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Joint Asymptotics for Estimating the Fractal Indices of Bivariate Gaussian Processes*

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Abstract: Multivariate (or vector-valued) processes are important for modeling multiple variables. The fractal indices of the components of the underlying multivariate process play a key role in characterizing the dependence structures and statistical properties of the multivariate process.

In this paper, under the infill asymptotics framework, we establish joint asymptotic results for the increment-based estimators of bivariate fractal indices. Our main results quantitatively describe the effect of the cross-dependence structure on the performance of the estimators.

Keywords and phrases: Fractal Indices, Bivariate Gaussian Process, Bivariate Matérn Field, Joint Asymptotics.

1. Introduction

The fractal index of a stochastic process is useful for measuring the roughness of its sample paths (e.g., it determines the Hausdorff dimension of the trajectories of the process), and it is an important parameter in geostatistical models. The problem of estimating the fractal index of a real-valued Gaussian or non-Gaussian process has attracted the attention of many authors in past decades. Hall and Wood [21] studied the asymptotic properties of the box-counting estimator of the fractal index. Constantine and Hall [11] constructed estimators of the effective fractal dimension based on the variogram. Kent and Wood [25] developed increment-based estimators for stationary Gaussian processes on \mathbb{R} , which can achieve improved performance under infill asymptotics (namely, asymptotic properties of statistical procedures as the sampling points grow dense in a fixed domain, see, e.g., [8, 12]). Chan and Wood [6, 7] extended the method to a class of stationary Gaussian random fields defined on \mathbb{R}^2 and their transformations, which are non-Gaussian in general. Zhu and Stein [42] expanded the work of Chan and Wood [6] by considering the fractional Brownian surface. More recently, Coeurjolly [10] introduced a new class of consistent estimators of the fractal dimension of locally self-similar Gaussian processes on \mathbb{R} using sample quantiles and derived the almost sure convergence and asymptotic normality for these estimators. Bardet and Surgailis [5] provided estimators of the fractal index based on increment ratios for several classes of real-valued

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processes with rough sample paths, including Gaussian processes, and studied their asymptotic properties. Loh [29] constructed estimators from irregularly spaced data on \mathbb{R}^d with $d = 1$ or 2 via higher-order quadratic variations. We refer to [20] and the references therein for further information on various types of estimators and their assessments.

In recent years, multivariate (or vector-valued) Gaussian processes and random fields have become popular in modeling multivariate spatial datasets (see, e.g., [17, 36]). Several classes of multivariate spatial models were introduced in [4, 13, 14, 19, 26, 30, 33]. Two of the challenges in multivariate modeling are to specify the cross-dependence structures and to quantify the effect of the cross-dependence on the estimation and prediction performance. We refer to [18] for an excellent review of the recent developments in multivariate covariance functions. They also raised many open questions and called for theoretical development of estimation and prediction methodology in the multivariate context. To the best of our knowledge, only a few authors have worked in this direction; see, for example, [16, 28, 31, 34, 41]. While with respect to focusing on estimating the fractal indices of a multivariate Gaussian process, we are only aware of the work by Amblard and Coeurjolly [3], in which they constructed estimators for the fractal indices of a class of multivariate fractional Brownian motions using discrete filtering techniques and studied their joint asymptotic distribution. By estimating the fractal index of each component separately, they found that the quality of these estimates was almost independent of the cross correlation of the multivariate fractional Brownian motion.

In this work, we consider a class of bivariate stationary Gaussian processes $\mathbf{X} \triangleq \{(X_1(t), X_2(t))^\top, t \in \mathbb{R}\}$ (the operator $(\cdot)^\top$ means the transpose of a vector or a matrix) and study the joint asymptotic properties of the estimators for the fractal indices of the components X_1 and X_2 under the infill asymptotics framework. Our main purpose is to clarify the effect of cross covariance on the performance of the joint estimators.

More specifically, we assume that \mathbf{X} has mean $E\mathbf{X}(t) = \mathbf{0}$ and matrix-valued covariance function

$$\mathbf{C}(t) = \begin{pmatrix} C_{11}(t) & C_{12}(t) \\ C_{21}(t) & C_{22}(t) \end{pmatrix}, \quad (1.1)$$

where $C_{ij}(t) := E[X_i(s)X_j(s+t)]$, $i = 1, 2$. Further, we assume that the following conditions are satisfied

$$\begin{aligned} C_{11}(t) &= \sigma_1^2 - c_{11}|t|^{\alpha_{11}} + o(|t|^{\alpha_{11}}), \\ C_{22}(t) &= \sigma_2^2 - c_{22}|t|^{\alpha_{22}} + o(|t|^{\alpha_{22}}), \\ C_{12}(t) &= C_{21}(t) = \rho\sigma_1\sigma_2(1 - c_{12}|t|^{\alpha_{12}} + o(|t|^{\alpha_{12}})), \end{aligned} \quad (1.2)$$

where $\alpha_{11}, \alpha_{22} \in (0, 2)$, $\sigma_1, \sigma_2 > 0$, $|\rho| \in (0, 1)$ and $c_{11}, c_{22}, c_{12} > 0$ are constants. Under the assumption (1.2), in order for (1.1) to be a valid covariance function, it is necessary to impose some restrictions on the parameters $(\alpha_{11}, \alpha_{22}, \alpha_{12})$. In

this paper, we assume

$$\begin{aligned} \frac{\alpha_{11} + \alpha_{22}}{2} < \alpha_{12}, \quad \text{or} \\ \frac{\alpha_{11} + \alpha_{22}}{2} = \alpha_{12} \quad \text{and} \quad c_{12}^2 \rho^2 \sigma_1^2 \sigma_2^2 < c_{11} c_{22}. \end{aligned} \tag{1.3}$$

This is a mild assumption. See Appendix A for justification.

Henceforth, we refer to (1.2) and (1.3) as Condition **(A1)**. Under the assumption (1.2), it is well known (see, e.g., [2], Theorem 8.4.1) that the fractal dimensions of the trajectories of each component X_1 and X_2 are given by

$$\dim \text{Gr}X_1([0, 1]) = 2 - \frac{\alpha_{11}}{2}, \quad \text{a.s.}$$

and

$$\dim \text{Gr}X_2([0, 1]) = 2 - \frac{\alpha_{22}}{2}, \quad \text{a.s.},$$

respectively. Above, for $i \in \{1, 2\}$, $\text{Gr}X_i([0, 1]) = \{(t, X_i(t)) : t \in [0, 1]\}$ is the trajectory (or graph set) of the real-valued process $X_i = \{X_i(t), t \in \mathbb{R}\}$ over the interval $[0, 1]$. A bivariate stationary Gaussian process \mathbf{X} with matrix-valued covariance function (1.1) that satisfies Condition **(A1)** has richer fractal properties. For example, we consider the trajectory of \mathbf{X} on $[0, 1]$, which is $\text{Gr}\mathbf{X}([0, 1]) = \{(t, X_1(t), X_2(t))^\top : t \in [0, 1]\} \subseteq \mathbb{R}^3$. For notational convenience, we further assume that $\alpha_{11} \leq \alpha_{22}$ (otherwise we may relabel the components of \mathbf{X}). Then, we can apply Theorem 2.1 in [37] to show that, with probability 1,

$$\begin{aligned} \dim \text{Gr}\mathbf{X}([0, 1]) &= \min \left\{ \frac{2 + \alpha_{22} - \alpha_{11}}{\alpha_{22}}, 3 - \frac{\alpha_{11} + \alpha_{22}}{2} \right\} \\ &= \begin{cases} \frac{2 + \alpha_{22} - \alpha_{11}}{\alpha_{22}}, & \text{if } \alpha_{11} + \alpha_{22} \geq 2, \\ 3 - \frac{\alpha_{11} + \alpha_{22}}{2}, & \text{if } \alpha_{11} + \alpha_{22} < 2. \end{cases} \end{aligned} \tag{1.4}$$

This result shows that the indices α_{11} and α_{22} determine the fractal dimension of the trajectory of the bivariate Gaussian process \mathbf{X} . Furthermore, one can characterize many other fractal properties of \mathbf{X} explicitly in terms of these indices. See [38] for a recent overview. Hence, analogous to the univariate case, it is natural to call $(\alpha_{11}, \alpha_{22})$ the fractal indices of \mathbf{X} . For the reader's convenience, we include a proof of (1.4) in Appendix B.

Although the parameters α_{11} and α_{22} can be estimated separately from observations of the coordinate processes X_1 and X_2 , (1.4) suggests that, in doing so, one might miss some important information about the structures of the bivariate process \mathbf{X} . For example, although the estimator of $\dim \text{Gr}\mathbf{X}([0, 1])$ can be obtained by plugging the estimators of α_{11} and α_{22} into (1.4), say $(\hat{\alpha}_{11}, \hat{\alpha}_{22})$, we cannot evaluate the estimation efficiency without the joint asymptotic properties of $(\hat{\alpha}_{11}, \hat{\alpha}_{22})$. Hence, it is necessary to study these estimators jointly and to quantify the effect of the cross-covariance on their performance.

In this paper, we consider the increment-based estimators of α_{11} and α_{22} , denoted by $\hat{\alpha}_{11}$ and $\hat{\alpha}_{22}$, respectively, and study the bias, mean square error

matrix and joint asymptotic distribution under the infill asymptotics framework. The main results are given in Theorems 3.3 ~ 3.5. In particular, we prove that $\sqrt{n}\hat{\alpha}_{11}$ and $\sqrt{n}\hat{\alpha}_{22}$ are asymptotically uncorrelated if $(\alpha_{11} + \alpha_{22})/2 < \alpha_{12}$, while they are asymptotically correlated if $(\alpha_{11} + \alpha_{22})/2 = \alpha_{12}$. Our results are applicable to a wide class of bivariate Gaussian processes, including the bivariate Matérn model introduced by Gneiting, Kleiber and Schlather [19], the bivariate powered exponential model and bivariate Cauchy model of Moreva and Schlather [30], and a class of bivariate models introduced by Du and Ma [14].

This paper raises several open questions. First, the method of joint asymptotics developed in this paper and the recent work by Loh [29] on constructing estimators for the univariate fractal index given irregularly spaced data make it possible to study the joint asymptotics in estimating bivariate fractal indices when data are observed irregularly on \mathbb{R}^2 . This problem is interesting from both theoretical and application viewpoints, but it appears to be challenging. Second, the work by Ruiz-Medina and Porcu [34], which established conditions for the equivalence of Gaussian measures of multivariate random fields, makes it promising to generalize the consistency and asymptotic normality results of maximum likelihood estimators for a univariate random field to the case of multivariate Gaussian fields. The existing results in the univariate case were established under the assumption that the smoothness parameter is known (see, e.g., [15, 23, 40]). It would be interesting to study whether the asymptotic properties hold in either the univariate or multivariate case while plugging in estimators of the smooth parameters.

The rest of this paper is organized as follows. We follow Chan and Wood [6] and Kent and Wood [25] and formulate the increment-based estimators for $(\alpha_{11}, \alpha_{22})$ in Section 2. In Section 3, we state the main results of the joint asymptotics of the bivariate estimators. An application to the non-smooth bivariate Matérn processes is given in Section 4. In Section 5, we present a simulation study on the efficiency of the estimators. The proofs of our main results are given in Section 6. Finally, some auxiliary results and their proofs are included in the Appendix.

We end the introduction with some notation. \mathbb{Z}^+ denotes the set of all positive integers, and $\mathcal{B}(\mathbb{R})$ is the collection of all Borel sets on \mathbb{R} . For any real-valued sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $a_n \sim b_n$ means $\lim_{n \rightarrow \infty} b_n/a_n = 1$, $a_n \gtrsim b_n$ means that there exists a constant $c > 0$ such that $a_n \geq c b_n$ for all n sufficiently large and $a_n \asymp b_n$ means $a_n \gtrsim b_n$ and $b_n \gtrsim a_n$. Similar notation is used for functions of continuous variables.

An unspecified positive and finite constant will be denoted by C_0 . More specific constants are numbered as C_1, C_2, \dots .

