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Development and Application of Difference and Fractional Calculus on Discrete Time Scales

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DEVELOPMENT AND APPLICATION OF DIFFERENCE AND FRACTIONAL CALCULUS ON DISCRETE TIME SCALES

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The purpose of this dissertation is to develop and apply results of both discrete calculus and discrete fractional calculus to further develop results on various discrete time scales. Two main goals of discrete and fractional discrete calculus are to extend results from traditional calculus and to unify results on the real line with those on a variety of subsets of the real line. Of particular interest is introducing and analyzing results related to a generalized fractional boundary value problem with Lidstone boundary conditions on a standard discrete domain $\mathbb{N}_a$. We also introduce new results regarding exponential order for functions on quantum time scales, along with extending previously discovered results. Finally, we conclude by introducing and analyzing a boundary value problem, again with Lidstone boundary conditions, on a mixed time scale, which may be thought of as a generalization of the other time scales in this work.

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Contents

List of Figures

Chapter 1

Introduction

The study of both discrete calculus and discrete fractional calculus provides perhaps a more complete, beautiful, and general view of calculus than most get from a traditional study of calculus. It provides much added insight into the ideas of derivatives and integrals as it shows the orders of derivatives and integrals need not be restricted to whole numbers but can, in fact, be sensibly, consistently, and continuously defined for any positive number (see [5], [38], and [39] for some insight). In this work, we focus on analogues of calculus and of fractional calculus on discrete time scales, i.e., discrete calculus and discrete fractional calculus.

Discrete fractional calculus is a relatively young field of study that begins with the analysis of calculus restricted to any generic closed subset of the real line. An attractive feature of discrete fractional calculus is how many of the results can be seen simply as generalizations of familiar results from calculus on the entire real line. A small set of definitions opens the door to many different statements and theorems that apply widely across various closed subsets. While many of these results are similar across these various, chosen domains, many of the results interestingly prove to look and behave quite differently with a simple domain change. Often, one must keep close track of the specified domains involved, as certain operators serve to shift or otherwise alter the domain of a given problem–an issue that does not arise in traditional calculus. Two main features of calculus on time scales are the unification of results from continuous and discrete domains and the extension of those results. Time scales calculus itself can be used to model insect (or other) populations which have a continuous growing season and then a dying out or dormancy season [19]. Fractional calculus has been shown to be suitable in the descriptions and applications of properties of real materials such as polymers and rocks. Fractal theory and dynamical systems also make use of fractional derivatives as do some biological applications. Some other areas which make use of fractional calculus are rheology, viscoelasticity, electrochemistry, and electromagnetism (see [2], [11], [36], and [39] for more about these aforementioned applications). Where calculus concepts from the entire real line show up, discrete calculus concepts and applications are not far behind.

We restrict ourselves here to analyzing results with respect to the delta difference, though much of the work can carry over similarly when working with the nabla difference (see [4], [8], and [30] for work with the nabla operator, while [28] and [29] highlight some results with the delta-nabla operator). Outside of this introductory chapter and any preliminary sections in other chapters, all work can be considered to have been developed originally unless otherwise indicated (however, most of the results in Section 3.2 are not new and can be found in sources such as [33], but these results were developed independently of outside sources and later compared and contrasted). In this chapter, we provide many well-known results that provide a foundation for the results and applications in the following chapters. While much of the general necessary background material for this work is presented in this chapter, other, more specific, foundational material will be presented in later chapters. Much of this can be found in [7], [19], [27], and [32], while other background, foundational, and related material can be found in [23] and [34].

Definition 1.1. A time scale, $\mathbb{T}$, is any closed, nonempty subset of $\mathbb{R}$.

Example 1.2. Some examples of time scales are as follows:

(i) $\mathbb{R}$;

(ii) $[0,1] \cup [5,6]$;

(iii) the set of integers $\mathbb{Z}$;

(iv) the Cantor set;

(v) $\mathbb{N}_a := \{a, a+1, a+2, \ldots \,|\, a \in \mathbb{R}\}$;

(vi) $\mathbb{N}_a^b := \{a, a+1, a+2, \ldots\, b |\, a, b \in \mathbb{R}\ s.t.\ b - a \in \mathbb{N}\}$;

(vii) $aq^{\mathbb{N}_0} := \{a, aq, aq^2, \ldots\}$ for a fixed $a > 0, q > 1$;

(viii) $\{1/n \,|\, n \in \mathbb{N}\} \cup \{0\}$.

We may note that $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$, $\mathbb{C}$, and $(0,1)$ are not time scales [19].

