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High Cognitive Demand Examples in Precalculus: Examining the Work and Knowledge Entailed in Enactment

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HIGH COGNITIVE DEMAND EXAMPLES IN PRECALCULUS: EXAMINING THE WORK AND KNOWLEDGE ENTAILED IN ENACTMENT

by

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Historically, pass rates in undergraduate precalculus courses have been dismally low and the teaching practices and knowledge of university instructors have been understudied. To help improve teaching effectiveness and student outcomes in undergraduate precalculus courses, I have studied the cognitive demand of enacted examples. The purpose of this dissertation is to examine the pedagogical work and mathematical knowledge entailed in the enactment of high cognitive demand examples in a three-part study. To answer my research questions, I conducted classroom observations as well as pre- and post-observation interviews with seven graduate student instructors at a large public R1 university in the Midwest and used grounded theory to analyze my data. In the first component of the dissertation, I examine what high cognitive demand examples look like and identify three roles that instructors take on when enacting high cognitive demand examples: modeling, facilitating, and monitoring. In the second component, I decomposed the work of enacting high cognitive demand examples into five teaching tasks: attending to the mathematical point, making connections, providing clear verbal explanations, articulating cognitive processes, and supporting student understanding. Finally, in the third component, I examined the mathematical knowledge for teaching entailed in enacting examples and found that there are five domains of knowledge that support the maintenance of cognitive demand: knowledge of connections, representations, unpacking, students, and sequencing. These findings suggest ways in which we can help novice instructors enact high cognitive demand examples by focusing on the work and knowledge entailed in maintaining the cognitive demand.

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CHAPTER 1: INTRODUCTION

I was teaching college algebra for the first time as the primary instructor. Thankfully, my department offered lots of teaching support for graduate students. College algebra was a coordinated course, which meant that I was provided with lesson guides, online homework assignments that were self-graded, and exams written by the course convener. In addition, I was given a course release during my first semester (which means I taught only one section instead of two) in return for taking a class titled *Teaching and Learning Mathematics at the Post-Secondary Level* (which I will refer to as the pedagogy course). Each week we met to discuss educational research, the ways in which students learn, and the experiences we were having in our individual classrooms. Most of the graduate instructors teaching college algebra were in this course, so it also provided me with a community to discuss and share my teaching experiences with. Yet, with all of these supports, there was much I had to learn about teaching.

The college algebra students had just finished taking their second exam, which covered quadratic functions. In the pedagogy course that night, we discussed the exam. One problem in particular was stuck on my mind. The problem started off by giving a quadratic equation in standard form that modeled the height, as a function of time, of a bottle rocket that was being launched into the air. Part (a) asked students, “For how long is the rocket in the air?” Part (b) asked, “What is the maximum height the rocket reaches during its flight?” Finally, Part (c) asked, “How long has the rocket been in the air when it reaches its maximum height?” For each part, students were instructed to “show your work using algebraic methods; just typing the function into a calculator is not sufficient work.”

Thinking critically about this problem made me realize how complex it was. First, students have to recognize that to find how long the rocket is in the air, they need to figure out when the height is equal to zero. To do this algebraically, they next need to recognize that they should set the equation equal to zero and solve for t by factoring. Finally, they need to interpret which zero tells them how long they are in the air for. In addition, to find the maximum height and the time it takes to reach the maximum height, they need to realize those numbers are associated with the vertex. Next, they need to identify an algebraic method that will help them find the vertex. Students could use several methods, such as finding the midpoint between the zeros, using the completing the square algorithm, or equating coefficients in standard and vertex form. In class, I had focused on asking my students to just memorize the completing the square algorithm, but I had also mentioned that they could use the zeros and some students were familiar with the idea of equating coefficients. While all of these methods are valid, and completing the square was the method we preferred students know how to use, an astute student might realize that since they had already found the zeros in Part (a), the midpoint method was actually the most efficient.

Recognizing all of the complexity involved in this problem made me question the way in which I had taught the material. Yes, I had taught my students the completing the square algorithm, but had I focused too much on *memorizing* the steps and not enough on *comprehending* when it should be used? Since many of the problems that I used during class *explicitly* asked students to find the vertex of a quadratic, I had not given them the opportunity to learn how to interpret a problem as *implicitly* asking this same question. While I mentioned the midpoint method in passing, did I spend time helping them recognize when one method might be better than another? And how exactly do I teach my students to understand a procedure, recognize when it's appropriate to use, and select the procedure that best fits the task? I even wondered, "How did *I* learn to do those things?" and realized that I had no idea.

While this may be a personal anecdote, talking to other instructors had made me realize that it is not unique to my experience. Often, as instructors, what we teach is more complex than it first appears and is something that we mastered so long ago that we are divorced from the experience of learning it for the first time. In addition, learning a topic for oneself does not qualify one to teach it. Even advanced mathematics courses, which I had taken plenty of, did not prepare me for teaching what I previously considered a "simple" topic. So what else did I need to know and do to teach my students more effectively?

Defining the Problem

Traditionally, mathematics departments have operated under the assumption that earning a Ph.D. in mathematics and with experience teaching is what qualifies one to become a university professor (Committee on the Undergraduate Program in Mathematics, 1967). Yet many mathematics departments have struggled with low pass rates in first-year courses and failed to both attract and retain students in their degree programs (Bressoud, Mesa, & Rasmussen, 2015). While university mathematics professors are often considered content experts and may be provided with professional development opportunities concentrated on teaching, these efforts alone seem to not be enough. To help improve student success rates and teaching quality in first-year undergraduate mathematics courses, my dissertation focuses on identifying the knowledge and practices that help support high quality teaching in precalculus.

Mathematician and educator Hyman Bass pointed out that "knowing something for oneself or for communication to an expert colleague is not the same as knowing it for explanation to a student" (p. 19). Seemingly in contrast to this view, studies in the late twentieth-century found that content knowledge is not a predictor of teaching quality and student outcomes (Begle, 1972; Greenwald, Hedges, & Laine, 1996; Hanushek, 1981, 1996). In response to this finding, one could assume that perhaps the missing piece is pedagogical training. However, Lee Shulman proposed in 1986 that teachers should know more than just the content they are expected to teach and general pedagogical knowledge. Rather, Shulman identified the importance of *pedagogical content knowledge*, "which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching" (p. 9). The following year, Shulman called for researchers and practitioners to pay more attention to *professional* knowledge of teaching, including pedagogical content knowledge. Since then, researchers have found supporting evidence for Shulman's claim that there *is* content knowledge that matters for teaching (Baumert et al., 2010; Hill, Rowan, & Ball, 2005).

However, to date, the majority of the work on content knowledge for teaching mathematics has been conducted at the K-12 level. While universities have begun providing professional development on teaching (Ellis, 2015), there is still much that we need to learn about how to best prepare university professors for their responsibilities as teachers. Currently, professors have a strong grasp of general content knowledge, which is "the knowledge, understanding, skill, and disposition that are to be learned by [the students in their courses]" (Shulman, 1987, p. 9). However, there is still a need to better understand and identify content knowledge *for teaching* mathematics at the undergraduate level.

Current Status of the Field

Following the recommendation of Shulman (1987), educational researchers began looking into professional knowledge for teaching mathematics (e.g., Ball, Thames, & Phelps, 2008; Baumert & Kunter, 2013; McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012; Rowland, Huckstep, & Thwaites, 2005). Ball and Bass (2003) introduced the term mathematical knowledge for teaching (MKT), which Ball and her colleagues defined as the "mathematical knowledge 'entailed by teaching'—in other words, mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (Ball et al., 2008, p. 395) (p. 395).

Although previous researchers had found that content knowledge was not a predictor of teaching effectiveness (Begle, 1972; Greenwald et al., 1996; Hanushek, 1981, 1996), MKT researchers found that there *was* content knowledge that mattered for teaching and that a focus on this content benefited teaching and learning (Baumert et al., 2010; Hill et al., 2005). A natural question that arises given their findings is "How can content knowledge both matter and not matter in teaching?" The difference lies in the content knowledge being focused upon. Begle (1979) had shown that there was little relationship between student outcomes and the number of mathematics courses the teacher had taken past calculus. However, Hill, Rowan and Ball (2005) showed that elementary "teacher's content knowledge *for teaching mathematics* was a significant predictor of student gains" (emphasis added, p. 396).

To illustrate what I mean by mathematical knowledge *for teaching*, we can examine items that were developed by researchers to assess MKT. In their study of content knowledge for teaching at the elementary level, Hill et al. (2005) found that the task of appraising non-standard solution strategies to see if they are generalizable as mathematical knowledge that is specific to the work of teaching.

> *To respond to this situation, teachers must draw on mathematical knowledge: inspecting the steps shown in each example to determine what was done, gauging whether or not this constitutes a "method," and, if so, determining whether it makes sense and whether it works in general. Appraising nonstandard solution methods is not a common task for adults who do not teach. Yet, this task is entirely mathematical, not pedagogical; to make sound pedagogical decisions, teachers must be able to size up and evaluate the mathematics of these alternatives--often swiftly and on the spot. (p. 388)*

In another study of content knowledge for teaching at the secondary level, Baumert et al. (2010) found that the task of identifying parallelograms for which students might fail to apply the standard area formula as mathematical knowledge that is specific to the work of teaching. While this task requires general content knowledge (i.e., knowing the area of a parallelogram), it also requires mathematical knowledge that is specific to the work of teaching (i.e., knowing common student misconceptions or potential pitfalls). In both of these examples, the mathematical knowledge that is specific to the work of teaching is not *usually* taught in general undergraduate mathematics courses. Therefore, using the number of mathematics courses taken beyond calculus as a measure for content knowledge is not the same as measuring content knowledge for teaching.

Identifying the Gap

While research on MKT has been conducted at the K-12 level (Krauss, Baumert, & Blum, 2008; McCrory et al., 2012), there still are relatively few studies that focus on MKT at the undergraduate level. Speer, Smith, and Horvath (2010) conducted a literature review to search for empirical research on the practices of postsecondary teachers of mathematics. While some may argue that we can just use research conducted at the K-12 level to study postsecondary teaching, the authors pointed out that “there are important differences between college and pre-college teachers and teaching” (p. 100), such as level and depth of content and pedagogy knowledge. In another article, Speer, King, and Howell (2015) focused on the danger of assuming that research on MKT at the K-12 level can be extended to MKT at the postsecondary level. The authors claimed that relatively little attention has been paid to the ways in which existing frameworks and theories for MKT may or may not apply to teachers at the secondary and postsecondary level. (p. 106). Therefore, the purpose of my dissertation is to study MKT at the undergraduate level from the perspective of practice, instead of relying on exiting frameworks or theories.

However, studying MKT at the undergraduate level holistically would be beyond the scope of a dissertation project, so I chose to focus on the knowledge and work entailed in enacting high cognitive demand examples. While some research has been done on the knowledge and work entailed in enacting high cognitive demand tasks (Charalambous, 2010; Henningsen & Stein, 1997), these studies focus primarily on the mathematical tasks that *students* engage in during class, which I consider to be different from the examples that *instructors* choose to use during class. In particular, I conceptualize examples as a subset of tasks that are done in a whole-class setting for illustrative purposes. However, since research has identified that giving students opportunities to engage with high cognitive demand tasks is related to teaching quality (Stein, Remillard, & Smith, 2007), then it is reasonable to assume that there might be a similar relationship between teaching quality and the cognitive demand of examples.

Study Overview

The purpose of this dissertation is to investigate the pedagogical work and mathematical knowledge entailed in enacting high cognitive demand examples, which I will define later. While there are various ways one could go about researching pedagogical work and mathematical knowledge for teaching at the undergraduate level, I chose to study both aspects from the perspective of practice. Speer et al. (2015) called for researchers of undergraduate mathematics teaching to approach their work "through the same kinds of careful study of mathematical demands of teaching that sparked the early work on mathematical knowledge for teaching (Ball & Bass 2000b)" (p. 119). Ball and Bass (2000a) chose to study mathematical knowledge for teaching from the perspective of practice instead of looking at the content, curriculum, or standards. To this end, Ball et al. (2008) advocated for asking the following two questions: "What are the recurrent tasks and problems of teaching mathematics?" and "What mathematical knowledge, skills, and sensibilities are required to manage these tasks?" (p. 395). Instead of studying recurrent tasks at large, my study focuses on the work and knowledge entailed in enacting examples in the classroom.

To do this, I observed undergraduate precalculus courses, conducted video stimulated-recall interviews with instructors, and analyzed my data using the lens of cognitive demand. The participants I recruited for my study were experienced graduate student precalculus instructors at a large Midwestern university. I also conducted a pre-observation interview with the instructor to talk about the examples in their intended lesson plan. During the observation, I recorded the enacted examples and took detailed field notes. From the observation video recordings, I selected clips to use in the video stimulated-recall post-observation interviews with the teachers in order to better understand the pedagogical work and mathematical knowledge that was entailed in enacting high cognitive demand examples. Finally, I analyzed both the observations and the interview data in order to decompose the work and identify the mathematical knowledge entailed in enactment.

Importance

Traditionally, university mathematics professors have been trained as research mathematicians, but many of them spend their professional lives teaching (Bass, 1997). Yet, they "receive virtually no professional preparation or development as educators, apart from the role models of their mentors" (p. 19). While mentoring is better than no training, there is much that cannot be learned from mentoring alone. As Bass put it, "imagine learning to sing arias simply by attending operas, learning to cook by eating, learning to write by reading. Much of the art of teaching—the thinking, the dynamic observations and judgments of an accomplished teacher—is invisible to the outside observer" (p. 19).

Many universities are responding to the need to better train professors as teachers by providing graduate students (who make up the future work force) and current instructors with professional development focused on teaching. In order to provide effective teaching professional development, it is imperative to have a good understanding of what contributes to teaching quality. As I mentioned previously, part of effective teaching is knowing the content you are teaching, effective pedagogical techniques, and content that is specific to the work of teaching (i.e., MKT). My dissertation focuses on gaining a better understanding of this last piece, with particular emphasis on undergraduate precalculus courses.

So why is it important to study precalculus courses? With the emergence of technology, the demand for mathematically skilled workers has increased and placed a higher burden on mathematics departments to train a larger and more diverse pool of students (Bass, 1997). Approximately 2,000,000 college students take introductory level mathematics courses each year and drop, fail, withdraw (DFW) rates are typically around 50% (Gordon, 2006, p. 108). There are many factors that contribute to success rates, such as the effectiveness of placement exams, students' prior experiences with mathematics, and teaching quality, which is why I feel it is important to study MKT first-year undergraduate courses.

Focusing on examples will help improve undergraduate mathematics instruction for many reasons. First, explanations are a foundational aspect of teaching. Also, studies have shown that explanations can support student learning (Borko et al., 1992; Weiss & Pasley, 2004), improve metacognition (Leinhardt, 2001, 2010), and cultivate productive habits of mind (Schoenfeld, 2010). Furthermore, all mathematically literate people should be able to "use representations to model situations and communicate about mathematical ideas" (Thames & Ball, 2013, p. 2). Thus, using examples to explain and model content, practices, and strategies is important to undergraduate mathematics teaching at large.

Intended Audience

My intended audience is twofold: university mathematics department and mathematics education researchers. By learning the work and knowledge entailed in enacting high cognitive demand examples, mathematics departments can help their graduate students and current instructors improve their teaching quality and student success. Second, by carefully decomposing pedagogical work and examining MKT at the undergraduate level, mathematics education researchers can join me in thinking critically about how undergraduate teaching differs from elementary and secondary teaching.

Research Questions

The purpose of the multiple case study is to examine high cognitive demand examples that are enacted in precalculus classrooms. Here I define cognitive demand as the level and kind of thinking required in order to successfully engage with a mathematical task (Stein, Henningsen, Smith, & Silver, 2009, p. 11). The central research question that guides my dissertation study is: *What do instructors know and do that supports their ability to enact high cognitive demand examples?* To focus this question, I came up with the following subquestions that I will use to guide my study:

RQ1. What do high cognitive demand examples look like in precalculus courses?

RQ2. What are the different roles that instructors can take on when enacting high cognitive demand examples?

RQ3. What pedagogical work is entailed in enacting high cognitive demand examples and how does it relate to the role of the teacher?

RQ4. What mathematical knowledge is entailed in enacting high cognitive demand examples and how does it relate to the role of the teacher?

The first three chapters of my dissertation are designed to give a broad introduction to my study, an overview of the related literature, and a detailed description of the methods that I used. In Chapter 4: Examining the Role of the Instructor, I answer RQ1 and RQ2. In this chapter, I examine what high cognitive demand examples and the ways in which they are presented, which helps identify what high quality teaching might look like in undergraduate precalculus classrooms. Chapter 5: Decomposing the Pedagogical Work Entailed in Enacting High Cognitive Demand Examples answers RQ3 and identifying what instructors *do* in order to maintain high quality teaching. Next, in Chapter 6: Identifying the Mathematical Knowledge Entailed in Enacting High Cognitive Demand Examples", I answer RQ4 and look at the knowledge that instructors draw upon in order to maintain high quality teaching[1]. Finally, in Chapter 7 I tie the dissertation together into a single narrative and illustrate how my research might be used to help improve student outcomes and teaching quality in first-year undergraduate courses.

Assumptions

Since this project is qualitative and depends primarily on observational and interview data, there are two major assumptions that my research hinges on. First, in collecting observational data, I am assuming that the work and knowledge entailed in teaching is observable. This is a reasonable assumption to make, since other educational researchers have depended heavily on observational data in conducting their research (e.g., Ball et al., 2008; Heid, Wilson, & Blume, 2015; K. Jackson, Garrison, Wilson, Gibbons, & Shahan, 2013; McCrory et al., 2012; Sleep, 2012). Second, in conducting interviews with instructors, I am assuming that their responses are honest, truthful, and accurate. To encourage my participants to be honest and truthful, I had them chose pseudonyms to protect their identity and keep my data confidential. Since my post-observation interviews focus primarily on digging into the instructors' thinking, I need to be concerned with the accuracy of their recall. To aid in this, I used video-stimulated recall (Bloom, 1953), which is used to help reposition the interviewee back in the

[1] As a note are, Chapters 4-6 are designed to stand alone as publishable research articles, so each of them also includes a brief overview of the literature and methods.

moment as a way to tap into their in-the-moment thinking. Another large assumption that I am making is that MKT exists at the undergraduate level. While it has been pointed out that MKT may look very different at the undergraduate level in comparison to the K-12 level (N. M. Speer et al., 2015), it is reasonable to assume that MKT exists at the undergraduate level because the nature of teaching is still the same.

Limitations

Like all case studies, one limitation of my research is that it is not generalizable. While it may be suggestive of general findings, I would need to conduct additional research utilizing a broader sample of instructors and universities to verify this. However, case study is still an appropriate methodology to use because undergraduate teaching is not well understood. Therefore, any insight into what it may look like is beneficial. Also, as with most qualitative research studies, my results may not be replicable. While I used pseudonyms in order to protect the identity of my participants, it is possible that participants were not able to accurately share or describe their experiences. In a perfect circumstance, it would have been desirable to have participants from different universities with a variety of experiences and collect observational data from multiple perspectives, however time, money, accessibility, and human resources limited me. Finally, it is important to mention the possible limitations that may have resulted from researcher bias. While I am aiming to study undergraduate teaching from the perspective of practice, I am familiar with existing decomposition and MKT frameworks that were formed at the K-12 level, which may have colored my view. Also, as an instructor myself, I am much more familiar with the college algebra curriculum than the trigonometry curriculum (which together make up the precalculus courses), which may have influenced my data analysis.

Delimitations

As I stated previously, the purpose of this dissertation is to investigate the pedagogical work and mathematical knowledge entailed in enacting high cognitive demand examples in undergraduate precalculus classrooms. While there are many other problems I could have studied for this dissertation, I chose to study MKT at the undergraduate level because it has been understudied and can contribute towards improving instruction in undergraduate mathematics courses. I chose to focus on undergraduate precalculus courses because they impact a large number of students. In particular, they are often taken primarily by students who are non-STEM intending, which I believe is a population that we also need to focus on. While I could have studied undergraduate teaching in many different ways, I chose to examine it from the perspective of practice. For that reason, my data collection is rooted in observing and digging into examples that are enacted in the classroom. To narrow my scope further, I am also focusing on just examples that are enacted in the classroom, as opposed to studying undergraduate teaching at large.

Since undergraduate teaching is not well understood, my research questions are well suited to qualitative research. Since little research has focused on the cognitive demand of examples and the work and knowledge entailed in maintaining high levels of cognitive demand, the methodological perspective of collective case study is also well suited to my purpose. Also, I am utilizing the analytic frameworks of task unfolding and cognitive demand in order to analyze my data. I decided to use these frameworks primarily because they have been used in MKT research at the K-12 level. Finally, the inclusion criteria that I used when selecting participants is that they were instructors of precalculus courses and had taught precalculus courses for at least two semesters at the university prior to the beginning of the study.

Defining Key Terms

Cognitive demand

The level and kind of thinking required in order to successfully engage with a mathematical task (Stein et al., 2009, p. 11)

Enacted example

The actual implementation of an example in the classroom (Stein et al., 2007, p. 321)

Entailed

“A necessary accompaniment” (Thames, 2009, p. 173)

Example (mathematical)

A whole-class activity, the purpose of which is to solve a mathematical problem “for illustrative purposes” (Good, 1959, p. 211)

Example unfolding

The temporal phases an example goes through as it is transformed from the written example to the intended example to the enacted example (Stein et al., 2007, p. 321)

High cognitive demand tasks

“Involve making connections, analyzing information, and drawing conclusions” (Van de Walle, Karp, & Bay-Williams, 2013, p. 36)

Intended example

The teachers’ plans for using the example during classroom instruction (Stein et al., 2007, p. 321)

Knowledge

“(1) The accumulated facts, truths, principles, and information to which the human mind has access; (2) the outcome of specified, rigorous inquiry which originated within the framework of human experience and functions in human experience” (Good, 1959, p. 308)

Lesson

“A short period of instruction devoted to a specific limited topic, skill, or idea” (Good, 1959, p. 316)

Lesson guide

In this study, the written lesson guides were developed internally by members of the mathematics department and provided instructors with suggested sequencing, examples, and timing for each class period.

Lesson plan

The instructors intended plan for instruction; “a detailed plan, usually drawn up by the teacher, encapsulating the content and sequence of the lesson” (Wallace, 2008, p. 162)

Low cognitive demand task

"Involve stating facts, following known procedures (computation), and solving routine problems" (Van de Walle et al., 2013, p. 36)

Mathematical knowledge for teaching (MKT)

The "mathematical knowledge 'entailed by teaching'—in other words, mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (Ball et al., 2008, p. 395)

Task (mathematical)

"A classroom activity, the purpose of which is to focus students' attention on a particular mathematical idea" (Stein, Grover, & Henningsen, 1996, p. 460)

Teaching

"The act of providing activities, materials, and guidance that facilitate learning, in either formal or informal situations" (Good, 1959, p. 552)

Work of teaching

"The core tasks that teachers must execute to help pupils learn" (Ball & Forzani, 2009, p. 497)

Written example

The example as "represented in curriculum materials or other teaching resources" (Stein et al., 2007, p. 340)

Chapter 2: Literature Review

This chapter provides a broad overview of the literature related to my dissertation study In particular, I reviewed the literature on cognitive demand, instructional examples, decompositions of teaching, and mathematical knowledge for teaching.

Cognitive Demand of Mathematical Tasks

Since the purpose of my study is to examine what teachers do and know that helps maintain the cognitive demand of the examples that they enact in the classroom, I chose to first provide a review of the literature on cognitive demand. Although it is based on the work of Walter Doyle in the 1980s, it was Mary K. Stein and Margaret Smith that together developed a strong framework for analyzing the cognitive demand of tasks. In this subsection, I will review the origins of cognitive demand and examine how the study of cognitive demand has developed over the years.

Academic Tasks

Doyle's (1983) work focused on examining academic tasks, their intellectual demands, and the ways in which they are enacted in the classroom. Doyle's conception of academic tasks comprised three parts: the products, the process, and the resources. In other words, "academic tasks…are defined by the answers students are required to produce and the routes that can be used to obtain these answers" (p. 161). In particular, Doyle emphasized that academic tasks are important because they are the medium through which students engage with the content and they have a large impact on the students' opportunities to learn.

Acknowledging that not all academic tasks provide students with equal opportunities to engage and learn, Doyle (1983) presented four categories of tasks, organized by the cognitive operations required to accomplish the task: memory, procedural or routine, comprehension or understanding, and opinion. Memory tasks were defined as "tasks in which students are expected to recognize or reproduce information previously encountered" (p. 162). Procedural or routine task were defined as "tasks in which students are expected to apply a standardized and predictable formula or algorithm to generate an answer" (p. 163). While Doyle presented a domain-generic definition of comprehension or understanding tasks, he identified that these are "tasks in which students are expected to…apply procedures to new problems or decide from among several procedures those which are applicable to a particular problem" or "draw inferences from previously encountered information or procedures" (p. 163). Finally, opinion tasks were defined as "tasks in which students are expected to state a preference for something" (p. 163).

Table 1. Doyle's (1983) Task Categories

Task	Definition	Examples
Memory	"Tasks in which students are expected to recognize or reproduce information previously encountered." (p. 162)	Memorize a list of spelling words or lines from a poem
Procedural /Routine	"Tasks in which students are expected to apply a standardized and predictable formula or algorithm to generate an answer." (p. 163)	Solve a set of subtraction problems
Comprehension/ Understanding	"Tasks in which students are expected to (a) recognize transformed or paraphrased versions of information previously encountered (b) apply procedures to new problems or decide from among several procedures those which are applicable to a particular problem...or (c) draw inferences from previously encountered information or procedures." (p. 163)	Solve "word problems" in mathematics; make predictions about chemical reactions; devise an alternate formula for squaring a number
Opinion	"Tasks in which students are expected to state a preference for something." (p. 163)	Select a favorite story

Doyle (1988) examined the impact that mathematics tasks have on the ways students think about the content. In particular, Doyle emphasized that "the work students do, which is defined in large measure by the tasks teachers assign, determines how they think about a curriculum domain and come to understand its meaning" (p. 167). Doyle identified cognitive demand as one way to characterize the academic work that occurs in a mathematics classroom. Here, Doyle defined cognitive demand as the "the cognitive processes students are required to use in accomplishing [a task]" (p. 170). Referring to his four task categories, Doyle emphasized that if the majority of the mathematical tasks that students engage with are based primarily on memorization and procedures, then this is how they will perceive the domain of mathematics. Therefore, it is important for teachers to provide students with opportunities to engage with higher-level cognitive tasks that focus on "comprehension, interpretation, flexible application of knowledge and skills, and assembly of information from several different sources to accomplish work" (p. 171).

Cognitive Demand

Building upon Doyle's (1983, 1986, 1988) work, Stein, Grover, and Henningsen (1996) examined the task features and cognitive demand of 144 different mathematical tasks used in reformed classrooms. The task features that they attended to were the number of solutions strategies, the number and kinds of representations, and communication requirements. Although Stein et al. do build upon the work of Doyle, the four types of tasks that they identified do differ from Doyle's (1983) categories. In particular, Stein et al. (1996) categorized tasks as:

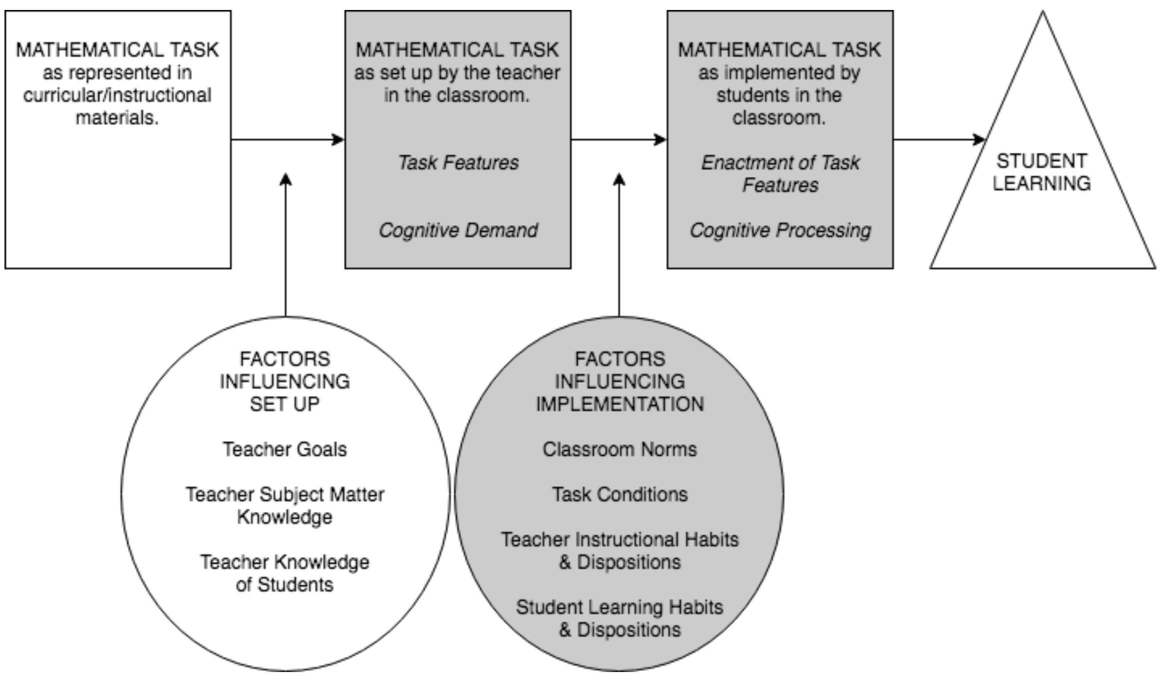
> *Memorization, the use of formulas, algorithms, or procedures* without connection *to concepts, understanding, or meaning, the use of formulas, algorithms, or procedures* with connection *to concepts, understanding, or meaning, and cognitive activity that can be characterized as "doing mathematics," including complex mathematical thinking and reasoning activities such as making and testing conjectures, framing problems, looking for patterns, and so on. (Emphasis in original, p. 466)*

Tasks categorized as memorization or procedures without connections were considered to require low levels of cognitive demand, while tasks categorized as procedures with connections and doing mathematics were considered to require high levels of cognitive demand[2].

[2] It is important to note that while Stein et al. first introduced their task categories in the 1996 publication, formal definitions or descriptions were provided in a later publication (Smith & Stein, 1998), which I will review shortly.

While Stein et al. (1996) were interested in the cognitive demand of the tasks that were implemented in reformed classrooms, they also wanted to examine how mathematical tasks might change during the implementation stage. To do this, the authors built a framework to describe the temporal phases that tasks go through as they are transformed from their representation in curricular/instructional materials, to the task as set up in the classroom, to the task as implemented by students in the classroom. Ultimately, the goal of this task unfolding is to impact student learning, but at each stage of the unfolding, there are various factors that can influence the task features and cognitive demand (see Figure 1).

Figure 1. Stein et al.'s (1996) Framework for Task Unfolding



In their study, Stein et al. (1996) focused on the final phase of task unfolding (represented by the shaded shapes in Figure 1) and examined the factors that influence the implementation. While they analyze both task features and cognitive demand, I will focus primarily on the results they found related to cognitive demand, since they are relevant to my work. Through their analysis, they found that nearly 74% of the 144 mathematical tasks were *set up* at a high level of cognitive demand. However, the cognitive demand of tasks tended to decline during the implementation stage and only 33% of the tasks were *implemented* at a high level of cognitive demand.

Curious to examine why these high level tasks declined during implementation, Stein et al. (1996) identified six factors that were judged to contribute to the decline: challenges become nonproblems, inappropriateness of the task for students, focus shifts to correct answer, too much or too little time, lack of accountability, and classroom management problems. On the other hand, the authors also identified seven factors that were judged to contribute to the maintenance of high levels of cognitive demand: task builds on students' prior knowledge, appropriate amount of time, high-level performance modeled, sustained pressure for explanation and meaning, scaffolding, students self-monitoring, and teacher draws conceptual connections.

One surprising result of Stein et al.'s (1996) work is that tasks could be set up as doing mathematics, but decline to procedures without connections, unsystematic exploration, or even no mathematical activity. To explore these types of decline, Henningsen and Stein (1997) focused on identifying classroom-based factors that support and inhibit students' engagement with doing mathematics tasks. First, they examined tasks that were set up *and* implemented at the level of doing mathematics and found that these tasks were successful because they built on students' prior knowledge, provided appropriate scaffolding, were allotted an appropriate amount of time, included modeling of high-level performance, and sustained pressure for explanation and meaning.

For the tasks that were set up as doing mathematics but declined to lower levels of cognitive demand or no mathematical activity, Stein et al. (1996) found that there were different factor profiles for each type of decline. Tasks that declined to procedures without connections tended to allot too much or too little time, make challenges into nonproblems, and shift in focus to the correct answer. Similarly, tasks that declined to unsystematic exploration tended to allot too much or too little time, shift in focus to the correct answer, and be inappropriate. Finally, tasks that declined to no mathematical activity tended to be inappropriate, run into classroom management problems, and allot too much or too little time.

In order to clarify what the authors meant by memorization, procedures without connections, procedures with connections, and doing mathematics tasks, Smith and Stein (1998) published a paper that listed the characteristics of the four different types of tasks (see Table 2). These descriptions not only provided researchers with a clear conceptualization with each category, but also provided teachers with a framework for thinking about the cognitive demand of tasks. In particular, Smith and Stein illustrated how the Task Analysis Guide can be used in professional development activity where participants sort tasks into each category and talk about the reasoning behind their categorizations. Additional, Stein and Smith (1998) talked about how the framework could be used as tool for reflection when teachers observe teachers or even when teachers reflect on their own teaching.

Table 2. Smith and Stein's (1998) Task Analysis Guide

Category	Description	Example
Memorization	• Involve either reproducing previously learned facts, rules, formulas, or definitions or committing facts, rules, formulas or definitions to memory • Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure • Are not ambiguous. Such tasks involve the exact reproduction of previously seen material, and what is to be reproduced is clearly and directly stated. • Have no connection to the concepts or meaning that underlie the facts, rules, formulas, or definitions being learned or reproduced.	What is the rule for multiplying fractions?
Procedures Without Connections	• Are algorithmic. Use of the procedure either is specifically called for or is evident from prior instruction, experience, or placement of the task. • Require limited cognitive demand for successful completion. Little ambiguity exists about what needs to be done and how to do it. • Have no connection to the concepts or meaning that underlie the procedure being used. • Are focused on producing correct answers instead of on developing mathematical understanding. • Require no explanations or explanations that focus solely on describing the procedure that was used.	Multiply the following expressions: $\frac{2}{3}\times\frac{3}{4}$ $\frac{5}{6}\times\frac{7}{8}$ $\frac{4}{9}\times\frac{3}{5}$
Procedures With Connections	• Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas. • Suggest explicitly or implicitly pathways to follow that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts. • Usually are represented in multiple ways, such as visual diagrams, manipulatives, symbols, and problem situations. • Making connections among multiple representations helps develop meaning. • Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with conceptual ideas that underlie the procedures to complete the task successfully and that develop understanding.	Find $1/6$ of $1/2$. Use pattern blocks. Draw your answer and explain your solution.
Doing Mathematics	• Require complex and nonalgorithmic thinking—a predictable, well-rehearsed approach or pathway is not explicitly suggested by the task, task instructions, or a worked-out example. • Require students to explore and understand the nature of mathematical concepts, processes, or relationships. • Demand self-monitoring or self-regulation of one's own cognitive processes. • Require students to access relevant knowledge and experiences and make appropriate use of them in working through the task. • Require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions. • Require considerable cognitive effort and may involve some level of anxiety for the student because of the unpredictable nature of the solution process required.	Create a real-world situation for the following problem: $\frac{2}{3}\times\frac{3}{4}$. Solve the problem you have created without using the rule, and explain your solution.

Researchers have continued to build upon and refine the task unfolding (Figure 1) and cognitive demand (Table 2) frameworks developed by Smith, Stein, and their colleagues. In particular, researchers have focused on how to plan, set up, implement, and conclude tasks so that they maintain a high level of cognitive demand. To facilitate the design of lesson plans that would support high cognitive demand tasks, Smith, Bill, and Hughes (2008) developed the "Thinking Through a Lesson Protocol" (TTLP). Teachers using the protocol are provided with a set of questions to consider when planning their lesson. In the first part, they focus on selecting and setting up a mathematical task, then ask questions related to supporting students exploration of the task, and finally consider how the teacher plans to share and discuss the task. While the authors don't suggest that teachers answer all of the questions included in the protocol every time they plan a lesson, they do suggest that teachers use the TTLP periodically and in collaboration with other teachers.

Moving from the planning to the set up stage, Jackson, Shahan, Gibbons, and Cobb (2012) examined four crucial elements of launching complex tasks: discussing the key contextual features, discussing the key mathematical ideas, developing a common language to describe the key features, and maintaining the cognitive demand. Similar to the TTLP, the authors provide teachers with a set of planning questions that teachers can use to reflect on what they need to do to launch a complex task effectively. In another paper, Jackson, Garrison, Wilson, Gibbons, and Shahan (2013) examined how the launch of tasks related to the opportunities to learn mathematics in the concluding whole-class discussion. As a result, they found that attending to the crucial elements of developing a common language to describe the key features and maintaining the cognitive demand of the task during the launch resulted in higher quality opportunities for students to learn in the concluding mathematics discussion.

Instructional Examples

Much of the research on cognitive demand has focused on mathematical tasks that students complete during class. However, I am interested in studying the cognitive demand of the examples that instructors enact. While examples are one type of mathematical task, they are different from a task that a teacher might give for students to work on. In particular, I define an example as a whole-class activity, the purpose of which is to solve a mathematical problem for illustrative purposes. For example, to help students understand why trigonometric equations can have infinite families of solutions, an instructor might use the example of $\sin\theta = -1/2$. Or if an instructor is teaching the completing the square algorithm, they might introduce it by working through several examples before asking students to work through related problems. In this subsection, I review some of the literature on examples and examine how they are different from other types of mathematical tasks.

Bills, Dreyfus, Mason, Tsamir, Watson, and Zaslavsky (2006) gave a general overview of how exemplification has been treated in mathematics education. First, the authors claimed that it is important to study examples and exemplification in mathematics for several reasons. First, examples play a central role in the development of mathematics as a discipline and the teaching and learning of mathematics. Second, "examples offer insight into the nature of mathematics through their use in complex tasks to demonstrate methods, in concept development to indicate relationships, and in explanations and proofs" (pp. 126-127). The authors identified different ways in which examples might be presented, which can range from worked-out examples, where "the procedure being applied is performed by the teacher, textbook author or programmer, often with some sort of explanation or commentary," to exercises, "where tasks are set for the learner to complete" (p. 127).

For the purposes of their review, Bills et al. (2006) defined examples as "anything used as raw material for generalising, including intuiting relationships and inductive reasoning; illustrating concepts and principles; indicating a larger class; motivating; exposing possible variation and change, etc. and practising technique" (p. 127)[3]. Exemplification, on the other hand, is a term they use "to describe any situation in which something specific is being offered to represent a general class to which learners' attention is being drawn" (p. 127). They also classified examples as a foundational device that mathematics instructors use to explain mathematics concepts (p. 133). However, just because examples are fundamental to mathematics teaching does not mean that they are a trivial part of instruction. On the contrary, Bills et al. highlighted several studies that have found that the art of constructing examples is a highly demanding task of teaching.

While examples can be presented in a variety of ways, Bills et al. (2006) emphasized that "providing worked-out examples with no further explanations or other conceptual support is usually insufficient", as "learners often regard such examples as specific (restricted) patterns which do not seem applicable to them when solving problems that require a slight deviation from the solution presented in the worked-out

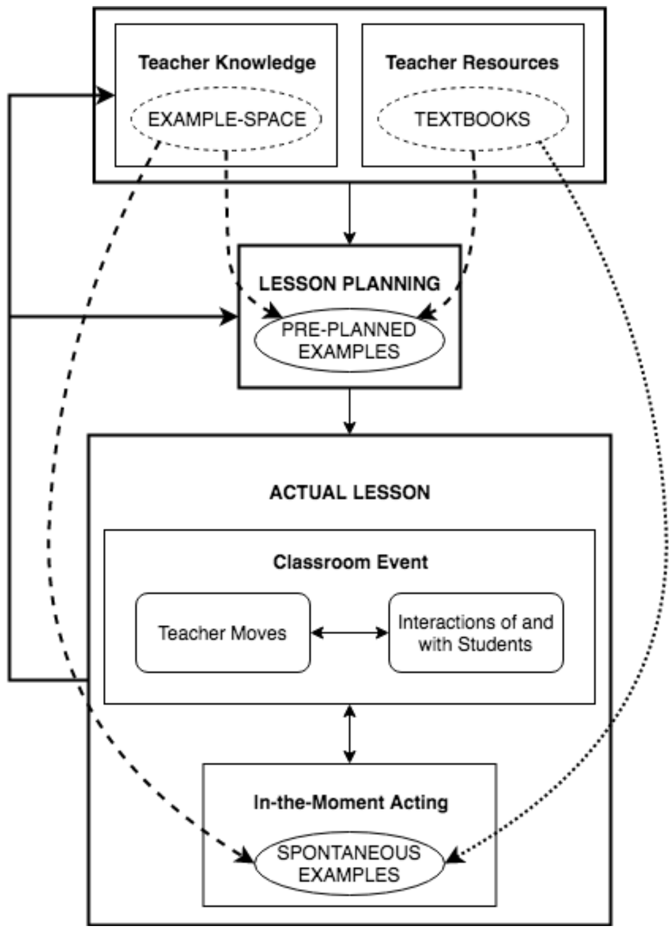
[3] It is important to note that the authors' definition of example is different from the definition of example that I chose to use. In particular, my definition could be viewed as a restriction of their definition, in that it only includes examples that are done in a whole-class setting and not exercises that are just given to students to work through.

example (Reed *et al.* 1985; Chi *et al.* 1989)" (p. 140). Therefore, the authors emphasize that it is important for worked-out examples to include explanations and reasoning.

In the past ten years, several researchers have focused on studying how teachers use examples in their classrooms. While each researcher conceptualizes "examples" in different ways, they all do consider examples as tasks used for illustrative purposes, which fits with my definition of example. In studying the purpose, design, and use of mathematical examples in elementary classrooms, Rowland (2008) found that teachers need to attend to variables, sequencing, representations, and learning objectives when choosing what examples to use in the classroom. Similarly, Muir (2007) found that teachers need to attend carefully to the examples that they choose to use when teaching numeracy in order to "avoid the likelihood of students developing common misconceptions about important mathematical concepts" (p. 513). Finally, Zodik and Zaslavsky (2008) examined different characteristics of how teachers choose mathematics examples and developed a framework that captures the teachers' choice and generation of examples (Figure 2).

The final piece of literature on mathematical examples that I have included in my review is by Mesa, Suh, Blake, and Whittemore (2012). This article is particularly relevant to my dissertation, because they looked at the opportunities to learn that were provided by examples included in college algebra textbooks. Mesa et al. claimed that it is important to consider the examples included in textbooks because instructors often draw upon these activities when planning the examples they want to include in their lesson (p. 78). In particular, the authors examined the cognitive demand of the examples included in textbooks because research has shown that "engaging students in activities that are high in cognitive demand…can indeed foster students' development of mathematical proficiency" (p. 79).

