f_m \in \vec{E}_\Lambda \setminus \vec{E}_{\Lambda\setminus T}$ and $\mathsf{defl}(f) = \varepsilon$, or else $m = 0$, in which case $t(y) = i(a)$ and $f = e_k \in \vec{E}_{\Lambda\setminus T}$ is the last letter of y. For the rest of Case 2 we consider the situation that $m > 0$; the proof for $m = 0$ is similar.

Note that $y' = \mathsf{nf}(g') = \mathsf{nf}(yw) = \mathsf{nf}(\mathsf{infl}(y)w)$. Applying Lemma 3.4 to the words $\tilde{y} := \mathsf{infl}(y)$ and $w \in A^*_{i(f)}$ shows that

$$\mathsf{infl}(y') = \mathsf{nf}_{\widetilde{\mathcal{N}}}(y') = \mathsf{prune}(w'_0e_1x_1e_2 \cdots e_kx_kf_1w_1f_2 \cdots f_{m-1}w_{m-1})$$

for some words $w'_0 \in \mathcal{N}_0$, $x_i \in \mathcal{N}_{\mathcal{T},e_i}$ and $w_j \in \mathcal{N}_{\mathcal{T},f_j}$. Since the word being pruned away is freely reduced and represents a path in T_Λ, the suffix that is removed is $\mathsf{path}_{T_\Lambda}(t(y'), t(f_{m-1}))$. For the letter $a' \in A_{i(f)} = A_{t(f_{m-1})}$ we have $\mathsf{infl}(y'a') = \mathsf{infl}(y')\mathsf{path}_{T_\Lambda}(t(y'), i(f_{m-1}))$, and so $\rho(\mathsf{infl}(y'a')) = e_1 \cdots e_kf_1 \cdots f_{m-1}$; that is, $\alpha_1(e') = k + m - 1$. Therefore $\alpha(e) > \alpha(e')$ in this case.

Case 3. Suppose that $a \in A_f$ and either $f = 0$, or else ($f \in \vec{E}_\Lambda$ and either $\mathsf{suf}_{i(a)}(y) \neq \varepsilon$ or $a \notin B_f$). This case is split into two subcases.

Case 3.1. Suppose that $f = 0$. In this case the word $y \in \mathcal{N}_0$ for the flow function Φ_0 on the vertex group G_{v_0}, and the letter $a \in A_{v_0}$, the generating set for this vertex group. In this case the edge e lies in the Cayley graph $\Gamma_{A_{v_0}}(G_{v_0})$ of the vertex group (considered as a subgraph of Γ), and $\phi(y, a) = \phi_0(\mathsf{suf}_{i(a)}(y), a) = \phi_0(y, a)$. Then the edge e' on $\Phi(e)$ is also an edge on $\Phi_0(e)$, and so $\alpha_1(e) = 0 = \alpha_1(e')$. Moreover, the descending chain lengths of these edges satisfy $\mathsf{dcl}_0(e) > \mathsf{dcl}_0(e')$, and so $\alpha_2(e) > \alpha_2(e')$. Therefore $\alpha(e) > \alpha(e')$.

Case 3.2. Suppose that $f \in \vec{E}_\Lambda$. In this case we again factor $\phi(y,a) = \phi_f(\mathsf{suf}_{i(a)}(y), a) = wa'w' \in A^*_{t(f)}$ such that $e' = e_{g',a'}$ for the element $g' =_G yw$ and letter $a' \in t(f)$. We write $\mathsf{infl}(y) = w_0 e_1 t_1 e_2 t_2 \cdots e_k t_k$ and $\mathsf{path}_{T_\Lambda}(t(y), i(a)) = f_1 \cdots f_m$ with $m \geq 0$, and $\alpha_1(e) = k+m$. Applying Lemma 3.4 to the words $\tilde{y} := \mathsf{infl}(y)$ and $w, a' \in A^*_{t(f)} = A^*_{t(f_m)}$ shows that $\alpha_1(e') = k + m = \alpha_1(e)$ in this case.

Next consider $\alpha_2(e) = \mathsf{idcl}_f(e^f_{\mathsf{suf}_{i(a)}(y),a})$. Since $f \neq 0$, then $\mathsf{suf}_{i(a)}(y)$ lies in $\mathcal{N}_{\mathcal{T},f}$, and so $\mathsf{suf}_{i(a)}(y)$ can only represent an element of $h_f(G_f)$ if $\mathsf{suf}_{i(a)}(y) = \varepsilon$, in which case $a \notin B_f$. Hence in Case 3.2 we have $e = e_{g,a}$ satisfies $e^f_{\mathsf{suf}_{i(a)}(y),a} \in \vec{E}_{t(f)} \setminus \vec{E}_{sub,f}$, and $\mathsf{idcl}(e^f_{\mathsf{suf}_{i(a)}(y),a}) \geq 1$.

Let h be an element of $h_f(G_f)$ achieving the minimum for this invariant descending chain length; that is, $\alpha_2(e) = \mathsf{idcl}(e^f_{\mathsf{suf}_{i(a)}(y),a}) = \mathsf{dcl}(e^f_{h\mathsf{suf}_{i(a)}(y),a})$. Since Φ_f is the bounded flow function of an autostackable structure for $G_{t(f)}$ respecting the subgroup $h_f(G_f)$, then Φ_f is $h_f(G_f)$–translation invariant, and so for this element h, we have $\mathsf{label}(\Phi_f(e^f_{h\mathsf{suf}_{i(a)}(y),a})) = wa'w'$ as well. Then the edge $e^f_{h\mathsf{suf}_{i(a)}(y)w,a'}$ lies on the path $\Phi_f(e^f_{h\mathsf{suf}_{i(a)}(y),a})$. Now the descending chain lengths satisfy $\mathsf{dcl}(e_{h\mathsf{suf}_{i(a)}(y)w,a'}) < \mathsf{dcl}(e_{h\mathsf{suf}_{i(a)}(y),a}) = \alpha_2(e)$.

In order to compute $\alpha_2(e')$, we first note that Lemma 3.4 also shows that $\mathsf{last}(y', a') = f$ and so $\alpha_2(e') = \mathsf{idcl}_f(e^f_{\mathsf{suf}_{i(a)}(y'),a'})$. However, by definition, $\mathsf{idcl}_f(e^f_{\mathsf{suf}_{i(a)}(y'),a'}) \leq \mathsf{dcl}(e^f_{\mathsf{suf}_{i(a)}(y'),a'}) < \mathsf{dcl}(e_{h\mathsf{suf}_{i(a)}(y),a}) = \alpha_2(e)$, as desired. Hence, $\alpha(e') < \alpha(e)$ in this last case as well.

Autostackability:

Next we show that the graph of the stacking function ϕ associated to the flow function Φ is a regular subset of $(A^*)^3$ in the case that the flow function Φ_0 and each of the flow functions Φ_f associated to the directed edges $f \in \vec{E}_\Lambda$ gives an autostackable structure; that is, the sets

$$\begin{aligned}
\mathsf{graph}\,(\Phi_0) &= \{(y, a, \phi_0(y,a)) \mid y \in \mathcal{N}_0, a \in A_0\} \text{ and} \\
\mathsf{graph}\,(\Phi_f) &= \{(y, a, \phi_f(y,a)) \mid y \in \mathcal{N}_f, a \in A_{t(f)}\}
\end{aligned}$$

are regular.

We begin by noting that Lemma 3.2 and Proposition 3.3 together show that the set $\mathcal{N}$ of normal forms associated to the spanning tree in Γ for Φ is a regular language, and moreover the set $\widetilde{\mathcal{N}}$ is also a regular language of normal forms for $\pi_1(\mathcal{G})$, with $\mathcal{N} = \mathsf{defl}(\widetilde{\mathcal{N}})$.