Definition 1.3. For a time scale $\mathbb{T}$, the forward jump operator, σ, is defined as

$$\sigma(t) := \inf\{s \in \mathbb{T} \,|\, s > t\}.$$

If $\sigma(t) = t$ (and $\sigma(t) \neq \sup \mathbb{T}$), we say that t is right-dense. Otherwise t is right-scattered. We also define $\sigma^n(t)$, for $n \in \mathbb{N}_0$, as

$$\sigma^n(t) := \begin{cases} \sigma(\sigma^{n-1}(t)), & n \in \mathbb{N}, \\ t, & n = 0. \end{cases}$$

Definition 1.4. For a time scale $\mathbb{T}$, the backward jump operator, ρ, is defined as

$$\rho(t) := \sup\{s \in \mathbb{T} \,|\, s < t\}.$$

If $\rho(t) = t$ (and $\rho(t) \neq \inf \mathbb{T}$), we say that t is left-dense. Otherwise t is left-scattered. We also define $\rho^n(t)$, for $n \in \mathbb{N}_0$, as

$$\rho^n(t) := \begin{cases} \rho(\rho^{n-1}(t)), & n \in \mathbb{N}, \\ t, & n = 0. \end{cases}$$

Remark 1.5. $\mathbb{R}$ is a time scale such that for every $t \in \mathbb{R}$, t is dense, i.e., both right-dense and left-dense.

Definition 1.6. For a time scale $\mathbb{T}$, we define the graininess function $\mu(t) : \mathbb{T} \to [0, \infty)$ by

$$\mu(t) = \sigma(t) - t.$$

Remark 1.7. The time scales considered throughout this work will be ones in which all elements are isolated points. In other words, for any time scale $\mathbb{T}$ here and for all $t \in \mathbb{T}$, we have $\mu(t) > 0$.

Since in the chapters that follow, we will only deal with time scales whose elements are all isolated points, i.e., points which are neither left nor right dense, we now define the delta difference on a time scale of a function at an isolated point. This operation can be thought of as an analogue to differentiation on $\mathbb{R}$. In fact, if a point in a time scale is right dense, the delta difference of a function at that point is defined to coincide with the traditional definition of derivative at that point [19].

Definition 1.8. Consider a function $f : \mathbb{T} \to \mathbb{R}$. We define the delta difference of f at a point $t \in \mathbb{T}$ as

$$\Delta f(t) := \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

We also define $\Delta^n f(t)$ for, $n \in \mathbb{N}_0$, as

$$\Delta^n f(t) := \begin{cases} \Delta(\Delta^{n-1} f(t)), & n \in \mathbb{N}, \\ f(t), & n = 0. \end{cases}$$

Remark 1.9. Note that on the time scale $\mathbb{N}_a$, the focus of the following chapter, we arrive at the definition

$$\Delta f(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = f(t+1) - f(t).$$

Since we have a notion of a delta difference which serves as an analogue to differentiation, we now define the delta definite integral, an analogue to definite Riemann integration on $\mathbb{R}$. Here, we will define the definite integral on $\mathbb{N}_a$. The definite integral on other time scales in this work will be defined in the appropriate chapter.

Definition 1.10. For $f : \mathbb{N}_a \to \mathbb{R}$ and $c, d \in \mathbb{N}_a$, we make the definition

$$\int_c^d f(s)\,\Delta s := \begin{cases} \sum_{s=c}^{d-1} f(s)\mu(s), & d > c, \\ 0, & d \leq c. \end{cases}$$

Remark 1.11. Note that the definite integral here is really just a left-hand Riemann sum. The definite integral in future chapters will be defined similarly to give a left-hand sum evaluated at points from the time scale.

Remark 1.12. The definite integral above helps define an antidifference of f on $\mathbb{N}_a$, namely $\int_a^t f(s)\,\Delta s$. For

$$\begin{aligned}\Delta \int_a^t f(s)\,\Delta s &= \Delta \sum_{s=a}^{t-1} f(s) \\ &= \sum_{s=a}^{t} f(s) - \sum_{s=a}^{t-1} f(s) \\ &= f(t).\end{aligned}$$

As it will show up repeatedly in the next chapter, we present Euler's Gamma Function along with some of its properties.

Definition 1.13. For $z \in \mathbb{C}$ such that $\operatorname{Re} z > 0$, Euler's Gamma Function is defined by the improper integral on $\mathbb{R}$

$$\Gamma(z) := \int_0^\infty e^{-t} t^{z-1}\,dt.$$

Remark 1.14. As can be found readily in many sources, the following are useful properties of Euler's Gamma Function which will be used extensively in the next chapter. The improper integral in the definition above converges for all $z \in \mathbb{C}$ such that $\operatorname{Re} z > 0$. Using property (ii) below, we extend the definition of Euler's Gamma Function to all $z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$.