Figure 2. Zodik and Zaslavsky's (2008) Example Cycle



Mesa et al. (2012) chose to examine ten college algebra textbooks that were commonly used in community colleges and universities in the state of Michigan around the time of the study. Of the 488 examples that were included in the textbooks, 445 (91%) of them were coded as procedures without connections, with individual textbooks ranging from 75%-100% in this category. Of the remaining examples, 41 (8%) were coded as procedures with connections, two (<1%) were coded as doing mathematics, and none were coded as memorization. While the authors recognized that procedures without connections examples help develop procedural fluency, they also warned that "concentrating only on these less cognitively demanding examples can restrict students' perceptions about the nature of mathematics" (pp. 96-97) and that an over-emphasis on procedural fluency is detrimental for students' learning.

Teacher Content Knowledge

Proxies for Measuring Teacher Content Knowledge

As mentioned in the introduction, numerous studies have investigated the effects of content knowledge on teaching. However, many of these studies used proxies, such as general content knowledge tests, teacher education, and number of years of teacher experience, for measuring teacher content knowledge. Using these proxies, there have been mixed reports concerning whether or not teacher content knowledge is positively correlated with student achievement.

In 1972, Edward Begle published a report that investigated the relationship between teachers' content knowledge and student achievement in algebra. In order to "search for characteristics which distinguish effective teachers," Begle focused on examining "the degree to which the teacher understands the material being taught" (p. 4). While this seemed to be a natural variable which would distinguish effective teachers from non-effective teachers, Begle found that reviewing recent studies produced "little empirical evidence to substantiate any claims that, for example, training in mathematics for mathematics teachers will have a payoff in increased mathematics achievement for their students" (p. 4). Thus, Begle set out to investigate further whether or not this curious problem existed at the high school level.

In his study, Begle (1972) used two sets of tests, one for teachers and one for their students. The teacher test was designed to measure two levels of algebra understanding: “that of the algebra of the real number system, to which the ninth grade high school algebra course is largely devoted” and that “of the abstract algebra of groups, rings, and fields” (p. 6). Given the assumption that a deeper understanding of the content should lead to better student achievement, Begle hypothesized that the second level would be more closely correlated with student achievement than the first. The student tests were administered at the end of the ninth grade: “One was devoted to algebraic computation and the other to understanding of algebraic concepts” (p. 7). In order to distinguish between differences in students, a mathematics achievement test and basic mental ability test were administered at the beginning of the ninth grade.

To analyze the effects of teacher content knowledge on student achievement, Begle (1972) conducted a regression analysis. Consistent with the studies Begle had reviewed, the analysis showed that teacher content knowledge had relatively few effects on student achievement. In particular, teacher understanding of modern algebra had no significant correlation with student achievement in either algebraic computation or understanding of ninth grade algebra. Teacher understanding of the algebra of the real number system was significantly correlated with student understanding of ninth grade algebra, but not with student understanding of algebraic computation. However, Begle reported that the significant correlation between teacher understanding of algebra of the real number system and student understanding of ninth grade algebra was “so small as to be educationally insignificant” (p. 13). Based upon his review of previous findings, Begle reported that, "these results were not completely unexpected but were nevertheless surprising" (p. 13).

Nine years later, Eric Hanushek (1981), an economist, reviewed studies of teacher effectiveness and found similar results. Recognizing that educational outcomes are often viewed as significant and long lasting, Hanushek approached his analysis from an industry perspective. In order to assess "the current state of knowledge in policy-related research", Hanushek focused on "research in areas where governmental actions might directly affect education goals our outcomes" (p. 194). In particular, Hanushek focused on articles that published results on the relationship between inputs (e.g., school factors, family background, and student body characteristics) and outputs (e.g., standardized test scores). The proxy that Hanushek used for teacher knowledge was the amount of graduate education the teacher had finished. Hanushek identified 101 studies that tested the statistical significance ($\alpha = 0.05$) of the relationship between teacher education and student achievement and found that six reported a statistically significant positive relationship, four reported a statistically significant negative relationship, and 90 did not report a statistically significant relationship. Other input factors, such as teacher experience, were also shown to have similar counterintuitive results. In response to these findings, Hanushek argued that most likely they are due to the narrow way in which studies have measured teacher effectiveness.

> *The research indicates that it is not possible to identify and measure a set of homogeneous input factors that enter into the production process, even though differences in teacher inputs are very important. The reason seems to be that teaching is a very complicated process.... Because of the complexity of the task*

> *and an incomplete understanding of the separate elements of effective teaching, it is not possible to single out a small set of factors that uniformly contribute to good performance. (p. 205)*

While Begle (1972) and Hanushek (1981) reported a growing mound of evidence that teacher content knowledge (as measured by general content tests and graduate education) did not have an effect on student achievement, other literature reviews indicated the opposite. In particular, Greenwald, Hedges, and Laine (1996) and Wayne and Youngs (2003) found that there was a larger body of literature that found statistically significant correlations. Although Greenwald et al. (1996) did not look at teacher content knowledge directly, they followed a similar approach to Hanushek (1981) and considered studies which measured teacher education. Greenwald et al. (1996) identified 38 studies that measured teacher education and its effect on student achievement and analyzed them using combined significance testing and effect magnitude estimation. To analyze combined significance, the authors conducted two one-tailed hypothesis tests, one in which the null hypothesis stated "that no *positive* relation exists between the resource input and student outcomes for the population coefficients" and another in which the null hypothesis stated "that no *negative* [emphasis added] relation exists between the resource input and student outcome for the population coefficient" (p. 365). The authors found that using teacher education as the resource input resulted in rejecting the null hypothesis in both cases (p. 369), which implied "that there is evidence of both some positive and some negative relations" (p. 366). In relation to effect sizes, the authors found that "the pattern of effect sizes for the newer (post-1970) studies...appear to be somewhat more positive..." (p. 375).

In their review, Greenwald et al. (1996) addressed the question of "Why have previous reviews failed to detect positive effects?" (p. 381). In particular, the authors single out the vote counting methods employed by Hanushek as erroneous and misleading. The authors claimed that "when individual studies have relatively low statistical power, only a small proportion of studies would be expected to obtain statistical significance, even if each study were estimating the same (nonzero) effect.... Hence, a large proportion of significant results would not be expected...and not counting would be expected to miss effects" (p. 381). However, within the same issue, Hanushek (1996) published his own rebuttal to Greenwald et al.'s (1996) findings. In particular, Hanushek (1996) criticized Greenwald et al. (1996) for presenting "a distorted and misleading view of the potential implications of school resource policies" (p. 397).

> *Ultimately, the fundamental problem with their analysis derives from a flawed statistical approach for investigating issues of how and when resources affect student performance. Their specialized meta-analytic approach to combining data is applicable to circumstances very different from the present ones. They assume that all of the schooling situations are identical, when in fact most people believe for good reason that they are very heterogeneous. They further assume that all of the studies should receive equal weight, when in fact the studies are also heterogeneous.... By forcing homogeneity onto the data about effectiveness, they both introduce powerful biases into their analysis of the results and distract decision makers from the important issues. (Hanushek, 1996, p. 398)*

Whether the methods used by Hanushek (1981), Greenwald et al. (1996), or others present the most accurate evidence of effects of content knowledge on teaching is yet to be decided. However, given the disparity of results and disagreement of interpretations, it is clear that measuring teacher content knowledge using proxies yields inconclusive findings.

Mathematical Knowledge for Teaching

One explanation for why studies found mixed results concerning whether or not teacher content knowledge has a positive effect on student outcomes is because the proxies used to measure content knowledge were misaligned. Instead of measuring content that was specific to the work of teaching, proxies either used general measures of teacher content knowledge or were based upon the assumption that advanced-level content knowledge was adequate for teaching lower levels.

In his 1985 presidential address to the *American Educational Research Association*, Lee Shulman identified "the missing paradigm" in educational research. Shulman (1986) claimed that in recent history, there existed a "sharp distinction between content and [pedagogy]" (p. 6) as evidenced by teacher examinations in the 1970s (which largely focused on content and ignored pedagogy) and 1980s (which largely focused on pedagogy and ignored content). However, Shulman claimed that this distinction was not always made. In medieval universities, "the purpose of the [ceremony of doctoral examination was] to demonstrate that the candidate possess[ed] the highest levels of subject matter competence in the domain for which the degree is awarded. How did one demonstrate such understanding in medieval times? By demonstrating the ability to *teach* the subject (Ong, 1985)...." (p. 7). In contrast to the famous quote by George Bernard Shaw (1999)—"He who can, does. He who cannot, teaches."— Shulman (1986) claimed that "what distinguishes the man who knows from the ignorant man is an ability to teach" (p. 7).

The missing paradigm Shulman (1986) identified concerned research on the content knowledge used in teaching. Shulman called for researchers to begin asking questions such as "What are the sources of teacher knowledge? What does a teacher know and when did he or she come to know it? How is new knowledge acquired, old knowledge retrieved, and both combined to form a new knowledge base?" (p. 8). Shulman proposed that content knowledge for teaching could be broken down into three categories: subject matter content knowledge, pedagogical content knowledge, and curricular knowledge. He defined content knowledge as "the amount and organization of knowledge per se in the mind of the teacher" (p. 9). In particular, subject matter knowledge comprised knowledge of the substantive and syntactic structure of the discipline (Schwab, 1978) . Finally, Shulman (1986) defined pedagogical content knowledge as content knowledge "which goes beyond knowledge of subject matter per se to the dimension of subject matter *for teaching*" (p. 9). For example, pedagogical content knowledge (as defined by Shulman) included familiarity with different representations and understanding of what makes learning topics easy or difficult.

Building upon his *AERA* presidential address, Shulman (1987) wrote about changes that must occur in teacher education in order to address the specialized knowledge that teachers must possess and use. He identified some categories of the knowledge base that teachers must possess, including content knowledge, general pedagogical knowledge, curriculum knowledge, pedagogical content knowledge, knowledge of learners and their characteristics, knowledge of educational contexts, and knowledge of educational ends. Again, Shulman emphasized the importance of blending content and pedagogy and, in particular, focuses on pedagogical content knowledge.

> *Pedagogical content knowledge is of special interest because it identifies the distinctive bodies of knowledge for teaching. It represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction. Pedagogical content knowledge is the category most likely to distinguish the understanding of the content specialist from that of the pedagogue. (p. 8)*

Deborah Ball: Mathematical Knowledge for Teaching. Many researchers answered Shulman's (1986, 1987) call and began investigating content knowledge as it is used in teaching. Starting with her dissertation (Ball, 1988), Deborah Ball began studying the mathematical knowledge that is needed for elementary teachers to teach mathematics effectively. Specifically, she was interested in studying what was entailed in subject matter knowledge for teaching (Ball, 1990). While it seemed common sense to "claim that teachers need substantive knowledge of mathematics—of particular concepts and procedures (rectangles, functions, and the multiplication of decimals, for example)" (p. 458), Ball argued that teachers needed to know more. In particular, Ball claimed that it was important for teachers to also know about mathematics: "This includes understanding about the nature of mathematical knowledge and of mathematics as a field" (p. 458).

The main reason why Ball (1990) claimed that there was "more" mathematics that teachers needed to know was due to the nature of teaching.

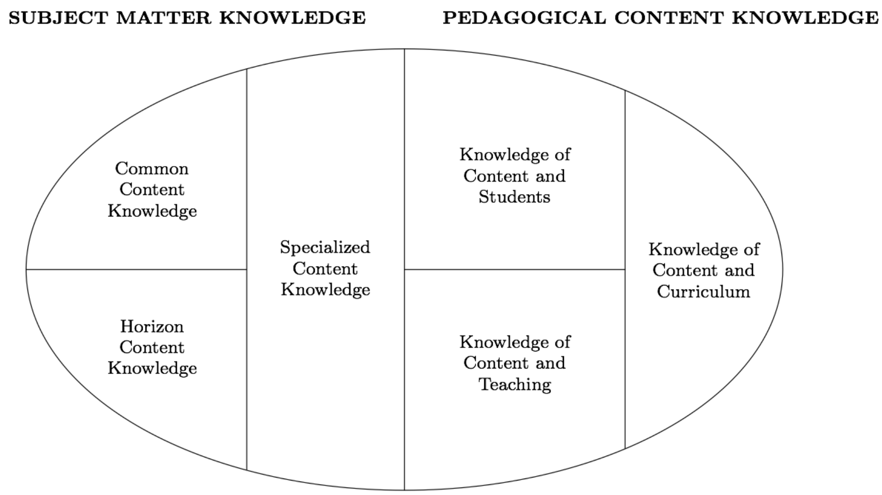
> *In order to help someone else understand and do mathematics, however, being able to `do it' oneself is not sufficient. Teachers must not only be able to describe the steps for following an algorithm but also discuss the judgments made and the meanings of and reasons for certain relationships or procedures. (pp. 458-459)*

Echoing Shulman's concerns regarding teacher examinations, Ball critiqued the current trend in teacher preparation programs. She claimed that "despite the fact that subject matter knowledge is logically central to teaching (Bachmann, 1984), it rarely figures prominently in teacher preparation" (p. 462). Moreover, Ball felt that "the fact that the subject matter preparation of [elementary] teachers is left to precollege and `liberal arts' college mathematics" was problematic (p. 462). In particular, Ball felt that three common assumptions concerning subject matter knowledge for teaching were erroneous. First, people assume that traditional school mathematics content is simple. Contrary to this belief, Ball cited various studies which show that even elementary mathematics is complex and difficult to teach (Duckworth, 1987; Lampert, 1985, 1986, 1989). Second, people assume that elementary and secondary school mathematics classes can prepare teachers to teach mathematics (Ball, 1990, p. 463). While it is true that most teachers have taken and passed the classes that they are teaching, Ball claimed that is not enough preparation for *teaching* that content. And third, people assume that majoring in mathematics ensures subject matter knowledge. While it may seem logical to assume that deeper understanding equates to better teaching, I will later review several studies that have shown that this is not the case.

In 2008, Ball, Thames, and Phelps published the formal theoretical framework for mathematical knowledge for teaching (MKT) that Ball and her colleagues had been working to develop over the past two decades. Even though several researchers had responded to Shulman's original call to study content knowledge for teaching, the authors recognized that "this bridge between knowledge and practice was still inadequately understood and the coherent theoretical framework Shulman (1986, p. 9) called for remained underdeveloped" (p. 389). In order to develop this framework, the authors used the work that had been produced by Ball and her colleagues at the University of Michigan under the *Mathematics and Learning to Teach Project* and *Learning Mathematics for Teaching Project*. In order to study content and its role in teaching, Ball and her colleagues chose to focus on the work of teaching. Instead of examining curriculum and standards, or asking experts mathematicians and educators to identify core ideas and skills, or reviewing research on students' learning, the authors began with practice. The aim of their analysis was to "develop a practice-based theory of mathematical knowledge as it is entailed by and used in teaching (Ball, 1999; Thames, 2008)" (p. 396).

Ball et al. (2008) defined MKT as "the mathematical knowledge needed to carry out the work of teaching mathematics" (p. 395). In the framework developed by Ball et al., MKT is broken down into the subdomains of subject matter knowledge (SMK) and pedagogical content knowledge (PCK), which purposefully reflect the domains of content knowledge that were initially identified by Shulman (1986). Subject matter knowledge is further broken down into common content knowledge (CCK), horizon content knowledge (HCK), and specialized content knowledge (SCK). Pedagogical content knowledge is further broken down into knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC). This decomposition of MKT is often referred to as "the egg", due to the diagram that it was originally illustrated with (see Figure 3).

Figure 3. Ball et al.'s (2008) Decomposition of Domains of MKT



Ball et al. (2008) defined common content knowledge as "the mathematical knowledge and skill used in settings other than teaching" (p. 399). For example, this would include the ability to correctly solve mathematical problems. As a point of clarification, the authors did no intend "common" to indicate that everyone has this knowledge: "Rather, we mean to indicate that this is knowledge of a kind used in a wide variety of settings--in other words, not unique to teaching" (p. 399). In contrast, specialized content knowledge was defined as "the mathematical knowledge and skill unique to teaching" (p. 400). For example, knowledge of how to decompress/unpack mathematical ideas to make them accessible to students (e.g., explaining why dividing by a fraction is equivalent to multiplying by its reciprocal) is an example of SCK. An example of knowledge of content and students is knowing common student conceptions and misconceptions. An example of knowledge of content and teaching is knowing how to sequence content for instruction.

Although it is not a focus of the paper, the authors do give a definition of horizon content knowledge: "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (Ball et al., 2008, p. 403). Knowledge of content and curriculum is not defined, but rather they identified it as being synonymous with Shulman's (1986) conceptualization of curricular knowledge. Here, the authors noted that "we have placed Shulman's third category, curricular knowledge, within pedagogical content knowledge. This is consistent with later publications from members of Shulman's research team (Grossman, 1990)" (Ball et al., 2008, pp. 402–403). The authors also noted that there are several limitations of their framework. First, because it was developed by examining practice, the framework "brings in some of the natural messiness and variability of teaching and learning. As we ask about the situations that arise in teaching that require teachers to use mathematics, we find that some situations can be managed using different kinds of knowledge" (p. 403). Also, splitting up the domain into categories makes it appear static.

A third problem related to the categorization of the domain is that "it is not always easy to discern where one of our categories dives from the next, and this affects the precision (or lack thereof) of our definitions" (p. 403). However, Ball et al. felt that the categories provide a useful structure for "studying the relationship between teachers' content knowledge and their students' achievement", studying "whether and how different approaches to teacher development have different effects on particular aspects of teachers' pedagogical content knowledge", and "inform[ing] the design of support materials for teachers as well as teacher education and professional development" (p. 405).

Table 3. Definitions of Ball's Sub-Domains of MKT

Term	Abbreviation	Definition	Examples
Common Content Knowledge	CCK	"The mathematical knowledge and skill used in settings other than teaching." (Ball et al., 2008, p. 399)	Knowing the material you are expected to teach; recognizing when students give wrong answers; identifying inaccurate textbook definitions
Specialized Content Knowledge	SCK	"The mathematical knowledge and skill unique to teaching." (Ball et al., 2008, p. 400)	Looking for patterns in student errors; determining whether a non-standard approach is generalizable; understanding different interpretations of operations
Horizon Content Knowledge	HCK	"An awareness of how mathematical topics are related over the span of mathematics included in the curriculum." (Ball et al., 2008, p. 403)	Awareness of how what you teach sets up other topics in later years; seeing connections to much later mathematical ideas
Knowledge of Content and Students	KCS	"Knowledge that combines knowing about students and knowing about mathematics." (Ball et al., 2008, p. 401)	Knowing common student errors or misconceptions; recognizing what students might struggle with in a problem; anticipating what students are likely to think
Knowledge of Content and Teaching	KCT	Knowledge that "combines knowing about teaching and knowing about mathematics." (Ball et al., 2008, p. 401)	Sequencing content for instruction; evaluating the instructional advantages and disadvantages of representations; identifying what different methods afford instructionally
Knowledge of Content and Curriculum	KCC	Knowledge of "the materials and programs that serve as 'tools of the trade' for teachers." (Shulman, 1987, p. 8)	Knowledge of the full range of programs available; knowledge of instructional materials

The Knowledge Quartet. In addition to the framework for mathematical knowledge for teaching developed by Ball et al. (2008), other frameworks have also been developed. Rowland, Huckstep, and Thwaites (2005) published a framework for elementary teachers' mathematical subject knowledge called the Knowledge Quartet (KQ). Like Ball, they sought to develop an empirically based conceptual framework of content knowledge for teaching by analyzing videotapes of teaching. Specifically, they used grounded theory to analyze the practice of a group of preservice teachers who were at the end of their initial training in order to identify the mathematics-related knowledge the teachers used during their practice (p. 255). While analyzing the videos, they focused on “aspects of trainees' actions in the classroom that seemed to be significant in the limited sense that it could be construed to be informed by a trainee's mathematics content knowledge or their mathematical pedagogical knowledge" (p. 258). Following an inductive process, they generated a set of 18 codes, which they later categorized into four broad dimensions: foundation, transformation, connection, and contingency.

Rowland et al. (2005) defined foundation as "the foundation of the trainees' theoretical background and beliefs. It concerns trainees' knowledge, understanding and ready recourse to their learning *in the academy*, in preparation (intentionally or otherwise) for their role in the classroom" (p. 260). The authors claimed that foundation is closely related to Shulman's idea of propositional form (Shulman, 1986, p. 10) and Shulman's first aspect of pedagogical reasoning, comprehension (Shulman, 1987, p. 14).

Table 4. Definitions of Rowland's Dimensions of the Knowledge Quartet

Term	Definition	Contributing Codes (Rowland, 2014)
Foundation	"The trainees' theoretical background and beliefs. It concerns trainees' knowledge, understanding and ready recourse to their learning *in the academy*, in preparation (intentionally or otherwise) for their role in the classroom." (Rowland et al., 2005, p. 260)	Awareness of purpose; identifying errors; overt subject knowledge; theoretical underpinning of pedagogy; use of terminology; use of textbook; reliance on procedures
Transformation	"The capacity of a teacher to *transform* [emphasis added] the content knowledge he or she possess into forms that are pedagogically powerful." (Shulman, 1987, p. 15)	Teacher demonstration; use of instructional materials; choice of representation; choice of examples
Connection	"Concerns the *coherence* of the planning or teaching displayed across an episode, lesson or series of lessons." (Rowland et al., 2005, p. 262)	Making connections between procedures; making connections between concepts; anticipation of complexity; decisions about sequencing; recognition of conceptual appropriateness
Contingency	"Concern[ing] classroom events that are almost impossible to plan for. In commonplace language it is the ability to \`think on one's feet': it is about *contingent action*." (Rowland et al., 2005, p. 263)	Responding to students' ideas; deviation from agenda; teacher insight; (un)availability of resources

The final three dimensions follow from foundational knowledge, but are also markedly different in that they "focus on knowledge-in-action" (Rowland et al., 2005, p. 261). The second dimension, transformation, is defined using the words of Shulman (1987): "The capacity of a teacher to *transform* [emphasis added] the content knowledge he or she possess into forms that are pedagogically powerful" (p. 15). This includes the ability to use teacher resources to choose examples "to assist concept formation, to demonstrate procedures, and [to select] exercise examples for student activity" (Rowland et al., 2005, p. 262).

The third dimension, connection, "concerns the *coherence* of the planning or teaching displayed across an episode, lesson or series of lessons" (Rowland et al., 2005, p. 262). This includes sequencing mathematical content based upon not only the mathematical structure, but also the "relative cognitive demands of different topics and tasks" (p. 263). The authors described the final dimension, contingency, as "concern[ing] classroom events that are almost impossible to plan for. In commonplace language it is the ability to `think on one's feet': it is about *contingent action*" (p. 263). For example, teachers must be able to respond to student ideas and, when appropriate, deviate from the planned agenda.

Knowledge of Algebra for Teaching. While the previous two frameworks of mathematical knowledge for teaching were developed by observing elementary teachers, the next framework was specifically developed to apply to teaching algebra at the secondary level. McCrory, Floden, Ferrini-Mundy, Reckase, and Senk (2012) developed their framework, Knowledge of Algebra for Teaching (KAT), in order to better understand "both what knowledge matters and how it matters" (p. 585). The authors began their framework development by analyzing domains of mathematical knowledge from research and policy documents. This stage of analysis was used to inform the first dimension of their framework, which encapsulated what knowledge mattered. The second dimension of their framework, which encapsulated how that knowledge mattered, was formed by analyzing textbooks, teaching videos, and interviews with teachers.

In developing the first dimension of KAT that describes *what* knowledge matters, McCrory et al. (2012) primarily drew upon three documents: *The Mathematical Education of Teachers* (Conference Board of the Mathematical Sciences, 2001), "Mathematical Proficiency for All Students: Toward a Strategic Research and Development Program in Mathematics Education" (RAND Mathematics Study Panel, 2002), and "Teachers' Mathematics: A Collection of Content Deserving to be a Field" (Usiskin, 2001). The first category in this dimension is knowledge of school algebra, which is defined to include "the content that typically would be taught and tested in U.S. high school courses conventionally called Algebra I and Algebra II" (McCrory et al., 2012, p. 596) and came from analysis of the CBMS book (2001) and RAND report (2002). The second category is knowledge of advanced mathematics, which "includes other mathematical knowledge, in particular college-level mathematics, that gives a teacher some perspective on the trajectory and growth of mathematical ideas beyond school algebra" (McCrory et al., 2012, p. 597) and came from the CBMS book (2001). The third category is mathematics-for-teaching knowledge, which the authors defined as "mathematics that is useful in teaching, but is not typically taught in conventional mathematics classes either at the high school or postsecondary levels" (McCrory et al., 2012, p. 598) and came from Usiskin's presentation (2001). Note that the authors also identified this final category as similar to what Ball and colleagues identified as specialized content knowledge.

The second dimension of the KAT framework describes how the knowledge described in the first dimension of mathematical knowledge is *used* in teaching. The first category, decompressing, "describes the need for teachers to decompress their knowledge in the practice of teaching" (McCrory et al., 2012, p. 1) and is related to the idea of unpacking that is talked about by Ball and Bass (2000b) and Cohen (2011). The second category, trimming, refers to the idea that "teachers may find it useful to `trim' the mathematical content in a way that matches students' current level of sophistication while treating the mathematics with integrity" (McCrory et al., 2012, p. 604) and is related to Bruner's (1960) idea of intellectually honest teaching and Ball and Bass's (2000a) idea of maintaining mathematical integrity. The last category, bridging, is defined as "efforts to connect and link mathematics across topics, courses, concepts, and goals, including connecting the ideas of school algebra to those of abstract algebra and real analysis, and linking one area of school mathematics to another" (McCrory et al., 2012, p. 607). When taken together, the two dimensions of mathematical knowledge form an array that can be used to analyze the teaching algebra at the secondary level.

Table 5. Definitions in McCrory's Framework for Knowledge of Algebra for Teaching

Term	Definition	Examples
Knowledge of School Algebra	Includes "the content that typically would be taught and tested in U.S. high school courses conventionally called Algebra I and Algebra II." (McCrory et al., 2012, p. 596)	The ability work flexibly and meaningfully with formulas; a structural understanding of basic operations; a robust understanding of the notion of function
Knowledge of Advanced Mathematics	"Includes other mathematical knowledge, in particular college-level mathematics, that gives a teacher some perspective on the trajectory and growth of mathematical ideas beyond school algebra." (McCrory et al., 2012, p. 597)	Knowledge of calculus, linear algebra, number theory, abstract algebra, real and complex analysis, and mathematical modeling; knowing alternate definitions; extensions and generalizations of familiar theorems; applications of school mathematics
Mathematics-for-Teaching Knowledge	"Mathematics that is useful in teaching, but is not typically taught in conventional mathematics classes either at the high school or postsecondary levels." (McCrory et al., 2012, p. 598)	Knowledge of different definitions of a particular mathematical object and the affordances that these definitions provide for mathematics to follow
Decompressing	"Describes the need for teachers to decompress their knowledge in the practice of teaching." (McCrory et al., 2012, p. 1)	Unpacking algorithms used in long division, finding common denominators, and dividing by a fraction, solving equations, simplifying expressions, etc.
Trimming	"Teachers may find ii useful to `trim' the mathematical content in a way that matches students' current level of sophistication while treating the mathematics with integrity" (McCrory et al., 2012, p. 604)	Adapting the textbook's treatment of an idea; modifying textbook problems; revoicing questions or comments; responding to student thinking about particular mathematical ideas; creating examples and problems
Bridging	"Efforts to connect and link mathematics across topics, courses, concepts, and goals, including connecting the ideas of school algebra to those of abstract algebra and real analysis, and linking one area of school mathematics to another" (McCrory et al., 2012, p. 607)	Recognizing that understanding fractions is essential for developing algebraic skills and concepts; linking polynomial expressions through expression of base 10 numbers in expanded form; relating fraction addition to rational function addition

COACTIV. In Germany, another framework for mathematical knowledge for teaching was developed under the COACTIV study (Baumert & Kunter, 2013; Krauss et al., 2008). Like the other frameworks described previously, the COACTIV framework was also developed as a response to the call by Shulman (1986, 1987). However, the COACTIV model does encompass a broader range of teacher professional competence, such as general pedagogical, organizational, and counseling knowledge (Baumert & Kunter, 2013). Since the focus of my study is primarily on content knowledge and pedagogical content knowledge, I will only describe those aspects of their model. Like KAT, the COACTIV model was developed to apply to the secondary level. The COACTIV model distinguished four levels of understanding of content being taught: (1) academic research, (2) a profound understanding of the mathematical content taught in school, (3) a command of the mathematical content covered at the level being taught, and (4) everyday mathematical knowledge that all adults who graduated from high school should have (p. 33).

In the COACTIV model, the content knowledge needed for teaching mathematics as synonymous with the second level: a profound understanding of the mathematics content taught in school. Baumert et al. (2010) cited this conceptualization of content knowledge for teaching as aligning with the National Council of Teachers of Mathematics (2000) and the National Mathematics Advisory Panel (2008). In regards to pedagogical content knowledge, the COACTIV model identified three dimensions: knowledge of mathematical tasks, knowledge of students' mathematical thinking, and explanatory knowledge (Baumert & Kunter, 2013). The first dimension, knowledge of mathematical tasks, is defined as "knowledge of the didactic and diagnostic potential of tasks, their cognitive demands and the prior knowledge they simplicity require, their effective orchestration in the classroom, and the long-term sequencing of learning content in the curriculum" (p. 33). The next two dimensions are based directly upon Shulman's (1986) general categorization of pedagogical content knowledge. Knowledge of students' mathematical thinking is defined as "knowledge of student cognitions (misconceptions, typical errors, strategies) and ways of assessing student knowledge and comprehension processes (Baumert & Kunter, 2013, p. 33). The last dimension, explanatory knowledge, is defined as "knowledge of explanations and multiple representations" (p. 33).

Table 6. Definitions in COACTIV's Framework

Term	Description	Examples
Research Knowledge	Academic research knowledge	Knowledge of Galois theory and functional analysis
Advanced Knowledge	"A profound mathematical understanding of the mathematics taught at school." (Baumert & Kunter, 2013, p. 33)	Knowledge of advanced concepts and procedures covered in lower secondary education—up to quadratic equations and first steps in similarity geometry
Basic Knowledge	"A command of school mathematics covered at the level taught." (Baumert & Kunter, 2013, p. 33)	Knowing what you expect your students to know or learn at the current level
Elementary Knowledge	"The mathematical everyday knowledge that all adults should have after leaving school." (Baumert & Kunter, 2013, p. 33)	Knowledge of basic mathematical operations and geometry that is covered in elementary school or familiar from everyday life
Knowledge of Mathematical Tasks	"Knowledge of the didactic and diagnostic potential of tasks, their cognitive demands and the prior knowledge they implicitly require, their effective orchestration in the classroom, and the long-term sequencing of learning content." (Baumert & Kunter, 2013, p. 33)	Knowing multiple ways to solve a mathematical task; comparing qualitatively different ways of solving a task; recognizing a tasks' potential for multiple solutions based on different representations
Knowledge of Student Cognition	"Knowledge of student cognitions (misconceptions, typical errors, strategies) and ways of assessing student knowledge and comprehension processes." (Baumert & Kunter, 2013, p. 33)	Recognizing, analyzing, and conceptually categorizing student errors; identifying parallelograms that students might have a hard time using base times height to find the area of; identifying possible student thinking
Knowledge of Explanations and Representations	"Knowledge of explanations and multiple representations." (Baumert & Kunter, 2013, p. 33)	Explaining why $-1 * -1 = 1$; identifying didactic values of considering multiple formulas for finding the surface area of a trapezoid

Chapter 3: Research Methods

Rationale

Studying Mathematical Knowledge for Teaching

In their 2008 article on mathematical knowledge for teaching, Ball and her colleagues acknowledged that there are several ways researchers could approach the question, "What mathematics do teachers need to know in order to teach effectively?" First, one could examine curriculum and standards—the material that usually dictates the content that teachers are expected to convey to their students—and make a list of what teachers need to know. Second, one could ask content specialists—such as mathematicians—to identify the core ideas and skills that are required to teach the curriculum. However, the disadvantage of both of these approaches is that they rely on knowledge that is unattached from the very act of teaching itself. While the curriculum delineates *what* content they need to teach, it doesn't uncover *how* that content must be understood or *what else* the teacher needs to know in addition. Mathematicians may have an advanced understanding of the content, but their understanding is for their *personal* use, not situated in the context of teaching *others* to build understanding.

Instead of choosing either of these routes, Ball et al. (2008) approached their question from the perspective of practice. Instead of speculating about what teachers needed to know, they chose to investigate what mathematical knowledge teachers used *in* and *for* teaching. It is commonly accepted that teachers need to know the content for which they are responsible to teach to students, however, Ball et al. were interested in examining what else teachers needed to know beyond that. By examining teaching, they conducted a sort of "job analysis" in order to better ascertain "the mathematical knowledge needed to carry out the work of teaching mathematics" (p. 395).

It is my intention to follow this approach in conducting my dissertation research. The framework by Ball et al. (2008) is the product of over twenty years of practice-based research that Ball and her colleagues have been conducting on teacher knowledge. Several other researchers have also attempted to answer the question of "What mathematical knowledge do teachers need to know in order to teach effectively?" however, not everyone has continued to follow the approach heralded by Ball. Some researchers have attempted to take the practice-based theory developed by Ball et al., which was developed using a collection of records of elementary teaching, and extend it to higher grade levels. However, Speer, King, and Howell (2015) claimed that such an extension is not necessarily appropriate. Instead, the authors challenged researchers to explore "the types of knowledge entailed in the work of teaching...through the same kinds of careful study of the mathematical demands of teaching that sparked the early work on mathematical knowledge for teaching (Ball and Bass 2000)" (p. 119).

Grounded Theory

In reading the works of Ball, I found it difficult to ascertain exactly what methodological approach she was using. However, there was a recurrent theme that sprung up in her descriptions:

- "In order to ground our inquiry, we analyze data from elementary classroom teaching of mathematics" (Ball & Bass, 2000a, p. 198),
- "We seek to identify patterns, themes, mathematical issues and lacunae, and to support the identification of those with evidence in the records" (Ball & Bass, 2000a, p. 201),
- "...when theoretical ideas emerge from observations of patterns across the data, we can use them as a sense for viewing other records, of other teachers' practices, and either reinforce or modify or reject our theoretical ideas in line with their adaptability to the new data.... This would permit the discussion of theoretical ideas to be grounded in a publicly shared body of data, inherently connected to actual practice" (Ball & Bass, 2003, pp. 5–6).

In these quotes, we can see that Ball and her colleagues aimed to *develop theory* that is *grounded in data* through the *identification of patterns*.

It is for this reason that I believe that Ball's approach to studying the practice of teaching agrees with the purpose and tenants of grounded theory. Strauss and Corbin (1994) defined grounded theory as "...a *general methodology* for developing theory that is *grounded in data* systematically gathered and analyzed" (emphasis added, p. 273). Herein we see the connection to Ball's method of studying the practice of teaching. Both approached theory development from the perspective that it should be intrinsically connected to data. In addition, not only do both agree on the approach to developing theory, but they also agree that the purpose of theory is to identify plausible relationships through systematic analysis of data, not to uncover "preexisting reality" (p. 279). Instead, Strauss and Corbin proposed that "...grounded theories...are systematic statements of plausible relationships" (p. 279). Similarly, Ball (1999) cited that her aim is to "...produce plausible analyses of teaching and learning that interplay mathematical perspective with pedagogy, with an eye to expand the range of mathematical possibility that might be seen, heard, located, and, in turn, nurtured, in teaching and learning" (p. 31).

Even if I temporarily put aside the methods used by Ball, I believe that grounded theory is still the best tool to use in answering my research questions. If I want to focus my analysis on the work of teaching, then staying grounded in data when developing my theory is of upmost importance. Also, the coding process for grounded theory is iterative and focused upon the development of theory. First, the researcher begins with open coding, which "...is the interpretive process by which data are broken down analytically" (Corbin & Strauss, 1990, p. 423). The purpose of using open coding is to help the analyst situate themselves in the data and to break "...through standard ways of thinking about (interpreting) phenomena reflected in the data" (p. 423). As the researcher begins to identify categories that emerge during open coding, they next engage in axial coding, wherein "...categories are related to their subcategories and those relationships are tested against data" (p. 423). Finally, in order to reach the point of saturation, which is "...when no new information seems to emerge during coding..." (Strauss & Corbin, 1998, p. 136) the grounded theory researcher must conduct theoretical sampling and coding. Theoretical sampling is defined as “data gathering driven by concepts derived from the evolving theory and based on the concept of ‘making comparisons’," (p. 191). By conducting theoretical sampling, the analyst will be able to conduct the final round of selective coding, which “...is the process by which all categories are unified around a central `core' category and categories that need further explanation are filled in with descriptive detail" (Corbin & Strauss, 1990, p. 424).

It is evident that Anselm Strauss and Juliet Corbin heavily influence my conceptualization of the grounded theory methodology. However, I feel that it is important for me to acknowledge that there are several approaches to grounded theory. Grounded theory was first introduced by Barney Glaser and Anselm Strauss in *The Discovery of Grounded Theory* (1999\1967). Since then, Glaser and Strauss have developed separate approaches to grounded theory and more recently Kathy Charmaz (2006) has developed an approach called constructivist grounded theory. Glaser's (1992) approach to grounded theory is a purely inductive process focused on theory development and is less structured. Strauss and Corbin (1998) use both inductive and deductive processes, stress the importance of verification, and use a more structured analytic process. Finally, Charmaz (2006) focus on the influence of the perspective of the researcher as they are involved in constructing the theory. The reason I chose to use Strauss and Corbin's (1998) approach is because I want to use the power of both induction and deduction (which as I mentioned previously, is similar to abduction) and because I like the structure that is provided using their method.

One question that naturally arises as a result of my choosing grounded theory as my methodological framework is why am I qualified to do grounded theory research? Strauss and Corbin (1998) cited six characteristics that a grounded theorist must possess.

First, a grounded theorist must have "the ability to step back and critically analyze situations" and "the ability to recognize the tendency toward bias" (p. 7). As I have taken courses in qualitative research methodology and been involved in several projects that have used grounded theory, I have begun to develop the ability to critically analyze situations. However, Strauss and Corbin emphasize that the ability to analyze must be coupled with the ability to step back and recognize the tendency toward bias. Since I am familiar with the literature surrounding mathematical knowledge for teaching, I realize that this might be one area I struggle with. However, Erickson (1986) argued that there is a way to combat bias and preconceived notions in order to make sure that the data is speaking for itself.

> *One can argue that there are no pure inductions. We always bring to experience frames of interpretation, or schemata. From this point of view, the task of fieldwork is to become more and more reflectively aware of the frames of interpretation of those we observe, and of our own culturally learned frames of interpretation we brought with us to the setting. (p. 140)*

Acknowledging that it is natural to bring hypotheses to any research, Erickson claimed, "...`observing without any preconceptions'...is a misleading characterization. Preconceptions and guiding questions are present from the outset, but the researcher does not presume at the outset to know where, specifically, the initial questions might lead next" (p. 143).

Next, Strauss and Corbin (1998) stated that a grounded theorist must have "the ability to think abstractly" (p. 7). In the coursework I have completed for my undergraduate and graduate degree, I have been trained to think abstractly in the field of mathematics[4]. While this field does differ from social science in many ways, I believe that I can draw upon my experience with reasoning abstractly in mathematics in order to reason abstractly in conducting grounded theory research.

Strauss and Corbin (1998) also claimed that grounded theorists must have "the ability to be flexible and open to helpful criticism" (p. 7). Unlike the previous traits, this is a characteristic I have more recently begun to develop. I often excelled in school and sought to please my parents at home, so criticism is something I was rarely exposed to. However, I am finding that in learning to become an educational researcher, I must learn to be open to "helpful criticism." While I'm not opposed to "helpful criticism" at face value, I often find that I want to work privately until I can present what I view to be as my finished or perfected work. Yet, I realize that by limiting what I share and, therefore, what feedback I receive, I am limiting my personal growth. Thus, this is a characteristic I feel I am still developing, but conscious of my need for.

Next, Strauss and Corbin claimed that grounded theorists need "sensitivity to the words and actions of respondents" (p. 7). While on one hand I am naturally a "listener," I'm also developing my sense of "sensitivity" to what people say. While reading in preparation for writing my literature review, I realized from conversations with others that often the way I would summarize a paper was heavily influenced by the main point that I took away from it. Once I recognized that I was confounding what I took away from the paper with the main argument the paper made, I felt like I was able to begin

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[4] It is important to note that although I am completing my dissertation research in the field of mathematics education, I am still earning a doctoral degree in mathematics. In particular, I have taken all of coursework required for doctoral students in the mathematics department who study pure and applied mathematics and have passed both qualifying and comprehensive exams in mathematics. Therefore, I view myself as a mathematician who is trained to conduct research in mathematics education.

parsing the two apart. The final characteristic that Strauss and Corbin identified as something that grounded theorist must have is "a sense of absorption and devotion to the work process" (p. 7). While perhaps I am making too broad of a generalization, I believe that it is safe to say that most people who pursue a Ph.D. must possess such devotion. Therefore, I believe that although I may not be a grounded theorist specialist and I recognize that I still have much to learn, I am able to complete grounded theory research in an appropriate manner.

Stimulated-Recall Interviews

The ideal study of the pedagogical work and knowledge entailed in teaching would collect data on the exact work done and knowledge used while teaching. However, since such data mostly occurs in the mind of the teacher and often is not observable, such instantaneous data collection is impossible. Thus, the practice of teaching must be studied through other means. This could be done by asking teachers, "What knowledge *do* you use when...?" However, hypothetical questions are not situated *in practice*, which is a scenario I want to avoid. Alternatively, we could a priori ask, "What knowledge *did* you use when...?" However, retrospective questions depend upon the interviewer being able to correctly recall the situation and reconstruct the knowledge they used. A third alternative would be to use the method of think-aloud protocol (Lewis, 1982), which asks the interviewee to verbalize their thoughts as they work through an activity. However, using a think-aloud protocol with observations would necessitate that the teacher think-aloud while teaching, which would be unnatural and disruptive for the students.

An alternative to the previous methods is to use stimulated-recall interviews. Bloom (1953) is often cited as the founder of the technique, although he drew inspiration for his interview methodology from other researchers. In his study, Bloom audio recorded lectures on tape and them played them back to students during interviews in order to revive "memories after the class in order to determine the thoughts which occurred during the class" (p. 161). According to Bloom, "the basic idea underlying the method of stimulated recall is that a subject may be enabled to relive an original situation with vividness and accuracy if he is presented with a large number of cues or stimuli which occurred during the original situation" (p. 161). Bloom admitted that even stimulated-recall is bound to include some elements of retrospective thought. However, he "found that as high as 95 per cent [*sic*] accurate recall of such overt, checkable events within two days" (p. 162). Thus, Bloom suggested that researchers using stimulated-recall could anticipate "that the accuracy of the recall of conscious thoughts is high enough for most studies of learning situations--if the interviews are made within a short time after the event" (p. 162).

Like Bloom (1953), other researchers have discussed the limitations and problems associated with using stimulated-recall. Yinger (1986) pointed out "that the participant is not likely to know if a thought is recalled or constructed. A researcher is even less likely to be able to untangle these two very different types of reports" (p. 270). Also, when the interviewer is asked to view a video or audio recording of themselves and then report what they were thinking in that moment, they are tasked with the cognitive demand of understanding and interpreting their past behavior, which is not easily done. Additionally, Calderhead (1981) pointed out that "some areas of a person's knowledge have never been verbalized and may not be communicable in verbal form" (p. 213). In particular, he points out that this may be especially true for experienced teachers, since they most likely have reached a level of cognitive atomization. Since my plan is to observe and interview experienced teachers, this is a limitation of my study that I will have to keep in consideration.

