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Physics, Chapter 43: X-Rays

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X-Rays

43-1 Discovery of X-Rays

The study of the electric discharge through gases led directly to the discovery of x-rays by W. C. Roentgen in 1895. While operating a gas-discharge tube, Roentgen observed that a platinum-barium cyanide screen at some distance from the tube fluoresced. He shielded the tube so that no visible radiation could reach the screen, but the fluorescence could still be observed. On interposing various materials between the tube and the screen, he found that the intensity of the fluorescence could be diminished, but that it was not completely obliterated. He interpreted these observations as being due to radiation coming from the walls of the tube which penetrated the absorbing screens and caused the screen to fluoresce, and he called the new radiation *x-rays* to indicate their unknown character. The x-rays were produced when the cathode rays struck the glass walls of the electric-discharge tube.

Ever since their discovery, x-rays have played an important part in the investigations of atomic physics and have added immeasurably to our knowledge of the structure of the atom. Almost immediately after their discovery, they were used by physicians as aids in diagnosis, and later in therapy. Industry turned to the use of x-rays for the study of the internal structure of materials and for the examination of castings to determine the presence of internal voids.

X-rays are produced whenever a stream of electrons strikes some substance and are rapidly accelerated. The electrons may come from the cathode of a gas-discharge tube, or, more commonly, from a heated filament. In the Coolidge-type x-ray tube, shown in Figure 43-1, a high potential difference, ranging from a few thousand volts to about a million volts, is applied between the cathode and anode. The electrons are accelerated and strike the target, that is, the anode, causing it to emit x-rays. For many purposes it is desirable to limit the region from which x-rays are

emitted, called the focal spot, and this may be done by surrounding the filament with a metallic cup which shapes the lines of force in such a way as to focus the electrons into a small region on the target. It is difficult to maintain potential differences greater than a million volts between two electrodes separated by the glass walls of a vacuum tube. The *betatron*,
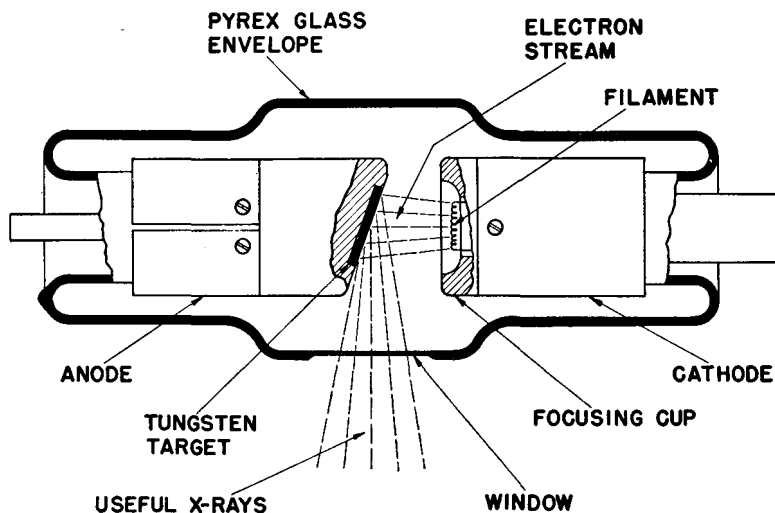

Fig. 43-1 A modern Coolidge-type x-ray tube. (Courtesy of General Electric X-ray Department.)

developed by D. W. Kerst in 1941, utilizes Faraday's law of electromagnetic induction as the means of accelerating electrons. The betatron is pictured in Figure 43-2 and is shown schematically in Figure 43-3.

In Figure 43-3 we see that electrons from a heated filament F are accelerated by a small difference in potential applied to an injector grid G inside a doughnut-shaped vacuum tube. The electrons are made to travel in a circular path by a magnetic field perpendicular to the plane of the paper, at the position of the orbit. The electrons are constrained to move within a ring of small cross section by the shape of the magnetic field. Electrons moving within the vacuum tube may be thought of as though they were moving in a circular loop of wire or in the coil of a transformer. The magnetic flux is caused to change rapidly through the central portion of the doughnut. This changing flux induces an emf such that each time the electron completes a circular turn, its energy has increased by the product of the electron's charge by the emf, as in the case of the transformer. As the energy of the electron is increased, it is necessary to adjust the magnetic field intensity at the orbit to keep the electrons rotating in a circle

of fixed radius.   In many betatrons the changing magnetic field is produced
by supplying 60-cycle alternating current to the flux coils of the electro-
magnet.   Electrons are injected into the tube for a very short time at the
beginning of a cycle and continue traveling around the circular orbit until
the magnetic flux reaches its maximum value, in 1/240 sec.   Each electron
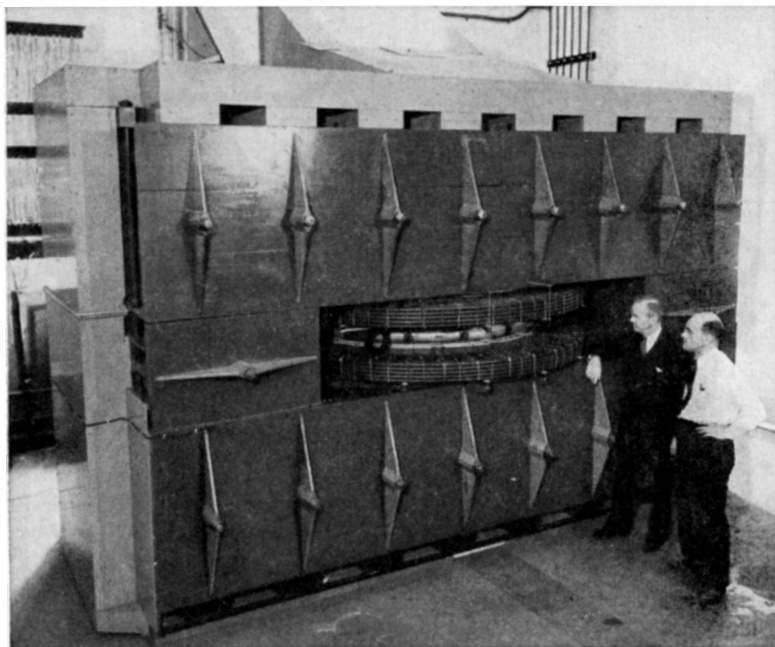makes hundreds of thousands of revolutions during this quarter cycle,

Fig. 43-2   The 100,000,000 electron-volt betatron.   The betatron tube is in the center
between the poles of the electromagnet.   (Courtesy of General Electric Company.)

gaining energy in each revolution.   When the electron has gained its
maximum energy, current is sent through an additional set of coils, expand-
ing the radius of the electron orbits so that the electrons strike a target,
generating x-rays.