(i) For $0 < z \in \mathbb{R}$, $\Gamma(z) > 0$;

(ii) $\Gamma(z+1) = z\Gamma(z)$;

(iii) for $n \in \mathbb{N}_0$, $\Gamma(n+1) = n!$.

The following figure presents a graph of the Gamma function:

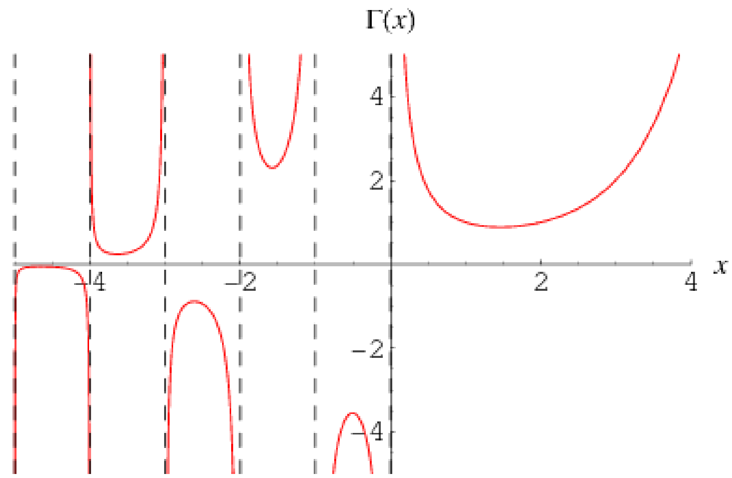

Figure 1.1: The Real Gamma Function on $(-5, 4]$

A common application of the Gamma function in discrete calculus is the use of "falling" notation.

Definition 1.15. For $t \in \mathbb{R}$, $t^{\underline{\nu}}$, read as "t to the ν falling," is defined as

$$t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t-\nu+1)}$$

for any $\nu \in \mathbb{R}$ such that the right-hand side makes sense. By convention, when $t-\nu+1$ is a nonpositive integer and $t+1$ is not,

$$t^{\underline{\nu}} := 0$$

since for $n \in \mathbb{N}_0$,

$$\lim_{t \to -n} |\Gamma(t)| = \infty.$$

Remark 1.16. We may make the following notes regarding the falling function above:

(i) when $t, \nu \in \mathbb{N}$,

$$t^{\underline{\nu}} = \frac{t!}{(t-\nu)!} = t(t-1)(t-2)\cdots(t-\nu+1);$$

(ii) $\nu^{\underline{\nu}} = \nu^{\underline{\nu-1}} = \Gamma(\nu+1)$;

(iii) for $a \in \mathbb{R}$, $\Delta(t-a)^{\underline{\nu}} = \nu(t-a)^{\underline{\nu-1}}$;

(iv) for $a \in \mathbb{R}$, $\Delta(a-t)^{\underline{\nu}} = -\nu(a-\sigma(t))^{\underline{\nu-1}}$;

(v) $t^{\underline{\nu+1}} = (t-\nu)t^{\underline{\nu}}$.

Properties (iii) and (iv) above constitute the power rule for the delta difference operator.

We now turn our attention to defining a fractional sum. Before we can define that, however, we define the n^{th}-order sum on $\mathbb{N}_a$. First, though, we note that on $\mathbb{R}$, the unique solution to the n^{th}-order initial value problem

$$\begin{cases} y^{(n)}(t) = f(t), \quad t \in [a, \infty), \\ y^{(i)}(a) = 0, \end{cases}$$

where $i = 0, 1, 2, \ldots, n-1$ is given by n repeated definite integrals of f, i.e.,

$$\begin{aligned} y(t) &= \int_a^t \int_a^{\tau_{n-1}} \int_a^{\tau_{n-2}} \cdots \int_a^{\tau_1} f(\tau_1)\, d\tau_1\, d\tau_2 \cdots d\tau_{n-1} \\ &= \int_a^t \frac{(t-\tau_{n-1})^{n-1}}{(n-1)!} f(\tau_{n-1})\, d\tau_{n-1} \\ &= \frac{1}{\Gamma(n)} \int_a^t (t-\tau_{n-1})^{n-1} f(\tau_{n-1})\, d\tau_{n-1}. \end{aligned}$$