We proceed by breaking down the graph of Φ using the three cases in the piecewise definition of its stacking function ϕ:

$$\begin{aligned}
\mathsf{graph}\,(\Phi) = & \left(\cup_{a \in \vec{E}_\Lambda \setminus \vec{E}_{\Lambda \setminus T}} \mathcal{N} \times \{a\} \times \{a\}\right) \\
& \bigcup \left(\cup_{f \in \vec{E}_\Lambda} \cup_{a \in B_f} \; L_{f,a} \times \{a\} \times \{\mathsf{defl}(f^{-1}\widehat{a}_f f)\}\right) \\
& \bigcup \left(\cup_{f \in \vec{E}_\Lambda} \cup_{a \in A_{t(f)}} \cup_{w \in \mathsf{im}(\phi_f)} L'_{f,a,w} \times \{a\} \times \{w\}\right) \\
& \quad \bigcup \left(\cup_{a \in A_{v_0}} \cup_{w \in \mathsf{im}(\phi_0)} L'_{0,a,w} \times \{a\} \times \{w\}\right)
\end{aligned}$$

where

$$\begin{aligned}
L_{f,a} &:= \{y \in \mathcal{N} \mid \mathsf{last}(y,a) = f \text{ and } \mathsf{suf}_{i(a)}(y) = \varepsilon\}, \\
L'_{f,a,w} &:= \{y \in \mathcal{N} \mid \mathsf{last}(y,a) = f,\ \mathsf{suf}_{i(a)}(y) \neq \varepsilon \text{ and } \phi_f(\mathsf{suf}_{i(a)}(y),a) = w\} \text{ if } a \in B_f, \\
L'_{f,a,w} &:= \{y \in \mathcal{N} \mid \mathsf{last}(y,a) = f \text{ and } \phi_f(\mathsf{suf}_{i(a)}(y),a) = w\} \text{ if } a \in A_{t(f)} \setminus B_f, \text{ and} \\
L'_{0,a,w} &:= \{y \in \mathcal{N} \mid \mathsf{last}(y,a) = 0 \text{ and } \phi_0(\mathsf{suf}_{i(a)}(y),a) = w\}.
\end{aligned}$$

Since the graph Λ is finite and each of the flow functions Φ_0 and Φ_f are bounded, this decomposition of $\mathsf{graph}\,(\Phi)$ is a finite union of subsets. The closure properties of regular languages in Theorem 2.1 imply that in order to show that $\mathsf{graph}\,(\Phi)$ is regular, it suffices to show that the languages $L_{f,a}$ and $L'_{f,a,w}$ are regular.

Let $f \in \vec{E}_\Lambda$ and $a \in A_{t(f)}$. Since $\mathsf{last}(y,a)$ is the last letter of the image of $\mathsf{infl}(ya)$ under the monoid homomorphism ρ, any word y in the normal form set $\mathcal{N}$ satisfying $\mathsf{last}(y,a) = f$ either satisfies $t(y) = u$ for some vertex u such that $\mathsf{path}_{T_\Lambda}(u, i(a))$ ends with f, or else has inflation with a suffix in $fA^*_{t(f)}$. Let V_f be the set of vertices u of Λ such that the last edge of the path $\mathsf{path}_{T_\Lambda}(u, i(a))$ is f. For each vertex v of Λ, let C_v be the (finite) set of all letters c in A with $t(c) = v$; that is, C_v is the union of A_v with all of the edges in $\vec{E}_\Lambda \setminus \vec{E}_{\Lambda\setminus T}$ whose terminal vertex is v. Then the set of normal form words y with $\mathsf{last}(y,a) = f$ is

$$R_{f,a} := \{y \in \mathcal{N} \mid \mathsf{last}(y,a) = f\} = \left(\cup_{u\in V_f} \mathcal{N} \cap A^*C_u\right) \bigcup R'_{f,a} \bigcup \mathsf{defl}(\widetilde{\mathcal{N}} \cap A^*fA^*_{t(f)})$$

where $R'_{f,a} = \{\varepsilon\}$ if $\mathsf{path}_{T_\Lambda}(v_0, i(a))$ ends with the letter f, and $R'_{f,a} = \emptyset$ otherwise. Since $\mathcal{N}$, $\widetilde{\mathcal{N}}$, and $R'_{f,a}$ are regular, as are the concatenations A^*C_u and $A^*fA^*_{t(f)}$, Theorem 2.1 implies that $R_{f,a}$ is also a regular language.

Next suppose that $a \in A_{i(a)}$. The set of normal form words y whose maximal suffix in $A^*_{i(a)}$ is empty is

$$S_a := \{y \in \mathcal{N} \mid \mathsf{suf}_{i(a)}(y) = \varepsilon\} = \{\varepsilon\} \cup (\mathcal{N} \cap A^*(A \setminus A_{i(a)})).$$

This union of a finite set with an intersection of regular languages is regular. Now for all $f \in \vec{E}_\Lambda$ and $a \in B_f$, the language $L_{f,a}$ is the intersection $L_{f,a} = R_{f,a} \cap S_a$, and hence $L_{f,a}$ is regular.

Finally, suppose that either $f \in \vec{E}_\Lambda$ and $a \in A_{t(f)}$ or else $f = 0$ and $a \in A_{v_0}$, and let w be a word in the image of ϕ_f. The set of normal forms y for which the image of the edge $e_{\mathsf{suf}_{i(a)}(y),a}$ under Φ_f has label w is

$$Q_{f,a,w} := \{y \in \mathcal{N} \mid \phi_f(\mathsf{suf}_{i(a)}(y), a) = w\} = \mathcal{N} \cap \Big(S_a \cdot p_1(\mathsf{graph}\,(\Phi_f) \cap (A^*_{i(a)} \times \{a\} \times \{w\}))\Big)$$

where p_1 denotes projection on the first coordinate. Again applying Theorem 2.1 and closure properties of regular languages, the set $Q_{f,a,w}$ is regular.

Now if $a \in B_f$ then $L'_{f,a,w} = R_{f,a} \cap (\mathcal{N} \setminus S_a) \cap Q_{f,a,w}$, and if $a \in A_{t(f)} \setminus B_f$ then $L'_{f,a,w} = R_{f,a} \cap Q_{f,a,w}$. Finally if $f = 0$ and $a \in A_{v_0}$, then $\mathsf{last}(y,a) = 0$ for a normal form y if and only if y lies in the regular language $\mathcal{N}_0$, and so $L'_{0,a,w} = \mathcal{N}_0 \cap Q_{0,a,w}$. Thus in all three cases $L'_{f,a,w}$ is regular.

Therefore $\mathsf{graph}\,(\Phi)$ is regular and $\pi_1(\mathcal{G})$ is autostackable. □

4. Extensions and autostackability respecting subgroups

In this section we record two results on autostackability respecting subgroups, for finite extensions and finite index supergroups, which will be used in Section 6 in our analysis of Seifert-fibered pieces of 3-manifolds. The first uses the proof of the closure of autostackability under group extensions in [9, Theorem 3.3].

Theorem 4.1. *Let $1 \to K \xrightarrow{i} G \xrightarrow{q} Q \to 1$ be a short exact sequence of groups and group homomorphisms, and let H be a subgroup of G containing K. If Q is autostackable [respectively, stackable] respecting $q(H)$ and K is autostackable, then G is also autostackable [respectively, stackable] respecting H.*

Proof. Let $\mathcal{N}_K$, Φ_K, and ϕ_K be the regular prefix-closed normal form set, bounded flow function, and associated stacking map for K over a (finite inverse-closed) generating set A_K, and similarly let $\mathcal{N}_Q = \mathcal{N}_{q(H)}\mathcal{N}_{\mathcal{T}}$, Φ_Q, and ϕ_Q be the regular prefix-closed normal form set, bounded flow function, and associated stacking map for Q over a generating set C of Q, respecting the subgroup $q(H)$ with generating set $D \subset C$ and regular normal forms $\mathcal{N}_{q(H)}$. By slight abuse of notation, we will consider the homomorphism i to be an inclusion map, and $A, K, H \subseteq G$, so that we may omit writing $i(\cdot)$.

For each $c \in C$, let $\hat{c}$ be an element of G satisfying $q(\hat{c}) = c$ (and choose these elements such that $\widehat{c^{-1}} = \hat{c}^{-1}$), and let $\hat{C} := \{\hat{c} \mid c \in C\}$. We note that $A_K \cap \hat{C} = \emptyset$, since an element of the generating set C for the autostackable structure on Q does not represent the identity in Q. Define the monoid homomorphism $\mathsf{hat} : C^* \to \hat{C}^*$ by $\mathsf{hat}(c) := \hat{c}$ for all $c \in C$. Let $A := A_K \cup \hat{C}$ and $B := A_K \cup \hat{D}$, and let

$$\mathcal{N}_G := \mathcal{N}_K \mathsf{hat}(\mathcal{N}_Q) = \mathcal{N}_K \mathsf{hat}(\mathcal{N}_{q(H)}) \mathsf{hat}(\mathcal{N}_{\mathcal{T}}).$$

Then $\mathcal{N}_G$ is a prefix-closed regular language of normal forms for the group G.