Betatrons are now being operated at energies up to 300 million electron
volts (abbreviated Mev) and modified forms of betatrons are operating at
higher energies.   Since the electrons are injected at the beginning of a
cycle, the x-ray output occurs in short pulses, at the peak of each cycle,
when the electrons have reached maximum energy.   The electrons are
moving with speeds very close to the velocity of light, and at these high
speeds the emitted x-rays are practically confined to a narrow cone of
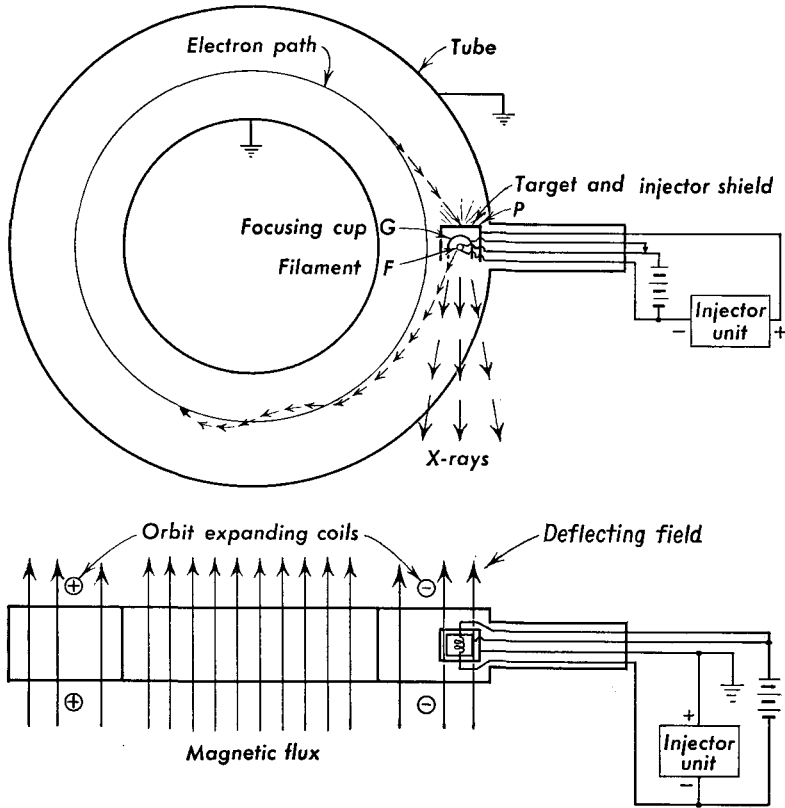aperture ranging from 2° to 15° in the forward direction.

Fig. 43-3   Path of an electron in a betatron tube.

43-2   Diffraction and Interference of X-Rays

Early attempts were made to determine the character of x-rays, but the results were quite puzzling. The rays were not deflected by an electric or magnetic field and so did not consist of charged particles. Although it is known today that x-rays are refracted very slightly by matter, early attempts to show refraction failed. The rays passed through electrical conductors, contrary to our expectations for the behavior of an electro-magnetic wave. Early attempts to produce interference effects also failed. In 1912 von Laue argued that the reason that interference effects had not been demonstrated was the very short wavelength of x-rays, and suggested that x-rays might have a wavelength of the order of the spacing of the atoms in the lattice of a crystalline solid. With his collaborators he was able to produce diffraction effects by passing a beam of x-rays through a thin crystal.

If a narrow pencil of x-rays is sent through a small thin crystal, such as a piece of rock salt, and received on a photographic plate a short distance away, as shown in Figure 43-4, the photograph will show a series of small spots arranged in a definite pattern. As shown in Figure 43-5, this pattern,
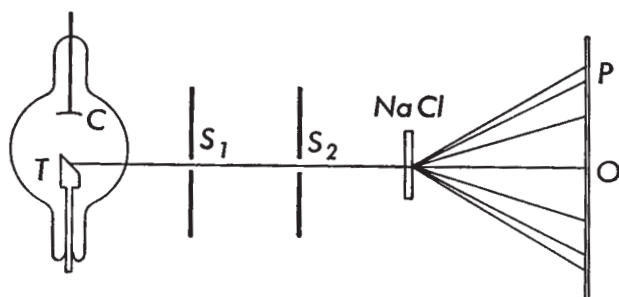

Fig. 43-4 Arrangement of apparatus for producing a Laue diffraction pattern using a rock salt crystal. S_1 and S_2 are pinholes and P is the photographic plate.

as obtained with rock salt, has a fourfold symmetry characteristic of a cubic crystal. The pattern may be interpreted by thinking of each ion of the crystal as a scattering center for the incident beam of x-rays. The crystal thus acts as a three-dimensional diffraction grating. The pattern

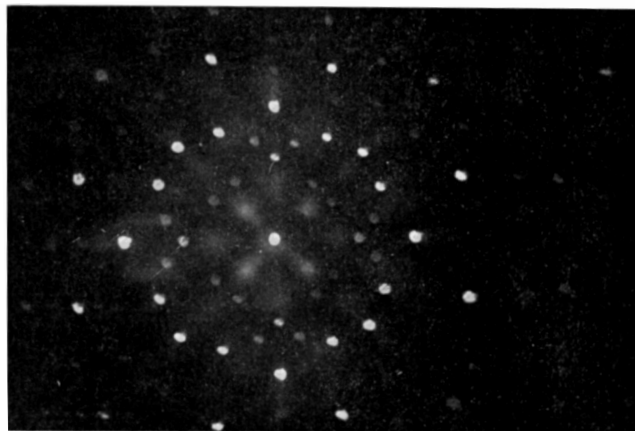**Fig. 43-5** Photograph of Laue diffraction pattern of rock salt. (From photograph by J. G. Dash.)

obtained on the photographic plate is called a Laue pattern. From the distribution and intensities of the points on the photographic plate, the arrangement of the ions in the crystal can be deduced.

A slightly different arrangement of the x-ray beam and crystal, used by Bragg, gives a simpler pattern which may be interpreted more easily.

This is shown in Figure 43-6, in which x-rays coming from the target T of the x-ray tube pass through two narrow slits, and are then incident upon the face of a crystal which is mounted on a spectrometer table. The crystal scatters the x-rays in all directions, but the photographic plate is set so as to receive only that part which comes from the face of the crystal.
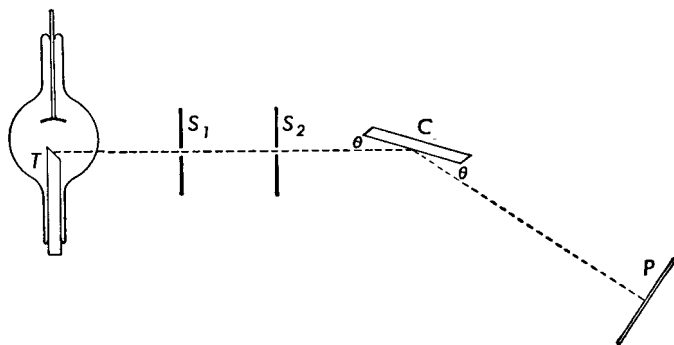

Fig. 43-6   The single crystal x-ray spectrometer with photographic plate. S_1 and S_2 are narrow slits, C is the crystal, and P is the photographic plate.

The angle θ between the incident beam and the face of the crystal is changed slowly by rotating the crystal. In general, the photograph will show a series of sharp lines against a continuous background. If an ionization chamber is used as a detector, in place of a photographic plate, it is found that the intensity of the x-rays is a maximum when the scattered beam makes an angle θ with the face of the crystal equal to the angle that the incident beam makes with it. For this reason the beam is sometimes said to be "reflected" from the crystal.