Similarly, on $\mathbb{N}_a$, the unique solution to the n^{th}-order initial value problem

$$\begin{cases} \Delta^n y(t) = f(t), \quad t \in \mathbb{N}_a, \\ \Delta^i y(a) = 0, \end{cases}$$

where $i = 0, 1, 2, \ldots, n-1$ is given by n repeated finite sums of f, i.e., for $t \in \mathbb{N}_a$

$$\begin{aligned} y(t) &= \int_a^t \int_a^{\tau_{n-1}} \int_a^{\tau_{n-2}} \cdots \int_a^{\tau_1} f(\tau_1)\, \Delta\tau_1\, \Delta\tau_2 \cdots \Delta\tau_{n-1} \\ &= \sum_{\tau_{n-1}=a}^{t-1} \sum_{\tau_{n-2}=a}^{\tau_{n-1}-1} \sum_{\tau_{n-3}=a}^{\tau_{n-2}-1} \cdots \sum_{\tau_1=a}^{\tau_2-1} f(\tau_1) \\ &= \sum_{\tau_{n-1}=a}^{t-n} \frac{(t-\sigma(\tau_{n-1}))^{\underline{n-1}}}{(n-1)!} f(\tau_{n-1}) \\ &= \frac{1}{\Gamma(n)} \sum_{\tau_{n-1}=a}^{t-n} (t-\sigma(\tau_{n-1}))^{\underline{n-1}} f(\tau_{n-1}), \end{aligned}$$

which we will call the n^{th}-order sum of f and denote as $\Delta_a^{-n} f(t)$.

The n^{th}-order sum above serves to motivate the definition of a ν^{th}-order fractional sum. Despite use of the word "fractional," ν may be any nonnegative real number here.

Definition 1.17. For $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu > 0$, the ν^{th}-order fractional sum of f (based at $a \in \mathbb{R}$) is given by

$$(\Delta_a^{-\nu} f)(t) = \Delta_a^{-\nu} f(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} f(s),$$

where $t \in \mathbb{N}_{a+\nu}$. Additionally, we define $\Delta_a^{-0} f(t) := f(t)$ for $t \in \mathbb{N}_a$.

Remark 1.18. Notice that the domain of the fractional sum of f above is shifted by

ν from the domain of f. We will also see that the domain of the fractional difference of f will be similarly shifted. For more on discrete fractional initial value problems, see [9].

We are now able to define a ν^{th}-order fractional difference. Though it is originally defined in terms of a fractional sum, we can also arrive at a formula similar to, but independent of, a fractional sum.

Definition 1.19. For $f : \mathbb{N}_a \to \mathbb{R}$, $\nu \geq 0$, and $N \in \mathbb{N}$ such that $N - 1 < \nu \leq N$, the ν^{th}-order fractional difference of f (based at $a \in \mathbb{R}$) is given by

$$(\Delta_a^\nu f)(t) = \Delta_a^\nu f(t) := \Delta^N \Delta_a^{-(N-\nu)} f(t),$$

where $t \in \mathbb{N}_{a+N-\nu}$.

Remark 1.20. If $\nu \in \mathbb{N}_0$, we see that the definition above coincides with the definition of a whole-order difference from Definition 1.8 as, for $t \in \mathbb{N}_a$,

$$\Delta_a^\nu f(t) = \Delta^N \Delta_a^{-(N-\nu)} f(t) = \Delta^N \Delta_a^{-0} f(t) = \Delta^N f(t).$$

Additionally, we note that whereas whole-order differences are not based at any certain point a, fractional-order differences do. However, as demonstrated in [32], this dependence on the base a vanishes as $\nu \to N \in \mathbb{N}_0$. Other domain issues, concerns, and consequences of the definitions above which are not immediately important to this work may be found in [32].

Remark 1.21. When analyzing ν^{th}-order fractional difference equations, we should be aware of $N \in \mathbb{N}_0$ such that $N - 1 < \nu \leq N$, since a well-posed ν^{th}-order fractional difference equation requires N initial conditions, e.g., the equation $\Delta_{-0.8}^{5.2}y(t) = f(t)$ needs 6 initial conditions to determine $y(t)$.

We have a version of Leibniz' Rule in the following theorem (whose short proof will be shown here) which is used in [32] to prove a theorem which unifies the definitions of fractional sums and differences.

Theorem 1.22. *For $g : \mathbb{N}_{a+\nu} \times \mathbb{N}_a \to \mathbb{R}$,*

$$\Delta\left(\sum_{s=a}^{t-\nu} g(t,s)\right) = \sum_{s=a}^{t-\nu} \Delta_t g(t,s) + g(t+1, t+1-\nu),$$

noting that the subscript in "Δ_t" is simply there to signify that the difference is being taken with respect to t.

