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ANALYTIC FOURIER-FEYNMAN TRANSFORMS AND CONVOLUTION

TIMOTHY HUFFMAN, CHULL PARK, AND DAVID SKOUG

ABSTRACT. In this paper we develop an L_p Fourier-Feynman theory for a class of functionals on Wiener space of the form $F(x) = f(\int_0^T \alpha_1 dx, \dots, \int_0^T \alpha_n dx)$. We then define a convolution product for functionals on Wiener space and show that the Fourier-Feynman transform of the convolution product is a product of Fourier-Feynman transforms.

1. INTRODUCTION AND PRELIMINARIES

The concept of an L_1 analytic Fourier-Feynman transform was introduced by Brue in [1]. In [3] Cameron and Storvick introduced an L_2 analytic Fourier-Feynman transform. In [6] Johnson and Skoug developed an L_p analytic Fourier-Feynman transform theory for $1 \leq p \leq 2$ which extended the results in [1, 3] and gave various relationships between the L_1 and the L_2 theories.

In this paper we first develop an L_p Fourier-Feynman theory for a class of functionals not considered in [1, 3, 6]. We next define a convolution product for functionals on Wiener space and then show that the Fourier-Feynman transform of the convolution product is a product of Fourier-Feynman transforms.

In [3, 6] all of the functionals F on Wiener space and all the real-valued functions F on \mathbb{R}^n were assumed to be Borel measurable. But, as was pointed out in [7, p. 170], the concept of scale-invariant measurability in Wiener space and Lebesgue measurability in \mathbb{R}^n is precisely correct for the analytic Fourier-Feynman theory.

Let $C_0[0, T]$ denote Wiener space; that is, the space of real-valued continuous functions x on $[0, T]$ such that $x(0) = 0$. Let \mathcal{M} denote the class of all Wiener measurable subsets of $C_0[0, T]$, and let m denote Wiener measure. $(C_0[0, T], \mathcal{M}, m)$ is a complete measure space and we denote the Wiener integral of a functional F by

$$\int_{C_0[0, T]} F(x) m(dx).$$

A subset E of $C_0[0, T]$ is said to be scale-invariant measurable [4, 7] provided $\rho E \in \mathcal{M}$ for each $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m(\rho N) = 0$ for each $\rho > 0$. A property

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that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). If two functionals F and G are equal s-a.e., we write $F \approx G$.

Let \mathbb{C} , \mathbb{C}_+ , and \mathbb{C}_+^\sim denote respectively the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with non-negative real part. Let F be a \mathbb{C} -valued scale-invariant measurable functional on $C_0[0, T]$ such that

$$J(\lambda) = \int_{C_0[0, T]} F(\lambda^{-1/2}x)m(dx)$$

exists as a finite number for all $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in \mathbb{C}_+ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic Wiener integral of F over $C_0[0, T]$ with parameter λ and for $\lambda \in \mathbb{C}_+$ we write

$$\int_{C_0[0, T]}^{anw_\lambda} F(x)m(dx) = J^*(\lambda).$$

Let $q \neq 0$ be a real number, and let F be a functional such that

$$\int_{C_0[0, T]}^{anw_\lambda} F(x)m(dx)$$

exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the analytic Feynman integral of F with parameter q and we write

$$\int_{C_0[0, T]}^{anf_q} F(x)m(dx) = \lim_{\lambda \rightarrow -iq} \int_{C_0[0, T]}^{anw_\lambda} F(x)m(dx)$$

where $\lambda \rightarrow -iq$ through \mathbb{C}_+ .

Notation. (i) For $\lambda \in \mathbb{C}_+$ and $y \in C_0[0, T]$ let

$$(1.1) \quad (T_\lambda(F))(y) = \int_{C_0[0, T]}^{anw_\lambda} F(x+y)m(dx).$$

(ii) Given a number p with $1 \leq p \leq +\infty$, p and p' will always be related by $1/p + 1/p' = 1$.

(iii) Let $1 < p \leq 2$, and let $\{H_n\}$ and H be scale-invariant measurable functionals such that for each $\rho > 0$,

$$(1.2) \quad \lim_{n \rightarrow \infty} \int_{C_0[0, T]} |H_n(\rho y) - H(\rho y)|^{p'} m(dy) = 0.$$

Then we write

$$(1.3) \quad \text{l. i. m.}(w_s^{p'}) (H_n) \approx H$$

and we call H the scale invariant limit in the mean of order p' . A similar definition is understood when n is replaced by the continuously varying parameter λ . We are finally ready to state the definition of the L_p analytic Fourier-Feynman transform [6] and our definition of the convolution product.