A simple explanation of the effect of the crystal in this case may be given with the aid of Figure 43-7, in which the ions of the crystal are arranged in layers parallel to the surface of the crystal. The distance d between atomic layers is shown greatly enlarged. The x-ray beam incident upon the crystal at an angle θ with its face penetrates the crystal and is scattered by the ions in all directions. Consider two rays I and II, very close together, and consider only that part of the scattered beam which makes an angle θ with the surface of the crystal. Ray I strikes the upper surface at A and is reflected; ray II strikes the next layer at B and is reflected. These two rays, which are so close together that they interact with the same grain of photographic emulsion, have traveled different distances. From the figure it is evident that ray II has traveled a longer distance than ray I. If originally they started out in phase, they will now differ in phase because of the difference in paths of $CB + BD$. If this difference in path is an integral number of wavelengths $n\lambda$, the two rays

will reinforce each other.   Thus we have constructive interference when

$$CD + BD = 2d \sin \theta = n\lambda. \qquad \text{(43-1)}$$

Equation (43-1), called the *Bragg equation*, is of fundamental importance in determining x-ray wavelengths and in the analysis of the structure of crystals.   As in the case of the diffraction grating, we refer to the bright spot when $n = 1$ as the first-order spectrum, when $n = 2$ we have the second-order spectrum, and so on.   When x-rays of a particular wave-
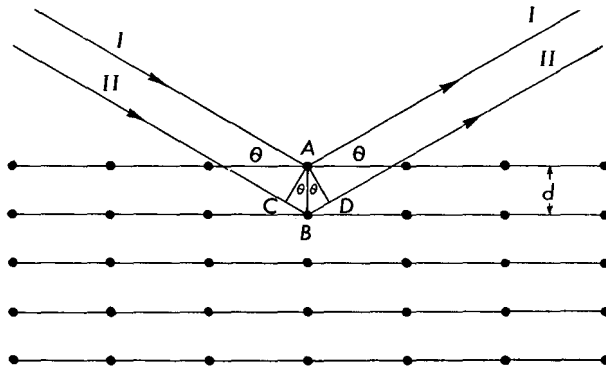
Fig. 43-7   Reflection of x-rays from atomic planes.

length are incident upon a crystal, reflection occurs only at angles where the Bragg equation is satisfied.   Although the above development considered atomic planes parallel to crystal face, we may consider any set of planes drawn through the atoms of the crystal as capable of reflecting x-rays. If we consider the first-order diffraction pattern produced when a continuous spectrum of x-rays is incident upon a crystal, as in the case of the Laue pattern, we see that a bright spot is produced in diffraction for those wavelengths and angles fulfilling the conditions of the Bragg equation. If a crystal has a fourfold symmetry with respect to the direction of the incident beam, the spots produced by one set of crystal planes will appear four times in the pattern, corresponding to the symmetry of the crystal. For each set of planes there is a different value of the grating space d.

Illustrative Example.   An x-ray line of wavelength 1.541 A is reflected from a set of planes in a quartz crystal in which the distance d between planes is 4.255 A.   Determine the angle between the x-ray beam and the atomic planes in the second order.

From the Bragg equation we find

$$\sin \theta = \frac{n\lambda}{2d},$$

and, substituting numerical values, we have

$$\sin \theta = \frac{2 \times 1.541 \times 10^{-8} \text{ cm}}{2 \times 4.255 \times 10^{-8} \text{ cm}} = 0.362,$$

$$\theta = 21° 14'.$$

43-3   X-Rays and Crystal Structure

In using the Bragg equation, it is essential that we know independently either the x-ray wavelength λ or the spacing d between atomic or ionic planes. At the time of the discovery of x-ray diffraction, the crystal grating
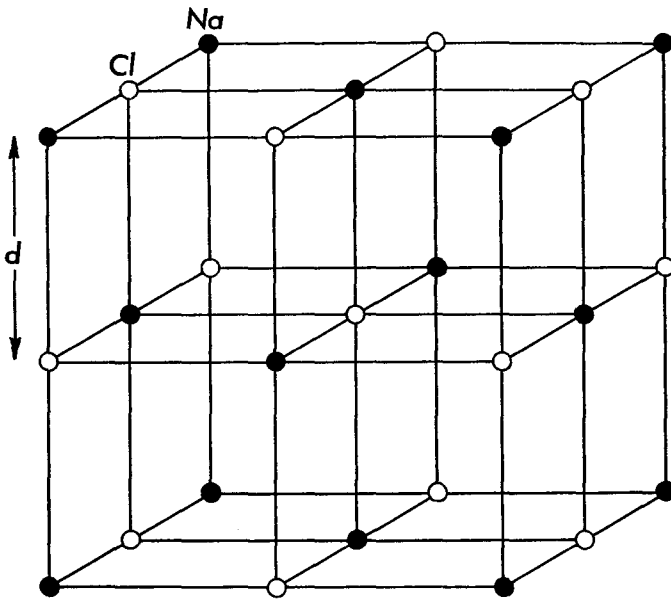
Fig. 43-8   The arrangement of sodium (Na) ions and chlorine (Cl) ions in a crystal of salt.

space of rock salt was determined from other data, and the value of d so obtained was used to measure x-ray wavelengths. Figure 43-8 shows the arrangement of the ions in a rock-salt crystal. From crystallographic studies, the crystal is known to be a cube; the centers of the ions are at the corners of the cubes. Each sodium ion (Na^+) is surrounded by 6 chlorine ions (Cl^-), and each chlorine ion is surrounded by 6 sodium ions. If d is the length of the smallest cube which can be drawn through neighboring ions, then the volume of each cube is $V = d^3$. This is the volume associated with each ion. If M is the gram molecular weight of sodium chloride

and ρ is its density, then the volume v of 1 mole of sodium chloride is

$$v = \frac{M}{\rho}.$$

Now there are $2N_0$ ions in each mole of sodium chloride, where N_0 is the Avogadro number; hence the volume associated with each ion is

$$V = \frac{v}{2N_0} = \frac{M}{2\rho N_0}.$$

The distance d between ions is therefore

$$d = \left(\frac{M}{2\rho N_0}\right)^{\frac{1}{3}}. \tag{43-2}$$

Let us calculate the grating space d of rock salt. The gram molecular weight M is the sum of the gram atomic weights of sodium and chlorine and is

$$M = 22.997 + 35.457 = 58.454 \text{ gm};$$

the density is $\rho = 2.164 \text{ gm/cm}^3$ at 18°C,

and $N_0 = 6.0248 \times 10^{23};$

hence we find $d = 2.820 \times 10^{-8} \text{ cm} = 2.820 \text{ A, at 18°C.}$

With the value of the grating space of rock salt known, it is now possible to measure x-ray wavelengths, using a rock salt crystal as a
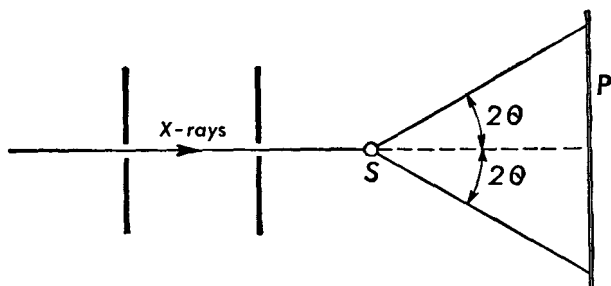
Fig. 43-9 Method of obtaining x-ray diffraction patterns using a powder.

spectrometer. The crystal used must be sufficiently large and well developed, say of the order of 1 to 2 cm in length and width. Such large crystals, sometimes called single crystals, may be found in nature and may sometimes be grown from saturated solutions or by the slow cooling of molten material in the case of metals. Once the wavelength of an x-ray beam is known, that beam may be used to determine the structure of other crystals.