Proof. By direct computation,

$$\begin{aligned}
\Delta\left(\sum_{s=a}^{t-\nu} g(t,s)\right) &= \sum_{s=a}^{t+1-\nu} g(t+1,s) - \sum_{s=a}^{t-\nu} g(t,s) \\
&= \sum_{s=a}^{t-\nu}[g(t+1,s) - g(t,s)] + g(t+1,t+1-\nu) \\
&= \sum_{s=a}^{t-\nu} \Delta_t g(t,s) + g(t+1,t+1-\nu).
\end{aligned}$$

□

The following well-known result unifies the definition of a fractional difference with that of a fractional sum.

Theorem 1.23. *For $f : \mathbb{N}_a \to \mathbb{R}$, $\nu \geq 0$, and $N \in \mathbb{N}$ such that $N-1 < \nu \leq N$, the*

ν^{th}-order fractional difference of f (based at $a \in \mathbb{R}$) is given by

$$\Delta_a^{\nu} f(t) = \begin{cases} \dfrac{1}{-\nu} \displaystyle\sum_{s=a}^{t+\nu} (t-\sigma(s))^{\underline{-\nu-1}} f(s), & N-1 < \nu < N, \\ \Delta^N f(t), & \nu = N. \end{cases}$$

We now present some fractional power rules involving both a fractional sum and difference, which may also be found in [27].

Theorem 1.24. *For $a \in \mathbb{R}$, $\mu \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$, $\nu > 0$, and $N \in \mathbb{N}$ such that $N-1 < \nu \leq N$, the following hold:*

(i) $\Delta_{a+\mu}^{-\nu}(t-a)^{\underline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1+\nu)}(t-a)^{\underline{\mu+\nu}}$, *for* $t \in \mathbb{N}_{a+\mu+\nu}$;

(ii) $\Delta_{a+\mu}^{\nu}(t-a)^{\underline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-\nu)}(t-a)^{\underline{\mu-\nu}}$, *for* $t \in \mathbb{N}_{a+\mu+N-\nu}$.

This completes the necessary background material needed to provide a foundation for the following chapters in which we will analyze the Green's Function of fractional boundary value problems with Lidstone boundary conditions on both $\mathbb{N}_a$ and a mixed time scale and investigate many of these previous results and others on q-time scales.

Chapter 2

Green's Functions on $\mathbb{N}_a$ with Lidstone Boundary Conditions

In this chapter, we wish to develop a fractional discrete analogue to an ordinary boundary value differential equation with Lidstone boundary conditions, which has the form

$$\begin{cases} (-1)^n y^{(2n)}(t) = h(t), \\ y^{(2i)}(0) = 0 = y^{(2i)}(1), \end{cases} \tag{2.0.1}$$

where $i = 0, 1, 2, \ldots, n-1$ and $t \in [a, b]$. In [1] and [3], we may find some properties and numerical applications of differential equations with Lidstone boundary conditions. This particular boundary value problem can be shown to have the solution

$$y(t) = \int_0^1 \int_0^1 \cdots \int_0^1 G(t, \tau_n) G(\tau_n, \tau_{n-1}) \cdots G(\tau_2, \tau_1) h(\tau_1)\, d\tau_1\, d\tau_2 \cdots d\tau_n,$$

where G is a Green's function given by

$$G(t,s) := \begin{cases} (1-t)s, & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1, \end{cases}$$

as can be found in [21]. For more on determining Green's functions for fractional boundary value problems, see [10], [27], and [32].

2.1 Preliminaries

In the last section of this chapter (and in the last chapter), we will make use of the following two theorems: a fixed point theorem attributed to Krasnosel'skii, which may be found in [27] and [32], and the Banach contraction mapping theorem, which may be found in [27], [32], and [35]. First, we define a cone as a subset of a Banach space.

Definition 2.1. If $\mathcal{B}$ is a real Banach space and $\mathcal{K} \subseteq \mathcal{B}$, then $\mathcal{K}$ is a cone if $\mathcal{K}$ satisfies both of the following conditions:

1. if $x \in \mathcal{K}$ and $\lambda \ge 0$, then $\lambda x \in \mathcal{K}$, and
2. if $x \in \mathcal{K}$ and $-x \in \mathcal{K}$, then $x = 0$.

Theorem 2.2. *Let $\mathcal{B}$ be a Banach space, and let $\mathcal{K} \subseteq \mathcal{B}$ be a cone. Suppose that Ω_1 and Ω_2 are bounded open sets contained in $\mathcal{B}$ such that $0 \in \Omega_1$ and $\overline{\Omega_1} \subseteq \Omega_2$. Then a completely continuous operator $T : \mathcal{K} \cap (\overline{\Omega_2} \setminus \Omega_1) \to \mathcal{K}$ has at least one fixed point in $\mathcal{K} \cap (\overline{\Omega_2} \setminus \Omega_1)$ if either*

1. $\|Ty\| \le \|y\|$ *for* $y \in \mathcal{K} \cap \partial\Omega_1$ *and* $\|Ty\| \ge \|y\|$ *for* $y \in \mathcal{K} \cap \partial\Omega_2$, *or*
2. $\|Ty\| \ge \|y\|$ *for* $y \in \mathcal{K} \cap \partial\Omega_1$ *and* $\|Ty\| \le \|y\|$ *for* $y \in \mathcal{K} \cap \partial\Omega_2$.