Data Collection

Participants

The graduate student instructors involved in my study had experience teaching precalculus courses. Here, I defined experienced as any graduate student instructor who has taught precalculus for at least two semesters previously. The reason why I chose to study experienced instructor is twofold. First, the mathematics department that my participants taught in required that precalculus instructors use specific instructional methods. Instruction is centered on group work and very little time is allocated each lesson for lecturing. Given the fact that this is atypical, although increasing in prevalence, for undergraduate courses, my assumption is that novice first-year instructors will not have taught using primarily group work before.

Second, precalculus instructors are given and asked to follow specific lesson guides for each day. While the standardization of lesson guides is beneficial in the sense that it provides the teacher with suggested sequencing, examples, and timing, novice instructors will still be teaching the content for the first time. Therefore they may struggle with not knowing the content as well as they need to in order to teach it or the expansive amount of variation associated with the content, such as student conceptions and misconceptions or different approaches to teaching procedures and concepts. So taking these two factors into consideration, I chose to observe experienced rather than novice graduate student instructors.

Sampling

During the first semester that I observed instructors, I asked them to provide me with three different dates, spread out through the semester, where I could come into their classroom to observe. Since the instructor was picking these dates, I observed lessons at random. Also, there were several lessons that I only observed part of because even if they were spread out over multiple days, I only asked to observe one day at a time. During the second semester that I observed instructors, I chose specific lessons that I wanted to observe and verified that these dates would work with the instructors. The lessons that I chose were more procedural in nature, because I thought that these would give me an opportunity to observe examples that could be enacted as either high or low cognitive demand, since procedural tasks can be enacted with or without an emphasis on connections. Several of these lessons were spread out over two days, so I would come to the classroom both days to observe.

Pre-Observation Interview

Before each classroom observation, I meet with the instructor to discuss their lesson plan. In particular, I focused on the examples that they chose to include and unpacked *why* they chose to include them. Typically, we met the morning before they taught the lesson, although occasionally this did not work with the instructors' schedule, so we would meet the day before. The full semi-structured pre-observation interview protocol can be found in Appendix B.

Observation

During the classroom observations, I videotaped the examples that the instructor enacted. Since my dissertation primarily focuses on the instructor, I only used one video camera on a tripod to capture what the instructor was doing. I also took detailed field notes in my observation protocol, which can be found in Appendix B. During each example, I would capture both what was said and what was written on the board. Then, if there was time between the end of one example and the beginning of another, I would capture my thoughts related to the cognitive demand of the example. After each observation, I would fill in more details about each example and reflect on the lesson as a whole. The full observation protocol that I used can be found in Appendix B.

Post-Observation Interview

Usually within 24 hours after each classroom observation I conducted a stimulated-recall interview with the instructor, lasting between 30 minutes to an hour. Occasionally this timeframe did not work out with the instructor, but we were always able to meet within 48 hours of the class. Before the interview, I will complete a pre-analysis of the video observation and tag moments to unpack with the instructor. In particular, I chose moments that related to decompressing, bridging, trimming, eliciting and interpreting student thinking, and using multiple representations[5].

[5] I chose to focus on decompressing, bridging, and trimming because these three teaching practices were identified by McCrory, Floden, Ferrini-Mundy, Reckase, and Senk (2012). Since instructors presented examples in different ways, I wanted to know how they were gaging whether or not students were following or understanding the example, which is why I asked about eliciting and interpreting student thinking. Finally, many high cognitive demand examples involve multiple representations, which is why I wanted to unpack this aspect.

During the interview, I showed the instructor 30 seconds to one minute of the moments that I tagged to help them recall what was happening. I then asked the instructor about the pedagogical work and the mathematical knowledge they used during the example enactment. As I brought up earlier, the use of interviews is standard in grounded theory but not always recommended by researchers who are studying knowledge used in teaching. My initial belief was that using stimulated-recall interviews would aid me in understanding the mathematical knowledge the instructor used *in* teaching. However, it is possible that the teachers' reflections were inaccurate or contrived.

Data Analysis

Since my dissertation follows a three-paper structure, I have reserved discussion of the analytical frameworks I used in my analysis for each individual paper (Chapters 4-6). However, in the following sections I will discuss the general procedures that I followed during my data analysis.

Primary Coding Stages

There were four stages of coded that I conducted for my data analysis.

Cognitive demand. First, I used my modified framework for cognitive demand (see Table 7) to code the cognitive demand of the example. The purpose of conducting this stage of coding was to identify examples that were enacted at a high level of cognitive demand for me to analyze in my subsequence stages of coding.

Roles of instructors. Next, I went through each high cognitive demand example and segmented it into times where the instructor was modeling content, practices, and strategies for students; facilitating whole class discussions; and monitoring students as they worked through parts of the example individually or in small groups. The purpose of doing this stage of analysis was to help answer RQ2 and the second parts of RQ3 and RQ4.

Pedagogical work. In my third stage, I conducted open, axial, and selective coding (which are described in the next section) of the pedagogical work entailed in enacting high cognitive demand examples. The purpose of doing this stage of analysis was to help answer RQ3.

Mathematical knowledge. In my final primary stage of coding, I conducted open, axial, and selective coding of the mathematical knowledge entailed in enacting high cognitive demand examples. The purpose of doing this stage of analysis was to help answer RQ4.

Secondary Coding Stages

Strauss and Corbin (1998) described grounded theory coding as consisting of three stages: open coding, axial coding, and selective coding.

Open coding. Open coding requires the analyst to break down the data and examine it for the purpose of comparing for similarities and differences and identifying emergent categories. By comparing data to bring to light similarities and differences, the analyst should begin to identify patterns that emerge from the data. Given these patterns, conceptually similar ideas are grouped together to form categories. In identifying emergent categories, it is also important to define their properties and how they vary dimensionally. Strauss and Corbin (1998) suggested three different ways of doing open coding: line-by-line analysis, sentence or paragraph analysis, and entire document analysis. In order to uncover both specific and general categories, open coding should be done at each level.

Axial coding. Following the identification of emergent categories, axial coding is used to uncover how the categories are related to their subcategories. In open coding, the data is broken down into discrete pieces of information to ensure that it can be closely examined. The purpose of axial coding is to "begin the process of reassembling data that were fractured during open coding" (Strauss & Corbin, 1998, p. 124). To identify the relationships between categories and subcategories, the analyst must both consider the structure and the process by which they are connected. The structure is manifested in the conditions that answer the questions of why, where, how come, and when. The process is manifested in the actions and interactions "made by individual or groups to issues, problems, happenings, or events that arise under those conditions" (p. 128) and results in consequences. Each of these aspects (conditions, actions/interactions, and consequences) must be identified through axial coding in order to establish the relationships between the categories and subcategories. Once saturation has been reached, which occurs "when no new information seems to emerges during coding" (p. 136), then we are ready to move on to the final stage of coding.

Selective coding. The final stage of coding for grounded theory is necessary for theory development to have conceptual density, which "...refers to richness of concept development and relationships—which rests on great familiarity with associated data and are checked out systematically with these data" (Strauss & Corbin, 1994, p. 274). In the process of selective coding, categories are refined and integrated. In particular, identification of a central "core" category is important. The "core" must appear frequently in the data, be logically and consistently related to the categories, sufficiently abstract, possess explanatory power, and yet account for variation. The analyst must also refine the theory by evaluating for internal consistency and logic, filling in underdeveloped categories, trimming overdeveloped categories, and validating. Instead of functioning as a separate stage of coding, selective coding rather is used to strength both the categories identified through open coding and the relationships established during axial coding.

For an overview of my study and to see how my research questions, data collection, and data analysis align, I have included a study diagram in Figure 16 in Appendix C.

CHAPTER 4: EXAMINING THE ROLE OF THE INSTRUCTOR

The cognitive demand of mathematical tasks is something that has been widely studied in the literature (Boston & Smith, 2009; K. J. Jackson et al., 2012; Kisa & Stein, 2015; Smith & Stein, 1998; Stein et al., 1996) . Studies have found that high cognitive demand tasks provide students with more opportunities to learn (Floden, 2002; K. Jackson et al., 2013; Smith & Stein, 1998; Stein et al., 2007). Researchers have also found that high cognitive demand tasks are difficult for instructors to enact (Henningsen & Stein, 1997; Rogers & Steele, 2016) and are related to mathematical knowledge for teaching (Charalambous, 2010). But what would it mean to have a high cognitive demand mathematical example? Examples are different from mathematical tasks that are primarily worked on by students. Examples may involve input from students or opportunities for students to work independently or in groups on parts of the example, but usually the teacher plays a leading role in working out or explaining the mathematics.

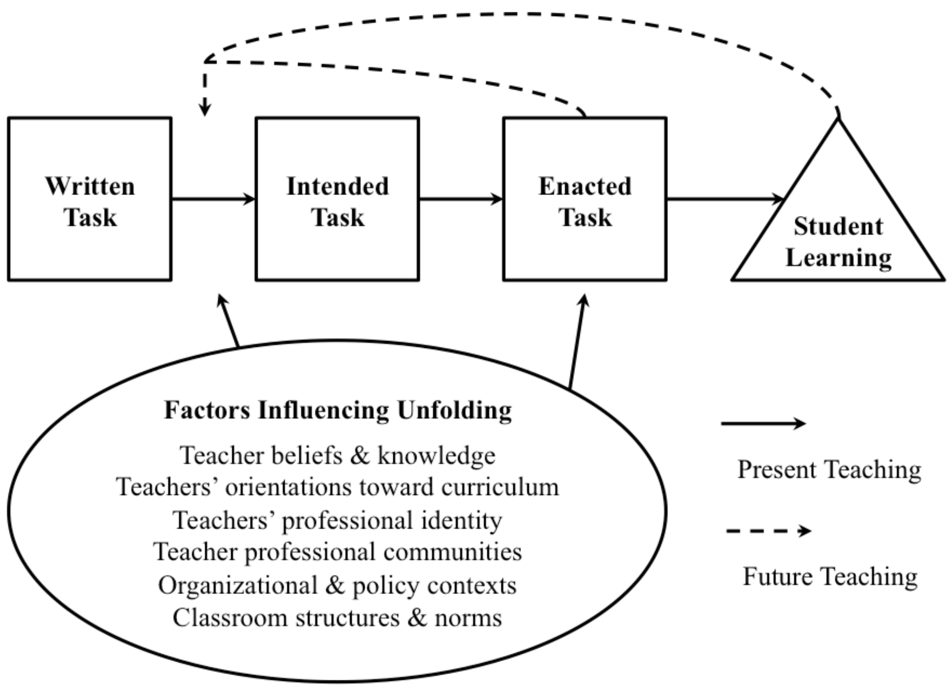
While studies have shown that students do not learn as much from observing a worked out example as they do from actively engaging in the problem solving process (Richey & Nokes-Malach, 2013), the examples that teachers use still play an important role in the learning process (Chick, 2007; Muir, 2007; Rowland, 2008; Zaslavsky & Zodik, 2007). In particular, Ball and her colleagues (TeachingWorks, 2017) identified "explaining and modeling content, practices, and strategies" as a high-leverage practice that is part of the core fundamentals of teaching. In addition, we create a dissonance in our classrooms if we expect our students to successfully engage with high cognitive demand tasks, but only ever present low cognitive demand examples. The purpose of this paper is to modify the Task Analysis Guide developed by Smith and Stein (1998) so that it can be used to analyze examples. In addition, I illustrate how high cognitive demand examples can differ in terms of the role and participation of the teacher and the students.

Conceptual Frameworks

Task Unfolding

Stein et al. (1996) defined a mathematical task as "a classroom activity, the purpose of which is to focus students' attention on a particular mathematical idea" (p. 460). They also described the phases involved in the unfolding of a mathematical task and the factors that influence this unfolding. In 2007, Stein, Remillard, and Smith generalized task unfolding to apply to curriculum unfolding more generally, but the underlying process remained the same. In Figure 4, the rectangle boxes represent the three phases of task unfolding. The written task describes how the mathematical task is represented in the written curriculum or instructional materials. The intended task describes the teacher's plan for implementing the task during instruction. Finally, the enacted task captures how the mathematical task is actually implemented during instruction. While each phase has an impact on student learning (represented by the triangle in Figure 4), studies have shown that the enacted task has the greatest impact (Carpenter & Fennema, 1991). The bottom oval identifies some factors that influence how teachers plan out a task for implementation in the classroom and how the task is actually implemented in the classroom. Finally, it is important to note that the return arrows from the enacted task and student learning represent the impact that these will have on future teaching actions.

Figure 4. The Phases and Factors Influencing Task Unfolding



Cognitive Demand of Tasks

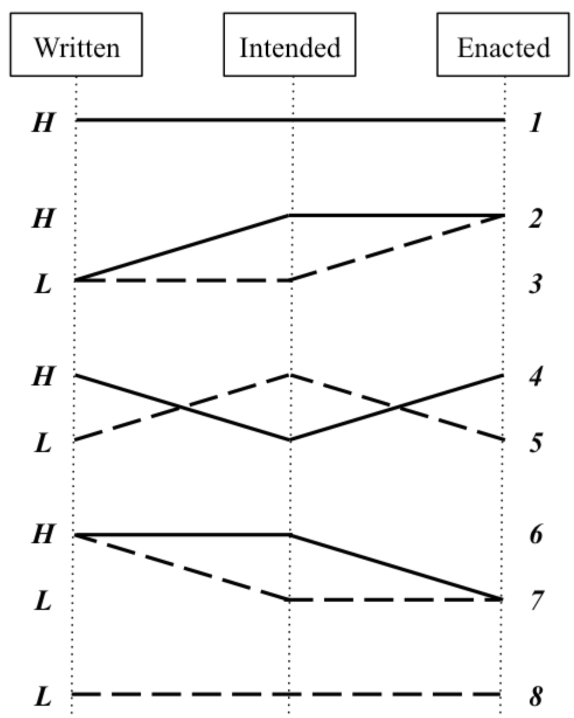
In order to differentiate between tasks of different types, Smith and Stein (1998) analyzed the cognitive demand of a task. They defined lower-level demand tasks as "tasks that ask students to perform a memorized procedure in a routine manner" and higher-level demand tasks as "tasks that require students to think conceptually and that stimulate students to make connections" (p. 269). Each of these categories was then broken down into two subcategories: memorization, procedures without connections, procedures with connections, and doing mathematics. Smith and Stein differentiated procedures *with* and *without* connections as representing differing levels of cognitive demand. They separated these two types of tasks in order to categorize mathematical tasks that “use procedures, but in a way that *builds connections to the mathematical meaning*” of the underlying concept as a higher-level demand task. Doing mathematics tasks are categorized as higher-level demand tasks that require “students to explore and understand the nature of relationships” (p. 347).

To aid in differentiating between the different types of tasks, Smith and Stein (1998) developed the Task Analysis Guide, which lists characteristics of the four types of mathematical tasks. Later, when utilizing the Task Analysis Guide to code the third phase of task unfolding, Stein et al. (1996) added a third type of lower-level demand task called unsystematic exploration. This type of task, which applies to only the third phase of task unfolding, describes declines in cognitive demand that are characterized by “motivated student engagement, well-intentioned teacher goals for complex work, and well-managed work” but “the cognitive activity…was not at a high enough level to be characterized as engagement in complex mathematical thinking and reasoning” (p. 478).

Categorizing Task Unfolding Using Cognitive Demand

In their 1996 study, Stein et al. used the Task Analysis Guide to analyze a sample of 144 tasks that were implemented in reform-oriented classrooms. They focused on the transition from the second to the third phase of task unfolding and found that the majority of the tasks were coded as maintaining or declining in cognitive demand. They also found that “the higher the cognitive demands of tasks at the set-up phase, the lower the percentage of tasks that actually remained that way during implementation" (p. 476). This finding provides confirming evidence for the claim that tasks with high cognitive demand are difficult to enact (National Council of Teachers of Mathematics, 2014, p. 17). In 2010, Charalambous conducted a similar case study, but explicitly categorized task unfolding by the type of path they follow (Figure 5). In his categorization, Charalambous used the Task Analysis Guide to code cognitive demand as high or low at each phase in task unfolding, which resulted in eight possible types that a task unfolding could follow. It is worth noting that Charalambous only observed five of the eight possible types of task unfolding (Types 1, 5, 6, 7, and 8 in Figure 5) in the cases he studied and I added in the type numberings for ease of reference.

Figure 5. Categorization of Possible Types of Task Unfolding



H High-Level Tasks: *Doing mathematics* and *Procedures with connections*
L Low-Level Tasks: *Procedures without connections, Memorization,* and *Unsystematic exploration* (the latter code applies only to task enactment)

Purpose

The purpose of this study is to propose a revised framework for assessing the cognitive demand of examples and examine the roles that instructor take when enacting high cognitive demand examples. First, I will spend some time explaining how I have view examples as different from mathematical tasks that students are responsible to work on during class. Then I propose that we modify the language used in the Task Analysis Guide in order to allow for different ways of enacting high cognitive demand examples. Finally, I describe three different roles (modeling, facilitating, and monitoring) that instructors might take on when enacting high cognitive demand examples and provide narrative descriptions of what these role profiles look like in undergraduate precalculus classrooms.

Assessing the Cognitive Demand of Examples

Before introducing my modified framework for analyzing the cognitive demand of examples, I first spend some time differentiating between examples and exercises, which are two types of mathematical tasks. I distinguish between the two types of tasks because while Stein et al.'s (1996) definition of mathematical tasks is broad enough to encompass both examples and exercises, the Task Analysis Framework (Stein & Smith, 1998) seems to apply more to exercises than examples. The main difficulty that I found in using the Task Analysis Framework to analyze examples is that the language that Smith and Stein use often specifies that students are doing the mathematical work. However, examples can be presented in a variety of formats, which might include the teacher modeling how to work through the mathematics in the example as students take notes. Therefore, I propose a modified framework for analyzing the cognitive demand of examples that focuses on *what* mathematics is included in the examples instead of *who* is doing the mathematics.

While it is still important to provide students with opportunities to work on exercises, I argue that examples also provide students with opportunities to learn. Opportunities to learn are defined as "whether or not…students have had the opportunity to study a particular topic or learn how to solve a particular type of problem presented on a test" (Husén, 1967, pp. 162–163). While studying actual student learning is important, studies have found that differences in actual learning are related to differences in opportunities to learn (Husén, 1967; National Research Council, 2002). Therefore, it is important to understand what high cognitive demand examples look like, since they provide students with an opportunity to learn how to solve high cognitive demand tasks on their own. Also, I argue it is important to differentiate between opportunities to learn and opportunities for students to struggle, since examples that presented by just the instructor can still bring explicit attention to concepts. Finally, I illustrate how an example related to the Law of Sines might be transformed to different levels of cognitive demand to illustrate each category in my modified framework.

Differentiating Exercises from Examples

Stein et al.'s (1996) definition of mathematical tasks is broad and has been interpreted in many ways. In my work, I differentiate between mathematical tasks that are given to students to work on (e.g., exercises) and mathematical tasks that are completed as a whole class activity (e.g., examples). In particular, I define examples as mathematical problems that are completed as whole-class activities and solved for illustrative purposes. For example, to help students understand why trigonometric equations can have infinite families of solutions, an instructor might use the example of $\sin\theta = -1/2$. Or if an instructor is teaching the completing the square algorithm, they might introduce it by working through several examples before asking students to work through related problems. While some authors have found that it is not necessary to differentiate between tasks in this way when analyzing the cognitive demand (Mesa et al., 2012), I found it difficult to use the Task Analysis Guide (Smith & Stein, 1998) to analyze the cognitive demand of examples. In particular, Smith and Stein's framework for the cognitive demand of mathematical tasks makes it clear that they assume that students are the ones responsible for *doing the mathematics* in a mathematical task. While it may be true that some instructors ask *students* to do the mathematics involved in an example, there are also times when *instructors* work out the mathematics for the students as part of the example.

I found that it was important to conceptualize examples *independent* of who is doing the mathematical work due to the fact that some instructors choose to model examples for students, while others involve students more in working out the mathematics. While these different approaches to presenting examples may provide students with different opportunities to learn, both approaches can be used to illustrate concepts, practices, and strategies. In either approach, one important feature of examples is that they include explanations. Bills, Dreyfus, Mason, Tsamir, Watson, and Zaslavsky (2006) emphasized that "providing worked-out examples with no further explanations or other conceptual support is usually insufficient", as "learners often regard such examples as specific (restricted) patterns which do not seem applicable to them when solving problems that require a slight deviation from the solution presented in the worked-out example (Reed *et al.* 1985; Chi *et al.* 1989)" (p. 140).

Analyzing the Cognitive Demand of Examples

While I originally planned to use the Task Analysis Guide (1998) to analyze the cognitive demand of examples, I ended up needing to create a modified framework .The biggest difference between my modified framework and the Task Analysis Guide is the language that is used concerning *who* is expected to be doing the mathematics. For example, in the original framework, students are situated as the doers of mathematics. This makes sense, as the framework was developed to analyze mathematical tasks that students engage with during instruction. However, examples may involve some work done by students and other work done by the instructor. Still, many of the same metrics can be used to measure the cognitive demand. Below, I go into more detail concerning how I modified each of the cognitive demand categories to fit with the context of examples.

Table 7. Modified Framework for Analyzing the Cognitive Demand of Examples

Lower Level
Memorization • Involve either reproducing previously learned facts, rules, formulae, or definitions OR committing facts, rules, formulae, or definitions to memory. • Cannot be solved by using procedures because a procedure does not exist or because the time frame in which the example is being completed is too short to use a procedure. • Are not ambiguous—such examples involve exact reproductions of previously seen material and what is to be reproduced is clearly and directly stated. • Does not make connections to the meaning that underlies the facts, formula, or definitions being learned or reproduced.
Procedures Without Connections • Are algorithmic. Use of the procedure is either specifically called for or its use is evident based on prior instruction, or placement of the example. • They can be solved by applying well-established procedures.

- Require limited cognitive demand for students to follow. There is little ambiguity about what needs to be done and how to do it.
- Have no connection to the concepts or meaning that underlie the procedure being used.
- Are focused on producing correct answers rather than developing mathematical understanding.
- Require no explanations or explanations that focus solely on describing the procedure that was used (e.g., the instructor or students simply describe the steps they followed in solving a problem).

Higher Level

Procedures With Connections

- Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas (i.e., the example can be solved using a procedure but the procedure is connected to the underlying mathematical concept).
- Suggest pathways to follow (explicitly or implicitly) that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts.
- Usually are represented in multiple ways (e.g., visual diagrams, manipulatives, symbols, problem situations). Making connections among multiple representations helps to develop meaning.
- Require some degree of cognitive effort for students to follow. Although general procedures may be followed, they cannot be followed mindlessly. Students' attention needs to be focused on the conceptual ideas that underlie the procedures in order to develop understanding.

Doing Mathematics

- Require complex and nonalgorithmic thinking (i.e., there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the example, example instructions, or previously worked-out examples).
- Require the instructor or the students to explore the nature of mathematical concepts, processes, or relationships.
- Involve explicit self-monitoring or self-regulations of cognitive processes.
- Require the instructor or students to access relevant knowledge and experiences and make appropriate use of them in working through the example.
- Require the instructor or students to analyze the example and actively examine example constraints that may limit possible solution strategies and solutions.
- Require considerable cognitive effort for students to follow and may involve some level of anxiety for the students due to the unpredictable nature of the solution process required.
- May include, but are not limited to, making and testing conjectures, framing problems, looking for patterns, examining constraints, knowing when the problem is solved, justifying, and explaining.

In each of the category descriptions, I first replaced the word "tasks" with "examples." The first category, memorization examples, exactly mirrors the original description of memorization tasks. The second one, procedures without connections examples, is similar, but required slight modifications. Here, I will address the modifications I made to the language. However, later I address a general modification that I made concerning how cognitive demand is interpreted, but this modification applies across categories. The primary modification I made to the language of the second lower-level demand category was in the final descriptor concerning explanations. Since it is possible that either the instructor or students may be giving explanations during an example, I modified the descriptor to reflect that it is not important *who* is explaining, but rather *what* they are explaining.

In the third category, procedures with connections examples, I left the first descriptor as is because even if the instructor is working out the example, they should still be focusing students' attention on the use of procedures for developing deeper understanding. I also modified the final descriptor to match this language and reflect the fact that while students may not be responsible for successfully completing the example, their attention should be focused on developing understanding.

The final category, doing mathematics examples, was modified the most. I modified the second descriptor so that it includes the instructor or the students working through the example. I also removed "understand" from this descriptor, since examples are primarily used for explaining concepts and not as opportunities for students to demonstrate understanding. That is not to say that examples can never be used in this way, but rather a way to highlight that they are usually employed as tools for building, not testing, understanding. Since the purpose of an example is to explain or model content, practices, and strategies, I changed the language used in the third descriptor to reflect the fact that cognitive processes should be made explicit as the example is being worked out. I modified the fourth and fifth descriptors so that they included the phrase "instructor or students". The sixth descriptor talks about cognitive demand, which I will talk about more below, but I did add the phrase "for students to follow", which can apply to examples where both the instructor and students are responsible for working through the mathematics. Finally, the last descriptor essentially remained the same.

In the procedures without connections, procedures with connections, and doing mathematics example descriptions, the phrases “cognitive demand” and “cognitive effort” are used. In the original framework, it is obvious that cognitive demand is dependent upon the students, since they are the ones completing the mathematical tasks. However, what would it mean for an example to require cognitive effort if the instructor is the one who is working through the mathematics? While it may be the case that the instructor would find the example cognitively demanding themselves, this is not as likely. However, students may still find the example to be cognitively demanding, even if they were not responsible for doing the mathematics. To capture this difference, I modified the language concerning cognitive demand to make it clear that this is a student, not instructor, dependent variable.

Differentiating Students’ Opportunities to Struggle

In my framework, I have attempted to define cognitive demand in a way that is independent of *who* is working through the mathematics. However, since much of the work on cognitive demand has been situated in the context of mathematical tasks that are given to *students*, a natural question that arises is, “How can an example be cognitive demanding if *students* are not the ones working through the mathematics?” To answer this question, I will explain how my modified framework for cognitive demand differentiates between the *mathematical* cognitive demand of an example and *student struggle*.

In my work, I conceptualize high cognitive demand examples as examples that bring explicit attention to concepts and provide students with *opportunities* to struggle with important mathematics. This conceptualization builds upon Hiebert and what

(2007) key features of teaching that promotes conceptual development: explicit attention to concepts and student struggle with important mathematics. The authors identify explicit attention to concepts as “treating mathematical connections in an explicit and public way” (p. 383) and student struggle with important mathematics as “the engagement of students in struggling or wrestling with important mathematical ideas” (p. 387). In their work, the authors are careful to define struggle as meaning that “students expend effort to make sense of mathematics, to figure something out that is not immediately apparent” and not “needless frustration or extreme levels of challenge created by nonsensical or overly difficult problems” (p. 387).

While it is important to consider whether or not students actually *engage* in this struggle, my modified framework for cognitive demand focuses on providing students with *opportunities* to struggle. If one instructor chooses to work through all of the mathematics at the board, while another instructor lets students work through parts of the example, then the opportunities to struggle may be different. In the first case, the students’ opportunities to struggle are mostly internal and may only be observable if they ask questions. In the second case, the students’ opportunities to struggle are more observable as they work through the mathematics.

This conceptualization of opportunities to struggle is based upon the work of Stein, Correnti, Moore, Russel, and Kelly (2017). Starting with Hiebert and Grouws’ (2007) two key features, Stein et al. (2017) built a matrix that reflected how explicit attention to concepts (EAC) and students’ opportunities to struggle (SOS) can interact and be enacted at different levels (Figure 6). According to Stein et al., teaching that falls in Quadrant 2 “can take a variety of forms, [but] often involves teacher demonstration of a general procedure for solving a problem with time taken to explain concepts as they relate to procedures and to encourage and entertain student questions" (p. 4). However, this quadrant still captures high cognitive demand tasks, since it still involves multiple representations, explaining concepts, and drawing connections. Even though students' opportunities to struggle are limited, "that does not mean, however, that students can mindlessly follow the pathway, but rather, they have to think about what they are doing and why" (p. 4).

Figure 6. Stein et al.'s (2017) Matrix Comparing High and Low SOS and EAC

		Students' Opportunity to Struggle	
		High	Low
Explicit Attention to Concepts	High	Quadrant 1 High EAC High SOS	Quadrant 2 High EAC Low SOS
	Low	Quadrant 3 Low EAC High SOS	Quadrant 4 Low EAC Low SOS

In my conceptualization of cognitive demand, I argue that cognitive demand is more dependent upon high levels of explicit attention to concepts than high levels of students' opportunities to struggle. In particular, the first three characteristics of procedures with connections examples all focus on concepts ("developing deeper understanding of mathematical concepts and ideas", "broad general procedures that have close connections to underlying conceptual ideas", and "making connections among multiple representations…to develop meaning"). It is the final characteristic, "requiring some degree of cognitive effort for students to follow” that captures opportunities for students to struggle. Similarly, the first two characteristics of doing mathematics examples focus on concepts, while the next three focus on explicit attention to the cognitive processes involved in solving the problem, and only the sixth characteristic focuses on opportunities for students to struggle. Therefore, high cognitive demand examples might fall in either Quadrant 1 or Quadrant 2. While these two quadrants do provide students with different opportunities to struggle, they both “involve making connections, analyzing information, and drawing conclusions” (Van de Walle et al., 2013, p. 36), which are some of the essential features of high cognitive demand tasks.

It is important to note that although I claim that high cognitive demand examples can be enacted with either high or low levels of opportunities for students to struggle, I am not claiming that it is not important to provide opportunities for students to struggle. In particular, my dissertation focuses only on the examples used during class, and not other activities such as group work, where students might be provided with higher levels of opportunities to struggle. Also, my work is focused on identifying what high cognitive demand examples might look like and does not examine the impact of these examples on actual student learning. However, it is interesting to note that Stein et al. (2017) found that students in Quadrant 2 classrooms performed well, but not quite as well as students in Quadrant 1 classrooms, which may suggest that “there could be affordances for learning associated with struggle but that some forms of bounded struggle might be worth exploring” (p. 16).

Transforming the Cognitive Demand of an Example

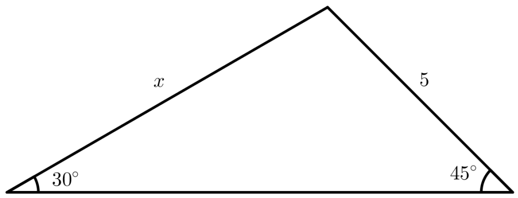
The purpose of this section is to demonstrate how an example can be presented at different levels of cognitive demand. The example that I present here comes from one of the examples that I observed an instructor enact in his classroom. The example was couched in a lesson that introduced the Law of Sines and Cosines and was situated as the first example to be completed after going through the proofs of each law. In Table 8, I include the four examples at the different levels of cognitive demand, and then I will explain how each one exemplifies the descriptors associated with that category.

The first version of the example is a memorization example because it involves reproducing the Law of Sines, which would have already been presented earlier in the lesson. Also, it cannot be solved using a procedure, is not ambiguous because it clearly and directly states what is to be reproduced, and does not make connections to the underlying meaning. The second version of the example is what was provided to the instructor in the written lesson guides. This version is a procedures without connections example because it is algorithmic, the procedure is specifically called for, it can be solved by applying a well-established procedure, requires limited cognitive demand for students to follow, is not ambiguous about what needs to be done or how to do it, has no connections to the concepts or meaning that underlie the procedure being used, is focused on producing the correct answer rather than developing mathematical understanding, and requires no explanation.

Table 8. Transforming the Cognitive Demand of an Example

Lower Level

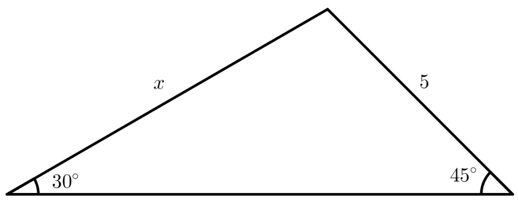
Memorization



Use the Law of Sines and the given triangle to fill in the missing information below.

$$\frac{x}{\sin(45)} = \frac{?}{\sin(30)}$$

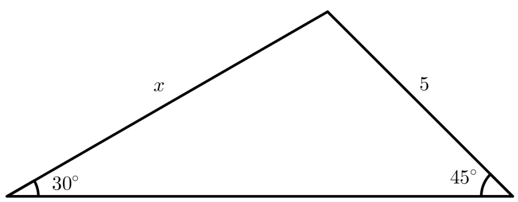
Procedures Without Connections



Use the Law of Sines to solve for the unknown side length, x, in the given triangle.

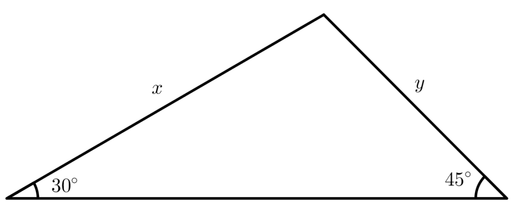
Higher Level

Procedures With Connections



Identify and use a procedure that can help us solve for the unknown side length, x, in the given triangle.

Doing Mathematics



Explain how the unknown side lengths, x and y, in the given triangle are related.

The third version of the example is close to how the instructor modified the written example to be included in his intended lesson plan. This version differs from the previous one in that it does not suggest a solution pathway. Rather, figuring out a solution strategy is part of the example itself. Also, the example focuses specifically on helping students to identify a procedure that can be used. In order to do this, the instructor should focus on why the Law of Sines is appropriate to use here and perhaps even why the Law of Cosines is not appropriate. Since students often struggle with identifying what procedure is appropriate to use in order to solve a problem, this would require some cognitive effort on the part of students. However, as illustrated in the lesson that I observed, the instructor would need to carefully attend to these features in order to maintain the higher level of cognitive demand.

In the final version of the example, the focus is no longer on following a procedure or producing a correct answer. Rather, the purpose of this example is illustrates how we can make connections between variables, even if they are unknown. This is a doing mathematics example because it requires nonalgorithmic thinking, exploration of mathematical relationships, accessing relevant knowledge and making appropriate use of it, analyzing the figure and examining constraints that may limit possible solution strategies (i.e., non-right triangle), and considerable cognitive effort on the part of the students. As the instructor is working through this example, it is important that they make cognitive processes explicit and also attend to students' level of anxiety.

Methods

In order to examine the roles that instructors take when enacting high cognitive demand examples, I conducted semi-structured interviews and observations of examples enacted in undergraduate precalculus classrooms. For the purposes of this study, precalculus courses included college algebra, trigonometry, and the combined college algebra + trigonometry, which were all taught in one semester.

Participants

The instructors that I observed were all experienced graduate students who were teaching a precalculus course. These graduate student instructors were experienced in two ways. First, they were in at least their third year in their graduate studies, which means that they had earned their M.S. in Mathematics and were working towards their Ph.D. Second, they were all teaching their respective course for at least the third time. The population of precalculus instructors that I had access to were mostly first-time graduate student instructors, so these instructors were experienced in comparison to many of their peers. Table 9 below provides a descriptive profile for each of the graduate student instructors that I observed. Instructors were asked to pick a pseudonym in order to conceal and protect their identity.

Table 9. Descriptive Profiles of Participants

Instructor	Course	Year in Graduate Program
Alex	College Algebra + Trigonometry	4
Dan	College Algebra + Trigonometry	4
Emma	College Algebra + Trigonometry	3
Greg	Trigonometry	5
Juno	College Algebra + Trigonometry	3
Kelly	College Algebra + Trigonometry	3
Selrach	College Algebra + Trigonometry	5

Data Sources

There are three primary sources of data that were collected for the purposes of this study. First, I conducted semi-structured pre-observation interviews with instructors before observing them. These typically occurred within 24 hours of the class that I was observing, but occasionally had to occur earlier due to scheduling issues. During these interviews, I asked questions regarding what topics they had covered in the previous class and what topic they were covering in the next class as well as what examples they planned to use and why. The full interview protocol for these interviews can be found in Appendix B. I also collected the lesson guides provided to the instructors, the lesson plans they created and planned to use, and the student workbook pages that they planned to use during class.

Second, I collected video observation data of the examples that the instructor enacted during class. During the observation I took field notes to record how the examples were enacted and how they fit into the larger lesson. I also recorded ways in which the enacted example differed from the intended example and whether or not any examples were added to the lesson that were not present in the lesson plan. The full field note guide that I used to record notes during and after the observation can be found in Appendix B.

Finally, I conducted semi-structured post-observation interviews with the instructors. These typically occurred within 24 hours of the observation. Between each observation and post-observation interview, I watched the video and selected one or two examples to discuss with the instructor. I tagged interesting moments during these examples and used these clips as video-stimulated recall during the post-observation interview. The full post-observation interview protocol that I used can be found in Appendix B.

Coding Procedures

Each enacted example was first coded using my modified framework for the cognitive demand of examples (Table 7). Next, I open coded the high cognitive demand examples to examine the roles the instructors took in enacting high cognitive demand examples. Three roles emerged out of this open coding (modeling, facilitating, and monitoring), which I will unpack more in the following section. I then went back and recoded each example using the final coding scheme for instructor roles.

Analysis Procedures

In order to better understand the different ways in which instructors modeled, facilitated, and monitored while enacting high cognitive demand examples, I analyzed the role profiles for each instructor. This involved calculating the aggregated amount of time that each instructor spent modeling, facilitating, and monitoring during the high cognitive demand examples that I observed. Next, I examined each example individually to see how instructors switched back and forth between these roles.

Sampling

For this study, I observed 24 different lessons over the course of a year. Every instructor, except for Greg, only taught their respective course during one semester, so I observed three lessons for each of them. Greg taught trigonometry both semesters, so I was able to observe six of his lessons. In the first semester I asked participants to choose three dates (spread out over September-December) that worked best for them, so my lesson sampling this semester was random. During the second semester I chose specific lessons that I wanted to observe and confirmed that the corresponding dates worked for the instructors. So my sampling here was more purposeful. The lessons that I chose were more procedural, because I thought they would provide me with an opportunity to see whether instructors chose to present examples as Procedures With or Without Connections. Also, I only observed one day of instruction in the first semester, regardless of whether or not the lesson was spread out over two days. However, if a lesson was spread out over two days in the second semester, I observed both days of instruction.

Role Profiles of HCD Examples

A full description of the examples used in the 24 lessons that I observed can be found in Table 34 in Appendix C. The 24 lessons spanned 33 days and included 93 different examples. Of those, 25 were high cognitive demand (HCD) examples.

Table 10. Overview of Examples by Instructor

Instructor	Number of Lessons	Number of Days	Number of Examples	Number of HCD Examples
Alex	3	3	5	3
Dan	3	6	18	3
Emma	3	3	9	1
Greg	6	8	25	10
Juno	3	5	14	4
Kelly	3	3	7	4
Selrach	3	5	15	0
Totals	24	33	93	25

When enacting HCD examples, instructors used a variety of approaches. Some instructors modeled content, practices, and strategies for their students, which required minimal contributions from students. In these cases, the instructor primarily worked through the example independently and expected students to follow along and copy the example into the their notes. Other instructors facilitated whole class discussions[6] as they worked through examples. The types of student contributions in these situations varied from providing simple computational answers to providing ideas of what to do next or justification for why a step or answer was reasonable. Other instructors placed even more responsibility on students and required students to work through parts of the example in small groups or independently while the instructor monitored their progress.

Table 11. Definitions of Modeling, Facilitating, and Monitoring

Term	Definition
Modeling	An instructor is modeling content, practices, and strategies if they are working through a problem independently and expecting students to follow take notes.
Facilitating	An instructor is facilitating a whole class discussion if they work through a problem together with their students.
Monitoring	An instructor is monitoring if they are requiring students to work through a problem independently or in small groups.

It is important to note that while some instructors primarily used one format of enacting high cognitive demand examples, others transitioned back and forth between different formats. For the high cognitive demand examples that I observed, Dan and Emma chose to just model the content, practices, and strategies for students. Juno incorporated both facilitating and modeling in the HCD examples that I observed and Kelly incorporated both monitoring and facilitating. Finally, Alex and Greg used all three formats for enacting HCD examples. Table 12 illustrates the different HCD example role profiles of each instructor that I observed. Table 12-Table 18 illustrate the different role profiles of each example that I observed, broken down by instructors.

[6] Here, a whole class discussion is interpreted broadly as any time when both the instructor and the students are working through part of the example.