The ordinary solid is not a large crystal but is made up of many very small crystals called *microcrystals*. Even when the solids are in powder form, they consist of many microcrystals. A very powerful method of x-ray analysis was developed by A. W. Hull, and independently by P. Debye and P. Scherrer. This method consists in sending a narrow pencil of x-rays of a single wavelength through a very small sample of the powder or solid, as shown in Figure 43-9. Since the powder consists of a great many

Fig. 43-10 X-ray powder diffraction pattern of aluminum. (Reproduced with the permission of A. W. Hull.)
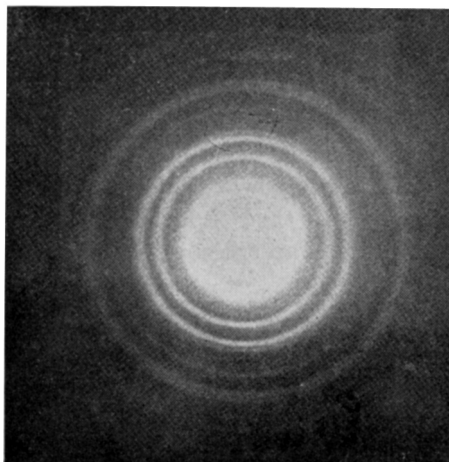
microcrystals oriented at random, there is some probability that one of these microcrystals will be oriented so that its atomic planes make an angle θ with the incident radiation which will satisfy the Bragg equation. If the crystals are randomly oriented, the radiation reflected from this plane will be symmetrically distributed with respect to the incident beam and will form a conical shell centered at the sample S. A photographic plate, placed at right angles to the direction of the incident beam at a convenient distance from the sample, will record the diffraction pattern as a set of circles, as shown in Figure 43-10. Each circle corresponds to a particular set of atomic planes whose spacing may then be determined. The planes we draw through the atoms of the crystal lattice will have different numbers of atoms per square centimeter of plane. Furthermore, in a crystal, such as sodium chloride, it may be possible to draw planes containing atoms of one kind, as all sodium atoms or all chlorine atoms. These planes reflect x-rays with different intensities. When the resulting diffraction pattern is interpreted, each of these details must be accounted for.

In a variation of this method, the photographic plate is replaced by a photographic film bent in the form of a cylinder with the sample at its center. Holes are cut in the film so that the direct pencil of x-rays can

enter and leave this camera without blackening the film. When this film is unwrapped and developed, we obtain the type of pattern shown in Figure 43-11. The x-ray powder diffraction pattern has lines whose position and intensity are determined by the crystal itself. Thus x-ray patterns may be used to identify unknown crystals. The procedure followed is to

Fig. 43-11 X-ray powder diffraction pattern of tungsten obtained with a photographic film bent in the form of a circular cylinder. X-rays from a copper target were used in making this photograph. (From a photograph made by L. L. Wyman and supplied by A. W. Hull.)

prepare patterns from a large number of known substances, and to code these patterns in terms of the *d* spacings of the three most intense lines obtained in the powder pattern. The pattern obtained with an unknown substance may then be identified, in much the same way that the atomic composition of an unknown is determined from its optical spectrum.

43-4 X-Ray Spectra and Atomic Structure

When x-rays from any target are analyzed with a crystal spectrometer, the spectrum is found to consist of a series of sharp lines superimposed on a continuous background of radiation, as shown in Figure 43-12. The energy in the continuous portion of the spectrum is found to depend upon the voltage across the tube, the current passing through the tube, and the atomic number of the element which constitutes the target; the higher the atomic number, the greater is the energy, the other quantities remaining constant. The sharp lines which are superimposed on the continuous radiation are found to be characteristic of the target.

The first systematic study of the characteristic x-ray spectra of the elements was made by Moseley in 1913. He used a modification of the Bragg method in which the crystal spectrometer and the photographic plate were placed in an evacuated chamber to avoid absorption in air of the x-rays of long wavelength. Each element investigated was used as the target of an x-ray tube. He found that all the elements gave similar types of spectra; the lines emitted by each element were classified into two groups or series: a group of short wavelengths called the *K* series and a group of comparatively long wavelength called the *L* series. These two series are widely separated from one another in wavelength, as illustrated in Figure 43-13 for the case of silver. Other investigators have found two other series of still longer wavelengths in the heavier elements, classified as *M* series and *N* series. One of the important results of Moseley's work was the discovery that the square root of the frequency of the K_α lines of the

elements he investigated was proportional to the atomic number Z. Moseley's work provided certain evidence that elements were missing from
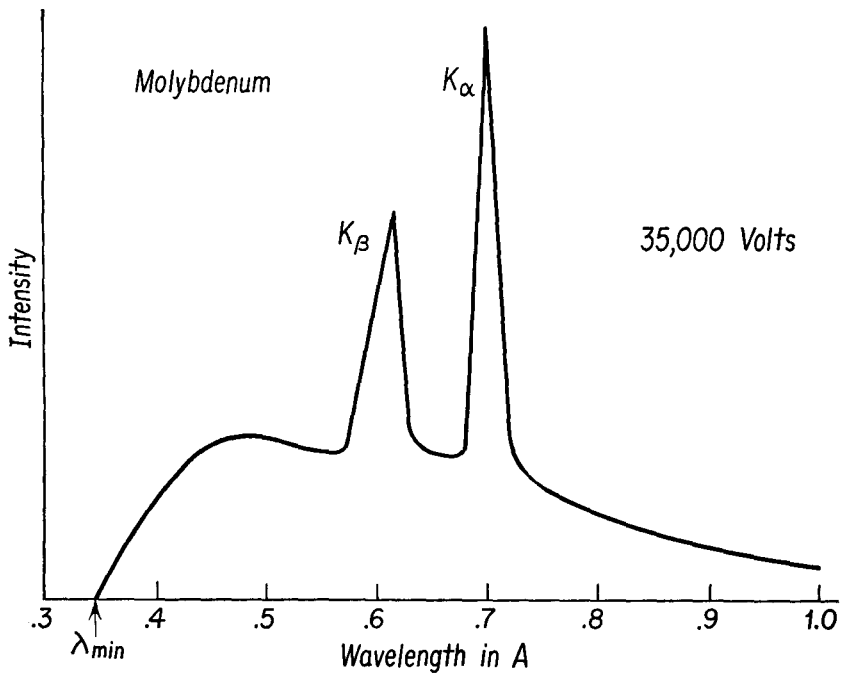
Fig. 43-12　Characteristic K_α and K_β lines superposed on the continuous x-ray spectrum of molybdenum.　Note the sharp cutoff at the wavelength λ_{min}.
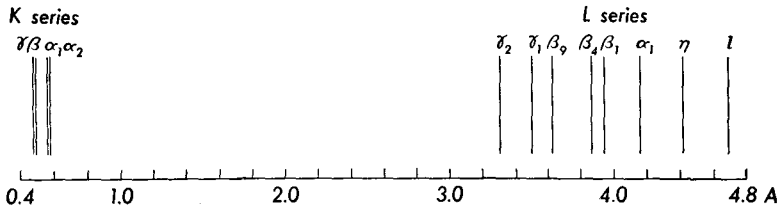
Fig. 43-13　Relative positions of the K and L x-ray series lines of silver.

the periodic table, as known at that time, and that the atomic number rather than the atomic weight was the key to the properties of the elements.

