1996

Pension Funding by Normal Costs or Amortization of Unfunded Liabilities

Keith P. Sharp
University of Waterloo, Canada, 73231.16@compuserve.com

Journal of Actuarial Practice Vol. 4, No. 2, 1996

Pension Funding by Normal Costs or Amortization of Unfunded Liabilities

Keith P. Sharp*

Abstract†

We discuss the extent of the actuary's freedom in choosing the funding method for defined benefit pension plans. In particular, we look at funding through a combination of normal costs, amortization of an unfunded liabilities, and fund of assets. The IRS constraint on "reasonable funding methods" is considered, with particular mention of the aggregate entry age normal method. In addition, an algebraic development is performed of year-to-year changes in the status of a plan's funding.

Key words and phrases: *reasonable funding*

1 Introduction

There are many methods used by actuaries to evaluate the funding of defined benefit pension plans. The choice of funding method is influenced by several factors, including:

- The plan's benefit design; in particular, whether the pension benefit is related to final salary;
- The plan sponsor's objectives;

*Keith Sharp, F.S.A., Ph.D., is an associate professor at the University of Waterloo, Canada. He has worked as an actuary for pension consulting firms in Canada and Australia and for an insurance company in Britain. His papers have appeared in various journals, including *Transactions of the Society of Actuaries, Insurance: Mathematics and Economics* and *Journal of Risk and Insurance.*

Dr. Sharp's address is: Department of Statistics & Actuarial Science, University of Waterloo, Waterloo ON N2L 3G1, CANADA. Internet address: *73231.16@compuserve.com*

†The author acknowledges the valuable contributions of the anonymous referees. All errors remain his responsibility.

- The requirements under the appropriate regulatory environment; and
- The traditions of the geographic area and of the actuary's firm.

This paper explores the extent of the actuary's freedom in devising methods for funding benefits and in adjusting contributions to take account of gains and losses and plan improvements. A particular constraint considered is the IRS requirement for "reasonable funding methods" to be used. Details of the mathematical characteristics of such reasonable funding methods are given in Appendices A through E.

Appendix A considers the definition of accrual (actuarial) liability for benefit allocation methods and shows the equivalence of the present value of accrued benefits and the $AL = PVFB - PVFNC$ definitions. Appendix B considers the frozen initial liability and aggregate methods, with their definitions of unfunded liability. Appendix C gives a more thorough confirmation that the benefit allocation methods adhere to the zero-gain criterion. Appendix D indicates that the individual level cost methods, too, satisfy the zero-gain criterion. Appendix E discusses the non-individual methods: in other words those in which the numerator and the denominator defining normal cost are separately summed over plan members. Thus, the frozen initial liability and aggregate methods are considered in Appendix E. Finally, Appendix F contains a numerical example.

2 Benefit Value as a Sum of Components

Fundamental to the actuarial valuation of a pension plan is that the actuary must ensure that the present value of projected future benefits at any time $t (PVFB_t)$ be balanced by the sum of the plan's assets of various types. Available assets (tangible and intangible) for a valuation at time t are:

- F_t, the fund of tangible invested assets at actuarial value, possibly a smoothed market value at time t;
- $PVFNC_t$, the present value at time t of future normal costs for plan members at the valuation at t, based on their normal costs (NC_t) calculated at that valuation at time t;
- UAL_t, the unfunded actuarial liability at time t. It is based on the initial unfunded actuarial liability UAL_0, which is amortized by level dollar annual payments $UAL_0 / \ddot{a}_{\overline{n}|}$. As a result, the UAL can

be regarded as the intangible asset of the present value of future amortization payments.

It follows that the equation of value that must be satisfied by any method for the entire pension plan is

$$PVFB_t = PVFNC_t + UAL_t + F_t. \tag{1}$$

Substituting $UAL_t = AL_t - F_t$ gives the usual expression for the plan's accrued liability at t:

$$AL_t = PVFB_t - PVFNC_t.$$

Thus, for a funding arrangement to be satisfactory, it is necessary but not sufficient for equation (1) to hold. This is considered in more detail in Section 3.

Because we need notation to allow for the various versions of quantities at any given time, the notation described in Table 1 is used for quantities at time t. Note, all quantities refer to the sum over plan members.

Some actuaries may prefer that calculations be done based on calculating the cost of plan modifications on the *revised* assumptions rather than on the previous assumptions. The results of this paper can be readily modified by regarding M as denoting modified assumptions and R as denoting a revised plan.

Consider column (2) of Table 2. The time $t-1$ plan normal cost NC_{t-1}^R is based on the time $t-1$ revised assumptions and on the time $t-1$ plan document with any amending modifications. Making the assumption that normal costs are payable at the beginning of the year, we have for the whole plan

$$PVFNC_{t-1}^R \times (1+i) = NC_{t-1}^R \times (1+i) + PVFNC_t^E \tag{2}$$

where $PVFNC_t^E$ is the notation for the present value at time t of the normal costs expected at time $t-1$. The quantities denoted by E at time t are the same as those denoted by R at time $t-1$. The validity of equation (2) is apparent for any predefined series of payments, including changes in the membership.

