1994

The Markov Chain Interest Rate Scenario Generator Revisited

Sarah L.M. Christiansen
Principal Financial Group

The Markov Chain Interest Rate Scenario Generator Revisited

Sarah L.M. Christiansen*

Abstract

This paper furthers the development of the Markov chain interest rate generator. Though the basic technique remains essentially unchanged, there are still many significant changes to the model. For example: (i) the long (key) rates are now are generated by a mean reversionary process; (ii) the number of shapes is increased from seven to 11; (iii) the limitation of changing by only two shape codes per year is removed; and (iv) the random walk matrix that determines the shapes is revised to be more realistic. An algorithm is developed to determine the shape code of the original yield curve, thus eliminating an input and assuring consistency. Flexibility in the choice of the key rate is introduced. Implications of the choice of the key rate are discussed.

Key words and phrases: *curve shape, yield curve, key rate, shape codes*

1 Introduction

The Markov chain interest rate generator (MCG)[1] was introduced by Christiansen (1992) as a model of the underlying term structure of interest rates. The MCG interest rate generator produces scenarios of

*Sarah L.M. Christiansen, Ph.D., F.S.A. (1991), M.A.A.A., received her B.A. in mathematics from the University of California, Riverside (1966) and her M.A. (1968) and Ph.D. (1971) from the University of New Mexico. Before starting her actuarial career in 1985, she taught mathematics at Drake University, Des Moines Area Community College, and Grandview College. She is currently an assistant corporate actuary at the Principal Financial Group, vice-chair of the Society of Actuaries' committee responsible for the F-480 examination entitled *Derivative Securities: Theory and Application*, and vice-chair of the Education and Research Section Council.

Dr. Christiansen's address is: The Principal Financial Group, 711 High Street, Des Moines IA 50392-0001, USA.

The author is grateful to Warren Adams, David Brown, Don Sanning, the editor, and the anonymous reviewers for their helpful suggestions.

[1]An interest rate generator is an algorithm or an equation that produces scenarios of future interest rates.

complete yield curves for up to 30 years into the future. As a result it can be used to produce scenarios of spot yield curves of interest rates. These interest rates then can be used for cash flow testing, for asset/liability studies, for pricing purposes, to compare different investment strategies, or to test pricing assumptions for some annuity products. The MCG most often is used with rates that already incorporate the spreads earned over Treasury (risk-free) rates, but can be used with Treasury rates.

The objective of this paper is to present improvements to Christiansen's (1992) model. But before discussing these improvements, the basics of the original model are reviewed and summarized briefly in Section 2. (We hope that this review makes the current paper self-contained.) These improvements (discussed in Section 3) have increased the model's flexibility, while improving the realism of the yield curves and their distribution. Section 4 shows how the revised model can be implemented. The revised MCG model also has several other desirable characteristics (discussed in Section 5) such as producing reasonable rates and spot and forward rates that are never negative. Section 6 contains the conclusions and the appendices follow.

2 Review of Original Model

In the original model, the sequence of interest rates for the kth scenario[2] is generated for n years and for different maturities using a particular long rate (or key rate). Without loss of generality, the scenarios were generated for $n = 30$ years with a 20-year long rate, and for maturities of 1/2, 1, 2, 3, 5, 7, 11, 15, and 20 years. The sequence of interest rates for the kth scenario is determined in six steps.

Step 1, Get Sample of Uniform Random Numbers: For scenario k, we obtain a sample of 30 uniform random numbers between 0 and $(R_{max} - R_{min} - 0.03)$

$$U_{k,t} \sim U(0, (R_{max} - R_{min} - 0.03))$$

where $t = 1, 2, \ldots, 30$; $U(a, b)$ is the uniform distribution on (a, b); R_{max} = the overall maximum rate to be permitted by the generator; and R_{min} = the minimum rate to be permitted.

Step 2, Determine MCG Key Interest Rate: Let

$$\bar{I}_{k,t} = \text{MCG 20-year interest rate at time } t \text{ for scenario } k;$$

[2] One hundred different scenarios were run.

$\bar{I}_{k,0}$ = Current 20-year interest rate; and

ρ = Maximum annual proportional change allowed.

The MCG interest rate at time t for scenario k is defined to be

$$\bar{I}_{k,t} = \max\{(1-\rho)\bar{I}_{k,t-1},\ \min[(1+\rho)\bar{I}_{k,t-1},\ 0.015 + U_{k,t}]\} \quad (1)$$

for $t = 1, 2, \ldots, 30$. The parameters R_{max}, R_{min}, and ρ are determined by the user. Christiansen (1992, p. 127) uses the following values:[3]

$$\rho = 0.20, \quad R_{max} = 0.25, \quad R_{min} = 0.03.$$

Shape Codes: The shape of the spot yield curve is determined by the assumed shape of the curve. Seven shapes are identified from historical spot curves, and a symmetric envelope of shapes is identified. The shapes are coded from 1 for a steep normal (upward sloping) yield curve to 7 for a steeply inverted yield curve (see Figure 1). The user determines the code that best represents the shape of the original curve. The shape codes for each scenario then are determined iteratively. The original yield curve is used for all scenarios for time 0.

Step 3, Get Random Walk Matrix: Get the original random walk matrix of transition probabilities of moving from one shape yield curve to another shape yield curve from Table 1.

