2012

On the Relation between the Feynman Paradox and the Aharonov–Bohm Effects

Scot McGregor
University of Nebraska - Lincoln

Ryan Hotovy
University of Nebraska - Lincoln

Adam Caprez
University of Nebraska-Lincoln, acaprez2@unl.edu

Herman Batelaan
University of Nebraska - Lincoln, hbatelaan@unl.edu

New Journal of Physics
The open-access journal for physics

On the relation between the Feynman paradox and the Aharonov–Bohm effects

Scot McGregor, Ryan Hotovy, Adam Caprez and Herman Batelaan[1]
Department of Physics and Astronomy, University of Nebraska—Lincoln, 208 Jorgensen Hall, Lincoln, NE 68588-0299, USA
E-mail: hbatelaan2@unlnotes.unl.edu

New Journal of Physics **14** (2012) 093020 (21pp)
Received 6 July 2012
Published 13 September 2012
Online at http://www.njp.org/
doi:10.1088/1367-2630/14/9/093020

Abstract. The magnetic Aharonov–Bohm (A–B) effect occurs when a point charge interacts with a line of magnetic flux, while its reciprocal, the Aharonov–Casher (A–C) effect, occurs when a magnetic moment interacts with a line of charge. For the two interacting parts of these physical systems, the equations of motion are discussed in this paper. The generally accepted claim is that both parts of these systems do not accelerate, while Boyer has claimed that both parts of these systems do accelerate. Using the Euler–Lagrange equations we predict that in the case of unconstrained motion, only one part of each system accelerates, while momentum remains conserved. This prediction requires a time-dependent electromagnetic momentum. For our analysis of unconstrained motion, the A–B effects are then examples of the Feynman paradox. In the case of constrained motion, the Euler–Lagrange equations give no forces, in agreement with the generally accepted analysis. The quantum mechanical A–B and A–C phase shifts are independent of the treatment of constraint. Nevertheless, experimental testing of the above ideas and further understanding of the A–B effects that are central to both quantum mechanics and electromagnetism could be possible.

[1] Authors to whom any correspondence should be addressed.

Contents

1. Introduction

The question of whether or not forces are present for physical systems that display the Aharonov–Bohm (A–B) effect has been debated for decades. The general consensus is that there are no forces, which is considered to be a defining property of the famous effect. The best-known version of the effect occurs when a current carrying solenoid (or more generally a magnetic flux) is enclosed by an electron interferometer. When the current is changed the consequence is that the observed electron fringes in the interferometer shift. Given that the solenoid is thought to produce no discernible magnetic (or electric) field external to its structure, and that is where the electron passes, there is no force on the electron. It is rare, if not unique, to encounter a response of a physical system without the presence of forces, which illuminates a part of the appeal of the A–B effect.

Central to the A–B effects is the interaction between a magnetic moment and a charge. This interaction is associated with a classical relativistic paradox [1]. Recently [2], Aharonov and Rohrlich stated that: 'The paradox is crucial to clarifying the entirely quantum interactions of "fluxons" and charges—the generalized Aharonov–Bohm effect.' The central problem of the paradox is the following. When a point charge moves in the vicinity of a tube that contains magnetic flux, the momentum in the electromagnetic field changes. Outside of the flux tube there is no electric or magnetic field and the charge does not change its momentum. The tube carries no net charge, it may thus not experience a Lorentz force and it appears not to change its momentum. These cursory observations would, if true, violate momentum conservation and give the appearance that the A–B effect is paradoxical in nature.

In this paper, we give a description of the magnetic A–B effect and its reciprocal [3] based on the Darwin Lagrangian. Our approach resolves the paradox, is consistent with all the experiments to date and can, in principle, be differentiated experimentally from previous theoretical approaches. We find that for constrained motion both parts of the physical system do

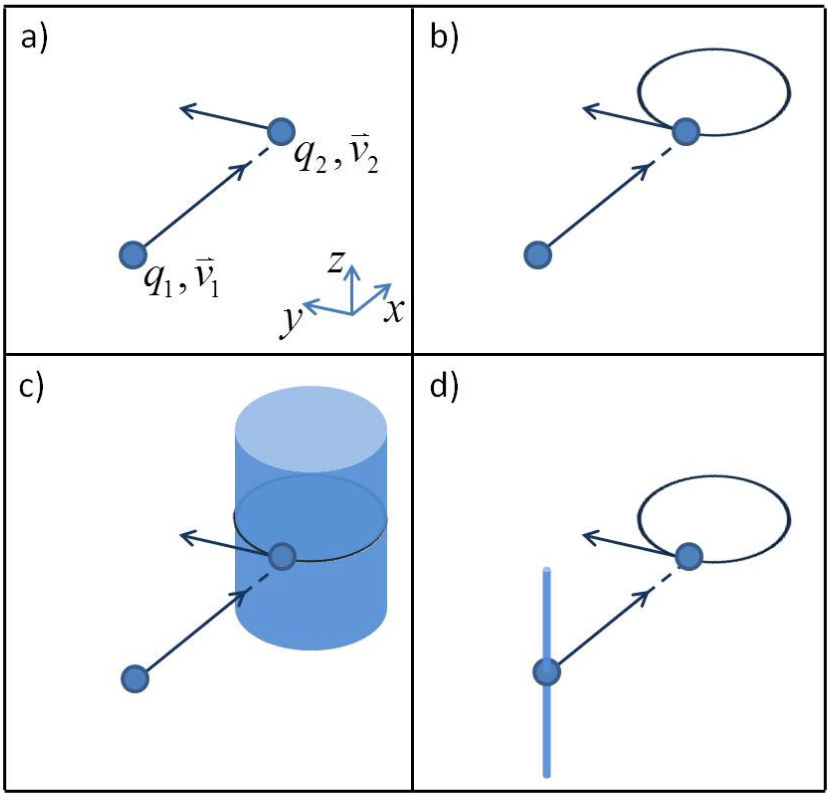

Figure 1. The Feynman paradox. The coordinate system used for the analysis of the Feynman paradox (see text) is given.

not accelerate, consistent with the generally accepted prediction; however, we also find that for unconstrained motion the magnetic part does accelerate and the charged part does not. The apparent violation of Newton's third law is typical of the 'Feynman paradox.' The relation between the Feynman paradox and the A–B effects has, to our knowledge, not been pointed out earlier. Building on the Feynman paradox, the difference between constrained and unconstrained motion is delineated. We argue that an appropriate description of physical systems, which are used for the demonstration of the A–B effects, is not known to be constrained or unconstrained.

Feynman explains a paradox in his famous *Lectures* where two particles interact in such a way that the momentum of one particle changes by a certain amount that is not the same as the momentum change of the other particle [4]. The specific scenario is that two charged particles are placed on the x-axis, with one charged particle moving initially along the x-axis, while the other moves along the y-axis. From the Lorentz force it is clear that the magnetic part of the force is not balanced (figure 1(a)). A relativistic treatment of this problem does not change this conclusion [5]. This is indeed an example where the interpretation of Newton's third law as conservation of mechanical momentum (as opposed to canonical momentum) breaks down.

In this work, a Lagrangian approach is chosen. The Lagrangian offers ways to conveniently impose constraints on the particle motion. A Hamiltonian can be obtained from it that can be compared with other approaches [6]. Finally, a path integral method can be used to obtain the quantum mechanical phase shifts that can be compared with the known A–B and A–C phase shifts. For the interaction of charged particles no Lagrangian exists that is manifestly invariant and obeys the Lorentz symmetry [7] to all orders in v/c. The Darwin Lagrangian is the best-known choice that is valid to $(v/c)^2$. This approximation will turn out to be sufficient to treat the Feynman paradox and the A–B and A–C problem in such a way that momentum is conserved, the equations of motion for both parts of the system are obtained and the method used for all the systems is the same. Note that the inclusion and the physical effect of higher order terms are potentially interesting but unknown.

2. Relativistic classical analysis

2.1. Preamble and assumptions: building the physical systems

It is from the constituents of the physical system presented in the Feynman paradox (figure 1(a)) that the Mott–Schwinger system (figure 1(b)), the A–B system (figure 1(c)) and the Aharonov–Casher system (figure 1(d)) can be constructed. The neutron in the Mott–Schwinger system can be modeled as a current loop. Such a loop may be thought of as many circulating charge elements. Thus, the transition from the Feynman paradox to the Mott–Schwinger system may be done by integration over the charges in the loop. Similarly, a solenoid may be constructed via the addition of non-interacting current loops, and a charged wire constructed by the addition of non-interacting point charges. Consequently, a transition from the Mott–Schwinger system to the A–B or Aharonov–Casher system may be done by integration of current loops or point charges, respectively.