Theorem 2.3. *Let (X, d) be a complete metric space. If $f : X \to X$ is a contraction mapping, where $\alpha \in [0, 1)$ is a constant such that for all $x, y \in X$,*

$$d\big(f(x), f(y)\big) \leq \alpha d(x, y),$$

then the following hold:

(i) f *has a unique fixed point* $x_0 \in X$*;*

(ii) $\lim_{n\to\infty} f^n(x) = x_0$ *for all* $x \in X$*;*

(iii) $d(f^n(x), x_0) \leq \frac{\alpha^n}{1-\alpha} d(f(x), x)$ *for all* $x \in X$ *and* $n \in \mathbb{N}$.

2.2 Derivation of the Green's Function

In the following theorem, we show how to develop and define the Green's function for a boundary value problem with Lidstone boundary conditions on $\mathbb{N}_a$. This problem serves as an analogue to the BVP (2.0.1) from above. In [17] and [37], we may gain insight into related higher-order equations on time scales, while here we discuss the solutions to higher-order fractional equations on time scales.

Theorem 2.4. *Let the domains*

$$S_j := \{(t, s) \in \mathbb{N}_{j(\nu-2)}^{b+j(\nu-2)} \times \mathbb{N}_{(j-1)(\nu-2)}^{b+(j-1)(\nu-2)} \mid s \leq t - \nu\},$$

and

$$T_j := \{(t, s) \in \mathbb{N}_{j(\nu-2)}^{b+j(\nu-2)} \times \mathbb{N}_{(j-1)(\nu-2)}^{b+(j-1)(\nu-2)} \mid t - \nu + 1 \leq s\},$$

and let the functions

$$u_j(t,s) := \frac{1}{\Gamma(\nu)} \frac{(b-\sigma(s)+j(\nu-2))^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}}(t-(j-1)(\nu-2))^{\underline{\nu-1}},$$

and

$$x(t,s) := \frac{1}{\Gamma(\nu)}(t-\sigma(s))^{\underline{\nu-1}}.$$

Then if $\nu \in (1,2]$ *and* $y : \mathbb{N}_{n(\nu-2)}^{b+n(\nu-2)} \to \mathbb{R}$ *(or* $y : \mathbb{N}_{n(\nu-2)} \to \mathbb{R}$*), the solution for the fractional boundary value problem*

$$\begin{cases} (-1)^n \Delta_{\nu-2}^{\nu} \Delta_{2\nu-4}^{\nu} \cdots \Delta_{n(\nu-2)}^{\nu} y(t) = h(t), \qquad t \in \mathbb{N}_0^b, \ \ n \in \mathbb{N}, \\ y\left(n(\nu-2)\right) = 0 = y\left(b+n(\nu-2)\right), \\ \Delta_{(n-(i-1))(\nu-2)}^{\nu} \Delta_{(n-(i-2))(\nu-2)}^{\nu} \cdots \Delta_{(n-1)(\nu-2)}^{\nu} \Delta_{n(\nu-2)}^{\nu} y\left((n-i)(\nu-2)\right) = 0, \\ \Delta_{(n-(i-1))(\nu-2)}^{\nu} \Delta_{(n-(i-2))(\nu-2)}^{\nu} \cdots \Delta_{(n-1)(\nu-2)}^{\nu} \Delta_{n(\nu-2)}^{\nu} y\left(b+(n-i)(\nu-2)\right) = 0, \end{cases}$$

where $i = 1, 2, 3, \ldots, n-1$, *has solution*

$$y(t) = \sum_{\tau_n=(n-1)(\nu-2)}^{b+(n-1)(\nu-2)} \sum_{\tau_{n-1}=(n-2)(\nu-2)}^{b+(n-2)(\nu-2)} \cdots \sum_{\tau_1=0}^{b} G_n(t,\tau_n) G_{n-1}(\tau_n,\tau_{n-1}) \cdots G_1(\tau_2,\tau_1) h(\tau_1),$$

for

$$G_j(t,s) := \begin{cases} u_j(t,s) - x(t,s), & (t,s) \in S_j, \\ u_j(t,s), & (t,s) \in T_j. \end{cases}$$

Proof. Consider $n = 1$. Then the problem is reduced to the following:

$$\begin{cases} -\Delta_{\nu-2}^{\nu} y(t) = h(t), \\ y(\nu-2) = 0 = y(b+\nu-2), \end{cases}$$

noting the domain of y is $\mathbb{N}_{\nu-2}$. From [32], the general solution of the problem is then

$$y(t) = \alpha_0(t-a)^{\underline{\nu-2}} + \alpha_1(t-a)^{\underline{\nu-1}} - \Delta_a^{-\nu}h(t),$$

where $t \in \mathbb{N}_{a+\nu-2}$. Since the domain of y here is $\mathbb{N}_{\nu-2}$, we have that $a = 0$.