Table 12. Role Profiles of Instructors' Observed HCD Examples

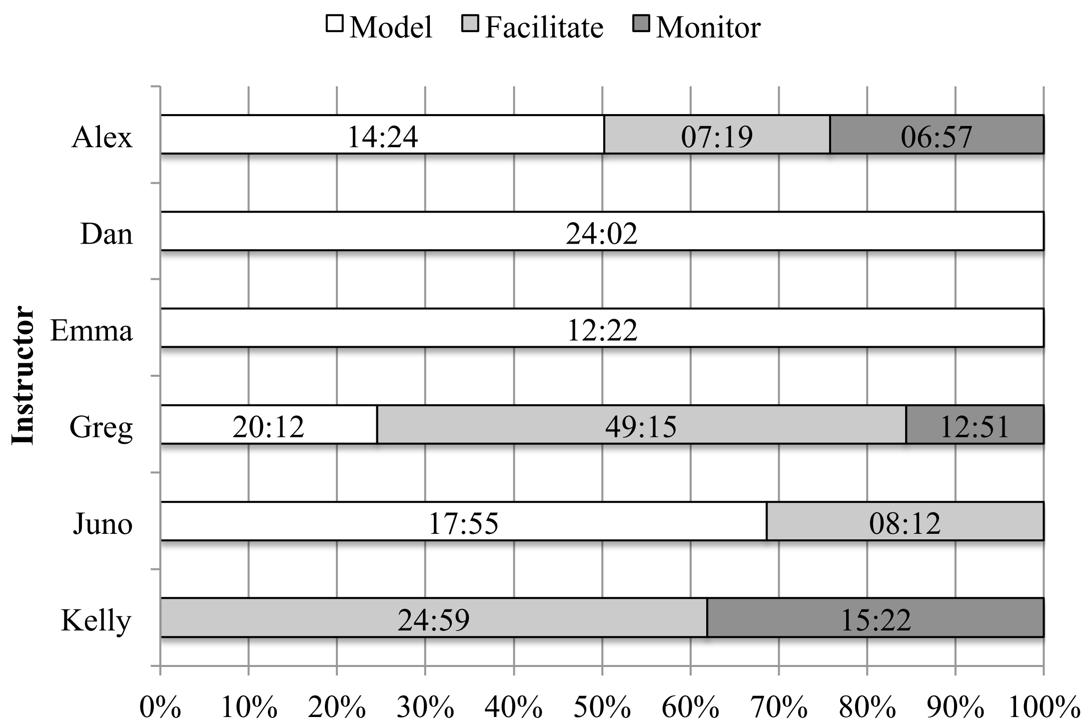


Table 13. Role Profiles of Alex's Observed HCD Examples

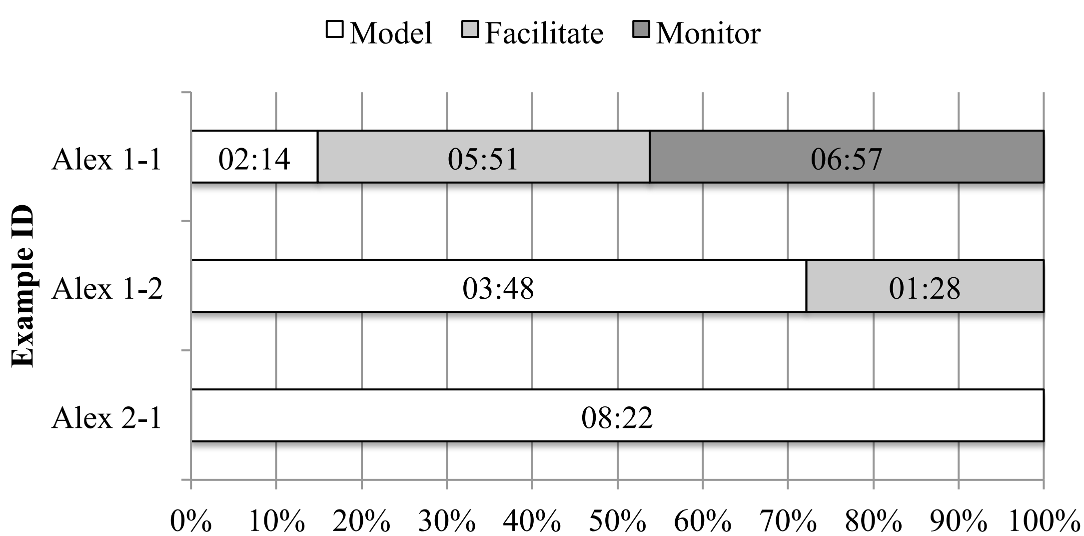


Table 14. Role Profiles of Dan's Observed HCD Examples

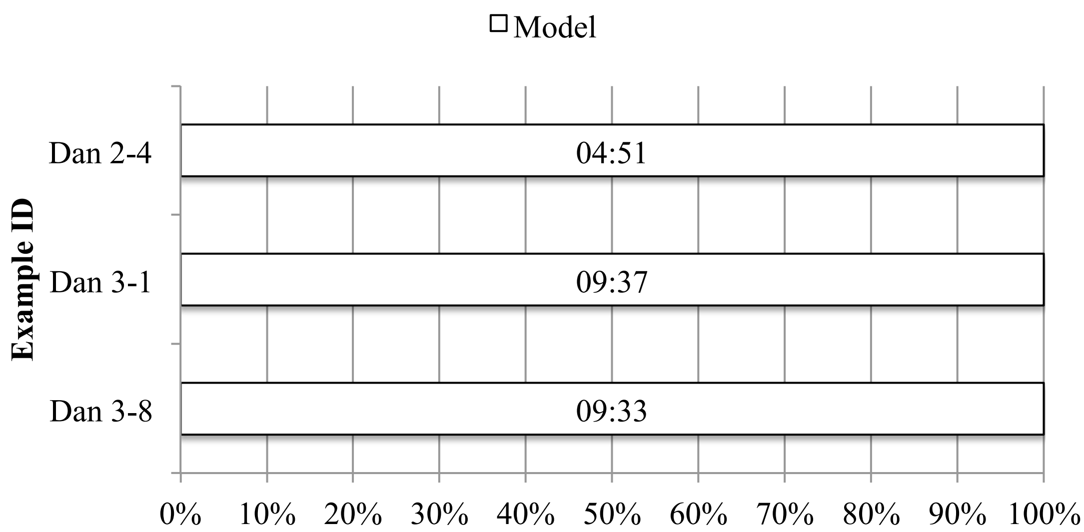


Table 15. Role Profiles of Emma's Observed HCD Examples

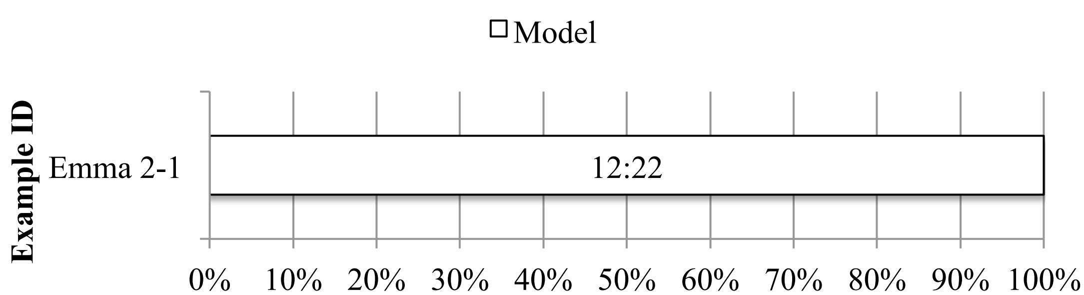


Table 16. Role Profiles of Greg's Observed HCD Examples

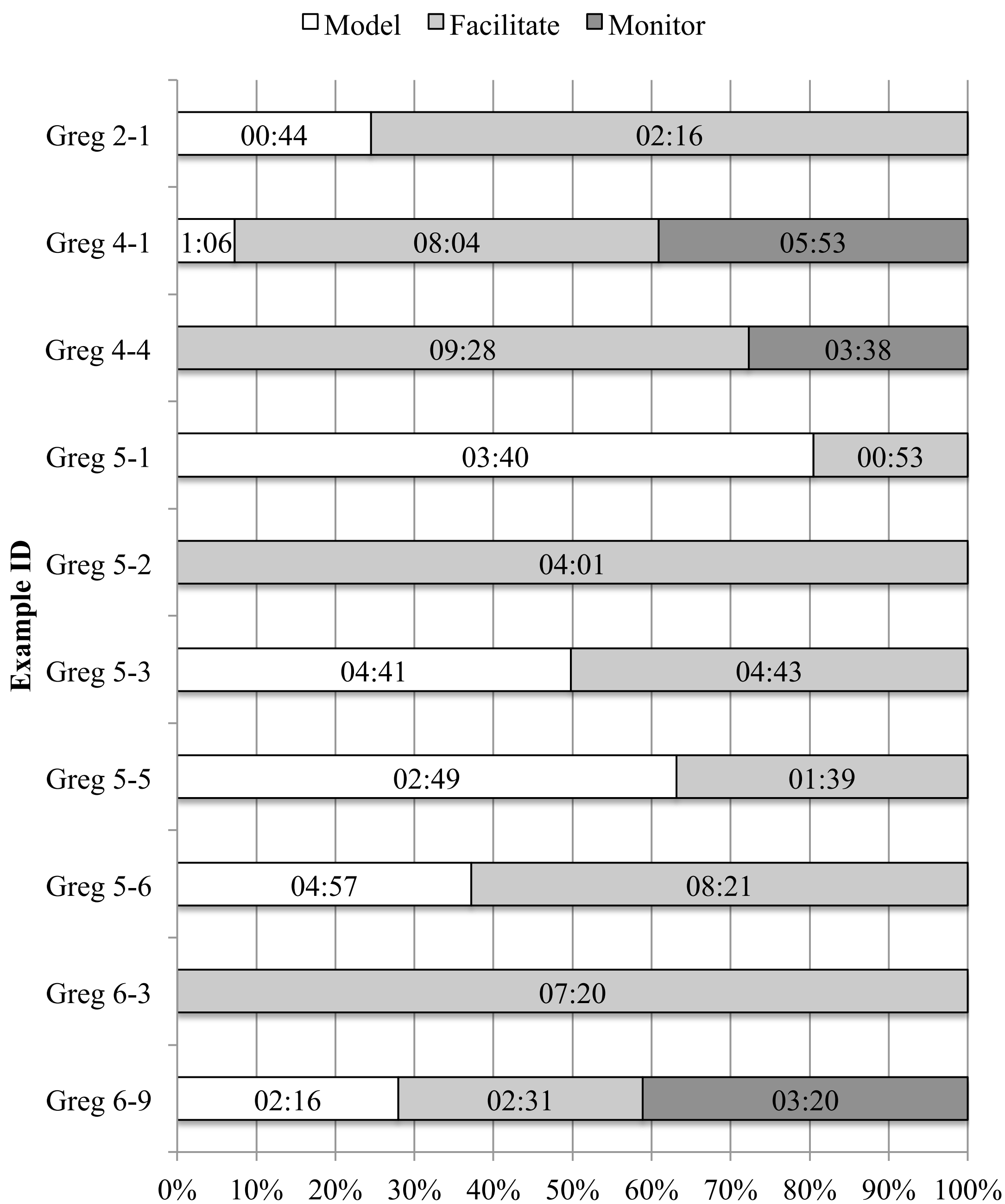


Table 17. Role Profiles of Juno's Observed HCD Examples

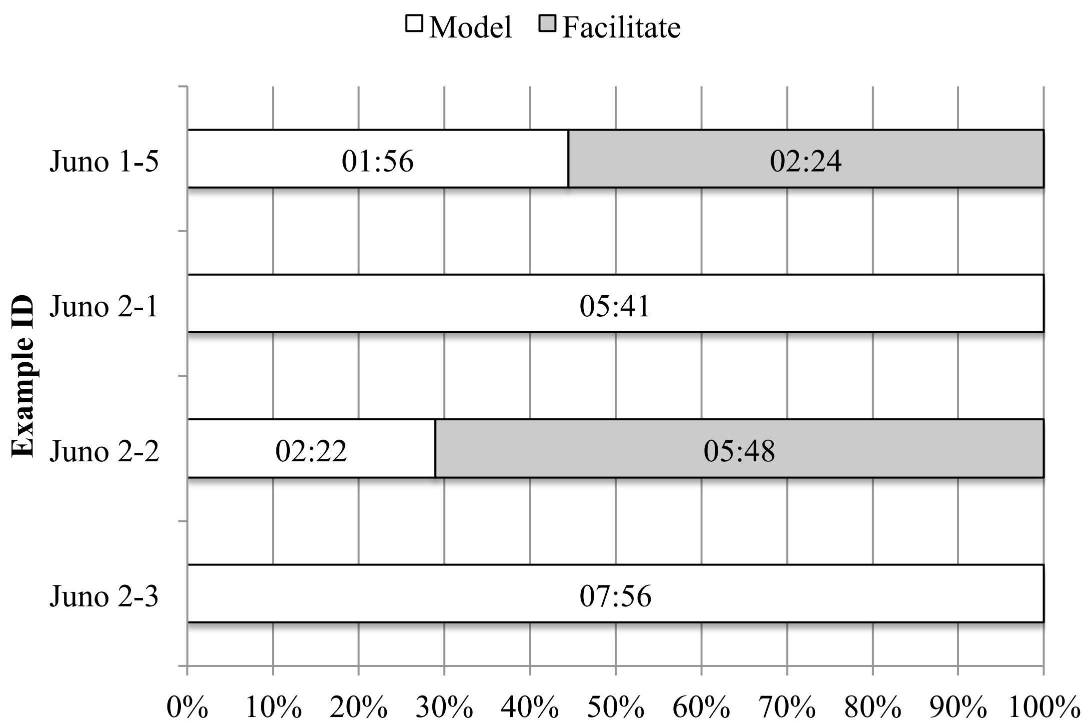
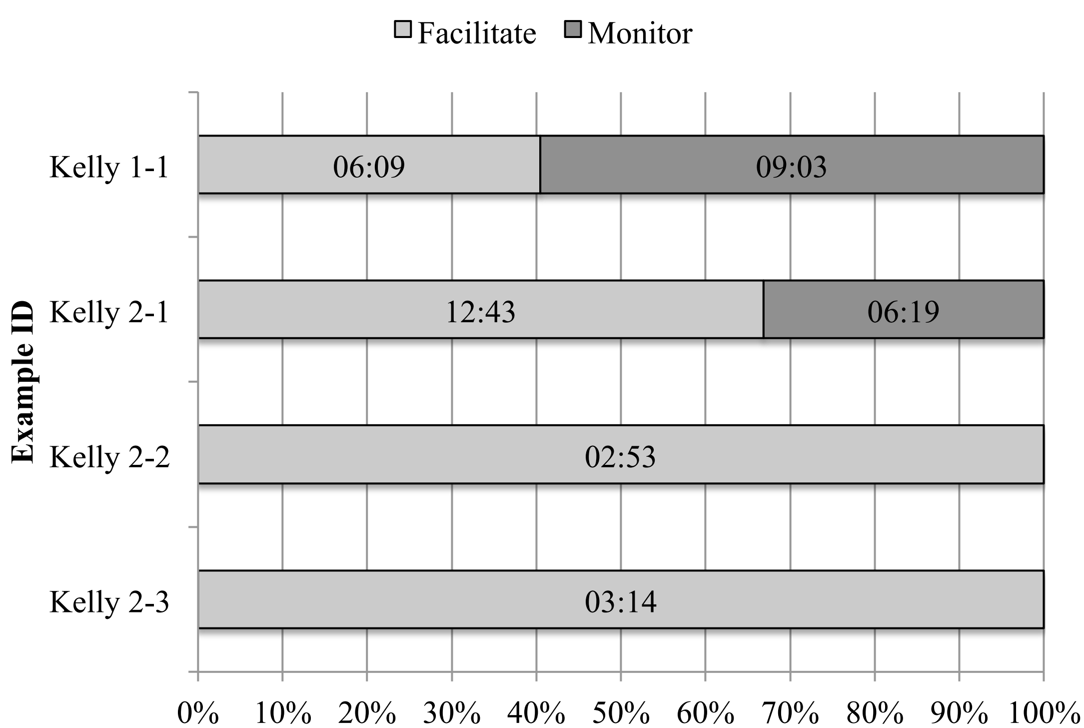


Table 18. Role Profiles of Kelly's Observed HCD Examples



Modeling

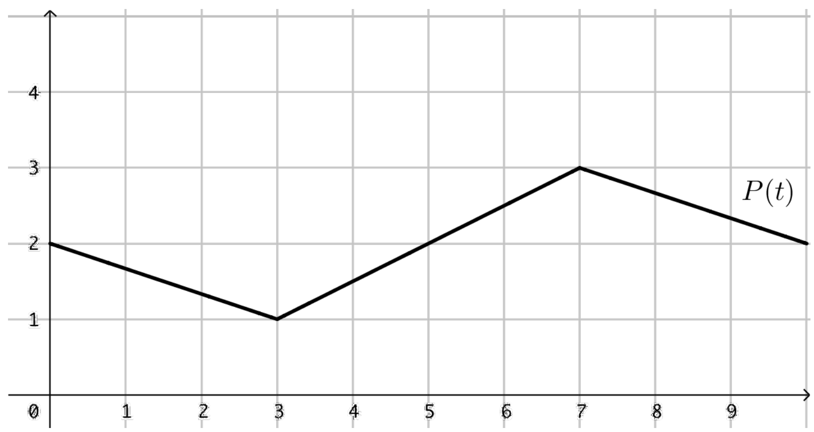
Many instructors chose to use different formats of presenting examples, but some chose to just model examples for their students. While students do not have an opportunity to struggle with the mathematics in this type of setting, they do have an opportunity to have high cognitive demand processes modeled for them. For the high cognitive demand examples that I observed, Dan and Emma only modeled and Alex and Juno used this presentation format for some of their examples In order to maintain the cognitive demand while modeling, instructors focused on making their cognitive processes explicit and attending to student understanding. In the following narrative, I illustrate how Emma modeled an example for students while still maintaining a high level of cognitive demand.

Emma. The example that I observed Emma enact at a high level of cognitive demand was situated at the end of a chapter on function transformations. Emma chose the example because it was a question on the chapter quiz that many of the students had struggled with[7]. In particular, she wanted to reemphasize the connection between order of operations and order of transformations and explain how to check their work using an alternative method. The example gave the graph of a piecewise linear function, shown in

Figure 7, and asked students to sketch a graph of $3P(t+1)-2$ for $0 \leq t \leq 9$ on a provided grid.

[7] Throughout my study, I talk about the reasons why instructors chose to do things, which I determined based upon the pre- and post-observation interviews I conducted with them. Since these types of references come up so often, I chose to not include citations linking them to the data sources. However, it is important to note that these claims are backed up by the data.

Figure 7. Original Function Given in Emma's Function Transformation Example



Emma worked through the example by first identifying the order of transformations based upon the transformed equation, $3P(t + 1) - 2$. First, she identified that there was only one horizontal transformation (a shift left by 1 unit), but two vertical transformations (a vertical stretch by a factor of 3 and a shift down by 2 units). Since many of her students had struggled with doing the vertical transformations in the wrong order, she next focused on explaining the order of vertical transformations. To help her students understand why the stretch had to occur before the shift, she explained how the process of transforming a graph is connected to order of operations, which students are familiar with. Next, she discussed how horizontal and vertical transformations are independent of each other, so order did not matter.

Next, Emma explained what points they could pick to transform (endpoints and corners) and how the graph in-between these points will just be a straight line. From there, she worked out a step-by-step transforming each of the endpoints and corners. Since the transformed graph was supposed to be drawn on the domain $0 \leq t \leq 9$, Emma then talked about what to do with the transformed point that was outside of this domain and how to find the new endpoint within this domain. Finally, she graphed her transformed endpoints and corners and connected them to draw the final transform graph. At this point, Emma paused and asked students for questions. One student asked about how she had found the new endpoint in the domain and Emma explained how she had used the original graph, as well as alternate strategies that students could use.

Next, Emma talked about a different method that students could use to solve the problem if they were not sure about the order of transformations. For this second method, Emma constructed an input-output table where the input values were $t = 0,1, \dots 9$ and the output values were found using the transformed equation $3P(t + 1) - 2$. She then explained how they could just solve the problem by inputting a value for t, using the graph to find the corresponding output value of $t + 1$, multiplying that output by 3, and then subtracting 2. Emma concluded the example by asking if any students had questions, and a new student asked a similar question as the one asked before concerning how she had found $P(2) = 4/3$. Several students piped up in agreement that they did not understand this step, so Emma explained how to use the slope of the first line segment to find the output value. In her explanation, she focused on not only calculating the slope, but also interpreting how it relates to finding the output values between integers.

I coded this as a procedures with connections example because of the following reasons. First, Emma focused students' attention on the use of procedures for the purpose of developing deeper understanding of mathematical concepts and ideas. To help her students remember the order of vertical transformations, she focused on the underlying mathematical concept of order of operations. Also, to help her students find exact output values, she focused on the underlying concept of slope and how to interpret it in a way that is helpful for calculating non-integer values. In her example, Emma presented two different pathways that students could follow to solve the problem (using order of transformations to move points or using an input-output table). In explaining each pathway, Emma focused on the underlying conceptual ideas (order of operations and evaluating function compositions), instead of the narrow algorithms. The example involved graphical, algebraic, and tabular representations and Emma often made connections between each of them. Finally, the number of student questions and the prevalence of student struggle on the problem when it was presented on the quiz are evidence that the example required some degree of cognitive effort for students to follow.

Facilitating

Of the six instructors that I observed enacting high cognitive demand examples, only Greg and Kelly chose to present an entire example as a whole class discussion. Interactions were coded as whole class discussions if instructors engaged students in the problem solving process in some significant way. For some instructors, this just involved asking students questions about computations. Other instructors had students engage in making more meaningful contributions, such as discovering patterns and making connections. Below I illustrate two narrative case descriptions, one in which the students were asked to make more superficial contributions and one in which the students were asked to make more significant contributions. The purpose of including both of these narratives is to compare and contrast how instructors maintain the cognitive demand of the example in each case.

Greg. There were two examples that I observed Greg enact at a high level of cognitive demand where he chose to facilitate the presentation of the example as a whole class discussion. In the example that I will focus on, Greg is explaining how to find all solutions to a trigonometric equation that has standard unit circle angle values. This topic was presented after students had learned about using inverse trigonometric functions to find solutions within the interval $0 \leq \theta \leq 2\pi$. Now, Greg was turning their focus to finding all solutions to a trigonometric equation. To introduce this topic, Greg used Demos to project the graph of $y = \cos\theta$ and $y = d$, where d was a slider set equal to 0.3. His purpose for starting with this visual representation was to spend more time thinking about the relationship between solutions to equations and the intersections of graphs and to illustrate why trigonometric equations might have infinitely many solutions. In particular, Greg talked about how even though there are infinitely many solutions, the periodicity of trigonometric equations means that these solutions repeat in a predictable way.

The first example that Greg chose to use involved finding all solutions to $\cos\theta = \sqrt{3}/2$. Greg chose this example to start with because it was simple enough that students didn't have to deal with the more technical aspects associated with sinusoidal equations and non-standard unit circle angles. Greg started the discussion by asking students what the initial solutions are in the first period ($0 \leq \theta \leq 2\pi$) using the unit circle. One student responded immediately with $\theta = \pi/6$, but the class seemed to be struggling with finding the second initial solution, as no one volunteered another answer. Greg responded by explaining that because the value of cosine is positive ($\sqrt{3}/2$), the corresponding angles on the unit circle will be in the first and fourth quadrant. Following his explanation, a student volunteered the answer $\theta = 11\pi/6$.

Drawing upon the earlier discussion, Greg reminded his class that the infinite families of solutions to trigonometric equations can be written as

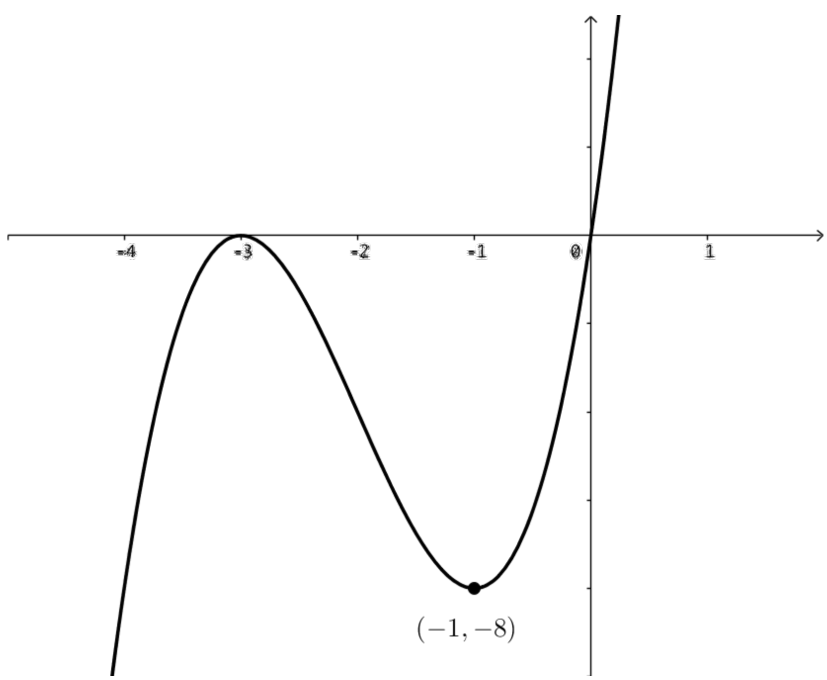
$$\text{(initial)} + \text{(period)}k \quad k = \text{any integer.}$$

He then explained that there would be two families of solutions corresponding to each of the initial solutions they had found. Next he asked his students, "What is the period of $\cos\theta$?" Students responded with 2π and Greg emphasized that we knew this was true because there was not a horizontal stretch or compression factor in the original equation. Next Greg used Demos to project the graphs of $y = \cos\theta$ and $y = \sqrt{3}/2$. He then made connections between the intersection points of the graphs and the solution families that they had found.

I coded this as a procedures with connections example because of the following reasons. First, Greg focused students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas. Instead of presenting the example algorithmically, Greg focused on how we can use solutions in the first period and the periodicity of trigonometric functions to help find all solutions to a trigonometric equation. Second, the solution strategy that Greg used (finding initial solutions and then adding on multiples of the period) is a broad general procedure that is closely connected to the underlying conceptual ideas. Greg used both algebraic and graphical representations and made connections between them to help students develop understanding of what it means to have an infinite family of solutions. While students were able to easily answer most of the questions that he asked, the general procedure that he was describing could not be followed mindlessly. In particular, students had to attend to the conceptual ideas of determining the number of initial solutions and the period of the function.

Kelly. Of the four high cognitive demand examples that I observed Kelly enact, she chose to present two by facilitating a whole class discussion. This example was situated in a unit on polynomial functions. The day before she introduced long-run behavior and earlier in class on this day she introduced short-run behavior (i.e., whether or not the graph of a function bounces or crosses the x-axis at zeros of the function). In previous examples, Kelly had worked with students to determine how short-run behavior is connected to multiplicities and how to graph a polynomial given its equation. For this example, Kelly challenged students to think backwards and find a formula for a polynomial with the lease degree possible based upon a given graph. The graph she provided at the beginning of the example is shown below in Figure 8.

Figure 8. Polynomial Graph from Kelly's Example



To start off the example, Kelly asked her students how they could find a formula for the graph. One student responded and said, “We will need $(x - 0)$ times $(x + 3)$, right?” Kelly then simplified this equation to $x(x + 3)$ and asked, “What else do we know?” The same student responded by saying that we need $(x + 3)$ to be raised to the second power. Kelly responded by asking if everyone saw why that was true. To make sure her students understood, she asked explicitly, “So where does the 2 come from? Or why do we need the 2 there?” Several students responded simultaneously, “Because it bounces at -3.” Kelly then directed her students’ attention to the other factor, x, and asked if we needed to change the exponent there. Her students responded by saying no, and Kelly went on to reiterate that we are looking for a polynomial of lease degree, so we want to use the smallest exponents possible.

At this point, Kelly asked, “What else do we have to do here? Is this our equation?” A student responded by saying, “No. If you plug in -1, you don’t get -8.” So Kelly asked, “So what else do we need here?” and a student responded by saying, “A coefficient out front.” Kelly then wrote $y = ax(x + 3)^2$ on the board and asked, “How can we find a?” A student suggested that we could plug in $x = -1$ and $y = -8$, and Kelly worked through the algebra to find that $a = 2$. Finally, Kelly asked if their equation made sense in terms of the long-run behavior that the graph is exhibiting and her students agreed that the equation and the graph both acted like $y = 2x^3$ in the long-run.

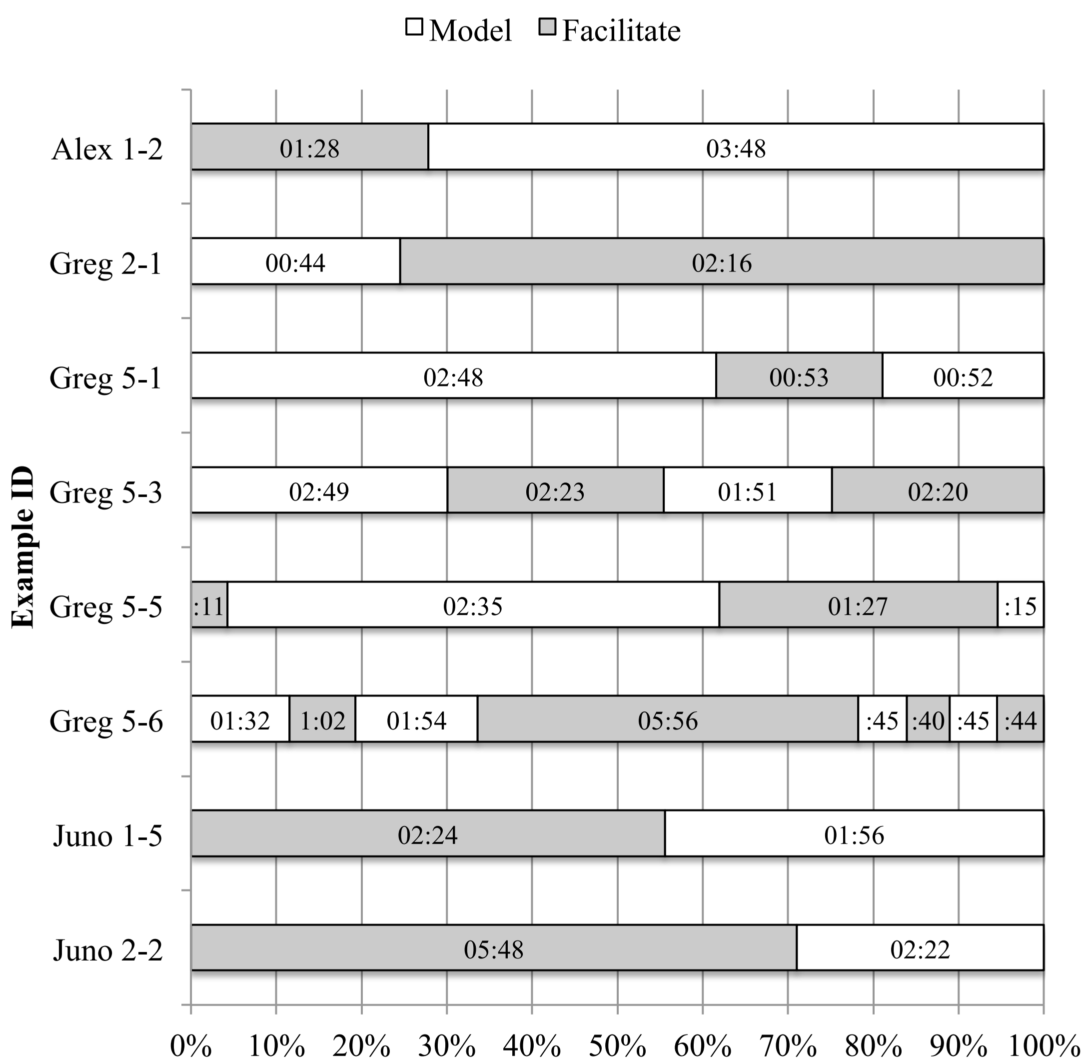
I coded this example as a doing mathematics example because of the following reasons. First, while the example could have been solved using an algorithm, this algorithm was never presented formally. So Kelly and her students worked together to construct an algorithm, as opposed to following a predictable, well-rehearsed approach or pathway. Also, in order to construct the equation, students had to explore the nature of graphical properties of polynomials and think about how they are connected to the algebraic properties of their equations. To do this, Kelly asked her students to access relevant knowledge (e.g., the connection between zeros and the factored form, the connection between short-run behavior and multiplicities, the implication of what it means for a point to lie on a graph) and make appropriate use of it while working through the example. She also asked students to analyze the example (e.g., points on the graph, the long-run behavior) and actively examine example constraints (e.g., the degree) in order to limit possible solution strategies and solutions. While her students were able to work through the example successfully, it did require cognitive effort and was unpredictable in that it was student, not teacher, lead. Finally, Kelly asked her students to examine constraints, justify, explain, and determine when the problem was solved.

Comparison. Comparing and contrasting the two different ways in which Greg and Kelly worked through a high cognitive demand examples shows that instructors can facilitate whole class discussions in very different ways. In Greg's example, students were primarily responsible for doing the less cognitively demanding work. On the other hand, Kelly relied on students to guide the entire problem solving process. However, it is important to note that in both cases, there was an emphasis on explaining content, processes, and strategies and making connections between representations, which is why they both illustrate what it might look like to facilitate a high cognitive demand examples.

Modeling and Facilitating

None of the instructors that I observed enacting high cognitive demand examples chose to only monitor students while they worked through the problem. However, there were several instructors that chose to present an example by both modeling and facilitating. Alex, Greg and Juno all integrated these two roles when presenting some of their examples. Table 19 illustrates the different ways in which these instructors presented examples by modeling and facilitating.

Table 19. Role Profiles of Examples that were Modeled and Facilitated

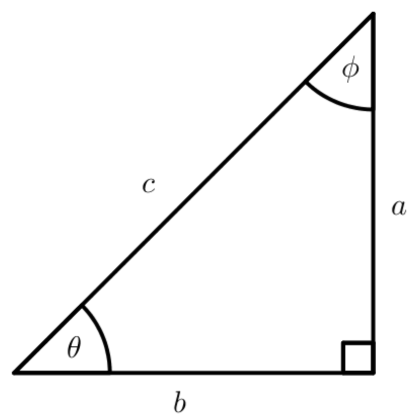


Half of the examples (4/8) were split into two different chunks of time, one in which the instructor modeled content, practices, and strategies for students, and another in which the instructor facilitated a whole-class discussion of the example. However, the other four examples involved a back and forth between these two forms of presentation, with Greg's Example #5-6 having the highest number of switching points. Also, half of the examples (4/8) began with a whole-class discussion, however in Greg's Example #5-5, this quickly morphed into him modeling for his students. In the large majority of the examples (5/8), more time was spent on the whole-class discussion than on the modeling. However, on average 52% of the example enactment time was dedicated to facilitating and 48% was dedicated to modeling. Below I present two narrative accounts of Juno's example that was segmented into almost equal segments of facilitating and modeling and Greg's example that had a high number of switching points.

Juno. This example that Juno enacted at a high level of cognitive demand was situated in a lesson on the tangent and reciprocal trigonometric functions. Now that students were exposed to all of the main trigonometric functions, including sine, cosine, tangent, cotangent, secant, and cosecant, Juno introduced the idea of cofunctions. Two functions are called cofunctions if they are equal on complementary angles. Juno started off by doing two examples to show that $\sin(\pi/6) = \cos(\pi/3)$ and $\sin(15°) = \cos(75°)$. As a final example, Juno chose to prove that sine and cosine are cofunctions. While proving is not a main component of this course, it does come up in some of the trigonometry lessons. So Juno wanted to go beyond just demonstrating that sine and cosine are equal for some complementary angles and prove that it was true for any complementary angles.

Juno begins by drawing and labeling the right triangle shown in Figure 9. She then asks her students to use the SOH-CAH-TOA definition to find what $\cos\theta$ is equal to. A student responds with "Adjacent over hypotenuse," which Juno then interprets as b/c. Juno then asks, "What is $\sin\phi$ equal to?" and a student responds with b/c. Finally, Juno asks what we know about the two angles, θ and ϕ, and a student responds with "They have to add up to 90. Juno explains why, using the fact that the sum of all of the interior angles of a triangle must equal $180°$, and since we know that one of the angles is equal to $90°$, the sum of the other two must equal $90°$. Therefore θ and ϕ must be complementary angles.

Figure 9. Right Triangle from Juno's Example #1-5



Juno then transitions from facilitating a whole-class discussion about the example to modeling content, practices, and strategies for students. First, she explains that the work they have done tells them that no matter what angle θ is, $\cos\theta = \sin(\pi/2 - \theta)$, where $\pi/2 - \theta = \phi$. Finally, Juno explains that while they had previously checked that $\cos\theta = \sin(\pi/2 - \theta)$ for $\theta = \pi/6$ and $\theta = 15°$, the work they have now done proves that this is true for all angle θ[8]. She pauses here to ask if there are any questions, but none of her students pipe up. Before she moves on, Juno goes back to the list of trigonometric functions and points out that if sine and cosine are cofunctions, then it would make sense for tangent and cotangent, secant and cosecant to also be cofunctions, assuming that they were named properly. Finally, she directs her students to start working on one of the problems in the workbook, which asks them to explore cofunctions graphically.

Even though doing proofs is a form of doing mathematics, I coded this as a procedures with connections example for the following reasons. First, Juno wanted to give a proof because it shows that $\cos\theta = \sin(\pi/2 - \theta)$ for any angle θ, not just $\theta = \pi/6$ and $\theta = 15°$. So her focus was on using the procedure (i.e., the proof) for the purpose of developing deeper levels of understanding of the mathematical concepts and ideas. The proof technique that she chose was closely connected to the underlying concept of the right triangle definition of sine and cosine. Also, Juno used multiple representations (pictorial and algebraic) and made connections between the representations to help develop meaning. Finally, students could not follow the proof mindlessly, but rather needed to make connections between the different representations in order to develop understanding.

Greg. The high cognitive demand example where Greg switched back and forth between modeling and facilitating was situated in the second lesson on finding all solutions to trigonometric equations. After spending a day exploring the structure of the infinite families of solutions and working through simpler problems that did not involve

[8] [8] It is important to note that Juno's proof assumes that $0 \leq \theta \leq \frac{\pi}{2}$, which is not implicit in the definition of complementary angles that she was using. While this would have been a fruitful topic to dig into during the post-observation interview, it unfortunately was not something that I recognized at the time.

shifts and stretches, Greg introduced more complicated sinusoidal functions. First, Greg did two examples that only involved vertical transformations. For his final example, Greg chose to find all solutions to $\sin(3\theta - 1) = 1/4$. Greg chose this function for several reasons. First, he wanted his students to learn how to find all solutions when the period is not equal to 2π. Second, he wanted to give an example with both a horizontal shift and a period change because he knew that problems of this type would come up on the online homework as well as the exam. Finally, he didn't want to use a standard unit circle angle and instead force students to use arcsine.

Greg started by first modeling content, practices, and strategies for students. After writing the problem on the bard he describing how this problem was different from the previous two examples they had worked through in class that day, which were, "Find all solutions to $1 + 2\cos\theta = 4/3$ and $3\tan\theta - 1 = 4$." In particular, he emphasized that they no longer had a linear equation in terms of sine, but rather a linear equation inside of sine. To make the equation more clear and appear less complicated, Greg decided to set $X = 3\theta - 1$. In the first example, Greg had set $X = \cos\theta$ and talked about how they could set $X = \tan\theta$ if they wanted to. His idea for doing this was to remove the part of the equation that looks unfamiliar and highlight that first they needed to isolate the input of sine. Greg also explained to his students that this step was not necessary, so they could skip it if they knew what they needed to do.

Next, Greg switched to facilitating a whole class discussion. First, he asked how they could proceed from $\sin(X) = 1/4$ to solve for X. A student suggested that they could use arcsine, so Greg wrote $X = \sin^{-1}(1/4)$ and explained that this gives us the first solution. When Greg asked where the second solution is, no students responded immediately, so Greg explained that currently they were generating initial solutions for the problem and that they had found one solution for this sine equation, but still needed to find the second solution. A student piped up and said "π minus…", which Greg interpreted to mean $X = \pi - \sin^{-1}(1/4)$.

From here, Greg switched to modeling. First he explained that they had started with θs, so they needed to end with θs and swap out the Xs. Doing this resulted in the following two equations: $3\theta - 1 = \sin^{-1}(1/4)$ and $3\theta - 1 = \pi - \sin^{-1}(1/4)$. Before solving for θ, Greg paused to explain that this problem "was a little bit more involved than the other [examples] because we generate our initial solutions and then we have to keep working to…find the initial solutions just in terms of θ." From here, Greg works through the algebra to solve for θ and ends with the following two equations: $\theta = 1/3(\sin^{-1}(1/4) + 1)$ and $\theta = 1/3(\pi - \sin^{-1}(1/4) + 1)$.

At this point, Greg switches back to facilitating by saying, "I'm going to pause here and ask who is lost? Who has a question? It's totally reasonable to be lost. There's a lot that goes into these. So just let me know where you are lost." Greg phrased his question this way because he had noticed that asking, "Are there any questions?" was not eliciting responses from students. But explicitly saying "it's totally reasonable to be lost" made students more comfortable asking questions. A student did pipe up and asked, "Why divide by 3? Where did the $1/3$ come from?" Greg first asked if the student was ok with everything that had come before, then went on to explain the algebraic step the student was stuck on. Next a student asked, "Will we still involve adding the period times k at the end?" Greg explained that was the next step and reiterated that the work they had done so far was all to get the initial solutions.

At this point, there was a hushed conversation going on between two students, so Greg asked if they had any questions. At first no one said anything, and then Greg encouraged the students whispering to share their questions with the whole class, because most likely other students had similar questions. One of the students piped up and said, "I was just gone yesterday, so I had no clue what's going on…so I was just kind of catching up." Greg then moved on to talk about all possible solutions and reminded the class that they should be of the form $(\text{initial}) + (\text{period})k$. So he asked, "What is the period of $[\sin(3\theta - 1)]$?" None of his students responded, so he reminded them that they could identify the period of a function by considering the B-value associated with $\sin(B(\theta - h))$. Still, no students offered an answer, so Greg asked specifically if they could identify what is B by mapping $\sin(3\theta - 1)$ to $\sin(B(\theta - h))$ to see what the coefficient is on the variable.

At this point, one student responded by saying B is the period. Greg responded by saying, "B is related to the period. It's not directly the period." Another student spoke up and said, "Isn't the period $2\pi/3$?" Greg then asked, "Why is it $2\pi/3$?" and she responded with, "Because that's the way you find the period when you have B." Greg agrees that the period is given by $2\pi/|B|$ and then goes on to say that the student must be using the fact that $B = 3$ in order to say that the period is $2\pi/3$. The student who had volunteered the answer $2\pi/3$ then asked, "Is that right? Even though there's not any parentheses around the θ and the 1?... So if you put the parentheses around the θ-1, does it still make the $B = 3$?" Greg then explains that B would still be 3 in that case, but just adding parentheses would result in a different function with a different horizontal shift.

He then shows how to rewrite the equation as $\sin(3(\theta - 1/3))$, but emphasizes that they don't have to factor the 3 out to know what B is equal to.

At the end, Greg switched frequently back and forth between modeling and facilitating. He first modeled how to use the two initial solutions they had found, as well as the period, in order to write out the two families of solutions. Then several students piped up to ask questions about whether or not this was a problem that could show up on a test, clarification of the general process was for solving the problem, and whether or not the parentheses around the period are required. Greg then took the time to summarize the whole process and highlighted the following steps.

1. Get sine all by itself.
2. Set X equal to inside of sine.
3. Use inverse sine to get two initial solutions.
4. Replace X with original θ expression.
5. Solve for θ.
6. Find the period.
7. Find all solutions.

Finally, Greg wrapped up the example by allowing students to ask questions.

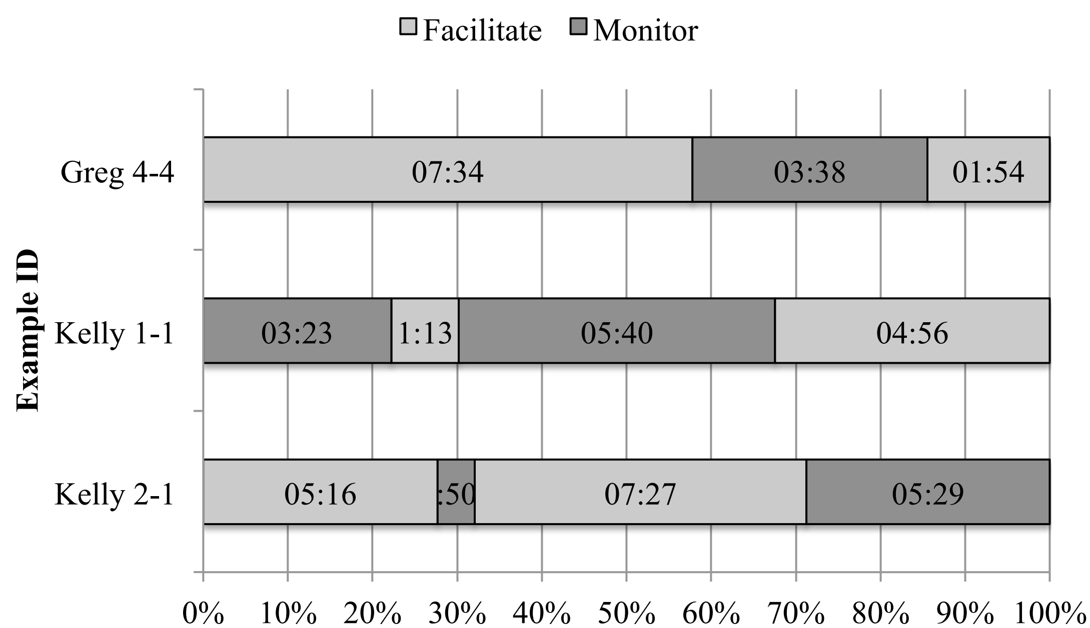
I coded this example as procedures with connections for the following reasons. While parts of this example strayed into lower cognitive demand tasks, the majority of the problem was focused on the broad general procedure of using the initial solutions and the periodicity of sinusoidal functions to find all solutions. First, Greg consistently focused students' attention on the underlying structure of solutions to trigonometric equations: $(\text{initial}) + (\text{period})k$. While there was a lot of algebra involved in getting the initial solutions and students struggled to find the period, Greg always brought the focus back to this underlying concept. While the example was algebraic, Greg emphasized the connections between the general form of solutions and the specific families of solutions that they had found. For example, Greg emphasized that $1/3(\sin^{-1}(1/4) + 1)$ represents one initial solution and $2\pi/3$ represents the period. Also, the number of questions asked by students is one form of evidence to support the claim that this example required some degree of cognitive effort for students to follow.

