2015

Thresholds for Correcting Errors, Erasures, and Faulty Syndrome Measurements in Degenerate Quantum Codes

Ilya Dumer
University of California, Riverside, dumer@ee.ucr.edu

Alexey Kovalev
University of Nebraska - Lncoln, alexey.kovalev@unl.edu

Leonid P. Pryadko
University of California at Riverside, leonid@ucr.edu

Dumer, Ilya; Kovalev, Alexey; and Pryadko, Leonid P., "Thresholds for Correcting Errors, Erasures, and Faulty Syndrome Measurements in Degenerate Quantum Codes" (2015). *Faculty Publications, Department of Physics and Astronomy*. 143.
http://digitalcommons.unl.edu/physicsfacpub/143

Thresholds for Correcting Errors, Erasures, and Faulty Syndrome Measurements in Degenerate Quantum Codes

Ilya Dumer,[1] Alexey A. Kovalev,[2] and Leonid P. Pryadko[3]
[1]*Department of Electrical Engineering, University of California, Riverside, California 92521, USA*
[2]*Department of Physics and Astronomy and Nebraska Center for Materials and Nanoscience, University of Nebraska, Lincoln, Nebraska 68588, USA*
[3]*Department of Physics and Astronomy, University of California, Riverside, California 92521, USA*

(Received 30 December 2014; published 31 July 2015)

We suggest a technique for constructing lower (existence) bounds for the fault-tolerant threshold to scalable quantum computation applicable to degenerate quantum codes with sublinear distance scaling. We give explicit analytic expressions combining probabilities of erasures, depolarizing errors, and phenomenological syndrome measurement errors for quantum low-density parity-check codes with logarithmic or larger distances. These threshold estimates are parametrically better than the existing analytical bound based on percolation.

DOI: 10.1103/PhysRevLett.115.050502 PACS numbers: 03.67.Pp, 03.67.Lx, 64.60.ah

Quantum computers process coherent superpositions of exponentially many basis states instead of one binary string at a time. In theory, this parallelism makes quantum computers faster than the classical ones. However, quantum superpositions are fragile; without quantum error correction, decoherence would make computations unfeasible [1]. Furthermore, unlike in a classical setup restricted to transmission errors, any quantum error-correcting code (QECC) requires complicated measurements prone to errors. This syndrome extraction from a system of qubits requires fault tolerance (FT): all operations have to limit error propagation. Then, an arbitrarily large quantum computation is possible with a polynomial complexity if physical qubits and elementary gates exceed some accuracy threshold (threshold theorem) [2–7].

For years, out of many existing families of QECCs [8,9], FT threshold was established for only two code families, concatenated [2] and surface [5] codes (also, related color codes [10]). However, both families have asymptotically zero code rates [11] and therefore require substantial hardware overhead. A new alternative is offered by quantum low-density parity-check (LDPC) codes [12], which can combine finite rates with a nonzero FT threshold. These are stabilizer codes [8,13] with a limited number of qubits in each stabilizer generator (operators to be measured during QEC). Several families of such codes have finite code rates [14–18]. The threshold existence has been proven [19] by two of us using ideas from percolation theory. Subsequently, a related approach of Gottesman [20] demonstrated that such codes can achieve scalable quantum computation with a finite overhead per logical qubit.

While Ref. [19] gives a finite threshold for certain quantum LDPC codes, the actual threshold value and its dependence on the parameters are both far off. The technique [19] also fails to give a finite threshold whenever a single qubit is shared by many stabilizer generators.

Here, we present an approach resulting in parametrically better lower bounds for the thresholds, for both a quantum channel and a phenomenological error model with a FT setting. We consider infinite sequences of long quantum LDPC codes of increasing length n, whose distances d scale with n at least logarithmically,

$$d \geq D \ln n, \qquad D > 0. \tag{1}$$

A superlogarithmic scaling of the distance (including a power law $d \geq A n^{\alpha}$ with $A, \alpha > 0$) gives $D \to \infty$. At the same time, we limit all stabilizer generators to some fixed number of w or fewer qubits. For any sequence of such codes, we give an analytical lower (existence) bound combining uncorrelated qubit erasures, depolarizing errors, and syndrome measurement errors. We also give a similar bound tailored for Calderbank-Shor-Steane (CSS) codes. These bounds no longer require that every qubit be included in a limited number of stabilizer generators. Tying our lower bound on erasure threshold to other results [21,22], we restrict the parameters of LDPC codes with certain properties.