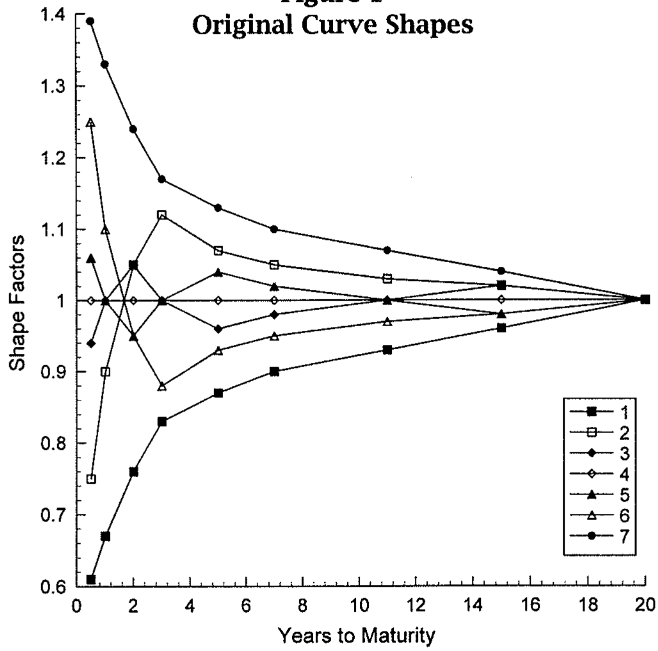
Step 4, Determine Shape Code: The random walk matrix then is translated into a *look-up matrix* containing the shape code for the previous curve; see Table 2. The shape code for the current curve under scenario k is determined as follows: let $\mathbf{B}_k = (b_{k,1}, b_{k,2}, \ldots, b_{k,30})$ be a 1×30 vector[4] of random integers between 1 and 10 inclusive and $\alpha_{k,t}$ be the shape code for scenario k at time t. In addition, let $\mathbf{L}$ be the look-up matrix, i.e., $\mathbf{L}$ is a 10×7 matrix with the ith row of $\mathbf{L}$ corresponding to the ith row of Table 2. For example, from Table 2 two elements of $\mathbf{L}$ are $L_{1,7} = 5$ and $L_{10,4} = 6$. The shape code, $\alpha_{k,t}$, is found from the look-up matrix at the intersection of the column headed by the shape code for time $t-1$ and the row containing $b_{k,t}$, that is,

$$\alpha_{k,t} = L_{i,j} \text{ where } i = b_{k,t} \text{ and } j = \alpha_{k,t-1}. \quad (2)$$

[3]In this paper ρ is used instead of *ADJRMAX*, which is the original notation of Christiansen.

[4]Throughout this paper, only vectors and matrices are denoted in bold uppercase letters. Their elements are written in corresponding nonbold lower case letters. Only scalars are written in nonbold lower case letters.

Figure 1
Original Curve Shapes



Legend: (1) = steep normal; (2) = early peak; (3) = oscillating starting up; (4) = level; (5) = oscillating starting down; (6) = early valley; and (7) = steeply inverted.

Step 5, Determine the Intermediate Yield Curve $\hat{\mathbf{I}}_{k,t}$: Each shape code $\alpha_{k,t}$ is associated with a vector of shape factors corresponding to points of maturity on the yield curve. In particular, $\alpha_{k,t}$ generates the row number of the factor shape table.

Let Ω be the set of maturity points given by

$$\Omega = \{1/2, 1, 2, 3, 5, 7, 11, 15, \text{ and } 20 \text{ years}\}$$

and let $\mathbf{G}_{k,t}$ be the 1×9 vector of shape factors generated by $\alpha_{k,t}$ for the maturities of Ω, i.e.,

$$\mathbf{G}_{k,t} = (g_{k,t}(1/2), g_{k,t}(1), \ldots, g_{k,t}(m), \ldots, g_{k,t}(20)) \qquad (3)$$

for $m \in \Omega$. The factors for 20-year rates are given in Table 3. The product of these factors and the 20-year rate is the intermediate yield curve (a vector of rates), $\hat{\mathbf{I}}_{k,t}$, i.e.,

$$\hat{\mathbf{I}}_{k,t} = \bar{I}_{k,20}\,\mathbf{G}_{k,t}.$$

Thus, for a specific maturity $m \in \Omega$, we have the intermediate rate is given by

$$\hat{I}_{k,t}(m) = \bar{I}_{k,20} \times g_{k,t}(m). \tag{4}$$

Step 6, Determine the Final Yield Curve $\mathbf{I}_{k,t}$: The yield curve of final interest rates, $\mathbf{I}_{k,t}$ are determined after checking the following conditions: (i) for each maturity m, the rates do not vary from the previous years' rate by more than a preset fraction ($\delta(m)$) of the previous rate; and (ii) the rates are subject to the overall maximum and minimum of R_{max} and R_{min}, respectively. For each maturity, the final interest rates are given by

$$\begin{aligned} I_{k,t}(m) &= \max\{R_{min}, (1-\delta(m))I_{k,t-1}(m), \min[\hat{I}_{k,t}(m), \\ &\quad (1+\delta(m))I_{k,t-1}(m), R_{max}]\} \end{aligned} \tag{5}$$

for $m \in \Omega$. (Note that in Christiansen's original model, $\delta(m)$ is a constant, i.e., independent of m.)

Table 1
Transition Probabilities of the Original Random Walk Matrix

Old Shape	New Shape 1	2	3	4	5	6	7
1	0.5	0.3	0.2	0	0	0	0
2	0.4	0.3	0.2	0.1	0	0	0
3	0.2	0.3	0.2	0.2	0.1	0	0
4	0	0.2	0.3	0.2	0.2	0.1	0
5	0	0	0.3	0.3	0.2	0.1	0.1
6	0	0	0	0.3	0.3	0.2	0.2
7	0	0	0	0	0.3	0.5	0.2

From Christiansen, S.L. "A Practical Guide to Interest Rate Generators for C-3 Risk Analysis." *Transactions of the Society of Actuaries* XLIV (1992): 101-134.

Table 2

Look-up Matrix

Random Number	Previous shape 1	2	3	4	5	6	7
1	1	1	1	2	3	4	5
2	1	1	1	2	3	4	5
3	1	1	2	3	3	4	5
4	1	1	2	3	4	5	6
5	1	2	2	3	4	5	6
6	2	2	3	4	4	5	6
7	2	2	3	4	5	6	6
8	2	3	4	5	5	6	6
9	3	3	4	5	6	7	7
10	3	4	5	6	7	7	7

From Christiansen, S.L. "A Practical Guide to Interest Rate Generators for C-3 Risk Analysis." *Transactions of the Society of Actuaries* XLIV (1992): 101–134. Copyright ©1992, the Society of Actuaries. Reprinted with permission.