In the construction phase the issue of constraints comes into play. The construction of the Mott–Schwinger system may be performed in two ways. Either the Lagrangian for the Feynman system can be integrated directly, or alternatively, the forces resulting from the Lagrangian can be integrated. These two methods imply inherent assumptions regarding the freedom of the relative motion of the charges that constitute the current loop. If the forces resulting from the Lagrangian are integrated, the net force on the overall system, and thus the equation of motion of the current loop, is determined. Because the forces were computed without applying any restrictions on the relative motion, the charge elements are free to move independently (i.e. the motion of the charge elements is unconstrained). If, on the other hand, the Lagrangian is integrated directly, the Euler–Lagrange equations give the equation of motion for the current loop. The derivatives of the Euler–Lagrange equations are taken with respect to the position and velocity of the current loop. This method stipulates that the charge elements move relative to each other in such a way that the initial shape of the charge distribution is preserved and the loop merely undergoes translation (i.e. the motion of the charge elements is constrained).

It appears obvious that the motion of the conduction electrons in a solenoid should be treated as constrained. Simple estimates can be made to investigate this statement. Consider an electron passing a solenoid in a certain interaction time. During this time the motion of solenoidal conduction electrons can be investigated and their distance traveled can be compared with the solenoid wire thickness. If the distance traveled is much longer, then constraints are certainly important, whereas if the distance traveled is much shorter the role the constraints play is unclear. Our argumentation hinges on the veracity of the latter and justifies an investigation of the comparison of motion for the unconstrained system versus the constrained system. *We do not claim that the system is either of them*, but we consider both fully unconstrained and constrained systems to be interesting limiting cases.

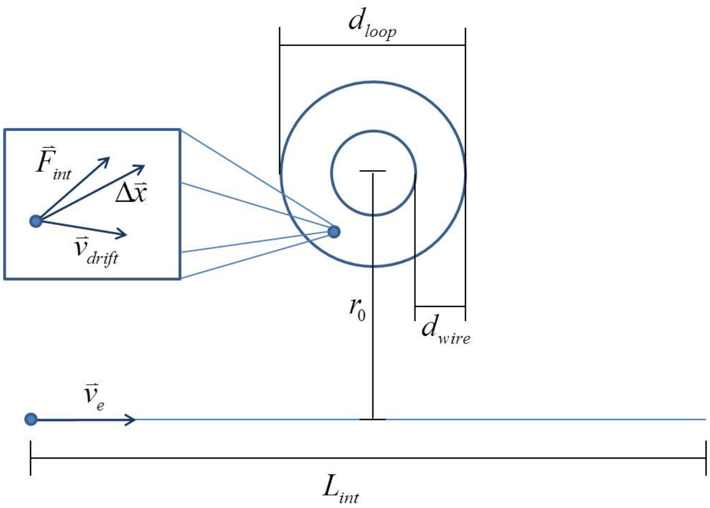

Figure 2. The Mott–Schwinger system. The coordinate system for the analysis of a charged particle interacting with a current loop (see text) is given.

In the A–B experiments such as that of Möllenstedt and Bayh [10], the interaction time of an electron passing a solenoid at 40 keV is roughly 1 ps (see figure 2), assuming an interaction length of three times the loop diameter ($3 \times 36\,\mu$m). The electron velocity has a drift velocity of $v_{\text{drift}} = I/nAq = 80\,\mu\text{m}\,\text{s}^{-1}$, where I is the current, n is the number of atoms per unit volume of the wire, A is the cross-sectional area of the wire and q is the charge of an electron. The electron has a far larger thermal component $v_{\text{thermal}} = \sqrt{2k_{\text{B}}T/m_{\text{e}}} = 9.5 \times 10^5\,\text{m}\,\text{s}^{-1}$. The thermal drift displacement during the interaction time is $\Delta x_{\text{thermal}} = 87\,\text{nm}$, which is much smaller than the solenoid wire diameter of $5\,\mu$m. The displacement of electrons within the coil due to the magnetic field of the passing electron can also be approximately determined by using the Lorentz force. The result is $\Delta x_{\text{int}} = 3.7 \times 10^{-20}\,\text{m}$ using the thermal velocity. Note that the inclusion of the effective electron mass of the Drude–Sommerfeld model has little effect on the estimates, as the effective mass of a conduction electron in tungsten is only 2–3 times that of a free electron [11]. The potential that restricts the charge to the wire may be thought of as having negligible curvature over such short distances. Additionally, the centripetal force required for the electrons in the solenoid to move in a circle with a drift velocity of $80\,\mu\text{m}\,\text{s}^{-1}$ is of the order of 10^{-34} N, whereas the Lorentz force due to the passing electron charge is of the order of 10^{-32} N. It appears reasonable to at least consider the scenario of unconstrained motion.

Objections can be raised to these estimates. For example, electron–phonon interaction may, in principle, lead to a back-action force. Another example is that the interaction time is much slower than the plasmonic response time of tungsten (0.44 fs) [12]. This motivates the inclusion of electron–electron interaction within the wire during the interaction time. An interesting attempt has been made to include such interactions and some constraints [13], that support the controversial idea that both parts of the A–B system experience a force. However, arguably [14], a recent experiment may rule out the presence of force on the passing electron [15]. To date, no detailed models have been analytically or numerically solved, which motivates the study of the simpler case of constrained and unconstrained motion.

For neutrons in the A–C system, this type of estimate gives a completely different result. The neutron could be modeled as a current loop of radius 10^{-15} m. (This simplistic classical model ignores quantum mechanical addition of quark angular momentum and magnetic moment.) In order for such a loop to generate a magnetic moment of $10^{-26}\,\mathrm{J\,T^{-1}}$, the constituent charges would circulate with a period of the order of 10^{-23} s. The interaction time in the experiment by Cimmino *et al* [16] was of the order of 10^{-5} s and thus the motion of the charged constituents of the neutron is constrained. For completeness it is still interesting to analyze the A–C system in terms of constrained and unconstrained motion as described above. Furthermore, the A–C phase shift may be observable for other larger magnetic particles, for which the constraints are not clear.

A case has been made in favor of the effective presence of constraints on the basis of the following lemma: any finite stationary distribution of matter has zero total momentum [17]. The term 'stationary' is defined by $\partial_0 T^{\mu\nu} = 0$, where $T^{\mu\nu}$ is the electromagnetic stress tensor. The assumption of a stationary distribution along with the conservation law $\partial_\mu T^{\mu\nu} = 0$ gives the result $\partial_j T^{j0} = 0$. Using the divergence theorem the total momentum may be written as a surface integral [18]

$$p^i = \frac{1}{c}\int T^{i0}\,\mathrm{d}\tau = \frac{1}{c}\int \left[\partial_j\left(x_i T^{j0}\right) - x_i \partial_j T^{j0}\right]\mathrm{d}\tau = \frac{1}{c}\oint x_i T^{j0}\,\mathrm{d}S_j. \qquad (1)$$

The assumption of a finite distribution of matter ensures that the elements of the stress tensor must fall off as $1/r^{4+\delta}$ ($\delta \geqslant 0$). Consequently, the above surface integral is zero, proving the lemma

$$p^i = \frac{1}{c}\oint x_i T^{j0}\,\mathrm{d}S_j = 0. \qquad (2)$$

The presence of electromagnetic momentum for a stationary charge-current distribution, taken together with the validity of the lemma, demands that there is another opposite and equal form of momentum. This 'hidden momentum' results from internal motion of a stationary system. One textbook example is that of a current-carrying loop of wire, bathed in a uniform external electric field [19] (figure 3). Relevant for our present discussion, the electric field could be thought of as arising from the presence of a distant point charge.