Definition. Let $q \neq 0$ be a real number. For $1 < p \leq 2$ we define the L_p analytic Fourier-Feynman transform $T_q^{(p)}(F)$ of F by the formula ($\lambda \in \mathbb{C}_+$)

$$(1.4) \quad (T_q^{(p)}(F))(y) = \text{l.i.m.}_{\lambda \rightarrow -iq} (w_s^{p'}) (T_\lambda(F))(y)$$

whenever this limit exists. We define the L_1 analytic Fourier-Feynman transform $T_q^{(1)}(F)$ of F by the formula

$$(1.5) \quad T_q^{(1)}(F)(y) = \lim_{\lambda \rightarrow -iq} (T_\lambda(F))(y)$$

for s-a.e. y . We note that for $1 \leq p \leq 2$, $T_q^{(p)}(F)$ is defined only s-a.e. We also note that if $T_q^{(p)}(F_1)$ exists and if $F_1 \approx F_2$, then $T_q^{(p)}(F_2)$ exists and $T_q^{(p)}(F_2) \approx T_q^{(p)}(F_1)$.

Definition. Let F_1 and F_2 be functionals on $C_0[0, T]$. For $\lambda \in \mathbb{C}_+^\sim$ we define their convolution product (if it exists) by

$$(1.6) \quad (F_1 * F_2)_\lambda(y) = \begin{cases} \int_{C_0[0, T]}^{\text{an}w_\lambda} F_1 \left(\frac{y+x}{\sqrt{2}} \right) F_2 \left(\frac{y-x}{\sqrt{2}} \right) m(dx), & \lambda \in \mathbb{C}_+, \\ \int_{C_0[0, T]}^{\text{anf}_q} F_1 \left(\frac{y+x}{\sqrt{2}} \right) F_2 \left(\frac{y-x}{\sqrt{2}} \right) m(dx), & \lambda = -iq, q \in \mathbb{R}, q \neq 0. \end{cases}$$

Remark. Our definition of convolution is different than the definition given by Yeh in [9]. For one thing, our convolution product is commutative; that is to say $(F_1 * F_2)_\lambda = (F_2 * F_1)_\lambda$. Next we briefly describe a class of functionals for which we establish the existence of $T_q^{(p)}(F)$. Let n be a positive integer, and let $\alpha_1, \alpha_2, \dots, \alpha_n$ be an orthonormal set of functions in $L_2[0, T]$. For $1 \leq p < \infty$ let $\mathcal{A}_n^{(p)}$ be the space of all functionals F on $C_0[0, T]$ of the form

$$(1.7) \quad F(x) = f \left(\int_0^T \alpha_1 dx, \dots, \int_0^T \alpha_n dx \right)$$

s-a.e. where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is in $L_p(\mathbb{R}^n)$ and the integrals $\int_0^T \alpha_j(t) dx(t)$ are Paley-Wiener-Zygmund stochastic integrals. Let $\mathcal{A}_n^{(\infty)}$ be the space of all functionals of the form (1.7) with $f \in C_0(\mathbb{R}^n)$, the space of bounded continuous functions on \mathbb{R}^n that vanish at infinity. It is quite easy to see that if F is in $\mathcal{A}_n^{(p)}$, then F is scale-invariant measurable. If $p > 1$ the Feynman integral above should be interpreted as the scale-invariant limit in the mean of the analytic Wiener integral.

2. THE TRANSFORM OF FUNCTIONALS IN $\mathcal{A}_n^{(p)}$

In this section we show that the L_p analytic Fourier-Feynman transform $T_q^{(p)}(F)$ exists for all F in $\mathcal{A}_n^{(p)}$ and belongs to $\mathcal{A}_n^{(p')}$. We start with some preliminary lemmas.

Lemma 2.1. *Let $1 \leq p \leq \infty$, and let $F \in \mathcal{A}_n^{(p)}$ be given by (1.7). Then for all $\lambda \in \mathbb{C}_+$,*

$$(2.1) \quad (T_\lambda(F))(y) = g \left(\lambda; \int_0^T \alpha_1 dy, \dots, \int_0^T \alpha_n dy \right)$$

where

$$(2.2) \quad \begin{aligned} &g(\lambda; w_1, \dots, w_n) \\ &\equiv g(\lambda; \vec{w}) = \left(\frac{\lambda}{2\pi} \right)^{n/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n (u_j - w_j)^2 \right\} d\vec{u}. \end{aligned}$$

Proof. For $\lambda > 0$, using a well-known Wiener integration theorem we obtain

$$\begin{aligned} (T_\lambda(F))(y) &= \int_{C_0[0, T]} F(\lambda^{-1/2}x + y) m dx \\ &= \int_{C_0[0, T]} f \left(\lambda^{-1/2} \int_0^T \alpha_1 dx + \int_0^T \alpha_1 dy, \dots, \lambda^{-1/2} \right. \\ &\quad \left. \times \int_0^T \alpha_n dx + \int_0^T \alpha_n dy \right) m(dx) \\ &= \left(\frac{\lambda}{2\pi} \right)^{n/2} \int_{\mathbb{R}^n} f \left(v_1 + \int_0^T dy, \dots, v_n + \int_0^T \alpha_n dy \right) \\ &\quad \times \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n v_j^2 \right\} d\vec{v} \\ &= \left(\frac{\lambda}{2\pi} \right)^{n/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n \left(u_j - \int_0^T \alpha_j dy \right)^2 \right\} d\vec{u} \\ &= g \left(\lambda; \int_0^T \alpha_1 dy, \dots, \int_0^T \alpha_n dy \right) \end{aligned}$$

where g is given by (2.2). Now by analytic continuation in λ , (2.1) holds throughout \mathbb{C}_+ . \square