Experience may differ from assumed in various areas including the number of terminations and the amount of salary increases. Taking this into account, the time t present value of future normal costs with gains or losses is given by

$$PVFNC_t^G = PVFNC_t^E + {}^{NC}G_t \tag{3}$$

Table 1
Summary of Notation

EOV_{t-1}	=	Equation of value at $t-1$.
${}^{inv}G$	=	Investment component of gain;
Δ^M	=	Change resulting from modification to the plan document;
Δ^R	=	Change resulting from revisions to assumptions;
F_t	=	Invested assets; there may be cash contributed to fund modifications or revisions to assumptions resulting in a new fund F^R.

Superscript Notation

E	=	Expected outcome if time $t-1$ assumptions are realized;
G	=	Actual outcome at time t with inclusion of gains (or losses) since the previous valuation, assumed to be at $t-1$;
M	=	Includes modifications effective at time t to the plan document;
R	=	Includes revisions effective at time t to the actuarial assumptions; and
A	=	Any one of the above E, G, M, or R;

where ${}^{NC}G_t$ is the portion of gain related to changes in the payroll on which the normal cost is calculated. Plan modifications at time t may cause another change $\Delta^M PVFNC_t$ to give the quantity $PVFNC^M$ including modifications:

$$PVFNC_t^M = PVFNC_t^G + \Delta^M PVFNC_t. \tag{4}$$

Similarly, including the effect of assumptions revised as of time t, we have

$$PVFNC_t^R = PVFNC_t^M + \Delta^R PVFNC_t. \tag{5}$$

Thus, we have confirmed column (2) of Table 2.

Column (3) of Table 2 indicates the development of the unfunded accrued (actuarial) liability over time. Changes in the unfunded may result from:

Table 2
Development of Asset Components of *PVFB*
For the Entire Pension Plan

	Assets		
Description (1)	Intangible (2)	Intangible (3)	Invested (4)
EOV_{t-1}	$PVFNC^R_{t-1}$	UAL^R_{t-1}	F^R_{t-1}
$(1+i)EOV_{t-1}$	$PVFNC^R_{t-1}(1+i)$	$UAL^R_{t-1}(1+i)$	$F^R_{t-1}(1+i)$
Contribution:	$-NC^R_{t-1}(1+i)$	$-({}^iC_{t-1} - NC^R_{t-1}(1+i))$	$+{}^iC_{t-1}$
Benefits:			$-{}^iB_{t-1}$
Sub-Total:	$PVFNC^E_t$	UAL^E_t	F^E_t
Gain	$+{}^{NC}G_t$	$-{}^{tot}G_t$	$+{}^{inv}G_t$
Sub-Total:	$PVFNC^G_t$	UAL^G_t	F^G_t
Modifications:	$+\Delta^M PVFNC_t$	$+\Delta^M UAL_t$	$+\Delta^M F_t$
Sub-Total:	$PVFNC^M_t$	UAL^M_t	F^M_t
Revisions:	$+\Delta^R PVFNC_t$	$+\Delta^R UAL_t$	$+\Delta^R F_t$
Total:	$PVFNC^R_t$	UAL^R_t	F^R_t

EOV_{t-1} = Equation of value at $t-1$; Column (2) = Equations (2) to (5); Column (3) = Equation (11); Column (4) = Equation (12).

- Experience gains or losses;
- Plan amendments;
- Plan inception (a special case of a plan amendment); or
- Changes in assumptions

as of $t-1$ or an earlier date. The unfunded liability may be under amortization and be regarded as an intangible asset equal to the present value of the scheduled amortization payments.

Following the notation used by Aitken (1994, p. 150), ${}^iC_{t-1}$ denotes the actual contributions for the year $[t-1, t)$ accumulated to t at the assumed rate i. Also, ${}^iC_{t-1} - NC^R_{t-1}(1+i)$ is regarded as a supplemental cost (McGill and Grubbs, 1989), which reduces the unfunded liability:

$$UAL^E_t = UAL^R_{t-1}(1+i) - ({}^iC_{t-1} - NC^R_{t-1}(1+i)). \qquad (6)$$

The unfunded liability at t may be the sum of several previous unfunded liabilities that are being amortized over different periods. The unfunded liability may consist only of the n year level dollar amortization of an unfunded liability detected at time T. Then we may have level contributions

$$^{i}C_t = (\frac{UAL^T}{\ddot{a}_{\overline{n}|}} + NC_t)(1+i) \tag{7}$$

and the amortization of equation (6) proceeds as

$$UAL_t^T = (UAL_{t-1}^T - \frac{UAL^T}{\ddot{a}_{\overline{n}|}})(1+i). \tag{8}$$

This special case thus gives the familiar formula for a level dollar amortization of the component UAL^T of the unfunded liability:

$$UAL_t^T = UAL_T^T \frac{\ddot{a}_{\overline{n-t+T}|}}{\ddot{a}_{\overline{n}|}}. \tag{9}$$

Let us now consider the gain ^{tot}G.[1] Thus the end of year unfunded is $UAL_{t-1}(1+i)$ reduced by the degree $(^{i}C_{t-1} - NC_{t-1}^R(1+i))$ to which actual contributions exceed that normal cost.