We consider QECCs defined on the n-qubit Hilbert space $\mathcal{H}_2^{\otimes n}$. Any operator acting in $\mathcal{H}_2^{\otimes n}$ can be represented as a linear combination of Pauli operators, elements of the n-qubit Pauli group $\mathcal{P}_n$ of the size 2^{2n+2},

$$\mathcal{P}_n = i^m \{I, X, Y, Z\}^{\otimes n}, m = 0, \ldots, 3, \tag{2}$$

where X, Y, and Z are the usual Pauli matrices, I is the identity matrix, and i^m a phase factor. The weight $\mathrm{wgt}(E)$ of a Pauli operator $E \in \mathcal{P}_n$ is the number of nonidentity terms in its expansion (2). A stabilizer code $\mathcal{Q}$ $[[n, k, d]]$ is a 2^k-dimensional subspace of the Hilbert space $\mathcal{H}_2^{\otimes n}$, a common

$+1$ eigenspace of operators in an Abelian stabilizer group $\mathcal{S} = \langle G_1, \ldots, G_r \rangle$ with generators G_i,

$$\mathcal{Q} = \{|\psi\rangle : S|\psi\rangle = |\psi\rangle \quad \forall\, S \in \mathcal{S}\}, \qquad -\mathbb{1} \notin \mathcal{S}. \tag{3}$$

A narrower set of CSS codes [23,24] contains codes where each stabilizer generator is a product of only Pauli X or Pauli Z operators. For a stabilizer group with r independent generators, the dimension of the quantum code is $k = n - r$; for a CSS code with r_μ independent generators of type $\mu = X, Z$, respectively, $k = n - r_X - r_Z$.

Error correction is done by measuring the stabilizer generators G_i, $i = 1, \ldots, r$; the corresponding eigenvalues $(-1)^{s_i}$, $s_i \in \{0, 1\}$ form the syndrome $\mathbf{s} \equiv (s_1, s_2, \ldots, s_r)$ of the error. Measuring the syndrome projects any state $|\psi\rangle \in \mathcal{H}_2^{\otimes n}$ into one of the 2^r subspaces $\mathcal{Q}_\mathbf{s}$ equivalent to the code $\mathcal{Q} \equiv \mathcal{Q}_\mathbf{0}$. A detectable error $E \in \mathcal{P}_n$ anticommutes with some generator(s) of the stabilizer; otherwise, it is called undetectable. Then, for any $|\psi\rangle \in \mathcal{Q}$, the syndrome measured in the state $E|\psi\rangle$ is nonzero for a detectable error, and it is zero otherwise. While operators in the stabilizer group are undetectable, they act trivially on the code; such errors can be ignored. Any two Pauli errors E_1, E_2 which differ by a phase and an element of the stabilizer, $E_2 = e^{i\alpha} E_1 S$, $S \in \mathcal{S}$, are called degenerate. Mutually degenerate errors act identically on the code, they cannot (and need not) be distinguished.

The distance d of a code $\mathcal{Q}$ is the minimum weight of an undetectable Pauli error $E \in \mathcal{P}_n$, which is not a part of the stabilizer, $E \notin \mathcal{S}$ (up to a phase). A code with distance d detects nontrivial Pauli errors of a weight up to $d - 1$, and it corrects such errors of a weight up to $\lfloor d/2 \rfloor$.

A code is called degenerate if its stabilizer includes a nontrivial operator $S \in \mathcal{S}$ with a weight smaller than the distance, $0 \neq \mathrm{wgt}(S) < d$. Degenerate codes are nice since generators of a small weight are easier to measure; all codes with a known FT threshold are degenerate. The ultimate case of degeneracy are w-limited quantum LDPC codes, where every stabilizer generator has weight w or smaller.