Lemma 2.2. *Let $F \in \mathcal{A}_n^{(1)}$ be given by (1.7), and let $g(\lambda; \vec{w})$ be given by (2.2). Then*

- (i) $g(\lambda; \cdot) \in C_0(\mathbb{R}^n)$ for all $\lambda \in \mathbb{C}_+^\sim$;
- (ii) $g(\lambda; \vec{w})$ converges pointwise to $g(-iq; \vec{w})$ as $\lambda \rightarrow -iq$ through \mathbb{C}_+ ; and
- (iii) as elements of $C_0(\mathbb{R}^n)$, $g(\lambda; \vec{w})$ converges weakly to $g(-iq; \vec{w})$ as $\lambda \rightarrow -iq$ through values in \mathbb{C}_+ .

Proof. We first note that for all $(\lambda, \vec{w}) \in \mathbb{C}_+^\sim \times \mathbb{R}^n$, $|g(\lambda; \vec{w})| \leq |\frac{\lambda}{2\pi}|^{n/2} \|f\|_1$. Then (i) follows from a standard argument and the dominated convergence theorem establishes (ii). To establish (iii) let $\mu \in M(\mathbb{R}^n)$, the dual of $C_0(\mathbb{R}^n)$.

By the dominated convergence theorem,

$$\begin{aligned} & \lim_{\lambda \rightarrow -iq} \int_{\mathbb{R}^n} g(\lambda; \vec{w}) d\mu(\vec{w}) \\ &= \lim_{\lambda \rightarrow -iq} \int_{\mathbb{R}^n} \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp\left\{-\frac{\lambda}{2} \sum_{j=1}^n (u_j - w_j)^2\right\} d\vec{u} d\mu(\vec{w}) \\ &= \int_{\mathbb{R}^n} \left(\frac{-iq}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp\left\{\frac{iq}{2} \sum_{j=1}^n (u_j - w_j)^2\right\} d\vec{u} d\mu(\vec{w}) \\ &= \int_{\mathbb{R}^n} g(-iq; \vec{w}) d\mu(\vec{w}). \quad \square \end{aligned}$$

Our first theorem, which is a direct consequence of Lemma 2.2, shows that the analytic L_1 Fourier-Feynman transform exists for all F in $\mathcal{A}_n^{(1)}$.

Theorem 2.1. *Let $F \in \mathcal{A}_n^{(1)}$ be given by (1.7). Then $T_q^{(1)}(F)$ exists for all real $q \neq 0$ and*

$$(2.3) \quad (T_q^{(1)}(F))(y) \approx g\left(-iq; \int_0^T \alpha_1 dy, \dots, \int_0^T \alpha_n dy\right) \in \mathcal{A}_n^{(\infty)}$$

where g is given by (2.2).

Remark. When $1 < p \leq 2$ and $\text{Re } \lambda = 0$, the integral in (2.2) should be interpreted in the mean just as in the theory of the L_p Fourier transform [8].

Theorem 2.2. *Let $1 < p \leq 2$, and let $F \in \mathcal{A}_n^{(p)}$ be given by (1.7). Then the L_p analytic Fourier-Feynman transform of F , $T_q^{(p)}(F)$ exists for all real $q \neq 0$, belongs to $\mathcal{A}_n^{(p')}$ and is given by the formula*

$$(2.4) \quad (T_q^{(p)}(F))(y) \approx g\left(-iq; \int_0^T \alpha_1 dy, \dots, \int_0^T \alpha_n dy\right)$$

where g is given by (2.2).

Proof. We first note that for each $\lambda \in \mathbb{C}_+^\sim$, $g(\lambda; \vec{w})$ is in $L_{p'}(\mathbb{R}^n)$ [5, Lemma 1.1, p. 98]. Furthermore by [5, Lemma 1.2, p. 100]

$$(2.5) \quad \lim_{\lambda \rightarrow -iq} \|g(\lambda; \cdot) - g(-iq; \cdot)\|_{p'} = 0.$$

Now to show that $T_q^{(p)}(F)$ exists and is given by (2.4) it suffices to show that for each $\rho > 0$

$$\lim_{\lambda \rightarrow -iq} \int_{C_0[0, T]} \left| g \left(\lambda; \rho \int_0^T \alpha_1 dy, \dots, \rho \int_0^T \alpha_n dy \right) - g \left(-iq; \rho \int_0^T \alpha_1 dy, \dots, \rho \int_0^T \alpha_n dy \right) \right|^{p'} m(dy) = 0.$$