2. The increment-based estimators

Assume that the values of the bivariate process \mathbf{X} are observed regularly on an interval I , say $I = [0, 1]$. More specifically, we have n pairs of observations

$(\mathbf{X}(1/n), \dots, \mathbf{X}(1))^\top$. By applying the increment-based method introduced by Kent and Wood [25] for estimating the fractal index of a real-valued locally self-similar Gaussian process (see also [6, 7] for further development), we can estimate the fractal indices $(\alpha_{11}, \alpha_{22})$ of \mathbf{X} . Our emphasis is on studying the joint asymptotic properties of the estimators. In particular, we study the effect of cross-covariance on their joint performance.

Let $m \geq 2$ be a fixed integer. For each component X_i , $i = 1, 2$ and integer $u \in \{1, \dots, m\}$, we define the dilated filtered discretized process with second difference (see, e.g., [25]),

$$Y_{n,i}^u(j) := n^{\frac{\alpha_{ii}}{2}} \left(X_i\left(\frac{j-u}{n}\right) - 2X_i\left(\frac{j}{n}\right) + X_i\left(\frac{j+u}{n}\right) \right), \quad j = 1, \dots, n.$$

Denote by $a_{-1} = 1, a_0 = -2, a_1 = 1$. $Y_{n,i}^u$ can be rewritten as

$$Y_{n,i}^u(j) = n^{\frac{\alpha_{ii}}{2}} \sum_{k=-1}^1 a_k X_i\left(\frac{j+ku}{n}\right).$$

As in Kent and Wood [25], one can verify that, under (1.2), $Y_{n,i}^u(j)$ is a Gaussian random variable with mean 0, and its variance converges to $c_{ii}(8 - 2^{\alpha_{ii}+1})u^{\alpha_{ii}}$ (this follows from (3.2) below). Let $Z_{n,i}^u(j) := (Y_{n,i}^u(j))^2$ and define

$$\bar{Z}_{n,i}^u := \frac{1}{n} \sum_{j=1}^n Z_{n,i}^u(j). \quad (2.1)$$

For $i = 1, 2$, it follows from [25] that, under certain regularity conditions on the covariance function $C_{ii}(t)$, we have

$$\bar{Z}_{n,i}^u \xrightarrow{p} C_{ii} u^{\alpha_{ii}},$$

where \xrightarrow{p} represents convergence in probability and $C_i = c_{ii}(8 - 2^{\alpha_{ii}+1})$. Hence,

$$\ln \bar{Z}_{n,i}^u \xrightarrow{p} \alpha_{ii} \ln u + \ln C_i, \quad i = 1, 2,$$

where \ln represents natural logarithm. Consequently, the fractal indices α_{ii} ($i = 1, 2$) can be estimated by linear regression of $\ln \bar{Z}_{n,i}^u$ on $\ln u$ for $u = 1, \dots, m$.

In this paper, we employ Chan and Wood [6]'s linear estimators for α_{ii} based on $\ln \bar{Z}_{n,i}^u$, that is,

$$\hat{\alpha}_{ii} = \sum_{u=1}^m L_{u,i} \ln \bar{Z}_{n,i}^u, \quad (2.2)$$

where $\{L_{u,i}, u = 1, \dots, m\}$ ($i = 1, 2$) are finite sequences of real numbers such that

$$\sum_{u=1}^m L_{u,i} = 0 \quad \text{and} \quad \sum_{u=1}^m L_{u,i} \ln u = 1. \quad (2.3)$$

Both the ordinary least squares and generalized least squares estimators introduced by Kend and Wood [25] are examples of the above estimators. We remark that due to the first condition in (2.3), the estimators $\hat{\alpha}_{ii}$ ($i = 1, 2$) defined in (2.2) can be computed from the observed values $(\mathbf{X}(1/n), \dots, \mathbf{X}(1))^\top$ and do not depend on the unknown indices α_{ii} .

3. Joint asymptotic properties

For $i = 1, 2$, let

$$\bar{\mathbf{Z}}_{n,i} = (\bar{Z}_{n,i}^1, \dots, \bar{Z}_{n,i}^m)^\top$$

and denote

$$\bar{\mathbf{Z}}_n = (\bar{\mathbf{Z}}_{n,1}^\top, \bar{\mathbf{Z}}_{n,2}^\top)^\top.$$

Under the infill asymptotics framework, we first study the asymptotic properties of $\bar{\mathbf{Z}}_n$ in Section 3.1. In Section 3.2, the joint asymptotic properties of the estimators $(\hat{\alpha}_{11}, \hat{\alpha}_{22})^\top$ are obtained.

3.1. Variance of $\bar{\mathbf{Z}}_n$ and asymptotic normality

First, given $u, v = 1, \dots, m$, we consider the covariance matrix of $(Y_{n,1}^u, Y_{n,2}^v)^\top$. For $i = 1, 2$, it follows from Kent and Wood [25] that the marginal covariance function for $Y_{n,i}^u$ and $Y_{n,i}^v$ is

$$\begin{aligned} \sigma_{n,ii}^{uv}(h) &:= \mathbb{E}[Y_{n,i}^u(\ell)Y_{n,i}^v(\ell+h)] \\ &\rightarrow -c_{ii} \sum_{j,k=-1}^1 a_j a_k |h+kv-ju|^{\alpha_{ii}} \triangleq \sigma_{0,ii}^{uv}(h), \end{aligned} \quad (3.1)$$

as $n \rightarrow \infty$. In particular, we derive that the variance of $Y_{n,i}^u(\ell)$ satisfies

$$\sigma_{n,ii}^{uu}(0) \rightarrow C_i u^{\alpha_{ii}}, \quad \text{as } n \rightarrow \infty, \quad (3.2)$$

where $C_i = c_{ii}(8 - 2^{\alpha_{ii}+1})$.

Under the assumption (1.2), the cross covariance between $Y_{n,1}^u$ and $Y_{n,2}^v$ can be derived as follows.

$$\begin{aligned} \sigma_{n,12}^{uv}(h) &:= \mathbb{E}[Y_{n,1}^u(\ell)Y_{n,2}^v(\ell+h)] \\ &= n^{\frac{\alpha_{11}+\alpha_{22}}{2}} \sum_{j,k=-1}^1 a_j a_k C_{12} \left(\frac{h+kv-ju}{n} \right) \\ &\rightarrow \begin{cases} 0, & \text{if } \frac{\alpha_{11}+\alpha_{22}}{2} < \alpha_{12}, \\ -\rho\sigma_1\sigma_2c_{12} \sum_{j,k=-1}^1 a_j a_k |h+kv-ju|^{\alpha_{12}}, & \text{if } \frac{\alpha_{11}+\alpha_{22}}{2} = \alpha_{12} \end{cases} \\ &\triangleq \sigma_{0,12}^{uv}(h). \end{aligned} \quad (3.3)$$

Therefore, if $(\alpha_{11} + \alpha_{22})/2 < \alpha_{12}$, the covariance matrix of $(Y_{n,1}^u(\ell), Y_{n,2}^v(\ell+h))^\top$ satisfies

$$\text{Var} \begin{pmatrix} Y_{n,1}^u(\ell) \\ Y_{n,2}^v(\ell+h) \end{pmatrix} \rightarrow \begin{pmatrix} \sigma_{0,11}^{uu}(0) & 0 \\ 0 & \sigma_{0,22}^{vv}(0) \end{pmatrix}, \text{ as } n \rightarrow \infty.$$

If $(\alpha_{11} + \alpha_{22})/2 = \alpha_{12}$, the covariance matrix of $(Y_{n,1}^u(\ell), Y_{n,2}^v(\ell+h))^\top$ satisfies

$$\text{Var} \begin{pmatrix} Y_{n,1}^u(\ell) \\ Y_{n,2}^v(\ell+h) \end{pmatrix} \rightarrow \begin{pmatrix} \sigma_{0,11}^{uu}(0) & \sigma_{0,12}^{uv}(h) \\ \sigma_{0,12}^{uv}(h) & \sigma_{0,22}^{vv}(0) \end{pmatrix}, \text{ as } n \rightarrow \infty.$$

We adapt the method of derivation in Section 3 of Kent and Wood [25] to find the covariance matrix of the random vector $\bar{\mathbf{Z}}_n$. Using the fact that if $(U, V) \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \xi \\ \xi & 1 \end{pmatrix}\right)$ is a bivariate normal random vector, then $\text{cov}(U^2, V^2) = 2\xi^2$, we obtain

$$\text{cov}(Z_{n,i}^u(\ell), Z_{n,j}^v(\ell+h)) = 2(\sigma_{n,ij}^{uv}(h))^2, \quad i, j = 1, 2;$$

hence,

$$\phi_{n,ij}^{uv} := \text{cov}(\bar{Z}_{n,i}^u, \bar{Z}_{n,j}^v) = \frac{1}{n} \sum_{h=-n+1}^{n-1} \left(1 - \frac{|h|}{n}\right) \times 2(\sigma_{n,ij}^{uv}(h))^2.$$

Denote by $\Phi_{n,ij} = (\phi_{n,ij}^{uv})_{u,v=1}^m$ the covariance matrix of $\bar{\mathbf{Z}}_{n,i}$ and $\bar{\mathbf{Z}}_{n,j}$. Then, the covariance matrix of $\bar{\mathbf{Z}}_n$ can be written as

$$\Phi_n = \begin{pmatrix} \Phi_{n,11} & \Phi_{n,12} \\ \Phi_{n,21} & \Phi_{n,22} \end{pmatrix}.$$

To study the asymptotic properties of Φ_n and $\bar{\mathbf{Z}}_n$, we impose an additional regularity condition on the fourth derivative of the functions $C_{ij}(t)$ in (1.1) around the origin, which is analogous to the condition (A_4) in [25] and will be called Condition **(A2)**:

$$\begin{aligned} C_{11}^{(4)}(t) &= -\frac{c_{11}\alpha_{11}!}{(\alpha_{11}-4)!} |t|^{\alpha_{11}-4} + o(|t|^{\alpha_{11}-4}), \\ C_{22}^{(4)}(t) &= -\frac{c_{22}\alpha_{22}!}{(\alpha_{22}-4)!} |t|^{\alpha_{22}-4} + o(|t|^{\alpha_{22}-4}), \\ C_{12}^{(4)}(t) &= C_{21}^{(4)}(t) = -\rho\sigma_1\sigma_2 \frac{c_{12}\alpha_{12}!}{(\alpha_{12}-4)!} |t|^{\alpha_{12}-4} + o(|t|^{\alpha_{12}-4}). \end{aligned}$$

Above, for any $\alpha > 0$, $\alpha!/(\alpha-4)! = \alpha(\alpha-1)(\alpha-2)(\alpha-3)$.

For $i, j = 1, 2$, let $\phi_{0,ij}^{uv} = 2 \sum_{h=-\infty}^{\infty} (\sigma_{0,ij}^{uv}(h))^2$, which is convergent, $\Phi_{0,ij} = (\phi_{0,ij}^{uv})_{u,v=1}^m$, and let

$$\Phi_0 = \begin{pmatrix} \Phi_{0,11} & \Phi_{0,12} \\ \Phi_{0,21} & \Phi_{0,22} \end{pmatrix}.$$

The following theorems describe the asymptotic properties of the random vector $\bar{\mathbf{Z}}_n$.

Theorem 3.1. *If Conditions (A1) and (A2) hold, then*

$$n\Phi_n \rightarrow \Phi_0, \text{ as } n \rightarrow \infty. \quad (3.4)$$

Moreover, if $(\alpha_{11} + \alpha_{22})/2 < \alpha_{12}$, then $\Phi_{0,12} = \Phi_{0,21} = \mathbf{0}$.

Theorem 3.2. *If Conditions (A1) and (A2) hold, then*

$$n^{1/2}(\bar{\mathbf{Z}}_n - \mathbb{E}[\bar{\mathbf{Z}}_n]) \xrightarrow{d} \mathcal{N}_{2m}(\mathbf{0}, \Phi_0), \text{ as } n \rightarrow \infty,$$

where $\mathcal{N}_{2m}(\mathbf{0}, \Phi_0)$ is the $(2m)$ -dimensional normal distribution with mean $\mathbf{0}$ and covariance matrix Φ_0 .

