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A Criterion for Nuclear-Energized Pulsational Instability

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This Note will present briefly some general properties of nuclear-energized pulsations of stars as treated in the linear quasi-adiabatic theory. In particular, the results will be derived from a number of detailed calculations of stellar models.

It is well known that an upper limit for the fundamental eigenfrequency of radial pulsation ω_0 is given by

$$\omega_{\max}^2 = \int_0^1 (3\Gamma_1 - 4) \frac{q}{x} dq \Big/ \int_0^1 x^2 dq ,$$

where

$$\omega^2 = \left(\frac{2\pi}{\text{Period}}\right)^2 \frac{R^3}{GM}$$

(e.g., Ledoux and Walraven 1958). According to this expression, $\omega_{\max}^2$ increases both with decreasing radiation pressure (i.e., increasing Γ_1) and with increasing central condensation as measured by the ratio of integrals

$$J = \int_0^1 \frac{q}{x} dq \Big/ \int_0^1 x^2 dq .$$

(Here q represents mass fraction, x radius fraction, and J a non-dimensional ratio of gravitational potential energy to moment of inertia.) Ledoux and his associates have shown that $\omega_{\max}^2$ provides a reasonable estimate of ω_0^2 if the central condensation is not too high.

It is also well known that a decrease in radiation pressure or an increase in central condensation gives rise to a more rapid drop of pulsation amplitudes from the stellar surface inward (Ledoux and Walraven 1958; Simon and Stothers 1969). Now a sharp drop of the amplitudes contributes to pulsational stability. There are two reasons for this. First, the pulsation is relatively weaker in interior energizing regions; second, damping in the outer portions of the star is enhanced through the flux-divergence equation.

We should therefore expect a relationship between the net gain or loss rate of pulsational energy, L_P^*, and the eigenfrequency, ω_0^2, such that an increase in ω_0^2 causes a decrease in L_P^*, indicating enhanced stability. In Figure 1, the normalized rate of gain or loss of pulsational energy L_P^*/L (where L is the unperturbed luminosity) is plotted against ω_0^2 for twenty-one stellar models. Sixteen of these are core helium-burning models with thin hydrogen-burning shells; they are described in Simon and Stothers (1969). The remaining five are homogeneous main-sequence stars, four of which are discussed in Schwarzschild and Härm (1958, 1959), the fifth being the 15 $M_\odot$ model of Stothers (1965). In all cases considered the stellar *core* is found to be the energizing source, the relevant nuclear temperature exponent being $\nu \sim 18$ for the inhomogeneous models (helium cores) and $\nu \sim 13$ for the homogeneous models (hydrogen cores). The *homogeneous* models may be characterized uniquely by a parameter $A = 0.802\ \mu^4 (M/M_\odot)^2$ which is given as an upper scale in Figure 1. The limits of the range in ω_0^2 refer to polytropes of index 3, the lower limit having $\Gamma_1 = \frac{4}{3}$ ($A = \infty$) and the upper limit $\Gamma_1 = \frac{5}{3}$ ($A = 0$), as calculated, for example, by Schwarzschild (1941).

The plotted models represent a fairly wide range of stellar configurations with differing compositions, nuclear temperature exponents, and energy sources, including models with hydrogen-burning shells. In view of this diversity the linear nature of the plot is quite striking and can fairly be said to represent a general characteristic of the linear pulsation theory.

Such a characteristic becomes very useful for those cases in which $\omega_{\max}^2$ does not depart too much from the true ω_0^2. In that event, since the former quantity can be calculated very simply from the equilibrium model, it becomes possible to get a good idea of the stability of a given model without integrating the pulsation equations.