But

$$\begin{aligned} & \int_{C_0[0, T]} \left| g \left(\lambda; \rho \int_0^T \alpha_1 dy, \dots, \rho \int_0^T \alpha_n dy \right) - g \left(-iq; \rho \int_0^T \alpha_1 dy, \dots, \rho \int_0^T \alpha_n dy \right) \right|^{p'} m(dy) \\ &= \rho^{-n} \int_{\mathbb{R}^n} |g(\lambda; \vec{u}) - g(-iq; \vec{u})|^{p'} \exp \left\{ -\frac{1}{2\rho^2} \sum_{j=1}^n u_j^2 \right\} d\vec{u} \\ &\leq \rho^{-n} \|g(\lambda; \cdot) - g(-iq; \cdot)\|_{p'}^{p'} \end{aligned}$$

which goes to zero as $\lambda \rightarrow -iq$ by (2.5). Thus $T_q^{(p)}(F)$ exists, belongs to $\mathcal{A}_n^{(p')}$, and is given by (2.4). \square

The following example generates an interesting set of functionals belonging to $\mathcal{A}_n^{(p)}$.

Example. Let $1 \leq p \leq +\infty$ be given, and let $\alpha_1, \alpha_2, \dots$ be an orthonormal set of functions from $L_2[0, T]$. Let $F \in L_p(C_0[0, T])$, and for each n define f_n by

$$f_n \left(\int_0^T \alpha_1 dx, \dots, \int_0^T \alpha_n dx \right) \equiv E \left[F(x) \mid \int_0^T \alpha_1 dx, \dots, \int_0^T \alpha_n dx \right].$$

Then, by the definition of conditional expectation, $f_n(\xi_1, \dots, \xi_n)$ is a Borel measurable function, and $\|f_n\|_p \leq \|F\|_p$, where

$$\|f_n\|_p = E \left[\left| f_n \left(\int_0^T \alpha_1 dx, \dots, \int_0^T \alpha_n dx \right) \right|^p \right],$$

and

$$\|F\|_p^p = E[|F(x)|^p].$$

Thus $f_n \in \mathcal{A}_n^{(p)}$, and so the analytic Fourier-Feynman transform $T_q^{(p)}(f_n)$ exists for all real $q \neq 0$.

We finish this section by obtaining an inverse transform theorem for F in $\mathcal{A}_n^{(p)}$.

Theorem 2.3. Let $1 \leq p \leq 2$, and let $F \in \mathcal{A}_n^{(p)}$. Let $q \neq 0$ be given. Then (i) for each $\rho > 0$,

$$\lim_{\lambda \rightarrow -iq} \int_{C_0[0, T]} |T_{\bar{\lambda}} T_{\lambda}(F)(\rho y) - F(\rho y)|^p m(dy) = 0,$$

and (ii) $T_{\bar{\lambda}} T_{\lambda} F \rightarrow F$ s-a.e. as $\lambda \rightarrow -iq$ through \mathbb{C}_+ .

Proof. Proceeding as in the proof of Lemma 2.1, we obtain for all $\lambda \in \mathbb{C}_+$,

$$\begin{aligned} (T_{\bar{\lambda}}(T_{\lambda}(F)))(y) &= \left(\frac{\bar{\lambda}}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} g(\lambda; \vec{w}) \exp \left\{ -\frac{\bar{\lambda}}{2} \sum_{j=1}^n \left(w_j - \int_0^T a_j dy \right)^2 \right\} d\vec{w} \\ &= \left(\frac{\bar{\lambda}}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n (u_j - w_j)^2 \right\} \\ &\quad \times \exp \left\{ -\frac{\bar{\lambda}}{2} \sum_{j=1}^n \left(w_j - \int_0^T \alpha_j dy \right)^2 \right\} d\vec{u} d\vec{w} \\ &= k \left(\lambda, \bar{\lambda}; \int_0^T \alpha_1 dy, \dots, \int_0^T \alpha_n dy \right) \end{aligned}$$

where $g(\lambda; \vec{w})$ is given by (2.2) and

$$\begin{aligned} k(\lambda, \bar{\lambda}; v_1, \dots, v_n) &\equiv k(\lambda, \bar{\lambda}; \vec{v}) \\ &= \left| \frac{\lambda}{2\pi} \right|^n \int_{\mathbb{R}^{2n}} f(\vec{u}) \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n (u_j - w_j)^2 - \frac{\bar{\lambda}}{2} \sum_{j=1}^n (w_j - v_j)^2 \right\} d\vec{u} d\vec{w}. \end{aligned}$$

But [2, p. 525]

$$\begin{aligned} &\int_{\mathbb{R}} \exp \left\{ -\frac{\lambda}{2} (u_j - w_j)^2 - \frac{\bar{\lambda}}{2} (w_j - v_j)^2 \right\} dw_j \\ &= \left(\frac{\pi}{\operatorname{Re} \lambda} \right)^{1/2} \exp \left\{ -\frac{|\lambda|^2}{4 \operatorname{Re} \lambda} (u_j - v_j)^2 \right\}. \end{aligned}$$