For any $g \in G$, write $\mathsf{nf}(g) = r_g s_g t_g$ with $r_g \in \mathcal{N}_K$, $s_g \in \mathsf{hat}(\mathcal{N}_{q(H)})$, and $t_g \in \mathsf{hat}(\mathcal{N}_{\mathcal{T}})$; similarly, for $y \in \mathcal{N}_G$, write $y = r_y s_y t_y$ with $r_y \in \mathcal{N}_K$, $s_y \in \mathsf{hat}(\mathcal{N}_{q(H)})$, and $t_y \in \mathsf{hat}(\mathcal{N}_{\mathcal{T}})$ For any nonempty word w, let $\mathsf{last}(w)$ denote the last letter of w.

Define a function $\phi : \mathcal{N}_G \times A \to A^*$ by

$$\phi(y, a) = \begin{cases} \phi_K(y, a) & \text{if } a \in A_K \text{ and } s_y t_y = \varepsilon \\ \mathsf{last}(y)^{-1} r_{\mathsf{last}(y) a \mathsf{last}(y)^{-1}} \mathsf{last}(y) & \text{if } a \in A_K \text{ and } s_y t_y \neq \varepsilon \\ r_{a(\mathsf{hat}(\phi_Q(q(s_y t_y), q(a))))^{-1}} \mathsf{hat}(\phi_Q(q(s_y t_y), q(a))) & \text{if } a \in \hat{C}. \end{cases}$$

Let $\Gamma := \Gamma_A(G)$ be the Cayley graph of G with respect to A, and let $\vec{E}$ and $\vec{P}$ be the sets of directed edges and directed paths in Γ. Also define $\Phi : \vec{E} \to \vec{P}$ by $\Phi(e_{g,a}) :=$ the path in Γ starting at g and labeled by $\phi(\mathsf{nf}(g), a)$.

Now the proof of [9, Theorem 3.3] shows that Φ is a bounded flow function for G over A, and that the graph of Φ is regular; that is, Φ gives an autostackable structure for G over A.

Let T be the spanning tree of Γ determined by the normal form set $\mathcal{N}_G$. Since the normal form set $\mathcal{N}_G$ of this structure is the concatenation of $\mathcal{N}_K \mathsf{hat}(\mathcal{N}_{q(H)})$, which is a set of normal forms for the subgroup H over B, with the set $\mathsf{hat}(\mathcal{N}_{\mathcal{T}})$, which is a set of normal forms over A for a right transversal of H in G, the tree T of Γ associated to the set $\mathcal{N}_G$ is the union of a spanning tree in the Cayley subgraph $\Gamma_B(H)$ with an H–orbit of a transversal tree for H in G, as required.

Suppose that $h \in H$ and $b \in B$, and consider the edge $e := e_{h,b}$ of $\Gamma_B(H)$ contained in Γ. If $h \in K$ and $b \in A_K$, then the first case of the definition of ϕ shows that $\mathsf{label}(\Phi(e)) \in A_K^* \subseteq B^*$. If $h \in H \setminus K$ and $b \in A_K$, then the second case of the definition of ϕ applies, and since $\mathsf{last}(\mathsf{nf}(h)) \in \hat{D} \subseteq B$ and $r_{\mathsf{last}(y) a \mathsf{last}(y)^{-1}} \in B^*$, again the label of $\Phi(e)$ is in B^*. Finally, if $b \in B \setminus A_K$, then the third case applies. In this case, since Φ_Q arises from an autostackable structure respecting $q(H)$, and since $q(b) \in D$, the word $\phi_Q(q(h), q(b))$ labeling $\Phi_Q(e_{q(h),q(b)})$ in $\Gamma_C(Q)$ must be in D^*. Then $\mathsf{hat}(\phi_Q(q(h), q(a))) \in \hat{D}^*$, and since $r_{a(\mathsf{hat}(\phi_Q(q(y),q(a))))^{-1}} \in A_K^*$, we have $\mathsf{label}(\Phi(e)) \in B^*$. Thus Φ satisfies the subgroup closure property for H.

On the other hand, suppose that $e = e_{g,a}$ is an edge of Γ that does not lie in $\Gamma_B(H)$, and that $h \in H$. If $a \in A_K$, then $g \notin H$, and so $\mathsf{nf}(g) \notin \mathcal{N}_K \mathsf{hat}(\mathcal{N}_{q(H)})$. In this case, then $t_g \neq \varepsilon$ and so the letter $\ell := \mathsf{last}(\mathsf{nf}(g))$ lies in $\hat{C} \setminus \hat{D}$. Since $\mathsf{nf}(hg) = \mathsf{nf}(h) t_g$, then $\mathsf{last}(\mathsf{nf}(hg)) = \ell$ as well, and case 2 of ϕ shows that $\mathsf{label}(\Phi(e_{hg,a})) = \ell^{-1} r_{\ell a \ell^{-1}} \ell = \mathsf{label}(\Phi(e))$. Suppose instead that $a \notin A_K$. Then either $a \in \hat{D}$ and $g \notin H$, or else $a \in \hat{C} \setminus \hat{D}$. Hence either $q(g) \notin q(H)$ or else $q(a) \notin D$, and so the edge $e_{q(g),q(a)}$ is not in the Cayley subgraph $\Gamma_D(q(H))$ of $\Gamma_C(Q)$.

For this case $\mathsf{label}(\Phi(e)) = r_{a(\mathsf{hat}(\phi_Q(q(s_y t_y),q(a))))^{-1}} \mathsf{hat}(\phi_Q(q(s_y t_y), q(a)))$. Note that $q(H)$–translation invariance of the autostackable structure Φ_Q for Q respecting $q(H)$, together with the fact that the H–coset representatives for y and hy satisfy $t_y = t_{hy}$, implies that $\phi_Q(q(s_y t_y), q(a)) = \phi_Q(q(s_{hy} t_{hy}), q(a))$, and so $\mathsf{label}(\Phi(e)) = \mathsf{label}(\Phi(e_{hg,a}))$ in this case as well. Hence Φ is also H–translation invariant. □

The next result is shown by the first two authors and Johnson in [9]. Although the theorem as stated in that paper did not use the phrase "respecting H", the proof does imply this extra property.

Proposition 4.2. [9, Theorem 3.4] *Let H be an autostackable [respectively, stackable] group, and let G be a group containing H as a subgroup of finite index. Then G is autostackable [respectively, stackable] respecting H.*

5. Strongly coset automatic groups and relative hyperbolicity

In this section we show, in Theorem 5.1, an extension to coset automaticity and autostackability respecting autostackable subgroups, of the result by the first two authors and Holt [8, Theorem 4.1] that every automatic group with respect to prefix-closed normal forms is autostackable. We then apply this result to relatively hyperbolic groups. See Subsections 2.3 and 2.5 for notation and definitions.

Theorem 5.1. *Let G be a finitely generated group and H a finitely generated autostackable subgroup of G. If the pair (G, H) is strongly prefix-closed coset automatic, then G is autostackable respecting H.*

Proof. Since (G, H) is strongly prefix-closed automatic, Definition 2.3 says that there is an inverse-closed generating set C for G, a prefix-closed regular set $\mathcal{N}_\mathcal{T} \subseteq C^*$ of unique representatives of the right cosets Hg of H in G, and a constant $K \geq 0$ such that the language $\mathcal{N}_\mathcal{T}$ satisfies the H–coset K–fellow traveler property. Autostackability of H gives a bounded flow function Φ_H for H over a finite inverse-closed generating set B, with a bound K_H, such that $\mathsf{graph}\,(\Phi_H)$ is regular. Let $\mathcal{N}_H$ be the set of normal forms for H over B that are the labels of the non-backtracking paths in the spanning tree of the Cayley graph $\Gamma_B(H)$ associated to Φ_H, and let $\phi_H : \mathcal{N}_H \times B \to B^*$ be the stacking function obtained from Φ_H. Then $\mathcal{N}_H$ is prefix-closed, and since $\mathcal{N}_H$ is the projection on the first coordinate of the regular language $\mathsf{graph}\,(\Phi)$, Theorem 2.1 shows that $\mathcal{N}_H$ is also regular.