Table 3

Factors for Shapes: 20-Year Rate

SC	Maturities 0.5	1	2	3	4	5	7	10	15	20
1	0.61	0.67	0.76	0.83	0.85	0.87	0.90	0.92	0.96	1
2	0.75	0.90	1.05	1.12	1.09	1.07	1.05	1.03	1.02	1
3	0.94	1.00	1.05	1.00	0.98	0.96	0.98	1.00	1.02	1
4	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1
4	1.06	1.00	0.95	1.00	1.03	1.04	1.02	1.00	0.98	1
6	1.25	1.10	0.95	0.88	0.91	0.93	0.95	0.97	0.98	1
7	1.39	1.33	1.24	1.17	1.15	1.13	1.10	1.07	1.04	1

From Christiansen, S.L. "A Practical Guide to Interest Rate Generators for C-3 Risk Analysis." *Transactions of the Society of Actuaries* XLIV (1992): 101–134. Copyright ©1992, the Society of Actuaries. Reprinted with permission.

Note: SC = Shape Code.

3 Improvements to the Original Model

We now discuss the improvements that have been made to Christiansen's (1992) model since the original research was done in 1989–1990.

3.1 Determination of the Key Rate Scenarios

The assumption of a uniform distribution of the 20-year rates is replaced by an updated form of the log-normal mean-reverting process. This process is used for the composite generator in Christiansen (1992). Though the uniform distribution appears to be effective in keeping rates in bounds without clustering at the bounds, there is no evidence in favor of a uniform distribution of interest rates. Doll (1991) does provide support, however, for the use of a mean reversionary interest rate generator.

All mean-reverting interest rate generators require a mean rate for the rate being modeled. It is considered desirable to have this mean be independent of the model's user. Murphy (1990, chapter 9, page 77) implies that the best goal is the current rate. This choice ensures that the mean rate is independent of the user. It also ensures that key rates are not biased upward when rates appear to be low or downward if recent history has periods of high rates.

For the kth scenario, the correction factor[5] at time t, $CF_{k,t}$, for the mean-reversionary process selected is a parameterization of the one used by Jetton (1990). It is given by:

$$CF_{k,t} = \text{sign}(D) \times SF_{k,t} \times \min(0.5|D|, 0.015|D^3|)$$

where

$$\text{sign}(D) = \begin{cases} 1 & \text{if } D > 0; \\ 0 & \text{if } D = 0; \\ -1 & \text{if } D < 0, \end{cases}$$

D = mean rate − previous rate, and $SF_{k,t}$ is the strength factor[6] at time t. The goal rate and the previous rate must be expressed as percentages,

[5]A correction factor is used to bring outliers partially back toward the mean and to prevent the repeated use of normal random variables with mean zero and variance one ($N(0,1)$) from becoming $N(0,n)$, which leads to unreasonable interest rates.

[6]The strength factor is a number from 0 to 1 which impacts how fast the correction factor pulls back toward the mean. If the strength factor is too high, there is insufficient volatility in the rates, while a strength factor of 0 eliminates the mean-reversionary process and leaves a pure log-normal interest rate generator.

i.e., 6 percent is entered as 6 and not as 0.06. The strength factor may be constant or vary with time. The result is used as the mean-reversionary correction factor in the key rate calculation. Thus, for the kth scenario, the current mean-reversionary interest rate, $MRI_{k,t+1}$, is given by

$$MRI_{k,t+1} = (MRI_{k,t} + CF_{k,t})e^{\sigma Z_{k,t}} \tag{6}$$

where $MRI_{k,t}$ is the previous mean-reversionary interest rate, σ is the volatility assumption, and $Z_{k,t}$ is a random number from the standard normal distribution.

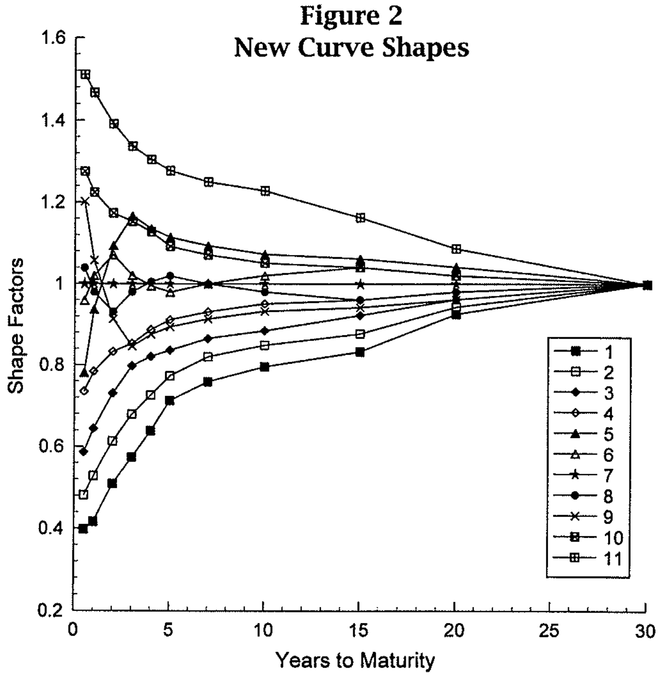
3.2 The Number of Shapes

The original seven shape factors include both a steep normal curve and a steep inverted curve and use maturities of 1/2, 1, 2, 3, 5, 7, 11, 15, and 20 years. Originally, there were no codes or probabilities for normal or inverted curves that were not steep, as it was thought that these shapes would result from the annual limitation on the permitted annual change. It became desirable to have shape codes and factors for the more usual forms of the normal curves and inverted curves, however, thus the increase to nine shapes. Also, the maturities used to specify the yield curve were changed to 1/2, 1, 2, 3, 4, 5, 7, 10, 15, 20, and 30 years because Barra[7] changed the points that it specifies on its curves. In addition, there was a desire for the 30-year rate.

As the current curve steepened in 1993, it became clear that the steep normal curve is not sufficiently steep. Concern arose when it was noticed that the shortest rates always increased in the first year in all of the random scenarios. It was discovered that the product of the shape factor for the (then) steepest curve with the lowest possible long rate was above the current short rate! Part of the cause of this problem is the increase in the length of the curve from a 20-year maturity to a 30-year maturity and using a 30-year key rate. Two possible solutions were considered: increasing the permitted percentage change in the long rate or increasing the number of shapes. Increasing the permitted change permitted almost no decreases in the shortest rate, so this appeared not to be a viable solution and was discarded.