Hence

$$\begin{aligned} k(\lambda, \bar{\lambda}; \vec{v}) &= \left| \frac{\lambda}{2\pi} \right|^n \int_{\mathbb{R}^n} f(\vec{u}) \left(\frac{\pi}{\operatorname{Re} \lambda} \right)^{n/2} \exp \left\{ -\frac{|\lambda|^2}{4 \operatorname{Re} \lambda} \sum_{j=1}^n (u_j - v_j)^2 \right\} d\vec{u} \\ &= (f * \phi_{\varepsilon})(v_1, \dots, v_n) \end{aligned}$$

where

$$\phi(v_1, \dots, v_n) \equiv (2\pi)^{-n/2} \exp \left\{ -\frac{1}{2} \sum_{j=1}^n v_j^2 \right\}, \quad \varepsilon \equiv \frac{\sqrt{2 \operatorname{Re} \lambda}}{|\lambda|},$$

and

$$\phi_{\varepsilon}(v_1, \dots, v_n) = \frac{1}{\varepsilon^n} = \frac{1}{\varepsilon^n} \phi \left(\frac{v_1}{\varepsilon}, \dots, \frac{v_n}{\varepsilon} \right).$$

Now

$$\int_{\mathbb{R}^n} \phi(v_1, \dots, v_n) dv_1 \cdots dv_n = 1 \quad \text{and} \quad \phi(v_1, \dots, v_n) > 0,$$

so using [8, Theorem 1.18, p. 10] it follows that

$$\begin{aligned}
 & \lim_{\lambda \rightarrow -iq} \int_{\mathbb{R}^n} |k(\lambda, \bar{\lambda}; v_1, \dots, v_n) - f(v_1, \dots, v_n)|^p d\vec{v} \\
 (2.6) \quad & = \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^n} |(f * \phi_\varepsilon)(v_1, \dots, v_n) - f(v_1, \dots, v_n)|^p d\vec{v} \\
 & = \lim_{\varepsilon \rightarrow 0^+} \|f * \phi_\varepsilon - f\|_p^p = 0
 \end{aligned}$$

since $\varepsilon \rightarrow 0^+$ as $\lambda \rightarrow -iq$ through \mathbb{C}_+ . But now (i) of the theorem follows easily since for each fixed $\rho > 0$,

$$\begin{aligned}
 & \int_{C_0[0, T]} |T_{\bar{\lambda}} T_\lambda(F)(\rho y) - F(\rho y)|^p m(dy) \\
 & = \rho^{-n} \int_{\mathbb{R}^n} |k(\lambda, \bar{\lambda}; \vec{v}) - f(\vec{v})|^p \exp\left\{-\frac{1}{2\rho^2} \sum_{j=1}^n v_j^2\right\} d\vec{v} \\
 & \leq \rho^{-n} \|f * \phi_\varepsilon - f\|_p^p.
 \end{aligned}$$

Finally, (ii) of the theorem follows since by [8, Theorem 1.25, p. 13] it follows that the function $k(\lambda, \bar{\lambda}; v_1, \dots, v_n = (f * \phi_\varepsilon)(v_1, \dots, v_n)$ converges pointwise to the function $f(v_1, \dots, v_n)$ as $\lambda \rightarrow -iq$ through \mathbb{C}_+ . \square

Note that in the case $p = 2, p' = 2$, and so for F in $\mathcal{A}_n^{(2)}$, $T_q^{(2)}(F)$ is in $\mathcal{A}_n^{(2)}$ by Theorem 2.2. Hence we have the following theorem.

Theorem 2.4. *Let $F \in \mathcal{A}_n^{(2)}$ be given by (1.7). Then for all real $q \neq 0$,*

$$T_{-q}(T_q(F)) \approx F.$$

3. CONVOLUTIONS AND TRANSFORMS OF CONVOLUTIONS

Our first lemma gives an expression for $(F_1 * F_2)_\lambda$ for $\lambda \in \mathbb{C}_+$.