We consider three simple error models [25]: the quantum depolarizing channel, where with probability p an incoming qubit is replaced by a qubit in a random state; independent X or Z errors, where Pauli operators X and Z are applied to each qubit with probabilities p_X and p_Z, respectively, and the quantum erasure channel, where with probability y each qubit is replaced by an erasure state $|2\rangle$ orthogonal to both $|0\rangle$ and $|1\rangle$. We also address FT using a phenomenological error model where measurement errors happen independently with probability q. Such an error affects the syndrome bits but not the qubit states. Our thresholds are as follows.

Theorem 1.—Any sequence of long quantum codes (1) with stabilizer generators of weights w or less can be decoded with a vanishing error probability if channel probabilities (y, p) of erasures and depolarizing errors satisfy the restriction $2(w-1)\Upsilon(y, p) < e^{-1/D}$, where parameter D is defined in Eq. (1) and

$$\Upsilon(y, p) \equiv y + (1 - y)\left\{\frac{2p}{3} + 2\left[\frac{p}{3}(1 - p)\right]^{1/2}\right\}. \tag{4}$$

Theorem 2.—Any sequence of long CSS codes (1) with generator weights not exceeding w_X, w_Z can be decoded with a vanishing error probability if channel probabilities (y, p_X, p_Z) of erasures and independent X and Z errors satisfy the restrictions $(w_X - 1)\Upsilon_{\mathrm{CSS}}(y, p_Z) < e^{-1/D}$, $(w_Z - 1)\Upsilon_{\mathrm{CSS}}(y, p_X) < e^{-1/D}$, where

$$\Upsilon_{\mathrm{CSS}}(y, p) \equiv y + 2(1 - y)[p(1 - p)]^{1/2}. \tag{5}$$

The FT case gives weaker versions of Theorems 1 and 2.

Theorem 3.—If phenomenological syndrome measurement errors occur with probability q, vanishing error rates are achieved by (a) stabilizer codes of Theorem 1 if

$$4[q(1 - q)]^{1/2} + 2w\Upsilon(y, p) < e^{-1/D}, \tag{6}$$

(b) CSS codes of Theorem 2 if

$$\begin{aligned} &4[q(1 - q)]^{1/2} + w_X \Upsilon_{\mathrm{CSS}}(y, p_Z) < e^{-1/D}, \\ &4[q(1 - q)]^{1/2} + w_Z \Upsilon_{\mathrm{CSS}}(y, p_X) < e^{-1/D}. \end{aligned} \tag{7}$$

Our analysis is based on counting irreducible undetectable operators.

Definition 1.—For a given stabilizer code $\mathcal{Q}$, an undetectable operator is called irreducible if it cannot be decomposed as a product of two undetectable Pauli operators with support on nonempty disjoint sets of qubits.

This definition implies the following.

Lemma 1.—Any undetectable operator $E \in \mathcal{P}_n$ can be written as $E = \prod_i J_i$, where undetectable operators $J_i \in \mathcal{P}_n$, $\mathrm{wgt}(J_i) \neq 0$ are irreducible and pairwise disjoint.

For a given code, let $\mathcal{U} \subset \mathcal{P}_n \backslash \mathcal{S}$ denote the set of all nontrivial irreducible undetectable Pauli operators.

Given some error probability function $P(E)$, consider a syndrome-based decoder which returns the Pauli operator $E \in \mathcal{P}_n$ that maximizes $P(E)$ for a given syndrome. Notice that this is not a maximum-likelihood (ML) decoder since we ignore contributions of errors degenerate with E. Using a statistical-mechanical analogy [5,7,26], ML decoding corresponds to minimizing the free energy; here, we ignore entropy contribution resulting from degenerate errors and just minimize the energy $\varepsilon(E) \equiv -\ln P(E)$. Such a procedure is intrinsically suboptimal; thus, a lower bound for decoding threshold is also a lower bound for the syndrome-based ML decoding.