By taking separate copies of letters representing the same group element, if necessary, we may assume that $B \cap C = \emptyset$. Let $A := B \sqcup C$ and let $\Gamma := \Gamma_A(G)$ be the Cayley graph of G with respect to the generating set A. Then the set

$$\mathcal{N}_G := \mathcal{N}_H \mathcal{N}_\mathcal{T}$$

is a prefix-closed regular language of normal forms for G over A. Let T be the spanning tree in Γ consisting of the edges that lie on paths starting at 1 and labeled by words in $\mathcal{N}_G$.

For any element $g \in G$, we can write its normal form in $\mathcal{N}_G$ uniquely as $\mathsf{nf}(g) = x_g z_g$ with $x_g \in \mathcal{N}_H$ and $z_g \in \mathcal{N}_\mathcal{T}$. Similarly, for each $y \in \mathcal{N}_G$, write $y = x_y z_y$ with $x_y \in \mathcal{N}_H$ and $z_y \in \mathcal{N}_\mathcal{T}$.

The flow function: Next we construct a function $\phi : \mathcal{N}_G \times A \to A^*$. We begin with some more notation.

Fix a total ordering on C, and for any letter $b \in B$, denote the shortlex least word over C representing the same element of G as b by $\mathsf{sl}_C(b)$.

Recall that for any word w in A^* and integer $i \geq 0$, the symbol $w(i)$ denotes the prefix of w of length i if $\ell(w) \geq i$ and $w(i) = w$ if $\ell(w) < i$. Let $w(i)'$ denote the suffix of w with the first i letters removed; that is, if $w = a_1 \cdots a_k$ with each $a_j \in A$, then $w(i) = a_1 \cdots a_i$

and $w(i)' := a_{i+1} \cdots a_k$ if $\ell(w) > i$, and $w(i)' := \varepsilon$ if $\ell(w) \le i$. Note that $w = w(i)w(i)'$. The symbol $\mathsf{red}(w)$ represents the resulting freely reduced word obtained from w after all subwords of the form aa^{-1} are (iteratively) removed.

Also recall from Proposition 2.5 that for each element $h \in H \cap B_{\Gamma_C(G)}(K)$ and $c \in C$, there is a finite state automaton $M_{h,c}$ accepting the set of all pairs (z, z') with $z, z' \in \mathcal{N}_\mathcal{T}$ and $zc =_G hz'$, with state set $\widetilde{Q}$, initial state (q_0, h), accept states $P \times \{c\}$, and transition function $\widetilde{\delta}$. Note that the set of states $\widetilde{Q}$ and the transition function $\widetilde{\delta}$ of $M_{h,c}$ in the proof of Proposition 2.5 do not depend upon h or c. Hence all of these automata have the same number of states; we denote this number by μ. On the other hand, the set $P \times \{c\}$ of accept states does depend on c, although it is independent of h. For each state $\widetilde{q}$ of $M_{h,c}$ for which there is a path from $\widetilde{q}$ to an accept state (p, c) of $M_{h,c}$ (viewing the finite state automaton as a graph with labeled directed edges), there must also be a simple path of length at most μ from $\widetilde{q}$ to an accept state. Such a state is called a *live* state of $M_{h,a}$. Fix a choice of a pair of words $(v_{c,\widetilde{q}}, w_{c,\widetilde{q}}) \in C^* \times C^*$ such that $\ell(v_{c,\widetilde{q}}), \ell(w_{c,\widetilde{q}}) \le \mu$ and $\widetilde{\delta}(\widetilde{q}, (v_{c,\widetilde{q}}, w_{c,\widetilde{q}}))$ is an accept state of $M_{h,c}$.

Let $y \in \mathcal{N}_G$ and $a \in A$. The stacking function ϕ is given by

$$\phi(y,a) := \begin{cases} \phi_H(y,a) & \text{if } a \in B \text{ and } z_y = \varepsilon \\ \mathsf{sl}_C(a) & \text{if } a \in B \text{ and } z_y \neq \varepsilon \\ a & \text{if } a \in C \text{ and either } \mathsf{nf}(ya) = ya \text{ or } y \in A^* a^{-1} \\ \mathsf{red}(z_y^{-1} x_{z_y a} z_{z_y a}) & \text{if } a \in C,\ \mathsf{nf}(ya) \neq ya,\ y \notin A^* a^{-1} \text{ and } \ell(z_y) \le \mu \\ (z_y(j)')^{-1} v_{a,\widetilde{q}}\, a (w_{a,\widetilde{q}})^{-1} z_{z_y a}(j)' & \text{if } a \in C,\ \mathsf{nf}(ya) \neq ya,\ y \notin A^* a^{-1}, \ell(z_y) > \mu, \\ & \quad j := \ell(z_y) - \mu - 1, \text{ and } \widetilde{q} := \widetilde{\delta}((q_0, x_{z_y a}), (z_y(j), z_{z_y a}(j))). \end{cases}$$

Let $\Phi : \vec{E} \to \vec{P}$ be defined by $\Phi(e_{g,a}) :=$ the path in $\Gamma_A(G)$ starting at g labeled by $\phi(\mathsf{nf}(g), a)$, for all $g \in G$ and $a \in A$.

Property (F1): It follows immediately from this definition that $\Phi(e_{g,a})$ has the same initial and terminal vertices as $e_{g,a}$ for all $g \in G$ and $a \in B$. Suppose instead that $a \in C$. If $\ell(z_g) \le \mu$, then since $z_g a =_G x_{z_g a} z_{z_g a}$, again Φ fixes the endpoints of $e_{g,a}$.

On the other hand, suppose that $\ell(z_g) \ge \mu + 1$ and let $y := \mathsf{nf}(g)$ and $j := \ell(z_y) - \mu - 1$. In this case since a is a single letter in C and $z_y, z_{z_y a} \in \mathcal{N}_\mathcal{T}$ satisfy $z_y a =_G x_{z_y a} z_{z_y a}$, the H–coset K–fellow traveler property implies that the element h of H represented by $x_{z_y a} = x_h$ lies in $B_{\Gamma_C(G)}(K)$. Hence the (padded word corresponding to the) pair $(z_y, z_{z_y a})$ is accepted by the automaton $M_{h,a}$, and the state $q_{y,a} := \widetilde{\delta}((q_0, h), (z_y, z_{z_y a}))$ is an accept state of $M_{h,a}$. Factor the word z_y as $z_y = z_y(j) z_y(j)'$, and note that $z_y(j)'$ is the suffix of z_y of length $\mu+1$. Similarly factor $z_{z_y a} = z_{z_y a}(j) z_{z_y a}(j)'$. Now the state $\widetilde{q} := \widetilde{\delta}((q_0, h), (z_y(j), z_{z_y a}(j)))$ of the automaton $M_{h,a}$ satisfies $\widetilde{\delta}(\widetilde{q}, (z_y(j)', z_{z_y a}(j)')) = q_{y,a}$, and so $\widetilde{q}$ is live in $M_{h,a}$. Now the pair $(z_y(j) v_{a,\widetilde{q}}, z_{z_y a}(j) w_{a,\widetilde{q}})$ is accepted by $M_{h,a}$, and so we have $z_y(j) v_{a,\widetilde{q}} a =_G h z_{z_y a}(j) w_{a,\widetilde{q}} =_G x_{z_y a} z_{z_y a}(j) w_{a,\widetilde{q}}$. Hence $(z_y(j)')^{-1} v_{a,\widetilde{q}} a (w_{a,\widetilde{q}})^{-1} z_{z_y a}(j)' =_G a$, and Φ fixes the endpoints of $e_{g,a}$ in this last case as well.