Lemma 3.1. *Let $1 \leq p \leq \infty$, and let $F_j \in \bigcup_{1 \leq p \leq \infty} \mathcal{A}_n^{(p)}$ for $j = 1, 2$ be given by (1.7). Then for all $\lambda \in \mathbb{C}_+$,*

$$(3.1) \quad (F_1 * F_2)_\lambda(y) = h\left(\lambda; \int_0^T \alpha_1 dy, \dots, \int_0^T \alpha_n dy\right)$$

where

(3.2)

$$h(\lambda; w_1, \dots, w_n) \equiv h(\lambda; \vec{w}) = \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f_1\left(\frac{\vec{w} + \vec{u}}{\sqrt{2}}\right) f_2\left(\frac{\vec{w} - \vec{u}}{\sqrt{2}}\right) \times \exp\left\{-\frac{\lambda}{2} \sum_{j=1}^n u_j^2\right\} d\vec{u}.$$

Proof. For $\lambda > 0$, using a well-known Wiener integration formula we obtain

$$\begin{aligned} (F_1 * F_2)_\lambda(y) &= \int_{C_{0|0, T}} F_1\left(\frac{y + \lambda^{-1/2}x}{\sqrt{2}}\right) F_2\left(\frac{y - \lambda^{-1/2}x}{\sqrt{2}}\right) m(dx) \\ &= \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f_1\left(2^{-1/2} \left[\int_0^T \alpha_1 dy + u_1\right], \dots, \right. \\ &\qquad\qquad\qquad \left. 2^{-1/2} \left[\int_0^T \alpha_n dy + u_n\right]\right) \\ &\qquad \times f_2\left(2^{-1/2} \left[\int_0^T \alpha_1 dy - u_1\right], \dots, 2^{-1/2} \left[\int_0^T \alpha_n dy - u_n\right]\right) \\ &\qquad \times \exp\left\{-\frac{\lambda}{2} \sum_{j=1}^n u_j^2\right\} d\vec{u} \\ &= h\left(\lambda; \int_0^T \alpha_1 dy, \dots, \int_0^T \alpha_n dy\right) \end{aligned}$$

where h is given by (3.2), so (3.1) holds for $\lambda > 0$. Now by analytic continuation in λ , we see that (3.1) holds for all λ in \mathbb{C}_+ . \square

Our next theorem establishes an interesting relationship involving convolutions and analytic Wiener integrals.

Theorem 3.1. *Let $1 \leq p \leq \infty$, and let $F_j \in \bigcup_{1 \leq p \leq \infty} \mathcal{A}_n^{(p)}$ for $j = 1, 2$ be given by (1.7). Then for all $\lambda \in \mathbb{C}_+$,*

$$(3.3) \quad (T_\lambda(F_1 * F_2)_\lambda)(z) = (T_\lambda(F_1))(2^{-1/2}z)(T_\lambda(F_2))(2^{-1/2}z).$$

Proof. It will suffice to establish (3.3) for $\lambda > 0$ since $T_\lambda(F_1 * F_2)_\lambda$, $T_\lambda(F_1)$, and $T_\lambda(F_2)$ all have analytic extensions throughout \mathbb{C}_+ . So let $\lambda > 0$ be given.

Then by (3.1) and (3.2),

$$\begin{aligned}
 (T_\lambda(F_1 * F_2)_\lambda)(z) &= \int_{C_0[0, T]} (F_1 * F_2)_\lambda(\lambda^{-1/2}x + z)m(dx) \\
 &= \int_{C_0[0, T]} h\left(\lambda; \int_0^T \alpha_1 d[\lambda^{-1/2}x + z], \dots, \int_0^T \alpha_n d[\lambda^{-1/2}x + z]\right) m(dx) \\
 &= \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} h\left(\lambda; v_1 + \int_0^T \alpha_1 dz, \dots, v_n + \int_0^T \alpha_n dz\right) \\
 &\qquad \qquad \qquad \times \left\{ -\frac{\lambda}{2} \sum_{j=1}^n v_j^2 \right\} d\vec{v} \\
 &= \left(\frac{\lambda}{2\pi}\right)^n \int_{\mathbb{R}^{2n}} f_1\left(2^{-1/2}\left[v_1 + u_1 + \int_0^T \alpha_1 dz\right], \dots, 2^{-1/2}\left[v_n + u_n + \int_0^T \alpha_n dz\right]\right) \\
 &\qquad \qquad \qquad \times f_2\left(2^{-1/2}\left[v_1 - u_1 + \int_0^T \alpha_1 dz\right], \dots, 2^{-1/2}\left[v_n - u_n + \int_0^T \alpha_n dz\right]\right) \\
 &\qquad \qquad \qquad \times \exp\left\{-\frac{\lambda}{2} \sum_{j=1}^\infty [u_j^2 + v_j^2]\right\} d\vec{u} d\vec{v}.
 \end{aligned}$$