Now, let $E \in \mathcal{P}_n$ be an actual error, and E' be the same-syndrome Pauli operator which minimizes the energy $\varepsilon(E')$.

The product $E'E^\dagger$ is undetectable, it satisfies Lemma 4, which gives a decomposition $E'E^\dagger = \prod_i J_i$ into irreducible undetectable operators, $J_i \in \mathcal{S}\cup\mathcal{U}$. Since the operators J_i are mutually disjoint, none of them can decrease the energy of E', $\varepsilon(J_i E') \geq \varepsilon(E')$. Otherwise E' would not be the smallest-energy error with the same syndrome. The minimal-energy error E' is correct if and only if $E'E^\dagger$ is trivial, which implies that every irreducible component needs to be in the stabilizer, $J_j \in \mathcal{S}$ (up to a phase).

Otherwise, there is an irreducible operator $U \in \mathcal{U}$ which does not increase the energy of the original error E, $\varepsilon(UE) \leq \varepsilon(E)$. Let $\mathcal{B}(U) \equiv \{E \in \mathcal{P}_n : \varepsilon(UE) \leq \varepsilon(E)\}$ be the full set of such bad errors for a given $U \in \mathcal{U}$. Minimum-energy decoding gives a vanishing error rate if

$$\mathrm{Prob}[E : E \in \textstyle\bigcup_{U\in\mathcal{U}} \mathcal{B}(U)] \to 0, \qquad n \to \infty. \tag{8}$$

Then, the union bound for all $\mathcal{B}(U)$'s gives the following sufficient condition for error-free decoding:

$$\sum_{U\in\mathcal{U}} \mathrm{Prob}[E : E \in \mathcal{B}(U)] \to 0, \qquad n \to \infty. \tag{9}$$

For uncorrelated errors, only the qubits in the support of U affect the probabilities in Eq. (9). With uniform error distributions, these probabilities depend only on the weights $f \equiv \mathrm{wgt}(U)$ of the operators $U \in \mathcal{U}$. For example, if erasures occur with a single-qubit probability y, a bad error must cover the entire support of U, which gives simply $\mathrm{Prob}[E : E \in \mathcal{B}(U)] = y^{\mathrm{wgt}(U)}$. Let N_f denote the number of operators $U \in \mathcal{U}$ of weight $f \equiv \mathrm{wgt}(U)$. Since members of the stabilizer group are excluded from $\mathcal{U}$, $N_f = 0$ for $f < d$. Thus, in the case of the erasure channel, condition (9) is equivalent to

$$\sum_{f\geq d} N_f y^f \to 0, \qquad n \to \infty. \tag{10}$$

To construct an upper bound for N_f, we use a simplified version of the cluster-enumeration algorithm originally designed for finding the distance of a quantum LDPC code [27,28]. First, fix an arbitrary order of the r stabilizer generators G_i, $1 \leq i < r$. Start by placing any of $\{X, Y, Z\}$ at a position $j \in \{0, \ldots, n-1\}$ and place the corresponding Pauli operator as the only element of the list of the components of the operator being constructed. At every subsequent step, take the generator G_i corresponding to a nonzero syndrome bit with the smallest index i, and choose any position j in the support of G_i that is not yet selected; there are up to $\mathrm{wgt}(G_i) - 1$ choices. Choose a single-qubit Pauli operator different from the term present at the position j in the expansion (2) of G_i, and add it to the list. This sets the syndrome bit s_i to zero without modifying any of the existing entries on the list. At every step of the recursion, zero syndrome means a completed undetectable cluster; no position available to correct a chosen syndrome bit means that recursion got stuck. In either case, we need to go back one step by removing the element last added to the list. The procedure stops when we exhaust all choices.