Comparison. The two narrative accounts I have given where Juno and Greg both modeled and facilitated illustrate how these role profiles may look very different. Juno started off by asking students questions and having them contribute to the problem solving process. However, once she got to the main point of her example, she switched to modeling. Greg, on the other hand, switched back and forth between modeling and facilitating throughout his example. In particular, he allowed students to ask questions throughout the example and used these opportunities to make sure that students understood. However, in both cases, it was primarily the instructor who modeled the more challenging aspects of the problem and guided its unfolding.

Facilitating and Monitoring

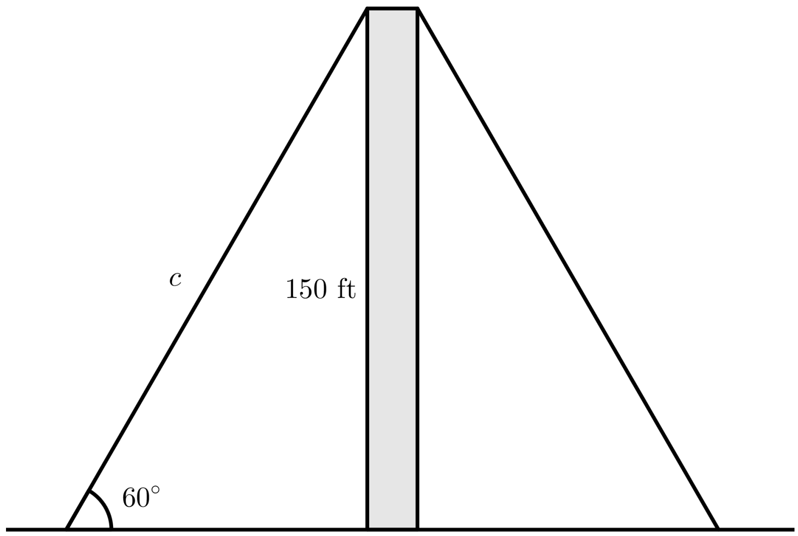
None of the observed high cognitive demand examples were enacted by modeling and monitoring, which is perhaps not surprising because monitoring student work time was always followed up by facilitating a whole class discussion. However, Greg and Kelly both enacted high cognitive demand examples where they only relied upon facilitating and monitoring. Greg did this once, where he gave his students time to work through a problem in the middle of a whole-class discussion of the example. Kelly did this two different times in two different ways. In one instance, she started off by having students work on the problem, then switched back and forth between facilitating and monitoring. In the other example, she started off by first facilitating a whole-class discussion of the example and then gave students time to work through parts of the example. Below I will present and compare two different example narratives, one where Greg started by facilitating a whole-class discussion before modeling and another where Kelly did the opposite.

Table 20. Role Profiles of Examples that were Facilitated and Monitored



Greg. The one example where Greg both facilitated and monitored was situated in a lesson tangent and reciprocal trigonometric functions. The example that Greg chose to do involved using a given angle and side length to find another side length of a triangle. However, this problem was couched within the context of finding the amount of cable needed to help stabilize a cell or radio tower. Greg chose this example because he wanted students to see a real life application of how trigonometric functions can be used outside of mathematics class. The example explained that a 150 foot tower is stabilized by cables that make an angle of 60° with the ground. Greg then drew the following illustration shown in Figure 10 and asked, "How long are the cables? (Give an exact answer.)"

Figure 10. Greg's Illustration of a Tower Stabilized by Cables



At this point, Greg asked if anyone could think of a way to relate c to the other pieces of information they had in the problem so far. One student suggested, "Couldn't you like cross-multiply the angles? If you like know the angles, can't we set up an equation to compare the length of sides with the angles?" Greg interpreted her response as referring to finding side lengths of similar triangles and explained how they could use information about a larger tower to find the height of a smaller tower that had a similar setup. Greg then solicited students to think of other ways to combine the information they had. A new student piped up and said, "$\sin\theta = 150/c$." Greg agreed and explained that if they felt stuck in this class, then looking for a triangle or a circle, since that is what this class is about. He then focused students' attention on the triangle formed by the ground, the tower, and the left cable. Using this triangle, he showed how we get $\sin(60°) = 150/c$ and then continued to solve for c with the help of his students.

Next, Greg told his students to figure out how far the cables must be from the base of the tower. He also announced that once they figured out how to do that, they should begin working on a problem from the workbook. Greg gave students 3 minutes 38 seconds (henceforth notated as 3:38) to work through this second part of the example individually or in small groups. During this time Greg walked around the room and monitored students' who were working through the problem.[9] As class was about to end, Greg mentioned that he saw a couple of different approaches and asked his students to share their different approaches. One student piped up and suggested that they could use the Pythagorean theorem, which Greg then worked through quickly. Another student suggested that we could also use tangent and Greg quickly worked through that method as well.

I coded this example as procedures with connections for the following reasons. First, Greg did not set up or encourage that students use only one method for solving this problem. Rather, he focused on making connections between the information they were given and the trigonometric equations that they had been studying. In particular, he focused students' attention on using the triangle definition of trigonometric equations in order to solve for unknown variables. He represented the problem in words, algebraically, and pictorially and focused on making connections between the algebra and the picture. Finally, deciding what procedure to use required some degree of cognitive effort on the part of students.

[9] My IRB did not include taking video of students, so I was not able to capture what instructors did as they moved around and interacted with students.

Kelly. The example where Kelly began by monitoring students was situated at the beginning of the lesson introducing exponential functions. Kelly gave her students 3:23 to work through the following questions.

> *Suppose you have* 100 *dollars to deposit into a savings account. If you put your money into Bank A, they will deposit an additional* 10 *dollars per year into your account. If you put your money into Bank B, they will increase your balance by* 10% *per year. How much money would you have after one year if you put your money into Bank A? How about Bank B? How much money would you have after two years if you put your money into Bank A? How about Bank B? After three years? Which bank should you use?*

During this time, she asked a group to write the balances in Banks A and B after one year on the board.

Kelly then brought the class back together to see if everyone agreed with what the students had written on the board for the balances after one year. She then asked a student to volunteer the balances after two years and wrote those on the board. Kelly then asked her students what the balances would be after three years, which students were able to calculate on the spot. Kelly then asked, “Which one would you chose?” A choral of students said “Bank B,” and Kelly explained why that was correct. At this point, Kelly gave her students a similar problem to work on: *Suppose you are investing* $\$500$ *at an annual rate of* 4.5%. Before moving on, she paused to make sure that everyone understood what “annual” meant and explained that they were just putting $\$500$ in a bank account and leaving it alone to see how it grows. Then she asked her students to fill in a table that shows what the balance is after $t = 0,1,2,3$ years. In addition, she asked her students to come up with a formula that would model how much money is in the bank account after t years.

Kelly gave her students 5:40 to work through this problem. During this time, she walked around the room and interacted with students as they worked. After 40 seconds, Kelly made an announcement to the whole class to clarify that $t = 0$ is the very beginning, when $\$500$ is deposited into the account. She also encouraged students to check with each other to make sure they are getting the same numbers. About half way through their work time, Kelly reminded the class that once they figured out the balances at the end of the first three years, they needed to find a general formula that would give the balance after t years.

Next, Kelly brought the whole class back together for a discussion of the general formula. Some students said they didn't have a formula yet, but Kelly assured them that they would figure it out together. First, Kelly asked students to volunteer the answers they got to fill in the table and verified that everyone had gotten the same answers. Then Kelly asked, "So how are we getting these numbers?" One student explained that they were using the formula $a(1 + r)^t$ and Kelly acknowledged that this was what they were going towards, but she wanted them to figure out how we could come up with that formula using the numbers in the table.

To help start the discussion, Kelly asked, "How did we get from $\$500$ to $\$522.50$?" Another student responded with, "Times 500 by 0.045." Kelly responded by explaining how we could times 500 by 0.045 and then add 500, but asked if anyone knew an easier way of doing that. A new student piped up and said, "Times 500 by 1.045." Kelly responded by explaining how we could factor out a 500 from both terms in

$500 * 0.045 + 500$ and get $500(0.045 + 1)$. Next Kelly asked how they had found that $\$546.01$ was the balance after two years. A student responded with, "522.5 times 1.045," which Kelly agreed with. Kelly asked, "What's another way of writing 522.5?" After waiting a few seconds and receiving no response, Kelly wrote added $= 522.5$ to the end of the equation $500(1.045)$ written on the board and said, "Maybe I will suggestively write that." After repeating her question a second time, she still received no response until a question asked, "Can you repeat your question?"

Kelly then explained how $522.5(1.045) = 546.01$ and checked to see if everyone understood why that was true. Then she asked, "So how can we rewrite this 522.5?" Finally, a student responded immediately to her question by saying, "Couldn't we write $500(1.045)$?" Kelly agreed and explained that this was where 522.5 had come from. So then to get 546.01, we needed to multiply that again by 1.045 to end up with $500(1.045)(1.045) = 546.01$. After writing this all on the board, Kelly asked her students if they saw a pattern and if they could guess what the formula for t years would be. A student responded with $500(1.045)^t$. Kelly then encouraged her class to plug in $t = 3$ and verify that the value agreed with what they found in their table. Kelly asked for any final questions, with no response, and then asked, "So what kind of formula is that?" A student responded with exponential and Kelly explained that this is what the new chapter was all about.

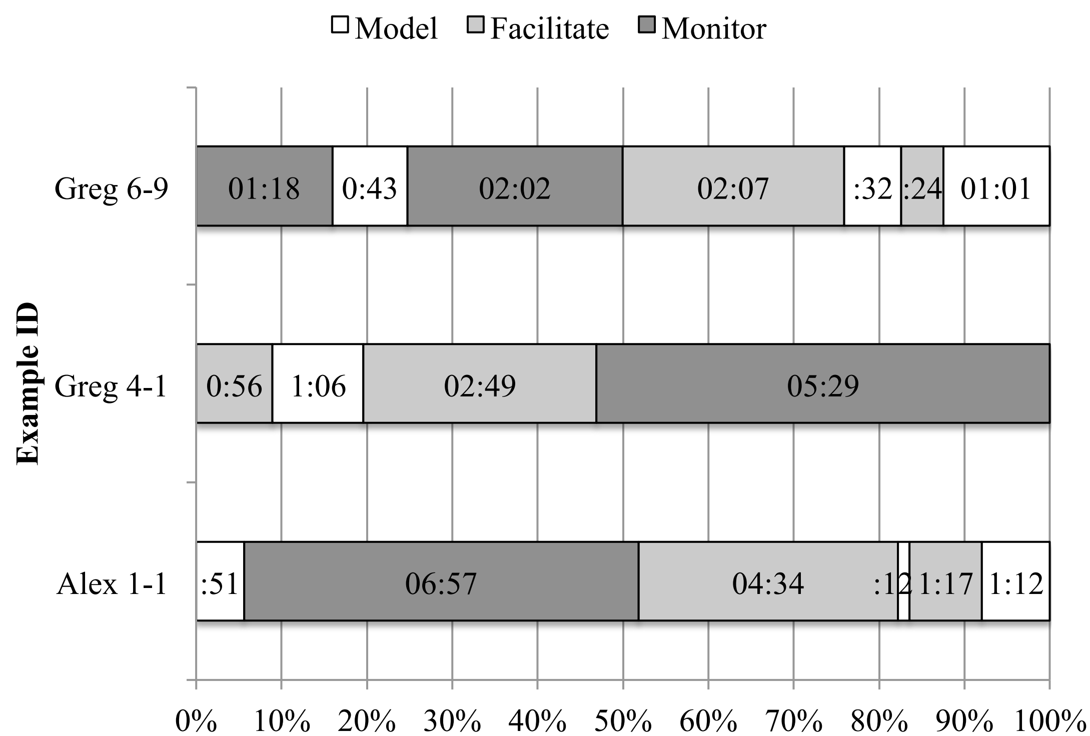
I coded this as a procedures with connections example for the following reasoning. First, Kelly expected her students to be familiar with exponentials and know how to work with them computationally, but she really focused the example on the underlying concept of multiplicative growth. Students were not provided with any specific pathways to follow and Kelly encouraged them to solve the problem in different ways in order to check their work. Kelly also used tabular and algebraic representations of the problem. Finally, not every student was able to come up with a formula during their small group time, so we know that it required some degree of cognitive effort for students to complete.

Comparison. While Kelly and Greg both enacted high cognitive demand examples by monitoring and facilitating, they did so in different ways. Greg first worked through one part of his example with the whole class, and then gave them time to work through another part individually or in small groups. While some students chose to do these two parts in different ways, Greg expected that they would have used the same method. So he set up his example in a way where they could first see a problem worked out, and then try a similar problem on their own. Kelly, on the other hand, asked her students to dive in and work on the mathematics from the start. Instead of working through a similar problem for them, Kelly relied on her students to provide answers and ideas of what to do next.

Modeling, Facilitating, and Monitoring

The final type of role profile associated with the high cognitive demand examples that I observed is when instructors incorporated modeling, facilitating, and monitoring into parts of their example. Alex and Greg were the only two instructors who enacted high cognitive demand examples in this way. There were no clear patterns that emerged from these examples, so I just selected one to describe in more detail below.

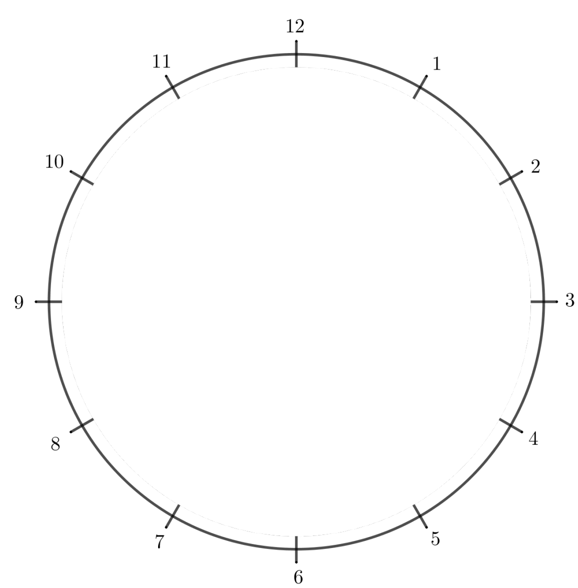
Table 21. Role Profiles of Examples that were Modeled, Facilitated, and Monitored



Greg. One of the high cognitive demand examples that Greg enacted by monitoring, facilitating, and modeling was done during a class where they were reviewing for the final exam. Greg had done a series of examples that were all related. The overarching problem context was situated in real-life Ferris wheel problem. The students were told that the Ferris wheel was 65 feet above the ground at its highest point and that people boarded the Ferris wheel on a 10-foot platform at the 6 o'clock position. Also, one revolution of the Ferris wheel took 6 minutes. In the first few examples, Greg constructed a formula that modeled the distance off the ground of someone riding the Ferris wheel as a function of time; found all times between $t = 0$ and $t = 18$ when the person is 15 feet off the ground; and found how far the person traveled after 1 minute, 30 seconds, and 13 seconds. In the final example related to this problem, Greg asked them to modify their formula to represent the trip starting at the 4 o'clock position instead of at the 6 o'clock position.

Greg started by asking students talk amongst themselves about the problem for three minutes. After 1:18, Greg announced to the whole class that he heard several groups say that starting at a different position is the same as applying a horizontal shift, so they just needed to figure out what the horizontal shift is. He also said that it is easiest to keep the B-value the same and write the equation as $g(t) = -27.5\cos(2\pi/3(t-h)) + 37.5$, where $|h|$ is the amount they should shift left or right by. Greg then suggested that if students were not sure if they wanted to shift right or left, they should graph both of them and see what the difference is in shifting right versus left on this problem. Greg then gave students another 2:02 to work on the problem and monitored their progress. Towards the end of this time, he drew the following figure on the board.

Figure 11. Illustration Used by Greg to Talk About Position on Ferris Wheel



Greg brought the class back together by asking, "How much do we want to shift by? Let's start by deciding that." No one responded immediately, so Greg drew line segments from the center of the circle in Figure 11 to the 6 o'clock, 5 o'clock, and 4 o'clock positions. He explained that t is measured in minutes and so the shift should also be measured in minutes. Then he asked, "If we are starting at 4 o'clock instead of 6 o'clock, how many minutes have we shifted by?" Greg looked around the room and said, "Some people said 2, others are giving me 1." He then turned back to his drawing and asked, "How much time does it take me to get from 6 o'clock to 5 o'clock?" One student responded with 60 minutes, to which Greg responded with, "Sixty minutes to get from 6 o'clock to 5 o'clock?... Well, so if I get the whole way around in 6 minutes, each of these is $1/12$ of…" At this point, a chorus of students started speaking and the student who gave the response of 60 minutes said, "Oh, I meant that to get from 6 o'clock to 5 o'clock it takes 60 minutes." At this point, Greg realized that his students had been thinking of a literal clock, as opposed to using the times as positions. So he recognized that it was slightly confusing, but since it takes 6 minutes to get all the way around and going from one position to the next is the same as going $1/12$ of the way around, moving one position (like from 6 o'clock to 5 o'clock) takes 30 seconds. So we want to shift by 1 to change from the 6 o'clock position to the 4 o'clock position.

Now that they had the shift amount figured out, Greg asked if anyone had graphed both a shift left and a shift right to see what the difference between them is. When no one responded, Greg started sketching each graph on the board. At this point, a student spoke up and said they wanted to shift to the left and Greg asked why. The student responded by saying, "Because if you board at the 8 o'clock position, you shift to the right." Next, Greg pointed out that we actually don't know which direction the Ferris wheel is turning. The student piped up and said, "Sorry, I messed up. Just flip around whatever I just said."

Greg responded by returning back to his point that we actually don't know which direction the Ferris wheel is turning (clockwise or counterclockwise). So, in short, both answers were actually right because we didn't know which direction the wheel is turning. In conclusion, Greg explained that shifting to the left corresponds with turning counterclockwise and shifting to the right corresponds with turning clockwise.

I coded this as a procedures with connections example for the following reasons. First, Greg chose this example because it was more difficult and he wanted his students to learn how everything they had learned throughout the semester worked together. In particular, this example focused students' attention on how to identify the amount and direction a graph is shifted by when they are only given a verbal description of the shift. The problem itself did not explicitly tell students that they needed to use a horizontal shift, but rather this was something they had to figure out as part of the problem. Greg also relied heavily upon the clock diagram that he drew to talk about moving the start position and how long that would take given what they know about how long it takes to travel around the whole Ferris wheel. Finally, it was clear from the students' questions and responses that the example required some degree of cognitive effort for students to follow.

Discussion

In this part of my study, I examined what high cognitive demand examples look like in precalculus courses and identified three different roles that instructors took on when enacting high cognitive demand examples. Originally, I sought to use the Task Analysis Guide developed by Smith and Stein (1998) to code the cognitive demand of enacted examples. However, the Task Analysis Guide includes language that specifies that students are the ones doing the mathematical work (e.g., “require students to explore…students need to engage…”). However, while some instructors did involve students explicitly in working out examples, others chose to do most of the mathematical work themselves. Therefore, I created a modified framework for analyzing the cognitive demand of examples (Table 7) that removed any language concerning *who* is doing the mathematical work.

Using this modified framework, I analyzed 93 examples that were enacted in precalculus classrooms and found that 25 of them were enacted at a high level of cognitive demand. In these examples, I found that there were three roles that instructors took on during the enactment: modeling, facilitating, and monitoring (Table 11). Using these three different ways of working through an example, I was able to construct role profiles of the instructors involved in my study as well as of the high cognitive demand examples that they enacted. As a result, I found that while some instructors chose to just model examples for their students (e.g., Dan and Emma), others chose to switch between different roles. Juno also modeled examples for her students, but often asked for student involvement and switched to facilitating. On the other hand, Alex and Greg switched back and forth between all three roles, while Kelly chose to never model and instead just facilitated a whole class discussion or monitored her students as they worked on parts of the example independently or in small groups.

In each of these examples, the instructors presented the material in a variety of ways. While the students in Dan and Emma’s class did not have the opportunity to struggle with the mathematics involved in the example, they still had the opportunity to learn from the high cognitive demand examples that Dan and Emma did enact. On the other hand, the students in Alex, Greg, Juno, and Kelly's class all had the opportunity to contribute to the mathematical work entailed in solving these high cognitive demand examples.

A natural question that arises is, "Why is it reasonable to assume that modeling, facilitating, and monitoring capture every type of role that an instructor might take on when enacting an example during class?" One way of thinking about these three roles is in terms of the continuum of student-centered and teacher-centered instruction. According to Felder and Brent (1996), "student-centered instruction is a broad teaching approach that includes substituting active learning for lectures, holding students responsible for their learning, and using self-paced and/or cooperative (team based) learning" (p. 43). In my framework, monitoring is a form of student-centered instruction while modeling is a form of teacher-centered instruction. On the other hand, facilitating whole class discussions exists somewhere in the continuum between the two. So altogether, the three roles are intended to cover the entire spectrum of student- and teacher-centered instruction.

Limitations

One limitation of this study is that the data I collected focused on the instructor and did not incorporate the student perspective. Therefore, I had to assess the cognitive demand of each example based upon the questions that students asked and the mathematical content of each example. While I tried to define the four different levels of cognitive demand so that a classroom observer could categorize examples, it was still difficult at times to determine whether or not an example required cognitive efforts for students to follow or understand.

Another limitation is that even though I did conduct a multiple case study, all of the instructors that I observed were teaching precalculus in the same mathematics department. In particular, all of the instructors were provided with lesson guides from the department, so they all had access to and drew from the same curriculum. While instructors could modify these lesson guides, many of them stuck to them and used the examples that were provided. So the cognitive demand of the enacted examples was probably influenced by the cognitive demand of the examples included in the lesson guides.

Another limitation of this study was that is difficult to determine when an instructor is switching between modeling and facilitating. In particular, facilitating still requires contributions from the teacher, so it can be difficult to determine exactly when an instructor stopped modeling and started facilitating a whole-class discussion. Therefore, the role profiles should be interpreted as having a margin of error any time an instructor switched between modeling and facilitating.

Implications

The modified framework that I developed for analyzing the cognitive demand of examples is useful for both researchers and practioners. First, this framework gives researchers a way to analyze the cognitive demand of tasks independent of *who* is doing the mathematical work. This is especially important for examples, since instructors can present them in a variety of ways. While it is similar in many ways to the Task Analysis Guide (Smith & Stein, 1998), I modified their original framework by removing any references to *who* is doing the mathematical work. The framework is also useful for practioners as a planning and reflection tool. As teachers plan and reflect on their teaching, they can use this framework to assess the cognitive demand of the examples they use.

The three roles that I identified (modeling, facilitating, and monitoring) are also useful for both researchers and practioners. First, these roles provide researchers with a way to identify what teachers do when they present examples and what they expect their students to do. In particular, researchers can construct role profiles for teachers and the examples that they enact and see how these profiles might correspond with student engagement and opportunities to learn. On the other hand, practioners may find it helpful to think about what role they plan to take on when enacting examples and why it might be useful to model, facilitate, or monitor in different circumstances. Also, being aware of these different roles can help instructors reflect in-the-moment on whether or not they should switch from modeling to facilitating or pause and monitor students as they work through part of an example.

Conclusion

The purpose of this paper was to identify what high cognitive demand examples look like in undergraduate precalculus classrooms and to examine the roles that instructors take on when presenting examples. While I originally intended to use the Task Analysis Guide (Smith & Stein, 1998) to analyze examples, I found that it was difficult to use because some of the language seemed to specify that students must be the ones doing the mathematical work. Since examples are different than tasks in that sometimes the instructor models them for students, I developed a modified framework for analyzing the cognitive demand of examples (Table 7) that focuses more on the cognitive demand of the mathematics involved in the example and less on *who* is doing the example. While this modified framework was useful for my study, other researchers who want to study cognitive demand independent of who is doing the mathematical work can also use it. Also, practioners can use this framework as a way to examine the cognitive demand of the examples that they use in their classrooms. In my study, I also found that instructors took on three different roles when presenting examples: modeling, facilitating, and monitoring. To help illustrate what these roles look like, I provided narrative descriptions of different examples that were presented by instructors in different ways. Using these three roles, other researchers can construct role profiles for teachers and the examples that they use and study how these role profiles might afford different opportunities for students to learn and struggle. Also, being aware of these roles can help instructors think about *who* is doing the mathematical work in their classrooms and what opportunities they are giving their students to learn and struggle with the content.

Chapter 5: Decomposing the Pedagogical Work Entailed in Enacting High Cognitive Demand Examples

This paper decomposes the pedagogical work of enacting high cognitive demand examples by identifying the teaching tasks entailed in enactment. In this chapter, I argue that instructors must attend to the mathematical point, make connections, provide clear verbal explanations, articulate cognitive processes, and support student understanding when enacting high cognitive demand examples. This case study was conducted using a thematic analysis of 25 high cognitive demand examples that were enacted by instructors in undergraduate precalculus classrooms. This paper contributes to the corpus of literature that decomposes the practice of teaching so that novice teachers can more easily see and replicate the work that teachers do.

Introduction

Examples are often used in mathematics classrooms as a way to explain and model content, practices, and strategies, which is a basic fundamental of teaching (TeachingWorks, 2017). Explaining and modeling goes beyond just working out an example at the board and should include the teacher thinking aloud and demonstrating complex academic practices and strategies. One way of classifying the complexity of an example is by examining the cognitive demand (Stein et al., 1996). Stein, Grover, and Henningsen defined cognitive demand as “the kind of thinking processes entailed in solving [a] task” (p. 461) and identified four categories to describe the cognitive demand of a task: memorization, procedure without connections, procedure with connections, and doing mathematics.

Despite the importance of explaining and modeling complex academic practices and strategies, I have found that the examples teachers often use do not involve high cognitive demand tasks (Chapter 4). Of the 93 examples that I observed in my study, only 25 (27%) of them were enacted at a high level of cognitive demand. While it may be true that students in these classrooms had the opportunity to engage with high cognitive demand tasks in other contexts (such as small group work and homework exercises), it is troubling that the problems that the teachers chose primarily focused on explaining and modeling memorization and procedures without connections tasks. In particular, while the teachers may have expected students to engage with high cognitive demand tasks in other settings, they often did not demonstrate the type of thinking entailed in solving these complex problems.

The paucity of high cognitive demand examples may be attributed to several factors. Teachers may have viewed small group work or homework as a more appropriate opportunity for students to engage with high cognitive demand tasks. Stein, Grover, and Henningsen (1996) also pointed out that high cognitive demand tasks are "often less structured, more complex, and longer than tasks to which students are typically exposed" (p. 462), which makes them more difficult to enact. In their study, the authors found that even tasks that were set up at a high level of cognitive demand could decline into low level due to the inappropriateness of the task for students, the focus shifting to the correct answer, too much or too little time, and several other factors (p. 479).

Another factor that may contribute to the cognitive demand of examples is the teachers' mathematical knowledge for teaching (MKT). Charalambous (2010) found evidence of this connection in a study of elementary school teachers, which used the *Learning Mathematics for Teaching* test (Hill, Sleep, Lewis, & Ball, 2007) to measure teachers' MKT. Since no similar measure exists at the secondary or undergraduate level, I built upon Charalambous' (2010) finding and examined the MKT entailed in enacting high cognitive demand examples (Chapter 6).

Finally, another reason why instructors may struggle to enact high cognitive demand examples is because they are not aware of the work that goes into setting up and enacting them. While most people have experienced years of sitting in a classroom and observing their teachers, much of the work of teaching is not observable or difficult to recognize. In fact, Clark and Lampert (1986) pointed out that teachers have to do many complex things at once, and yet need to make it all look effortless in order to maintain credibility with their students. While this hidden work of teaching is vital, it is often difficult for novices to recognize and reproduce.

The purpose of this paper is to examine the work of teaching entailed in enacting high cognitive demand examples. Through my analysis of the examples that instructors were able to enact at a high level of cognitive demand, I decomposed the work of teaching entailed in explaining and modeling content, practices, and strategies. The purpose of this decomposition is to create a framework that breaks down teaching with examples so that novice instructors can both see the work involved and model their own teaching practices after it. In my work, I define an example as a mathematical task that an instructor facilitates with the entire class for illustrative purposes. While students may be asked to work individually or in small groups on parts of the example, the majority of the example is done together as a whole class.

In the next subsection, I provide a narrative of an example that declined in cognitive demand during enactment, even though the instructor intended for the example to be of higher cognitive demand. The purpose of this narrative is to illustrate how examples can quickly decline in cognitive demand if the instructor does not attend to the work of maintaining the cognitive demand.

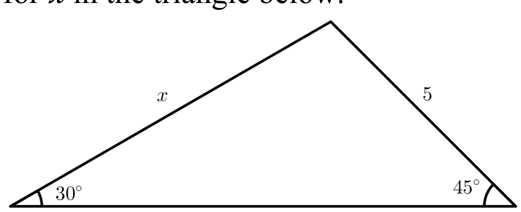
Greg's Law of Sines Example

Greg is teaching trigonometry, which meets on Tuesdays and Thursdays for 50 minutes each day. It is the beginning of the semester and Greg is finishing the first chapter that introduces trigonometric functions. In this lesson, Greg is introducing the Laws of Sines and Cosines as a way to talk about finding the side lengths and angle measures of non-right triangles. At the beginning of the lesson, Greg planned to first derive the Laws of Sines and Cosines. Next, Greg wanted to give students an opportunity to practice applying these Laws. However, instead of just giving them a problem and telling them what procedure to use, Greg wanted to provide students with an opportunity to think critically about what procedure would be appropriate.

Greg's students had worksheets with problems that they worked through during every class period. The first problem on their worksheet for this lesson is given below in Figure 12. In the written example, the students were told that the goal of the problem was to study how to use the Law of Sines to solve for x. However, Greg knew that identifying the correct procedure to use to solve a problem is what students struggle with. So he decided to use this worksheet problem as an example, but without the goal statement. Instead, Greg planned to just draw the triangle on the board and ask students to figure out how we can use the information given to solve for x. Greg wanted to help get his students used to looking at a problem, identifying the given information, and then identifying what tools they have that can take that given information and produce what we want.

Figure 12. Greg's Law of Sines Written Example

Goal: To study how to use the Law of Sines to find the length of an unknown side.
Exercise: Solve for x in the triangle below.



Greg thought that this problem, in particular, was appropriate for helping students learn how to "choose the right tool" because it's a simpler case. In the example, the students are given a lot of information and a specific outcome. In later problems on the worksheet, there were problems where students are only given side lengths and asked to fill in every piece of information that they can. But with this example, students just have to figure out one piece of information. Greg anticipated that students might try to solve the problem using the Law of Cosines, since they will have talked about both Laws before working through this example.

During class, Greg set up the example as he had intended by just drawing the triangle and not giving the goal statement. However, he told students that it was the first problem on their worksheet so many students began flipping through their workbook to find the right page. Greg then asks, "Based on what we have done today…we are going to use our new tools…. What information allows us to combine side lengths and opposite angles?" A student immediately responded with sines, which Greg interpreted as meaning Law of Sines. Greg then asked, "What does the Law of Sines tell us in this case?" One student responded with, "Break this into two triangles," which Greg interpreted as referring the method that they had used previously to derive the Law of Sines.

Greg went on to explain that they could re-deduce the Law of Sines, but it would be easier if they just used the final equation that they had come up with in the first place: $\sin(A)/a = \sin(B)/b$, where A is the angle opposite the side of length a and B is the angle opposite the side of length b. A student then suggested that they could write $\sin(30)/5 = \sin(45)/x$. Greg then worked through the algebra of solving for x. He then paused and asked if there were any questions of what they did in the example or why they did it. Two students asked questions about some of the algebraic steps involved in solving for x, which Greg explained by writing out some of the steps he had skipped.

I coded the intended example as procedures with connections, because Greg said that his main focus for doing this example was on understanding why a procedure is appropriate to use based upon the given information and problem solving goals. So Greg's plan was to focus students' attention on the use of the procedure for the purpose of developing deeper understanding of why it is appropriate to use in this case. Greg also planned to remove the goal statement, so the use of the procedure was no longer specifically called for. However, the cognitive demand of the example changed during the enactment of the example. First, Greg specifically told students that the example was in their workbook, so if students flipped to the right page, they could easily read off the goal statement and therefore know right away what procedure they were supposed to use. Second, Greg's cued that they should be using one of the Laws when he told students that the example was "based on what we have done today" and that "we are going to use our new tools." Finally, instead of taking time to pause and unpack why it's appropriate to use the Law of Sines, Greg quickly moved on to finding the right answer. Therefore I coded the enacted example as procedures without connections.

In later sections, I will provide a decomposition of the work entailed in enacting high cognitive demand examples. The purpose of providing this narrative is to illustrate that even if an instructor intends to enact an example at a high level of cognitive demand, what they do during class has a large impact on whether or not the cognitive demand is maintained. Later, when I provide examples of the work that instructors did to enact high cognitive demand examples, I will highlight how Greg did not attend to these things in this first narrative.

Goals of the Study

The first research question that guides this study is, "What do instructors do when enacting examples to help maintain the cognitive demand?" In answering my first research question, I aim to decompose the work entailed in enacting high cognitive demand examples. The second research question that guides this study is, "How does this work relate to the roles that instructors and students take on when enacting high cognitive demand examples?" In answering my second research question, I aim to draw connections between the results of this study and the results I found in my previous study (Chapter 3), which examines how instructors model, facilitate, and monitor.

Theoretical Foundations

Decompositions of practice were first identified as a critical aspect of professional education by Grossman, Compton, Igra, Ronfeldt, Shahan, and Williamson (2009). In order to develop a framework to describe and analyze the teaching of practice, Grossman et al. examined the professional preparation of clergy, teachers, and clinical psychologists. Through their cross-professional analysis, they found that there are three key concepts that influence the construction of understanding of pedagogies of practice: representations, decomposition, and approximations of practice. In their work, representations of practice involved "the different ways that practice is represented in professional education and what these various representations make visible to novices" (p. 2058). A decomposition of practice "involves breaking down practice into its constituent parts for the purposes of teaching and learning" and an approximation of practice "refer[s] to the opportunities for novices to engage in practices that are more or less proximal to the practices of a profession" (p. 2058).

Since Grossman et al.'s (2009) study, there have been several researchers who have focused on developing decompositions of the practice of teaching mathematics. Sleep (2012) decomposed the work entailed in steering instruction toward the mathematical point. Through her analysis of preservice teachers, Sleep identified seven central subtasks, along with strategies and problematic issues associated with each subtask. The main contribution of Sleep's work is that she provided "an articulation of a key aspect of the work of teaching at a grain size that is directly usable in the design of practice-based teacher education" (p. 965). Other examples of decompositions of mathematics teaching include Smith, Bill, and Hughes' protocol for thinking through a lesson (2008); Jacobs, Lamb, and Philipp's framework for professional noticing of children's mathematical thinking (2010); Smith and Stein's *5 Practices for Orchestrating Productive Mathematics Discussions* (2011); Herbst's decompositions of promoting and managing students' discourse (2011a), explaining concepts and propositions (2011b), setting norms for mathematical work (2011c), explaining procedures (2013), and assigning and reviewing students' work (2014); and the LESRA mathematics instruction framework (Wisconsin Department of Public Instruction, 2013).

In developing my decomposition of practice, I drew upon the curriculum framework developed by Stein, Remillard, and Smith (2007). Instructors are often given or seek out curriculum resources, which the authors refer to as the *written curriculum*. Drawing upon these resources, the instructors create their lesson plans, which make up the *intended curriculum*. Finally, what actually occurs during class is described as the *enacted curriculum*. *Between* the phases of written curriculum and intended curriculum and *within* the phase of enacted curriculum, there are several factors that contribute to the transformation of the curriculum. For example, teachers may change the written curriculum based upon their beliefs or knowledge or classroom structures and norms may influence how the curriculum is enacted.

Smith, Bill, and Hughes (2008) published a lesson planning protocol that decompose the work of planning high cognitive demand tasks. Although very similar to the work that I aim to do, Smith et al. focused on the first stage of transformation that a task goes through as it is taken from the written curriculum and transformed in the teacher's lesson plan. In their protocol, Smith et al. identified three parts of the lesson planning process: selecting and setting up a mathematical task, supporting students' exploration of the task, and sharing and discussing the task. Within each part, the authors provided several guiding questions for teachers to consider as they are planning their lesson. While they admitted that thinking through all of the questions in the protocol might be overwhelming for teachers to do with every task in their lesson plan, the authors argued that it could be used as a tool for collaborative planning. They also highlighted how teachers have used the protocol in pieces until it becomes a more natural part of their thinking process when lesson planning.

Data and Methods of Analysis

To decompose the work of enacting high cognitive demand examples, I analyzed 25 examples that graduate student instructors were able to enact at a high level of cognitive demand. These examples came from a larger set of 92 examples that I observed seven different instructors enact over the period of a year (Chapter 3).

Participants

The six graduate student instructors that participated in my study came from a larger sample of seven graduate student instructors (hereafter called instructors) that I observed. The one instructor, Selrach, that was not included in the analysis for this study was removed from the data set because I did not observe any examples that he enacted at a high level of cognitive demand. The six instructors were doctoral mathematics graduate students in at least their third year of study at a large public university in the Midwestern United States. Graduate instructors were selected based on their level of experience teaching their course (they had to be teaching their respective course for at least the third time) and willingness to participate in the study. All of the instructors were provided with essentially the same written curriculum, which was developed by the mathematics department they were teaching in. However, they were using slightly different versions, as the curriculum was still in development and undergoing revisions.

Data

Each instructor was observed teaching three mathematics lessons, which spanned either one or two days each, and was interviewed both before and after teaching. Recordings were made of all of the interviews (audio) as well as the classroom observations (video and field notes). Any curriculum materials used and lesson plans created by the instructors were captured.

Pre-observation interviews. The semi-structured pre-observation interviews focused on providing context for the observation and motivation for the examples. The instructor was first asked about the topic of the previous lesson(s), whether or not the instructor had taught this topic before using the same or different s, and what curriculum materials the instructor used to develop their lesson plan. Next, the instructor was asked to talk about each example included in their intended lesson plan and identify the mathematics they intended for students to learn through the example and why they chose to use this particular example.

Lesson observations. Table 22 shows the instructors pseudonyms as well as a short description of the topics of each lesson where I observed them enact a high cognitive demand example. All of the instructors, except for Greg, were teaching college algebra and trigonometry, which met five days a week. Greg, on the other hand, was teaching trigonometry, which met two days a week. The examples enacted during each lesson were video recorded and I took observational field notes. As I observed each example, I attended to whether or not the instructor made any changes to their intended lesson plan and what the instructor did while enacting the examples.

Table 22. Descriptions of Observed High Cognitive Demand Examples

Example ID	Lesson	Example Description	Cognitive Demand
Alex 1-1	Introduction to Exponentials	Exploring the notions of exponential vs. linear growth	High
Alex 1-2	Introduction to Exponentials	Building an exponential function from a word problem	High
Alex 2-1	Function Compositions	Exploring the notion of function compositions through unit conversions	High
Dan 2-4†	Function Compositions	Decomposing function compositions into any two functions	High
Dan 3-1†	Trig Equations & Inverse Functions*	Graphing solutions to trig equations as points of intersection	High
Dan 3-8†	Trig Equations &	Finding all solutions in a given	High

Example ID	Lesson	Example Description	Cognitive Demand
	Inverse Functions*	interval to sinusoidal equations	
Emma 2-1	Function Transformations	Transforming the graph of a function	High
Greg 2-1	Trig Equations & Inverse Functions	Illustrate why sine and cosine may have 2 solutions/period, but tangent has 1	High
Greg 4-1	Tangent & Reciprocal Trig Functions*	Exploring the behavior of tangent using standard unit circle angles	High
Greg 4-4	Tangent & Reciprocal* Trig Functions*	Solving real-life problems using tangent	High
Greg 5-1†	Trig Equations & Inverse Functions*	Using graphs to identify how many solutions are in a single period	High
Greg 5-2†	Trig Equations & Inverse Functions*	Finding all solutions to trig equations with standard unit circle angles	High
Greg 5-3†	Trig Equations & Inverse Functions*	Finding all solutions to trig equations with non-standard unit circle angles	High
Greg 5-5†	Trig Equations & Inverse Functions*	Finding all solutions to tangent equation with non-standard unit circle angles	High
Greg 5-6†	Trig Equations & Inverse Functions*	Finding all solutions to sinusoidal equations with non-standard unit circle angles	High
Greg 6-3†	Review*	Finding a sinusoidal equation given a description of a real-life context	High
Greg 6-9†	Review*	Finding the horizontal shift of a sinusoidal function	High
Juno 1-5	Tangent & Reciprocal Trig Functions*	Proving that sine and cosine are cofunctions	High
Juno 2-1†	Trig Equations & Inverse Functions*	Graphing solutions to trig equations as points of intersection	High
Juno 2-2†	Trig Equations & Inverse Functions*	Finding all solutions to trig equations with standard unit circle angles	High
Juno 2-3†	Trig Equations &	Finding all solutions to trig equations	High

Example ID	Lesson	Example Description	Cognitive Demand
	Inverse Functions*	with non-standard unit circle angles	
Kelly 1-1	Introduction to Exponentials	Exploring the notions of exponential vs. linear growth	High
Kelly 2-1	Polynomials & Rational Functions	Exploring the behavior of polynomials near the roots	High
Kelly 2-2	Polynomials & Rational Functions	Graphing polynomials given the equation in factored form	High
Kelly 2-3	Polynomials & Rational Functions	Constructing polynomial equations given the graph	High

Note: The example ID represents the instructor, the observation number, and the enacted example number.
*These lessons were purposefully sampled because of their focus on procedures.
†These examples were spread out over two days of instruction.

Post-observation interviews. After each observation, I met with the instructor (typically the next day) to conduct a post-observation interview. Between the observation and the interview, I analyzed the video recordings of the examples that the instructor enacted during class and selected one to talk about with the instructor. When possible, I selected an example that was enacted at a high level of cognitive demand. If all of the examples were enacted at a low level of cognitive demand, then I either chose one that declined in cognitive demand (i.e., was intended, but not enacted, at a high level of cognitive demand) or selected a random example to unpack.

I began the post-observation interview by first asking the instructor if the class had went as planned and why they chose to either add or skip examples. Next, we watched pre-selected video clips (30-60 seconds in length) together and I probed their thinking regarding what they were doing at specific moments and their reasoning behind their actions. Given that some time had elapsed between the observation and the interview (usually somewhere between 12 and 24 hours), I used the video clips to help with video-stimulated recall (reference). The interviews usually included specific conversations centered around 5-10 video clips and lasted 45-60 minutes.

Analysis

I began my analysis of the work of enacting high cognitive demand examples by first identifying general themes based on literature, my own experiences teaching, and my observations during data collection. The full list of these codes can be found in Appendix D. I organized these themes into general categories, which were used in the initial coding stage. As I analyzed the set of high cognitive demand examples I observed, the categories and subcategories were reorganized and added based upon what I observed in my data. This refined coding scheme can be found in Table 35 in Appendix D. I then conducted an axial coding in order to organize my decomposition codes into broad tasks and tasks. Finally, I used this semi-final set of codes and recoded all 25 examples. The final tasks and subtasks that resulted from this analysis will be discussed in the following section and can be found in Appendix D.

Results: Pedagogical Work of Enacting High Cognitive Demand Examples

Based upon my analysis, I found that the work of enacting high cognitive demand examples can be broken town into five broad tasks:

1. Attending to the mathematical point,
2. Making connections,
3. Providing clear verbal explanations,
4. Articulating cognitive processes, and
5. Supporting student understanding.

Each of these tasks will be discussed in turn, with narrative descriptions of how the instructors that I observed did these things. I will also talk about the first narrative that I presented in the introduction of Greg and his Law of Sines example in order to illustrate how he did not attend to these things during the example enactment.