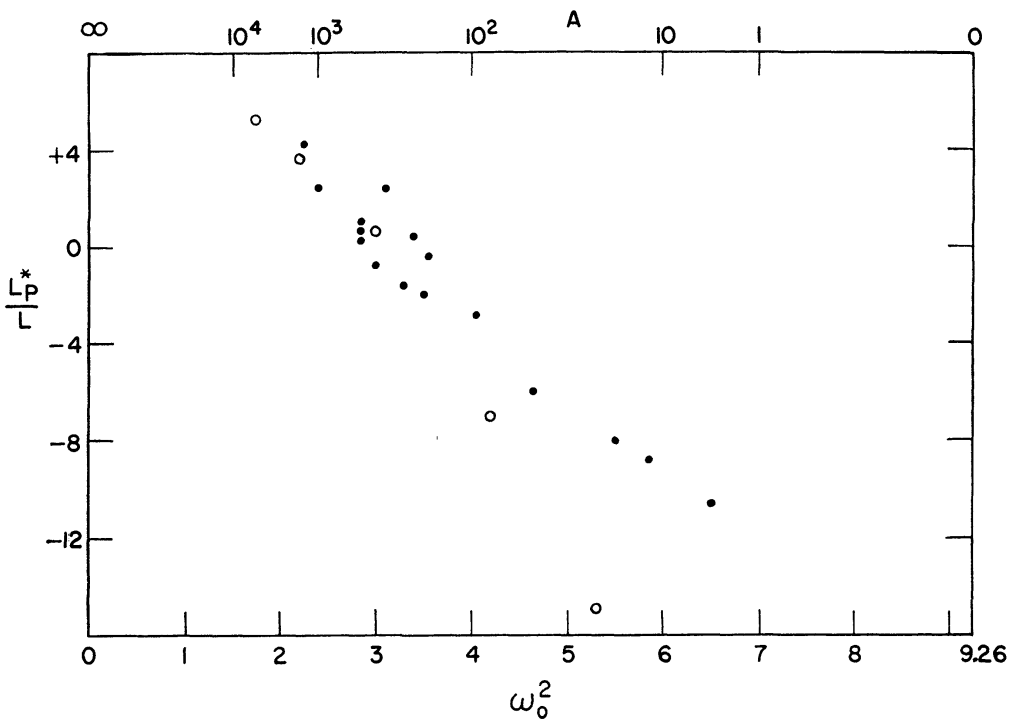

FIG. 1.—Fundamental eigenfrequency (squared) versus the net gain or loss rate of pulsational energy for stellar models. Open circles refer to homogeneous hydrogen-burning stars (characterized by the parameter A), and filled circles to stars with helium-burning cores and hydrogen-burning envelopes.

In Figure 2 the quantity $\omega_{\max}^2$ is plotted against ω_0^2. It is apparent that for a large majority of cases $\omega_{\max}^2$ approximates ω_0^2 quite well, the error ranging from 5 to 13 per cent over a range of values of $\omega_{\max}^2$ up to 5. The approximation breaks down completely for three models shown on the plot due to their extremely high degree of central condensation. According to our data, the approximation begins to fail at values of J exceeding 25 or 30. This corresponds to $\rho_c/\langle\rho\rangle$ greater than 100. (We note that J is very nearly proportional to $\rho_c/\langle\rho\rangle$, the constant of proportionality ranging from $\sim\frac{1}{3}$ at $\rho_c/\langle\rho\rangle = 20$ to $\sim\frac{1}{4}$ at $\rho_c/\langle\rho\rangle = 100$.) However, in the region of greatest interest in Figure 1, the error incurred in using $\omega_{\max}^2$ is much less than the horizontal spread of the relationship in that figure.

We may conclude that for $\omega_{\max}^2$ much exceeding 4, a star is quite stable, while instability is indicated for $\omega_{\max}^2 < 3$. When $3 \leq \omega_{\max}^2 \leq 4$ the case is marginal, and pulsation integrations are probably necessary to determine the stability.

It may be noted that Figure 2 gives a reasonably tight correlation between $\omega_{\max}^2$ and ω_0^2 for all the plotted points. Hence we may use $\omega_{\max}^2$ to estimate ω_0^2 (and therefore L_P^*/L) even for models lying outside the range of approximate equality.