The end of year unfunded is further reduced by any total (i.e., investment, decrements etc.) gain $^{tot}G_t$ to give

$$UAL_t^G = (UAL_{t-1}^R + NC_{t-1}^R)(1+i) - {}^{i}C_{t-1} - {}^{tot}G_t. \tag{10}$$

Equation (10) corresponds to the top four cells of column (3) of Table 2. It is often expressed as a formula for the gain, taken to the left side. It can be approached from various directions. (See e.g., Aitken, 1994, p. 157, and Anderson, 1992, p. 13.) The expression UAL_t^G indicates the time t balance after gains or losses but before any time t amendments or changes in assumptions. Such changes add amounts $\Delta^M UAL_t$ and $\Delta^R UAL_t$ respectively to give column (3) of Table 2:

$$\begin{aligned} UAL_t^R &= UAL_{t-1}^R(1+i) - (^{i}C_{t-1} - NC_{t-1}^R(1+i)) - {}^{tot}G_t \\ &\quad + \Delta^M UAL_t + \Delta^R UAL_t. \end{aligned} \tag{11}$$

[1]The gain ^{tot}G can be regarded as the amount by which the actual end of year unfunded is less than the expected (if all assumptions were realized) end of year unfunded. In addition, the normal cost can be defined as the contribution that would result in the unfunded normally growing with interest. Here *normally* is interpreted as all assumptions being realized.

Table 3
Development of Components of *PVFB*
For the Entire Pension Plan

(1)	(2)	(3)
EOV_{t-1}	$= PVFB^R_{t-1}$	AL^R_{t-1}
$(1+i)EOV_{t-1}$	$PVFB^R_{t-1}(1+i)$	$AL^R_{t-1}(1+i)$
Contribution:	$+0$	$+NC^R_{t-1}(1+i)$
Benefits:	$-{}^iB_{t-1}$	$-{}^iB_{t-1}$
Sub-Total:	$PVFB^E_t$	AL^E_t
Gain	$-({}^{tot}G_t - {}^{inv}G_t - {}^{NC}G_t)$	$-({}^{tot}G_t - {}^{inv}G_t)$
Sub-Total:	$PVFB^G_t$	AL^G_t
Modifications:	$+\Delta^M PVFB_t$	$+\Delta^M AL_t$
Sub-Total:	$PVFB^M_t$	AL^M_t
Revisions:	$+\Delta^R PVFB_t$	$+\Delta^R AL_t$
Total:	$PVFB^R_t$	AL^R_t

EOV_{t-1} = Equation of value at $t-1$; Column (2) = Columns (3) + (4) of Table 1; Column (3) = Columns (2) + (3) + (4) of Table 1.

Column (4) of Table 2 indicates the fund being increased by contributions and reduced by benefits (and expenses if paid by the fund). Interest to the end of the year is calculated at the annual rate. The necessary correction for reality is the investment gain ${}^{inv}G_t$. This investment gain is identical to the excess of interest earned on a savings account over the amount that would have been earned at some assumed rate i. One can allow for the possibility of a lump sum contribution of amount $\Delta^M F_t + \Delta^R F_t$ to give column (4) of Table 2:

$$F^R_t = F^R_{t-1}(1+i) + {}^iC_{t-1} - {}^iB_{t-1} + {}^{inv}G_t + \Delta^M F_t + \Delta^R F_t. \qquad (12)$$

Column (2) of Table 3 shows the breakdown of the change from $PVFB^R_{t-1}$ to $PVFB^R_t$ and, consistently, is the total of columns (2), (3), and (4) of Table 2. Column (3) of Table 3 gives the development of the accrued liability AL and equals column (2) of Table 3 less column (2) of Table 2; $AL = PVFB - PVFNC$.

3 Desirable Characteristics of a Funding Method

Legal requirements of the applicable jurisdiction must be satisfied together with the code of conduct requirements of the actuary's professional body. Other considerations will also come into play. Among the matters to consider in choosing a funding method are:

- Rate of funding of accruing benefits;
- Speed at which the cost of plan improvements, including plan inception, is amortized;
- Degree to which cost to the employer is level and predictable, perhaps as a percentage of payroll; and
- Degree to which a surplus or unfunded liability is produced.

In recent years there has been much focus on the question of pension plan surplus. The employer may be required to make up, for example, any shortfall of assets on plan termination. But in some jurisdictions (e.g., Ontario, Canada) the employer may have difficulty in recovering any surplus. In view of this one-way bet, some affected employers may tend to favor low rates of contribution even though this reduces the security of benefits.

The speed of funding may have significant consequences. Consider an extreme example that may not be allowed under IRS regulations. Membership includes a highly compensated individual age 64 at valuation at t. Pensions are paid by annuity purchase rather than monthly withdrawals from the fund. An assumed age 65 retirement could, under equation (1), be balanced by normal costs payable over an extended future period. But on the retirement there may be insufficient invested assets to purchase the large required annuity. Thus, in reality, the intangible assets $PVFNC_t^A$ and UAL_t^A cannot always substitute for the invested asset F_t^A. Attention must be paid to the incidence and not only to the present value of the normal cost and amortization payment streams.