Remark 3.1. Theorem 3.2 extends Theorem 2 in Kent and Wood [25] to the bivariate case, and shows that $\sqrt{n}\bar{\mathbf{Z}}_{n,1}$ and $\sqrt{n}\bar{\mathbf{Z}}_{n,2}$ are asymptotically independent when $(\alpha_{11} + \alpha_{22})/2 < \alpha_{12}$. The proofs of Theorems 3.1 and 3.2 are given in Appendix D.

Remark 3.2. The class of matrix-valued covariance functions whose properties around the origin satisfy (A1) and (A2) is large, including such significant examples as the bivariate Matérn model of Gneiting, Kleiber and Schlather [19], the bivariate powered exponential model and bivariate Cauchy model of Moreva and Schlather [30], the bivariate Wendland-Gneiting covariance function of Daley, Procu and Bevilacqua [13] and a class of bivariate models introduced by Du and Ma [14], such as Example 3. Since the matrix-valued covariance functions in these references have explicit closed forms, Conditions (A1) and (A2) can be verified directly by using Taylor's expansion or L'Hospital's rule.

Remark 3.3. Another way to verify Conditions (A1) and (A2) is to make use of the spectral representation of C_{ij} :

$$C_{ij}(t) = \int_{\mathbb{R}} \cos(t\xi) F_{ij}(d\xi),$$

where F_{ij} is the spectral measure of C_{ij} . Writing

$$C_{ij}(0) - C_{ij}(t) = \int_{\mathbb{R}} (1 - \cos(t\xi)) F_{ij}(d\xi),$$

one can see that Condition (A1) may follow from an Abelian-type theorem and the tail behavior of the spectral measure F_{ij} at infinity (see, for example, [32]).

To verify Condition (A2), we may assume that F_{ij} has a density function $f_{ij}(\xi)$ which decays faster than certain polynomial rate as $|\xi| \rightarrow \infty$. A change of variable yields that for $t \neq 0$,

$$C_{ij}(0) - C_{ij}(t) = \frac{1}{t} \int_{\mathbb{R}} (1 - \cos \xi) f_{ij}\left(\frac{\xi}{t}\right) d\xi.$$

Then we can differentiate $C_{ij}(t)$ as follows:

$$\begin{aligned} C'_{ij}(t) &= \frac{1}{t^2} \int_{\mathbb{R}} (1 - \cos \xi) f_{ij}\left(\frac{\xi}{t}\right) d\xi + \frac{1}{t} \int_{\mathbb{R}} (1 - \cos \xi) f'_{ij}\left(\frac{\xi}{t}\right) \frac{\xi}{t^2} d\xi \\ &= \frac{C_{ij}(0) - C_{ij}(t)}{t} + \frac{1}{t} \int_{\mathbb{R}} (1 - \cos \xi) f'_{ij}\left(\frac{\xi}{t}\right) \frac{\xi}{t^2} d\xi. \end{aligned}$$

Consequently, the asymptotic behavior of $C'_{ij}(t)$ as $t \rightarrow 0$ can be derived from (A1) and another application of the Abelian-type theorem in [32] to the second integral. Iterating this procedure three more times, we can verify Condition (A2).

3.2. Asymptotic properties of $(\hat{\alpha}_{11}, \hat{\alpha}_{22})^\top$

This section contains the main results of this paper. We make a stronger assumption by specifying the remainder terms in Assumption (1.2). Suppose that for some constants $\beta_{11}, \beta_{22}, \beta_{12} > 0$,

$$\begin{aligned} C_{11}(t) &= \sigma_1^2 - c_{11}|t|^{\alpha_{11}} + O(|t|^{\alpha_{11}+\beta_{11}}), \\ C_{22}(t) &= \sigma_2^2 - c_{22}|t|^{\alpha_{22}} + O(|t|^{\alpha_{22}+\beta_{22}}), \\ C_{12}(t) &= C_{21}(t) = \rho\sigma_1\sigma_2(1 - c_{12}|t|^{\alpha_{12}} + O(|t|^{\alpha_{12}+\beta_{12}})). \end{aligned} \quad (3.5)$$

We label the three conditions in (3.5), together with (1.3), as Condition (A3).

Let $\hat{\alpha} = (\hat{\alpha}_{11}, \hat{\alpha}_{22})^\top$ be the estimators of the fractal indices $\alpha = (\alpha_{11}, \alpha_{22})^\top$, as defined in (2.2). The theorems below establish the asymptotic properties of $\hat{\alpha}$, including the bias, mean square error matrix and their joint asymptotic distribution.

Theorem 3.3 (Bias). *Assume Conditions (A2) and (A3) hold. Then, for the estimators $\hat{\alpha}_{ii}, i = 1, 2$, we have*

$$\mathbb{E}[\hat{\alpha}_{ii} - \alpha_{ii}] = O(n^{-1}) + O(n^{-\beta_{ii}}), \quad i = 1, 2.$$

Theorem 3.4 (Mean square error matrix). *Assume (A2) and (A3) hold. If $(\alpha_{11} + \alpha_{22})/2 = \alpha_{12}$, then*

$$\begin{aligned} &\mathbb{E}[(\hat{\alpha} - \alpha)(\hat{\alpha} - \alpha)^\top] \\ &= \begin{pmatrix} O(n^{-1}) & O(n^{-1}) \\ O(n^{-1}) & O(n^{-1}) \end{pmatrix} + \begin{pmatrix} O(n^{-\psi(\beta_{11}, \beta_{11})}) & O(n^{-\psi(\beta_{11}, \beta_{22})}) \\ O(n^{-\psi(\beta_{11}, \beta_{22})}) & O(n^{-\psi(\beta_{22}, \beta_{22})}) \end{pmatrix}. \end{aligned} \quad (3.6)$$

Here and below, $\psi(x_1, x_2) := \min\{1 + x_1, 1 + x_2, x_1 + x_2\}$.

If $(\alpha_{11} + \alpha_{22})/2 < \alpha_{12}$, then

$$\begin{aligned} &\mathbb{E}[(\hat{\alpha} - \alpha)(\hat{\alpha} - \alpha)^\top] \\ &= \begin{pmatrix} O(n^{-1}) & o(n^{-1}) \\ o(n^{-1}) & O(n^{-1}) \end{pmatrix} + \begin{pmatrix} O(n^{-\psi(\beta_{11}, \beta_{11})}) & O(n^{-\psi(\beta_{11}, \beta_{22})}) \\ O(n^{-\psi(\beta_{11}, \beta_{22})}) & O(n^{-\psi(\beta_{22}, \beta_{22})}) \end{pmatrix}. \end{aligned} \quad (3.7)$$

Remark 3.4. The constants β_{11}, β_{22} and β_{12} from (3.5) appear in both the bias and mean square error matrix (Theorems 3.3 and 3.4) because the remainder terms $O(|t|^{\alpha_{ii}+\beta_{ii}})$ in the covariance function are ignored in the estimation procedure, which might strongly affect the efficiency of the estimators (see, e.g., [25]). The statistical performance of the estimators $(\hat{\alpha}_{11}, \hat{\alpha}_{22})^\top$ can be significantly improved if more detailed information on the remainder term is available. In Section 4, we show that this is indeed the case when \mathbf{X} is a nonsmooth bivariate Matérn process.

Finally, we study the asymptotic distribution of $\hat{\alpha}$ by applying multivariate delta methods (see, e.g., [3, 27]). By (2.1), (3.1), and (3.5), we have

$$\mathbb{E}\bar{Z}_{n,i}^u = \mathbb{E}[(Y_{n,i}^u(0))^2] = \tau_{u,i}(1 + O(n^{-\beta_{ii}})), \quad (3.8)$$

where $\tau_{u,i} = c_{ii}(8 - 2^{\alpha_{ii}+1})u^{\alpha_{ii}}$. Let

$$\tilde{\mathbf{L}}_i = (L_{1,i}/\tau_{1,i}, \dots, L_{m,i}/\tau_{m,i})^\top, \quad i = 1, 2.$$

The following theorem provides the joint asymptotic distribution of $\hat{\alpha}$.

Theorem 3.5 (Asymptotic distribution). *Assume (A2) and (A3) hold with $\beta_{11}, \beta_{22} > 1/2$. Then, $\sqrt{n}(\hat{\alpha} - \alpha)$ follows the asymptotic properties below.*

$$\sqrt{n} \begin{pmatrix} \hat{\alpha}_{11} - \alpha_{11} \\ \hat{\alpha}_{22} - \alpha_{22} \end{pmatrix} \xrightarrow{d} \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \tilde{\mathbf{L}}_1^\top \Phi_{0,11} \tilde{\mathbf{L}}_1 & \tilde{\mathbf{L}}_1^\top \Phi_{0,12} \tilde{\mathbf{L}}_2 \\ \tilde{\mathbf{L}}_2^\top \Phi_{0,21} \tilde{\mathbf{L}}_1 & \tilde{\mathbf{L}}_2^\top \Phi_{0,22} \tilde{\mathbf{L}}_2 \end{pmatrix} \right).$$

Specifically, if $(\alpha_{11} + \alpha_{22})/2 < \alpha_{12}$, then $\sqrt{n}\hat{\alpha}_{11}$ and $\sqrt{n}\hat{\alpha}_{22}$ are asymptotically independent.

Remark 3.5. The current estimation procedure and asymptotic properties are derived for nonsmooth bivariate Gaussian models, that is, the smoothness parameters $\alpha_{ii} \in (0, 2)$ for $i = 1, 2$. If the sample function of the component X_i is almost surely differentiable, then the corresponding index $\alpha_{ii} \geq 2$ in (1.2). In this case, one may extend the idea of Kent and Wood [24] and consider the covariance functions with the following local properties

$$\begin{aligned} C_{11}(t) &= \sigma_1^2 - \sum_{k=1}^q b_{1,k} t^{2j} - c_{11}|t|^{\alpha_{11}} + o(|t|^{\alpha_{11}}), \\ C_{22}(t) &= \sigma_2^2 - \sum_{k=1}^q b_{2,k} t^{2j} - c_{22}|t|^{\alpha_{22}} + o(|t|^{\alpha_{22}}), \\ C_{12}(t) &= C_{21}(t) = \rho\sigma_1\sigma_2 \left(1 - \sum_{k=1}^q b_{12,k} t^{2j} - c_{12}|t|^{\alpha_{12}} + o(|t|^{\alpha_{12}}) \right), \end{aligned}$$

where q is a positive integer and $\alpha_{ii} \in (2q, 2q + 2)$. Then, the q th derivative process $\mathbf{X}^{(q)} := (X_1^{(q)}, X_2^{(q)})^\top$ would satisfy Condition (A1) with smoothness parameters $(\alpha_{11} - 2q, \alpha_{22} - 2q)^\top$. Thus, the framework proposed in our paper can be extended to smooth bivariate Gaussian fields via estimating the fractal indices of their derivative processes.

4. An example: nonsmooth bivariate Matérn processes on \mathbb{R}

The Matérn correlation function $M(h|\nu, a)$ on \mathbb{R}^N , where $a > 0, \nu > 0$ are scale and smoothness parameters, is widely used to model covariance structures in spatial statistics. It is defined as

$$M(h|\nu, a) := \frac{2^{1-\nu}}{\Gamma(\nu)} (a|h|)^\nu K_\nu(a|h|), \quad h \in \mathbb{R}^N,$$

where K_ν is a modified Bessel function of the second kind. Recently, Gneiting, Kleiber and Schlather [19] introduced the full bivariate Matérn field $\mathbf{X} = \{(X_1(s), X_2(s))^\top, s \in \mathbb{R}^N\}$, which is an \mathbb{R}^2 -valued Gaussian random field on \mathbb{R}^N with zero mean and matrix-valued covariance function:

$$\mathbf{C}(h) = \begin{pmatrix} C_{11}(h) & C_{12}(h) \\ C_{21}(h) & C_{22}(h) \end{pmatrix}, \quad (4.1)$$

where $C_{ij}(h) := \mathbb{E}[X_i(s+h)X_j(s)]$ are specified by

$$\begin{aligned} C_{11}(h) &= \sigma_1^2 M(h|\nu_{11}, a_{11}), \\ C_{22}(h) &= \sigma_2^2 M(h|\nu_{22}, a_{22}), \\ C_{12}(h) &= C_{21}(h) = \rho\sigma_1\sigma_2 M(h|\nu_{12}, a_{12}). \end{aligned}$$

A necessary and sufficient condition for $C(h)$ in (4.1) to be valid is given by [19]. We assume that the parameters $\nu_{ij}, a_{ij}, \sigma_i, (i, j = 1, 2)$ and ρ satisfy the condition in Theorem 3 of [19], as well as our condition (1.3).