The second possible solution, increasing the number of shapes, involved a study of what the steepest shape should be. A historical database of curves was consulted, and the curves were examined on a factor basis. As historically there are steeper curves than the steepest shape in the generator, two more very steep positively sloped curve

[7]Barra is an international financial consulting firm.



Legend: (1) = very steep normal; (2) = another very steep normal; (3) steep normal; (4) = normal; (5) = early peak; (6) = oscillating starting up; (7) = level; (8) = oscillating starting down; (9) = early valley; (10) = inverted; and (11) = steeply inverted.

shapes have been added to the generator, bringing the total number of curve shapes to 11. Adding new curve shapes solved the problem, i.e., the short rates now are allowed to decrease in the first year. A decrease in short rates does not happen often, as it still requires a drop in the long rate along with a shape code for the following rate that does not exceed that of the original curve. The possible shapes are no longer perfectly symmetric, as it was neither necessary nor desirable to add more steeply inverted curves. Figure 2 displays these new curves.

3.3 Permitted Annual Changes

The annual permitted change in rates originally consisted of two parameters: the permitted annual change in the long (key) rate and the permitted annual change in any rate. The smaller parameter is the permitted annual change in the key rate, and the larger parameter is the permitted annual change in any rate. The new version has a permitted annual change for each rate, which, like the new volatility assumption, is a non-increasing function of the maturity. The requirement for an appropriate volatility assumption for any choice of the key rate led to the new volatility assumption.

3.4 Overall Maximum and Minimum Rates

The overall minimum and maximum rates are specified as parameters. Due to the lower interest rates in 1992 and 1993, regulators expressed concern that the minimum interest rate was too high. The regulators proposed a new dynamic minimum rate for the New York seven interest rate scenarios.[8] The revised Markov chain generator made minor modifications to the new dynamic minimum rate for these scenarios. For assets other than Treasuries, the minimum is the lesser of 3.5 percent or 50 percent of the original rate for that maturity. For Treasury curves the minimum is the minimum of 2.5 percent and 50 percent of the original rate. The maximum rate remains level.

3.5 Frequency of Shapes

When the historical database of shapes was examined for steepness and for the changes of shape, it was discovered that the yield curve shifted in 1981 from an inverted yield curve to a normal curve and back again on a quarterly basis. Thus, it is too restrictive to permit only an annual change in shape codes of two or less. Therefore, in addition to increasing the number of shapes, the random walk matrix is restructured so that it is possible to move from one shape to any other shape. Many investment officers feel that some of the shapes (early peak and early valley) occur very rarely. Because it is desired to be as realistic as possible, the random walk matrix currently reflects the fact that these shapes are more rare than inverted curves.

The sum of probabilities assigned to the inverted curve shapes (8, 9, 10, and 11) is approximately the probability of an inverted curve as determined by Becker (1992). Becker determines inversions by using

[8]The New York seven interest rate scenarios are the scenarios specified in New York Regulation 126. They are the following parallel shift scenarios: (1) no change; (2) rates rise 0.5 percent per year for the next ten years and then remain constant; (3) rates rise 1 percent per year for five years, then fall 1 percent per year for the next five years, and remain constant thereafter; (4) rates rise 3 percent in the first year and then remain constant. Scenarios 5-7 are the opposite of 2-4 rates falling instead of rising and rising instead of falling.

the ratio of the shortest rate to the 10-year rate; he considers a curve to be inverted if this ratio is at least 1.05. Because Becker does not look at the shape of the entire curve, it is difficult to determine the reasonableness of each individual curve type based on his data.

An algorithm to determine the shape code (at time $t = 0$) for the current curve has been developed. The algorithm requires that all rates be expressed as multiples of the key rate and exploits the idea that the short-term rates are more important in determining the shape than the longer term rates. Details are given in Appendix B.

Salomon Brothers Treasury bond equivalent yield curves,[9] in yield to maturity form, are converted as closely as possible to equivalent spot yield curves. (It is necessary to use one 29-year rate instead of the 30-year rate.) These yield curves are added to the data obtained from Barra in spot yield form. Each of these 338 curves[10] is run through the algorithm to determine its shape code. Eleven of these curves are graphed (in factor form). Figure 3 illustrates real rather than theoretical shapes. Table 4 shows the frequency of curve types contained in the sample of 338 curves.

Table 4

Curve Types and Frequencies

CS	1	2	3	4	5	6	7	8	9	10	11
FQ	3	16	30	107	14	39	69	20	2	36	2

Note: CS = Curve Shape, and FQ = Frequency.

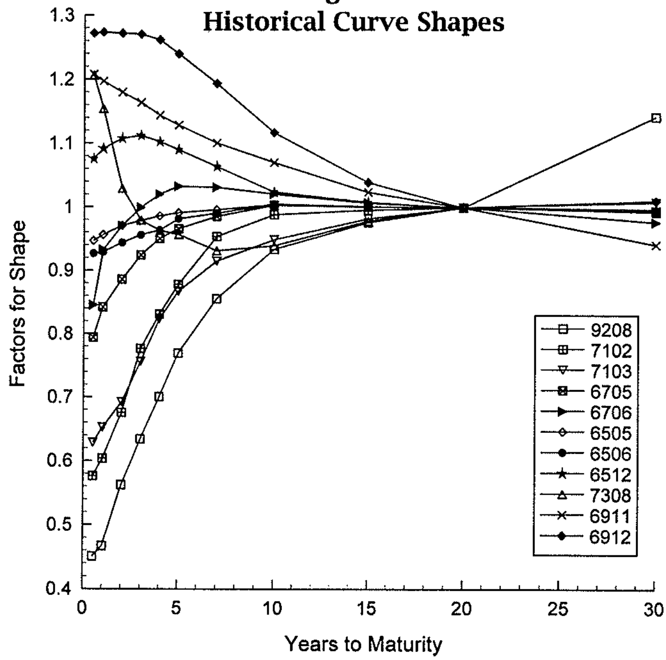
3.6 Key Rate

The key rate is the one for which the original set of rates is generated by the mean-reversionary log-normal process. There are situations where the model is being used and a particular rate other than the long rate may be most important. In such a situation, it is possible to use that particular rate as the key rate. This flexibility requires the imposition of a volatility assumption for each maturity, which impacts both the rate generation process at the key rate (where the volatility assumption is used directly) and the adjustment process (where the permitted deviation from last year's rate varies by the maturity or duration of the rate studied).