4 IRS Reasonable Funding Method

According to §1.412(c)(3) - 1(c)(2) of regulations under the Internal Revenue Code, under a reasonable funding method no experience gains or losses are produced if each actuarial assumption is exactly realized. Below we consider which classes of methods satisfy this zero-gain criterion.

Let us examine the gain for methods that satisfy

$$PVFNC_t^R + UAL_t^R + F_t^R = PVFB_t^R. \tag{13}$$

We assume all plan assumptions are realized; so, for example, the equation

$$[PVFNC_{t-1}^R - NC_{t-1}^R](1+i) = PVFNC_t^G,$$

which corresponds to column (2) of Table 2, is satisfied.

We start from the standard definition of total gain (Aitken, 1994, p. 50, and Anderson, 1992, p. 20)

$$\begin{aligned}
{}^{tot}G_t &= (UAL_{t-1}^R + NC_{t-1}^R)(1+i) - UAL_t^G - {}^iC_{t-1} \\
&= (PVFB_{t-1}^R - PVFNC_{t-1}^R - F_{t-1}^R + NC_{t-1}^R)(1+i) \\
&\quad -(PVFB_t^G - PVFNC_t^G - F_t^G) - {}^iC_{t-1} \\
&= {}^iB_t - {}^iC_{t-1} - F_{t-1}^R(1+i) + F_t^G \\
&= 0.
\end{aligned}$$

In the above expressions for the total gain, we have assumed that there are no plan modifications or revisions to assumptions, and we used column (2) of Table 2 and (2) of Table 3. Thus, we have shown that methods satisfying equation (13) satisfy the zero-gain criterion.

From column (4) of Table 2 one finds that a zero investment gain results if the assumed rate of interest is realized. It can therefore be useful to concentrate on the non-investment portion of gain. From column (3) of Table 3 we can find an expression for the non-investment (or liability) portion of the total gain:

$${}^{tot}G_t - {}^{inv}G_t = AL_{t-1}^R(1+i) + NC_{t-1}^R(1+i) - {}^iB_{t-1} - AL_t^G. \tag{14}$$

Equation (14) can be used to examine whether a method satisfies the zero-gain criterion.

5 Aggregate Entry Age Normal

The plan normal cost under the aggregate entry age normal method is defined (Aitken, 1994, p. 131 and Daskais, 1982) as

$$NC_t^R = n_t \times \frac{\sum_{\mathcal{M}_t} PV_{e_j} FB^R}{\sum_{\mathcal{M}_t} \ddot{a}^R_{e_j:\overline{r-e_j}|}} \tag{15}$$

where n_t is the number of active members at time t, e_j is the plan entry date of employee j and $\mathcal{M}_t$ is the set of active members at time t. $PV_{ej}FB^R$ is the present value at entry age of the benefits of employee j, including the effect of any plan revisions. It is necessary that equation (13) be obeyed so we have an unfunded liability given for the plan by

$$UAL_t^R = PVFB_t^R - PVFNC_t^R - F_t^R. \tag{16}$$

Calculation of $PVFNC_t^R$ is complicated in view of the equation (15) definition of plan normal cost and future changes in membership when retirements occur. (An example of the operation of the method is given in Appendix F.)

Equation (15) is somewhat unusual. In both the numerator and the denominator terms of the form PV_{e_j} are summed over participants. The present values are taken at the entry date of each individual. The participants will in general have different entry dates. Thus the summation is of present values taken at different dates; apples are being added to oranges.

Equation (13) can still be used; this sharing of cost between normal costs, unfunded liability, and fund can be made to continue to function despite the unusual definition of normal cost. A result, however, is that the normal costs calculated each year are projected to be nonlevel (as either dollars or percentage of salary). The costs are nonlevel even if the assumptions are realized and despite the level dollar appearance of equation (15). This allows equation (2) or column 2 of Table 2 to be valid, but is not an acceptable practical situation.

Attempts to fit aggregate entry age normal into a consistent framework while satisfying the zero-gain criterion are explored by Tino and Sypher (1995). Their paper gives a thorough critique of the aggregate entry age normal method and finds it unacceptable.

6 Conclusions

Equation (1) indicates only that the present value of future benefits is split between the present value of future normal costs, the fund, and a balancing item, the unfunded liability. Table 2 indicates concisely the year-to-year development of the three components. All three of these components can be, and often are, varied by making changes in method and assumptions. Then the choice of cost method and asset valuation method can be made to suit the circumstances of regulation and custom.

References

Aitken, W.H. *A Problem-Solving Approach to Pension Funding and Valuation.* Winsted, Conn.: Actex Publications, 1994.

Anderson, A.W. *Pension Mathematics for Actuaries*, 2nd edition. Winsted, Conn.: Actex Publications, 1992.