The applied electric field $\vec{E}$ gives rise to a change in velocity of the charges as they move along the vertical segments of the loop. Consequently, the velocity of the charges moving in the bottom segment, u_1, is smaller than the velocity in the top section, u_2. The result is that the charges in the loop carry a net relativistic mechanical momentum equal and opposite to the electromagnetic field momentum [19]. Proponents of using hidden momentum for the analysis of the A–B effects claim that in the case of dynamic systems for which equations of motion are being calculated, the hidden momentum has a direct effect on the equation of motion of the object in question. In the case of a current loop passing a charged wire (the A–C system), the 'hidden momentum' goes directly into the equation of motion so as to cancel the force on the loop. However, one should tread carefully when taking this approach considering that the lemma being applied requires a stationary system while the calculation of the equations of motion of a system requires the assumption of a non-stationary system. Such an analysis of the loop-wire system has been made with three different models of the current loop [17]: a gas of charged particles constrained to move inside a neutral tube, a gas of charged particles constrained to move inside a conducting tube and a charged (incompressible) fluid constrained to move inside

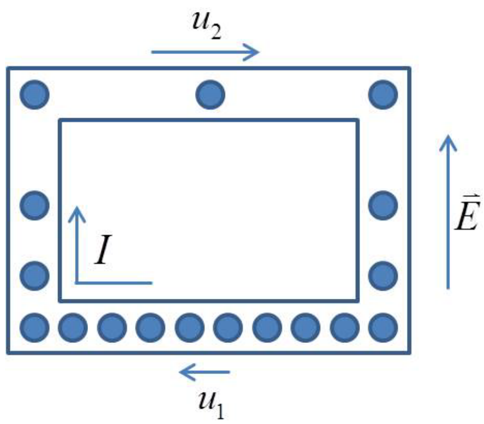

Figure 3. Building physical systems. (a) In the physical system presented in the Feynman paradox, particle 1 moves toward particle 2, and particle 2 moves with a velocity perpendicular to that of particle 1. The Lorentz forces are not balanced in this case. (b) The Mott–Schwinger system consists of a charged particle moving in the vicinity of a current loop [8, 9]. The current loop may be thought of as many circulating charge elements. Consequently, this system bears a resemblance to the Feynman system. (c) In the case of the A–B effect, a charged particle is moving near a current-carrying solenoid. Here the solenoid is depicted as constructed from current loops as they appear in the Mott–Schwinger system. (d) The Aharonov–Casher system involves a charged wire and a current loop. Similar to the solenoid in the A–B system, the charged wire is shown as constructed from charged particles as in the Mott–Schwinger system.

a neutral tube. Although these analyses all predict zero forces, this is not a general property for unconstrained motion as shown by the counterexample given in our present analysis.

2.2. *Unconstrained motion*

In section 2.2.1 the force and the equations of motion for two interacting charged particles are derived from the Darwin Lagrangian for the Feynman problem (figure 1(a)). In the following two sections, the force is integrated for the charge and current distributions that are relevant for the Mott–Schwinger, and the A–B and A–C effects, respectively.

2.2.1. Equations of motion for two interacting charged particles using the Darwin Lagrangian. The Darwin Lagrangian [18] is given by

$$L = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - \frac{q_1 q_2}{r} + \frac{q_1 q_2}{2rc^2}\left[\vec{v}_1 \cdot \vec{v}_2 + \frac{(\vec{v}_1 \cdot \vec{r})(\vec{v}_2 \cdot \vec{r})}{r^2}\right], \tag{3}$$

where $\vec{r} = \vec{r}_1 - \vec{r}_2$. The vector potential and scalar potential for a moving charged particle are given by

$$\vec{A} = \frac{q}{2rc}\left[\vec{v} + \frac{\vec{r}(\vec{v} \cdot \vec{r})}{r^2}\right], \tag{4}$$

$$\varphi = \frac{q}{r}. \tag{5}$$

The Euler–Lagrangian equations of motion [20] are $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \vec{v}_1} = \frac{\partial L}{\partial \vec{r}_1}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \vec{v}_2} = \frac{\partial L}{\partial \vec{r}_2}$, where

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \vec{v}_1} &= m_1\vec{a}_1 - \frac{q_1q_2(\vec{r}\cdot\dot{\vec{r}})}{2c^2r^3}\left[\vec{v}_2 + \frac{(\vec{v}_2\cdot\vec{r})\vec{r}}{r^2}\right] \\ &\quad + \frac{q_1q_2}{2c^2r}\left\{\vec{a}_2 - \frac{2(\vec{v}_2\cdot\vec{r})(\vec{r}\cdot\dot{\vec{r}})\vec{r}}{r^4} + \frac{\left[(\vec{a}_2\cdot\vec{r}) + (\vec{v}_2\cdot\dot{\vec{r}})\right]\vec{r} + (\vec{v}_2\cdot\vec{r})\dot{\vec{r}}}{r^2}\right\} \\ &= m_1\vec{a}_1 + \frac{q_1q_2}{2c^2r}\left\{\vec{a}_2 - \frac{(\vec{r}\cdot\dot{\vec{r}})\vec{v}_2}{r^2} - \frac{3(\vec{v}_2\cdot\vec{r})(\vec{r}\cdot\dot{\vec{r}})\vec{r}}{r^4} + \frac{\left[(\vec{a}_2\cdot\vec{r}) + (\vec{v}_2\cdot\dot{\vec{r}})\right]\vec{r} + (\vec{v}_2\cdot\vec{r})\dot{\vec{r}}}{r^2}\right\},\end{aligned} \tag{6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \vec{v}_2} = m_2\vec{a}_2 + \frac{q_1q_2}{2c^2r}\left\{\vec{a}_1 - \frac{(\vec{r}\cdot\dot{\vec{r}})\vec{v}_1}{r^2} - \frac{3(\vec{v}_1\cdot\vec{r})(\vec{r}\cdot\dot{\vec{r}})\vec{r}}{r^4} + \frac{\left[(\vec{a}_1\cdot\vec{r}) + (\vec{v}_1\cdot\dot{\vec{r}})\right]\vec{r} + (\vec{v}_1\cdot\vec{r})\dot{\vec{r}}}{r^2}\right\}, \tag{7}$$

$$\frac{\partial L}{\partial \vec{r}_1} = \frac{q_1q_2}{r^3}\vec{r} + \frac{q_1q_2}{2c^2}\left[\frac{-(\vec{v}_1\cdot\vec{v}_2)\vec{r}}{r^3} - \frac{3(\vec{v}_1\cdot\vec{r})(\vec{v}_2\cdot\vec{r})\vec{r}}{r^5} + \frac{(\vec{v}_1\cdot\vec{r})\vec{v}_2 + (\vec{v}_2\cdot\vec{r})\vec{v}_1}{r^3}\right], \tag{8}$$

$$\frac{\partial L}{\partial \vec{r}_2} = -\frac{q_1q_2}{r^3}\vec{r} - \frac{q_1q_2}{2c^2}\left[\frac{-(\vec{v}_1\cdot\vec{v}_2)\vec{r}}{r^3} - \frac{3(\vec{v}_1\cdot\vec{r})(\vec{v}_2\cdot\vec{r})\vec{r}}{r^5} + \frac{(\vec{v}_1\cdot\vec{r})\vec{v}_2 + (\vec{v}_2\cdot\vec{r})\vec{v}_1}{r^3}\right]. \tag{9}$$

Taking the conditions that define the Feynman paradox (figure 4):

$$\begin{aligned}&\vec{r}_1 = \vec{0}, \quad \vec{r}_2 = r\hat{x}, \\ &\vec{v}_1 = v\hat{x}, \quad \vec{v}_2 = v\hat{y}, \\ &q_1 = q_2, \quad m_1 = m_2, \\ &\vec{r} = -r\hat{x}, \quad \dot{\vec{r}} = v(\hat{x} - \hat{y}), \quad \hat{r} = -\hat{x}.\end{aligned} \tag{10}$$

The equations of motion obtained for particle 1 are

$$a_{1x} = \frac{-\frac{q^2}{mr^2}\left[\left(1 + \frac{v^2}{2c^2}\right) + \frac{q^2}{mc^2r}\left(1 - \frac{v^2}{c^2}\right)\right]}{1 - \frac{1}{m^2}\left(\frac{q^2}{c^2r}\right)^2} \approx -\frac{q^2}{mr^2}\left(1 + \frac{v^2}{2c^2}\right), \tag{11}$$

$$a_{1y} = \frac{-\frac{q^2v^2}{mc^2r^2}}{1 - \frac{1}{4m^2}\left(\frac{q^2}{c^2r}\right)^2} \approx -\frac{q^2v^2}{mc^2r^2}, \tag{12}$$