[9]Obtained from a database of Salomon Brothers yield curves from February 1965 through March 1993.

[10]One per month from the above database for 28 complete years from February 1965 through January of 1993 plus February and March 1993.

Figure 3
Historical Curve Shapes



Legend: Each four digit number represents the date of the curve in year-month format. The first two digits represent the calendar year, and the last two digits represent the month. For example, 7103 = 71:03 = March 1971; and 6912 = 69:12 = December 1969.

The choice of the key rate determines the rate that will not be biased up or down in the process of generating rates. The key rate sequence is, by construction, equally likely to show increasing rates as decreasing ones. Unless the original yield curve is essentially flat, however, the shape of the original yield curve will impact the other rates. For example, when starting with the 20-year rate as the key rate and a steeply normal yield curve, the shorter rates tend to be higher than the original shorter rate. In the spring/summer of 1992 the yield curve was steeply normal, and using the 20-year rate as the key rate the generator produces 20-year rates that are higher than the original 20-year rate about 50 percent of the time. The 7-year rates produced are higher than or equal to the original 7-year rate about 70 percent of the time, however; see Appendix C.

When using the generator to test pricing, the 7-year rate may be a critical rate. Because one does not want to bias the 7-year rates upwards, the 7-year rate is selected as the key rate. Each new shape factor is the old shape factor divided by that curve's key rate factor. Thus, the shape factors at the new key rate are all one.

When the 7-year rate was used as the key rate, however, the standard deviations of the resulting rates no longer are decreasing with increasing maturity; see Appendix C. This occurs even though the permitted percent change from one year to the next decreases with increasing time and the original volatility assumption is changed to depend on the key rate, and is non-increasing. When the longest rate is the key rate the standard deviations of the resulting rates generally decrease with increasing time until maturity. It does not appear to be possible to have both a key rate that is not the longest rate and to have decreasing standard deviations as the maturities lengthen. This appears to make intuitive sense from Figure 4. If the variation due to shape is squeezed out of one rate, it then must pop up in the others. Thus, there is a trade-off depending on which property is deemed to be the most important.

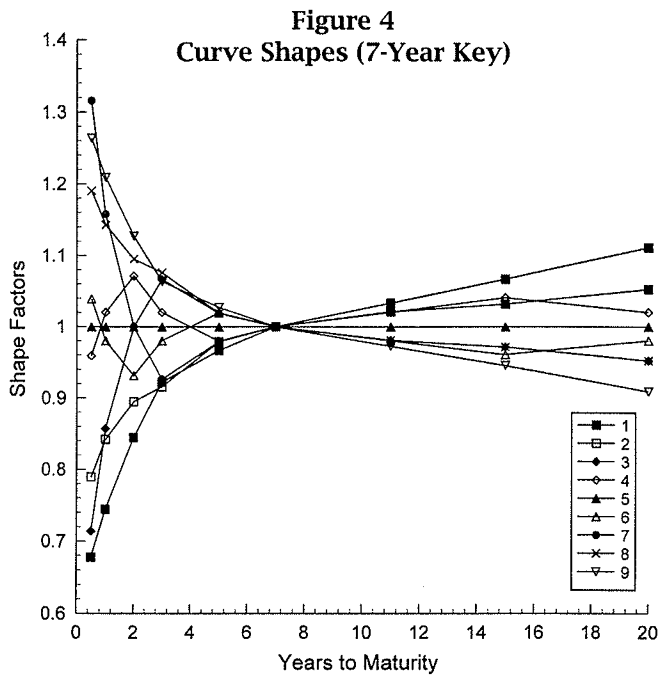
It appears historically that the long-term rate has been the most stable.[11] Murphy (1990, Chapter 17, page 181) implies that the longest maturities have the lowest volatility (over the same time interval). As it is desirable from many standpoints to have decreasing standard deviations with increasing maturity (duration), the longest rate being modeled usually will be the key rate. This does have the effect of biasing the shorter rates: they will tend to go up more than down if the original yield curve is normally sloped and down more than up in the inverted situation.

4 Using the Revised Markov Chain Generator

The steps to follow are essentially the same as those described in Section 2 for the original model. The major difference is that, instead of using uniform random numbers, we use a sequence of standard normal random numbers; the number of shape codes is now 11. Thus, the sequence of interest rates $\{I_{k,t}\}$ at time t for the kth scenario again is determined in six steps.

Step 1, Get Normal Random Sample: For the kth scenario, obtain a sample of 30 standard normal random numbers $Z_{k,t}$ for $t = 1, 2, \ldots, 30$. Using these standard normal random numbers, compute the se-

[11] In the Salomon database, the 30-year rate had the lowest standard deviation.



Legend: (1) steep normal; (2) = normal; (3) = early peak; (4) = oscillating starting up; (5) = level; (6) = oscillating starting down; (7) = early valley; (8) = inverted; and (9) = steeply inverted.

quence of mean-reversionary interest rate, $MRI_{k,t}$, using equation (??).

Step 2, Determine MCG Interest Rate: The MCG 30-year interest rate at time t for scenario k, $\bar{I}_{k,t}$, is defined to be

$$\bar{I}_{k,t} = \max\{(1-\rho)\bar{I}_{k,t-1},\ \min[(1+\rho)\bar{I}_{k,t-1},\ MRI_{k,t}]\} \quad (7)$$

for $t = 1, 2, \ldots, 30$, where $\bar{I}_{k,0}$ = the current 30-year interest rate.

Step 3, Get Random Walk Matrix: Get the new random walk matrix of transition probabilities from Table 5.