If the recursion has depth f, we only construct operators of weight up to f. There are $3n$ possible choices for the first step, and up to $2[\mathrm{wgt}(G_i) - 1]$ for each subsequent step. Then, a w-limited LDPC code yields at most

$$\bar{N}_f = 3n[2(w-1)]^{f-1} \tag{11}$$

recursion paths to construct operators of weight up to f. This algorithm returns only undetectable operators. While not all of them are irreducible, all irreducible operators of weight f are constructed with depth-f recursion; see Sec. I in the Supplemental Material [29]. These arguments give the upper bound $\bar{N}_f \geq N_f$ for the number N_f of the irreducible operators $U \in \mathcal{U}$ of weight $\mathrm{wgt}(U) = f$.

For CSS codes, let $\mathcal{U}_X \subset \mathcal{U}$ and $\mathcal{U}_Z \subset \mathcal{U}$ be the sets of nontrivial irreducible undetectable operators composed of only X and Z operators, respectively, and $N_f^{(\mu)}$ be the number of weight-f operators in $\mathcal{U}_\mu$, $\mu \in \{X, Z\}$. For the codes in Theorem 2, this gives improved bounds, e.g.,

$$N_f^{(X)} \leq \bar{N}_f^{(X)} \equiv n(w_Z - 1)^{f-1}. \tag{12}$$

We illustrate the cluster enumeration on the toric code $[[2L^2, 2, L]]$, a CSS code with $w_X = w_Z = 4$ generators local in two dimensions. The qubits are on the bonds of an $L \times L$ square lattice with periodic boundary conditions along both bond directions. The stabilizer generators are the plaquette and vertex operators, $A_\Box = \prod_{j\in\Box} X_j$ and $B_+ = \prod_{j\in+} Z_j$ [Fig. 1(a)]. A type-X cluster can be started by placing an X operator anywhere, which makes the two operators B_+ on the neighboring vertices unhappy (the corresponding syndrome bits are nonzero). Either can be corrected by placing an additional X operator on one of the remaining three open bonds adjoining the corresponding vertex. This produces an additional unhappy operator B_+ at the other end of the bond, etc. An undetectable cluster corresponds to a closed walk (cycle). Any cycle can be constructed this way. A topologically trivial cycle gives a member of the stabilizer group, while a cycle winding an odd number of times over one or both periodicity directions corresponds to a logical operator. Further, a cycle with

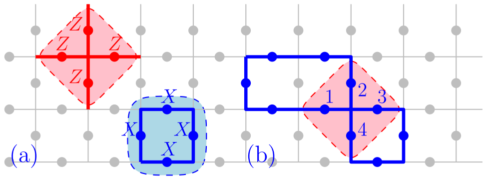

FIG. 1 (color online). (a) Toric code generators. Plaquette $A_\Box$ (shaded rounded square) and vertex B_+ (shaded diamonds) operators constructed as products of four Pauli X and Pauli Z operators, respectively. (b) A reducible cluster is counted as one or two clusters, depending on the order in which the numbered qubits are chosen.

self-intersections gives an operator which can be decomposed into a product of two or more disjoint irreducible operators [Fig. 1(b)].

Combining Eq. (10) and the bound $N_f \le \bar{N}_f$, see Eq. (11), we can prove a simplified version of Theorem 1 for erasures only. Namely, consider the sum

$$Q_d(y) \equiv \sum_f \bar{N}_f y^f = \frac{3ny[2y(w-1)]^{d-1}}{1-2y(w-1)}; \qquad (13)$$

it converges absolutely for $2y(w-1) < 1$. Asymptotically, $Q_d(y)$ converges to zero as long as $n[2y(w-1)]^d \to 0$. This is true for any $y < e^{-1/D}/2(w-1)$ for codes in Eq. (1). The sum (13) majors Eq. (10) term by term, which gives a lower bound for erasure threshold, $y_c \ge e^{-1/D}/2(w-1)$; cf. Theorem 1. With the distance scaling superlogarithmically (e.g., as a power law), the sum (13) vanishes anywhere within the convergence radius, $y < [2(w-1)]^{-1}$, and we may just set $e^{-1/D} \to 1$.

Theorems 1 and 2, which combine erasures and errors, can be proved similarly if we notice that the probabilities in Eq. (9) can be bounded as in Eq. (10), with some effective erasure rate $\Upsilon \ge y$ (see the Supplemental Material [29], Sec. II).