Daskais, R. *Aggregate Entry-Age Cost Methods of Pension Funding*: Study Note 360-21-91. Schaumburg, Ill.: Society of Actuaries, 1991.

McGill, D.M. and Grubbs, D.S. *Fundamentals of Private Pensions*, 6th edition. Homewood, Ill.: Irwin, 1989.

Tino, P. and Sypher, E. *Variation on Entry Age Normal Cost Methods*: Study Note 360-28-95. Schaumburg, Ill.: Society of Actuaries, 1995.

Appendix A Benefit Allocation Methods: Accrued Liability

Under unit credit methods, it is usual to define the accrued liability AL as the present value of benefits accrued up to the valuation date, equation (A.2) below. For other cost methods the usual definition given is (A.1), $AL_t^A = PVFB_t^A - PVFNC_t^A$. It is demonstrated below that the definitions (A.1) and (A.2) are equivalent in the special case of no pre-retirement decrements. Thus, the present value of future benefits is for benefits at only one age:

$$\begin{aligned}
AL_t^A &= \sum_{\mathcal{M}_t} (PVFB_t^{jA} - PVFNC_t^{jA}) \qquad \text{(A.1)}\\
&= \sum_{\mathcal{M}_t} \frac{D_y}{D_{x_j(t)}} \ddot{a}_y^{(12)} AB_y^j \\
&\quad - \sum_{\mathcal{M}_t} \sum_{s=t}^{y-1} \frac{D_{x_j(s)}}{D_{x_j(t)}} \frac{D_y}{D_{x_j(s)}} \ddot{a}_y^{(12)} (AB_{x_j(s)+1}^j - AB_{x_j(s)}^j) \\
&= \sum_{\mathcal{M}_t} \frac{D_y}{D_{x_j(t)}} \ddot{a}_y^{(12)} AB_y^j \\
&\quad - \sum_{\mathcal{M}_t} \frac{D_y}{D_{x_j(t)}} \ddot{a}_y^{(12)} \sum_{s=t}^{y-1} (AB_{x_j(s)+1}^j - AB_{x_j(s)}^j) \\
&= \sum_{\mathcal{M}_t} \frac{D_y}{D_{x_j(t)}} \ddot{a}_y^{(12)} AB_y^j - \sum_{\mathcal{M}_t} \frac{D_y}{D_{x_j(t)}} \ddot{a}_y^{(12)} (AB_y^j - AB_{x_j(t)}^j)
\end{aligned}$$

$$= \sum_{\mathcal{M}_t} \frac{D_y}{D_{x_j(t)}} \ddot{a}_y^{(12)} AB^j_{x_j(t)}. \tag{A.2}$$

We now consider the more general case of n decrements operating in all years till the latest retirement at age y. We use the notation:

$q^{(k)}_{x_j(z)}$ = Probability of decrement k operating in the year of age $x_j(z)$ through $x_j(z+1)$, conditional on being a plan member at age $x_j(z)$.

${}^{L}AB^{j,(k)}_{x_j(z)}(x_j(s))$ = Portion accrued by age $x_j(s)$ of the lump sum equivalent of the benefit payable on decrement k occurring in the year preceding age $x_j(z)$.

Below it is shown that the expression of accrued liability (A.3) is equal to equation (A.4), the present value of the accrued benefit.

$$
\begin{aligned}
AL^A_t &= \sum_{\mathcal{M}_t} (PVFB^{j,A}_t - PVFNC^{j,A}_t) \qquad\qquad (A.3)\\
&= \sum_{\mathcal{M}_t} \sum_{z=t}^{t+y-1-x_j(t)} \sum_{k=1}^{n} \frac{D_{x_j(z)}}{D_{x_j(t)}} \frac{q^{(k)}_{x_j(z)}\ {}^{L}AB^{j,(k)}_{x_j(z+1)}(x_j(z+1))}{(1+i)} \\
&\quad - \sum_{\mathcal{M}_t} \sum_{s=t}^{t+y-1-x_j(t)} \frac{D_{x_j(s)}}{D_{s_j(t)}} NC^j_{x_j(s)} \\
&= \sum_{\mathcal{M}_t} \sum_{z=t}^{t+y-1-x_j(t)} \sum_{k=1}^{n} \frac{D_{x_j(z)}}{D_{x_j(t)}} \frac{q^{(k)}_{x_j(z)}\ {}^{L}AB^{j,(k)}_{x_j(z+1)}(x_j(z+1))}{(1+i)} \\
&\quad - \sum_{\mathcal{M}_t} \sum_{s=t}^{t+y-1-x_j(t)} \frac{D_{x_j(s)}}{D_{x_j(t)}} \sum_{z=s}^{t+y-1-x_j(t)} \frac{D_{x_j(z)}}{D_{x_j(s)}} \times \\
&\quad \sum_{k=1}^{n} \frac{q^{(k)}_{x_j(z)}}{(1+i)} \left[{}^{L}AB^{j,(k)}_{x_j(z+1)}(x_j(s+1)) - {}^{L}AB^{j,(k)}_{x_j(z+1)}(x_j(s)) \right] \\
&= \sum_{\mathcal{M}_t} \sum_{k=1}^{n} \left\{ \sum_{z=t}^{t+y-1-x_j(t)} \frac{D_{x_j(z)}}{D_{x_j(t)}} \frac{q^{(k)}_{x_j(z)}\ {}^{L}AB^{j,(k)}_{x_j(z+1)}(x_j(z+1))}{(1+i)} \right. \\
&\quad - \sum_{s=t}^{t+y-1-x_j(t)} \sum_{z=s}^{t+y-1-x_j(t)} \frac{D_{x_j(z)}}{D_{x_j(t)}} \frac{q_{x_j(z)}}{(1+i)} \times
\end{aligned}
$$