Table 5

Random Walk Probabilities* for Shape Codes

Original	New Shape										
Shape	1	2	3	4	5	6	7	8	9	10	11
1	0.220	0.180	0.180	0.180	0.060	0.060	0.050	0.030	0.015	0.020	0.005
2	0.120	0.250	0.250	0.120	0.060	0.060	0.050	0.030	0.020	0.030	0.010
3	0.100	0.120	0.315	0.180	0.060	0.060	0.060	0.030	0.020	0.040	0.015
4	0.090	0.110	0.170	0.330	0.060	0.070	0.060	0.030	0.020	0.040	0.020
5	0.090	0.110	0.140	0.250	0.100	0.100	0.070	0.050	0.020	0.050	0.020
6	0.080	0.100	0.140	0.240	0.050	0.150	0.090	0.050	0.030	0.040	0.030
7	0.060	0.090	0.140	0.240	0.050	0.150	0.120	0.050	0.030	0.040	0.030
8	0.040	0.080	0.120	0.210	0.050	0.150	0.120	0.080	0.040	0.070	0.040
9	0.020	0.060	0.090	0.170	0.040	0.150	0.150	0.100	0.050	0.110	0.060
10	0.020	0.050	0.070	0.150	0.030	0.110	0.150	0.100	0.050	0.170	0.100
11	0.010	0.030	0.050	0.130	0.020	0.100	0.150	0.090	0.070	0.200	0.150

*The probabilities along the rows sum to one.

Step 4, Determine Shape Code: Let $\mathbf{B}_k = (b_{k,1}, b_{k,2}, \ldots, b_{k,30})$ again denote a vector of uniform random numbers between 0 and 1, and let $l = \alpha_{k,t-1}$ be the shape code for scenario k at time $t-1$.

Select row l from the random walk matrix found in Step 4. Let $\mathbf{J}(l) = (j_1(l), j_2(l), \ldots, j_{11}(l))$ denote the vector of cumulative sums of the 11 elements of row l. (Note $j_1(l)$ = first element of row l and $j_{11}(l) = 1$.) Given $b_{k,t}$, the new shape code $\alpha_{k,t}$ is given by

$$\alpha_{k,t} = \min\{\alpha : j_\alpha(l) \geq b_{k,t}\}. \tag{8}$$

The shape code for all scenarios at time $t = 0$ is determined by the algorithm given in Appendix B.

New Shape Factors and Set of Maturities: Here the shape factors associated with $\alpha_{k,t}$ for 30-year rates are given in Table A1 in Appendix A. The new set of maturities is now

$$\Omega = \{1/2, 1, 2, 3, 5, 7, 11, 15, 20 \text{ and } 30 \text{ years}\}.$$

Steps 5 and 6: Given the new shape factors and set of maturities, Ω, follow Steps 5 and 6 in Section 2.

5 Spot, Forward, and Yield to Maturity Rates

5.1 Positive Spot Rates and Forward Rates

Given the way this model is constructed, it produces only positive spot rates[12] because every rate on the yield curve is the product of a positive long-term rate and a positive factor. Tilley (1992) points out that positive spot rates imply positive yield to maturity (coupon) rates, but not necessarily positive forward rates.[13] Are any of the forward rates produced by this generator negative?

The relationship between spot rates and forward rates is as follows: let s_n denote the n-year spot rate and let f_n be the 1-year forward rate between years n and $n+1$. It follows that

$$f_n = \frac{(1+s_{n+1})^{n+1}}{(1+s_n)^n} - 1. \tag{9}$$

[12]The *n-year spot rate* is the yield of a zero coupon (pure discount) bond maturing in n years.

[13]The *m-year forward rate* beginning n years from now is the implied rate of interest to lend money n years from now to be repaid in a single payment of principal and interest $m + n$ years from now.

Clearly, for a flat yield curve, the forward rates are equal to the spot rates. When the yield curve has a positive slope between two maturities, the associated forward rate will be positive and greater than the spot rate associated with the shorter maturity, i.e., $f_n > s_n$, thus creating the expectation of rising rates with a positively sloped yield curve. Forward rates are lower than spot rates only when the rate for the shorter maturity is greater than the rate for the longer maturity, i.e., only in a negatively sloped portion of a yield curve.

If some spot rates are unknown, then the forward rate is defined differently. For example, assume that the spot rates s_k and s_n are known (with $k < n$) but the intervening spot rates are unknown. Then the assumption is made that the forward rates between k and n are all equal, giving

$$f_j = \left[\frac{(1+s_n)^n}{(1+s_k)^k}\right]^{1/(n-k)} - 1 \tag{10}$$

for $j = k, k+1, \ldots, n-1$.

It is important that interest rate generators do not produce *negative* forward rates. Testing the theoretical shapes associated with this Markov chain generator produced no negative forward rates. The test was conducted by setting up a spreadsheet that had the factors for the shapes of the curves. When a 30-year rate was input, the spreadsheet computed $(1+s_n)^n$ and successive differences of $(1+s_n)^n$, i.e., $(1+s_{n+1})^{n+1} - (1+s_n)^n$. From equation (??), it is clear that a negative forward rate implies that these successive difference must be negative. All successive differences tested, however, were positive, implying that all forward rates must be positive. The successive differences were smallest when the longest rate was at a minimum. All of the successive differences were positive, even with the 30-year rate as low as 2 percent. Because the curves that were considered to be problematic are those with negative slopes, the effect of any minimum was ignored for this test only.

Further tests were made by considering that the shapes could vary from the theoretical due to the adjustment process, and these tests still did not produce zero or negative forward rates. Using the same spreadsheet, a positively sloped yield curve was set up for time t, at a long-term rate of 3.5 percent, which is the minimum rate for this generator (ignoring minimums at other points). The following period it was assumed that the long rate had dropped by the maximum allowed and the curve had tried to invert. The resulting curve did not have a perfect shape, and this curve was tested to see it the successive powers of $(1+s_n)^n$ were increasing. They increased regardless of the level of the long-term rate (6 percent also was tried, and they increased faster), and each of the shapes of the positive curve was tried for the initial curve. Thus, it appears that this generator will produce only positive forward rates.