To apply the results in Section 3, we focus on the case of $N = 1$ and $0 < \nu_{11}, \nu_{22} < 1$. Then, $\mathbf{X} = \{(X_1(s), X_2(s))^\top, s \in \mathbb{R}\}$ is a stationary bivariate Gaussian process with nonsmooth sample functions. For simplicity, we call \mathbf{X} a bivariate Matérn process.

Recall that the Matérn correlation function has the following asymptotic expansion at $h = 0$,

$$M(h|\nu, a) = 1 - b_1|h|^{2\nu} + b_2|h|^2 + O(|h|^{2+2\nu}), \quad \text{as } |h| \rightarrow 0, \quad (4.2)$$

where b_1 and b_2 are explicit constants depending only on ν and a (Eq. (4.2) follows from (9.6.2) and (9.6.10) in [1]). Therefore, (A3) is satisfied with $\beta_{ij} = 2 - \nu_{ij}$ for $i, j = 1, 2$. Moreover, one can check that the regularity condition (A2) regarding the fourth derivatives of the covariance function is also satisfied (see the proof in Appendix C).

According to (4.2) and the fact that $\sum_{j,k=-1}^1 a_j a_k |k-j|^2 = 0$, we have

$$\begin{aligned} \sigma_{n,ii}^{uu}(0) &:= \mathbb{E}(Y_{n,i}^u(0))^2 = n^{2\nu_{ii}} \sum_{j,k=-1}^1 a_j a_k C_{ii}\left(\frac{(k-j)u}{n}\right) \\ &= -b_1 \sigma_{ii}^2 \sum_{j,k=-1}^1 a_j a_k |k-j|^{2\nu_{ii}} u^{2\nu_{ii}} + O(n^{-2}). \end{aligned} \quad (4.3)$$

Observe that, unlike (3.8), the constants $\beta_{ij} = 2 - \nu_{ij}$ do not appear in (4.3) because the related terms sum to 0. Consequently, we can prove the following results, which are stronger than what can be obtained by directly applying Theorems 3.3 ~ 3.5 to bivariate Matérn processes. Their proofs are modifications of those of Theorems 3.3 ~ 3.5 in Section 6 and will be omitted.

Proposition 4.1 (Bias). *For the bivariate Matérn process \mathbf{X} with $0 < \nu_{11}, \nu_{22} < 1$, the bias of $\hat{\nu}_{ii}$ is*

$$\mathbb{E}[\hat{\nu}_{ii} - \nu_{ii}] = O(n^{-1}), \quad i = 1, 2.$$

For the next proposition, we write $\boldsymbol{\nu} = (\nu_{11}, \nu_{22})^\top$ and $\hat{\boldsymbol{\nu}} = (\hat{\nu}_{11}, \hat{\nu}_{22})^\top$.

Proposition 4.2 (Mean square error matrix). *For the bivariate Matérn process \mathbf{X} with $0 < \nu_{11}, \nu_{22} < 1$, if $(\nu_{11} + \nu_{22})/2 = \nu_{12}$, then*

$$\mathbb{E}[(\hat{\boldsymbol{\nu}} - \boldsymbol{\nu})(\hat{\boldsymbol{\nu}} - \boldsymbol{\nu})^\top] = \begin{pmatrix} O(n^{-1}) & O(n^{-1}) \\ O(n^{-1}) & O(n^{-1}) \end{pmatrix};$$

if $(\nu_{11} + \nu_{22})/2 < \nu_{12}$, we have

$$\mathbb{E}[(\hat{\boldsymbol{\nu}} - \boldsymbol{\nu})(\hat{\boldsymbol{\nu}} - \boldsymbol{\nu})^\top] = \begin{pmatrix} O(n^{-1}) & o(n^{-1}) \\ o(n^{-1}) & O(n^{-1}) \end{pmatrix}.$$

Proposition 4.3 (Asymptotic distribution). *For the bivariate Matérn process \mathbf{X} with $0 < \nu_{11}, \nu_{22} < 1$,*

$$\sqrt{n} \begin{pmatrix} \hat{\nu}_{11} - \nu_{11} \\ \hat{\nu}_{22} - \nu_{22} \end{pmatrix} \xrightarrow{d} \mathcal{N} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \tilde{\mathbf{L}}_1^\top \boldsymbol{\Phi}_{0,11} \tilde{\mathbf{L}}_1 & \tilde{\mathbf{L}}_1^\top \boldsymbol{\Phi}_{0,12} \tilde{\mathbf{L}}_2 \\ \tilde{\mathbf{L}}_2^\top \boldsymbol{\Phi}_{0,21} \tilde{\mathbf{L}}_1 & \tilde{\mathbf{L}}_2^\top \boldsymbol{\Phi}_{0,22} \tilde{\mathbf{L}}_2 \end{pmatrix} \right).$$

Specifically, if $(\nu_{11} + \nu_{22})/2 < \nu_{12}$, $\sqrt{n}\hat{\nu}_{11}$ and $\sqrt{n}\hat{\nu}_{22}$ are asymptotically independent.

5. Simulation Study

In this section, we simulate data from a nonsmooth bivariate Matérn process and illustrate that when $(\nu_{11} + \nu_{22})/2 = \nu_{12}$, the decay rates of the bias and mean square error matrix for $\hat{\nu}_{11}$ and $\hat{\nu}_{22}$ are n^{-1} . Then, we compare with the case when $(\nu_{11} + \nu_{22})/2 < \nu_{12}$.

We take $\nu_{11} = 0.2$, $\nu_{22} = 0.7$, $\nu_{12} = 0.45$, $\rho = 0.5$, $\sigma_1^2 = \sigma_2^2 = 1$ and $a_{11} = a_{22} = a_{12} = 1$. We simulated the corresponding bivariate Matérn process on regular grids within the interval $[0, 1]$, where the length of the grid was set to $1/n$ with $n = 200, 210, 220, \dots, 1000$. For each n , we used generalized least squares (*abbr.* GLS) to obtain the estimators of the fractal indices, say $(\hat{\nu}_{11}, \hat{\nu}_{22})$ (see, e.g., [25]). Here, we fixed the number of dilations to $m = 50$. The weight matrix $\Omega_i = (\omega_i^{uv})_{u,v=1}^m$ of the GLS estimator with a Matérn covariance function is given by

$$\omega_i^{uv} = \frac{2}{(n - 2u + 1)(n - 2v + 1)} \frac{\sum_{h=u}^{n-u} \sum_{\ell=v}^{n-v} (\sum_{j,k=-1}^1 a_j a_k |h - \ell + kv - ju|^{2\nu_{ii}})^2}{(\sum_{j,k=-1}^1 a_j a_k |k - j|^{2\nu_{ii}})^2 u^{2\nu_{ii}} v^{2\nu_{ii}}},$$

which can be approximated by plugging in the ordinary least squares estimators of ν_{ii} , $i = 1, 2$. To evaluate the efficiency of the estimators, we repeated the above procedure 1000 times independently.

The 95% confidence intervals for $(\nu_{11}, \nu_{22})^\top$ with varying n are shown in FIG 1 (a). FIG 1 (c) and (e) show how the bias, marginal variances and cross covariance decrease when n increases from 200 to 1000. By fitting the natural logarithm of the absolute value of the bias, marginal variances and absolute

values of the cross covariance with respect to $\ln n$, we find the power of the decay rate for each is very close to -1 . This is consistent with the conclusions in Proposition 4.1 and Proposition 4.2 when $(\nu_{11} + \nu_{22})/2 = \nu_{12}$.

Further, we show how the decay rate changes if $(\nu_{11} + \nu_{22})/2 < \nu_{12}$. Fixing all previously assigned parameters but setting ν_{12} to 0.6, we rerun the simulation and repeat the estimation procedures. The results are shown on the right side of FIG 1, where we can see that the results are mostly the same, but the cross covariance decays much faster than n^{-1} . Indeed, the power of the decay rate is approximately -1.5 , which is consistent with the conclusion in Proposition 4.2.

6. Proof of the main results

To prove Theorems 3.3 ~ 3.5, we make use of the following key lemma.

Lemma 6.1. *For $u = 1, \dots, m$ and $i = 1, 2$, let $T_{n,i}^u = (\bar{Z}_{n,i}^u - \mathbb{E}\bar{Z}_{n,i}^u)/\mathbb{E}\bar{Z}_{n,i}^u$. Then, for any $k \in \mathbb{Z}^+$, there exist positive and finite constants C_3 and C_4 (which may depend on u and k) such that for all $n \geq 1$ and $\xi > 0$,*

$$\mathbb{E}[|\ln(1 + T_{n,i}^u)|^k; |T_{n,i}^u| > \xi] \leq C_3 e^{-C_4 \xi \sqrt{n}}.$$

The proof of Lemma 6.1 is given at the end of this section. Now, we proceed to prove our main theorems.

Proof of Theorem 3.3. Recall that $T_{n,i}^u = (\bar{Z}_{n,i}^u - \mathbb{E}\bar{Z}_{n,i}^u)/\mathbb{E}\bar{Z}_{n,i}^u$. Then,

$$\hat{\alpha}_{ii} = \sum_{u=1}^m L_{u,i} \ln(1 + T_{n,i}^u) + \sum_{u=1}^m L_{u,i} \ln \mathbb{E}\bar{Z}_{n,i}^u. \quad (6.1)$$

It follows from (3.8) that

$$\mathbb{E}\bar{Z}_{n,i}^u = \sigma_{n,ii}^{uu}(0) = C_i u^{\alpha_{ii}} (1 + O(n^{-\beta_{ii}})),$$

Hence, using the conditions on $L_{u,i}$ in (2.3), we conclude

$$\sum_{u=1}^m L_{u,i} \ln \mathbb{E}\bar{Z}_{n,i}^u = \alpha_{ii} + O(n^{-\beta_{ii}}). \quad (6.2)$$

Next, we estimate the first sum in (6.1). By Taylor's expansion, we obtain

$$\ln(1 + T_{n,i}^u) = T_{n,i}^u - \frac{1}{2}(T_{n,i}^u)^2 + R_{n,i}^u,$$

where $R_{n,i}^u$ is the residual term. Hence,

$$\mathbb{E}[\ln(1 + T_{n,i}^u)] = \mathbb{E}\left[T_{n,i}^u - \frac{1}{2}(T_{n,i}^u)^2 + R_{n,i}^u\right] = -\frac{1}{2}\mathbb{E}(T_{n,i}^u)^2 + \mathbb{E}R_{n,i}^u.$$