$$\left[{}^{L}AB^{j,(k)}_{x_j(z+1)}(x_j(s+1)) - {}^{L}AB^{j,(k)}_{x_j(z+1)}(x_j(s))\right] \Bigg\}$$

$$= \sum_{\mathcal{M}_t}\sum_{k=1}^{n}\left\{\sum_{z=t}^{t+y-1-x_j(t)} \frac{D_{x_j(z)}}{D_{x_j(t)}}\frac{q^{(k)}_{x_j(z)}}{(1+i)} \times \right.$$
$$\left[{}^{L}AB^{j,(k)}_{x_j(z+1)}(x_j(z+1)) - \sum_{s=t}^{z}\left[{}^{L}AB^{j,(k)}_{x_j(z+1)}(x_j(s+1))\right.\right.$$
$$\left.\left.\left. - {}^{L}AB^{j,(k)}_{x_j(z+1)}(x_j(s))\right]\right]\right\}$$

$$= \sum_{\mathcal{M}_t}\sum_{k=1}^{n}\left\{\sum_{z=t}^{t+y-1-x_j(t)} \frac{D_{x_j(z)}}{D_{x_j(t)}}\frac{q^{(k)}_{x_j(z)}}{(1+i)} \times \right.$$
$$\left[{}^{L}AB^{j,(k)}_{x_j(z+1)}(x_j(z+1)) - \left[{}^{L}AB^{j,(k)}_{x_j(z+1)}(x_j(z+1))\right.\right.$$
$$\left.\left.\left. - {}^{L}AB^{j,(k)}_{x_j(z+1)}(x_j(t))\right]\right]\right\}$$

$$= \sum_{\mathcal{M}_t}\sum_{k=1}^{n}\sum_{z=t}^{t+y-1-x_j(t)} \frac{D_{x_j(z)}}{D_{x_j(t)}}\frac{q^{(k)}_{x_j(z)}}{(1+i)}\, {}^{L}AB^{j,(k)}_{x_j(z+1)}(x_j(t)) \quad \text{(A.4)}$$

Appendix B Benefit Allocation Methods: Basic Funding Equations

Consider any cost method for which is valid the equation for the whole plan

$$AL^A_t = PVFB^A_t - PVFNC^A_t. \quad \text{(B.1)}$$

This includes the individual level cost methods because the accrued liability for those methods is defined by equation (B.1). It also includes benefit allocation methods because they too satisfy equation (B.1), as is shown in Appendix A.

Equation (B.1) is valid also for any method that satisfies

$$PVFNC^A_t + UAL^A_t + F^A_t = PVFB^A_t \quad \text{(B.2)}$$

and the equation

$$AL^A_t = UAL^A_t + F^A_t. \quad \text{(B.3)}$$

The entry age and attained age versions of the frozen initial liability method satisfy (B.2) as (B.2) is used to define their normal cost (Aitken, 1994, p. 117). Similarly the aggregate method uses (B.2) to define its normal cost with UAL_t set to zero. One could argue that (B.2) must be satisfied by any acceptable cost method. Similarly, for the frozen initial liability methods, equation (B.3) can be used to define the accrued liability (Aitken 1994, p. 117).

The aggregate method satisfies (B.3) when the definitions $UAL_t^A = 0$ and $AL_t^A = F_t^A$ are used. Thus, (B.1) is satisfied by all the usual cost methods; it is used as the usual definition and meaning of accrued (actuarial) liability.

Appendix C Benefit Allocation Methods: Reasonable Funding Method

Let us consider benefit allocation methods such as traditional unit credit and projected unit credit. Under all such methods we have, assuming that the only benefit is on normal retirement,

$$NC_t^G = \sum_{\mathcal{M}_t} \frac{D_y^G}{D_{x_j(t)}^G} \ddot{a}_y^{(12)} (AB_{x_j(t)+1}^{j,G} - AB_{x_j(t)}^{j,G}) \tag{C.1}$$

where $\mathcal{M}_t$ is the set of active members at time t and $AB_{x(t)}^{j,G}$ is the benefit accrued up to the plan year end nearest to age $x(t)$ for member j. The accrued actuarial liability is consistently defined (see Appendix A) as

$$AL_t^G = \sum_{\mathcal{M}_t} \frac{D_y^G}{D_{x_j(t)}^G} \ddot{a}_y^{(12)} AB_{x_j(t)}^{j,G} \tag{C.2}$$

if the only benefits paid are at retirement age y.