5.2 Spot Rates and Yield to Maturity Rates

The relationship between spot rates and yield to maturity rates depends upon the pattern of cash flows. Let $(y_1, y_2, \ldots, y_n)$ be the annual coupon payments from an interest-only bond maturing in n years at par ($1) and let $(s_1, s_2, \ldots, s_n)$ denote the sequence of spot rates. It follows that $y_1 = s_1$ and

$$1 = \frac{1 + y_k}{(1 + s_k)^k} + \sum_{j=1}^{k-1} \frac{y_k}{(1 + s_j)^j} \tag{11}$$

for $k = 2, 3, \ldots, n$. Equation (??) is based on the fact that an interest-only bond due at time k paying y_k annually and returning the principal at maturity should be valued at par using the spot yield curve. If the spot rates are known, the coupons easily can be found by solving equation (??) for y_k, which gives

$$y_k = \frac{[1 - (1 + s_k)^{-k}]}{\sum_{j=1}^{k}(1 + s_j)^{-j}}. \tag{12}$$

If, on the other hand, the coupons are known, then the spot rates can be found easily by solving equation (??) iteratively. Thus, assuming that $(s_1, s_2, \ldots, s_{k-1})$ are known, then

$$s_k = \left[\frac{(1 + y_k)}{1 - y_k \sum_{j=1}^{k-1}(1 + s_j)^{-j}}\right]^{1/k} - 1. \tag{13}$$

These formulas can be modified for interest rates and cash flows m times per year.

When equation (??) with the usual assumptions[14] is applied to the historical yield to maturity curves for curve of December 1981, it yields a complex number for the value of the 30-year spot rate (s_{30}). (Note that Tilley (1992) mentions this problem when generators initially create YTM curves.) Because that result does not make sense, the 29-year rate is substituted.

[14]The usual assumptions are (i) constant forward rates for intervals between specified points; (ii) semi-annual coupons; and (iii) compounding (adjusted for semi-annual payments and rates).

Using equation (??) is the only method available to convert curves when only one curve is available at a time. But the financial services and Wall Street firms do not determine the spot rate curves from the yield to maturity curves. Instead they determine the spot rate curves from the prices of on-the-run Treasuries, which are those priced closest to par. Their methods of fitting the spot rate curves are proprietary.

6 Conclusions

As with any other model, the Markov chain generator model continues to evolve as problems surface and questions arise. In order to further refine this model, a study is being made to reduce the number of scenarios from 1,000 to 50, while preserving the characteristics of the original set of scenarios. This study will test the results of the 50 scenarios and compare them to the original scenarios in cash flow testing.

References

Becker, D.N. "The Frequency of Inversions of the Yield Curve and Historical Data on the Volatility and Level of Interest Rates." *The Financial Reporter* (May 1992): 14-17.

Christiansen, S.L. "A Practical Guide to Interest Rate Generators for C-3 Risk Analysis." *Transactions of the Society of Actuaries* XLIV (1992): 101-134

Doll, D.C. "Interest Rate Mean Reversions: Fact of Fiction?" *The Financial Reporter* (April 1992): 1-2.

Jetton, M.F. "Interest Rate Scenarios." *Transactions of the Society of Actuaries* XL (1990): 423-437.

Murphy, J.E. *The Random Character of Interest Rates.* Chicago, Ill.: Probus Publishing Company, 1990.

Tilley, J.A. "Building Stochastic Interest Rate Generators." *Transactions of the Society of Actuaries* XLIV (1992): 509-538.

Appendix A

Table A1
Factors for Shapes Using 30-Year Rate Key

	Years to Maturity										
Code	0.5	1	2	3	4	5	7	10	15	20	30
1	0.40	0.42	0.51	0.57	0.64	0.71	0.76	0.80	0.83	0.93	1.00
2	0.48	0.53	0.61	0.68	0.73	0.77	0.82	0.85	0.88	0.94	1.00
3	0.59	0.64	0.73	0.80	0.82	0.84	0.87	0.88	0.92	0.96	1.00
4	0.74	0.78	0.83	0.85	0.89	0.91	0.93	0.95	0.96	0.98	1.00
5	0.78	0.94	1.09	1.17	1.14	1.11	1.09	1.07	1.06	1.04	1.00
6	0.96	1.02	1.07	1.02	0.99	0.98	1.00	1.02	1.04	1.02	1.00
7	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
8	1.04	0.98	0.93	0.98	1.00	1.02	1.00	0.98	0.96	0.98	1.00
9	1.20	1.06	0.91	0.85	0.88	0.89	0.91	0.93	0.94	0.96	1.00
10	1.28	1.22	1.17	1.15	1.13	1.09	1.07	1.05	1.04	1.02	1.00
11	1.51	1.47	1.39	1.34	1.30	1.28	1.25	1.23	1.16	1.09	1.00

Table A2
New Adjustment and Volatility Assumptions

Maturity	0.5 yr.	1 yr.	2 yr.	3 yr.	4 yr.	5 yr.
Max. Adjust.	0.400	0.350	0.330	0.300	0.270	0.250
Volatility	0.170	0.165	0.160	0.140	0.130	0.120
Maturity	5 yr.	7 yr.	10 yr.	15 yr.	20 yr.	30 yr.
Max. Adjust.	0.250	0.200	0.160	0.130	0.100	0.100
Volatility	0.120	0.115	0.110	0.100	0.100	0.095

Appendix B: Algorithm to Determine Curve Shape

Let α be a number denoting the shape of the curve **C**, where **C** is the 1×11 vector of the original rates. Also, let **M** be an 11×11 matrix of shape factors and $\mathbf{M}_i$ be the vector corresponds to curve i (the ith row of **M**) for $i = 1, 2, 3, \ldots 11$. The following is the algorithm to determine the shape of the original curve:

Step 1: Let $\mathbf{R} = \mathbf{C}/s_{30}$ where s_{30} is the 30-year rate.

Step 2: Make an element by element comparison of the vectors **R** and $\mathbf{M}_{11}$. If every element of **R** is greater the corresponding element $\mathbf{M}_{11}$, then set $\alpha = 11$ and go to Step 7. Otherwise, go to Step 3.