To see that the function Φ is bounded, we inspect each of the cases. In the first case of the piecewise definition of ϕ above, the length of the path $\Phi(e_{g,a})$ is at most the bound K_H of the flow function Φ_H, is it at most $\max\{\ell(\mathsf{sl}_A(b)) \mid b \in B\}$ in the second, 1 in the third, and $\max\{\ell(z^{-1} x_{za} z_{za}) \mid z \in \mathcal{N}_\mathcal{T}, \ell(z) \le \mu, \text{ and } a \in C\}$ in the fourth case. Since the two maxima are over finite sets, these are finite numbers. Now suppose that the fifth case holds. Since the set $\mathcal{N}_\mathcal{T}$ contains only one representative of each coset, there is a unique word z' such that (z_y, z') is accepted by $M_{h,a}$, namely $z' = z_{z_y a}$, and so the path in $M_{h,a}$ from $\widetilde{q}$ to $q_{y,a}$ labeled by $(z_y(j)', z_{z_y a}(j)')$ cannot have length greater than $\ell(z_y(j)') + \mu$, since it cannot repeat a state after the word $z_y(j)'$ is completed. To see this, if a state is repeated, then the definition of the transition function implies that $z_{ya}(i)^{-1}x_{z_ya}^{-1}z_y = z_{ya}(k)^{-1}x_{z_ya}^{-1}z_y$ for $j \le i \le k$. But then $z_{ya}(k)$ and $z_{ya}(i)$ represent the same H–coset; however, $\mathcal{N}_{\mathcal{T}}$ is prefix-closed and has unique coset representatives. Thus $i = k$. Therefore $\ell(z_{z_ya}(j)') \le 2\mu + 1$. Hence in this fifth case the length of $\Phi(e_{g,a})$ is at most $(\mu+1)+\mu+1+\mu+(2\mu+1) = 5\mu+3$. Therefore (F1) holds for Φ.

Property (F2): Since a directed edge $e_{g,a}$ from $g \in G$ labeled by $a \in A$ in Γ lies in the spanning tree T obtained from $\mathcal{N}_G$ if and only if either $\mathsf{nf}(ga) = \mathsf{nf}(g)a$ or $\mathsf{nf}(g) \in A^*a^{-1}$, it is immediate from the definition of Φ that any edge $e_{g,a}$ in T with $a \in C$ satisfies $\Phi(e_{g,a}) = e_{g,a}$. Suppose instead that $e_{g,a}$ is in T and $a \in B$. Since $\mathcal{N}_G = \mathcal{N}_H\mathcal{N}_{\mathcal{T}}$ with $\mathcal{N}_H \subset B^*$ and $\mathcal{N}_{\mathcal{T}} \subset C^*$, we must have $z_g = \varepsilon$, and so the fact that the flow function Φ_H satisfies property (F2) implies that $\Phi(e_{g,a}) = e_{g,a}$ in this case as well.

Property (F3): In order to show that there is no infinite sequence $e_1, e_2, \ldots \in \vec{E}$ of directed edges of $\Gamma_A(G)$ lying outside of the spanning tree T such that e_{i+1} is in the path $\Phi(e_i)$ for each i, we use the same technique as in the proof of Theorem 3.5. That is, we define a function $\alpha : \vec{E} \to \mathbb{N}^3$, and show that whenever $e, e' \in \vec{E}$ are not in T and e' is on the path $\Phi(e)$, then $\alpha(e) > \alpha(e')$ (using the lexicographic order on $\mathbb{N}^3$).

Since B is a subset of the generating set A of G, we can consider the Cayley graph $\Gamma_B(H)$ to be a subgraph of the graph $\Gamma_A(G)$; for each $h \in H$ and $b \in B$ we consider the edge $e_{h,b}$ to be an edge of both of these graphs. Let $\mathsf{dcl}_H(e_{h,b})$ be the descending chain length for that edge from the autostackable structure on H (that is, the maximum possible number of edges of $\Gamma_B(H)$ in a sequence $e_{h,b} = e_1, e_2, \ldots$ such that e_i is not in T and e_{i+1} is on $\Phi_H(e_i)$ for all i).

For $1 \le i \le 3$ we define functions $\alpha_i \colon \vec{E} \to \mathbb{N}$ by

$$\begin{aligned}
\alpha_1(e_{g,a}) &:= \begin{cases} \ell(z_g), & \text{if } a \in C \\ \max\{\ell(z_{g\mathsf{sl}_C(a)(i)}) \mid i < \ell(\mathsf{sl}_C(a))\} + 1 & \text{if } g \notin H \text{ and } a \in B \\ 0, & \text{if } g \in H \ , \end{cases} \\
\alpha_2(e_{g,a}) &:= \begin{cases} 1, & \text{if } g \in H \text{ and } a \in C \\ 0, & \text{otherwise,} \end{cases} \\
\alpha_3(e_{g,a}) &:= \begin{cases} \mathsf{dcl}_H(e_{g,a}), & \text{if } g \in H \text{ and } a \in B \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Now, let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

Let $e = e_{g,a} \in \vec{E}$ be an edge outside of the tree T, and let $e' = e_{g',a'}$ be an edge on $\Phi(e')$ that also is not in T. Let $y := \mathsf{nf}(g)$ and $y' := \mathsf{nf}(g')$. Consider the five cases of the definition of $\phi(y, a)$ in turn.

Case 1. Suppose that $a \in B$ and $z_y = \varepsilon$. In this case $g \in H$, $\alpha(e) = (0, 0, \mathsf{dcl}_H(e))$, and $\mathsf{label}(\Phi(e)) = \phi_H(y, a) \in B^*$. Then $a' \in B$ and $y' =_G yw$ for a prefix w of $\phi_H(y, a)$, so $g' \in H$ and $\alpha(e') = (0, 0, \mathsf{dcl}_H(e'))$. Since e' lies on $\Phi_H(e)$, the descending chain lengths satisfy $\mathsf{dcl}_H(e) > \mathsf{dcl}_H(e')$, and so $\alpha(e) > \alpha(e')$.

Case 2. Suppose that $a \in B$ and $z_y \neq \varepsilon$. In this case $g \notin H$ and $\alpha(e) = (\max\{\ell(z_{g\mathsf{sl}_C(a)(i)}) \mid i < \ell(\mathsf{sl}_C(a))\} + 1, 0, 0)$. We can factor $\mathsf{label}(\Phi(e)) = \mathsf{sl}_C(a) = wa'w'$ for some $w, w' \in C^*$ such that $g' =_G gw$ and $w = \mathsf{sl}_C(a)(i)$ for some $i < \ell(\mathsf{sl}_C(a))$. Note that $a' \in C$. Now either $g' \in H$, in which case $\alpha_1(e') = 0 < \alpha_1(e)$, or else $g' \notin H$, in which case $\alpha_1(e') = \ell(z_{g'}) = \ell(z_{g\mathsf{sl}_C(a)(i)}) < \alpha_1(e)$. Hence in both options $\alpha(e) > \alpha(e')$.

Case 3. Suppose that $a \in C$ and either $\mathsf{nf}(ya) = ya$ or $y \in A^*a^{-1}$. In this case e lies in the tree T, so this case can't occur.

Case 4. Suppose that $a \in C$, $\mathsf{nf}(ya) \neq ya$, $y \notin A^*a^{-1}$, and $\ell(z_y) \leq \mu$. In this case $\alpha(e) = (\ell(z_g), \alpha_2(e), 0)$; if $g \in H$, then $\alpha(e) = (0,1,0)$, and if $g \notin H$ then $\alpha_1(e) > 0$. We also have $\mathsf{label}(\Phi(e)) = \mathsf{red}(z_y^{-1}x_{z_ya}z_{z_ya})$. We can again factor the word $z_y^{-1}x_{z_ya}z_{z_ya} = wa'w'$ with $w, w' \in A^*$ and $g' =_G gw$. Note that since $x_{ya}z_{ya} =_G ya =_G x_yz_ya =_G x_yx_{z_ya}z_{z_ya}$, we have $x_{ya} =_H x_yx_{z_ya}$ and $z_{ya} = z_{z_ya}$ since each right coset of H in G has only one representative in $\mathcal{N}_\mathcal{T}$. If $a' \in C$, then the edge e' is either on the subpath starting at g labeled by z_y^{-1}, or the subpath ending at ga labeled by $z_{z_ya} = z_{ya}$; but these two paths are in the normal form tree T, giving a contradiction. So, we must have $a' \in B$ and e' is on the subpath of $\Phi(e)$ starting at $gz_y^{-1} =_G x_y \in H$ labeled by $x_{z_ya} \in B^*$. Hence $g' \in H$, and so $\alpha(e') = (0, 0, \mathsf{dcl}(e'))$. Therefore $\alpha(e) > \alpha(e')$.