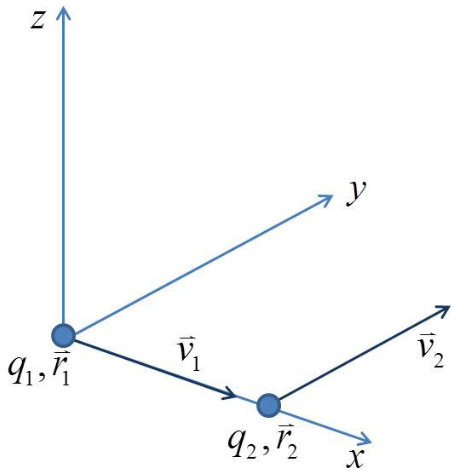

Figure 4. Motion of a conduction electron. An electron in a current loop with diameter d_{loop} and an electron passing at a distance r_0 interact via the Lorentz force. The electron in the loop experiences a force F_{int}. During the interaction time the electron in the loop moves a distance Δx. This movement is a combination of drift movement due to drift velocity v_{drift} and the displacement due to the Lorentz force.

and for particle 2

$$a_{2x} = \frac{\frac{q^2}{mr^2}\left[\left(1-\frac{v^2}{c^2}\right)+\frac{q^2}{mc^2r}\left(1+\frac{v^2}{2c^2}\right)\right]}{1-\frac{1}{m^2}\left(\frac{q^2}{c^2r}\right)^2} \approx \frac{q^2}{mr^2}\left(1-\frac{v^2}{c^2}\right), \tag{13}$$

$$a_{2y} = \frac{\frac{v^2}{2m^2r}\left(\frac{q^2}{c^2r}\right)^2}{1-\frac{1}{4m^2}\left(\frac{q^2}{c^2r}\right)^2} \approx 0. \tag{14}$$

The approximation in equations (11)–(14) is obtained by expansion to first order in q^2/mc^2r under the assumption that $q^2/mc^2r \ll v^2/c^2$. This is valid if the paths of the charged particles are approximately straight. A small deflection implies that the potential energy of the particle is always less than the kinetic energy (i.e. $q^2/r < mv^2/2$). Alternatively, the relativistic equation of motion is given by the Lorentz force law

$$\vec{F} = q\left(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right). \tag{15}$$

Expanding the Lorentz force in this equation to second order in v/c leads to the equations of motion

$$a_{1x} = -\frac{\gamma q^2}{mr^2} \approx -\frac{q^2}{mr^2}\left(1+\frac{v^2}{2c^2}\right), \tag{16}$$

$$a_{1y} = -\frac{\gamma q^2 v^2}{mc^2r^2} \approx -\frac{q^2v^2}{mc^2r^2}, \tag{17}$$

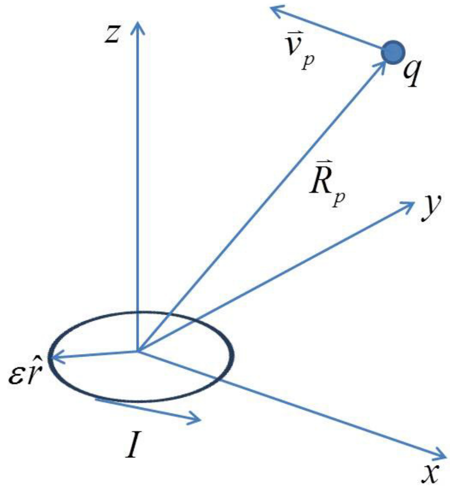

Figure 5. Hidden momentum. A conducting loop with current circulating clockwise is immersed in an external homogeneous electric field directed toward the top of the page. The electric field accelerates the charges moving toward the top of the loop and decelerates those moving toward the bottom of the loop. Consequently, there is a non-zero net relativistic total linear momentum of the charges contained in the loop [19]. This is the 'hidden momentum' and exactly cancels the momentum in the electromagnetic field.

$$a_{2x} = \frac{q^2}{\gamma^2 m r^2} = \frac{q^2}{mr^2}\left(1 - \frac{v^2}{c^2}\right), \tag{18}$$

$$a_{2y} = 0, \tag{19}$$

which agree with the Darwin Lagrangian approach as well as Feynman's resolution of the paradox [5] in the non-relativistic limit. As Feynman points out, Newton's third law does not hold for mechanical momentum; however, consideration of the change of electromagnetic momentum ensures the conservation of total and canonical momentum. Note that the use of the Darwin Lagrangian is a superfluous step. We could have limited ourselves to the forces occurring in the relativistic equation of motion. However, for a consistent treatment of the unconstrained and constrained motion an identical starting point is favored. For unconstrained motion we can now proceed to integrate over the forces acting on the constituent particles of an extended body.

2.2.2. Charged particle and current loop. The forces in a system consisting of two interacting point charges have now been determined. A system of a point charge and a loop consisting of many mutually non-interacting point charges can now be constructed by direct integration over the forces. Consider a system consisting of a charged particle moving in the x-direction in the vicinity of a current loop of radius ε centered at the origin (figure 5).

$\vec{R}_q = (x_q, y_q, z_q)$ is the position of the charged particle relative to the center of the loop, q is the charge, $\vec{v}_{\mathrm{p}}$ is the velocity of the particle and I is the current. The force on the current loop due to the charged particle in the limit $\varepsilon \to 0$ is

$$\vec{F}_\mu = \frac{1}{c}\int \vec{J} \times \vec{B}\, \mathrm{d}\tau = \frac{\mu q v_q}{c R_q^3}\left[\frac{3\left(x_q y_q \hat{x} + y_q^2 \hat{y} + y_q z_q \hat{z}\right)}{R_q^2} - \hat{y}\right]. \tag{20}$$

The force on the moving charge due to the current loop is

$$\vec{F}_q = \frac{q}{c}\vec{v}_q \times \vec{B} = \frac{\mu q v_q}{c R_q^3}\left[\frac{3\left(z_q^2 \hat{y} - y_q z_q \hat{z}\right)}{R_q^2} - \hat{y}\right], \tag{21}$$

where the magnetic moment is denoted by μ. Note that the forces are not equal and opposite after integration and thus total mechanical momentum is not conserved similar to the Feynman paradox. The same procedure will now be followed for the A–B and Aharonov–Casher systems (figures 1(c) and (d)).

2.2.3. The Aharonov–Bohm and Aharonov–Casher systems. The forces involved in the A–B (figure 1(c)) and Aharonov–Casher (figure 1(d)) systems can be determined by integration of the forces obtained for the loop/charge. For the A–B system the connection between the loop magnetic moment and the solenoid is made by substituting a differential magnetic moment element of the solenoid for the magnetic moment of the current loop:

$$\vec{\mu} \to \frac{c\Phi_B}{4\pi}\hat{z}\, \mathrm{d}z_\mu, \tag{22}$$

where Φ_B is the magnetic flux in the solenoid. The charged particle is assumed to move in the x-direction. By integrating equation (21) the force on the charged particle is

$$\vec{F}_q = \int_{\text{solenoid}} \mathrm{d}\vec{F}_q = 0. \tag{23}$$

This is obvious given that the particle is propagating through a region where there are no electric or magnetic fields. By integrating equation (20) the force on the solenoid is

$$\vec{F}_\mathrm{s} = \int_{\text{solenoid}} \mathrm{d}\vec{F}_\mu = \frac{q\phi_B v_q}{4\pi}\left\{\frac{2(x_q - x_\mathrm{s})(y_q - y_\mathrm{s})\hat{x} - [(x_q - x_\mathrm{s})^2 - (y_q - y_\mathrm{s})^2]\hat{y}}{\left[(x_q - x_\mathrm{s})^2 + (y_q - y_\mathrm{s})^2\right]^2}\right\}. \tag{24}$$

For the A–C system the connection between charged particle and the wire was made by substituting a differential charge element of the wire for the charge of the particle:

$$q \to \lambda\, \mathrm{d}z_q. \tag{25}$$

By integrating equation (21) the force on the wire is

$$\vec{F}_\mathrm{w} = \int_{\text{wire}} \mathrm{d}\vec{F}_q = 0 \tag{26}$$

and by integrating equation (20) the force on the current loop is

$$\vec{F}_\mu = \int_{\text{wire}} \mathrm{d}\vec{F}_\mu = \frac{2\lambda\mu v_\mathrm{w}}{c}\left\{\frac{2(x_\mathrm{w} - x_\mu)(y_\mathrm{w} - y_\mu)\hat{x} - [(x_\mathrm{w} - x_\mu)^2 - (y_\mathrm{w} - y_\mu)^2]\hat{y}}{\left[(x_\mathrm{w} - x_\mu)^2 + (y_\mathrm{w} - y_\mu)^2\right]^2}\right\}. \tag{27}$$

As stated in the introduction, it is unreasonable to describe the motion of constituents of a neutron as unconstrained during the typical interaction times for the A–C experiments. Moreover, the above simplistic reasoning foregoes the interesting physics that underlies the understanding of the neutron's magnetic moment as the sum of the magnetic moment of its parts and dynamics [21]. Nevertheless, for the completeness of our present argument, the unconstrained model is considered in the context of the A–C physical system and hopefully highlights the disparity in the nature of the solenoidal versus the neutron's magnetic moment. In this point of view, Aharonov and Casher's realization that a neutron passing by a charged wire accumulates a phase shift that can be interpreted as the reciprocal of the A–B effect is both beautiful and surprising.