By Theorem 3.1, it is easy to verify that

$$\mathbb{E}(T_{n,i}^u)^2 = O(n^{-1}). \quad (6.3)$$

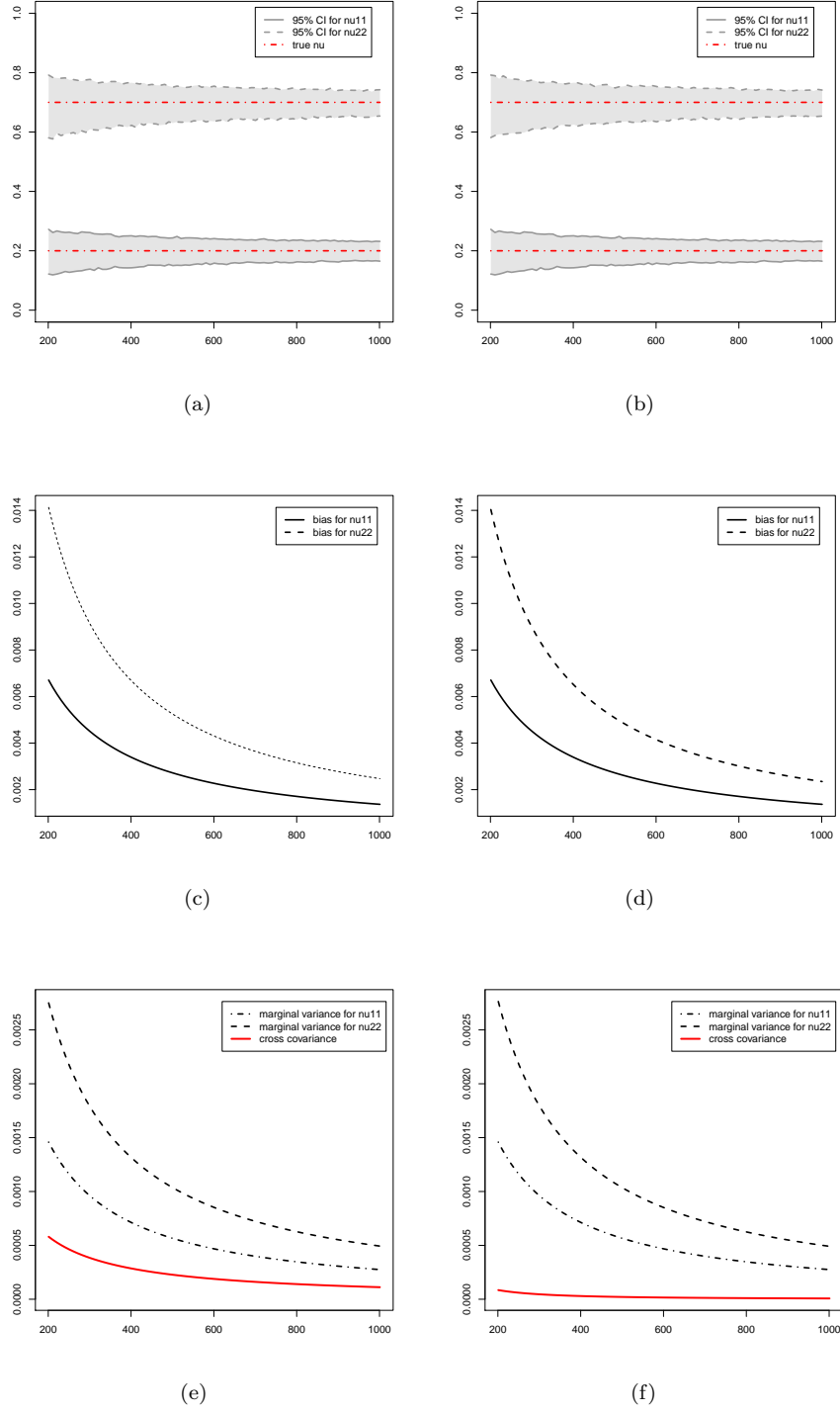


FIG 1. Confidence intervals, absolute value of the bias, marginal variances and absolute value of the cross covariance for $(\hat{\nu}_{11}, \hat{\nu}_{22})$ with varying n . The plots on the left side (i.e., a, c, e) show the results for $(\nu_{11} + \nu_{22})/2 = \nu_{12}$, whereas those on the right side (i.e., b, d, f) correspond to the situation where $(\nu_{11} + \nu_{22})/2 < \nu_{12}$.

Using the fact that if $|T_{n,i}^u| \leq \xi \leq 1/2$, then $|R_{n,i}^u| \leq \xi(T_{n,i}^u)^2$, we have

$$\mathbb{E}[|R_{n,i}^u|; |T_{n,i}^u| \leq \xi] \leq \xi \mathbb{E}[(T_{n,i}^u)^2; |T_{n,i}^u| \leq \xi] = O(n^{-1}), \quad (6.4)$$

where the last equality follows from (6.3). On the other hand, by applying Lemma 6.1, the Cauchy-Schwarz inequality and (6.3), we obtain

$$\begin{aligned} & \mathbb{E}[|R_{n,i}^u|; |T_{n,i}^u| > \xi] \\ & \leq \mathbb{E}[|\ln(1 + T_{n,i}^u)| + |T_{n,i}^u| + \frac{1}{2}(T_{n,i}^u)^2; |T_{n,i}^u| > \xi] = O(n^{-1}). \end{aligned} \quad (6.5)$$

By combining (6.3), (6.4) and (6.5), we obtain

$$\mathbb{E}[\ln(1 + T_{n,i}^u)] = O(n^{-1}). \quad (6.6)$$

This, together with (6.1) and (6.2), proves Theorem 3.3. \square

Proof of Theorem 3.4. For $i = 1, 2$, we expand $\mathbb{E}[(\hat{\alpha}_{ii} - \alpha_{ii})^2]$ as follows.

$$\begin{aligned} \mathbb{E}[(\hat{\alpha}_{ii} - \alpha_{ii})^2] &= \sum_{u=1}^m \sum_{v=1}^m L_{u,i} L_{v,i} \mathbb{E} \left[\left(\ln(1 + T_{n,i}^u) + \ln \mathbb{E} \bar{Z}_{n,i}^u - \alpha_{ii} \ln u \right) \right. \\ & \quad \left. \times \left(\ln(1 + T_{n,i}^v) + \ln \mathbb{E} \bar{Z}_{n,i}^v - \alpha_{ii} \ln v \right) \right] \\ &= \sum_{u=1}^m \sum_{v=1}^m L_{u,i} L_{v,i} \mathbb{E} \left[\ln(1 + T_{n,i}^u) \ln(1 + T_{n,i}^v) \right] \\ & \quad + \sum_{u=1}^m \sum_{v=1}^m L_{u,i} L_{v,i} \mathbb{E} \left[\ln(1 + T_{n,i}^u) \right] \left(\ln \mathbb{E} \bar{Z}_{n,i}^v - \alpha_{ii} \ln v \right) \\ & \quad + \sum_{u=1}^m \sum_{v=1}^m L_{u,i} L_{v,i} \left(\ln \mathbb{E} \bar{Z}_{n,i}^u - \alpha_{ii} \ln u \right) \mathbb{E} \left[\ln(1 + T_{n,i}^v) \right] \\ & \quad + \sum_{u=1}^m \sum_{v=1}^m L_{u,i} L_{v,i} \left(\ln \mathbb{E} \bar{Z}_{n,i}^u - \alpha_{ii} \ln u \right) \left(\ln \mathbb{E} \bar{Z}_{n,i}^v - \alpha_{ii} \ln v \right) \\ & \triangleq \text{I} + \text{II} + \text{III} + \text{IV}. \end{aligned} \quad (6.7)$$

By (6.2) and (6.6), we have

$$\text{II} = O(n^{-1-\beta_{ii}}), \quad \text{III} = O(n^{-1-\beta_{ii}}), \quad \text{IV} = O(n^{-2\beta_{ii}}). \quad (6.8)$$

To bound the first term I in (6.7), we take $\xi = 1/2$ and decompose the probability space into the union of the following four disjoint events, $\{|T_{n,i}^u| \leq \xi, |T_{n,i}^v| \leq \xi\}$, $\{|T_{n,i}^u| > \xi, |T_{n,i}^v| \leq \xi\}$, $\{|T_{n,i}^u| \leq \xi, |T_{n,i}^v| > \xi\}$, and $\{|T_{n,i}^u| > \xi, |T_{n,i}^v| > \xi\}$.

- i). For the event $\{|T_{n,i}^u| \leq \xi, |T_{n,i}^v| \leq \xi\}$, we use the elementary inequality $|\ln(1+x)| \leq 2|x|$ for all $|x| \leq \xi$ to derive

$$|\ln(1 + T_{n,i}^u) \ln(1 + T_{n,i}^v)| \leq 4|T_{n,i}^u| |T_{n,i}^v|.$$

It follows from the Cauchy-Schwarz inequality and Theorem 3.1 that

$$\mathbb{E}[\ln(1 + T_{n,i}^u) \ln(1 + T_{n,i}^v); |T_{n,i}^u| \leq \xi, |T_{n,i}^v| \leq \xi] = O(n^{-1}).$$

ii). By Lemma 6.1, we have

$$\begin{aligned} & \mathbf{E} \left[\left| \ln(1 + T_{n,i}^u) \ln(1 + T_{n,i}^v) \right|; |T_{n,i}^u| > \xi, |T_{n,i}^v| \leq \xi \right] \\ & \leq \ln 2 \mathbf{E} \left[\left| \ln(1 + T_{n,i}^u) \right|; |T_{n,i}^u| > \xi \right] \\ & = o(n^{-1}). \end{aligned}$$

iii). As in ii), we have

$$\begin{aligned} & \mathbf{E} \left[\left| \ln(1 + T_{n,i}^u) \ln(1 + T_{n,i}^v) \right|; |T_{n,i}^u| \leq \xi, |T_{n,i}^v| > \xi \right] \\ & \leq \ln 2 \mathbf{E} \left[\left| \ln(1 + T_{n,i}^v) \right|; |T_{n,i}^v| > \xi \right] \\ & = o(n^{-1}). \end{aligned}$$

iv). By Lemma 6.1 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \mathbf{E} \left[\left| \ln(1 + T_{n,i}^u) \ln(1 + T_{n,i}^v) \right|; |T_{n,i}^u| > \xi, |T_{n,i}^v| > \xi \right] \\ & \leq \sqrt{\mathbf{E} \left[\ln^2(1 + T_{n,i}^u); |T_{n,i}^u| > \xi \right]} \sqrt{\mathbf{E} \left[\ln^2(1 + T_{n,i}^v); |T_{n,i}^v| > \xi \right]} \\ & = o(n^{-1}). \end{aligned}$$

By combining i)-(iv) above, we see that

$$\mathbf{I} = O(n^{-1}). \quad (6.9)$$

By (6.8) and (6.9), we have

$$\mathbf{E} \left[(\hat{\alpha}_{ii} - \alpha_{ii})^2 \right] = O(n^{-1}) + O(n^{-2\beta_{ii}}).$$

Next, we study the cross term $\mathbf{E} \left[(\hat{\alpha}_{11} - \alpha_{11})(\hat{\alpha}_{22} - \alpha_{22}) \right]$, which can be written as

$$\begin{aligned} & \mathbf{E} \left[(\hat{\alpha}_{11} - \alpha_{11})(\hat{\alpha}_{22} - \alpha_{22}) \right] \\ & = \sum_{u=1}^m \sum_{v=1}^m L_{u,1} L_{v,2} \mathbf{E} \left[\ln(1 + T_{n,1}^u) \ln(1 + T_{n,2}^v) \right] \\ & \quad + \sum_{u=1}^m \sum_{v=1}^m L_{u,1} L_{v,2} \mathbf{E} \left[\ln(1 + T_{n,1}^u) \right] (\ln \mathbf{E} \bar{Z}_{n,2}^v - \alpha_{22} \ln v) \\ & \quad + \sum_{u=1}^m \sum_{v=1}^m L_{u,1} L_{v,2} (\ln \mathbf{E} \bar{Z}_{n,1}^u - \alpha_{11} \ln u) \mathbf{E} \left[\ln(1 + T_{n,2}^v) \right] \\ & \quad + \sum_{u=1}^m \sum_{v=1}^m L_{u,1} L_{v,2} (\ln \mathbf{E} \bar{Z}_{n,1}^u - \alpha_{11} \ln u) (\ln \mathbf{E} \bar{Z}_{n,2}^v - \alpha_{22} \ln v) \\ & \triangleq \text{I} + \text{II} + \text{III} + \text{IV}. \end{aligned} \quad (6.10)$$

Applying similar arguments as used in evaluating $\mathbf{E} \left[(\hat{\alpha}_{ii} - \alpha_{ii})^2 \right]$, we obtain

$$\text{II} = O(n^{-1-\beta_{22}}), \quad \text{III} = O(n^{-1-\beta_{11}}), \quad \text{IV} = O(n^{-\beta_{11}-\beta_{22}}). \quad (6.11)$$

In order to bound the term I, we distinguish two cases.