Case 5. Suppose that $a \in C$, $\mathsf{nf}(ya) \neq ya$, $y \notin A^*a^{-1}$, and $\ell(z_y) > \mu$. In this case $\alpha(e) = (\ell(z_g), 0, 0)$. Let $j := \ell(z_y) - \mu - 1$ and $\widetilde{q} := \widetilde{\delta}((q_0, x_{z_ya}), (z_y(j), z_{z_ya}(j)))$; then $\mathsf{label}(\Phi(e) = (z_y(j)')^{-1}v_{a,\widetilde{q}}a(w_{a,\widetilde{q}})^{-1}z_{z_ya}(j)'$. As in Case 4, the subpath of $\Phi(e)$ starting at g labeled $(z_y(j)')^{-1}$, and the subpath ending at ga and labeled $z_{z_ya}(j)' = z_{ya}(j)'$, both lie in the spanning tree T. Moreover, since the pair $(z_y(j)v_{a,\widetilde{q}}, z_{ya}(j)w_{a,\widetilde{q}})$ is accepted by the automaton $M_{h,a}$ for $h =_G x_{z_ya}$, the words $x_yz_y(j)v_{a,\widetilde{q}}$ and $x_{ya}z_{ya}(j)w_{a,\widetilde{q}}$ are also normal forms in $\mathcal{N}_G$, and so the edges in $\Phi(e)$ in the subpaths labeled $v_{a,\widetilde{q}}$ and $(w_{a,\widetilde{q}})^{-1}$ also lie in the tree. So we must have $a' = a$ and $g' =_G g(z_y(j)')^{-1}v_{a,\widetilde{q}}$. Then $a' \in C$, and $z_{g'} = z_y(j)v_{a,\widetilde{q}}$. Now $\alpha_1(e') = \ell(z_{g'}) = \ell(z_y(j)v_{a,\widetilde{q}}) = \ell(z_y) - \mu - 1 + \ell(v_{a,\widetilde{q}}) < \ell(z_y)$ since $\ell(v_{a,\widetilde{q}}) \leq \mu$. Hence $\alpha(e) > \alpha(e')$.

Then all of the properties (F1)–(F3) hold, and Φ is a flow function for G over A.

Respecting the subgroup H: The definition of the normal form set $\mathcal{N}_G$ as the concatenation $\mathcal{N}_H\mathcal{N}_\mathcal{T}$ of normal forms for H over B and normal forms for $H\backslash G$ over C implies the required structure for the spanning tree built from these normal forms. Further, subgroup closure of the flow function Φ is immediate from the first case of the definition of ϕ. The H–translation invariance of Φ follows from the fact that the word $\mathsf{label}(\Phi(e_{g,a})) = \phi(\mathsf{nf}(g), a)$ depends only upon the coset representative z_g from the transversal, and is independent of x_g, in the other four cases of the definition of ϕ.

Autostackability: Recall from the beginning of this proof that the sets $\mathcal{N}_G$, $\mathcal{N}_H$, $\mathcal{N}_\mathcal{T}$, and $\mathsf{graph}\,(\Phi_H)$ are all regular. Also recall that for any natural number j, the symbol $C^{\leq j}$ denotes the set of all words over C of length at most j.

Using the state set and transition function from the automata $M_{h,a}$ constructed in the proof of Proposition 2.5, we build more finite state automata as follows. For every $\widetilde{q} \in \widetilde{Q}$ and $\widetilde{P} \subset \widetilde{Q}$, let $M_{\widetilde{q},\widetilde{P}}$ be the automaton with state set $\widetilde{Q}$, start state $\widetilde{q}$, accept state set $\widetilde{P}$, and transition function $\widetilde{\delta}$. Let $L(M_{\widetilde{q},\widetilde{P}})$ be the set of all words accepted by this finite state automaton.

In order to show that $\mathsf{graph}\,(\Phi)$ is also regular, we separate the graph of Φ into five pieces using the five cases in the definition of its stacking function ϕ:

$$\begin{aligned}
\mathsf{graph}\,(\Phi) \quad = \quad & \mathsf{graph}\,(\Phi_H) \\
& \bigcup (\cup_{a\in B}(\mathcal{N}_G \setminus \mathcal{N}_H) \times \{a\} \times \{\mathsf{sl}_C(a)\}) \\
& \bigcup (\cup_{a\in C} L_a \times \{a\} \times \{a\}) \\
& \bigcup \left(\cup_{a\in C} \cup_{z\in \mathcal{N}_\mathcal{T}\cap B^{\leq\mu}} L'_{a,z} \times \{a\} \times \{\mathsf{red}(z^{-1}x_{za}z_{za})\}\right) \\
& \bigcup \left(\cup_{a\in C} \cup_{\widetilde{q}\in\widetilde{Q}} \cup_{(u,u')\in(C^{\mu+1}\times C^{\leq 2\mu+1})\cap L(M_{\widetilde{q},P\times\{a\}})} L''_{a,\widetilde{q},u,u'} \times \{a\} \times \{u^{-1}v_{a,\widetilde{q}}a(w_{a,\widetilde{q}})^{-1}u'\}\right)
\end{aligned}$$

where

$$\begin{aligned}
L_a &:= \{y \in \mathcal{N}_G \mid \mathsf{nf}(ya) = ya \text{ or } y \in A^*a^{-1}\} = (\mathcal{N}_G)_a \cup (\mathcal{N}_G \cap A^*a^{-1}), \\
L'_{a,z} &:= \{y \in \mathcal{N}_G \mid y \notin L_a \text{ and } y \in B^*z\} = (\mathcal{N}_G \setminus L_a) \cap B^*z, \text{ and} \\
L''_{a,\widetilde{q},u,u'} &:= \{y \in \mathcal{N}_G \mid y \notin L_a \text{ and } \exists\, z, z' \in C^* \text{ and } h \in H \cap B_{\Gamma_C(G)}(K) \text{ such that} \\
&\qquad\qquad y \in B^*zu \text{ and } \widetilde{\delta}((q_0, h), (z, z')) = \widetilde{q}\}.
\end{aligned}$$

Note that although the word u' does not appear in the definition of the set $L''_{a,\widetilde{q},u,u'}$, it follows from the fact that $(u, u') \in L(M_{\widetilde{q},P\times\{a\}})$ that if $\widetilde{\delta}((q_0, h), (z, z')) = \widetilde{q}$ for some $h \in H \cap B_{\Gamma_C(G)}(K)$ and $z, z' \in C^*$, then $(zu, z'u') \in L(M_{(q_0,h),P\times\{a\}})$, and $zua =_G hz'u'$ with $z'u' \in \mathcal{N}_{\mathcal{T}}$, and so uniqueness of coset representatives among the words in $\mathcal{N}_{\mathcal{T}}$ implies that z, u, a, and z' uniquely determine u'. Moreover, the equation $zua =_G hz'u'$ also shows that the element h must satisfy $h =_G x_{zua}$; that is, h is also uniquely determined by z, u, and a.

The closure of regular languages under products and unions implies that it suffices to show that the languages L_a, $L_{a,z}$ and $L''_{a,\widetilde{q},u,u'}$ are regular. Closure of regular languages under quotients (Theorem 2.1), unions, and intersections shows that the language L_a is regular, and closure under complementation and concatenation shows that $L'_{a,z}$ is also regular.

Analyzing the language $L''_{a,\widetilde{q},u,u'}$ further, we have

$$\begin{aligned}
L''_{a,\widetilde{q},u,u'} &= (\mathcal{N}_G \setminus L_a) \cap (\cup_{h \in H \cap B_{\Gamma_C(G)}(K)}\ B^* L'''_{h,\widetilde{q}}\ u) \qquad \text{where} \\
L'''_{h,\widetilde{q}} &:= \{z \in C^* \mid \exists\, z' \in C^* \text{ such that } \widetilde{\delta}((q_0, h), (z, z')) = \widetilde{q}\}.
\end{aligned}$$

Now $L'''_{h,\widetilde{q}} = p_1(L(M_{(q_0,h),\{\widetilde{q}\}}))$, where p_1 denotes projection on the first coordinate. Since $L(M_{(q_0,h),\{\widetilde{q}\}})$ is the language of a finite state automaton, it is regular, and so closure under projection shows that $L'''_{h,\widetilde{q}}$ is regular. Hence each set $L''_{a,\widetilde{q},u,u'}$ is regular.