The characteristic x-ray spectra of the elements fit neatly into our theory of atomic structure.　For example, the frequency of the most intense line of the K series can be written in the form

$$f = cR(Z - 1)^2 \left(\frac{1}{1^2} - \frac{1}{2^2}\right),$$ 　　　　**(43-3)**

in which R is the Rydberg constant, c is the velocity of light, and Z is the atomic number of the element emitting this line of frequency f. The interpretation of this equation is that this line is emitted when an electron goes from the L shell in which the principal quantum number is 2 to the K shell in which the principal quantum number is 1. All lines of the K series are associated with the transition of an electron from an outer orbit to the K shell.

To understand the appearance of the factor $(Z - 1)$ rather than Z in the equation, let us consider how the x-rays are produced. In the atoms of the elements investigated, the orbits of the principal quantum numbers 1 and 2 are completely filled. An electron from the cathode of the x-ray tube strikes the target with energy Ve, where V is the potential difference between the anode and cathode of the x-ray tube. When such an electron approaches an atom in the target, this electron may knock an electron out of the orbit for which $n = 1$. Since the other electrons in this atom are in their normal states, the electron which has been ejected from the K shell is removed outside the atom. The atom is thus ionized and left in an excited state. It is highly probable that an electron from the L shell may jump into the vacancy left behind in the K shell, thereby emitting radiation of frequency f, given by Equation (43-3). As for the factor $(Z - 1)$, since normally the orbit for which $n = 1$ has 2 electrons, when one is removed the other electron still remains in this orbit, and effectively "screens" the Z positive charges of the nucleus. Hence when an electron goes from the L shell to the K shell it moves in an electric field due to Z positive charges and 1 negative charge, or effectively $Z - 1$ positive charges.

The interpretation of the x-ray spectra is simplified by the use of an energy-level diagram such as that shown in Figure 43-14, Let us take the normal atom with all of its electrons in their normal states as our zero level of energy. When an electron is removed from the innermost orbit, a certain amount of energy must be supplied to do this; let us call this amount of energy $\mathscr{E}_K$, and we shall say that the atom is in the K state. If, however, an electron from the L shell is removed from the neutral atom, than a smaller amount of energy will be required; we shall call this amount of energy $\mathscr{E}_L$ and say that the atom is now in the L state. An atom in the M state is one in which an electron is removed from the M shell, for which $n = 3$, and so on. These energy states are plotted in the diagram.

Suppose that an atom has been raised to the K state by the removal of an electron from orbit $n = 1$. If an electron goes from orbit $n = 2$ to orbit $n = 1$, the atom will be left in the L state, and the frequency of the emitted radiation will be

$$f = \frac{\mathscr{E}_K - \mathscr{E}_L}{h}.$$ **(43-4)**

In the diagram this is called the K_α line. Similarly, the K_β line is emitted in the transition from the M shell to the K shell, and so on.

The continuous x-ray spectrum provides us with another important verification of the quantum character of electromagnetic radiation. In Figure 43-12 we note that the continuous spectrum exhibits a minimum
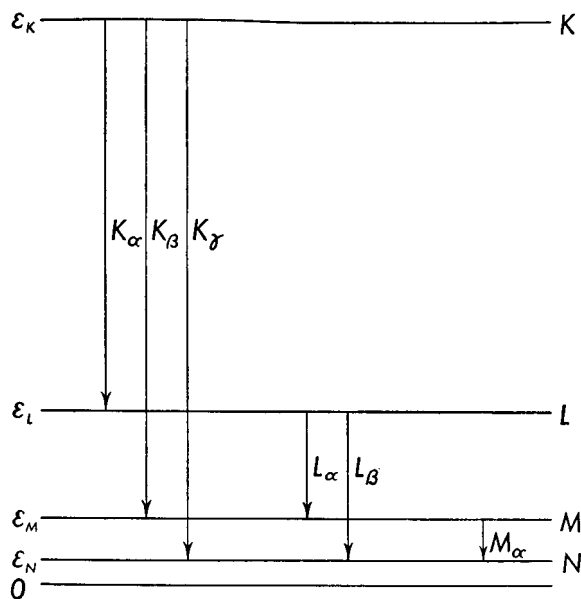

Fig. 43-14   Simplified x-ray energy-level diagram.

wavelength, λ_{min}. When the voltage across the tube is increased, the short-wavelength limit is shifted toward smaller values. The continuous spectrum results from the acceleration of the electrons when they strike the target. The energy may be radiated in a single quantum or in several quanta, but no single quantum can be emitted which has more energy than the kinetic energy of the incident electron. Thus we have

$$Ve = hf_{max} = \frac{hc}{\lambda_{min}}, \qquad (43\text{-}5)$$

where V is the voltage across the x-ray tube, e is the electronic charge, and h is Planck's constant. Ve represents the energy with which an electron strikes the target. Duane and Hunt carried out a series of careful experiments on the determination of the short-wavelength limit of the continuous x-ray spectrum for various voltages across the x-ray tube and found the

value of Planck's constant to be

$$h = 6.556 \times 10^{27} \text{ erg sec,}$$

in good agreement with other determinations made optically.

The information obtained by the study of optical and x-ray spectra has played an important part in establishing our present view of atomic structure, particularly the arrangement and distribution of electrons of the atom. Thus the energy-level diagram may be verified in the absorption of x-rays by means of the photoelectric effect. The interaction of x-rays with atoms is associated with the filled inner shells, while the interaction of visible light with matter is associated with the outermost electrons. The energy of the emitted photoelectron may be determined by means of a magnetic spectrometer. When the photoelectron is emitted from the K shell, we find that

$$\mathscr{E}_e = hf - \mathscr{E}_K,$$

where f is the frequency of the incident x-rays, $\mathscr{E}_K$ is the K level energy, and $\mathscr{E}_e$ is the kinetic energy of the emitted photoelectron.

43-5   Particles and Waves

As we have seen, radiation has a dual character, that of a wave motion and that of a corpuscular nature. The wave character of radiation has been amply verified by the phenomena of interference, diffraction, and polarization which we have outlined for x-rays and for visible light.

The corpuscular character of radiation was first introduced by Planck to explain the distribution of energy in the continuous spectrum of a black body at a high temperature. This idea was used by Einstein to explain the photoelectric effect and by Bohr to explain the spectrum of hydrogen, and was found applicable to the spectra of higher-atomic-number elements and to x-ray spectra. Another very convincing argument concerning the corpuscular nature of electromagnetic radiation is provided by the *Compton effect.*

From classical electromagnetic theory it is known that a beam of radiation has momentum as well as energy. As we have seen in our discussion of momentum and impulse, force is equal to the rate of change of momentum. When a light beam is incident upon a black plate and is absorbed by it, the light beam not only delivers energy to the plate but delivers momentum as well, thereby changing the momentum of the plate. The light beam thus exerts a force on the plate. This was verified experimentally by Nichols and Hull in 1903, who measured the force exerted by light from an arc lamp on a blackened disk suspended from a quartz fiber. The force per unit area, called the radiation pressure, measured by Nichols

and Hull was about 7×10^{-5} dyne/cm^2; this value was within 2 per cent of that calculated from electromagnetic theory.