- 1). If $(\alpha_{11} + \alpha_{22})/2 = \alpha_{12}$, then by a similar argument as that for proving (6.9) (using Lemma 6.1 and Theorem 3.1) and the fact that $\Phi_{0,12} \neq \mathbf{0}$, we have

$$\sum_{u=1}^m \sum_{v=1}^m L_{u,1} L_{v,2} \mathbb{E}[\ln(1 + T_{n,1}^u) \ln(1 + T_{n,2}^v)] = O(n^{-1}).$$

This, together with (6.10) and (6.11), implies

$$\begin{aligned} & \mathbb{E}[(\hat{\alpha}_{11} - \alpha_{11})(\hat{\alpha}_{22} - \alpha_{22})] \\ &= O(n^{-1}) + O(n^{-1-\beta_{11}}) + O(n^{-1-\beta_{22}}) + O(n^{-\beta_{11}-\beta_{22}}). \end{aligned}$$

- 2). If $(\alpha_{11} + \alpha_{22})/2 < \alpha_{12}$, then by an argument similar to that for proving (6.9) and the fact that $\Phi_{0,12} = \mathbf{0}$, we obtain

$$\sum_{u=1}^m \sum_{v=1}^m L_{u,1} L_{v,2} \mathbb{E}[\ln(1 + T_{n,1}^u) \ln(1 + T_{n,2}^v)] = o(n^{-1}).$$

Consequently,

$$\begin{aligned} & \mathbb{E}[(\hat{\alpha}_{11} - \alpha_{11})(\hat{\alpha}_{22} - \alpha_{22})] \\ &= o(n^{-1}) + O(n^{-1-\beta_{11}}) + O(n^{-1-\beta_{22}}) + O(n^{-\beta_{11}-\beta_{22}}). \end{aligned}$$

Therefore, we have proved (3.6) and (3.7). \square

Proof of Theorem 3.5. Recall (3.8) for $\mathbb{E}\bar{Z}_{n,i}^u$ and denote

$$\boldsymbol{\tau} = (\tau_{1,1}, \dots, \tau_{m,1}, \tau_{1,2}, \dots, \tau_{m,2})^\top.$$

Since $\beta_{ii} > 1/2$ for $i = 1, 2$, we have $\sqrt{n}(\mathbb{E}\bar{\mathbf{Z}}_n - \boldsymbol{\tau}) \rightarrow 0$ as $n \rightarrow \infty$. By Theorem 3.2 and Slutsky's theorem, we conclude that

$$n^{1/2}(\bar{\mathbf{Z}}_n - \boldsymbol{\tau}) \xrightarrow{d} \mathcal{N}_{2m}(\mathbf{0}, \Phi_0), \quad \text{as } n \rightarrow \infty.$$

Define a mapping $\mathbf{f} : \mathbb{R}^{2m} \rightarrow \mathbb{R}^2$, $\forall \mathbf{x} \triangleq (x_{1,1}, \dots, x_{m,1}, x_{1,2}, \dots, x_{m,2}) \in \mathbb{R}^{2m}$,

$$\mathbf{f}(\mathbf{x}) := \left(\sum_{u=1}^m L_{u,1} \ln x_{u,1}, \sum_{u=1}^m L_{u,2} \ln x_{u,2} \right)^\top.$$

Hence, it is easy to verify that $\mathbf{f}(\cdot)$ is continuously differentiable, $\hat{\boldsymbol{\alpha}} = \mathbf{f}(\bar{\mathbf{Z}}_n)$ and $\boldsymbol{\alpha} = \mathbf{f}(\boldsymbol{\tau})$. By applying the multivariate delta method [27, Theorem 8.22], we conclude that

$$\sqrt{n}(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}) \xrightarrow{d} \mathcal{N}(\mathbf{0}, \nabla \mathbf{f}(\boldsymbol{\tau})^\top \Phi_0 \nabla \mathbf{f}(\boldsymbol{\tau})),$$

where $\nabla \mathbf{f}(\boldsymbol{\tau}) = (\tilde{\mathbf{L}}_1^\top, \tilde{\mathbf{L}}_2^\top)^\top$. \square

Proof of Lemma 6.1. By Hölder's inequality, we have

$$\begin{aligned} & \mathbb{E}[|\ln(1 + T_{n,i}^u)|^k; |T_{n,i}^u| > \xi] \\ & \leq \sqrt{\mathbb{E}[\ln^{2k}(1 + T_{n,i}^u)]} \sqrt{\Pr(|T_{n,i}^u| > \xi)}. \end{aligned} \quad (6.12)$$

First, we establish an upper bound for $\Pr(|T_{n,i}^u| > \xi)$. By (3.1), we see that the covariance matrix of the random vector $\mathbf{Y}_{n,i} = (Y_{n,i}^u(1), \dots, Y_{n,i}^u(n))^\top$ is $\boldsymbol{\Sigma}_{n,i} = (\sigma_{n,ii}^{uu}(j-k))_{j,k=1}^n$. Let $\boldsymbol{\Lambda}_{n,i} = \text{diag}(\lambda_{j,i})_{j=1}^n$ be the diagonal matrix whose diagonal entries are the eigenvalues of $\boldsymbol{\Sigma}_{n,i}$, and let $\mathbf{U} = (U_1, \dots, U_n)^\top$, where $U_j \stackrel{iid}{\sim} \mathcal{N}(0, 1)$, $j = 1, \dots, n$. Then, we have

$$\bar{Z}_{n,i}^u = \frac{1}{n} \mathbf{Y}_{n,i}^\top \mathbf{Y}_{n,i} \stackrel{d}{=} \frac{1}{n} \mathbf{U}^\top \boldsymbol{\Lambda}_{n,i} \mathbf{U}.$$

Since $n\mathbb{E}\bar{Z}_{n,i}^u = \mathbb{E}(\mathbf{U}^\top \boldsymbol{\Lambda}_{n,i} \mathbf{U}) = \text{trace}(\boldsymbol{\Lambda}_{n,i})$, we apply the Hanson and Wright inequality [22] to the tail probability of the quadratic forms to obtain

$$\begin{aligned} & \Pr(|T_{n,i}^u| > \xi) = \Pr(|\mathbf{U}^\top \boldsymbol{\Lambda}_{n,i} \mathbf{U} - \text{trace}(\boldsymbol{\Lambda}_{n,i})| > \text{trace}(\boldsymbol{\Lambda}_{n,i})\xi) \\ & \leq \exp \left\{ -\min \left(C_5 \xi \frac{\text{trace}(\boldsymbol{\Lambda}_{n,i})}{\|\boldsymbol{\Lambda}_{n,i}\|_2}, C_6 \xi^2 \frac{(\text{trace}(\boldsymbol{\Lambda}_{n,i}))^2}{\|\boldsymbol{\Lambda}_{n,i}\|_F^2} \right) \right\}, \end{aligned} \quad (6.13)$$

where $\|\boldsymbol{\Lambda}_{n,i}\|_2$ and $\|\boldsymbol{\Lambda}_{n,i}\|_F$ are the ℓ_2 norm and Frobenius norm of $\boldsymbol{\Lambda}_{n,i}$, respectively, and C_5, C_6 are positive constants independent of $\boldsymbol{\Lambda}_{n,i}$, n and ξ .

Note that

$$\|\boldsymbol{\Lambda}_{n,i}\|_F^2 = \text{trace}(\boldsymbol{\Lambda}_{n,i}^2) = \text{trace}(\boldsymbol{\Sigma}_n^2) = \sum_{j=1, k=1}^n (\sigma_{n,ii}^{uu}(j-k))^2,$$

and

$$\phi_{n,ii}^{uu} = \text{Var}(\bar{Z}_{n,ii}^u) = \frac{2}{n^2} \sum_{j=1, k=1}^n (\sigma_{n,ii}^{uu}(j-k))^2.$$

By Theorem 3.1, we have

$$\|\boldsymbol{\Lambda}_{n,i}\|_F^2 = \frac{n^2}{2} \phi_{n,ii}^{uu} \asymp n \phi_{0,ii}^{uu}.$$

By combining the above with the facts that $\|\boldsymbol{\Lambda}_{n,i}\|_2 \leq \|\boldsymbol{\Lambda}_{n,i}\|_F$ and $\text{trace}(\boldsymbol{\Lambda}_{n,i})/n = \mathbb{E}\bar{Z}_{n,i}^u \rightarrow C_i u^{\alpha_{ii}}$ as $n \rightarrow \infty$, we have

$$\frac{\text{trace}(\boldsymbol{\Lambda}_{n,i})}{\|\boldsymbol{\Lambda}_{n,i}\|_2} = \sqrt{n} \frac{\text{trace}(\boldsymbol{\Lambda}_{n,i})/n}{\|\boldsymbol{\Lambda}_{n,i}\|_2/\sqrt{n}} \gtrsim u^{\alpha_{ii}} (\phi_{0,ii}^{uu})^{-1/2} \sqrt{n},$$

and

$$\frac{(\text{trace}(\boldsymbol{\Lambda}_{n,i}))^2}{\|\boldsymbol{\Lambda}_{n,i}\|_F^2} \asymp u^{2\alpha_{ii}} (\phi_{0,ii}^{uu})^{-1} n, \text{ as } n \rightarrow \infty.$$

Hence, when $n \rightarrow \infty$, (6.13) decays exponentially with rate \sqrt{n} . Consequently, when n is sufficiently large,

$$\Pr(|T_{n,i}^u| > \xi) \leq e^{-C_0 u^{\alpha_{ii}} (\phi_{0,ii}^{uu})^{-1/2} \xi \sqrt{n}}. \quad (6.14)$$

Next, we prove $E[\ln^{2k}(1 + T_{n,i}^u)]$ is bounded by $C_0 n$. It is easy to see that

$$E[\ln^{2k}(1 + T_{n,i}^u)] \leq 2^{2k-1} \left(E \ln^{2k} \bar{Z}_{n,i}^u + \ln^{2k}(E \bar{Z}_{n,i}^u) \right).$$

For any fixed $k \in \mathbb{Z}^+$, there exists $c_k > 1$ such that $\ln^{2k} x \leq x^2, \forall x > c_k$. Using the fact that $E \bar{Z}_{n,i}^u \rightarrow C_i u^{\alpha_{ii}}$ and $n \Phi_n \rightarrow \Phi_0$ as $n \rightarrow \infty$, we obtain that for all sufficiently large n ,

$$E(\ln^{2k} \bar{Z}_{n,i}^u; \bar{Z}_{n,i}^u > c_k) \leq E(\bar{Z}_{n,i}^u)^2 = (E \bar{Z}_{n,i}^u)^2 + \text{Var}(\bar{Z}_{n,i}^u) \asymp u^{2\alpha_{ii}}.$$

Therefore, the problem is reduced to proving $E(\ln^{2k} \bar{Z}_{n,i}^u; \bar{Z}_{n,i}^u \leq c_k) \leq C_0 n$. It is sufficient to show

$$E(\ln^{2k} \bar{Z}_{n,i}^u; \bar{Z}_{n,i}^u \leq 1) \leq C_0 n.$$

Let $U_{\min}^2 = \min_{1 \leq i \leq n} U_i^2$. Then,

$$\bar{Z}_{n,i}^u \stackrel{d}{=} \frac{1}{n} \sum_{j=1}^n \lambda_{j,i} U_j^2 \geq \frac{\text{trace}(\mathbf{\Lambda}_{n,i})}{n} U_{\min}^2 \geq \frac{1}{2} C_i u^{\alpha_{ii}} U_{\min}^2 \triangleq C U_{\min}^2. \quad (6.15)$$

where the second inequality holds for sufficiently large n .