In each of these systems one object feels a force while the other does not. This again is a system that exhibits the qualitative feature of the underlying Feynman system that total mechanical momentum is not conserved.

2.3. *Constrained motion*

In the following sections, the integrated Lagrangian will be used to obtain the equations of motion for the Mott–Schwinger, A–B and A–C systems. The derivatives in the Euler Lagrange equation will be made with respect to coordinates that describe the motion of complete objects, such as the current loop in the Mott–Schwinger system. This constrains the motion of the charge elements in the loop to experience the same acceleration.

2.3.1. Integration of the Lagrangian. An alternative to the unconstrained method of analysis described above for the Mott–Schwinger system (figure 1(b)) is the approach of assuming that the charge elements within the loop are fixed relative to one another and must accelerate identically along with a coordinate defining the location of the loop. This can be done by two possible methods. By the first method, the vector potential of the moving charge, appropriate for the Darwin Lagrangian, is taken to determine the resulting magnetic field. The vector potential and magnetic field of the moving charge are

$$\vec{A}_q = \frac{q}{2rc}\left[\vec{v}_q + \frac{\vec{r}(\vec{v}_q\cdot\vec{r})}{r^2}\right], \tag{28}$$

$$\vec{B}_q = \vec{\nabla}\times\vec{A}_q = \frac{q}{c}\frac{\vec{v}_q\times\vec{r}}{r^3}. \tag{29}$$

The magnetic and electric fields are coupled to the magnetic dipole and relativistic electric dipole to obtain the Lagrangian

$$\begin{aligned} L &= \frac{1}{2}m_q v_q^2 + \frac{1}{2}m_\mu v_\mu^2 + \vec{\mu}\cdot\vec{B} + \vec{d}\cdot\vec{E} \\ &= \frac{1}{2}m_q v_q^2 + \frac{1}{2}m_\mu v_\mu^2 + \frac{q}{c}\frac{\vec{\mu}\cdot[\vec{v}_q\times(\vec{r}_\mu-\vec{r}_q)]}{|\vec{r}_\mu-\vec{r}_q|^3} + \frac{q}{c}\frac{(\vec{v}_\mu\times\vec{\mu})\cdot(\vec{r}_\mu-\vec{r}_q)}{|\vec{r}_\mu-\vec{r}_q|^3} \\ &= \frac{1}{2}m_q v_q^2 + \frac{1}{2}m_\mu v_\mu^2 + \frac{q}{c}\frac{(\vec{v}_\mu-\vec{v}_q)\cdot[\vec{\mu}\times(\vec{r}_\mu-\vec{r}_q)]}{|\vec{r}_\mu-\vec{r}_q|^3}. \end{aligned} \tag{30}$$

The second method is integration of the vector potential over the charges in the current loop. Integration of the vector potential (equation (23)) as it appears in the Darwin Lagrangian (equation (3)) for a current loop with no net charge gives

$$\vec{A}_\mu = \frac{\vec{\mu} \times \vec{r}}{r^3}, \tag{31}$$

$$\varphi_\mu = \frac{1}{c}\vec{v}_\mu \cdot \vec{A}_\mu = \frac{\vec{v}_\mu \cdot (\vec{\mu} \times \vec{r})}{cr^3}. \tag{32}$$

Coupling these potentials to the point charge gives the Lagrangian

$$\begin{aligned} L &= \frac{1}{2}m_q v_q^2 + \frac{1}{2}m_\mu v_\mu^2 + \frac{q}{c}\vec{v}_q \cdot \vec{A}_\mu - q\varphi_\mu \\ &= \frac{1}{2}m_q v_q^2 + \frac{1}{2}m_\mu v_\mu^2 + \frac{q}{c}\frac{\vec{v}_q \cdot [\vec{\mu} \times (\vec{r}_q - \vec{r}_\mu)]}{|\vec{r}_q - \vec{r}_\mu|^3} - \frac{q}{c}\frac{\vec{v}_\mu \cdot [\vec{\mu} \times (\vec{r}_q - \vec{r}_\mu)]}{|\vec{r}_q - \vec{r}_\mu|^3} \\ &= \frac{1}{2}m_q v_q^2 + \frac{1}{2}m_\mu v_\mu^2 + \frac{q}{c}\frac{(\vec{v}_\mu - \vec{v}_q) \cdot [\vec{\mu} \times (\vec{r}_\mu - \vec{r}_q)]}{|\vec{r}_\mu - \vec{r}_q|^3}. \end{aligned} \tag{33}$$

These two methods give the same result due to the symmetry under permutation of particles of the Darwin Lagrangian and therefore only one should be taken for the computation of the equations of motion to avoid double counting. Applying the Euler–Lagrange equations gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \vec{v}} - \frac{\partial L}{\partial \vec{r}} = 0, \tag{34}$$

$$m_\mu \vec{a}_\mu = -\frac{q}{c}\left\{\frac{(\vec{v}_\mu - \vec{v}_q) \times \vec{\mu}}{|\vec{r}_\mu - \vec{r}_q|^3} + \frac{3[(\vec{r}_\mu - \vec{r}_q)\cdot\vec{\mu}][(\vec{r}_\mu - \vec{r}_q) \times (\vec{v}_\mu - \vec{v}_q)]}{|\vec{r}_\mu - \vec{r}_q|^5}\right\}, \tag{35}$$

$$m_q \vec{a}_q = \frac{q}{c}\left\{\frac{(\vec{v}_\mu - \vec{v}_q) \times \vec{\mu}}{|\vec{r}_\mu - \vec{r}_q|^3} + \frac{3[(\vec{r}_\mu - \vec{r}_q)\cdot\vec{\mu}][(\vec{r}_\mu - \vec{r}_q) \times (\vec{v}_\mu - \vec{v}_q)]}{|\vec{r}_\mu - \vec{r}_q|^5}\right\}. \tag{36}$$

These forces are equal in magnitude and opposite in direction and thus conserve total mechanical momentum. Therefore, this cannot be characterized as a Feynman-type paradox.

The forces acting on the individual components of the A–B (figure 1(c)) and A–C (figure 1(d)) systems can be determined by integrating the Mott–Schwinger Lagrangian (equation (30) or (33)). The Lagrangian obtained for the A–B system is

$$L = \frac{1}{2}m_s v_s^2 + \frac{1}{2}m_q v_q^2 + \frac{q\Phi_B}{2\pi}\frac{(\vec{v}_q - \vec{v}_s)\cdot[\hat{z} \times (\vec{r}_q - \vec{r}_s)]}{(x_q - x_s)^2 + (y_q - y_s)^2} \tag{37}$$

$$= \frac{1}{2}m_q v_q^2 + \frac{1}{2}m_s v_s^2 + \frac{q}{c}(\vec{v}_q - \vec{v}_s)\cdot\vec{A}_s. \tag{38}$$

Likewise, the Lagrangian obtained for the A–C system is

$$L = \frac{1}{2}m_\mu v_\mu^2 + \frac{1}{2}m_{\rm w} v_{\rm w}^2 + \frac{2\lambda}{c}\frac{(\vec{v}_{\rm w} - \vec{v}_\mu)\cdot[\vec{\mu}\times(\vec{r}_{\rm w} - \vec{r}_\mu)]}{(x_{\rm w} - x_\mu)^2 + (y_{\rm w} - y_\mu)^2} \tag{39}$$

$$= \frac{1}{2}m_\mu v_\mu^2 + \frac{1}{2}m_{\rm w} v_{\rm w}^2 + \frac{1}{c}(\vec{v}_\mu - \vec{v}_{\rm w})\cdot(\vec{\mu}\times\vec{E}_{\rm w}). \tag{40}$$

In both cases the application of the Euler–Lagrange equations of motion gives zero force acting on both elements of both the A–B and A–C systems.

The predictions for the unconstrained motion are very different from the predictions of the constrained motion (the latter coinciding with the generally accepted one). Can these two methods be distinguished by comparing their predicted phase shifts to the experimentally measured phase shifts?