If the concept of momentum applies to a light beam, it must apply to the quanta as well. It will be recalled that a quantity of energy $\mathcal{E}$ has a mass m given by

$$\mathcal{E} = mc^2, \tag{7-12b}$$

where c is the speed of light. Now the energy of a quantum of light, or a photon, is given by

$$\mathcal{E} = hf,$$

hence the mass of a photon is

$$m = \frac{hf}{c^2}. \tag{43-6}$$

Since momentum is the product of the mass of a particle by its velocity, the momentum p of a photon is

$$p = mc,$$

so that

$$p = \frac{hf}{c}. \tag{43-7}$$

Let us consider the scattering of a beam of x-rays by matter. From the point of view of electromagnetic waves, we would imagine that the electric vector of the electromagnetic wave caused the charged particles of the substance to vibrate, and that these particles would reradiate energy of the same frequency as the incident beam. The scattering would be due to the electrons of the substance, for their acceleration would be greater than the acceleration of the nucleus under the application of a given force. On examination of the wavelength of the scattered beam with an x-ray spectrometer, A. H. Compton found that the scattered beam contained not only x-rays of the original frequency f and wavelength λ but also radiation of a lower frequency f' and thus of longer wavelength λ'. The explanation of this effect, given by Compton, is that the x-ray photon makes an elastic collision with an electron and is scattered by it, and, if the electron is free to move, the photon gives up some of its energy to the electron. Using the principle of conservation of energy, we can write

$$hf = hf' + \tfrac{1}{2}mv^2, \tag{43-8}$$

where hf is the energy of the incident photon, hf' the energy of the scattered photon, and $\tfrac{1}{2}mv^2$ the kinetic energy of the scattered electron. Since this is a collision between two particles, we can apply the principle of conservation of momentum to this process. Figure 43-15 shows the incident photon with momentum hf/c, the photon scattered through an angle θ

with momentum hf'/c, and the electron recoiling as a result of the collision
with momentum mv in a direction making an angle ϕ with the original
direction of the incident radiation.   From the principle of conservation of
momentum, we get the equations

$$\frac{hf}{c} = \frac{hf'}{c} \cos \theta + mv \cos \phi \qquad \textbf{(43-9a)}$$

and
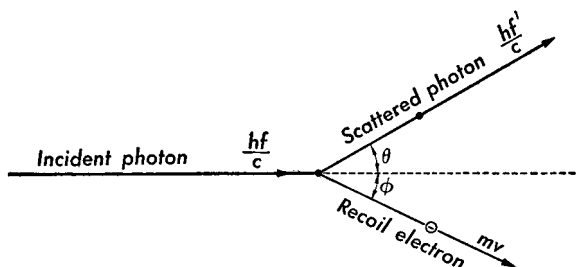$$0 = \frac{hf'}{c} \sin \theta - mv \sin \phi. \qquad \textbf{(43-9b)}$$

Fig. 43-15 The Compton effect; scattering of a photon and recoil of an electron as a
result of a collision.

Equation (43-9a) states that the initial momentum along the original direc-
tion of the x-ray beam is equal to the components in this direction of the
final momenta of the scattered photon and the recoil electron.   Equation
(43-9b) is a similar expression for the components of the momenta at right
angles to the original direction of motion.

    If the above three equations are solved for the frequency of the scat-
tered photon, and the result is then converted to their corresponding wave-
lengths, the result obtained is

$$\boxed{\lambda' - \lambda = \frac{h}{mc} (1 - \cos \theta),} \qquad \textbf{(43-10)}$$

where λ' is the wavelength of the ray scattered at an angle θ, and λ is the
original wavelength.   The change in wavelength depends only upon the
angle of scattering and not upon the substance.   The results predicted by
Equation (43-10) have been amply verified by many careful experiments.
The energy and momentum of the recoil electron have also been measured,
and the results have been found in agreement with the predictions made
from the solutions of the above equations.   Once again, radiation was
found to have a corpuscular character in its interaction with matter.

As has been previously stated (Section 42-6), Louis de Broglie, using the dual character of radiation as a guide, put forth the hypothesis that there should be a wave motion associated with every material particle. If m is the mass of a particle and v its speed, its momentum is $p = mv$. The wavelength λ associated with this particle should be given by the same relationship as that connecting wavelength and momentum of electromagnetic waves

$$\lambda = h/p,$$

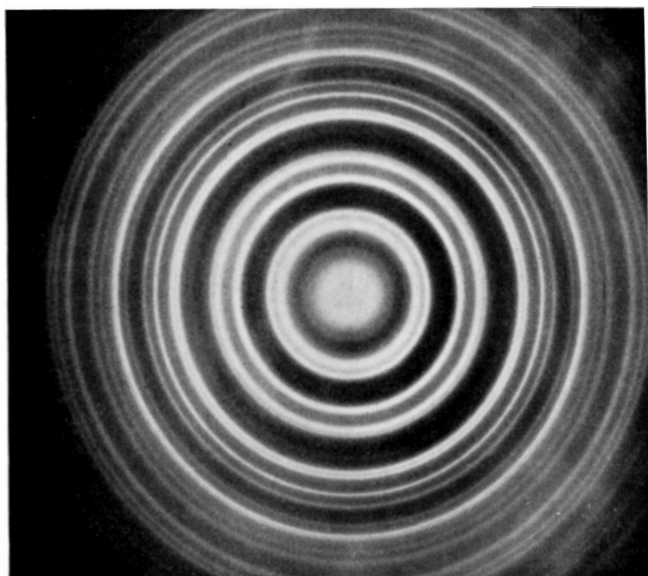so that $$\lambda = h/mv. \tag{43-11}$$

Fig. 43-16   Diffraction pattern obtained by passing a beam of electrons through gold foil; thickness of foil was about 250 A. (Reproduced from photograph by Oliver Row and N. R. Mukherjee.)

De Broglie's hypothesis was verified in two classic experiments—one performed by Davisson and Germer in 1927, and the other performed by G. P. Thomson in 1928. In the Davisson-Germer experiment, a stream of electrons was reflected from a nickel crystal. Intense maxima were observed at certain angles of reflection, which could be explained only in terms of constructive interference of electron waves from different atomic layers of the nickel crystal. The Bragg equation together with the de Broglie equation provide an explanation of the reflection maxima. Electrons, formerly thought to be wholly particlelike in character, were found to display wave properties.