Therefore $\mathsf{graph}\,(\Phi)$ is a regular language. Thus, G is autostackable respecting H. □

Remark 5.2. Suppose that (G, H) is a strongly shortlex coset automatic pair such that both of the groups G and H are also shortlex automatic. Holt and Hurt [18] have shown that there is an algorithm which, upon input of a finite presentation of G (over the relevant generating set) and the finite generating set of H, can compute the finite state automata $M_{h,a}$ of Proposition 2.5, together with a finite state automaton accepting the shortlex transversal, for the strongly shortlex coset automatic structure for (G, H). Since H is shortlex automatic, there is an algorithm to compute the shortlex automatic structure for H from a finite presentation with the associated generators as well (see [12, Chapters 5–6] for more details). Hence there also is an algorithm which can produce the automaton accepting the regular language $\mathsf{graph}\,(\Phi)$; that is, it is possible to algorithmically compute the autostackable structure on G respecting H in this case.

For hyperbolic groups, we obtain the following corollary which will be used in the following section.

Corollary 5.3. *Hyperbolic groups are autostackable respecting quasiconvex subgroups. In particular, a hyperbolic group is autostackable respecting any virtually cyclic subgroup.*

Proof. Let G be a hyperbolic group and let $H \le G$ be a quasiconvex subgroup. In [28, Chapter 10], Redfern proves that any hyperbolic group has the coset fellow traveler property with respect to any quasiconvex subgroup using the shortlex transversal for the right cosets (over any finite generating set, and with respect to any ordering on that finite set). Theorem 2.4 then shows that the pair (G, H) is strongly shortlex coset automatic. Since H is quasiconvex in G, then H is hyperbolic (see, for example, [5, Proposition III.Γ.3.7]), and so H is autostackable by [8, Theorem 4.1]. Now apply Theorem 5.1 to see that G is autostackable respecting H.

The last claim follows since virtually cyclic subgroups of a hyperbolic group are quasiconvex ([5, Corollaries III.Γ.3.6,III.Γ.3.10]). □

As discussed in Section 2.5 above, Antolin and Ciobanu [1, Corollary 1.8] showed that groups that are hyperbolic relative to a collection of abelian subgroups are shortlex biautomatic using a "nice" generating set. In the remainder of this section we extend their argument to obtain strong shortlex coset automaticity and autostackability of the group respecting any of its peripheral subgroups. This is critical in our analysis of fundamental groups of hyperbolic pieces for Section 6.

Theorem 5.4. *Let G be a group that is hyperbolic relative to a collection of subgroups $\{H_1, ..., H_n\}$ and is generated by a finite set A'. Suppose that for every index j, the group H_j is shortlex biautomatic on every finite ordered generating set. Then there is a finite subset $\mathcal{H}' \subseteq \mathcal{H} := \cup_{j=1}^n (H_j \setminus 1)$ such that for every finite generating set A of G with $A' \cup \mathcal{H}' \subseteq A \subseteq A' \cup \mathcal{H}$ and any ordering on A, and for any $1 \leq j \leq n$, the pair (G, H_j) is strongly shortlex coset automatic, and G is autostackable respecting H_j, over A.*

Proof. Theorem 2.14 says that there are constants $\lambda \geq 1$ and $\epsilon \geq 0$ and a finite subset $\mathcal{H}' \subseteq \mathcal{H}$ such that any finite A satisfying $A' \cup \mathcal{H}' \subseteq A \subseteq A' \cup \mathcal{H}$ is a (λ, ϵ)–nice generating set of G with respect to $\{H_1, ..., H_n\}$. Let $B := B(\lambda, \epsilon+\lambda+1)$ be the bounded coset penetration constant from Proposition 2.10.

Fix a total ordering on A. Since for each index j the set $A \cap H_j$ generates H_j (from Definition 2.13(2)), by hypothesis the group H_j is shortlex biautomatic on this ordered set. Then niceness of A implies that the group G is shortlex biautomatic on the generating set A. Let K be the fellow traveler constant associated to this biautomatic structure.

Now fix an index $j \in \{1, ..., n\}$, and let $L_{\mathcal{T}} \subset A^*$ be the set of shortlex least representatives of the right cosets of H_j in G. Suppose that $v, w \in L_{\mathcal{T}}$, $a \in A$, and $h \in H_j$ satisfy $va =_G hw$.

Let p be the path in $\Gamma_A(G)$ starting at 1 labeled by v, and let q be the path in $\Gamma_A(G)$ starting at h labeled by w. Then p and q are geodesics, and so have no parabolic shortenings. Consider the paths $\hat{p}$ and $\hat{q}$ in $\Gamma_{A\cup\mathcal{H}}$ derived from the paths p and q. By Definition 2.13(1), both $\hat{p}$ and $\hat{q}$ are (λ, ϵ)–quasigeodesics without backtracking. Since the words v, w are shortlex minimal in their right H_j cosets, no nonempty prefix of v or w can represent an element of H_j. As a consequence, the paths $\hat{p}$ and $\hat{q}$ cannot penetrate the coset $1H_j$.

Let e be the edge in $\Gamma_{A\cup\mathcal{H}}$ from 1 to h labeled by h; then the concatenation $e\hat{q}$ is a $(\lambda, \epsilon+\lambda+1)$–quasigeodesic (see, for example, [25, Lemma 3.5]) in $\Gamma_{A\cup\mathcal{H}}$ from 1 to $t(q)$. The edge e is an H_j–component of the path $e\hat{q}$ lying in the coset $1H_j$. Since $i(\hat{p}) = i(e\hat{q})$ and $d_{\Gamma_A(G)}(t(\hat{p}), t(e\hat{q})) \leq 1$ all of the hypotheses of the bounded coset penetration property are satisfied for the pair of paths $\hat{p}, e\hat{q}$, and so by Proposition 2.10 applied to Definition 2.9(2)(a), the component e satisfies $d_{\Gamma_A(G)}(i(e), t(e)) \leq B$. That is, $d_{\Gamma_A(G)}(1, h) \leq B$.

Write $h = a_1 \cdots a_m$ with each $a_i \in A$ and $m \leq B$. For each $0 \leq i \leq m$, let $w_i \in A^*$ be the shortlex least representative of $(a_1 \cdots a_i)^{-1}va$ in G, and let r_i be the path in $\Gamma_A(G)$ from the vertex $i(r_i) =_G a_1 \cdots a_i$ to the vertex $t(r_i) =_G ua$ labeled by w_i. Now the paths p, r_0 have the same initial point and terminal vertices that are a distance 1 apart in $\Gamma_A(G)$, and so shortlex biautomaticity implies that these paths K-fellow travel. Similarly each pair of paths r_{i-1}, r_i (with $1 \leq i \leq m$) start a distance 1 apart and terminate at the same vertex, and so they K-fellow travel. Now r_m is the original path q, and so the paths p and q must $\widetilde{K}$–fellow travel, for the constant $\widetilde{K} = (B+1)K$. That is, for all $i \geq 0$ we have $d_{\Gamma_A(G)}(v(i), hw(i)) \leq \widetilde{K}$.

Hence the pair (G, H_j) satisfies the H_j–coset $\widetilde{K}$–fellow traveler property using the shortlex normal forms of the cosets, and Theorem 2.4 shows that (G, H_j) is strongly shortlex coset automatic. Now Theorem 5.1 shows that G is also autostackable respecting H_j on the same generating set A. □

Finally we consider the special case that G is hyperbolic relative to abelian subgroups. As noted in Remark 2.15, Holt has shown that finitely generated abelian groups satisfy the property that they are shortlex biautomatic on every ordered generating set. Then Theorem 5.4 and Remark 5.2 give the following.