Arguments used so far require ideal syndrome measurements. For quantum codes, it is more important to consider the FT case where errors can occur in any quantum gate during syndrome measurements [2,4,30–34]. Such a complete analysis is beyond the scope of this Letter. Instead, we give a simplified estimate based on a phenomenological error model, which assumes that measured syndrome bits can have errors, but otherwise there is no effect on the qubits [5,10,19]. Error correction involves repeated syndrome measurement cycles and an auxiliary code which combines the syndromes measured in subsequent cycles. We only consider the simplest case where repetition code is used for combining the syndromes. For a CSS code, with equal uncorrelated qubit and syndrome errors $q = p_X = p_Z$, the net effect is equivalent to increasing the weights of stabilizer generators in Eq. (12) and in Theorem 2 by two, $w \to w+2$. With the surface codes, decoding corresponds to minimal-weight matching of chains in three dimensions [5]. For a more general result, we have to bound the number of weight-f clusters N_{f,f_q}, which include f_q qubit Pauli operators and $f - f_q$ binary syndrome errors. Theorem 3 follows from the bound $N_{f,f_q} \le \bar{N}_{f,f_q}$,

$$\bar{N}_{f,f_q} \equiv 3nm2^f \binom{f}{f_q} w^{f_q}, \qquad (14)$$

where m is the number of measurement cycles (same as the code distance d; see the Supplemental Material [29], Sec. III).

How tight are these bounds? The toric code ($w = 4$) has an erasure threshold $y_c = 0.5$ and the ML threshold for independent X and Z errors $p_{Zc} = p_{Xc} \approx 0.11$, compared to $y_c^* = 1/3$ and $p_{Zc}^* \approx 0.029$ of Theorem 2. Bound (12) was also verified by counting irreducible clusters numerically (see Sec. IV of the Supplemental Material [29]) and fitting with $\ln N_f = A + \zeta_w f$, where $\zeta_w \le w-1$ for CSS codes with row weight w was expected from Eq. (12). In particular, we got $\zeta_6 \approx 4.76$, $\zeta_7 \approx 5.74$, and $\zeta_8 \approx 5.79$, indicating that our bounds for N_f are relatively tight.

In conclusion, we constructed lower bounds on the thresholds of weight-limited quantum LDPC codes with sublinear distances scaling logarithmically or faster with the code length n. These bounds are based on estimating the number of logical operators which cannot be decomposed into a product of disjoint undetectable operators. The resulting analytical expressions combine probabilities of erasures, depolarizing errors (independent X/Z errors for CSS codes), and syndrome measurement errors using a phenomenological error model. These bounds are much stronger than those constructed previously [19], and they have a different dependence on the code parameters. In particular, we no longer require that each qubit be involved in a limited number of stabilizer generators. Qualitatively, the main difference is that the present analysis is not based on percolation theory.

This technique could carry over from LDPC codes to more general degenerate codes, where the corresponding scaling of N_f can be calculated numerically or analytically (e.g., in the case of concatenated codes). It would be interesting to see if a finite FT threshold exists for finite-rate and finite relative distance quantum LDPC codes constructed by Bravyi and Hastings [18]. A related open problem is the existence of FT threshold for subsystem codes, e.g., a subclass of those constructed in Ref. [35].

Our bounds also limit the parameters of quantum LDPC codes, in particular, their rate R. Indeed, Theorem 2 gives the erasure threshold $y_c^{(\mathrm{CSS})} \ge 1/(w-1)$ for CSS LDPC codes with superlogarithmic distance. Along with the trivial upper bound $y_c \le (1-R)/2$, this implies that no such codes exist if $R > 1 - 2/(w-1)$. For codes with $w = 4$, this gives $R \le 1/3$, whereas the only known example of such codes is $R = 0$ (toric codes). These can be further improved by using the tighter upper bounds constructed for quantum LDPC codes in Ref. [21].