Let $f_n(x)$ be the density function of U_{\min}^2 , that is $\forall x > 0$,

$$f_n(x) = \frac{n}{\sqrt{2\pi x}} e^{-x/2} \left(2 \int_{\sqrt{x}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \right)^{n-1}.$$

It is easy to verify that $f_n(x) \leq n/\sqrt{2\pi x}$. It follows from (6.15) that

$$\begin{aligned} E(\ln^{2k} \bar{Z}_{n,i}^u; \bar{Z}_{n,i}^u \leq 1) &\leq E[\ln^{2k}(C U_{\min}^2); U_{\min}^2 \leq 1/C] \\ &= \int_0^{\frac{1}{C}} \ln^{2k}(Cx) f_n(x) dx \leq \frac{n\sqrt{C}}{\sqrt{2\pi}} \int_0^1 y^{-\frac{1}{2}} \ln^{2k} y dy = C_0 n, \end{aligned}$$

for all sufficiently large n . Therefore, we have proven

$$E[\ln^{2k}(1 + T_{n,i}^u)] \leq C_0 n. \quad (6.16)$$

By (6.12), (6.14) and (6.16), we obtain that when n is large,

$$E[|\ln(1 + T_{n,i}^u)|^k; |T_{n,i}^u| > \xi] \leq C_7 n^{1/2} e^{-C_8 \xi \sqrt{n}} \leq C_7 e^{-C_9 \xi \sqrt{n}},$$

where C_7, C_8 and C_9 are independent of n and ξ and $C_9 < C_8$. □

7. Appendix

A. *Remark on Condition (A1).* Let F_{11} , F_{22} and F_{12} be the corresponding spectral measures of $C_{11}(\cdot)$, $C_{22}(\cdot)$ and $C_{12}(\cdot)$. By (1.2) and the Tauberian Theorem (see, e.g., [35]), we have that as $x \rightarrow \infty$,

$$F_{ij}(x, \infty) \sim C_{ij}(0) - C_{ij}(1/x) \sim \tilde{c}_{ij}|x|^{-\alpha_{ij}}, \quad i, j = 1, 2,$$

where $\tilde{c}_{ii} = c_{ii}$ for $i = 1, 2$ and $\tilde{c}_{12} = c_{12}\rho\sigma_1\sigma_2$.

According to Cramer's theorem ([9], [36], and [39] p.315), a necessary and sufficient condition for the matrix (1.1) to be a valid covariance function for $\mathbf{X}(t)$ is

$$(F_{12}(B))^2 \leq F_{11}(B)F_{22}(B), \quad \forall B \in \mathcal{B}(\mathbb{R}).$$

Hence, it is necessary to assume the following conditions on the parameters α_{ij} , c_{ij} , σ_i ($i = 1, 2$) and ρ :

$$\begin{aligned} \frac{\alpha_{11} + \alpha_{22}}{2} &< \alpha_{12}, \quad \text{or} \\ \frac{\alpha_{11} + \alpha_{22}}{2} &= \alpha_{12} \quad \text{and} \quad c_{12}^2\rho^2\sigma_1^2\sigma_2^2 \leq c_{11}c_{22}. \end{aligned} \quad (7.1)$$

This shows that (1.3) is only slightly stronger than (7.1) in the second case, which guarantees that the bivariate process \mathbf{X} is not degenerate and satisfies (7.2) below. \square

B. *Proof of (1.4).* In order to apply Theorem 2.1 in [37] to prove (1.4), it is sufficient to verify that there is a constant $c > 0$ such that

$$\det\text{Cov}(\mathbf{X}(s) - \mathbf{X}(t)) \geq c|s - t|^{\alpha_{11} + \alpha_{22}} \quad (7.2)$$

for all $s, t \in [0, 1]$ with sufficiently small $|s - t|$. Here, $\det\text{Cov}(\xi)$ denotes the determinant of the covariance matrix of the random vector ξ . Under Condition (A1), we see that for $i = 1, 2$,

$$\begin{aligned} \mathbb{E}[(X_i(s) - X_i(t))^2] &= 2C_{ii}(0) - 2C_{ii}(s - t) \sim 2c_{ii}|s - t|^{\alpha_{ii}}, \\ \mathbb{E}[(X_1(s) - X_1(t))(X_2(s) - X_2(t))] &= 2C_{12}(0) - 2C_{12}(s - t) \\ &\sim 2c_{12}\rho\sigma_1\sigma_2|s - t|^{\alpha_{12}} \end{aligned}$$

as $|s - t| \rightarrow 0$. Consequently,

$$\det\text{Cov}(\mathbf{X}(s) - \mathbf{X}(t)) \sim 4c_{11}c_{22}|s - t|^{\alpha_{11} + \alpha_{22}} - 4c_{12}^2\rho^2\sigma_1^2\sigma_2^2|s - t|^{2\alpha_{12}}.$$

This implies (7.2) and hence proves (1.4). \square

C. *Checking the condition (A2) for the bivariate Matérn process.* Without loss of generality, assume that $a = 1$ and $M_\nu(h) := M(h|\nu, 1)$. Denote by $\kappa_\nu = 2^{1-\nu}/\Gamma(\nu)$, which satisfies $\kappa_{\nu+1} = (2\nu)^{-1}\kappa_\nu$. Recall that the derivative of the

Bessel function of the second kind K_ν satisfies the following recurrence formula (see, e.g., [1], Section 9.6)

$$K'_\nu(z) = -K_{\nu+1}(z) + \frac{\nu}{z}K_\nu(z),$$

and for $\ell \in \mathbb{Z}^+ \cup \{0\}$, when $\ell < \nu < \ell + 1$, we have the following expansion for $M(\cdot)$

$$M_\nu(h) = \sum_{j=0}^{\ell} b_j h^{2j} - b|h|^{2\nu} + o(|h|^{2\ell+2}),$$

where b_0, \dots, b_ℓ are constants and $b = \Gamma(1 - \nu)/(2^{2\nu}\Gamma(1 + \nu))$ (see, e.g., [35], p. 32). Hence,

$$\begin{aligned} M'_\nu(h) &= \operatorname{sgn}(h)(\kappa_\nu \nu |h|^{\nu-1} K_\nu(|h|) + \kappa_\nu |h|^\nu K'_\nu(|h|)) \\ &= 2\nu \cdot \operatorname{sgn}(h) |h|^{-1} (M_\nu(h) - M_{\nu+1}(h)) \\ &= -2\nu b \cdot \operatorname{sgn}(h) |h|^{2\nu-1} + o(|h|^{2\nu-1}), \end{aligned}$$

where $\operatorname{sgn}(h)$ is the sign function. Similarly,

$$\begin{aligned} M''_\nu(h) &= (2\nu - 1) \operatorname{sgn}(h) |h|^{-1} M'_\nu(h) - 2\nu \cdot \operatorname{sgn}(h) |h|^{-1} M'_{\nu+1}(h) \\ &= -2\nu(2\nu - 1)b \cdot \operatorname{sgn}^2(h) |h|^{2\nu-2} + o(|h|^{2\nu-2}), \\ M_\nu^{(3)}(h) &= (2\nu - 2) \operatorname{sgn}(h) |h|^{-1} M''_\nu(h) - 2\nu \cdot \operatorname{sgn}(h) |h|^{-1} M''_{\nu+1}(h), \\ &= -2\nu(2\nu - 1)(2\nu - 2)b \cdot \operatorname{sgn}^3(h) |h|^{2\nu-3} + o(|h|^{2\nu-3}), \\ &\dots \\ M_\nu^{(q)}(h) &= (2\nu - q + 1) \operatorname{sgn}(h) |h|^{-1} M_\nu^{(q-1)}(h) - 2\nu \cdot \operatorname{sgn}(h) |h|^{-1} M_{\nu+1}^{(q-1)}(h) \\ &= -\frac{b(2\nu)!}{(2\nu - q)!} \operatorname{sgn}^q(h) |h|^{2\nu-q} + o(|h|^{2\nu-q}). \end{aligned}$$

When $q = 4$, the nonsmooth bivariate Matérn field \mathbf{X} satisfies the regularity condition **(A2)**. \square

D. Proof of Theorems 3.1 ~ 3.2. To prove Theorem 3.1 and Theorem 3.2, we make use of the following lemma.

Lemma 7.1. *If Conditions (A1) and (A2) hold, then as $|h| \rightarrow \infty$,*

$$\sigma_{n,ii}^{uv}(h) = O(|h|^{\alpha_{ii}-4}), \text{ uniformly for } n > |h|, i = 1, 2 \quad (7.3)$$

and

$$\sigma_{n,12}^{uv}(h) = n^{\frac{\alpha_{11} + \alpha_{22}}{2} - \alpha_{12}} O(|h|^{\alpha_{12}-4}) = O(|h|^{\frac{\alpha_{11} + \alpha_{22}}{2} - 4}) \quad (7.4)$$

uniformly for $n > |h|$.

We postpone the proof of Lemma 7.1 to the end of this section.

Proof of Theorem 3.1. Let

$$d_{n,ij}^{uv}(h) := \begin{cases} \left(1 - \frac{|h|}{n}\right) (\sigma_{n,ij}^{uv}(h))^2, & |h| < n \\ 0, & \text{otherwise.} \end{cases}$$

By (3.1) and (3.3), for any fixed h , we have $d_{n,ij}^{uv}(h) \rightarrow \sigma_{0,ij}^{uv}(h)$ as $n \rightarrow \infty$. By Lemma 7.1, we know

$$d_{n,ij}^{uv}(h) \leq C_0 |h|^{\alpha_{ii} + \alpha_{jj} - 8},$$

with the power $\alpha_{ii} + \alpha_{jj} - 8 < -4$. Therefore, $\sum_{h=-\infty}^{\infty} d_{n,ij}^{uv}(h)$ is bounded by a summable series, and (3.4) can be concluded by the dominated convergence theorem. \square

Proof of Theorem 3.2. The argument in the following generalizes Kent and Wood [24]'s method to the bivariate case. According to the Cramér-Wold theorem, it is equivalent to prove that for $\forall \gamma = (\gamma_{1,1}, \dots, \gamma_{m,1}, \gamma_{1,2}, \dots, \gamma_{m,2})^\top \in \mathbb{R}^{2m}$,

$$n^{1/2} \gamma^\top (\bar{\mathbf{Z}}_n - \mathbb{E}[\bar{\mathbf{Z}}_n]) \xrightarrow{d} \mathcal{N}(0, \gamma^\top \Phi_0 \gamma), \text{ as } n \rightarrow \infty.$$

Let $\gamma_i := (\gamma_{1,i}, \dots, \gamma_{m,i})^\top, i = 1, 2$ and

$$\mathbf{\Gamma}_n = \text{diag}(\underbrace{\gamma_1^\top, \dots, \gamma_1^\top}_{n \text{ times}}, \underbrace{\gamma_2^\top, \dots, \gamma_2^\top}_{n \text{ times}})^\top.$$

Therefore, $\mathbf{\Gamma}_n$ is a $(2mn) \times (2mn)$ matrix including n copies of γ_1 and γ_2 on the diagonal. Let

$$\mathbf{Y}_{n,i}(j) := (Y_{n,i}^1(j), \dots, Y_{n,i}^m(j))^\top, i = 1, 2, j = 1, \dots, n,$$

and

$$\mathbf{W}_n = (\mathbf{Y}_{n,1}^\top(1), \dots, \mathbf{Y}_{n,1}^\top(n), \mathbf{Y}_{n,2}^\top(1), \dots, \mathbf{Y}_{n,2}^\top(n))^\top.$$

Therefore, \mathbf{W}_n is a $(2mn)$ -dimensional vector. Then, we have

$$S_n \triangleq n^{1/2} \gamma^\top (\bar{\mathbf{Z}}_n - \mathbb{E}[\bar{\mathbf{Z}}_n]) = n^{-1/2} (\mathbf{W}_n^\top \mathbf{\Gamma}_n \mathbf{W}_n - \mathbb{E}(\mathbf{W}_n^\top \mathbf{\Gamma}_n \mathbf{W}_n)).$$

Denote by $\mathbf{V}_n = \mathbb{E}(\mathbf{W}_n \mathbf{W}_n^\top)$ the covariance matrix of \mathbf{W}_n and by $\mathbf{V}_n^{1/2}$, the Cholesky factor of \mathbf{V}_n , i.e., the lower triangular matrix satisfying $\mathbf{V}_n = \mathbf{V}_n^{1/2} (\mathbf{V}_n^{1/2})^\top$. Denote by $\mathbf{\Lambda}_n = \text{diag}(\lambda_{n,j})_{j=1}^{2mn}$ the diagonal matrix whose diagonal entries are eigenvalues of $2n^{-1/2} (\mathbf{V}_n^{1/2})^\top \mathbf{\Gamma}_n \mathbf{V}_n^{1/2}$. Then, for a $(2mn)$ -dimensional vector $\boldsymbol{\epsilon}_n = (\epsilon_{1,n}, \dots, \epsilon_{2mn,n})^\top$ of i.i.d. standard normal random variables, we obtain

$$n^{-\frac{1}{2}} \mathbf{W}_n^\top \mathbf{\Gamma}_n \mathbf{W}_n \stackrel{d}{=} \boldsymbol{\epsilon}_n^\top (n^{-\frac{1}{2}} (\mathbf{V}_n^{\frac{1}{2}})^\top \mathbf{\Gamma}_n \mathbf{V}_n^{\frac{1}{2}}) \boldsymbol{\epsilon}_n \stackrel{d}{=} \frac{1}{2} \boldsymbol{\epsilon}_n^\top \mathbf{\Lambda}_n \boldsymbol{\epsilon}_n.$$