To complete our general investigation of stability, we have calculated first-overtone pulsations for three extreme models. Two of these are among the core helium-burning models with hydrogen-burning shells previously discussed; the third is a homogeneous hydrogen-burning model, constructed to produce a highly unstable fundamental mode. Model and pulsational characteristics are given in Table 1.

The homogeneous model illustrates strikingly the overwhelming stability of the overtones. The reason for this is the extremely rapid drop of pulsation amplitudes near the surface, which characterizes overtone pulsations. Clearly it would be extremely difficult, if not impossible, to energize overtone pulsations during ordinary nuclear burning. The core helium-burning models were included to examine the effect of a burning shell on overtone pulsations. It has occasionally been suggested that when an energizing source arises in the outer layers of a star, an overtone pulsation becomes the preferred mode. We have found no indication that this is true. On the contrary, pulsational characteristics in this class of models always seem to be determined by radiation pressure and central condensation with the fundamental the overwhelmingly preferred unstable mode.

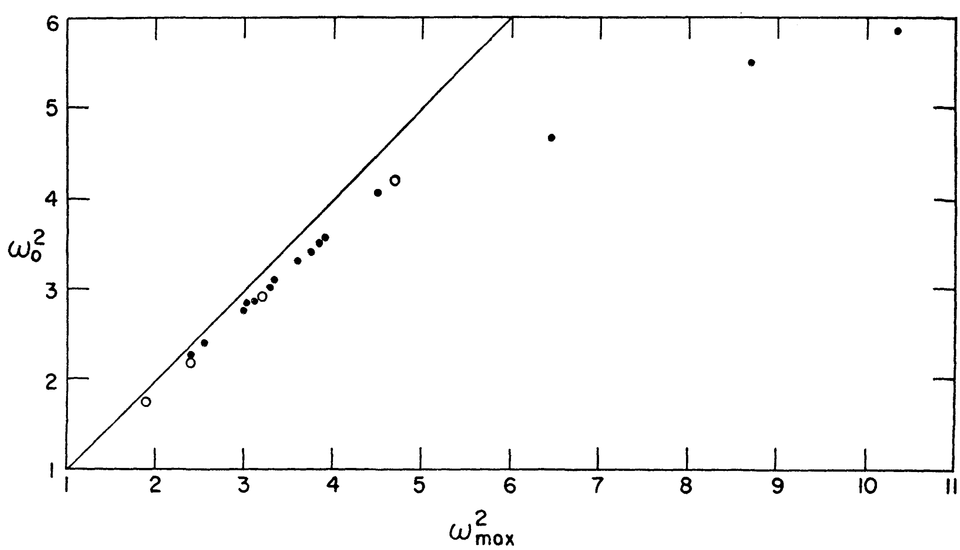

FIG. 2.—Estimated fundamental eigenfrequency (squared) versus actual value for stellar models of Fig. 1.

TABLE 1

PULSATIONAL CHARACTERISTICS OF SELECTED MODELS*

	60 $M\odot$		15 $M\odot$		400 $M\odot$	
X_R	0 15		0 05		0 70	
q_s	0 80		0 97		Homogeneous	
Mode	0	1	0	1	0	1
ω^2	6 52	13 9	3 08	12 0	1 39	10 5
Period (hours)	5 1	3 5	0 45	0 23	17 5	6 4
$(\delta r/r)_R$	1 000	1 000	1 000	1 000	1 000	1 000
$(\delta r/r)_s$	0 009	− 0 003	0 606	0 028	...	...
$(\delta r/r)_c$	0 005	− 0 002	0 467	− 0 029	0 680	− 0 045
L_P^*/L	−10 5	−36 2	2 39	−35 3	6 94	−22 8

* $Z = 0\ 03$; $Y = 0\ 97$ below q_s

Finally we should emphasize that our conclusions throughout apply only to *ordinary* nuclear-energized pulsations. In cases where a thermal instability arises under "flash" conditions (e.g., Rose 1967), such that pulsational instability also develops, strongly enhanced energizing is possible, perhaps even to the extent that an overtone might become excited.

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