3. Quantum mechanical phase shifts

3.1. Constrained

To compute the quantum mechanical phase shift for the charged particle and the neutron in the A–B and A–C effects, respectively, a closed-loop path integral over time is taken for the Lagrangian described for constrained motion. The phase for the constrained case is the generally accepted one and only a brief summary is given in this section. In these calculations, the charged wire and the solenoid are taken to be stationary ($v_{\rm w} = v_{\rm s} = 0$). Using the Lagrangian given by equation (38) the A–B phase is

$$\varphi_{\rm AB} = \frac{1}{\hbar}\oint\left(\frac{1}{2}m_q v_q^2 + \frac{q}{c}\vec{v}_q\cdot\vec{A}_{\rm s}\right){\rm d}t = \frac{q\Phi_B}{\hbar c}, \tag{41}$$

which has been experimentally verified [10, 22–24]. Using the Lagrangian given by equation (40) the A–C phase is

$$\varphi_{\rm AC} = \frac{1}{\hbar}\oint\left(\frac{1}{2}m_\mu v_\mu^2 + \frac{1}{c}\vec{v}_\mu\cdot(\vec{\mu}\times\vec{E}_{\rm w})\right){\rm d}t = \frac{4\pi\lambda\mu}{\hbar c}. \tag{42}$$

In either case the first term in the Lagrangian, $(mv^2/2)$, does not contribute to the phase. There is no force acting on the charged particle or the neutron and the effects are true A–B effects. An experimental test of the Aharonov–Casher effect by Werner and Klein [6] is in agreement with the standard quantum mechanical prediction, where the experimental-to-theoretical ratio is given by $\varphi_{\rm AC}^{\rm E}/\varphi_{\rm AC}^{\rm T} = 1.46 \pm 0.35$.

3.2. Unconstrained

In the path integral formulation [25], the wavefunction is propagated with the kernel, $K(b,a) = \exp(\frac{\rm i}{\hbar}\int_{t_a}^{t_b} L\,{\rm d}t)$, where L is the classical Lagrangian. For a free particle the kernel is $\exp(\frac{\rm i}{\hbar}\int\vec{p}\cdot{\rm d}\vec{r})$, where $p = mv$. Formally, the initial wavefunction should now be constructed and propagated. However, for the purpose of understanding the measured phase shift in an interferometer it is customary to consider the effect on plane waves. In this case the phase shift is given by $\frac{1}{\hbar}\int_{t_a}^{t_b} L\,{\rm d}t = \frac{1}{\hbar}\int_{t_a}^{t_b}(p\dot{x} - H)\,{\rm d}t$, where p is the canonical momentum $p = mv + qA$. In the case when the Hamiltonian is time independent, the phase shift becomes $\frac{1}{\hbar}\int_{x_a}^{x_b}\vec{p}\cdot{\rm d}\vec{x}$ [26].

For unconstrained motion in the case of the A–B effect, the phase may therefore be written as follows:

$$\varphi_{\text{total}} = \frac{1}{\hbar}\int \vec{p}\cdot \mathrm{d}\vec{x} = \frac{1}{\hbar}\int \left(m\vec{v} + \sum q\vec{A}_j\right)\cdot \mathrm{d}\vec{x}$$
$$= \frac{1}{\hbar}\int \left(m\vec{v} + q\sum \vec{A}_j\right)\cdot \mathrm{d}\vec{x} = \frac{1}{\hbar}\int (m\vec{v} + q\vec{A}_{\mathrm{s}})\cdot \mathrm{d}\vec{x}, \tag{43}$$

where $\vec{A}_{\mathrm{s}}$ is the vector potential generated by the solenoid and $\vec{A}_j$ is the vector potential generated by the charges that constitute the solenoid. This is identical to the phase integral for the A–B effect in the case of constrained motion.

In the case of the A–C effect, considering unconstrained motion as argued above is unreasonable. However, the existence of a larger particle with a magnetic moment cannot be excluded. Such a particle may have constituents that are best described by unconstrained motion. In our model, there are different forces acting on such constituents. How is the path integral phase shift defined for a composite object if the constituents experience different forces? The physical picture is that if the interaction does not lead to a change in the internal quantum states, then the two arms of the interferometer remain indistinguishable. The measured phase shift reflects only the effect in the center-of-mass coordinate or external quantum state. If the internal quantum states do change, then the contrast of the interferometer may be reduced. The initial wavefunction for an unconstrained composite particle with N mutually non-interacting constituents can be written as a product state of plane waves, $\psi_{\mathrm{C}} = \prod_{j=1}^{N} \exp(\mathrm{i}\vec{p}_j \cdot \vec{R}_j/\hbar)$. The phase accumulated by each plane wave along a path is $\varphi = \frac{1}{\hbar}\int \vec{p}\cdot \mathrm{d}\vec{x}$ and thus the phase of the composite wavefunction ψ_{C} picks up an overall phase factor of $\exp(\frac{\mathrm{i}}{\hbar}\sum\int \vec{p}_j \cdot \mathrm{d}\vec{x})$. This phase factor may be rewritten in terms of the total force, $\vec{F}_{\text{total}}$, on the current loop as computed in section 2.2.3,

$$\varphi_{\text{total}} = \frac{1}{\hbar}\sum\int \vec{p}_j \cdot \mathrm{d}\vec{x}$$
$$= \frac{1}{\hbar}\int \left(\sum \vec{p}_j\right)\cdot \mathrm{d}\vec{x} = \frac{1}{\hbar}\int \left[\sum\left(\vec{p}_{0j} + \int \vec{F}_j\,\mathrm{d}t\right)\right]\cdot \mathrm{d}\vec{x}$$
$$= \frac{1}{\hbar}\int \left(\sum \vec{p}_{0j}\right)\cdot \mathrm{d}\vec{x} + \frac{1}{\hbar}\int \left[\int\left(\sum \vec{F}_j\right)\mathrm{d}t\right]\cdot \mathrm{d}\vec{x}$$
$$= \frac{1}{\hbar}\int \left(\sum \vec{p}_{0j}\right)\cdot \mathrm{d}\vec{x} + \frac{1}{\hbar}\int \left[\int \vec{F}_{\text{total}}\,\mathrm{d}t\right]\cdot \mathrm{d}\vec{x}. \tag{44}$$

Note that the composite particle has no charge and the qA term does not contribute to the phase.

Integration of the total force (equation (27)) along a straight path gives the total phase

$$\varphi_{\text{total}} = \frac{1}{\hbar}\int \left(\sum \vec{p}_{0j}\right)\cdot \mathrm{d}\vec{x} + \frac{2\pi\lambda\mu}{\hbar c}\,\mathrm{sign}\left(y_\mu - y_{\mathrm{w}}\right). \tag{45}$$

The difference in phase between the two paths is $\Delta\varphi_{\text{total}} = \frac{4\pi\lambda\mu}{\hbar c}$, which is the appropriate ac phase shift. Thus, the constrained and unconstrained method cannot be distinguished by inspecting the phase.

4. Comparison to previous analyses

4.1. Hidden momentum

The approach taken by Vaidman [17] as applied to the A–C system is one in which internal motion of the system manifest itself in 'hidden momentum' that affects the motion of the neutron. The time derivative of this 'hidden momentum' or the hidden force, as one may refer to it, is applied directly to the equation of motion

$$m\vec{a} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} - \frac{\mathrm{d}\vec{p}_{\mathrm{hid}}}{\mathrm{d}t}. \tag{46}$$

As mentioned above, the justification for the use of the hidden momentum comes from a lemma that states that, for stationary and finite current and charge distributions, the total momentum is zero. A non-zero value of the electromagnetic field momentum then implies the presence of a hidden momentum of equal magnitude and opposite in direction:

$$\vec{p}_{\mathrm{hid}} = -\frac{1}{c^2}\int \varphi \vec{J}\,\mathrm{d}\tau = -\frac{1}{4\pi c}\int \vec{E}\times\vec{B}\,\mathrm{d}\tau = -\vec{p}_{\mathrm{em}}, \tag{47}$$

where φ is the electrostatic potential of the charged wire and $\vec{J}$ is the current density of the loop. The electric potential and current density result in an electric field $\vec{E}$ and magnetic field $\vec{B}$, respectively. Thus, the equation of motion explicitly depends on the change of the electromagnetic field momentum,

$$m\vec{a} = \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{4\pi c}\int \vec{E}\times\vec{B}\,\mathrm{d}\tau\right]. \tag{48}$$

The equation of motion for a current loop in the Aharonov–Casher system (figure 1(d)) determined by direct application of this method is

$$\begin{aligned} m\vec{a} &= \frac{\mathrm{d}\vec{p}}{\mathrm{d}t} - \frac{\mathrm{d}\vec{p}_{\mathrm{hid}}}{\mathrm{d}t} = \vec{\nabla}(\vec{\mu}\cdot\vec{B}) - \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}(\vec{\mu}\times\vec{E}) \\ &= -\frac{1}{c}\vec{\nabla}[\vec{\mu}\cdot(\vec{v}\times\vec{E})] - \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}(\vec{\mu}\times\vec{E}) \\ &= -\frac{1}{c}[(\vec{\mu}\cdot\vec{\nabla})(\vec{v}\times\vec{E}) - (\vec{v}\cdot\vec{\nabla})(\vec{\mu}\times\vec{E})] - \frac{1}{c}(\vec{v}\cdot\vec{\nabla})(\vec{\mu}\times\vec{E}) \\ &= -\frac{1}{c}(\vec{\mu}\cdot\vec{\nabla})(\vec{v}\times\vec{E}). \end{aligned} \tag{49}$$

This acceleration is zero for the geometry of the Aharonov–Casher effect. Thus the force on both objects in the Aharonov–Casher system is zero by this method.