Also, Pastawski and Yoshida pointed to us that our erasure thresholds can be combined with their upper bound [22] for codes which include nontrivial transversal logical gates from mth level of the Clifford hierarchy [36], $y_m \le 1/m$. Thus, e.g., only CSS codes with generators of weight $w \ge m+1$ may include such logical gates. We note that the analysis in Refs. [22,36] is largely based on the cleaning lemma [11,37] and the notion of correctable subsets, which complement our irreducible undetectable operators (Definition 1). It would be interesting to check to see if this relation could help extending the bounds from Ref. [11] to general LDPC codes.

This work was supported in part by the U.S. Army Research Office under Grant No. W911NF-14-1-0272 (L. P. P.) and by the NSF under Grants No. ECCS-1102074 (I. D.), No. PHY-1416578 (L. P. P.), No. PHY-1415600 (A. A. K.), and No. EPSCoR-1004094 (A. A. K.). L. P. P. also acknowledges hospitality by the Institute for Quantum Information and Matter, a NSF Physics Frontiers Center with support of the Gordon and Betty Moore Foundation.

[1] P. W. Shor, Scheme for reducing decoherence in quantum computer memory, Phys. Rev. A **52**, R2493 (1995).
[2] P. W. Shor, in *Proceedings of the 37th Annual Symposium on Foundations of Computer Science (FOCS 1996), Burlington, VT, 1996* (IEEE, New York, 1996), p. 56.
[3] A. M. Steane, Active Stabilization, Quantum Computation, and Quantum State Synthesis, Phys. Rev. Lett. **78**, 2252 (1997).
[4] D. Gottesman, Theory of fault-tolerant quantum computation, Phys. Rev. A **57**, 127 (1998).
[5] E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, Topological quantum memory, J. Math. Phys. (N.Y.) **43**, 4452 (2002).
[6] E. Knill, Scalable quantum computation in the presence of large detected-error rates, arXiv:quant-ph/0312190; E. Knill, Fault-tolerant postselected quantum computation: Threshold analysis, arXiv:quant-ph/0404104; P. Aliferis, D. Gottesman, and J. Preskill, Quantum accuracy threshold for concatenated distance-3 codes, Quantum Inf. Comput. **6**, 97 (2006); B. W. Reichardt, Error-detection-based quantum fault-tolerance threshold, Algorithmica **55**, 517 (2009).
[7] H. G. Katzgraber, H. Bombin, and M. A. Martin-Delgado, Error Threshold for Color Codes and Random Three-Body Ising Models, Phys. Rev. Lett. **103**, 090501 (2009).
[8] A. R. Calderbank, E. M. Rains, P. M. Shor, and N. J. A. Sloane, Quantum error correction via codes over GF(4), IEEE Trans. Inf. Theory **44**, 1369 (1998).
[9] M. Grassl, Bounds on the minimum distance of linear codes and quantum codes, http://www.codetables.de (July 28, 2011).
[10] H. Bombin and M. A. Martin-Delgado, Topological Quantum Distillation, Phys. Rev. Lett. **97**, 180501 (2006); H. Bombin and M. A. Martin-Delgado, Optimal resources for topological two-dimensional stabilizer codes: Comparative study, Phys. Rev. A **76**, 012305 (2007); A. J. Landahl, J. T. Anderson, and P. R. Rice, Fault-tolerant quantum computing with color codes, arXiv:1108.5738.
[11] S. Bravyi and B. Terhal, A no-go theorem for a two-dimensional self-correcting quantum memory based on stabilizer codes, New J. Phys. **11**, 043029 (2009); S. Bravyi, D. Poulin, and B. Terhal, Tradeoffs for Reliable Quantum Information Storage in 2D Systems, Phys. Rev. Lett. **104**, 050503 (2010).
[12] M. S. Postol, A proposed quantum low density parity check code, arXiv:quant-ph/0108131; D. J. C. MacKay, G. Mitchison, and P. L. McFadden, Sparse-graph codes for quantum error correction, IEEE Trans. Inf. Theory **50**, 2315 (2004).