Therefore, for $\forall \theta < \min_{1 \leq j \leq 2mn} \lambda_{n,j}^{-1}$, the cumulant generating function S_n is given by

$$k_n(\theta) \triangleq \ln \mathbb{E} e^{\theta S_n} = -\frac{1}{2} \sum_{j=1}^{2mn} (\ln(1 - \theta \lambda_{n,j}) + \theta \lambda_{n,j})$$

To obtain the limit of $k_n(\theta)$ as $n \rightarrow \infty$, we first prove

$$\text{trace}(\Lambda_n^4) = \sum_{j=1}^{2mn} \lambda_{n,j}^4 \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (7.5)$$

For $1 \leq i_1, i_2 \leq 2, 1 \leq j_1, j_2 \leq n, 1 \leq k_1, k_2 \leq m$, let

$$\begin{aligned} \ell_1 &= (i_1 - 1)mn + (j_1 - 1)m + k_1, \\ \ell_2 &= (i_2 - 1)mn + (j_2 - 1)m + k_2. \end{aligned}$$

The (ℓ_1, ℓ_2) entry of \mathbf{W}_n is

$$\mathbf{V}_n(\ell_1, \ell_2) = \mathbb{E}[Y_{n,i_1}^{k_1}(j_1)Y_{n,i_2}^{k_2}(j_2)] = \sigma_{n,i_1 i_2}^{k_1 k_2}(j_2 - j_1).$$

Therefore,

$$\begin{aligned} \text{trace}(\Lambda_n^4) &= \frac{16}{n^2} \text{trace}((\mathbf{V}_n \Gamma_n)^4) \\ &= \frac{16}{n^2} \sum_{\ell_1, \dots, \ell_4=1}^{2mn} (\mathbf{V}_n \Gamma_n)(\ell_1, \ell_2)(\mathbf{V}_n \Gamma_n)(\ell_2, \ell_3)(\mathbf{V}_n \Gamma_n)(\ell_3, \ell_4)(\mathbf{V}_n \Gamma_n)(\ell_4, \ell_1) \\ &= \frac{16}{n^2} \sum_{i_1, \dots, i_4=1}^2 \sum_{k_1, \dots, k_4=1}^m \gamma_{k_1, i_1} \gamma_{k_2, i_2} \gamma_{k_3, i_3} \gamma_{k_4, i_4} \Delta_n(k_1, \dots, k_4, i_1, \dots, i_4), \quad (7.6) \end{aligned}$$

where

$$\begin{aligned} &\Delta_n(k_1, \dots, k_4, i_1, \dots, i_4) \\ &:= \sum_{j_1, \dots, j_4=1}^n \sigma_{n, i_1 i_2}^{k_1 k_2}(j_2 - j_1) \sigma_{n, i_2 i_3}^{k_2 k_3}(j_3 - j_2) \sigma_{n, i_3 i_4}^{k_3 k_4}(j_4 - j_3) \sigma_{n, i_4 i_1}^{k_4 k_1}(j_4 - j_1). \end{aligned}$$

Letting $h_i = j_{i+1} - j_i, i = 1, 2, 3$, we have

$$\begin{aligned} &\Delta_n(k_1, \dots, k_4, i_1, \dots, i_4) \\ &= \sum_{j_1, \dots, j_4=1}^n \sigma_{n, i_1 i_2}^{k_1 k_2}(h_1) \sigma_{n, i_2 i_3}^{k_2 k_3}(h_2) \sigma_{n, i_3 i_4}^{k_3 k_4}(h_3) \sigma_{n, i_4 i_1}^{k_4 k_1}(h_1 + h_2 + h_3). \end{aligned}$$

Given fixed h_1, h_2 and h_3 , the cardinality of the set

$$\#\{(j_1, \dots, j_4) \mid 1 \leq j_1, \dots, j_4 \leq n\} \leq n.$$

Hence,

$$|\Delta_n(k_1, \dots, k_4, i_1, \dots, i_4)| \leq n \sum_{|h_1|, |h_2|, |h_3| \leq n-1} |\sigma_{n, i_1 i_2}^{k_1 k_2}(h_1) \sigma_{n, i_2 i_3}^{k_2 k_3}(h_2) \sigma_{n, i_3 i_4}^{k_3 k_4}(h_3) \sigma_{n, i_4 i_1}^{k_4 k_1}(h_1 + h_2 + h_3)|$$

Further, by Lemma 7.1, we have

$$\begin{aligned} & |\Delta_n(k_1, \dots, k_4, i_1, \dots, i_4)| \\ & \leq C_0 n \prod_{r=1}^3 \sum_{h_r=-n+1}^{n-1} h_r^{\frac{\alpha_{i_r i_r}}{2} + \frac{\alpha_{i_{r+1} i_{r+1}}}{2} - 4} \\ & \leq C_0 n \prod_{r=1}^3 \sum_{h_r=-\infty}^{\infty} h_r^{\frac{\alpha_{i_r i_r}}{2} + \frac{\alpha_{i_{r+1} i_{r+1}}}{2} - 4} \\ & = O(n). \end{aligned} \tag{7.7}$$

The last equality holds since $\alpha_{i_r i_r}/2 + \alpha_{i_{r+1} i_{r+1}}/2 - 4 < -2$. By (7.6) and (7.7), we have

$$\text{trace}(\mathbf{\Lambda}_n^4) = O(n^{-1}) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Now, we are ready to prove the asymptotic normality of S_n . By applying Taylor's expansion to $\ln(1 - \theta \lambda_{n,j})$ at $\theta = 0$, we obtain

$$k_n(\theta) = \frac{\theta^2}{4} \sum_{j=1}^{2mn} \lambda_{n,j}^2 + \frac{\theta^3}{6} \sum_{j=1}^{2mn} \lambda_{n,j}^3 + \frac{\theta^4}{8} \sum_{j=1}^{2mn} (1 - \theta_{n,j} \lambda_{n,j})^{-4} \lambda_{n,j}^4,$$

where $\theta_{n,j}$ is between 0 and θ .

Let us first consider the term $\sum_{j=1}^{2mn} \lambda_{n,j}^2/2$. Since

$$\begin{aligned} & \frac{1}{2} \sum_{j=1}^{2mn} \lambda_{n,j}^2 = \frac{1}{2} \text{trace}(\mathbf{\Lambda}_n^2) = \frac{2}{n} \text{trace}((\mathbf{V}_n \mathbf{\Gamma}_n)^2) \\ & = \frac{2}{n} \sum_{i_1, i_2=1}^2 \sum_{k_1, k_2=1}^m \sum_{j_1, j_2=1}^n \gamma_{k_1, i_1} \gamma_{k_2, i_2} (\sigma_{n, i_1 i_2}^{k_1 k_2} (j_2 - j_1))^2, \end{aligned}$$

and

$$\begin{aligned} & \gamma^\top \mathbf{\Phi}_n \gamma = \sum_{i_1, i_2=1}^2 \sum_{k_1, k_2=1}^m \gamma_{k_1, i_1} \gamma_{k_2, i_2} \phi_{n, i_1 i_2}^{k_1 k_2} \\ & = \frac{2}{n^2} \sum_{i_1, i_2=1}^2 \sum_{k_1, k_2=1}^m \sum_{j_1, j_2=1}^n \gamma_{k_1, i_1} \gamma_{k_2, i_2} (\sigma_{n, i_1 i_2}^{k_1 k_2} (j_2 - j_1))^2, \end{aligned}$$

it follows from Theorem 3.1 that

$$\frac{1}{2} \sum_{j=1}^{2mn} \lambda_{n,j}^2 = \gamma^\top (n\Phi_n) \gamma \rightarrow \gamma^\top \Phi_0 \gamma, \text{ as } n \rightarrow \infty. \quad (7.8)$$

Secondly, by (7.5), we have

$$\max_{1 \leq j \leq 2mn} |\lambda_{n,j}| \leq \left(\sum_{j=1}^{2mn} \lambda_{n,j}^4 \right)^{\frac{1}{4}} \rightarrow 0, \text{ as } n \rightarrow \infty, \quad (7.9)$$

which implies

$$\left| \sum_{j=1}^{2mn} \lambda_{n,j}^3 \right| \leq \max_{1 \leq j \leq 2mn} |\lambda_{n,j}| \sum_{j=1}^{2mn} \lambda_{n,j}^2 \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (7.10)$$

Thirdly, note that $\delta := \sup_{n \geq 1} \max_{1 \leq j \leq 2mn} |\lambda_{n,j}|$ is positive and finite by (7.9). If we restrict attention to $|\theta| \leq (2\delta)^{-1}$, we have $(1 - \theta_{n,j} \lambda_{n,j})^{-4} \leq 16$; hence, for $\theta \in (-(2\delta)^{-1}, (2\delta)^{-1})$,

$$\sum_{j=1}^{2mn} (1 - \theta_{n,j} \lambda_{n,j})^{-4} \lambda_{n,j}^4 \rightarrow 0, \text{ as } n \rightarrow \infty. \quad (7.11)$$

Therefore, by (7.8), (7.10) and (7.11), for $\forall \theta \in (-(2\delta)^{-1}, (2\delta)^{-1})$, we have

$$k_n(\theta) \rightarrow \frac{\theta^2}{2} \gamma^\top \Phi_0 \gamma,$$

which leads to

$$S_n := n^{1/2} \gamma^\top (\bar{\mathbf{Z}}_n - \mathbf{E} \bar{\mathbf{Z}}_n) \xrightarrow{d} \mathcal{N}(0, \gamma^\top \Phi_0 \gamma), \text{ as } n \rightarrow \infty.$$

This proves Theorem 3.2. \square

Finally, we prove Lemma 7.1.

Proof of Lemma 7.1. (7.3) comes directly from the proof of Theorem 1 in Kent and Wood [25]. We only need to prove (7.4). To this end, we expand $C_{12}((h + kv - ju)/n)$ in a Taylor series about h/n to the fourth order to obtain

$$\begin{aligned} \sigma_{n,12}^{uv}(h) &= n^{\frac{\alpha_{11} + \alpha_{22}}{2}} \sum_{j,k=-1}^1 a_j a_k C_{12} \left(\frac{h + kv - ju}{n} \right) \\ &= n^{\frac{\alpha_{11} + \alpha_{22}}{2}} \sum_{r=0}^3 \sum_{j,k=-1}^1 a_j a_k \frac{(kv - ju)^r}{r! n^r} C_{12}^{(r)} \left(\frac{h}{n} \right) \\ &\quad + n^{\frac{\alpha_{11} + \alpha_{22}}{2}} \sum_{j,k=-1}^1 a_j a_k \frac{(kv - ju)^4}{4! n^4} C_{12}^{(4)} \left(\frac{h_{kj}^*}{n} \right) \end{aligned}$$

$$= n^{\frac{\alpha_{11} + \alpha_{22}}{2}} \sum_{j,k=-1}^1 a_j a_k \frac{(kv - ju)^4}{4!n^4} C_{12}^{(4)} \left(\frac{h_{kj}^*}{n} \right), \quad (7.12)$$

where h_{kj}^* lies between h and $h + kv - ju$. Since $|kv - ju| \leq u + v \leq 2m$, $h_{kj}^* \leq 2|h|$ for all $|h| \geq 2m$. By applying Condition (A2) to the last terms in (7.12), we derive that

$$|\sigma_{n,12}^{uv}(h)| \leq C_0 |h|^{\alpha_{12} - 4} \cdot n^{\frac{\alpha_{11} + \alpha_{22}}{2} - \alpha_{12}} \leq C_0 |h|^{\frac{\alpha_{11} + \alpha_{22}}{2} - 4}$$

for all $|h| \geq 2m$ and all $n > |h|$. This concludes the proof. \square

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