However, for the Feynman paradox the equations of motion do not depend on the change in the electromagnetic field momentum. The inclusion of electromagnetic field momentum solves the paradox by offering a third physical entity that carries a changing momentum [5], while the forces on both objects are not zero, contrasting the Vaidman analysis of the Aharonov–Casher system. Why is there a difference between the two analyses? The reason is that the Feynman paradox concerns a physical system that is not a stationary charge distribution and the lemma does not hold. The question for the A–C system is whether it is well represented by a stationary charge and current distribution. Clearly, the neutron passes by the charged wire, and formally, the A–C system is not represented by a stationary distribution. The result that our constrained description gives is the same as the Vaidman approach, while it is interesting to consider the unconstrained result in relation to the Feynman paradox.

4.2. Newton's third law

The approach taken by Boyer is documented in a series of papers that extend over several decades [13, 14, 27, 28], and argue that the A–B effects are accompanied by a force. This point of view conflicts the generally accepted interpretation of the A–B effect. We will limit ourselves to commenting on two of the more recent papers in this series. Boyer considers a charged particle passing by a solenoid (represented by a line of magnetic dipoles) and calculates the Lorentz force on the solenoid [13]. This force is the same as that given in section 2.2.3 (equation (24)) and Boyer's work motivated that part of our calculation. Boyer continues his argument by invoking Newton's third law and noting that the back-acting force on the electron causes a displacement that through a semi-classical argument gives exactly the A–B phase shift. It is remarkable that such an argument can be given that provides exactly the necessary force, in view of the observation that an unperturbed solenoid has no external electromagnetic fields. The argument hinges on three assumptions. Firstly, the force on the solenoid is the total force that acts on the solenoid; secondly, Newton's third law holds, and thirdly, the semi-classical approximation is valid. Our work shows that the total force on the solenoid depends on the presence or absence of constraints. Additionally, Feynman's paradox illustrates that Newton's third law is not generally valid. (Boyer argues in another paper published in 2002 that the electromagnetic momentum is conserved during the interaction [13].) Finally, it is interesting to note that Boyer's force is dispersionless, implying that the group velocity of a wavepacket in a semiclassical approximation does not change. All these issues are interesting in their own right, and warrant further discussion. Additional forces in this context have been predicted to exist by Anandan [29, 30].

In a paper that comments on our experimental demonstration of the absence of force for a charged particle passing a solenoid [14], Boyer argues that charged particles in a solenoid that mutually interact and experience friction can provide a back-action on the passing particle. This line of reasoning considers a model that is more complex than those considered previously and in the present paper, because the mutual interactions between the constituents of magnetic dipoles are excluded.

4.3. Hamiltonian approach

An analysis based on a Hamiltonian approach by Werner and Klein [6] has been performed to determine the force on the neutron in the Aharonov–Casher system (figure 1(d)). The Hamiltonian used was

$$H = \frac{p^2}{2m} - \frac{1}{mc}\vec{\mu} \cdot (\vec{E} \times \vec{p}). \tag{50}$$

A direct application of Hamilton's equations of motion gives

$$\dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}}, \tag{51}$$

$$\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}}, \tag{52}$$

$$m\ddot{\vec{r}} = -\frac{1}{c}(\vec{\mu}\cdot\vec{\nabla})(\vec{v}\times\vec{E}). \tag{53}$$

In the Aharonov–Casher geometry, the electric field has no spatial dependence in the direction of the magnetic moment; therefore the force on the neutron is zero, by the above prescription. Note that this approach does not describe a closed system as it is a single-particle Hamiltonian. Because this approach is that of an open system, it does not address conservation of momentum. Thus, the criterion that total momentum must be conserved cannot be applied to this approach as a test of the validity of the Hamiltonian. Furthermore, this Hamiltonian is equivalent to our Lagrangian (equation (40)) for a stationary wire. Using the vector identity $(a\times b)\cdot c = a\cdot(b\times c)$ the equivalence is found to be

$$L = \frac{1}{2}mv^2 + \vec{d}\cdot\vec{E} = \frac{1}{2}mv^2 + \frac{1}{c}(\vec{v}\times\vec{\mu})\cdot\vec{E}, \tag{54}$$

$$\vec{p} = \frac{\partial L}{\partial\vec{v}} = m\vec{v} + \frac{1}{c}\vec{\mu}\times\vec{E}, \tag{55}$$

$$H = \vec{p}\cdot\vec{v} - L = \frac{1}{2m}\left(\vec{p} - \frac{1}{c}\vec{\mu}\times\vec{E}\right)^2 \approx \frac{p^2}{2m} - \frac{1}{mc}\vec{p}\cdot(\vec{\mu}\times\vec{E}). \tag{56}$$

This Hamiltonian can thus be classified as describing a constrained system as described in section 2.3.1.

4.4. Aharonov and Rohrlich

In their book published in 2005, Aharonov and Rohrlich discuss various momentum terms that can make up for the changing momentum in the electromagnetic field and ultimately conserve momentum. The missing momentum is stated to be the relativistic momentum of the charged particles that give rise to the magnetic flux. The contribution of the Lorentz force to momentum conservation is ignored. The statement 'We move it [passing particle] as slowly as we like, so that the charge scarcely induces a magnetic field . . . ' does not address this issue. Although the magnetic field and thus the Lorentz force scale linearly with velocity, the momentum exchange is independent of velocity as the interaction time scales inversely with velocity. In this paper, it is shown that (in the unconstrained description) the change of momentum due to the Lorentz force is identical in magnitude to the change of momentum in the electromagnetic field.

5. Conclusion

The relation between the Feynman paradox and the A–B effects is that an unconstrained treatment of the A–B effects shares with the Feynman paradox the property that momentum is stored in the electromagnetic field during the interaction and consequently that the forces on the two interacting mechanical parts of the system are not balanced. This implies that one part of the system experiences a force, which is a prediction that is in stark contrast with the usual understanding of the A–B effects. In the constrained description, the A–B effects are very different from the Feynman paradox. In this description, the usual prediction is made that both mechanical parts do not experience a force. Both of these scenarios are limited to the case when the constituents that make up the magnetic moment are assumed to not interact. Given the limited theoretical scope of the theoretical claims, experiments are important. However, as we will indicate now, there are very few options within the reach of current technology.

An experiment to test for the force on an electron in the A–B system (figure 1(c)) has been conducted by our group (see Caprez *et al* [15]). In that experiment a time delay was measured for an electron passing between two solenoids [15]. The time required for the electron to pass from the source to the detector was found to be independent of the magnetic flux contained in the solenoids and thus it appears that the A–B phase shift cannot be explained by a classical force on the electron. However, it has been pointed out that in this case a macroscopic solenoid was used and the qualitative characteristics of the system, such as whether or not there is a measurable delay, potentially depend on the size of the solenoid [14]. For larger solenoids the interaction time is greater than the plasma oscillation period. This is the case for all experimental tests of the A–B effect so far, and as such the force experiment and phase experiments are performed in the same regime. The issue considered in this paper is a different one. The above experiment does not discriminate between the unconstrained and the constrained description.

For the A–B system, an experiment to detect the predicted force on the solenoid (as predicted by the unconstrained model) appears impossible given the necessity to detect the force of a single electron on a macroscopic object.