[13] D. Gottesman, Ph.D. thesis, California Institute of Technology, 1997.
[14] J.-P. Tillich and G. Zemor, in *Proceedings of the IEEE International Symposium on Information Theory (ISIT), Seoul, 2009* (IEEE, New York, 2009), p. 799.
[15] A. A. Kovalev and L. P. Pryadko, *Improved Quantum Hypergraph-Product LDPC Codes* (IEEE, New York, 2012), p. 348.
[16] I. Andriyanova, D. Maurice, and J.-P. Tillich, New constructions of CSS codes obtained by moving to higher alphabets, arXiv:1202.3338.
[17] A. A. Kovalev and L. P. Pryadko, Quantum Kronecker sum-product low-density parity-check codes with finite rate, Phys. Rev. A **88**, 012311 (2013).
[18] S. Bravyi and M. B. Hastings, in *Proceedings of the 46th ACM Symposium on Theory of Computing (STOC 2014), New York, 2014* (ACM, New York, 2014).
[19] A. A. Kovalev and L. P. Pryadko, Fault tolerance of quantum low-density parity check codes with sublinear distance scaling, Phys. Rev. A **87**, 020304(R) (2013).
[20] D. Gottesman, What is the overhead required for fault-tolerant quantum computation?, arXiv:1310.2984.
[21] N. Delfosse and G. Zémor, Upper bounds on the rate of low density stabilizer codes for the quantum erasure channel, Quantum Inf. Comput. **13**, 793 (2013).
[22] F. Pastawski and B. Yoshida, Fault-tolerant logical gates in quantum error-correcting codes, Phys. Rev. A **91**, 012305 (2015).
[23] A. R. Calderbank and P. W. Shor, Good quantum error-correcting codes exist, Phys. Rev. A **54**, 1098 (1996).
[24] A. M. Steane, Simple quantum error-correcting codes, Phys. Rev. A **54**, 4741 (1996).
[25] C. H. Bennett, D. P. DiVincenzo, and J. A. Smolin, Capacities of Quantum Erasure Channels, Phys. Rev. Lett. **78**, 3217 (1997).
[26] A. A. Kovalev and L. P. Pryadko, Spin glass reflection of the decoding transition for quantum error-correcting codes, Quantum Inf. Comput. **15**, 0825 (2015).
[27] A. A. Kovalev, I. Dumer, and L. P. Pryadko, Linked-cluster technique for finding the distance of a quantum LDPC code, in *Proceedings of the 2013 Information Theory and Applications (ITA) Workshop, San Diego, 2013* (IEEE, New York, 2013), p. 1.
[28] I. Dumer, A. A. Kovalev, and L. P. Pryadko, in *Proceedings of the 2014 IEEE International Symposium on Information Theory (ISIT), Honolulu, 2014* (IEEE, New York, 2014), p. 1086.
[29] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.115.050502 for proofs and additional discussion.
[30] D. P. DiVincenzo and P. W. Shor, Fault-Tolerant Error Correction with Efficient Quantum Codes, Phys. Rev. Lett. **77**, 3260 (1996).
[31] E. Knill, R. Laflamme, and W. H. Zurek, Resilient quantum computation, Science **279**, 342 (1998).
[32] A. M. Steane, Efficient fault-tolerant quantum computing, Nature (London) **399**, 124 (1999).
[33] A. M. Steane, Overhead and noise threshold of fault-tolerant quantum error correction, Phys. Rev. A **68**, 042322 (2003).
[34] A. M. Steane and B. Ibinson, Fault-tolerant logical gate networks for Calderbank-Shor-Steane codes, Phys. Rev. A **72**, 052335 (2005).
[35] S. Bravyi, Subsystem codes with spatially local generators, Phys. Rev. A **83**, 012320 (2011).
[36] S. Bravyi and R. König, Classification of Topologically Protected Gates for Local Stabilizer Codes, Phys. Rev. Lett. **110**, 170503 (2013).
[37] S. Bravyi and A. Kitaev, Universal quantum computation with ideal Clifford gates and noisy ancillas, Phys. Rev. A **71**, 022316 (2005).