Next we make the transformation

$$w_j = 2^{-1/2}(v_j + u_j)$$

and

$$r_j = 2^{-1/2}(v_j - u_j)$$

for $j = 1, 2, \dots, n$. The Jacobian of this transformation is one and

$$\sum_{j=1}^n [w_j^2 + r_j^2] = \sum_{j=1}^n [u_j^2 + v_j^2].$$

Hence for $\lambda > 0$, using (2.1) and (2.2), we see that

$$\begin{aligned}
& (T_\lambda(F_1 * F_2)_\lambda)(z) \\
&= \left(\frac{\lambda}{2\pi}\right) \int_{\mathbb{R}^{2n}} f_1 \left(w_1 + 2^{-1/2} \int_0^T \alpha_1 dz, \dots, w_n + 2^{-1/2} \int_0^T \alpha_n dz \right) \\
&\quad \times \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n w_j^2 \right\} \\
&\times f_2 \left(r_1 + 2^{-1/2} \int_0^T \alpha_1 dz, \dots, r_n + 2^{-1/2} \int_0^T \alpha_n dz \right) \\
&\quad \times \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n r_j^2 \right\} d\bar{w} d\bar{r} \\
&= \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f_1 \left(w_1 + 2^{-1/2} \int_0^T \alpha_1 dz, \dots, w_n + 2^{-1/2} \int_0^T \alpha_n dz \right) \\
&\quad \times \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n w_j^2 \right\} d\bar{w} \\
&\times \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f_2 \left(r_1 + 2^{-1/2} \int_0^T \alpha_1 dz, \dots, r_n + 2^{-1/2} \int_0^T \alpha_n dz \right) \\
&\quad \times \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n r_j^2 \right\} d\bar{r} \\
&= \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f_1(\bar{w}) \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n \left(w_j - 2^{-1/2} \int_0^T \alpha_j dz \right)^2 \right\} d\bar{w} \\
&\quad \times \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f_2(\bar{r}) \exp \left\{ -\frac{\lambda}{2} \sum_{j=1}^n \left(r_j - 2^{-1/2} \int_0^T \alpha_j dz \right)^2 \right\} d\bar{r} \\
&= (T_\lambda(F_1))(2^{-1/2}z)(T_\lambda(F_2))(2^{-1/2}z).
\end{aligned}$$

Theorem 3.2. *The following hold for all $\lambda \in \mathbb{C}_+^\sim$.*

- (i) *If $F_1 \in \mathcal{A}_n^{(1)}$ and $F_2 \in \mathcal{A}_n^{(1)}$, then $(F_1 * F_2)_\lambda \in \mathcal{A}_n^{(1)}$.*
- (ii) *If $F_1 \in \mathcal{A}_n^{(2)}$ and $F_2 \in \mathcal{A}_n^{(2)}$, then $(F_1 * F_2)_\lambda \in \mathcal{A}_n^{(\infty)}$.*
- (iii) *If $F_1 \in \mathcal{A}_n^{(1)}$ and $F_2 \in \mathcal{A}_n^{(2)}$, then $(F_1 * F_2)_\lambda \in \mathcal{A}_n^{(2)}$.*
- (iv) *If $F_1 \in \mathcal{A}_n^{(1)}$ and $F_2 \in \mathcal{A}_n^{(1)} \cap \mathcal{A}_n^{(2)}$, then $(F_1 * F_2)_\lambda \in \mathcal{A}_n^{(1)} \cap \mathcal{A}_n^{(2)}$.*
- (v) *If $F_1 \in \mathcal{A}_n^{(1)}$ and $F_2 \in \mathcal{A}_n^{(\infty)}$, then $(F_1 * F_2)_\lambda \in \mathcal{A}_n^{(\infty)}$.*

Proof. (i) Assume F_1 and F_2 belong to $\mathcal{A}_n^{(1)}$ and are given by (1.7). It will suffice to show that $h(\lambda; \cdot)$ given by (3.2) is in $L_1(\mathbb{R}^n)$ for every $\lambda \in \mathbb{C}_+^\sim$. But this follows from the calculations

$$\begin{aligned} \int_{\mathbb{R}^n} |h(\lambda; \vec{w})| d\vec{w} &\leq \left| \frac{\lambda}{2\pi} \right|^{n/2} \int_{\mathbb{R}^{2n}} |f_1(2^{-1/2}(\vec{w} + \vec{u}))f_2(2^{-1/2}(\vec{w} - \vec{u}))| d\vec{w} d\vec{u} \\ &= \left| \frac{\lambda}{2\pi} \right|^{n/2} \int_{\mathbb{R}^n} |f_1(\vec{v})| 2^{n/2} \int_{\mathbb{R}^n} |f_2(\sqrt{2}\vec{w} - \vec{v})| d\vec{w} d\vec{v} \\ &= \left| \frac{\lambda}{2\pi} \right|^{n/2} \|f_1\|_1 \|f_2\|_1. \end{aligned}$$

(ii) In this case for f_1, f_2 in $L_2(\mathbb{R}^n)$ we first note that $h(\lambda; \cdot)$ is in $L_\infty(\mathbb{R}^n)$ since for all $\vec{w} \in \mathbb{R}^n$,

$$\begin{aligned} |h(\lambda; \vec{w})| &\leq \left| \frac{\lambda}{2\pi} \right|^{n/2} \int_{\mathbb{R}^n} |f_1(2^{-1/2}(\vec{w} + \vec{u}))| |f_2(2^{-1/2}(\vec{w} - \vec{u}))| d\vec{u} \\ &\leq \left| \frac{\lambda}{2\pi} \right|^{n/2} \left\{ \int_{\mathbb{R}^n} |f_1(2^{-1/2}(w + \vec{u}))|^2 du \right\}^{1/2} \\ &\quad \times \left\{ \int_{\mathbb{R}^n} |f_2(2^{-1/2}(\vec{w} - \vec{u}))|^2 d\vec{u} \right\}^{1/2} \\ &= \left| \frac{\lambda}{2\pi} \right|^{n/2} (\sqrt{2})^n \|f_1\|_2 \|f_2\|_2 \\ &= \left| \frac{\lambda}{\pi} \right|^{n/2} \|f_1\|_2 \|f_2\|_2. \end{aligned}$$

A standard argument now shows that h belongs to $C_0(\mathbb{R}^n)$.