Although experiments have been done to show the Aharonov–Casher phase shift, no experiments have tested for the presence of a force on the neutron. However, for the molecule thallium fluoride the phase shift was shown to be independent of velocity [31], which is a feature associated with the dispersionless nature of the A–B effect and provides a link to the absence of force [32–34]. The interaction between the applied electric field and the magnetic moment of the fluoride nucleus was responsible for the phase shift. Given the small size of a nucleus, or even an atom or molecule that may be used in such a type of experiments, the circulation time for the constituent charges that produce the magnetic moment is much less than the interaction time. It is likely then that the system must be modeled by constrained motion. Consequently, our present analysis would predict that there is, in fact, no force acting on the interfering particle, consistent with the thallium fluoride experiment.

Similarly, due to the small size of the neutron, the Mott–Schwinger effect for neutron scattering of nuclei is not a physical system that can provide an interesting test between the constrained and the unconstrained description. On the other hand, if the magnetic moment is present in a physical system that has a size between that of a neutron and a solenoid, the unconstrained description may be appropriate while still allowing an observation of the motion of the magnetic moment. This scenario suffers an additional difficulty. For a finite system of charge and current distribution, the electric and magnetic fields must approach zero at long distances from the charges and currents. Consider a charge and current loop that scatter from each other. When the charge and current loop are far apart, the electromagnetic field momentum tends to zero. The total mechanical momentum must thus be identical for the final and the initial state and Newton's third law holds. These statements imply that there is no difference between the constrained and the unconstrained approach as far as momentum exchange is concerned. This statement may appear to be at odds with our above argumentation, but is not. The result of the imbalance of forces, and the violation of Newton's third law during the interaction in close proximity to the two interacting parts of the system, is a displacement for the final states, not a momentum exchange. This is not a general property, but can be shown in the impulse approximation for our unconstrained (equation (20)) and constrained force (equation (35)) by integrating the force over time for a straight path. Effects that depend on the differential cross section, such as the Sherman function for the Mott–Schwinger effect, are thus not expected to depend on the effective constraint in such a classical treatment.

Although the testing of unconstrained forces for A–B systems appears to be out of reach, a test of the Feynman paradox may be possible with current technology. Such a test would provide the first demonstration of the violation of Newton's third law (as it applies to the instantaneous conservation of mechanical momentum). Consider two electrons that are cross fired at each other. The capability to generate femtosecond electron pulses from nanoscale sources [35–37] gives control over the initial conditions of the trajectories that these electrons will follow. For electrons of about 1 keV energy the point of closest approach is of the order of microns. The capability to influence the motion of electrons in flight with a focused, pulsed laser may provide a means of making a 'movie' of the electrons' trajectory. If momentum is stored in the electromagnetic field as Feynman states, then controlling and monitoring both electron trajectories should reveal this behavior. Even with current technology, this is a major experimental challenge and perhaps explains why the Feynman paradox has never been demonstrated.

Acknowledgment

This work was supported by the National Science Foundation under grant no. 0969506.

References

[1] Shockley W and James R P 1967 Try simplest cases discovery of hidden momentum forces on magnetic currents *Phys. Rev. Lett.* **18** 876

[2] Aharonov Y and Rohrlich D 2005 *Quantum Paradoxes: Quantum Theory for the Perplexed* (Weinheim: Wiley-VCH)

[3] Ionicioiu R 2003 Quantum gates with topological phases *Phys. Rev.* A **68** 034305

[4] Feynman R P, Leighton R B and Sands M 1963 *The Feynman Lectures on Physics* vol II (Reading, MA: Addison-Wesley) pp 26–35

[5] Caprez A and Batelaan H 2009 Feynman's relativistic electrodynamics paradox and the Aharonov–Bohm effect *Found. Phys.* **39** 295–306

[6] Werner S and Klein A 2010 Observation of Aharonov–Bohm effects by neutron interferometry *J. Phys. A: Math. Theor.* **43** 354006

[7] Jaen X, Llosa J and Molina A 1989 Lagrangian formalism and retarded classical electrodynamics *J. Math. Phys.* **30** 1502–4

[8] Mott N 1929 The scattering of fast electrons by atomic nuclei *Proc. R. Soc. Lond.* A **124** 425–42

[9] Schwinger J 1948 On the polarization of fast neutrons *Phys. Rev.* **73** 407

[10] Möllenstedt G and Bayh W 1962 Messung der kontinuierlichen Phasenschiebung von Elektronenwellen im kraftfeldfreien Raum durch das magnetische Vektorpotential einer Luftspule *Naturwissenschaften* **49** 81

[11] Artamonov O M, Bode M and Kirschner J 1994 The role of elastic electron scattering in coincidence spectroscopy of W(100) in back-reflection geometry *Surf. Sci.* **307–309** 912–6

[12] Tonks L and Langmuir I 1929 Oscillations in ionized gases *Phys. Rev.* **33** 195–210

[13] Boyer T H 2002 Classical electromagnetic interaction of a point charge and a magnetic moment: considerations related to the Aharonov–Bohm phase shift *Found. Phys.* **32** 1–39

[14] Boyer T H 2008 Comment on experiments related to the Aharonov–Bohm phase shift *Found. Phys.* **38** 498–505

[15] Caprez A, Barwick B and Batelaan H 2007 Macroscopic test of the Aharonov–Bohm effect *Phys. Rev. Lett.* **99** 210401

[16] Cimmino A *et al* 1989 Observation of the topological Aharonov–Casher phase shift by neutron interferometry *Phys. Rev. Lett.* **63** 380

[17] Vaidman L 1990 Torque and force on a magnetic dipole *Am. J. Phys.* **58** 978–83

[18] Jackson J D 1999 *Classical Electrodynamics* 3rd edn (New York: Wiley)

[19] Griffiths D J 1989 *Introduction to Electrodynamics* ed C F Zita de Schauensee (Englewood Cliffs, NJ: Prentice-Hall) p 532

[20] Goldstein H, Poole C and Safko J 2002 *Classical Mechanics* 3rd edn (Reading, MA: Addison-Wesley)

[21] Dubbers D and Schmidt M G 2011 The neutron and its role in cosmology and particle physics *Rev. Mod. Phys.* **83** 1111–71

[22] Chambers R G 1960 Shift of an electron interference pattern by enclosed magnetic flux *Phys. Rev. Lett.* **5** 3–5

[23] Osakabe N *et al* 1986 Experimental confirmation of Aharonov–Bohm effect using a toroidal magnetic field confined by a superconductor *Phys. Rev.* A **34** 815–22

[24] Bachtold A *et al* 1999 Aharonov–Bohm oscillations in carbon nanotubes *Nature* **397** 673–5

[25] Feynman R P 1948 Space–time approach to non-relativistic quantum mechanics *Rev. Mod. Phys.* **20** 367–87

[26] Berman P R (ed) 1997 *Atom Interferometry* (New York: Academic)

[27] Boyer T H 1973 Classical electromagnetic deflections and lag effects associated with quantum interference pattern shifts: considerations related to the Aharonov–Bohm effect *Phys. Rev.* D **8** 1679–93

[28] Boyer T H 1988 The force on a magnetic dipole *Am. J. Phys.* **56** 688

[29] Anandan J 2000 Classical and quantum interaction of the dipole *Phys. Rev. Lett.* **85** 1354–7

[30] Anandan J 1989 Electromagnetic effects in the quantum interference of dipoles *Phys. Lett.* A **138** 347–52

[31] Sangster K *et al* 1995 Aharonov–Casher phase in an atomic system *Phys. Rev.* A **51** 1776–86

[32] Badurek G *et al* 1993 Nondispersive phase of the Aharonov–Bohm effect *Phys. Rev. Lett.* **71** 307–11

[33] Peshkin M and Lipkin H J 1995 Topology, locality and Aharonov–Bohm effect with neutrons *Phys. Rev. Lett.* **74** 2847–50

[34] Peshkin M 1999 Force-free interactions and nondispersive phase shifts in interferometry *Found. Phys.* **29** 481–9

[35] Kruger M, Schenk M and Hommelhoff P 2011 Attosecond control of electrons emitted from a nanoscale metal tip *Nature* **475** 78–81

[36] Hommelhoff P, Kealhofer C and Kasevich M A 2006 Ultrafast electron pulses from a tungsten tip triggered by low-power femtosecond laser pulses *Phys. Rev. Lett.* **97** 247402

[37] Hilbert S A *et al* 2009 Exploring temporal and rate limits of laser-induced electron emission *J. Phys. B: At. Mol. Opt. Phys.* **42** 141001