Thus, expressing equation (14) as a sum over the members and noting that the basis $R(t-1)$ used for calculating D^R at time $t-1$ is the same as $G(t)$ used for D^G at time t, we have

$$\begin{aligned}
&{}^{tot}G_t - {}^{inv}G_t \\
&= AL_{t-1}^R(1+i) + NC_{t-1}^R(1+i) - {}^iB_{t-1} - AL_t^G \\
&= \sum_{\mathcal{M}_{t-1}} \frac{D_y^R}{D_{x_j(t-1)}^R} \ddot{a}_y^{(12)} AB_{x_j(t-1)}^{j,R}(1+i) \\
&\quad + \sum_{\mathcal{M}_{t-1}} \frac{D_y^R}{D_{x_j(t-1)}^R} \ddot{a}_y^{(12)} (AB_{x_j(t)}^{j,R} - AB_{x_j(t-1)}^{j,R})(1+i) \\
&\quad - \sum_{\mathcal{M}_{t-1}} {}^iB_{t-1}^j - \sum_{\mathcal{M}_t} \frac{D_y^G}{D_{x_j(t)}^G} \ddot{a}_y^{(12)} AB_{x_j(t)}^{j,G}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\mathcal{M}_{t-1}} \frac{D_y^R}{D_{x_j(t-1)}^R} \ddot{a}_y^{(12)} AB_{x_j(t)}^{j,R} (1+i) - \sum_{\mathcal{M}_{t-1}} {}^iB_{t-1}^j \\
&\quad - \sum_{\mathcal{M}_t} \frac{D_y^G}{D_{x_j(t)}^G} \ddot{a}_y^{(12)} AB_{x_j(t)}^{j,G} \\
&= \sum_{\mathcal{M}_{t-1}} \frac{D_y^{R(t-1)}}{D_{x_j(t)}^{R(t-1)}} (1 - q_{x_j(t-1)}^{R(t-1)}) \ddot{a}_y^{(12)} AB_{x_j(t)}^{j,R(t-1)} \\
&\quad - \sum_{\mathcal{M}_{t-1}} {}^iB_{t-1}^j - \sum_{\mathcal{M}_t} \frac{D_y^{G(t)}}{D_{x_j(t)}^{G(t)}} \ddot{a}_y^{(12)} AB_{x_j(t)}^{j,G} \\
&= 0 \qquad \text{(C.3)}
\end{aligned}
$$

if the assumptions at time $t-1$ are realized. The final step, equating to zero, is valid if

- $D^{R(t-1)} = D^{G(t)}$, which is true as mentioned above.
- At all $x(t) < y$, the set of active members $\mathcal{M}_t$ is $\mathcal{M}_{t-1}$ reduced in the proportion $(1 - q_{x_j(t-1)}^{R(t-1)})$, which is true if the assumptions are realized.
- For all active members j, $AB_{x_j(t)}^{j,R(t-1)} = AB_{x_j(t)}^{j,G}$, which is true if the assumptions are realized.
- For those who retire, ${}^iB_{t-1}^j = \ddot{a}_y^{(12)} AB_y^{j,R(t-1)}$, which is true if the assumptions are realized.

Appendix D Individual Level Cost Methods: Reasonable Funding Method

Under the individual level cost methods we have for some age a (e.g., entry if using entry age normal) for an individual j:

$$
NC_t^{j,R} = \frac{PV_{a_j} FB_{x(t)}^{j,R}}{\ddot{a}_{a_j:\overline{r-a_j}|x(t)}} \qquad \text{(D.1)}
$$

where $PV_{a_j} FB_{x(t)}^{j,R}$ is the present value at age a_j of employee j's benefits using the revised plan. Then using the retrospective definition of accrued liability, if no preretirement benefits are payable,

$$\begin{aligned} AL_t^{j,R} &= NC_{x(t)}^{j,R}\, \ddot{s}^{j,R}_{a_j:\overline{x(t)-a_j}|} \\ &= \frac{PV_{a_j} FB^{j,R}_{x(t)}}{\ddot{a}_{a_j:\overline{r-a_j}|x(t)}}\, \ddot{s}^{j,R}_{a_j:\overline{x-a_j}|} \\ &= PV_{x(t)} FB^{j,R}_{x(t)} \frac{\ddot{a}_{a_j:\overline{x(t)-a_j}|}}{\ddot{a}_{a_j:\overline{r-a_j}|}}. \end{aligned} \tag{D.2}$$