Note now that

$$\Delta_0^{-\nu}h(\nu-2) = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{(\nu-2)-\nu} (\nu - 2 - \sigma(s))^{\underline{\nu-1}} h(s) = 0$$

by our convention on sums. Now, using the first boundary condition, we have

$$\begin{aligned}
0 &= y(\nu-2) \\
&= \alpha_0(\nu-2)^{\underline{\nu-2}} + \alpha_1(\nu-2)^{\underline{\nu-1}} - \Delta_0^{-\nu}h(\nu-2) \\
&= \alpha_0\frac{\Gamma(\nu-1)}{\Gamma(1)} + \alpha_1\frac{\Gamma(\nu-1)}{\Gamma(0)} - \Delta_0^{-\nu}h(\nu-2) \\
&= \alpha_0\Gamma(\nu-1) + 0 - 0 \\
&\implies \alpha_0 = 0.
\end{aligned}$$

Using the second boundary condition, we have

$$\begin{aligned}
0 &= y(b+\nu-2) \\
&= \alpha_1(b+\nu-2)^{\underline{\nu-1}} - \Delta_0^{-\nu}h(b+\nu-2) \\
&\implies \alpha_1 = \frac{\Delta_0^{-\nu}h(b+\nu-2)}{(b+\nu-2)^{\underline{\nu-1}}}.
\end{aligned}$$

Now, since the maximum t-value on our considered domain is $b+\nu-2$, this implies that $t-\nu+1 \leq b+\nu-2-\nu+1 = b-1$ for all t that we are considering. Now when

$s = b - 1$, note that

$$\begin{aligned}(b-\sigma(s)+(\nu-2))^{\underline{\nu-1}} &= (b-b+\nu-2)^{\underline{\nu-1}}\\ &= (\nu-2)^{\underline{\nu-1}}\\ &= \frac{\Gamma(\nu-1)}{\Gamma(0)}\\ &= 0,\end{aligned}$$

and, similarly, when $s = b$, we have $(b-\sigma(s)+(\nu-2))^{\underline{\nu-1}} = (\nu-3)^{\underline{\nu-1}} = \frac{\Gamma(\nu-2)}{\Gamma(-1)} = 0$. So when $s = b-1, b$, we have $s \geq t-\nu+1$, which implies $G_1(t,s) = 0$ for these s-values for any t-value in consideration here.

So now we have

$$\begin{aligned}y(t) &= \frac{\Delta_0^{-\nu}h(b+\nu-2)}{(b+\nu-2)^{\underline{\nu-1}}}t^{\underline{\nu-1}} - \Delta_0^{-\nu}h(t)\\ &= \frac{t^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}}\frac{1}{\Gamma(\nu)}\sum_{s=0}^{(b+\nu-2)-\nu}(b+\nu-2-\sigma(s))^{\underline{\nu-1}}h(s)\\ &\qquad - \frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}(t-\sigma(s))^{\underline{\nu-1}}h(s)\\ &= \frac{1}{\Gamma(\nu)}\sum_{s=0}^{t-\nu}\left[\frac{(b+\nu-2-\sigma(s))^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}}t^{\underline{\nu-1}} - (t-\sigma(s))^{\underline{\nu-1}}\right]h(s)\\ &\qquad + \frac{1}{\Gamma(\nu)}\sum_{s=t-\nu+1}^{b-2}\left[\frac{(b+\nu-2-\sigma(s))^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}}t^{\underline{\nu-1}}\right]h(s)\\ &= \sum_{s=0}^{b-2}G_1(t,s)h(s) = \sum_{s=0}^{b}G_1(t,s)h(s).\end{aligned}$$

So the theorem holds for $n = 1$.