In the Thomson experiment, a stream of electrons of known velocity was sent through a very thin metal foil. Since a metal foil consists of many microcrystals oriented at random, the pattern obtained is similar to that obtained in the powdered crystal x-ray diffraction experiments. From the
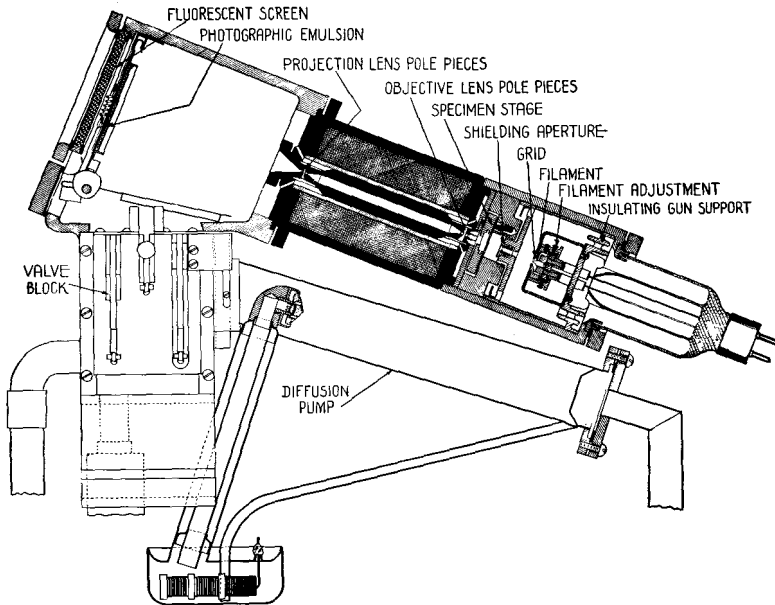

Fig. 43-17   A simplified cross section of a small electron-microscope unit. (Courtesy of RCA Laboratories.)

results of these experiments, G. P. Thomson was able to compute the grating space of metallic crystals in the thin foil, and the results agreed very well with those obtained by x-ray diffraction. The diffraction pattern obtained when a narrow beam of electrons is sent through a thin gold foil is shown in Figure 43-16.

The wave character of material particles is not limited to electrons but is characteristic of all matter. Diffraction patterns have been obtained by reflecting hydrogen and helium molecules from crystal surfaces. The wavelengths computed from the results of these experiments agreed with those calculated from Equation (43-11), where the velocity v of the molecules was determined from a knowledge of the temperature of the gas.

One of the practical results of the de Broglie hypothesis was the development of the *electron microscope*. We recall that the resolving power of a microscope is limited by the wavelength of the illuminating radiation. By using electrons of appropriate speed, the wavelength of these electron waves may be made any desired value. The electron beam may illuminate

a specimen by means of properly arranged electric and magnetic fields which form *electron lenses* and may then be focused onto a fluorescent screen or a photographic plate to produce an image of the specimen. Such an electron microscope is shown in Figure 43-17, while pictures of a specimen taken optically and with an electron microscope are shown in Figures 43-18 and 43-19. In each of the two pictures the actual resolving power is about a factor of 3 from the theoretical limit of resolution of the microscopes used. The highest resolution obtained with an electron microscope is about 20 A.
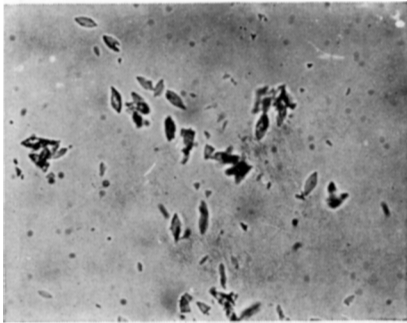
Fig. 43-18 Photograph with an optical microscope of crystals of monohydrated aluminum oxide. Magnification 500×. (Courtesy of J. Hillier, RCA Laboratories.)
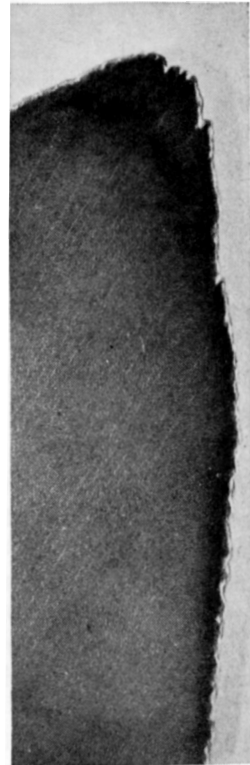
Fig. 43-19 Electron-microscope photograph of the tip of one of the crystals of Figure 43-18. Magnification 90,000×. (Courtesy of J. Hillier, RCA Laboratories.)

Illustrative Example. Find the de Broglie wavelength of an electron accelerated through a potential difference of 100 volts.

The energy acquired by an electron in passing through a potential difference of 100 volts appears as kinetic energy, so that

$$Ve = \tfrac{1}{2}mv^2.$$

Thus
$$mv = (2Vem)^{\frac{1}{2}}.$$

Now
$$Ve = 100 \text{ volts} \times 1.6 \times 10^{-19} \text{ coul} = 1.6 \times 10^{-17} \text{ joule},$$

or
$$Ve = 1.6 \times 10^{-10} \text{ erg}.$$

The mass of the electron is $m = 9.1 \times 10^{-28}$ gm, and $h = 6.62 \times 10^{-27}$ erg sec.

Thus $\qquad \lambda = \dfrac{h}{mv} = \dfrac{6.62 \times 10^{-27}}{(2 \times 1.6 \times 10^{-10} \times 9.1 \times 10^{-28})^{1/2}}$,

$$\lambda = 1.23 \times 10^{-8} \text{ cm} = 1.23 \text{ A.}$$

This wavelength is of the order of atomic dimensions.

43-6   Heisenberg's Uncertainty Principle

An interesting interpretation of the wave-particle duality of both matter and radiant energy has been given by Heisenberg. Heisenberg's *uncertainty principle* refers to the *simultaneous* determination of the position and momentum of a particle and states that *the uncertainty* Δx *involved in the measurement of the coordinate of a particle and the uncertainty* Δp_x *involved in the simultaneous determination of its corresponding momentum are governed by the relationship*

$$(\Delta x)(\Delta p_x) \geqq h, \qquad\qquad \text{(43-12)}$$

where h is Planck's constant.

Let us examine an idealized experiment to see how the wave concept acts as a limitation on the particle concept, giving rise to the uncertainty principle. One such idealized experiment was given by Bohr. Suppose it is desired to determine the position of an electron, using some instrument such as a microscope of very high resolving power. It may be shown that the resolving power of a microscope is given by

$$\Delta x = \frac{\lambda}{\sin \alpha} , \qquad\qquad \text{(43-13)}$$

where Δx represents the distance between two points which can just be resolved by the microscope, λ is the wavelength of the light used, and α is the semivertical angle of the cone of light coming from the illuminated object into the microscope objective. The uncertainty in the determination of the x coordinate of the electron is represented by Δx. To make Δx as small as possible, we must use light of very short wavelength, or x-rays or gamma rays. The minimum amount of light that can be used is a single quantum hf. When the electron scatters this quantum into the microscope, as shown in Figure 43-20, the electron will receive some momentum from the quantum, for the process by which the radiation is scattered is the Compton effect. Since the scattered quantum can enter the microscope anywhere within the semivertical angle α, the x component of the momentum which the electron acquires is uncertain by an amount

$$\Delta p_x = p \sin \alpha = \frac{h}{\lambda} \sin \alpha,$$

where h/λ is the momentum of the quantum. The product of the uncertainties in the determination of the simultaneous values of position and momentum is therefore

$$\Delta x \, \Delta p_x = \frac{\lambda}{\sin \alpha} \times \frac{h}{\lambda} \sin \alpha = h.$$