Corollary 5.5. *Let G be a finitely generated group that is hyperbolic relative to a collection $\{H_1, ..., H_n\}$ of abelian subgroups. Then there is a finite inverse-closed generating set A of G such that for any $1 \leq j \leq n$, the group G is autostackable respecting H_j over A. Moreover, there is an algorithm which, upon input of a finite presentation for G with generators A and a finite presentation for H_j with generators $A \cap H_j$, produces the autostackable structure.*

6. 3-Manifolds

In this section we prove that the fundamental group of any connected, compact 3-manifold with incompressible toral boundary is autostackable. Our proof follows the procedure from Thurston's Geometrization for decomposing a 3-manifold, discussed in Section 2.6.

We begin with an analysis of the autostackability of Seifert fibered 3-manifolds with incompressible toral boundary, that arise in the JSJ decomposition of prime, compact, nongeometric 3-manifolds with incompressible toral boundary.

Proposition 6.1. *Let M be a compact Seifert fibered 3-manifold with incompressible toral boundary. Let T be any component of ∂M, and let H be any conjugate of $\pi_1(T)$ in $\pi_1(M)$. Then $\pi_1(M)$ is autostackable respecting H.*

Proof. Note that if $\partial M = \emptyset$, then M is closed and geometric, and so $\pi_1(M)$ is autostackable by [8, Corollary 1.5]. For the remainder of this proof, we assume that $\partial M \neq \emptyset$.

Let X be the base orbifold of the Seifert fibered space M, and let $\pi_1^o(X)$ be the *orbifold* fundamental group of X. There exists a short exact sequence

$$1 \longrightarrow K \xrightarrow{i} \pi_1(M) \xrightarrow{q} \pi_1^o(X) \longrightarrow 1,$$

where $K \cong \mathbb{Z}$ is generated by a regular fiber [29, Lemma 3.2] and $K < \pi_1(T)$. Since K is normal in $\pi_1(M)$, the conjugate H of $\pi_1(T)$ also satisfies $K < H$, and $H \cong \mathbb{Z}^2$.

Since the infinite cyclic group K is autostackable, Theorem 4.1 implies that in order to prove that $\pi_1(M)$ is autostackable respecting H, it suffices to show that $\pi^o(X)$ is autostackable respecting the image $q(H)$ of H in this orbifold fundamental group.

Since M has nonempty boundary, the base orbifold X has nonempty boundary, as well. Comparing to the list of compact orbifolds in the classification given in [32, Theorem 13.3.6] (and noting that there are no singular fibers over points in the boundary of X), we find that no elliptic or bad orbifold occurs as the base of a Seifert fibered space with incompressible boundary, since all of these give a solid torus for M. The only Euclidean (called parabolic in [32]) compact orbifolds which occur are the annulus, the Möbius band, and the 2-disk with two fibers of multiplicity 2; all other base orbifolds are hyperbolic. We consider the Euclidean and hyperbolic cases separately.

Suppose first that X is a hyperbolic orbifold. In this case the group $\pi_1^o(X)$ is hyperbolic. Since H is abelian, the image $q(H)$ of H in $\pi_1^o(X)$ is an abelian subgroup of this hyperbolic group, and so $q(H)$ must be virtually cyclic. Thus, by Corollary 5.3, $\pi_1^o(X)$ is autostackable respecting $q(H)$, in any generating set for $\pi_1^o(X)$ containing generators for $q(H)$.

Suppose instead that X is a Euclidean orbifold. Then X is one of the three possible orbifolds listed above, all of which have orbifold fundamental group $\pi_1^o(X)$ that is virtually $\mathbb{Z}$. Since the kernel of the restriction of the map $q : \pi_1(M) \to \pi_1^o(X)$ to $H \cong \mathbb{Z}^2$ is contained in the cyclic group K, the image $q(H)$ is an infinite subgroup of $\pi_1^o(X)$, and hence is of finite index. Since $q(H)$ is a finitely generated abelian group, $q(H)$ is also autostackable. Now Proposition 4.2 shows that $\pi_1^o(X)$ is autostackable relative to $q(H)$. □

We are now ready to prove our theorem for compact 3-manifolds with incompressible toral boundary. Note that the proof is very direct, and produces an autostackable structure that can, in theory, be computed using software. This is in sharp contrast to the proof in [12] of the existence of an automatic structure on the fundamental group of a 3-manifold with no *Nil* or *Sol* pieces in its prime decomposition, which gives an automatic structure which would be difficult to explicitly produce.

Theorem 6.2. *Let M be a compact 3-manifold with incompressible toral boundary. Then $\pi_1(M)$ is autostackable. In particular, if M is closed, then $\pi_1(M)$ is autostackable.*

Proof. Let $\widetilde{M}$ be an orientable double cover of M in the case that M is not orientable; otherwise let $\widetilde{M} := M$. Then $\pi_1(\widetilde{M})$ is a finite index subgroup of $\pi_1(M)$, and so [9, Theorem 3.4] (restated above as Proposition 4.2) shows that it suffices to prove that $\pi_1(\widetilde{M})$ is autostackable. Further, $\widetilde{M}$ also has incompressible toral boundary.

The orientable 3-manifold $\widetilde{M}$ has a unique decomposition as a connected sum of prime manifolds, $\widetilde{M} = M_1 \# M_2 \# \cdots \# M_k$. Then the fundamental group is the free product $\pi_1(\widetilde{M}) = \pi_1(M_1) * \pi_1(M_2) * \cdots * \pi_1(M_k)$. As a free product of autostackable groups is autostackable (this is shown in [9, Theorem 3.2], but also follows as a special case of Theorem 3.5), it suffices to show that $\pi_1(M)$ is autostackable in the case that M is a prime, orientable, compact 3-manifold with incompressible toral boundary; for the remainder of this proof we assume that M satisfies these four properties.

Suppose that M is also geometric. If M is closed, then by [8, Corollary 1.5], $\pi_1(M)$ is autostackable. If M is not closed, then either M is Seifert fibered, in which case autostackability of $\pi_1(M)$ is shown in Proposition 6.1, or else the interior of M is a finite volume hyperbolic 3-manifold, and so the last sentence of Section 2.6 together with Corollary 5.5 show that $\pi_1(M)$ is autostackable.

On the other hand, if M is not geometric, then M admits a JSJ decomposition into finitely many compact Seifert fibered and hyperbolic pieces $\{M_v\}_{v \in V}$ also with incompressible toral boundary. Then $\pi_1(M)$ is the fundamental group of a graph of groups on a finite connected graph Λ with vertex set V, satisfying the property that for each $v \in V$ the vertex group is $\pi_1(M_v)$, and for each directed edge e in Λ the edge group $G_e \cong \mathbb{Z}^2$ maps via the homomorphism h_e to the image of the fundamental group of an incompressible torus T_e in the boundary of $M_t(e)$; that is, $h_e(G_e) = \pi_1(T_e)$.

Let $v \in V$ and let T be an incompressible torus in the boundary of M_v. In the case that M_v is Seifert-fibered, Proposition 6.1 shows that $\pi_1(M_v)$ is autostackable respecting $\pi_1(T)$. In the case that the interior of M_v is hyperbolic, the fundamental group $\pi_1(M_v)$ is hyperbolic relative to a (finite) collection of peripheral ($\mathbb{Z}^2$) subgroups corresponding to the boundary components of M_v [13, Theorem 5.1], and so Corollary 5.5 shows that $\pi_1(M_v)$ is autostackable respecting $\pi_1(T)$ in this case as well.

Therefore, by Theorem 3.5, $\pi_1(M)$ is autostackable. □

As a historical note, we remark that it is a consequence of Theorem 6.2 that all closed 3-manifold groups satisfy the tame combability condition of Mihalik and Tschantz [22], since every stackable group is tame combable [6]. Mihalik and Tschantz show that if M

is a closed irreducible 3-manifold and $\pi_1(M)$ is infinite and tame combable, then M has universal cover homeomorphic to $\mathbb{R}^3$; tame combability was introduced in part to establish a conjecture that all closed irreducible 3-manifolds with infinite fundamental group have universal cover $\mathbb{R}^3$. While Perelman's [23] subsequent proof of the geometrization theorem has proven this conjecture (and the proof of Theorem 6.2 relies on geometrization), the proof of tame combability for all closed 3-manifold groups shows the validity of Mihalik and Tschantz's earlier approach.

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