Attending to the Mathematical Point

The importance of attending to the mathematical point has been highlighted by the work of Sleep (2012). Sleep identified seven central tasks entailed in steering instruction toward the mathematical point: attending to and managing multiple purposes, spending instructional time on mathematical work, spending instructional time on the intended mathematics, making sure students are doing the mathematical work, developing and maintaining a mathematical storyline, opening up and emphasizing key mathematical ideas, and keeping a focus on meaning. While Sleep's work focused on attending to the mathematical point throughout a lesson as a whole, I found that it was also an important aspect of enacting high cognitive demand examples. While Sleep broke down attending to the mathematical point into seven subtasks, I examined this task of teaching at a larger grain size. In particular, I found that instructors introduced the mathematical point as a way to set the focus of the example, maintained the focus of the example on the mathematical point, and summarized the example to reiterate the mathematical point. In the following subsections, I address each of these subtasks separately and provide narrative descriptions of how the instructors in my study did these things.

Introducing. Out of the six instructors that I observed enacting a high cognitive demand example, five of them made sure to introduce the mathematical point as a way to set the focus of the example. However, these instructors chose to introduce the mathematical point at different times during the example. In two of the three high cognitive demand examples that I observed Alex enact, she introduced the mathematical point at the very beginning of the example. Before asking students to explore the notions of exponential versus linear growth, Alex drew a web diagram on the board that mapped out the different families of functions they had studied so far. In particular, she explained that today they were going to begin exploring the family of exponential functions.

Dan, on the other hand, sometimes chose to wait until the end of an example to introduce the mathematical point. After explaining how we can visualize solutions to equations by finding points of intersection on graphs, Dan brought attention to the fact that trigonometric equations can have infinitely many solutions. Dan is also very specific in the language that he uses and answers the question, "What is the point?" for two of the three high cognitive demand examples that he enacted. Juno and Greg also introduced the mathematical point as a way of interpreting the mathematics that they had been working through.

In the example narrative that I presented at the beginning of this chapter, Greg had a clear mathematical point in mind for the example he did on using the Law of Sines to solve for an unknown side. In particular, Greg wanted to give students an opportunity to think about the information they were given and decide what tools they could use to get the desired result. However, this point was never explicitly introduced during instruction. In fact, the mathematical point that Greg seemed to be making during class was that this was a problem we can use our new "tools" to solve. Since this change in direction reduced the cognitive demand of figuring out what tools are appropriate, not introducing the original mathematical point contributed to the decline of the cognitive demand.

Maintaining. Since many of the instructors chose to enact examples by either facilitating whole-class discussions or monitoring students as they worked individually or in small groups on parts of the example, it was important for instructors to make sure that the example stayed focused on the mathematical point. Even for instructors who chose to model the example for students, they still were explicit about making sure that students were focusing on the mathematical point and not getting lost in the arithmetic.

The work of maintaining the mathematical point was particularly important in examples of longer duration. For example, after taking the first five minutes to determine the order of transformations and move individual points on the graph, Emma reminded her students that the point of the example was to graph the transformed graph between $t = 0$ and $t = 9$, which meant that not all of the transformed points would be on the final graph. In another example, Greg first calculated $\sin\theta / \cos\theta$ for all θ in the first quadrant and asked students to calculate these values for the rest of the quadrants, but reminded his class that the real purpose of doing this was to see if $y = \sin\theta / \cos\theta$ is periodic.

When monitoring students as they worked through parts of the example individually or in small groups, Greg and Kelly maintained the mathematical point by asking students to make sure they were discussing a particular question. In Greg's example, he anticipated that students might be struggling with identifying whether or not they should shift a function to the right or the left, so he reminded them that they could graph both transformations and see which one fit the phenomenon they were trying to model. In Kelly's example, she asked students to fill in a table of input-output values and find a general formula that would give the output values for any input. After monitoring students as they worked on this problem for three minutes, she made an announcement to the whole class to remind them to try to find a general formula.

In the Law of Sines example that Greg enacted, the mathematical point turned quickly from focusing on identifying a procedure to computing the answer. If Greg had paused after the student quickly responded with the correct procedure and asked, "Why would hat be that an appropriate procedure to use?" or "Why do our problem conditions make that an ideal procedure to use?" or "Is that the only procedure we could use? Can we generate other strategies that would also work?", then he could have maintained the focus of the example on the original mathematical point.

Summarizing. The final subtask associated with attending to the mathematical point was summarizing the example. For the six instructors that I observed, every one of them did this during at least one of their high cognitive demand examples. Alex was the most consistent, as she summarized the mathematical point at the end of every high cognitive demand example that I observed her enact. Most of her high cognitive demand examples were also longer in length (5:17, 8:22, and 15:02), which may contribute to her tendency to always summarize at the end.

Dan also used summarizing the mathematical point as a way to conclude his examples. In particular, Dan highlighted the strategies that he used to solve problems. In his example where he explained how to decompose function compositions into two functions, Dan wrapped up by highlighting that students could use a similar strategy of identifying an "inside" and "outside" function when working on similar problems in the future. Emma summarized her high cognitive demand example in a similar way by highlighting the two different approaches that she had demonstrated and explaining that students could use whatever one made more sense to them. When finding all solutions to a sinusoidal equation, Greg also ended his example by summarizing what all the steps were that they had gone through to find their families of solutions.

In the Law of Sines example that Greg did, the majority of the example ended up being focused on the computations needed to solve for x. In our post-observation interview, Greg mentioned that he had not expected students to struggle with the computational aspect of the problem. However, he was responsive to their questions and made sure that everyone understood the algebra involved in the problem. One way that he could have wrapped up the example and brought his students attention back to the original mathematical point that he had intended would have been by summarizing. Even though his students seemed to struggle with the algebra more than the selection of the procedure, providing a summary of why that procedure was appropriate would have helped refocus his students attention on what he originally intended.

Making Connections

The importance of making connections is well studied in the mathematics education literature (Baki, Çatlıoğlu, Coştu, & Birgin, 2009; Elia, Gagatsis, & Heuvel-Panhuizen, 2014; Gainsburg, 2008; Sidney & Alibali, 2015). In order for students to build a deeper, more conceptual understanding of mathematics, it is important for them to see how mathematics is connected as a domain and to other domains. However, connections are not just important in building conceptual understanding, but also in building understanding of procedures. The Task Analysis Guide (Smith & Stein, 1998) highlights this importance by categorizing procedures without connections tasks as lower cognitive demand and procedures with connections tasks as higher cognitive demand. Therefore, it is not surprising that making connections was something that emerged from my data set.

While there are many ways one can make connections and many things one can make connections between, there were three primary subtasks that emerged from my analysis. First, instructors made connections to previously learned content, practices, and strategies. This finding is similar to what Stein, Grover, and Henningsen (1996) found in that one factor that influences the maintenance of the cognitive demand of a task is building upon prior student knowledge. Teachers also made connections between algebraic, graphical, tabular, pictorial, and verbal representations. Finally, instructors made connections between concepts, such as exponential and linear growth.

Prior knowledge. Making connections to previously learned content, practices, and strategies came up in almost every high cognitive demand example that I observed. Many instructors used connections to prior knowledge as a way to transition into a new topic. For example, Alex used a problem from a previous exam as a way to reintroduce the concept of function compositions. Her class had briefly explored function compositions at the beginning of the semester and was tested over it on their first exam. Later on in the semester, they took a deeper dive into topic, but Alex purposely used an example they had seen before as a way to help students make connections to their prior knowledge.

Instructors also made connections to prior knowledge as a way to help students recognize that they could use similar problem solving strategies. When explaining how to find all solutions to sinusoidal equations, Dan focused on the ways in which the problem solving process was the same as the one they had used for solving less-complex trigonometric equation. Again, the language that Dan used was very purposeful and explicit in expressing how these two problems were similar.

> *Now just like last time, this equation means something on the unit circle. That still doesn't change. Sine of whatever corresponds to y-coordinates on the unit circle. Which y-coordinate? Well, $-1/2$. So just like last time, we are going to draw in our picture of the unit circle. Just like last time, I'm interested in y-coordinates—that's sine—being equal to $-1/2$. Just like last time, this gives me two points on the unit circle.*

In the example where Juno proved that sine and cosine are cofunctions, Juno used prior knowledge to build up intuition before attacking a proof. Before attempting the proof, Juno did two examples to show that sine and cosine are equivalent on certain pairs of complementary angles. Her purpose of using these examples was to motivate her proof that sine and cosine are equal on all complementary angles. After proving that this was true, Juno pointed out that if sine and cosine are properly named, then we could also expect that tangent and cotangent and secant and cosecant are also cofunctions, which is what she asked her students to prove next.

In Greg's Law of Sine example, there is really only one time when connections are made to prior knowledge. When Greg asks, "What does the Law of Sines tell us in this case?", a student responds by saying, "Break this into two triangles." This student was referring to the method that Greg had used at the beginning of class to derive the Law of Sines. While this is a valid method, Greg explained that instead of going through all the work again, we could instead just use the final result. So while Greg did acknowledge that the student was making connections to prior knowledge, he also implicitly labeled that knowledge as unnecessary or unhelpful in solving this problem.

Representations. Since many of the high cognitive demand examples that I observed were coded procedures with connections, it is not surprising that making connections among multiple representations was a common theme. Again, the majority of the high cognitive demand examples that I coded involved making connections between representations. However, the types of representations that the instructors were making connections between (e.g. verbal and pictorial or algebraic and graphical) differed.

Although Emma only had one example that I observed her enact at a high level of cognitive demand, this one example was coded six different times with the representations code. In the example, Emma was explaining two different methods that could be used to sketch a graph transformation. The problem provided the graph of $f(x)$, which was piecewise linear, and then asked for the graph of $3f(x+1)-2$ on the interval $[0,9]$. Throughout the example Emma consistently made connections between the algebraic and graphical representations. After deciding what the order of transformations was, Emma went back to the graph and explained that if we move the corner points, we can then connect them with a line in our transformed graph. She also made connections between verbal and algebraic representations of function transformations as she transformed individual points. Finally, when her students struggled to understand the algebra behind finding outputs that did not land on integer values in the original graph, Emma consistently turned the conversation back to making connections between using the slope of the graph to calculate non-integer output values.

In the Law of Sines example that Greg enacted, few connections were made between representations. Once they were able to use the Law of Sines to set up inequalities, the original triangle was never referenced again. In particular, there were no connections drawn between the triangle given in the problem and the triangles that they had used previously to derive the Laws of Sines and Cosines. If Greg had wanted to focus on developing an understanding of why the Law of Sines was an appropriate procedure to use, then it seems like it would have made sense to draw connections between the values given in the two triangle diagrams. In particular, drawing these connections may have helped students develop a deeper understanding of when it might be appropriate to use the Law of Sines versus the Law of Cosines.

Concepts. Making connections between concepts was not as prominent of a theme as making connections between representations, but was common enough that I added it to my final coding scheme. One reason for this is because connections between representations and connections between concepts are not easily teased apart. For example, when introducing the concept of multiplicities of the zeros of polynomials, Kelly focused on explaining how multiplicity and degree are related concepts. However, in order to do this, she relied upon students' understanding of what polynomials look like graphically. In particular, Kelly used the example of parabolas and explained that depending on where they lie on the xy-plane, they will have zero, one, or two x-intercepts.

In her example where she focused on building an exponential function from a word problem, Alex asked her students to make connections between the equation they had derived and the standard form of an exponential. In particular, Alex focused her students' attention on making connections between the particular numbers involved in their equation and the different components (e.g., initial value, growth factor, and growth rate) of the standard form.

In the example that Greg did involving the Law of Sines, Greg focused more on the algebra than he did on making connections between concepts. In particular, Greg talked about making connections between the given information and the problem solving goals, but these connections were never explicitly focused on or verbalized during class. Given that a student identified right away that the Law of Sines was an appropriate tool to use to solve the problem, it is perhaps true that his students were already making these connections on their own. However, given that Greg told them the example came from the first problem on their worksheet and that many students flipped to this page as he was drawing the picture on the board, it's impossible to tell if they figured out this connection on their own or just read the goal statement printed on the worksheet.

Providing Clear Verbal Explanations

According to TeachingWorks (2017), "explaining and modeling are practices for making a wide variety of content, academic practices, and strategies *explicit* to students" (emphasis added). In order to provide equal access to education for all, there has been a recent focus on the importance of using explicit instruction (Archer & Hughes, 2010). Doabler and Fien (2013) identified teacher modeling as an essential element of explicit mathematics instruction. They identified that two key components of teacher modeling are using clear and consistent wording and providing unambiguous explanations and demonstrations. Similarly, I found that providing clear verbal explanations was a prominent task that teacher engaged in when enacting high cognitive demand examples.

The instructors that I observed provided explanations of a variety of aspects of each example, which I will explain in more detail in the following subsections.

Instructions. Since the examples that instructors used were often not included in the students' workbooks, instructors were careful to provide clear explanations of the example set up, constraints, and goal. This included providing clear instructions for what they expected students to do if they chose to monitor students as they worked through parts of the example individually or in small groups. While instructors often wrote the example instructions on the board, they also provided additional verbal explanations to explain the problem set up, constraints, and goal. For example, after writing up a long description of a word problem that asked students to compose a function that gives the temperature (in °F) of a kiln after t minutes with a function that transforms °F to °C, Alex went back and explained that what they were trying to do was come up with a function that would give the temperature (in °C) of a kiln after t minutes.

This practice often went hand-in-hand with attending to the mathematical point, but other times was focused more on making sure that students understood the example set up and constraints. Greg, for example, made sure that his students understood how to interpret a graph before engaging with trying to find the equation of the graph. In the example, a weight is suspended from the ceiling by a spring. The students were provided with a graph that showed the distance (in centimeters) from the ceiling to the weight as a function of time (in seconds). Greg anticipated that his students might have a hard time interpreting the graph, since whenever the graph is bigger the weight is further away from the ceiling. To make sure that his students understood the example set up, Greg explained that the graph is increasing when the weight moves downward, which is opposite of what his students might expect if they are thinking about the visual motion of the weight.

In the Law of Sines example that Greg enacted, the problem was clear enough that perhaps no explanation of the set up was needed. However, one way that Greg could have been more explicit in his explanation is by making it clear what the problem constraints were and relating that to a discussion of knowing which procedure is appropriate. For example, Greg could have emphasized that it made sense to use Law of Sines because it relates two pairs of side lengths and opposite angle measures (which is exactly what was provided in the diagram). He could have also discussed why using the Law of Cosines would be less ideal, because we would need to know the third side length of the triangle, which was not given in the initial example.

Content, practices, and strategies. Providing clear verbal explanations of content, practices, and strategies was the most common subtask used in this category and was something that every instructor attended to when enacting high cognitive demand examples. In many of his examples, Dan focused on explaining the problem solving strategy that he was using. When explaining how to decompose a function composition into two functions, Dan explained that he uses the strategy of thinking “outside” and “inside”. Dan had used similar language when demonstrating how to compose two functions, so he drew connections between the work they had done before and the work that they were doing now. In particular, he focused on explaining we can look at the function composition and try to identify an outside and inside function. Once they had done this, he then explained how to check their work by composing the two functions they had found and verifying that they ended up back at the given function composition.

In several of Juno's examples she elicited student thinking as she worked through parts of the example and made sure that it was clear to everyone what they were doing. When proving that sine and cosine are cofunctions, Juno asked students to identify the values for sine and cosine using the right triangle definitions. After each student response, Juno made sure that it was clear where the student had gotten their answer from by referring back to the right triangle that she had drawn. Other instructors asked students to provide verbal explanations, which I will discuss in the later section on articulating cognitive processes.

In Greg's example of using the Law of Sines, he did provide verbal explanations, but they were mostly of the algebraic steps involved in solving for x. However, since several of his students seemed confused by these steps, Greg was responding to his class and providing clear verbal explanations as a way to support student understanding. However, the explanations mostly focused on the computational aspects of the problem and not on the strategy that they had used to solve the problem or other higher-cognitive demand aspects.

Similarities and differences. Providing clear verbal explanations of similarities and differences between content, practices, and strategies was one way that instructors made connections. So these two tasks are also inherently linked. In explaining why the degree of a polynomial is always greater than or equal the number of zeros, Kelly talked about how moving around a parabola could give us zero, one, or two zeros. Similarly, she talked about how they could shift the graph of a quartic function and get more or less zeros depending upon its position. Other instructors focused on explaining differences between content, practices, and strategies. When explaining how to find all solutions to a trigonometric equation, Juno explained that they could use either $-\cos^{-1}\theta$ or $2\pi - \cos^{-1}\theta$ as the second initial solutions. She emphasized that graphically these represent two different intersection points but they also produce the same solution family once multiples of the period are added on.

Several of the high cognitive demand examples that I observed were focused on finding all solutions to sinusoidal equations. In these examples, Dan and Greg focused on identifying both similarities and differences between finding all solutions to trigonometric equations versus finding all solutions to sinusoidal equations. Earlier I highlighted how Dan used specific language to make connections to their prior knowledge of how to solve trigonometric equations. Greg used a similar strategy by identifying that periodic equations will always have solutions of the form (initial) $+$ $(\text{period})k$, where $k = \text{any integer}$. However, Dan and Greg emphasized that even though these two types of problems are similar, the process involved in finding the initial solutions of sinusoidal functions is more involved.

In Greg's example with the Law of Sines, he asked, "What information allows us to combine side lengths and opposite angles?" The root of this question is asking students to find similarities between the given problem and the tools that they had learned about that day. However, Greg did not spend time explaining these similarities after a student suggested the correct solution strategy. On the other hand, Greg did spend time explaining differences between a derivation and a procedure. When Greg asked, "What does the Law of Sines tell us in this case?", a student responded with, "Break this into two triangles." Greg recognized that the student thought using the Law of Sines meant going through the all the derivation steps, instead of just using the final end product. So

Greg took time to explain that they only had to do the derivation once, and from then on they could just use the final result.

Representations. Since most of the high cognitive demand examples that I observed were coded as procedures with connections, many of them involved multiple representations. Instructors often took the time to explain representations at the beginning of an example in order to make sure everyone understood the example setup. In order to illustrate why sine and cosine might have two solutions per period, but tangent will always have exactly one, Greg first took time to explain what the graphs of sine, cosine, and tangent look like. Greg first took time to make sure that students were comfortable with the graphs of these functions because he felt it would be easier for his student to understand initial solutions if he drew connections to visual representations of the functions.

Other times, instructors introduced representations later on in the example and then took the time to explain how they were to be used. In the example where Alex asked her students to explore the notions of exponential versus linear growth, Alex asked her students to come up with an equation that would model compound interest. Alex wanted to see if her students could use their intuition of how interest accumulates as a way to derive the exponential formula. However, most of her students approached the problem by building a recursive formula. To help her students see how exponential functions grow, Alex introduced an input-output table and focused on how the output values were changing. However, before using the table to solve the problem, Alex first took time to explain the setup of the table as a way to make sure that everyone understood the new representation that she was introducing.

In the Law of Sines example that Greg enacted, little time was spent explaining the pictorial representation that was given in the problem. In particular, Greg quickly brushed over the fact that the given triangle had two pairs of side lengths and opposite angle measures. However, it is possible that the simplicity of the pictorial representation made these additional explanations unnecessary.

Notation and vocabulary. Another subtask that teachers attended to when enacting high cognitive demand examples was providing clear verbal explanations of mathematical notation and vocabulary. Instructors attended to this subtask at various points during their instruction. In some examples, the instructor made sure to explain the notation and vocabulary they were using at the very beginning. Other times, instructors paused during the middle of instruction to make sure that students understood the notation and/or vocabulary that they had been using.

When Alex used an input output table in the exponential versus linear growth example, she introduced the notation $A(t)$ and $B(t)$ as functions modeling the balance in Bank A and Bank B after t years. Previously, students had been working through this problem individually or in small groups as Alex monitored their progress. Now that she was bringing everyone together to facilitate a whole-class discussion, she wanted to make sure that everyone understood the notation, even if it was different from the notation they had been using. Doing this was important because Alex was helping transition the class from individual/small group work to a whole-class discussion, so it was important that people could used a shared notation when talking about similar ideas.

Instructors were also careful to explain both formal and informal vocabulary that they were using. When explaining how to find all solutions to trig equations, Juno introduced the vocabulary of "initial" or "base" solutions. She then explained that these are the solutions in one period and that with sine or cosine, there are usually two of them. Juno was also careful with her wording and made sure to not make any claims regarding the uniqueness of initial solutions. Later, this became important in another example when Juno talked about the equivalence of using $-\cos^{-1}\theta$ and $2\pi-\cos^{-1}\theta$ as initial solutions. Dan also introduced the informal vocabulary of identifying "inside" and "outside" functions as a way to provide students with a way to talk about decomposing function compositions.

In the Law of Sines example that Greg enacted, I had a hard time identifying any ways in which Greg could have attended more to explaining notation and/or vocabulary. I think that this example, in particular, did not have any new or confusing notation or vocabulary for the students, which is why I don't think Greg needed to attend to explaining these things.

Checking your work. The final subtask associated with providing clear verbal explanations has to do with checking your work. Instructors talked about checking your work in different ways. First, they discussed how to check your work at the end of an example to make sure you had not made any computational errors along the way. Second, they also explained how you could check your work if you were unsure of your answer. In both context, the instructors focused on helping students determine what a reasonable answer might be and interpret their results in terms of the problem context. However, checking your work at the end was used as more of a method to find mistakes, whereas checking your work while problem solving was used more of a method of determine whether or not a solution strategy is correct.

In an example where Kelly asked her students to find an equation of a polynomial of least degree given the graph, Kelly asked her students to check their work at one point to see whether or not they were done. So far, they had used the zeros and multiplicities to set up the factors and the exponents. They had not yet attended to the leading coefficient, but instead of pointing this out, Kelly asked her students to check and see if they had found the final equation. One student responded with no, because the graph went through the point $(-1, -8)$, but plugging -1 into the equation they had did not result in -8. So Kelly asked her students how they could adjust their equation to meet this final constraint.

After asking her students to compare exponential and linear growth using compound and simple interest, Alex asked students to write an equation to model an exponential word problem. Both of these examples were done before presenting the standard form of an exponential, but Alex had used the first example (with interest) as a way to build up students' understanding of exponential growth. In this example, Alex asked her students to consider why the base of the exponent should be 1.25 instead of 0.25. One student piped up and said, "Because the 1 is kind of like the initial value and we want to show a 25% increase. So that's why you tack on the 1. Because if it were 0.25 then that would be saying it's a 75% reduction." Alex then took this opportunity to build upon what the student had said and explain that the word problem was clearly describing exponential growth, so we would want our function to also model growth. However, if we used 0.25 as the base of the exponent, then this would give us a decreasing function.

In the Law of Sines example that Greg enacted, Greg spent most of the time working through and explaining the algebraic manipulations that were required to solve for x, but never explained how they could check their work. Since students struggled to understand the algebra involved in the problem, it is reasonable to assume that they would have struggled with successfully completing the algebraic manipulations on their own. So one way that Greg could have responded to his students would have been by taking time to explain how they could check their work on problems when they are unsure about the algebra involved.

Articulating Cognitive Processes

According to TeachingWorks (TeachingWorks, 2017), explaining and modeling content, practices, and strategies might involve just simple verbal explanations, examples, and representations. However, more complex academic practices and strategies may require “thinking aloud and demonstrating”. It is this process of thinking aloud that I have coded as articulating cognitive processes. I differentiated between the two because I wanted to identify thinking aloud as something that the teacher might do when modeling or facilitating from asking students to provide justification or reasoning during a whole-class discussion. Yet, these two categories both relate to making cognitive processes more clear.

Thinking aloud. The process of thinking aloud involves the instructor doing more than just verbalizing what they are doing mathematically, but also making their metacognitive activities more explicit. When teaching his class how to find all solutions to sinusoidal functions, Dan talked through the logic involved in each step of the problem solving process. First, he explained that the first thing to do with these types of problems is isolate the trigonometric function. Then, Dan explained how the next few steps were exactly the same as the first few steps involved in solving a simpler trigonometric function (for a full description of this, see the subsection on Prior Knowledge). Next, Dan explained how the process differed with sinusoidal equations because they needed to isolate θ in order to find the initial solutions. In each step, Dan focused on articulating his thinking and explaining the process that he was going through to approach solving the problem.

Juno did one thing that was unique in terms of how she articulated her cognitive processes. While many instructors did this primarily verbally, Juno often took the time to capture her thinking in writing on the board. In the example where Juno graphed solutions to trigonometric equations as points of intersection, Juno took the time to write out on the board (in full sentences) what she was doing at each step in the process and why. In doing this, Juno not only made each step in the problem solving process clear, but also provide students with explicit, written explanations could help them decipher what they were doing and why they were doing it later while reviewing their notes.

In the Law of Sines example that Greg enacted, Greg mainly used the think-aloud strategy when working through the algebra. While this was still important for him to do, as his students struggled to understand what he was doing algebraically at each step, he did not spend time articulating the cognitive processes that had went into deciding what “tool” to use to find the value of x. In the next subsection, I will discuss how some instructors asked their students to provide justification and reasoning, which is another approach that Greg might have taken to help articulate the cognitive processes involved in solving the problem.

Student thinking. Instead of being the sole articulators of cognitive processes, some instructors asked their students to provide justification and reasoning to support the mathematics. However, it is interesting to note, but perhaps not surprising, that none of the instructors who presented examples by just modeling them for their students chose to use teacher think-alouds instead of asking students to provide justification or reasoning. Even the instructors who did chose to engage students in the example enactment by facilitating whole-class discussions or monitoring students as they worked on parts of the example individually or in groups sometimes chose to do all of the articulating themselves. However, there were still several examples of times when the instructor asked students to provide justification and reasoning.

During an example where Greg was explaining how to find all solutions to trig equations with non-standard unit circle angles, Greg would ask his students what they should do next at different points in the example. In one of these instances, a student responded that they should use arcsine in order to isolate θ. Another student spoke up immediately and asked, “How did we know to use arcsine right there?” Instead of answering the question himself, Greg asked the student who had offered the idea originally to explain why he had chosen to do this. Doing this not only helped make it clear to the second student why this step was appropriate, but also gave the first student the opportunity to verbally articulate his reasoning for choosing to do that as the next logical step.

Out of all of the instructors, Kelly was the most consistent in asking her students to provide justification and reasoning for their answers. During every example, Kelly facilitated whole-class discussions and monitored students as they worked through parts of the example. In fact, I never observed Kelly enacting a high cognitive demand example by just modeling for her students: she always included them actively in the example enactment. During whole-class discussions, Kelly consistently asked students to explain how they had gotten their answers. Even when doing computations, Kelly asked her students to articulate what they were doing. Also, Kelly would press her students to make connections to definitions as a way to provide justification for why statements were true. Finally, if Kelly was not sure that everyone understood a concept, she would pause her instruction and ask someone to articulate his or her reasoning for why it was true.

When Greg enacted the example that used the Law of Sines, his original intent was to focus students' attention on determine what an appropriate "tool" would be for solving this problem. However, when he enacted this example, he did not press his students to articulate their reasoning. If Greg had asked, "Why is Law of Sines an appropriate tool to use in this case," it would have been interesting to see if he students could articulate a justification beyond, "Because it says we should in the workbook." While it may be true that he students understood why this was an appropriate procedure, Greg did not take advantage of this moment as an opportunity to dig into their understanding and bring attention to their cognitive processes.

Supporting Student Understanding.

The final task that instructors attended to when enacting high cognitive demand examples was supporting student understanding. Since examples can be enacted in different ways that include different levels of student participation and opportunities to struggle, it is important for instructors to be attention to whether or not their students are following and understanding the example the instructor is enacting. Instructors did this in a variety of ways by providing students with opportunities to ask questions, recognizing when students were struggling to follow or understand, and providing scaffolding for students who were struggling.

Student questions. All of the instructors provided opportunities for students to ask questions, although they each did this in different ways. Dan ended every example by asking if there were any questions, however he was often met by silence and then quickly moved on. Occasionally a student would ask a question during the example, but Dan rarely paused for longer than five seconds and pressed them to ask questions during whole-class presentations. Dan did provide his students with opportunities to work on workbook problems individually or in small groups, so it is possible that students used this opportunity to ask questions. But for the most part, his students were silent during examples.

Alex, on the other hand, provided her students with opportunities to ask questions and used other informal ways of measuring their understanding. In several cases, Alex would ask her students to give thumbs up, down, or sidewise to express how well they understood. She would also stop frequently during examples in order to make sure that her students were following what they had done so far. Emma used a similar technique in her function transformation example and paused after working through critical pieces of the example to see how well students understood. However, instead of just asking for a thumb indicator of understanding, Emma asked explicitly for questions and several of her students piped up, asking for clarification.

Out of all of the instructors, Greg was the only one who communicated to his students that having questions was a good thing. Greg had found that in his experience asking, “Are there any questions?” often did not elicit responses from his students. So instead, Greg would ask, “What questions do you have?” in an attempt to communicate that it was completely fine to have questions. In one instance, Greg paused in the middle of a long example and said, “I’m going to pause here and ask who is lost? Who has a question? It’s totally reasonable to be lost. There’s a lot that goes into these. So just let me know where you are lost.” While this technique did not work every time, Greg’s students often were willing to speak up and ask questions or verbalize where they had gotten lost or identify what they were struggling to understand.

In the Law of Sines example that Greg enacted, Greg’s students did pipe up and ask questions, but mostly they were focused on the algebraic manipulations. One way that Greg could have attended to whether or not students understood the justification behind why they chose to use Law of Sines is by explicitly asking students if they had any questions regarding why this might be a good method to use.

Student struggle. Instructors used a variety of techniques to recognize when students were struggling to follow or understand. For many of them, they provided students with an opportunity to ask questions as a way to verbal their struggle. In other cases, it was students’ silence and not their questions or answers that cued instructors that they were struggling with a concept or idea. Some instructors actually used student struggle as a guide for choosing and designing examples. Finally, the instructors who monitored students as they worked through parts of the example also used this time to interact with students individually and in small groups and identify when they were struggling.

The one high cognitive demand example that I observed Emma enact originally was presented as a problem on a quiz. However, when grading Emma realized that most of her students had struggled with the problem in ways that she had not anticipated. So Emma decided to work through the problem the next day as an example at the beginning of class. In this case, Emma actually recognized that students were struggling with a concept and then purposefully incorporated an example into her lesson plan to help students overcome their struggles and misconceptions.

When students were working on trying to find an equation to model the balance in a bank account that earned simple and compound interest after t years, Alex took a quick survey of the class to see whether or not they needed more time. Based upon their responses, Alex decided to give them more time to work individually and in small groups on this part of the example. She also reminded them that the point of the example was to try to come up with equations to model the two different bank balances.

In the example that Greg enacted that used the Law of Sines, he had not anticipated that students would struggle as much with the computational aspect. Also, the one part that he had anticipated they might struggle with (identifying the right "tool" to use) seemed to be an easy task for them. Also, the way that Greg set up the example reduced the students' opportunities to struggle with identifying the right "tool" to use. In particular, the language that Greg used to set up the example cued his students that it was in their workbooks, which contained the goal statement that told them which procedure to use, and that it was directly related to what they had learned that day. So while Greg did think about ways in which his students might struggle with this example, he actually reduced their opportunity to struggle when he set it up.

Scaffolding. Providing scaffolding has been identified as one important feature of explicit instruction (Archer & Hughes, 2010; Doabler et al., 2012; Rosenshine, 2012). In the high cognitive demand examples that I observed, instructors provided scaffolding in two different ways. Some instructors purposefully designed their example(s) so that they were scaffolded to support student understanding. Other times, instructors incorporated in-the-moment scaffolding in response to recognizing that students were struggling to follow or understand.

Instructors incorporated scaffolding into their lesson plans in a variety of ways. In the lessons where I observed Dan, Greg, and Juno teach students how to find all solutions to trig equations, each of the instructors scaffolded their examples by ordering them in a certain way. Most of them started off by first doing an example of how to find all solutions to simple trigonometric equations with standard unit circle angles. Next, they would do a similar example, but with a non-standard unit circle angle in order to introduce students to the strategy of using inverse trigonometric functions to find initial solutions. After giving students some practice solving these simpler types of trigonometric equations, they would then move on and do some examples of sinusoidal functions with both standard and non-standard unit circle angles. While not all of these examples were enacted at a high level of cognitive demand, many of them were, which means that the instructors were able to scaffold in a way that did not decrease the cognitive demand.

In the example where Kelly asked her students to come up with an equation to model the balance in a bank account that earned 4.5% annual interest, her students had no problem using a recursive formula to find the balance after one, two, and three years.

However, her students struggled to see how to generate a formula that did not depend upon knowing the balance in the bank account the year before. To help scaffold her students understanding of exponential growth, Kelly focused in students attention on how they could rewrite the balance at the end of year two in terms of the balance at the beginning of year one. Then, she asked her students how they could use a similar idea to rewrite the balance at the end of year three in terms of balance at the beginning of year one. Finally, her students were able to come up with the final exponential equation that gave the balance at the end of year t in terms of the balance at the beginning of year one. By providing this scaffolding of looking at beginning cases and seeing how they related to subsequent cases, her students were able to transform their way of thinking about the example from viewing it as a recursive relationship to an exponential relationship.

Given that the Law of Sines example that Greg chose to use was simpler, it's hard to imagine what scaffolding might look like in this case.

Results: Relationships Between Tasks and Roles

In a previous paper (Chapter 3), I examine three different roles that instructors can take on when enacting high cognitive demand tasks. First, instructors can model content, practices, and strategies for students. Second, instructors can facilitate whole-class discussions of the example. And finally, instructors can monitor students as they work through parts of the example individually or in small groups. While some instructors enacted examples by just modeling, many of them switched back and forth between different roles.

In order to address my second research question, I examined how the five tasks entailed in enacting high cognitive demand examples overlapped with the three roles that instructors could take on. Table 23 illustrates the overlap of these two sets of codes. Table 24-Table 28 then break down each of the five main tasks and provide the role profiles of each individual subtask. One thing to note is that my IRB did not cover capturing video of the students, so I was not able to really capture what the instructors did while they monitored student work time. Therefore, the overlap between the Monitor role code and the decomposition codes is only representative of the how the instructor interacted with the class as a whole during these times.

Table 23. Role Profiles of Tasks Entailed in Enacting HCD Examples

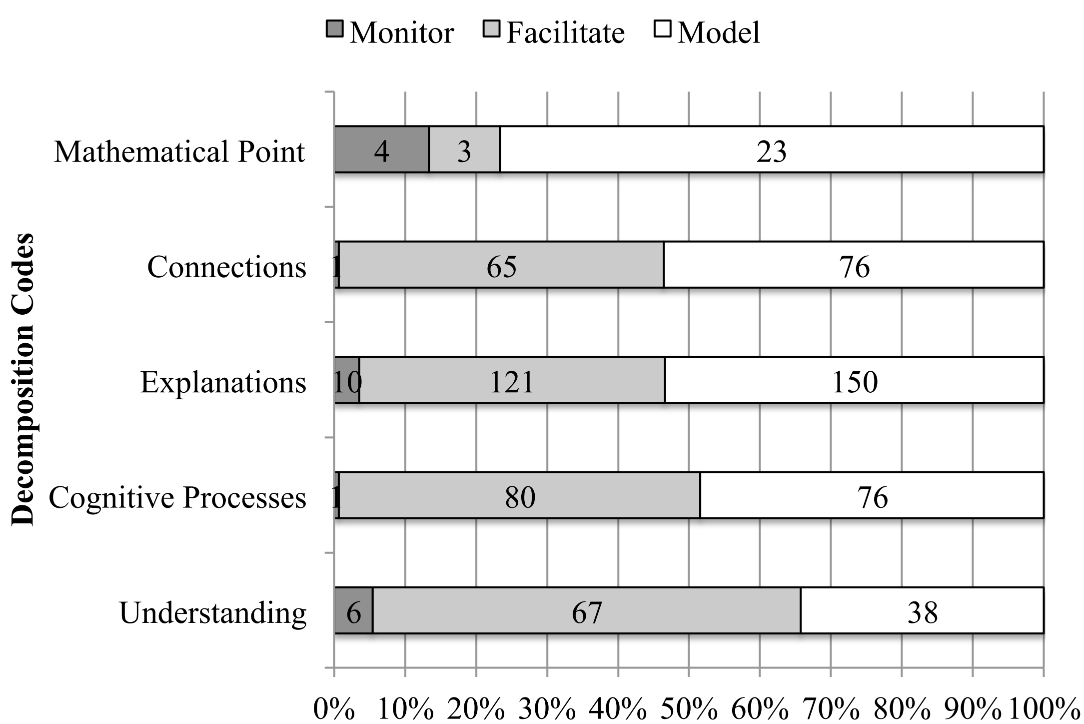


Table 24. Role Profiles of Mathematical Point Subtasks

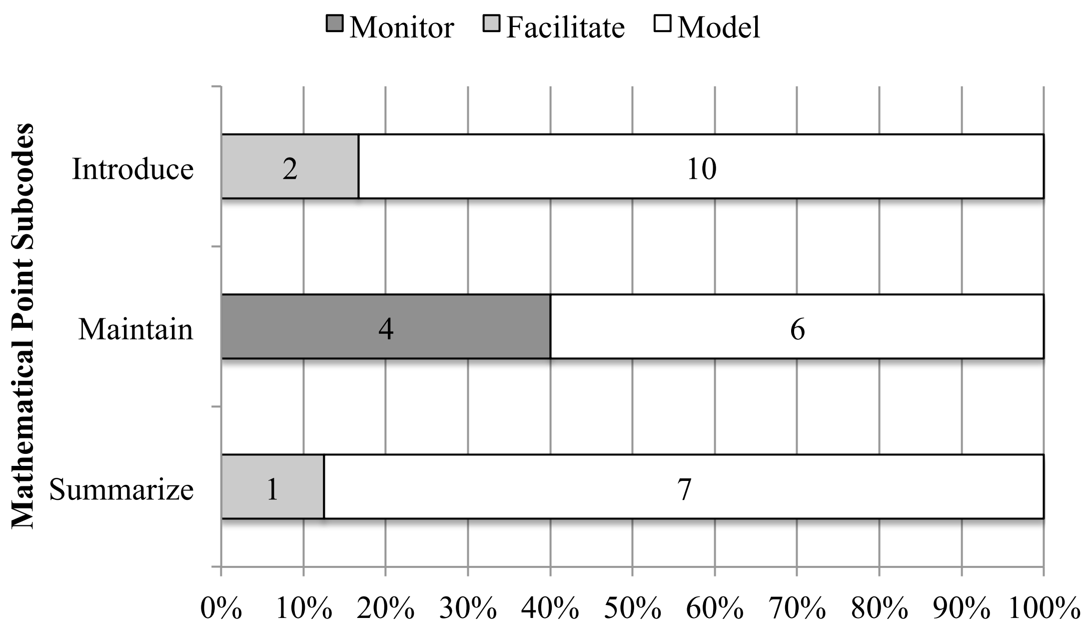


Table 25. Role Profiles of Connections Subtasks

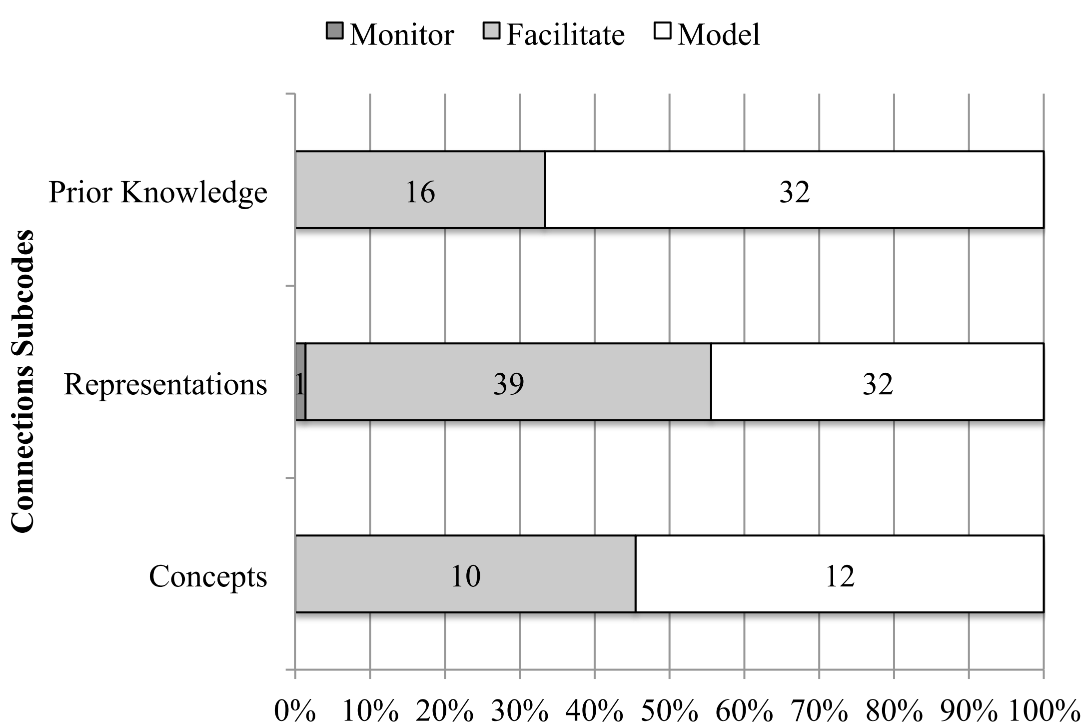


Table 26. Role Profiles of Explanations Subtasks

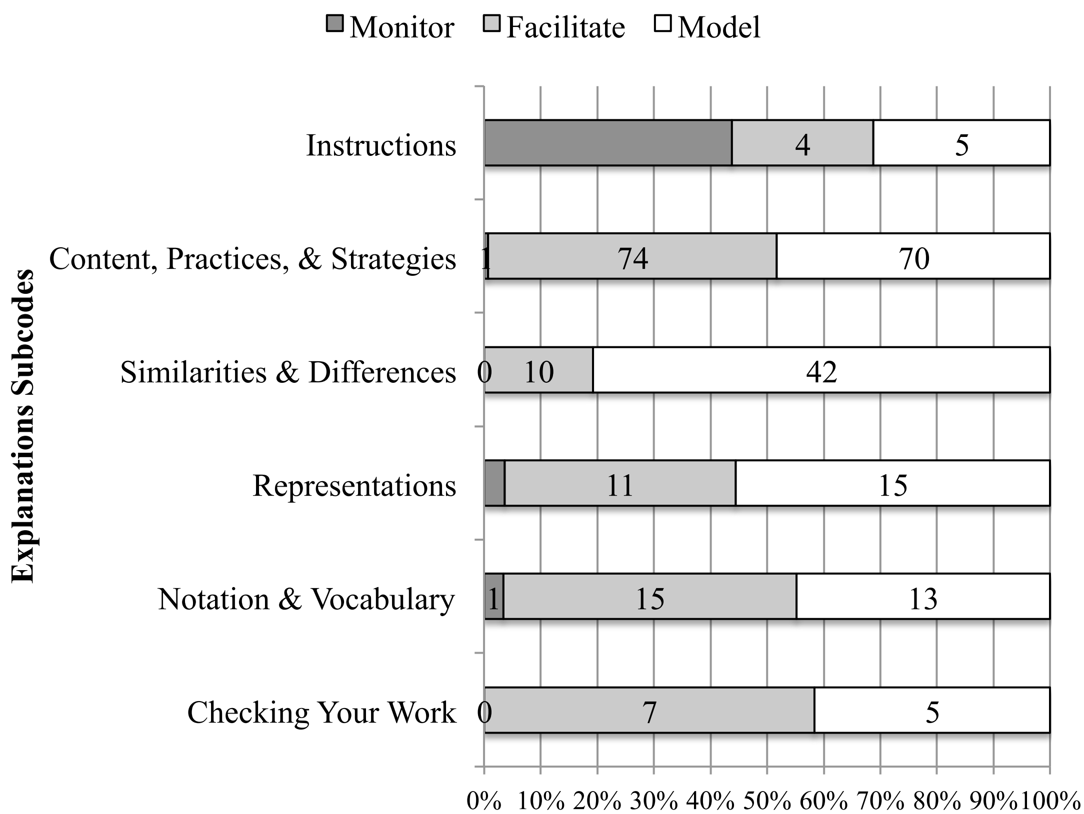


Table 27. Role Profiles of Cognitive Processes Subtasks

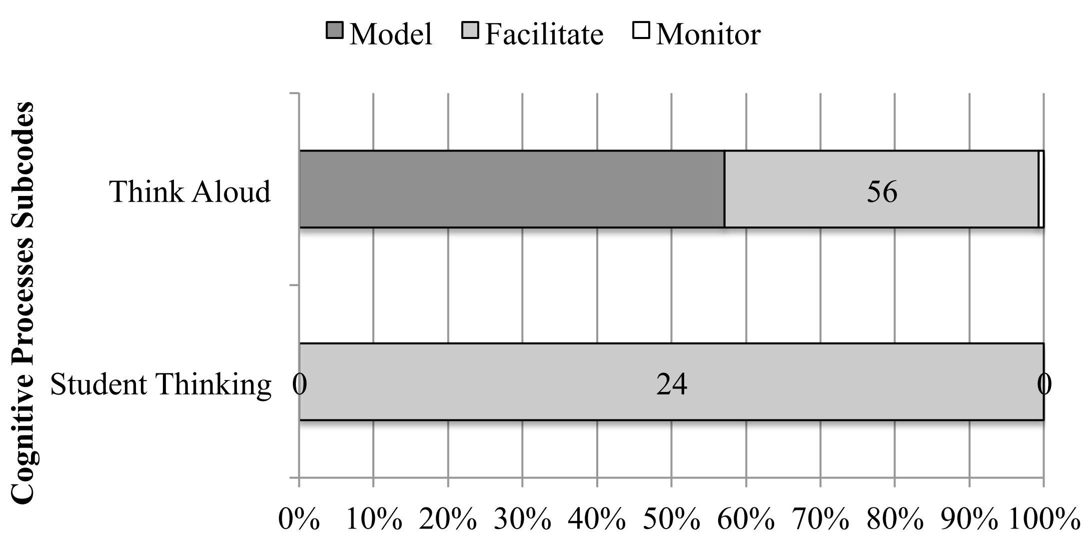
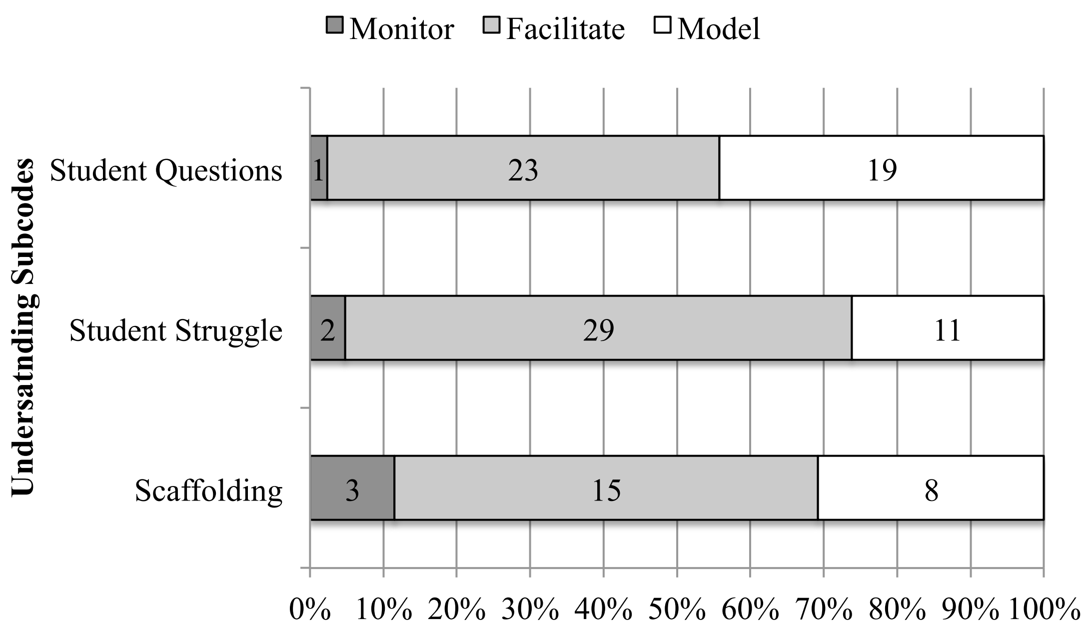


Table 28. Role Profiles of Understanding Subtasks



Discussion

In this paper, I argue that the work of enacting examples at a high level of cognitive demand can be decomposed into five main tasks: attending to the mathematical point, making connections, providing clear verbal explanations, articulating cognitive processes, and supporting student understanding. An instructor may attend to the mathematical point by introducing it as a way to set the focus of the example, maintaining the focus of the example on the mathematical point, and summarizing the example in order to reiterate the mathematical point. When making connections, they might connect to previously learned content or focus on making connections between representations and/or concepts. To help students understand the example, instructors provide clear verbal explanations of the example set up, constraints, and goal; of content, practices, and strategies; of similarities and differences; of representations; of notation and vocabulary; and of how to check your work. Depending on whether an instructor models or facilitates the presentation of the example, the instructor might make cognitive processes explicit by thinking aloud as they work through the example or asking students to provide justification and reasoning. Finally, instructors need to support student understanding by providing opportunities for students to ask questions, recognizing when students are struggling, and providing scaffolding to support struggling students.