Now equation (14) enables us to examine non-investment component of gain. We assume that the experience follows assumptions:

$$\begin{aligned} &\sum_{\mathcal{M}_{t-1}} [{}^{tot}G_t - {}^{inv}G_t] \\ &= \sum_{\mathcal{M}_{t-1}} (AL_{t-1}^{j,R} + NC_{t-1}^{j,R})(1+i) - \sum_{\mathcal{M}_{t-1}} {}^{i}B^j_{t-1} - \sum_{\mathcal{M}_t} AL_t^{j,G} \\ &= \sum_{\mathcal{M}_{t-1}} \left\{ \frac{PV_{a_j} FB^{j,R}_{x_j(t-1)}}{\ddot{a}_{a_j:\overline{r-a_j}|}} \left[\ddot{s}_{a_j:\overline{x_j(t)-1-a_j}|} + 1 \right] (1+i) - {}^{i}B^j_{t-1} \right\} \\ &\quad - \sum_{\mathcal{M}_t} PV_{a_j} FB^{j,G}_{x_j(t)} \frac{\ddot{s}_{a_j:\overline{x_j(t)-a_j}|}}{\ddot{a}_{a_j:\overline{r-a_j}|}} \\ &= \left[\sum_{\mathcal{M}_{t-1}} PV_{a_j} FB^{j,R}_{x_j(t-1)} (1 - q^{R(t-1)}_{x_j(t-1)}) - \sum_{\mathcal{M}_t} PV_{a_j} FB^{j,G}_{x_j(t)} \right] \frac{\ddot{s}_{a_j:\overline{x_j(t)-a_j}|}}{\ddot{a}_{a_j:\overline{r-a_j}|}} \\ &\quad - \sum_{\mathcal{M}_{t-1}} {}^{i}B^j_{t-1} \\ &= 0. \end{aligned} \tag{D.3}$$

Again, the final step of equating to zero is valid if the assumptions are realized in the year from $t-1$ to t. Because the assumptions are realized, the set $\mathcal{M}_{t-1}$ reduced in the proportion $(1 - q^{R(t-1)}_{x_j(t-1)})$ gives the set $\mathcal{M}_t$. Hence the two terms cancel in the numerator in the last stage of the above derivation. Also, assumptions $R(t-1)$ are the same as the assumption $G(t)$. Thus all individual level cost methods satisfy the zero-gain criterion which must be satisfied by a reasonable funding method.

Appendix E Non-Individual Methods

The frozen initial liability (entry age normal) and frozen initial liability (attained age normal) methods have, by definition, zero gain. Thus they satisfy the zero-gain criterion, though arguably through the use of a somewhat artificial procedure. These methods continue to obey equation (1) at all times because equation (1) is used to define the normal cost.

The aggregate method could be argued to give a non-zero gain by equation (10) if the actual contribution does not equal the normal cost. The subsequent forcing of the accrued liability to equal the fund is done to give the zero unfunded liability required under the aggregate method.

Appendix F An Example

Let us consider a numerical example of the operation of the aggregate entry age normal method for a two person pension plan when experience is as assumed:

Table F.1
Pension Plan Data

Membership Data	Member K	Member L
Date of plan inception	1/1/1999	1/1/1999
Date of birth	1/1/1936	1/1/1935
Date of hire	1/1/1981	1/1/1999
Retirement date	1/1/2001	1/1/2000
Annuity value	$1500	$100

Actuarial Assumptions and Method

Interest rate:	7%
$\ddot{a}_{\overline{20}\rceil}$	11.3356
Pre-retirement decrements:	None
Method:	Aggregate entry age normal

The following quantities are needed to determine the normal costs.

$$PVFB_{45}^{K} \quad = \quad \$387.628 = 1500 \times 1.07^{-20}$$

$$PVFB_{64}^{L} = \$93.458 = 100 \times 1.07^{-1}$$
$$PVFB_{63}^{K} = \$1310.158 = 1500 \times 1.07^{-2}$$

The calculation of plan normal costs from equation (15) is as follows for 1999 and 2000, where it is assumed that the actual contribution made equals the normal cost. The aggregate entry age normal method obeys the zero-gain criterion in general if equation (15) is used every year despite the resulting non-level normal cost. In other words, the experience follows the assumptions and zero gain results so the unfunded follows equation (10) with zero substituted for ${}^{tot}G$. Hence the unfunded grows only with interest: $1481.108 = 1384.213 \times 1.07 = 1293.657 \times 1.07^2$. However, the year-to-year use of equation (15) gives a non-level normal cost even if the termination and other experience is as assumed; this renders the method unacceptable. In this example the normal cost per person changes from \$39.00 to \$34.196.

In a practical situation it would be unacceptable also to have a negative fund after the members have both retired. The unfunded would be amortized by making amortization payments.

Table F.2
Pension Plan Calculations

(1) Jan. 1	(2) Normal Cost	(3) Annuity	(4) Fund[1]	(5) Fund[2]
1999	78.000	0	0	\$78.000
2000	34.196	\$100	-\$16.54	\$17.656
2001	0	\$1500	-\$1481.108	-\$1481.108

[1] Fund before (2) and after (3); [2] Fund after (2) and (3); Normal cost from equation (15), e.g., $78.000 = 2 \times (387.628 + 93.458)/(11.3356 + 1)$;

Table F.3
Calculation of the Unfunded

(1) Jan. 1	(6) *PVFB*	(7) *PVFNC*	(8) *UAL*
1999	\$1403.616	\$109.959	\$1293.657
2000	\$1401.869	\$34.196	\$1384.213
2001	\$0	\$0	\$1481.108

UAL = Columns(6) − (7) − (4) in Tables F.2 and F.3.