(iii) Let $F_1 \in \mathcal{A}_n^{(1)}$ and $F_2 \in \mathcal{A}_n^{(2)}$ be given by (1.7). It will suffice to show that $h(\lambda; \cdot)$ given by (3.2) is in $L_2(\mathbb{R}^n)$. But this follows from the calculations

$$\begin{aligned} \int_{\mathbb{R}^n} |h(\lambda; \vec{w})|^2 d\vec{w} &\leq \int_{\mathbb{R}^n} \left| \frac{\lambda}{2\pi} \right|^n \left[\int_{\mathbb{R}^n} |f_1(2^{-1/2}(\vec{w} + \vec{u}))f_2(2^{-1/2}(\vec{w} - \vec{u}))| d\vec{u} \right. \\ &\quad \left. \times \int_{\mathbb{R}^n} |f_1(2^{-1/2}(\vec{w} + \vec{u}))f_2(2^{-1/2}(\vec{w} - \vec{u}))| d\vec{v} \right] d\vec{w} \\ &= \left| \frac{\lambda}{2\pi} \right|^n \int_{\mathbb{R}^n} |f_1(\vec{r})| \int_{\mathbb{R}^n} |f_1(\vec{s})| \int_{\mathbb{R}^n} |f_2(\sqrt{2}\vec{w} - \vec{r}) \\ &\quad \times f_2(\sqrt{2}\vec{w} - \vec{s})| d\vec{w} d\vec{s} d\vec{r} \\ &\leq \left| \frac{\lambda}{2\pi} \right|^n \|f_1\|_1^2 \int_{\mathbb{R}^n} |f_2(\sqrt{2}\vec{w} - \vec{r})|^2 d\vec{r} \\ &= \left| \frac{\lambda}{2\pi} \right|^n (2)^{n/2} \|f_1\|_1^2 \|f_2\|_2^2. \end{aligned}$$

Hence $\|h\|_2 \leq |\lambda/\pi\sqrt{2}|^{n/2} \|f_1\|_1 \|f_2\|_2$.

Finally we note that (iv) follows directly from (i) and (iii) while (v) is immediate. \square

In our next theorem we show that the Fourier-Feynman transform of the convolution product is the product of transforms.

Theorem 3.3. (i) Let $F_1, F_2 \in \mathcal{A}_n^{(1)}$. Then for all real $q \neq 0$,

$$(3.4) \quad (T_q^{(1)}(F_1 * F_2)_q)(z) = (T_q^{(1)}(F_1))(2^{-1/2}z)(T_q^{(1)}(F_2))(2^{-1/2}z).$$

(ii) Let $F_1 \in \mathcal{A}_n^{(1)}$ and $F_2 \in \mathcal{A}_n^{(2)}$. Then for all real $q \neq 0$,

$$(3.5) \quad (T_q^{(2)}(F_1 * F_2)_q)(z) = (T_q^{(1)}(F_1))(2^{-1/2}z)(T_q^{(2)}(F_2))(2^{-1/2}z).$$

(iii) Let $F_1 \in \mathcal{A}_n^{(1)}$ and $F_2 \in \mathcal{A}_n^{(1)} \cap \mathcal{A}_n^{(2)}$. Then for all real $q \neq 0$,

$$(3.6) \quad (T_q^{(1)}(F_1 * F_2)_q)(z) = (T_q^{(1)}(F_1))(2^{-1/2}z)(T_q^{(1)}(F_2))(2^{-1/2}z)$$

and

$$(3.7) \quad (T_q^{(2)}(F_1 * F_2)_q)(z) = (T_q^{(1)}(F_1))(2^{-1/2}z)(T_q^{(2)}(F_2))(2^{-1/2}z).$$

Proof. Theorem 3.2 together with Theorem 2.2 assures us that all of the transforms on both sides of (3.4) through (3.7) exist. Equations (3.4) through (3.7) now follow from equation (3.3). \square

Remark. Throughout this paper, for simplicity we assumed that $\{\alpha_1, \dots, \alpha_n\}$ was an orthonormal set of functions in $L_2[0, T]$. However, all of our results hold provided that $\{\alpha_1, \dots, \alpha_n\}$ is a linearly independent set of functions from $L_2[0, T]$.

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