Table 29. Decomposition of the Work Entailed in Enacting HCD Examples

1. **Attend to the mathematical point**
   - **Introduce** the mathematical point as a way to set the focus of the example
   - **Maintain** the focus of the example on the mathematical point
   - **Summarize** the example in order to reiterate the mathematical point
2. **Make connections**
   - To **previously learned** content, practices, and strategies
   - Between **representations**
   - Between **concepts**
3. **Provide clear verbal explanations**
   - Of the **example set up, constraints, and goal**
   - Of **content, practices, and strategies**
   - Of **similarities and differences**
   - Of **representations**
   - Of **notation and vocabulary**
   - Of **how to check your work**
4. **Articulate cognitive processes**
   - By **thinking aloud** as you work through the example
   - By **asking students to provide justification and reasoning**
5. **Support student understanding**
   - By providing opportunities for students to **ask questions**
   - By **recognizing when students are struggling** to follow or understand
   - By **providing scaffolding** for struggling students without decreasing the cognitive demand

One thing that is important to note is that instructors did not have to attend to all five tasks and their associated subtasks in order for the example to be classified as high cognitive demand. In fact, some of the instructors only attended to three of the five main tasks during enactment. However, on average an instructor attended to at least four of the tasks, so these tasks do reflect a reasonable portrait of the pedagogical work that was entailed in maintaining the cognitive demand of an example.

Limitations

One limitation of this decomposition of the work entailed in maintaining the cognitive demand of examples is that the tasks are not necessarily independent. For example, in practice it may be hard to distinguish between making connections and explaining similarities and differences. Also, unless an observer speaks with the instructor beforehand, they may not know what the mathematical point of the example is, so it would be difficult for them to know if the instructor is introducing, maintaining, or summarizing.

Another limitation of this decomposition is that some of the tasks could be enacted at a superficial level, which may not end up contributing to the decline instead of maintenance of cognitive demand. For example, if an instructor says that an example is connected to a previously learned concept, but does not explain that connection, then they would not be building understanding of the underlying concepts. Similarly, an instructor might introduce the mathematical point as a way to set the focus of the example, but never return to this point or summarize. Therefore, it is important that instructors attend to multiple tasks instead of just one or two in isolation.

Implications

One reason why it is important to decompose the work of teaching is because it makes the hidden work of teaching more visible for novices. When novices observe teaching, it can be difficult for them to identify what the instructor *does* that creates the classroom experience. In addition, it is often difficult for experienced instructors to reflect on their teaching and unpacking everything that they do to support instruction. Therefore, it is important for researchers to decompose the work of teaching to make it more visible.

Doing this not only provides novice teachers with a way to see the work of teaching, but also provides teachers with a tool for reflection. Oftentimes, things do not go as planned in the classroom. In particular, if an instructor chose a high cognitive demand example to include in their intended lesson plan, but found that the cognitive demand declined during enactment, then it would be helpful for them to reflect on whether or not they engaged in pedagogical work associated with maintaining the cognitive demand. For example, upon reflection, a teacher might realize that an example became more algorithmic because they forgot to make connections or explain the cognitive processes involved in working through the problem.

Conclusion

In this paper I examined the pedagogical work entailed in enacting high cognitive demand examples. After conducting open and thematic coding of 25 HCD examples, I found that there are five main teaching tasks that instructors attend to when enacting HCD examples: attending to the mathematical point, making connections, providing clear verbal explanations, articulating cognitive processes, and supporting student understanding. While instructors may not attend to all of these tasks during a single example, I found that they do attend to many of them, which supports my claim that they are the main teaching tasks that contribute to the maintenance of cognitive demand. This decomposition of the work of enacting HCD examples is useful for both researchers who might be interested in studying factors that contribute to the maintenance or decline of cognitive demand and for practioners who want to reflect on their own teaching.

Chapter 6: Identifying the Mathematical Knowledge Entailed in Enacting High Cognitive Demand Examples

The purpose of this collective case study is to examine mathematical knowledge for teaching examples in precalculus. The instructors involved in the study were experienced graduate student instructors who were teaching their course for the third time. Utilizing a social constructivist and cognitive theory approach, I analyzed video recordings of enacted examples. The central question that guided this analysis was: What is the mathematical knowledge for teaching entailed in enacting high cognitive demand examples? The goal of this study is to examine undergraduate mathematical knowledge for teaching from the perspective of practice, instead of relying on existing frameworks. As a result of this study, I identified five domains of mathematical knowledge for teaching that support the maintenance of cognitive demand: knowledge of connections, representations, unpacking, students, and sequencing.

Introduction

Mathematical knowledge for teaching (MKT) has been defined as the "mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students" (Ball et al., 2008, p. 395). While MKT has been studied extensively at the elementary level (Ball et al., 2008; Carpenter & Fennema, 1991; Hill et al., 2007; Ma, 2010) and at the secondary level (Krauss et al., 2008; McCrory et al., 2012; Rowland et al., 2005), research on MKT at the undergraduate level is still a growing field (N. Speer et al., 2010). The goal of this study is to contribute to that field by building upon the link between MKT and high cognitive demand tasks (Charalambous, 2010) in order to study mathematical knowledge for teaching examples in precalculus from the perspective of practice.

Problem

Often, it is assumed that earning a degree in mathematics is what initially qualifies ones to teach at the undergraduate level. Historically, university instructors learned to teach by following the role model of mentors. However, Bass (1997) pointed out that there is much that cannot be learned through observations alone. To address lack of teaching preparation, many doctoral programs today offer teaching professional development for graduate student instructors (GSIs), who will make up the future workforce of university instructors (Bressoud et al., 2015; Ellis, 2014). While offering some teaching PD is better than none, the content of what is being taught is an important aspect to consider.

Of course, pedagogical knowledge is a component of teaching and should be included in GSI PD. However, studies have shown that despite their formal mathematical education, GSIs still lack mathematical knowledge that is needed for effective teaching (Kung & Speer, 2009; N. Speer & Hald, 2008). In these studies, the authors relied on existing frameworks for MKT that where developed at the K-12 level. While it is reasonable to assume that K-12 and undergraduate MKT are similar, Speer et al. pointed out that there are important differences between K-12 and university instructors that need to be attended to (N. M. Speer et al., 2015). Therefore, the goal of this study is to examine MKT at the undergraduate level from the perspective of practice, instead of relying on existing frameworks.

Significance

As previously stated, there is little research on MKT at the undergraduate level. But why is it important to study MKT to start with? First, studies have found that pure content knowledge is not a predictor of teaching quality and student achievement (Begle, 1972; Greenwald et al., 1996; Hanushek, 1981, 1996). However, studies at the K-12 level have shown that MKT is a predictor of teaching quality and student achievement (Hill et al., 2008, 2007; Krauss et al., 2008). This knowledge is not usually taught in content courses, hence why many GSIs seem to be lacking MKT. While no measures of MKT at the undergraduate level exist, it is reasonable to assume that this positive relationship still exists at the undergraduate level. Therefore, if we can identify what MKT at the undergraduate level looks like and integrate it into GSI professional development programs, we can have a positive impact on undergraduate education.

The other question that is reasonable to ask is why focus on precalculus? As the number of students needing to take introductory mathematics courses for their degree increases, the teaching burden of mathematics departments increases (Ellis, 2014). Approximately 1,000,000 college students take introductory level mathematics courses each year (Gordon, 2008). Of these, approximately 85-90% are non-STEM intending (Rasmussen & Ellis, 2013) and success rates are typically around 50% (Gordon, 2008). Even for STEM-intending students, studies have found that difficulty passing introductory-level courses is contributing to the “leaking pipeline” of students leaving STEM (Thompson et al., 2007). Therefore the instructional quality of precalculus has a large impact on undergraduate students.

Background

While research on MKT at the undergraduate level is sparse, there is a large body of research on K-12 MKT. While my goal is to examine MKT at the undergraduate level from the perspective of practice instead of using existing frameworks of MKT that were developed at the K-12 level, the two are bound to be closely related. In an effort to situate my study within the existing field of research on MKT and avoid the assumption that I am attempting to study MKT at the undergraduate level in an epistemological vacuum, I will first present a broad overview of existing research on MKT. Also, I chose to study MKT by building upon its relationship with the cognitive demand of tasks. This decision was motivated by Charalambous’ (2010) exploratory study, which found that MKT and the cognitive demand of enacted tasks are positively related.

Mathematical Knowledge for Teaching

Following the studies that showed that subject matter knowledge was not a predictor of teaching quality and student outcomes, Lee Shulman (1986, 1987) proposed that researchers begin studying pedagogical content knowledge. Shulman defined pedagogical content as going “beyond knowledge of subject matter per se to the dimension of subject matter knowledge for teaching” (1986, p. 9). Shulman situated pedagogical content knowledge in contrast to subject matter knowledge, which is “the knowledge, understanding, skill, and disposition” of a subject matter (1987, p. 8). Since then, mathematics education researchers have begun looking into professional knowledge for teaching mathematics. Hill, Rowan, and Ball (2005) found that elementary teacher’s MKT was a significant predictor of student gains. Similarly, Baumert et al. (2010) showed that secondary teachers’ MKT was a predictor of student outcomes. In both of these examples, the mathematical knowledge that is specific to the work of teaching is not usually taught in general undergraduate mathematics courses. Therefore, using the number of mathematics courses taken beyond calculus is not the same as measuring content knowledge for teaching.

Speer, Smith, and Horvath (2010) conducted a literature review to search for empirical research on the practices of undergraduate teachers of mathematics. As a result, the authors identified only five articles, indicating that “collegiate teaching practice remains a largely unexamined topic in mathematics education” (p. 100). Since then, more studies have been published specifically on MKT at the postsecondary level (Bargiband, Bell, & Berezovski, 2016; Callingham et al., 2012; Castro Superfine & Li, 2014; Firouzian & Speer, 2015; Hauk, Toney, Jackson, Nair, & Tsay, 2013; Jaworski, Mali, &

Petropoulou, 2017; Musgrave & Carlson, 2017; Rogers & Steele, 2016; Rogers, 2012; N. Speer & Wagner, 2009; Vincent & Sealy, 2015). However, some of these studies used existing frameworks for MKT that were developed at the K-12 level, which can be problematic (N. M. Speer et al., 2015). Therefore, the purpose of this study is to contribute to this growing body of research by examining MKT at the undergraduate level from the perspective of practice.

Cognitive Demand and Task Unfolding

Smith and Stein (1998) defined lower-level demand tasks as “tasks that ask students to perform a memorized procedure in a routine manner” and higher-level demand tasks as “tasks that require students to think conceptually and that stimulate students to make connections” (p. 269). Stein, Remillard, and Smith (2007) also created a framework to describe the temporal process of task unfolding and factors that contribute to this transformation. In this process, teachers use a written task to formulate their intended task, which in turn influences the enacted task. Each phase in this process is motivated by the goal of producing student learning and is influenced by factors, such as teacher’s beliefs and knowledge. In 2010, Charalambous found that there was a connection between elementary teachers’ MKT and their ability to enact tasks at a high level of cognitive demand. It is this relationship between MKT and cognitive demand that I plan to build upon in this study.

Purpose and Research Question

The purpose of this collective case study is to examine mathematical knowledge for teaching examples in precalculus. I will do this by first examining cognitive demand in order to identify examples that were enacted at a high level of cognitive demand. Building upon Charalambous' (2010) results, I believe that these examples will provide me with fertile ground for examining MKT. While I believe that MKT influences every stage in the process of example unfolding, this report will focus on the final stage of example unfolding. The central question that guides this study is: What is the mathematical knowledge for teaching entailed in enacting high cognitive demand examples? To narrow the focus of this study, I will primarily attend to answering the following subquestions:

1. What mathematical knowledge enables instructors to enact examples at a high level of cognitive demand?
2. How can we characterize this knowledge?
3. How does this knowledge related to specialized content knowledge and pedagogical content knowledge?
4. How does this knowledge relate to the roles that instructors take on when enacting high cognitive demand examples?

Methods

Setting and Participants

For the purposes of this study, precalculus courses are defined to include the college algebra, trigonometry, and combined college algebra + trigonometry courses. The participants from this study were all instructors at the same large public university in the Midwest. At the university involved in the study, second-year graduate students make up the majority of the instructors for precalculus. Since second-year graduate students are teaching their own class for the first time, I chose to exclude them from my data set and instead only recruited participants who were teaching a precalculus course for at least the third time. The participants in this study included one trigonometry instructor (Greg) and six college algebra + trigonometry instructors (Alex, Dan, Emma, Juno, Kelly, and Selrach). All of them were graduate students in their third, fourth, or fifth year, had already earned their M.S., and were working towards their Ph.D. in mathematics. Also, all of the instructors were teaching their respective course for at least the third time.

Design and Procedures

In order to answer my research questions, I am utilizing a collective case study design (Stake, 1995). In order to examine MKT more generally, I included multiple instructors and collected data on multiple examples. Since I have included a limited number of participants, there is little is known about mathematical knowledge for teaching precalculus, and I seek to propose new theoretical insight into MKT, I chose to use an exploratory case study (Yin, 2014). The unit of analysis I am focusing on is the examples enacted by precalculus instructors. Studying teaching from the perspective of practice can be difficult, so I used the frameworks of cognitive demand and task unfolding to help make the knowledge the teachers were using more visible. Building upon Charalambous' (2010) finding that MKT and cognitive demand are positively related, I used cognitive demand as a way to identify examples that would provide me with rich opportunities to examine MKT. Second, studying teaching through the task unfolding framework (Stein et al., 2007) allowed me to see the instructors' decision-making and examine how their mathematical knowledge enabled them to enacting examples.

Coding proceeded in two stages that concentrated on cognitive demand and then knowledge. In the first stage, I use my modified framework for analyzing the cognitive demand of examples (Table 7) to code the cognitive demand of enacted example. Examples that were coded as enacted at a high level of cognitive demand were then analyzed in the second stage, which has two cycles. In the first cycle, I used inductive descriptive coding (Miles, Huberman, & Saldaña, 2014) to identify mathematical knowledge that enabled the instructors to enact the example at a high level of cognitive demand. This round of coding would help me to answer my first research question. To answer my second research question, I conducted a second cycle of pattern coding in order to identify emergent themes and relationships between the codes that resulted from the first cycle. I then looked at the relationships between the knowledge domains I identified and SCK/PCK and the roles that instructors take on when enacting examples.

Results

In total, there were 93 examples that I observed the seven instructors enact. Of those, 25 were enacted at a high level of cognitive demand. It is also important to note that almost all of these high cognitive demand examples were coded as procedures with connections tasks (Smith & Stein, 1998). In the second stage of coding, five main domains of knowledge emerged: knowledge of connections, representations, unpacking, students, and sequencing. In the following subsections, I describe each of these domains and provide narratives of instances where instructors used this knowledge to maintain the cognitive demand of the example. In my analysis, I focus primarily on content knowledge that goes beyond what the instructors expected their students to learn and know. This includes both specialized content knowledge[10] and pedagogical content knowledge.

Knowledge of Connections

Given that procedures with connections examples focus on "developing deeper levels of understanding of mathematical concepts and ideas", "have close connections to underlying conceptual ideas", make "connections among multiple representations", it is not surprising the knowledge of connections was one of the main domains that emerged from my analysis. Here, I define knowledge of connections as knowledge of mathematical relationships between content, practices, and strategies. While this was a type of knowledge that instructors wanted their students to build, instructors also used knowledge of connections that went beyond what they expected their students to

[10] I recognize that this term is used by Ball, Thames, and Phelps (2008) in their framework for MKT, but I am using it somewhat differently in that I define specialized content knowledge as knowledge that is *essential* (but not necessarily unique) to the work of teaching.

necessarily know or learn. However, the connections that we want our students to learn are still different than the connections we should know, as their teachers. In particular, Ambrose, Bridges, DiPietro, Lovett, and Norman (2010) identified that one of the biggest differentiators between novices (e.g., students) and experts (e.g., instructors) is that experts have a much richer "density of connections among the concepts, facts, and skills they know" (p. 49).

Dan, Greg, and Juno. Of all of the knowledge domains that emerged from my analysis, knowledge of connections was the most prominent. Almost every instructor relied on their knowledge of connections when enacting high cognitive demand examples. Several of the high cognitive demand examples that I observed came from instructors who were teaching the same lesson on trig equations and inverse functions. This lesson was spread out over two days and focused on finding all solutions to trigonometric and sinusoidal equations using the unit circle and inverse trigonometric functions. In particular, I observed Dan, Greg, and Juno all enact high cognitive demand examples during this lesson. Since these three instructors drew upon their knowledge of connections in a similar way, I will talk about them collectively.

Previous to teaching this lesson, the instructors had taught their students how to find a solution to a trigonometric equation using either the unit circle or inverse trigonometric functions. However, up to this point they had not discussed how to find all solutions to these types of equations. To help their students transition from the routine problem of finding one or two solutions to finding infinite families of solutions, the instructors first drew upon their knowledge of a similar, but less complicated, problem:

finding solutions to quadratic equations. The particular quadratic that they used to begin their example was $x^2 - 2 = 1$.

In choosing to use this quadratic, the instructors drew upon their knowledge of connections in the following ways. First, they knew that this quadratic would provide them with two solutions. This is significant because the whole purpose of the example was to illustrate to the students that trigonometric equations had multiple solutions. Second, the instructors talked about how the students could use the graphical representation to help them recognize how many solutions they should anticipate to find algebraically. In particular, Juno talked about how looking at the graph might help students remember that they need to include both the positive and negative solution when taking a square root. Finally, the instructors used this quadratic to emphasize that the number of solutions they found algebraically should always match the number of intersection points between the graphs of $y = x^2 - 2$ and $y = 1$.

Building upon these connections, the instructors then introduced the idea that solutions to trigonometric equations could also be represented as intersection points of graphs. First, the instructors drew the graphs of a trigonometric function and the line $y = c$ (where $-1 \leq c \leq 1$) and emphasized that since these two graphs intersected an infinite number of times, the corresponding trigonometric equation must have an infinite number of solutions. They then made the connection that if they tried solving these equations in the way that they had done before, which involved finding just the solutions between 0 and 2π, then that would only get them some initial solutions. Finally, the instructors drew upon their knowledge of how the periodicity of trigonometric functions relates to the infinite families of solutions that they needed to find. In particular, Greg used the graph to show his students why these infinite families of solutions would all take the form

$$\text{(initial)} + \text{(period)}k \quad k = \text{any integer.}$$

In each of these high cognitive demand examples that I observed Dan, Greg, and Juno enact, the instructors used their knowledge of connections in planning which equations they were going to use. By choosing equations that illustrated similar concepts, but also highlighted differences, the instructors were able to maintain the cognitive demand of the examples so that they focused on developing deeper understanding of the underlying mathematics.

Knowledge of Representations

Since procedures with connections examples are "usually represented in multiple ways" (Smith & Stein, 1998, p. 348), it is also not surprising that representations emerged as a main domain of knowledge that supported the maintenance of cognitive demand. Here, I define knowledge of representations as knowledge of graphical, pictorial, tabular, algebraic, verbal, and written forms of mathematical content, practices, and strategies. In many cases, knowledge of connections and knowledge of representations went hand-in-hand, since instructors were utilizing multiple representations of the same ideas. Knowledge of representations has been studied in depth by Mitchell, Charalambous, and Hill (2014). Instead of providing a deep dive into this domain of knowledge, the purpose of this study is to highlight how this knowledge is used in maintaining the cognitive demand of examples.

Knowledge of representations was used in a variety of ways. In the narrative presented in the previous subsection, we can see how the instructors drew upon their knowledge of representations in order to illustrate why trigonometric equations have infinitely many solutions. When students seemed stuck, instructors often drew upon their knowledge of graphical and pictorial representations as a way to help students visualize the mathematics that they were working through algebraically. Another common theme that emerged from my data analysis was that instructors drew upon their knowledge of representations when they attended to the mathematical point of the example. In the narratives presented below, I give two examples of how instructors used their knowledge of representations to attend to the mathematical point in two different ways.

Greg. In the lesson where the concept of tangent was first introduced, Greg thought it was important to provide students with a real-life application problem in order to illustrate how tangent is useful in solving problems. In the example, Greg used a tower that was 150 feet tall and stabilized by cables that formed an angle of 60° with the ground. He then asked his students to calculate how long the cables needed to be and how far they needed to be anchored from the tower. As Greg verbalized the problem, he wrote it on the board as well. Immediately afterwards, Greg chose to draw a picture of the situation (Figure 13).

As they were figuring out how to solve for c, Greg reminded his students that many of the problems they worked with in trigonometry involved triangles and circles, so it would be helpful to identify a triangle in the picture they had drawn. After successfully doing this and using sine to solve for c, Greg then asked the students to work individually or in small groups on calculating how far the cables needed to be anchored from the tower. As the students worked, Greg drew a simpler version of the picture, with some information removed and other information filled in, on another board (Figure 14).

Figure 13. *Greg's Illustration of a Tower Stabilized by Cables*

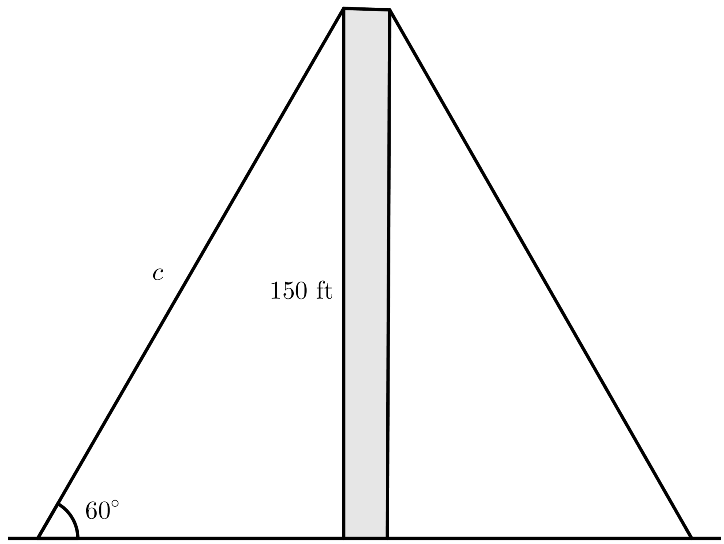
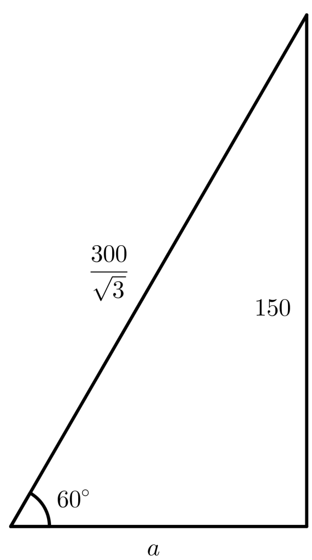


Figure 14. *Greg's Simpler Version of the Tower Triangle*



During the post-observation interview, I asked Greg why he had decided to represent the problem using two different diagrams. Greg explained that he wanted to draw the first picture because it would help his students recognize that they could use sine to solve for c. However, once they had engaged with a more accurate pictorial representation of the problem, he thought it was important to strip away the unnecessary aspects and just draw the simple triangle that they were working with. Greg also explained that he thought it was important to start with the more detailed picture because it provided students with a way to make connections between the written problem and the mathematical operations they were using to solve the problem. In particular, Greg thought it was important that students had the opportunity to see how to go from a written to a detailed picture to a basic picture when solving the problem.

Throughout this example, Greg used his knowledge of representations to help maintain the mathematical point. At first, he wanted students to recognize that the situation described in words gave us a right triangle, so we could use trigonometric functions to solve for the unknown length. Then, Greg wanted his students to just focus on using the given information about the triangle to solve for the final unknown side. By including written and two different pictorial representations of this problem, Greg was able to maintain the cognitive demand and keep the discussion focused on the mathematical point.

Alex. During her first lesson on exponential functions, Alex asked students to come up with equations that modeled simple and compound interest. Even though they had not talked about exponential equations explicitly, Alex expected that her students were familiar with these types of equations and could use their intuition of how to calculate interest in order to come up with an exponential equation. After introducing the example setup, Alex asked her students to work individually or in small groups to come up with an equation to model the balance in a bank account that starts with a $100 deposit and earns $10 in interest per year versus a bank account that starts with a $100 deposit and earns 10% in interest per year. In particular, Alex asked students to first compute how much money would be in each bank after one and two years, and then come up with a formula that would calculate the balance in each bank after t years.

As students worked through the problem, Alex walked around the room and monitored their progress. After allowing students seven minutes to work through the problem, she brought them together for a whole-class discussion. Alex had noticed that many of the students were thinking about the compound interest recursively, so Alex decided to capitalize on this and start by building a table of input and output values. As students helped her fill in the corresponding output values for each bank, Alex asked them to explain how they had calculated their answers. Even though different students responded, they all used the technique of calculating outputs recursively. When Alex asked what a general formula was for the simple interest bank, the students were able to quickly recognize that it was linear. However, the students struggled to move away from a recursive formula for the compound interest bank.

To help her students move to the mathematical point that Alex wanted to make, Alex went back to the table values that they had calculated for $t = 1$ and $t = 2$. In particular, she focused on rewriting the recursive calculations so that they only depended upon the starting value. After using the fact that $B(1) = B(0)(1.1)$ to rewrite $B(2) = B(1)(1.1)$ as $B(2) = B(0)(1.1)^2$, Alex showed how they could rewrite $B(t) = B(t-1)(1.1)$ as $B(t) = B(t-2)(1.1)^2$. A student then piped up and conjectured that they would eventually be able to rewrite $B(t) = B(0)(1.1)^t$, which is what Alex was hoping they would eventually get to.

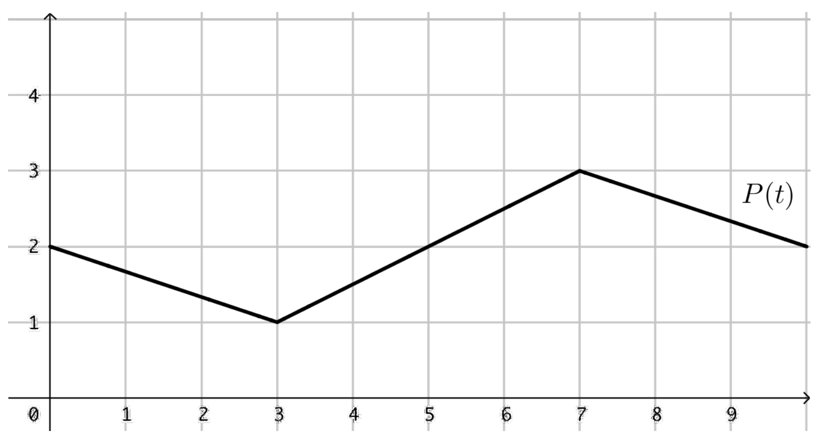
In this example, Alex had to find a way to help her students move from thinking about compound interest recursively to thinking about compound interest exponentially. In order to do this, Alex used her knowledge of representations and chose to introduce the table as a way to make a connection between the students' current way of thinking about the problem and her intended mathematical point. However, it is really the integration of representations that made this connection so powerful. Alex not only identified a representation (the table) that reflected her students' thinking (recursive relations), but also integrated other representations (algebraic) in order to help move their thinking towards the intended point (an explicit exponential formula).

Knowledge of Unpacking

In the literature, the idea of unpacking has been conceptualized several different ways. Ball, Thames, and Phelps (2008) claimed that teacher must hold unpacked mathematical knowledge because teaching involves making features of particular content visible to and learnable by students" (p. 400). McCrory, Floden, Ferrini-Mundy, Reckase, and Senk (2012) referred to unpacking as decompressing, but shift the emphasis from knowledge to work. Here, I define knowledge of unpacking as knowledge of the essential features of mathematical content, practices, and strategies. Not surprisingly, knowledge of unpacking was the most frequent knowledge domain that was coded with the teaching tasks of providing clear verbal explanations, making cognitive processes explicit, and supporting student understanding. In the narrative below, I talk about how Emma relied upon her knowledge of unpacking in the one example that I observed her enact at a high level of cognitive demand.

Emma. After giving students an end-of-the-chapter quiz on function transformations, Emma decided to review one of the quiz problems as an example at the beginning of the next class. Most of the students had struggled with the problem, so Emma wanted to address some of the misconceptions she had recognized during class and show them another way they could do the problem if they were struggling to remember the correct order for transformations. In the problem set, the students were given a piecewise linear graph (Figure 15) and asked to sketch a graph of $3P(t+1)-2$ for $0 \leq t \leq 9$ on a provided grid.

Figure 15. Original Function Given in Emma's Function Transformation Example



In grading the quizzes, Emma had realized that students were struggling to identify the correct order of transformations and to use the order of transformations to sketch the graph. In some cases, the students had done the vertical transformations in the wrong order, but correctly sketched their graph based upon the order that they had used. But in other cases, the students were not able to correctly use the order of transformations they had determined in order to sketch the graph. In order to set up the example in a way that would help clear up these misconceptions, Emma relied upon her knowledge of unpacking graph transformations to demonstrate two different methods.

In the first method, Emma broke down the essential features involved in determining and using the order of transformations. The first essential feature that she identified was determining the order of transformations from the provided function ($3P(t+1)-2$). In presenting this step, Emma focused on explaining why the order of horizontal versus vertical transformations does not matter, but the order of horizontal versus horizontal and vertical versus vertical does. She also focused on making connections to order of operations in order to help her students remember whether or stretch or shift first. The second essential feature that Emma identified was transforming individual points. In presenting this step, she focused on explaining how to use the order of transformations that they had identified in step one to move the endpoints and corners. The final essential feature that Emma identified was graphing the transformed points. While many of the transformed points fell within the desired interval ($0 \leq t \leq 9$), one endpoint did not, so Emma explained how to find the new endpoint for the transformed graph.

After working through this first method, Emma unpacked a second, alternative method for the problem. In this method, Emma identified that there were really only two essential features that the students had to understand, and both of these were things that the students should feel comfortable with. The first feature was input values. Since the example asked students to graph the transformed function on the interval $0 \leq t \leq 9$, Emma emphasized that students could start by focusing on the inputs $t = 0,1, \dots, 9$. The second essential feature, then, for this method was the function $3P(t+1)-2$. Since they were given a graph of $y = P(t)$, Emma explained that they could just plug in values for t and begin solving this equation using the graph of $y = P(t)$. Finally, Emma emphasized that this was a method that would have worked well for students who were unsure about what the correct order of transformations was.

In this example, Emma used her knowledge of unpacking to maintain the cognitive demand of the example in the following ways. First, she focused students attention on the mathematical point by breaking down each method into the essential feature required to solve in. In doing this, Emma drew the attention away from the algebra involved in the example and instead focused on building understanding of the underlying concepts. Second, Emma supported students understanding by using her knowledge of unpacking to provide scaffolding for struggling students. In particular, Emma set up the example on the board so that each essential feature of the first method was highlighted in a separate space. By providing students with this visual scaffolding, Emma was able to focus their attention on one essential feature at a time.

Knowledge of Students

Since I am interested in studying mathematical knowledge for teaching, knowledge of students involves an intersection of content and pedagogical knowledge. Lee Shulman first introduced the idea of pedagogical content knowledge (1986) and Ball and her colleagues (2008) identified knowledge of content and students as a component of pedagogical content knowledge. Here, I define knowledge of students as knowledge about how students interact with and think about mathematical content, practices, and strategies. This involves both knowledge that is used in the moment of teaching and during the planning stage when instructors anticipate how students will interact with and comprehend the lesson. Of all of the instructors that I observed, Alex drew upon her knowledge of students the most during her lesson planning and enactment. In the following narrative, I describe the different ways in which she drew upon this knowledge to maintain the cognitive demand of examples.

Alex. Alex relied upon her knowledge of students in primarily two ways. First, Alex considered how students might interact with and think about mathematical content, practices, and strategies as she planned out the examples. The lesson guides that Alex was using introduced function compositions briefly at the beginning of the semester, and then came back to them again towards the end for a deeper dive. At the beginning of the "deeper dive" lesson, Alex decided to use a problem from the first exam to reintroduce the idea of function compositions. Her reason for doing this was because she used an example that was familiar in order to focus students' attention on the underlying concepts. In particular, Alex wanted to use a function diagram and have students think critically about interpreting what each set and arrow represented in relation to the given story problem.

In choosing this example, Alex drew on her knowledge of students in multiple ways. First, Alex recognized that in order to focus her students' attention on the underlying concepts, she needed to build upon their prior knowledge. In this case, the students had worked with the example algebraically on a test, so Alex used that foundation to build a more conceptual understanding. Second, Alex knew that students struggled to interpret and understand both function diagrams and function compositions as a whole. Alex recognized that notation can be hard for students, but it is important and can be a roadblock for students if they go on to take calculus.

Second, Alex considered how students might interact with and think about mathematical content, practices, and strategies during the enactment of the examples. In one example, Alex asked her students to come up with the equation for an exponential function given a word problem. Previous to this example, students had worked through the simple and compound interest problem, but Alex still had not introduced the standard form of an exponential. So students were using their intuition of exponential growth to come up with an equation. The example that Alex chose to use involved both an initial value of 25 and a growth rate of 25%. Alex knew that some students might be struggling to identify the relationship between the two 25s in the equation and the two 25s in the story problem, so she asked her students to explain this correspondence.

Once the students had come up with the equation, Alex introduced the standard form of an exponential as well as the terms initial value, growth factor, and growth rate. Alex knew that students often struggle to differentiate between the growth rate and the growth factor of an exponential. So she focused the rest of the example on differentiating between these two terms as well as helping her students see how they were related. In the interviews, Alex talked about how she knew that students often struggled with this concept, so she made this the main mathematical focus of the example.

Knowledge of Sequencing

The final domain of knowledge that I identified as helping maintain the cognitive demand of an example is knowledge of sequencing. While sequencing the presentation of content and activities is a form of pedagogical *work*, I am more concerned with the knowledge that instructors rely upon when deciding how to sequence. While this knowledge is often activated in the lesson planning stage, it can also be used during class time. Here, I define knowledge of sequencing as knowledge of the difficulty and appropriateness of mathematical content, practices, and strategies in relationship to each other. While some of this sequencing was suggested by the lesson guides that the instructors used, instructors often chose to alter the way they presented the content and added or subtracted from the lesson guides. So I am interested in identifying the mathematical knowledge that instructors relied upon when making these decisions.

Kelly. In the lesson on the short-term behavior of polynomials, Kelly relied on her knowledge of sequencing when deciding how to present the content. Within the lesson, Kelly needed to introduce the ideas of multiplicities of zeros and short-term behavior (i.e., whether or not the graph bounces off of or crosses the x-axis at zeros) and draw connections between the degree, multiplicities, and number of zeros. While Kelly could have presented these ideas all separately, she chose to build upon their connectedness and introduce the all using one example.

The example that Kelly chose was $p(x) = x^2(x+3)(x-5)^3$. In choosing the polynomial to use for the example, Kelly had to consider whether or not it was robust enough to model everything that she wanted to address. In particular, Kelly often relied upon her students to recognize patterns and make connections, so the polynomial that she chose had to support their ability to do that. Kelly chose to start by introducing the idea of multiplicity and then asked students, “What could happen at the zeros?” After providing her students with some additional scaffolding in the form of questions, her students recognized that the graph of a polynomial would either bounce off the x-axis or cross it at each zero. Kelly then asked students to consider the graph of $y = p(x)$ and use that to see if they could figure out a pattern for when the graph bounces and when it crosses. Because Kelly had chosen a polynomial that had multiple zeros with even and odd multiplicities, her students were able to make this connection.

Next, Kelly asked her students to consider how degree and multiplicities are related. One student conjectured that the degree was equal to the product of the multiplicities, which was true in this case, but Kelly then explained how expanding out the factored form would lead us to add, not multiply, the multiplicities. Finally, Kelly asked her students to identify the relationship between the degree and the number of zeros, which again was a connection that her students were able to make.

In this example, Kelly drew upon her knowledge of sequencing in several ways. First, Kelly chose to present the four main ideas in the example in a way so that they naturally built upon each other. Also, Kelly could have presented the ideas first and then asked students to apply them to the specific example. However, Kelly wanted her students to make the connections and generate the relationships, so she chose to situate the ideas within and not before or after the example. Finally, Kelly integrated several representations, both algebraic and graphical, throughout the example as a way to help the students recognize patterns and make connections.

Table 30. Definitions and examples of subdomains of MKT

Term	Definition	Examples
Knowledge of Connections	Knowledge of mathematical relationships between content, practices, and strategies	Knowledge of how new mathematical content is connected to previously learned mathematical content; knowledge of how graphical and algebraic solutions are related; knowledge of similarities and differences between mathematical concepts
Knowledge of Representation	Knowledge of graphical, pictorial, tabular, algebraic, verbal, and written forms of mathematical content, practices, and strategies	Knowledge of the affordances and constraints associated with different representations; knowledge of how different representations can produce different solution strategies
Knowledge of Unpacking	Knowledge of the essential features of mathematical content, practices, and strategies	Knowledge of different solution strategies that can be used to solve a problem; knowledge of ways of scaffolding content without decreasing the cognitive demand
Knowledge of Students	Knowledge about how students interact with and think about mathematical content, practices, and strategies	Knowledge of concepts and strategies that students typically struggle with; knowledge of common student approaches to problems (both productive and unproductive); knowledge of typical student solutions and ways of thinking about a problem
Knowledge of Sequencing	Knowledge of the difficulty and appropriateness of mathematical content, practices, and strategies in relationship to each other	Knowledge of how content, practices, and strategies can be used to build upon each other; knowledge of how to both strip down a difficult concept to its' simplest form and then add in more complexity

Relationships

The five knowledge domains that I identified reflect what other researchers have found, but highlights the knowledge that supports the maintenance of high cognitive demand examples. I also wanted to see how these domains are connected to specialized and pedagogical content knowledge, as well as the instructor roles and decomposition of pedagogical work that I have identified in previous chapters (Chapters 4 and 5). So in the following subsections, I examine the relationships between each of the knowledge domains and SCK/PCK, instructors' roles, and the decomposition categories.