Fig. 43-20 Schematic diagram of a gamma-ray microscope. If we neglect the change in wavelength of the scattered gamma ray, we note that the electron recoils with a momentum whose x component ranges between the limits $\pm p \sin \alpha$ about some central value, depending on whether the gamma ray entering the microscope takes path 1, path 2, or some intermediate path.
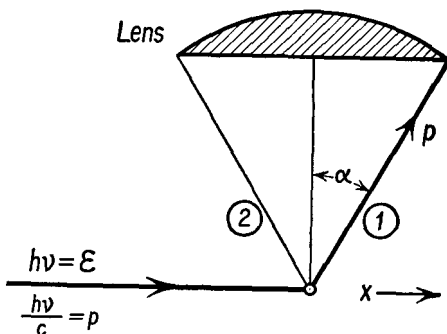
The uncertainty principle has had some striking implications in physics itself and in the philosophical extensions of physical concepts. In Newtonian mechanics the trajectory of a particle was completely determined if the initial position and momentum of the particle were known initially with any desired precision, and the forces acting on the particle could also be specified. Since the forces acting on a particle are generated by other particles through gravitation, through the electric and magnetic fields, and so on, a knowledge of the position and momentum of each particle in the universe at any one time made it possible to predict, at least in principle the position of each particle at every other time. Since each of us is composed of elementary particles, this state of affairs implied that our lives were completely predetermined. The uncertainty principle has taken away the physical foundation which was assumed to lie at the basis of this philosophy of determinism, for the wave-particle duality of both particles and waves implies that it is impossible to specify both the position and momentum of an elementary particle with sufficient precision to predict its future with certainty.

The wave-particle duality has been clarified by the principle of *complementarity* expressed by Bohr. According to Bohr, the wave and particle aspects of a phenomenon are never exhibited simultaneously. For some phenomena an electron may be interpreted as a wave, while for others it may be interpreted as a particle. The particle does not alter its characteristics, but rather, these two aspects of the electron are complementary rather than contradictory descriptions, for they represent limiting cases of a more complete description which may not yet have been achieved. In

such a description the electron is neither wave nor particle but is described mathematically by an equation or function which under some conditions is more nearly approximated by a wave, while under other conditions may be discussed as though it were a particle. We have seen evidence of this sort of thing in optics. In the phenomena of reflection and refraction, we may achieve a sufficient description of light by treating infinitely narrow rays, and as long as we are interested in the magnifications produced by lenses and mirrors and optical instruments, we may overlook the wave nature of light. In interference and diffraction phenomena we find it essential to treat light as a wave motion, and we must assert that it is impossible to restrict a light beam so that it is infinitely narrow.

Problems

43-1. A single crystal x-ray spectrometer with a quartz crystal ($d = 4.255$ A) is used to analyze the x-rays from a molybdenum target. The wavelength of the K_α line of molybdenum is 0.7078 A. (a) Determine the angle between the incident beam and the face of the crystal so that this line may be reflected in the first order. (b) Determine the angle between the incident beam and the reflected beam. (c) The atomic number of molybdenum is 42. As calculated from Equation (43-3), what is the percentage error in the calculated wavelength as compared to the experimental wavelength of this K_α line?

43-2. (a) What is the frequency of the K_α line of tungsten for which $Z = 74$? (b) What is the energy required to move a K shell electron just out of the atom? This may be inferred from Equation (43-3). (c) What voltage must be applied to a tungsten-target x-ray tube to excite the K series lines of tungsten?

43-3. A single crystal x-ray spectrometer with a calcite crystal is used to analyze the x-rays from a tube with a silver target. A very intense line is obtained when the beam makes an angle of $5°17'$ with the crystal. Determine the wavelength of this line. The grating space of calcite is 3.0294 A.

43-4. The K energy level of silver is 25,500 ev. X-rays of wavelength 0.40 A incident upon a silver foil eject electrons from the K levels of some of the silver atoms. (a) Determine the energy of the incident photons in electron volts. (b) Determine the kinetic energy of the ejected electrons in electron volts.

43-5. Assuming that Equation (43-3) may be extended to all the lines of the K series by proper adjustment of the quantum numbers, find the wavelengths of the K series lines of silver ($Z = 47$). Consult Table 5, Appendix A, to determine how many lines there are in this series.

43-6. Determine the energy, in electron volts, of the K_α line of sodium.

43-7. A high-voltage vacuum-tube rectifier with a tungsten filament and a nickel plate is used to rectify alternating voltages with a peak value of 100,000 volts. (a) Will x-rays be emitted from this tube? (b) If so, what will be the shortest wavelength generated by the tube? (c) What is the excitation energy of the K state of nickel? Assume that you may infer this value by appropriate adjustment of Equation (43-3). (d) Will the K_α line of nickel be emitted from

either the filament or the plate of this diode?   (e) If a continuous spectrum of x-rays is emitted from the tube, will it be emitted from the filament or the plate?

43-8. Determine the change in wavelength produced when an x-ray photon is scattered by a substance (a) through an angle of 90° and (b) through an angle of 180°.

43-9. X-rays of wavelength 0.500 A are incident upon a block of carbon. (a) Determine the energy of the photons of the incident beam. (b) Determine the change in energy of the photons which are scattered through an angle of 90°. (c) Assuming that this scattering is due to free electrons which originally had negligible energy and momentum, determine the energy of the recoil electrons.

43-10. Referring to Problem 43-9, determine (a) the momenta of the incident photons, (b) the momenta of the photons scattered through 90°, and (c) the momenta of the recoil electrons.

43-11. (a) Determine the wavelength associated with an electron whose energy is 1,600 ev. (b) Assuming that Bragg's law holds for the waves associated with electrons, determine the angle at which a stream of electrons should be directed toward the surface of a rock salt crystal to obtain maximum reflection, if the energy of these electrons is 1,600 ev.

43-12. (a) An electron is moving with a velocity of 2×10^8 cm/sec. Determine its momentum and wavelength. (b) A proton is moving with a velocity of 2×10^8 cm/sec. Determine its momentum and wavelength.

43-13. A stream of helium gas consists of helium atoms moving with an average speed of 3×10^5 cm/sec. Determine (a) the average momentum of a helium atom and (b) the wavelength associated with such an atom.

43-14. When we know that a particle is enclosed within a cubical container of side s, we may say that its position is given by the coordinates of the center of the box, with an uncertainty in each coordinate of $s/2$. Let us suppose that an electron is imprisoned in a box whose walls are equivalent in strength to a potential of -100 volts, the inside of the box being at ground potential or at zero potential. How small can the box be so that the electron will remain trapped in the box? We may assume that the electron will leave the box when the uncertainty in its energy is equal to the potential barrier represented by the walls.

43-15. A particle rests on the level surface of a yard provided with a high wall. (a) From a classical Newtonian viewpoint, how high does the wall have to to be to keep the particle inside? (b) If the yard is 20 ft wide, and a ball has a mass of 1 gm, how high must the wall be, according to the uncertainty principle? (c) If the "yard" is 10^{-6} cm wide and the particle has a mass of 10^{-10} gm, how high must the wall be? (d) If the "yard" is 10^{-13} cm wide and the particle has a mass of 10^{-27} gm, how high must the wall be? Assume that the "wall" is built in a uniform gravitational field in which $g = 980$ cm/sec^2.