Step 3: Define the 1×11 vector **P** and the 11×11 matrices **Q**, **V**, and **W** follows:

$$\mathbf{P} = (0.1, 0.075, 0.05, 0.04, 0.035, 0.03, 0.02, 0.01, 0.005, 0.0025, 0.0);$$

Q is a matrix where each row is **P**;
$\mathbf{V} = \mathbf{M} - \mathbf{Q}$ and $\mathbf{W} = \mathbf{M} + \mathbf{Q}$.

Step 4: Define a new 11×11 matrix **A** as follows:

$$a_{ij} = \begin{cases} 1 & \text{if } v_{ij} \le r_j \le w_{ij} \\ 0 & \text{otherwise} \end{cases}$$

for $i, j = 1, 2, \ldots, 11$.

Step 5: Define the 11×1 column vectors **D** and **E** as follows: $d_i = (12 - i)^3$, $i = 1, 2, \ldots, 11$ and **E** is the product $\mathbf{E} = \mathbf{AD}$.

Step 6: $\alpha = k$ where $k = \max\{i : e_i \geq e_j, \text{ for } j = 1, 2, \ldots, 11\}$, i.e., k is the index with the largest value of the elements of **E**.

Step 7: The shape code has been determined. Quit.

Appendix C: Summary Statistics

The following are summary statistics for all scenarios combined. The projection period is 30 years and all statistics are based on the entire period.

Table A3

Summary Statistics for Scenarios and Key Rates

	7-Year Key Rate		20-Year Key Rate	
	0–499	500–999	0–499	500–999
Mean	3.698	3.677	3.698	3.677
Median	2.000	2.000	2.000	2.000
Std. Dev.	2.686	2.661	2.686	2.661
Minimum	1.000	1.000	1.000	1.000
Maximum	9.000	9.000	9.000	9.000

Note: 0–499 = Scenarios 0–499; and 500–999 = Scenarios 500–999.

Note that:

- For the scenarios 0–499 and the 7-year key rate, the percent of 7-year rates that is greater than the original rate is 49.22 percent, while the percentage equal to the original rate is 3.23 percent;
- For the scenarios 500-999 and the 7-year key rate, the percent of 7-year rates that is greater than the original rate is 46.74 percent, while the percentage equal to the original rate is 3.23 percent;
- For the scenarios 0–499 and the 20-year key rate, the percent of 7-year rates that is greater than the original rate is 68.55 percent, while the percentage equal to the original rate is 3.23 percent; and
- For the scenarios 500-999 and the 20-year key rate, the percent of 7-year rates that is greater than the original rate is 66.46 percent, while the percentage equal to the original rate is 3.23 percent.

Key Rate: 7-Year Rate

Table A4

Scenarios 0–499: Statistics for Rates

Rate	Mean	Median	Std. Dev.	Minimum	Maximum
6 mo.	7.4702	6.9429	2.8535	2.9836	25.0000
1 yr.	7.6977	7.2715	2.7584	2.9836	25.0000
2 yr.	7.9792	7.5970	2.6741	3.4161	25.0000
3 yr.	8.1034	7.7652	2.6267	3.5000	25.0000
5 yr.	8.3570	8.0155	2.6622	3.5000	25.0000
7 yr.	8.4634	8.3282	2.6841	3.5000	25.0000
11 yr.	8.5630	8.4110	2.6947	3.5000	25.0000
15 yr.	8.6241	8.4625	2.6825	3.5000	25.0000
20 yr.	8.7408	8.6612	2.6428	3.5000	25.0000

Table A5

Scenarios 500–999: Statistics for Rates

Rate	Mean	Median	Std. Dev.	Minimum	Maximum
6 mo.	7.3108	6.7879	2.7803	2.9836	25.0000
1 yr.	7.5352	7.1258	2.6902	2.9836	25.0000
2 yr.	7.8172	7.4594	2.6187	3.4161	25.0000
3 yr.	7.9498	7.6245	2.5867	3.5000	24.2782
5 yr.	8.2021	7.9485	2.6299	3.5000	24.4737
7 yr.	8.3079	8.1680	2.6543	3.5000	25.0000
11 yr.	8.4083	8.2645	2.6647	3.5000	25.0000
15 yr.	8.4704	8.3297	2.6565	3.5000	25.0000
20 yr.	8.5976	8.5405	2.6280	3.5000	25.0000

Key Rate: 20-Year Rate

Table A6

Scenarios 0–499: Statistics for Rates

Rate	Mean	Median	Std. Dev.	Minimum	Maximum
6 mo.	8.2880	7.7566	2.8506	2.9836	25.0000
1 yr.	8.5445	8.0697	2.6599	2.9836	23.7613
2 yr.	8.8825	8.5735	2.4800	3.4161	23.5227
3 yr.	9.0386	8.7955	2.3625	3.5000	22.4026
5 yr.	9.3218	9.1213	2.3032	3.5000	22.1785
7 yr.	9.4352	9.2517	2.2665	3.5000	22.1785
11 yr.	9.5517	9.3750	2.2478	3.5000	22.4026
15 yr.	9.6199	9.4401	2.2387	3.5000	22.8506
20 yr.	9.7390	9.7140	2.2487	3.6004	22.4026

Table A7

Scenarios 500–999: Statistics for Rates

Rate	Mean	Median	Std. Dev.	Minimum	Maximum
6 mo.	8.1452	7.6545	2.7491	2.9836	25.0000
1 yr.	8.4010	8.0557	2.5637	2.9836	25.0000
2 yr.	8.7416	8.4416	2.3945	3.4161	24.7305
3 yr.	8.9069	8.6571	2.2933	3.5000	23.3344
5 yr.	9.1903	8.9933	2.2458	3.5000	22.5367
7 yr.	9.3034	9.1303	2.2124	3.5000	21.9435
11 yr.	9.4202	9.2455	2.1970	3.5000	22.4054
15 yr.	9.4900	9.3163	2.1912	3.5000	22.6364
20 yr.	9.6128	9.6045	2.2095	3.5000	23.0984