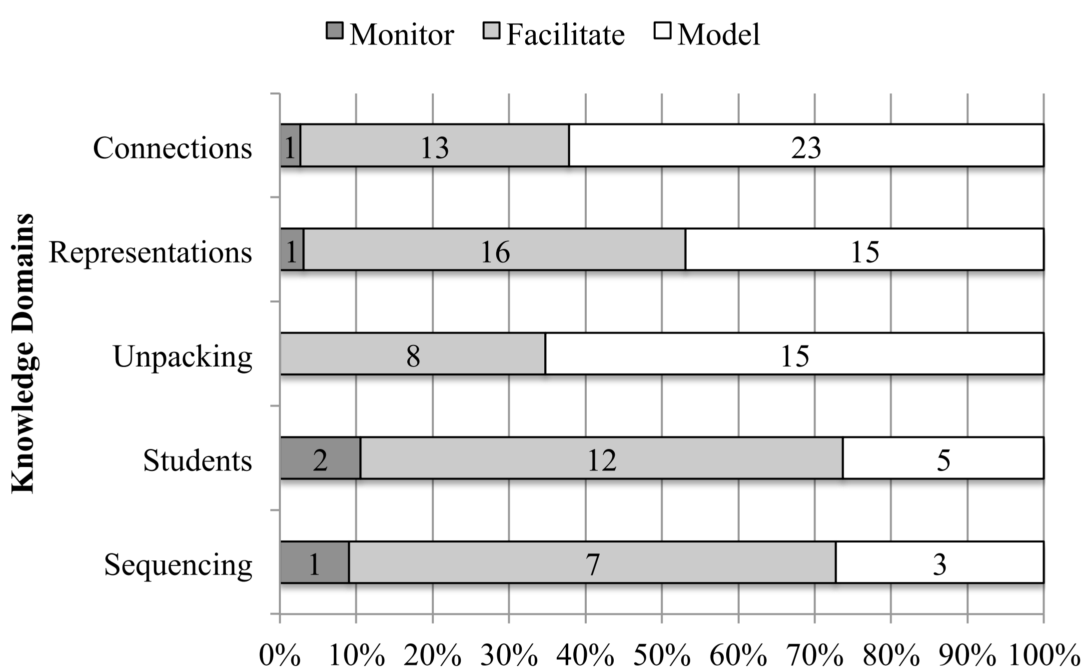
SCK and PCK. Each time I used one of the knowledge domain codes, I also decided whether or not that instance was representative of specialized content knowledge (SCK) and/or pedagogical content knowledge (PCK). While every knowledge domain was coded at least once in both categories, knowledge of connections, representations, and unpacking were primarily categorized as subdomains of SCK. Knowledge of students, on the other hand, was primarily categorized as a subdomain of PCK. However, knowledge of sequencing was almost an even split between the two categories.

Table 31. Overlap of Knowledge Domains and SCK/PCK

	Connections	Representations	Unpacking	Students	Sequencing
SCK	28	22	19	0	4
Both	2	0	1	3	0
PCK	4	5	0	16	6
Total	34	27	20	19	10

Instructor roles. In an earlier chapter (Chapter 3), I identified three different roles that the instructors took on when enacting high cognitive demand examples. First, some instructors modeled content, practices, and strategies for their students as they took notes. Others facilitated whole-class discussions where they worked through the example together with the students. Finally, some instructors chose to monitor students while they worked independently or in groups on parts of the example. Most of the instructors switched back and froth between different roles, although some chose to just use one way of presenting the example. In Table 32 I have constructed role profiles that show the relationship between the knowledge domains and the instructors' roles. It is important to note that my IRB did not allow me to video the students, so I was often not able to capture what the instructors did or said while they were monitoring. Therefore, the overlap between the different knowledge domains and the monitoring code is probably not representative of the actual occurrences.

Table 32. Role Profiles of Knowledge Domains



Decomposition of pedagogical work. In addition to identifying the different roles that instructors took on when enacting high cognitive demand examples, I also decomposed the work entailed in maintaining the cognitive demand of examples. Table 33 captures the overlap between these two coding schemes. It is not surprising that the making connections decomposition code overlaps primarily with the knowledge of connections domain. However, it was surprising to see that knowledge of unpacking was the dominant overlapping knowledge domain with the other four decomposition codes.

Table 33. Overlap of Knowledge Domains and Decomposition Codes

	Connections	Representations	Unpacking	Students	Sequencing
Mathematical Point	5	3	9	2	1
Connections	45	35	30	18	9
Explanations	63	57	78	39	9
Cognitive Processes	34	30	49	13	4
Understanding	14	14	31	18	1

Discussion

In analyzing the data, I found that knowledge of connections, representations, unpacking, students, and sequencing help instructors enact examples at a high level of cognitive demand. I also found that these knowledge domains overlap primarily with specialized content knowledge and that knowledge of unpacking was used heavily in almost all of the tasks of teaching that I identified in my decomposition.

Limitations

First, as noted previously, the five domains of knowledge are not assumed to be independent. From a quantitative standpoint, this is a limitation of the model, but I believe it accurately reflects the interconnected nature of teaching. Second, since almost all of the high cognitive demand examples were coded as procedures with connections tasks, this model may overemphasize knowledge of connections and representations. However, "doing mathematics" may not be well suited for examples and it may be reasonable to assume that most high cognitive demand examples are procedures with connections tasks. Also, since this study was a collective case study and all of the instructors were graduate students, it may not be generalizable.

One limitation of this study is that I focused on identifying *observable* knowledge. In particular, my analysis focused on identifying the knowledge that I could observe during the enactment stage of example unfolding. While analyzing knowledge from an observational perspective does require some assuming on the side of the observer, I relied upon the pre- and post-observational interview data to verify claims that I made concerning knowledge used during enactment. This is a limitation because there is knowledge that the instructors may have been relying upon that was not observable. Therefore, my results only highlight a portion and not the totality of knowledge entailed in enacting high cognitive demand examples. Another related limitation is that I only analyzed the knowledge entailed in enactment, so it is possible that instructors were drawing upon other forms of knowledge during the planning stage.

Another limitation of this work is that the instructors involved in the study were teaching a coordinated course. In this context, course coordination meant that the syllabus, course schedule, lesson guides, student workbook, online homework, chapter quizzes, and exams were the same for all sections. The department chose to provide this much structure for the graduate students who taught these precalculus courses because they wanted the instructors to focus on using active learning during class time and not worry as much about the curricular and assessment aspects of teaching. Since much of the pedagogical structure is laid out for the teachers in the lesson guides, this might be one reason why I mainly observed instructors using specialized content knowledge instead of pedagogical content knowledge.

Future Research

There is still much work that needs to be done to understand MKT at the undergraduate level, but this study provides a starting point for future investigations. In particular, it would be interesting to extend this study in several different directions. First, expanding the sample size and including instructors with a variety of backgrounds and teaching experience would test whether or not the model could be generalizable. Second, observing enacted examples that are doing mathematics examples (Smith & Stein, 1998) would help further refine the model and test whether or not procedures with connections examples had a large influence on the knowledge domains that emerged. Third, in order to better understand MKT at the undergraduate level at large, it would be beneficial to collect classroom data that focuses on more than just examples. Finally, my intention is to dig into the entire process of example unfolding and see what knowledge instructors use in the planning stage and use pre- and post-observation interview data to dig further into the knowledge used by instructors when teaching precalculus.

Conclusions

Given that examples are an important part of teaching, it is important to identify the mathematical knowledge that supports the maintenance of high cognitive demand examples. These knowledge domains can be used in designing teaching professional development opportunities for GSIs. In particular, professional development should be designed to help GSIs develop knowledge of connections, representations, unpacking, students, and sequencing. This chapter benefits the community of mathematics education by identifying the mathematical knowledge used by instructors when teaching examples in precalculus. While it is similar to other models of MKT, it is also different in several important ways. First, the domains of knowledge are inherently connected. Second, while knowledge of connections and representations are implicit in many of the other models, they are not explicitly emphasized.

CHAPTER 7: CONCLUSION

In summary, the purpose of my dissertation has been to examine the teaching tasks and mathematical knowledge entailed in enacting high cognitive demand examples. While high cognitive demand tasks have been studied extensively in the educational literature, many of these studies have focused on the tasks that students engage with either during class work time, in homework on assignments, or on assessments. However, little research has focused on the cognitive demand of the examples that the instructors choses to do during class.

Since examples differ from other mathematical tasks in that it is usually the teacher, not the students, who takes on the responsibility for doing the mathematical work, I first examined what it would mean for an example to be enacted at a high level of cognitive demand. While the Task Analysis Guide developed by Smith and Stein (1998) was general in many aspects, it also made explicit references to how the *students* were engaging with the mathematics. Since I wanted to analyze the cognitive demand of an example, regardless of whether or not the teacher or the students were engaging with the mathematics, I modified this framework to not include any language about *who* is doing the mathematics.

For initial data analysis, I conducted classroom observations and pre- and post-observation interviews with seven graduate student instructors who were teaching a precalculus course for at least the third time. After observing these instructors enact 93 different examples in their classrooms, I used my modified framework to analyze the cognitive demand of each example. As a result, I found that the instructors enacted 25 of the examples at a high level of cognitive demand. I also identified three different roles that instructors took on when enacting examples. First, some of them chose to model content, practices, and strategies for students while they took notes. Others chose to work through the example by facilitating a whole-class discussion. Finally, some chose to monitor students while they worked individually or in small groups on part of the example. In many cases, instructors chose to switch between these different roles as the presented examples. However, some instructors chose to just model or facilitate. Then, based on my analysis, I constructed role profiles to see how instructors distributed their time in the three different roles. I also examined the role profiles of some of the high cognitive demand examples in order to see how instructors switched back and forth between different roles.

Next, I focused on identifying the teaching tasks entailed in enacting high cognitive demand examples. Using open and axial coding, I found that there are five main tasks that teachers engage in to maintain the cognitive demand of examples. First, instructors attended to the mathematical point. To do this, instructors introduced the mathematical point as a way to set the focus of the example, maintained the focus of the example on the mathematical point, and summarized the example in order to reiterate the mathematical point. Second, instructors made connections to prior knowledge, between representations, and between concepts. Third, instructors provided clear verbal explanations. These explanations could focus on the example set up, constraints, and goal; the content, practices, or strategies; similarities and differences; representations; notations and vocabulary; or on how to check your work. Fourth, instructors articulated cognitive processes. Some instructor chose to think aloud as they worked through the example, while other asked students to provide justification and reasoning. Finally, instructors supported student understanding by providing students with opportunities to ask questions, recognizing when students were struggling to follow or understand, and scaffolding the example.

Finally, I examined the mathematical knowledge entailed in enacting high cognitive demand examples. Through open and axial coding, I identified five domains of mathematical knowledge for teaching that support the maintenance of the cognitive demand of examples. First, instructors used knowledge of connections, which I defined as knowledge of mathematical relationships between content, practices, and strategies. Second, instructors used knowledge of representations, which I defined as knowledge of graphical, pictorial, tabular, algebraic, verbal, and written forms of mathematical content, practices, and strategies. Third, instructors used knowledge of unpacking, which I defined as knowledge of the essential features of mathematical content, practices, and strategies. Fourth, instructors used knowledge of students, which I defined as knowledge about how students interact with and think about mathematical content, practices, and strategies. And finally, instructors used knowledge of sequencing, which I defined as knowledge of the difficulty and appropriateness of mathematical content, practices, and strategies in relationship to each other.

While these knowledge domains emerged from my data analysis, they are connected in many ways to the work that other researchers have done. In particular, knowledge of connections, representations, and unpacking overlap primarily with specialized content knowledge (Ball et al., 2008). Knowledge of students, on the other hand, overlaps primarily with pedagogical content knowledge (Shulman, 1986). However, knowledge of sequencing was split almost evenly between the two. I also analyzed how these knowledge domains overlapped with both the roles that instructors take on when enacting high cognitive demand examples and the teaching tasks entailed in maintaining the cognitive demand. Not surprisingly, knowledge of connections overlapped significantly with the teaching task of making connections. However, I was surprised to find that knowledge of unpacking was the knowledge domain that had the most overlap with the other four tasks of teaching.

Through my dissertation study, I have sought to identify the knowledge and work entailed in enacting high cognitive demand examples. In doing this, I aim to help our field move one step closer to improving student outcomes and teaching quality in first-year undergraduate mathematics courses. While there are many aspects of teaching, I chose to focus on examples because they are one of the essential components of instruction in mathematics classrooms. Also, while many studies have focused on the cognitive demand of the tasks that we give students to work on, few have looked at the cognitive demand of the examples that we use. Therefore, my dissertation contributes to this field by identifying what high cognitive demand examples might look like, examining the different roles instructors and students take on during the enactment of examples, decomposing the work entailed in maintaining the cognitive demand of examples, and examining the mathematical knowledge for teaching entailed in enacting high cognitive demand examples. While there is still a lot of work that needs to be done to improve undergraduate precalculus courses, this work provides both researchers and practioners with a way to think about the quality of the examples that we use.

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APPENDIX A: CHANGES MADE TO ORIGINAL STUDY

When I first started my dissertation, I was focused on deep procedural knowledge (DPK) and the teaching tasks of decomposing, bridging, and trimming. While these frameworks are still connected to what I am doing, I have moved away from talking about DPK and instead used Smith and Stein's framework for cognitive demand. In addition, instead of using the teaching tasks of decomposing, bridging, and trimming, I decided to study pedagogical work entailed in enacting high cognitive demand examples from a grounded theory perspective and use the framework of decompositions of practice. So while I have moved away from using this language in talking about my research, you will still find it in my data collection protocols. However, I want to note that this is mainly just a shift in language use.

Also, I collected data on two faculty members who were teaching precalculus. However, I chose to only analyze the data from the seven graduate student instructors that I observed, since together they made up a coherent population of instructors.

APPENDIX B: DATA COLLECTION PROTOCOLS

Pre-Observation Interview

General Information and Previous Experience

1. What is the topic of the lesson taught right before this lesson? (Add description to field notes.)
2. Have you previously taught the content that you are teaching today? Yes No
    a. *If so*, have you previously taught with this same exact lesson plan? Yes No
        i. *If not*, is this the first time you have used this lesson plan or is it a modified version of a lesson plan you have used previously?
            1. *If it was modified*, ask whether or not examples were modified and probe into those examples specifically later.
3. Is the lesson plan that you intend to use one that was given to you, one that you found from another source, or one that you created yourself?

Identifying Examples that May Afford Opportunities to Learn DPK

4. First, what was the mathematics that you intended the students to learn through the use of the given example?
    a. Why did you want your students to learn this?
    b. What about the example made you believe that it was an appropriate way to learn that mathematics?

I'm specifically interested in examples that involve procedures. The definition for procedure that I'm utilizing is "a series of steps, or actions, done to accomplish a goal" (Rittle-Johnson, Schneider, Star, 2015).

5. Do you think that this example involves a procedure? Yes No
   If yes...
   a. What is that procedure?
   b. Are there any other procedures that could be used in this example? Yes No

While procedural knowledge is often thought of as superficial, there is such a thing as deep procedural knowledge (DPK). In particular, deep procedural knowledge is defined as having three independent characteristics: comprehension, flexibility, and critical judgment. Comprehension is knowing why a procedure works, flexibility is characterized by knowledge of multiple procedures and the ability to select the most appropriate one, and critical judgment is knowing when it is appropriate to use a procedure.

   c. Do you think that this example affords an opportunity for students to learn deep procedural knowledge? Yes No
      If yes...
      i. What characteristics of DPK do you think this example affords an opportunity for students to learn and how does this example afford an opportunity for students to learn these characteristics of DPK?

Observation

General Information

Instructor ID: ___________________________ Date: ___________________________

Course ID: _____________________________ Observer: ________________________

Start Time: _____________________________ End Time: _______________________

Demographics

Number of Students

Total Enrolled:___________________ Males: _____________

Females: ___________

Total In Attendance: _______ Males: ______ Females: ___________

Room Setup

- ☐ Tables Number of Seats/Table: ___________
- ☐ Individual Desks Arranged in Groups Number of Desks/Group: __________
- ☐ Individual Desks

Room Diagram: (note location of camera & observer)

Researcher Positioning

Description of relationship between researcher & instructor: _________________________

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Perceived attitudes concerning the researcher's presence in the classroom: _____________

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Perceived effect of the video camera's presence in the classroom: ____________________

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Description of researcher's thoughts, feelings, and experiences prior to observation: _____

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Consideration of how these prior thoughts, feelings, and experiences may affect researchers' perception of the observation: ____

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Important events/occurrences: ____

____

Purpose of Sampling

What is the exact purpose of observing this particular instructor, course, and/or lesson?

____

____

Mathematical Context

Previous Lesson Topic: ____

Description of Previous Lesson (specifically include where left off): __________________

____

____

Today's Lesson Topic: ____

Description of Today's Lesson: ____

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____

Potential DPK Examples

List (and number) all examples in the lesson plan that were identified as affording opportunities for explaining and modeling concepts, practices, or strategies that require the use of DPK. *Attach a copy of the lesson plan used during class.*

General Observation Notes !!! START AUDIO RECORDING !!!

Anticipated Example #____

Pre-Observation Notes

Short description (see attachment for full text): ____

____

Origin of Identification:

☐ Researcher

☐ Instructor

Anticipated Use of DPK:

	Not Very	Likelihood		Extremely
☐ None				
☐ Flexibility	1	2	3	4
☐ Critical Judgment	1	2	3	4
☐ Comprehension	1	2	3	4

Details concerning anticipated use of DPK: ____

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____

Anticipated Use of Mathematical Teaching Practices:

	Not Very	Likelihood		Extremely
☐ None				
☐ Decompressing	1	2	3	4
☐ Trimming	1	2	3	4
☐ Bridging	1	2	3	4

Details concerning anticipated use of practices: ____

____

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Anticipated Example #____

Start Time: ____________________

Detailed observation notes: (Attend to how the instructor elicits & interprets student thinking.)

End Time: _____________________

Anticipated Example #____

Quick Post-Reflection During Observation

Use of DPK:

	Not Very	Likelihood		*Extremely*
☐ None				
☐ Flexibility	1	2	3	4
☐ Critical Judgment	1	2	3	4
☐ Comprehension	1	2	3	4

Eliciting & Interpreting Student Thinking:

- ☐ None
- ☐ Before Example
- ☐ During Example
- ☐ After Example

Mathematical Practices:

	Not Very	Likelihood		*Extremely*
☐ None				
☐ Decompressing	1	2	3	4
☐ Trimming	1	2	3	4
☐ Bridging	1	2	3	4

Preliminarily Nominate as Exemplary:

- ☐ Yes
- ☐ Maybe
- ☐ No

Anticipated Example #____

Full Post-Observation Reflection

Date: ___________________________ Time: ___________________________

General comments: ____

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DPK:	*Not Very*	Likelihood		*Extremely*
☐ Flexibility	1	2	3	4
☐ Critical Judgment	1	2	3	4
☐ Comprehension	1	2	3	4

Details concerning perceived use of DPK: ____

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Eliciting & Interpreting Student Thinking:

- ☐ Before Example
- ☐ During Example
- ☐ After Example

Details concerning eliciting & interpreting student thinking: ________________________

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Mathematical Practices:	*Not Very*	Likelihood		*Extremely*
☐ Decompressing	1	2	3	4
☐ Trimming	1	2	3	4
☐ Bridging	1	2	3	4

Details concerning perceived use of mathematical practices: ________________________

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Perceived MKT Used: ____

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Other Comments: ____

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Officially Nominate as Exemplary:

- ☐ Yes
- ☐ Maybe
- ☐ No

Unanticipated Example #____

Observation Notes

Start Time: ______________________

Short Description: ____

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Detailed observation notes: (Attend to how the instructor elicits & interprets student thinking.)

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End Time: ______________________

Unanticipated Example #____

Quick Post-Reflection During Observation

Use of DPK:

	Not Very	Likelihood		*Extremely*
☐ None				
☐ Flexibility	1	2	3	4
☐ Critical Judgment	1	2	3	4
☐ Comprehension	1	2	3	4

Eliciting & Interpreting Student Thinking:

- ☐ None
- ☐ Before Example
- ☐ During Example
- ☐ After Example

Mathematical Practices:

	Not Very	Likelihood		*Extremely*
☐ None				
☐ Decompressing	1	2	3	4
☐ Trimming	1	2	3	4
☐ Bridging	1	2	3	4

Preliminarily Nominate as Exemplary:

- ☐ Yes
- ☐ Maybe
- ☐ No

Full Post-Observation Reflection

Date: ___________________________ Time: ___________________________

Type:

- ☐ In original lesson plan, but not identified in lesson plan analysis.
- ☐ Not in original lesson plan, but add purposely by instructor before lesson began.
- ☐ Not in original lesson plan, but added spontaneously during the lesson.

General comments: ____

____

DPK: *Not Very* Likelihood *Extremely*

☐ Flexibility	1	2	3	4	
☐ Critical Judgment		1	2	3	4
☐ Comprehension	1	2	3	4	

Details concerning perceived use of DPK: ____

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Eliciting & Interpreting Student Thinking:

- ☐ Before Example
- ☐ During Example
- ☐ After Example

Details concerning eliciting & interpreting student thinking: ________________________

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Mathematical Practices:	*Not Very*	Likelihood		*Extremely*
☐ Decompressing	1	2	3	4
☐ Trimming	1	2	3	4
☐ Bridging	1	2	3	4

Details concerning perceived use of mathematical practices: ________________________

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Perceived MKT Used: ____

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Other Comments: ____

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Officially Nominate as Exemplary:

- ☐ Yes
- ☐ Maybe
- ☐ No

Details concerning why example was not anticipated: ______________________________

General Post-Observation Reflection

Date: ___________________________ Time: ___________________________

Post-Observation Identification of Exemplary Example

For use in the post-observation interview, identify one example that I feel is exemplary: _

Describe reasoning for choosing this example:

Reflection on Researcher Positioning

Changes in relationship between researcher & instructor: ___________________________

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____

Changes in perceived attitudes concerning the researcher's presence in the classroom: ___

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____

Changes in perceived effect of the video camera's presence in the classroom: __________

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____

Consideration of how the prior thoughts, feelings, and experiences may have affected researchers' perception of the observation: _______________________________________

____

____

Description of researcher's thoughts, feelings, and experiences after observation: _______

____

____

Consideration of how the thoughts, feelings, and experiences after observation may affect researchers' perception of the observation: _______________________________________

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____

Important events/occurrences: ____

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Reflection on Purpose of Sampling

Did the observation serve the intended purpose?

- ☐ Yes
- ☐ No

Comments: ____

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Reflections on Mathematical Context *Observation #*_____

Description of Lesson: ____

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Post-Observation Interview

Reviewing Planned Examples

You planned on using the following examples during class.

1. Did you use all of the examples you planned on covering? Yes No
    a. *If not*, why?

Examining Mathematical Knowledge for Teaching

I identified one example that you used in your lesson that I would like to concentrate on for the remainder of the interview.

2. Before we start talking about this example, I first want to know if there was a specific example that stands out in your mind as affording the best opportunity to learn DPK.
3. What about the example you identified made it stick out in your mind as special?

Intended Learning Outcome

4. During our pre-observation interview, you said that you wanted to use this example because you intended for students to learn.
    a. Do you believe that your students learned the mathematics that you intended? Yes No
        i. *If so*, how do you know?
        ii. *If not*, why not?

Opportunity to Learn Deep Procedural Knowledge

5. During our pre-observation interview, you said that you thought that this example provided an opportunity for your students to learn.
    a. When you used this example in class, do you believe it provided an opportunity to learn this? Yes No
        i. *If yes*, how do you know?
        ii. *If no*, why not?

Identifying the Intended Procedure

6. What are the steps of the specific procedure that you intended to use in this example?
    a. If the instructor lists steps...
    Based on what you did during the observation, I wrote down these

steps. Do you think they are the same as the steps you outlined? Yes No

1. If no...
    i. How do they differ?
    ii. Are these differences important?

b. If the instructor is not able to list steps...
Based on what you did during the observation, I wrote down these steps. Do you think this procedure matches what you did during the observation? Yes No

2. If no...
    i. How do they differ?
    ii. Are these differences important?

7. Are there any other procedures you could have followed? Yes No

3. If yes...
    a. What are the steps of that procedure?
    b. Did you consider using this procedure instead of the one you chose? Yes No
        i. Why or why not?
    c. Are there benefits to using one procedure over the other?
    d. Are there disadvantages to using one procedure over the other?

Decision Making

There are a couple of clips from the video of this example that I want to watch together and discuss. (For each clip...determine which of the following questions are appropriate to ask.)

Decompressing. During this part of the example, it seemed like you were unpacking the mathematics to make it comprehensible for your students.

1. What exactly were you trying to unpack?
2. What made you decide to unpack it?
3. How did you determine a way to unpack it?

Bridging. During this part of the example, it seemed like you were making mathematical connections across topics, assignments, representation, or domain.

1. How did you make connections?
2. What made you decide to make these connections?
3. How did you determine a way to make these connections?

Trimming down. During this part of the example, it seemed like you removed some mathematical complexity to make it more comprehensible for your students?

1. How did you determine what you could remove?
2. What made you decide to remove it?
3. How did you maintain the integrity of the problem?

Trimming up. During this part of the example, it seemed like you added some mathematical complexity to make it more challenging for your students?

1. How did you determine what you could add?
2. What made you decide to add it?
3. How did you maintain the integrity of the problem?

Eliciting and interpreting student thinking. During this part of the example, you elicited and interpreted student thinking.

1. What made you elicit the students' thinking?
2. What response did you anticipate?
3. What did you interpret the student to mean mathematically when he/she gave their response to your prompt?

Representations. During this part of the example, you used the following representation(s).

1. What made you decide to use this representation?
2. Were there any other representations that you considered using?

Other. Use this IF AND ONLY IF none of the other categories _t. Write out questions beforehand and associate them with a code and memo explaining why these questions needed to be asked and did not fit into any of the other categories.

APPENDIX C: SUPPLEMENTARY FIGURES & TABLES

Table 34. Full List of Observed Examples

Example ID	Lesson	Example Description	Cognitive Demand
Alex 1-1	Introduction to Exponentials	Exploring the notions of exponential vs. linear growth	High
Alex 1-2	Introduction to Exponentials	Building an exponential function from a word problem	High
Alex 2-1	Function Compositions	Exploring the notion of function compositions through unit conversions	High
Alex 2-2	Function Compositions	Finding the formula for a function composition	Low
Alex 3-1	Inverse Trig Functions	Finding all solutions to trig equations with standard unit circle angles	Low
Dan 1-1†	The Vertex of a Parabola*	Identifying the vertex of a parabola given the vertex-form of the function	Low
Dan 1-2†	The Vertex of a Parabola*	Writing the equation of a parabola given its vertex and a point	Low
Dan 1-3†	The Vertex of a Parabola*	Factoring quadratics that are perfect squares	Low
Dan 1-4†	The Vertex of a Parabola*	Using completing the square to write equations in vertex form	Low
Dan 1-5†	The Vertex of a Parabola*	Completing the square when the coefficient on x is odd	Low
Dan 1-6†	The Vertex of a Parabola*	Completing the square when the coefficient on x^2 is not 1	Low

Example ID	Lesson	Example Description	Cognitive Demand
Dan 2-1†	Function Compositions	Exploring the notion of function compositions using a word problem	Low
Dan 2-2†	Function Compositions	Evaluating function compositions	Low
Dan 2-3†	Function Compositions	Decomposing function compositions given the outside function	Low
Dan 2-4†	Function Compositions	Decomposing function compositions into any two functions	High
Dan 3-1†	Trig Equations & Inverse Functions*	Graphing solutions to trig equations as points of intersection	High
Dan 3-2†	Trig Equations & Inverse Functions*	Finding all solutions to trig equations with standard unit circle angles	Low
Dan 3-3†	Trig Equations & Inverse Functions*	Finding all solutions to trig equations with non-standard unit circle angles	Low
Dan 3-4†	Trig Equations & Inverse Functions*	Finding all solutions to trig equations with only one initial solution	Low
Dan 3-5†	Trig Equations & Inverse Functions*	Finding all solutions to trig equations with standard unit circle angles	Low
Dan 3-6†	Trig Equations & Inverse Functions*	Finding all solutions to sinusoidal equations with vertical transformations	Low
Dan 3-7†	Trig Equations & Inverse Functions*	Finding all solutions to sinusoidal equations with horizontal transformations	Low
Dan 3-8†	Trig Equations & Inverse Functions*	Finding all solutions in a given interval to sinusoidal equations	High
Emma 1-1	Introduction to Quadratics	Expanding a factored quadratic to standard form	Low
Emma 1-2	Introduction to Quadratics	Factoring a quadratic when the coefficient on x^2 is 1	Low
Emma 1-3	The Vertex of a Parabola	Using vertex form of a quadratic to graph a parabola	Low
Emma 2-1	Function Transformations	Transforming the graph of a function	High
Emma 2-2	Function Compositions	Evaluating and simplifying function compositions	Low

Example ID	Lesson	Example Description	Cognitive Demand
Emma 2-3	Function Compositions	Evaluating function compositions	Low
Emma 2-4	Function Compositions	Decomposing function compositions into any two functions	Low
Emma 3-1	Arc Length	Finding distance using arc length	Low
Emma 3-2	Sinusoidal Functions & Their Graphs	Using function transformations to graph sinusoidal functions	Low
Greg 1-1	Law of Sines & Cosines	Using the Law of Sines to solve for a side length	Low
Greg 2-1	Trig Equations & Inverse Functions	Explaining why sine and cosine may have 2 solutions/period, but tangent can only have 1	High
Greg 2-2	Trig Equations & Inverse Functions	Finding all solutions to sinusoidal equations	Low
Greg 3-1	Review	Sketching the graph of a sinusoidal function	Low
Greg 3-2	Review	Evaluating trig functions given value of sine and quadrant of θ	Low
Greg 3-3	Review	Evaluating trig functions using sum and difference formulas	Low
Greg 4-1	Tangent & Reciprocal Trig Functions*	Exploring the behavior of tangent using standard unit circle angles	High
Greg 4-2	Tangent & Reciprocal Trig Functions*	Using the unit circle definition to evaluate tangent	Low
Greg 4-3	Tangent & Reciprocal Trig Functions*	Using the triangle definition to evaluate sine, cosine, and tangent	Low
Greg 4-4	Tangent & Reciprocal* Trig Functions*	Solving real-life problems using tangent	High
Greg 5-1†	Trig Equations & Inverse Functions*	Using graphs to identify how many solutions are in a single period	High
Greg 5-2†	Trig Equations & Inverse Functions*	Finding all solutions to trig equations with standard unit circle angles	High
Greg 5-3†	Trig Equations & Inverse Functions*	Finding all solutions to trig equations with non-standard unit circle angles	High

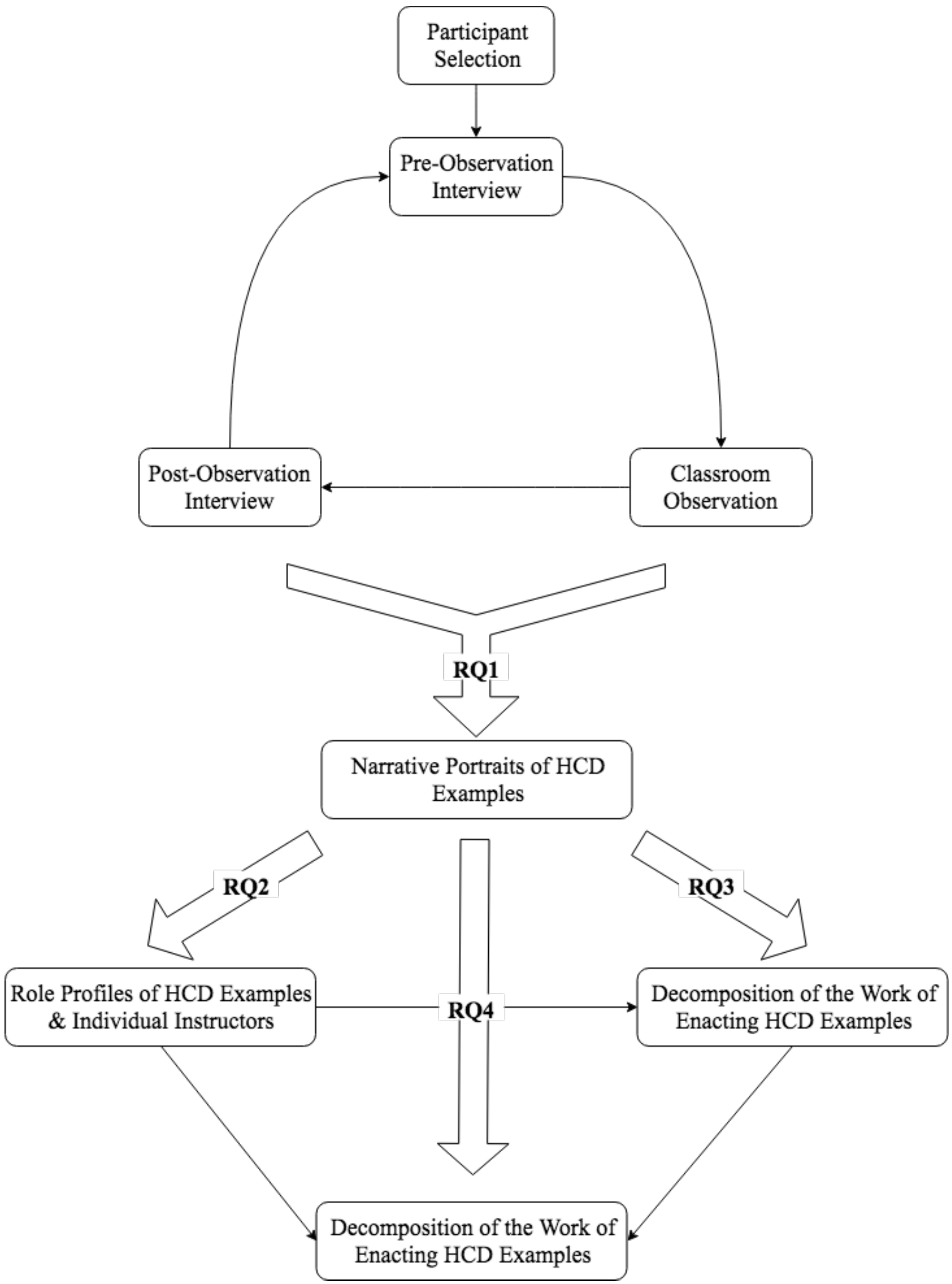
Example ID	Lesson	Example Description	Cognitive Demand
Greg 5-4†	Trig Equations & Inverse Functions*	Finding all solutions to trig equations with non-standard unit circle angles	Low
Greg 5-5†	Trig Equations & Inverse Functions*	Finding all solutions to tangent equation with non-standard unit circle angles	High
Greg 5-6†	Trig Equations & Inverse Functions*	Finding all solutions to sinusoidal equations with non-standard unit circle angles	High
Greg 6-1†	Review*	Finding the sign of trig functions given the quadrant of θ	Low
Greg 6-2†	Review*	Connecting outputs of tangent, sine, and cosine	Low
Greg 6-3†	Review*	Finding a sinusoidal equation given a description of a real-life context	High
Greg 6-4†	Review*	Finding the distance traveled given the distance function	Low
Greg 6-5†	Review*	Finding sinusoidal equation given a description of a real-life context	Low
Greg 6-6†	Review*	Finding all solutions to a sinusoidal equation	Low
Greg 6-7†	Review*	Finding the distance travelled using arc length	Low
Greg 6-8†	Review*	Finding the distance travelled using unit conversions	Low
Greg 6-9†	Review*	Finding the horizontal shift of a sinusoidal function	High
Juno 1-1	Tangent & Reciprocal Trig Functions*	Using unit circle definition to evaluate tangent	Low
Juno 1-2	Tangent & Reciprocal Trig Functions*	Using the right triangle definition to evaluate tangent	Low
Juno 1-3	Tangent & Reciprocal Trig Functions*	Evaluating sine and cosine on complementary angles	Low
Juno 1-4	Tangent & Reciprocal Trig Functions*	Evaluating sine and cosine on complementary angles	Low
Juno 1-5	Tangent & Reciprocal Trig Functions*	Proving that sine and cosine are cofunctions	High

Example ID	Lesson	Example Description	Cognitive Demand
Juno 1-6	Tangent & Reciprocal Trig Functions*	Proving that tangent and cotangent are cofunctions	Low
Juno 2-1†	Trig Equations & Inverse Functions*	Graphing solutions to trig equations as points of intersection	High
Juno 2-2†	Trig Equations & Inverse Functions*	Finding all solutions to trig equations with standard unit circle angles	High
Juno 2-3†	Trig Equations & Inverse Functions*	Finding all solutions to trig equations with non-standard unit circle angles	High
Juno 2-4†	Trig Equations & Inverse Functions*	Finding all solutions to sinusoidal equations	Low
Juno 2-5†	Trig Equations & Inverse Functions*	Finding all solutions in a given interval to sinusoidal equations	Low
Juno 3-1†	Review*	Graphing sinusoidal functions	Low
Juno 3-2†	Review*	Using Law of Sines to find unknown side lengths and angle measures	Low
Juno 3-3†	Review*	Finding points of intersection using polar coordinates	Low
Kelly 1-1	Introduction to Exponentials	Exploring the notions of exponential vs. linear growth	High
Kelly 1-2	Introduction to Exponentials	Differentiating exponential growth and decay given exponential function	Low
Kelly 2-1	Polynomials & Rational Functions	Exploring the behavior of polynomials near the roots	High
Kelly 2-2	Polynomials & Rational Functions	Graphing polynomials given the equation in factored form	High
Kelly 2-3	Polynomials & Rational Functions	Constructing polynomial equations given the graph	High
Kelly 3-1	Arc Length	Finding arc length on the unit circle	Low
Kelly 3-2	Arc Length	Finding arc length on a non-unit circle	Low
Selrach 1-1	Logarithms & Their Properties*	Finding the equation of an exponential function given two	Low

Example ID	Lesson	Example Description	Cognitive Demand
		points	
Selrach 1-2	Logarithms & Their Properties*	Solving equations using exponentials and logarithms	Low
Selrach 1-3	Logarithms & Their Properties*	Solving equations using exponentials and logarithms	Low
Selrach 1-4	Logarithms & Their Properties*	Solving equations using exponentials and logarithms	Low
Selrach 1-5	Logarithms & Their Properties*	Solving equations using properties of logarithms	Low
Selrach 1-6	Logarithms & Their Properties*	Solving equations using properties of logarithms	Low
Selrach 2-1†	Inverse Functions*	Examining what is an inverse & whether or not every function has an inverse	Low
Selrach 2-2†	Inverse Functions*	Examining functions that are not invertible	Low
Selrach 2-3†	Inverse Functions*	Finding an inverse function using function diagrams	Low
Selrach 2-4†	Inverse Functions*	Finding an inverse function algebraically	Low
Selrach 2-5†	Inverse Functions*	Evaluating inverse functions using a table	Low
Selrach 3-1†	Trig Equations & Inverse Functions*	Finding all solutions to sinusoidal equations	Low
Selrach 3-2†	Trig Equations & Inverse Functions*	Finding all solutions in a given interval to sinusoidal equations	Low
Selrach 3-3†	Trig Equations & Inverse Functions*	Finding all solutions to complex trig equations by factoring	Low
Selrach 3-4†	Trig Equations & Inverse Functions*	Finding all solutions in a given interval to sinusoidal equations	Low

Note: The example ID represents the instructor, the observation number, and the enacted example number.
*These lessons were purposefully sampled because of their focus on procedures.
†These examples were spread out over two days of instruction.

Figure 16. Study Diagram



Appendix D: Coding Schemes

Decomposition of Practice

Initial Coding Scheme

The initial coding scheme that I developed for capturing the work of enacting high cognitive demand examples was generated before I began data analysis. I generated this coding scheme based upon literature, my own experiences teaching, my observations during data collection, and my conversations with instructors during the pre- and post-observation interviews.

- Set up
    - Transition from the previous activity or beginning of class.
    - Explain the mathematical point of the example.
    - Make connections between the example and other content, practices, and strategies students are familiar with.
- Enactment
    - Modeling
        - Make connections to previously learned content, practices, and strategies.
        - Explain their thinking process.
        - Make connections between representations.
        - Monitor time spent on the example.
        - Steer the example towards the mathematical point.

    - Facilitating
        - Elicit, interpret, and respond to student thinking.
        - Gauge student understanding and engagement.
    - Monitoring
        - Give instructions on what they expect students to do.
        - Monitor and respond to student struggle.
        - Steer students towards the mathematical point.
- Wrap up.
    - Reiterate the mathematical point of the example.
    - Summarize the content, practices, and/or strategies that were used in the example.
    - Reiterate or make new connections to related content, practices, or strategies.
    - Transition to the next activity or the end of class.

Refined Coding Scheme

As I used my initial coding scheme to code the high cognitive demand examples, I began refining it to reflect what I was seeing in the data. One big difference I noticed between my initial and my refined coding scheme was that the hierarchical structure of the initial coding scheme did not fit the data as nicely as I had expected. So I decided to remove the hierarchy as I was coding. Table 35 shows the resulting refined coding scheme as well as the frequency with which I used each code.

Table 35. Refined (Non-Hierarchical) Codes with Frequency Counts

Code	Frequency
Explaining	474
Solution Strategy or Procedure	342
Modeling	308
Connect	293
Facilitating	238
Think-Aloud	236
Student Thinking	230
Representations	221
Concepts	97
Notation or Vocabulary	66
Student Understanding	65

Previous Knowledge or Examples	60
Scaffolding	59
Student Struggle	51
Differences	48
Maintain Mathematical Point	21
Transition	19
Introduce Mathematical Point	14
Monitoring	14
Student Work Time	14
Set Up the Example	13
Instructions	13
Mathematical Point	13
Similarities	11
Student Engagement	10
Connect	9
Multiple solution strategies	9
Abstract to Concrete	9
Transition	9
Summarize	7
Real Life	6
Checking final answer	5
Connect	4
Prioritize	3
Monitor Time	1
Ran Out of Time	0

Final Coding Scheme

After I finished my first round of coding, I realized that my original hierarchical structure was not working quite as I planned because instructors would do similar things when modeling as they did when facilitating or monitoring. So I decided to split off those three codes (Modeling, Facilitating, and Monitoring) into a separate category, which I now refer to as roles. Doing this allowed me to see how the remaining codes were related and create a new hierarchical structure for my final coding scheme. After using axial coding to create this semi-final coding scheme, I went back and re-coded all 25 examples. While coding, I made small tweaks until I had a stable final coding scheme that described the work of enacting high cognitive demand examples. For descriptions of these codes and video clips that were coded with these codes, read through my results section.

- Attend to the mathematical point
    - Introduce the mathematical point as a way to set the focus of the example
    - Maintain the focus of the example on the mathematical point
    - Summarize the example to reiterate the mathematical point
- Make connections
    - To previously learned content, practices, and strategies
    - Between representations
    - Between concepts
- Provide clear verbal explanations
    - Of the example set up, constraints, and goal
    - Of content, practices, and strategies
    - Of similarities and differences
    - Of representations
    - Of notation and vocabulary
    - Of how to check your work
- Articulate cognitive processes
    - By thinking aloud as you work through the example
    - By asking students to provide justification and reasoning
- Support student understanding
    - By providing opportunities for students to ask questions
    - By recognizing when students are struggling to follow or understand
    - By providing scaffolding for struggling students without decreasing the cognitive demand