To add some more insight into the specifics of this theorem, consider the case

$n = 2$, which results in the following fractional boundary value problem:

$$\begin{cases} \Delta^{\nu}_{\nu-2}\Delta^{\nu}_{2\nu-4}y(t) = h(t), \\ y(2\nu-4) = 0 = y(b+2\nu-4), \\ \Delta^{\nu}_{2\nu-4}y(\nu-2) = 0 = \Delta^{\nu}_{2\nu-4}y(b+\nu-2), \end{cases}$$

noting the domain of y is $\mathbb{N}_{2\nu-4}$. Let $w(t) = -\Delta^{\nu}_{2\nu-4}y(t)$. Then we may consider the problem

$$\begin{cases} -\Delta^{\nu}_{\nu-2}w(t) = h(t), \\ w(\nu-2) = 0 = w(b+\nu-2). \end{cases}$$

From the case $n = 1$, we have that

$$\begin{aligned} w(t) &= -\Delta^{\nu}_{2\nu-4}y(t) \\ &= \sum_{s=0}^{b} G_1(t,s)h(s). \end{aligned}$$

Noting that we still have boundary conditions $y(2\nu-4) = 0$ and $y(b+2\nu-4) = 0$, we can solve for y in terms of w. As in the $n = 1$ case, our general solution is given by

$$\begin{aligned} y(t) &= \alpha_0(t-a)^{\underline{\nu-2}} + \alpha_1(t-a)^{\underline{\nu-1}} - \Delta_a^{-\nu}w(t) \\ &= \alpha_0(t-\nu+2)^{\underline{\nu-2}} + \alpha_1(t-\nu+2)^{\underline{\nu-1}} - \Delta^{-\nu}_{\nu-2}w(t), \end{aligned}$$

where $t \in \mathbb{N}_{2\nu-4}$.

Note now that

$$\begin{aligned}\Delta_{\nu-2}^{-\nu} w(2\nu-4) &= \frac{1}{\Gamma(\nu)} \sum_{s=\nu-2}^{2\nu-4-\nu} (2\nu-4-\sigma(s))^{\underline{\nu-1}} w(s) \\ &= \frac{1}{\Gamma(\nu)} \sum_{s=\nu-2}^{\nu-4} (2\nu-4-\sigma(s))^{\underline{\nu-1}} w(s) = 0,\end{aligned}$$

by the convention on sums. Using the first boundary condition, we have

$$\begin{aligned}0 = y(2\nu-4) &= \alpha_0(\nu-2)^{\underline{\nu-2}} + \alpha_1(\nu-2)^{\underline{\nu-1}} - \Delta_{\nu-2}^{-\nu} w(2\nu-4) \\ &= \alpha_0(\nu-2)^{\underline{\nu-2}} + 0 - 0 \\ &\Longrightarrow \alpha_0 = 0.\end{aligned}$$

Using the second boundary condition, we have

$$\begin{aligned}0 = y(b+2\nu-4) &= \alpha_1(b+\nu-2)^{\underline{\nu-1}} - \Delta_{\nu-2}^{-\nu} w(b+2\nu-4) \\ &\Longrightarrow \alpha_1 = \frac{\Delta_{\nu-2}^{-\nu} w(b+2\nu-4)}{(b+\nu-2)^{\underline{\nu-1}}}.\end{aligned}$$

Since the maximum t-value on our considered domain is $b+2\nu-4$, this implies that $t-\nu+1 \leq b+2\nu-4-\nu+1 = b+\nu-3$ for all t that we are considering. Now when $s = b+\nu-3$, note that

$$\begin{aligned}(b-\sigma(s)+2(\nu-2))^{\underline{\nu-1}} &= (b-b-\nu+2+2\nu-4)^{\underline{\nu-1}} \\ &= (\nu-2)^{\underline{\nu-1}} = 0,\end{aligned}$$

and, similarly, when $s = b+\nu-2$, we have $(b-\sigma(s)+2(\nu-2))^{\underline{\nu-1}} = (\nu-3)^{\underline{\nu-1}} = 0$. Thus, when $s = b+\nu-3, b+\nu-2$, we have $s \geq t-\nu+1$, which implies $G_2(t,s) = 0$ for these s-values for any t-value in consideration here.

Therefore, we have

$$
\begin{aligned}
y(t) &= \frac{\Delta_{\nu-2}^{-\nu} w(b+2\nu-4)}{(b+\nu-2)^{\underline{\nu-1}}}(t-\nu+2)^{\underline{\nu-1}} - \Delta_{\nu-2}^{-\nu} w(t) \\
&= \frac{(t-\nu+2)^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}} \frac{1}{\Gamma(\nu)} \sum_{s=\nu-2}^{b+\nu-4} (b+2\nu-4-\sigma(s))^{\underline{\nu-1}} w(s) \\
&\qquad\qquad - \frac{1}{\Gamma(\nu)} \sum_{s=\nu-2}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} w(s) \\
&= \frac{1}{\Gamma(\nu)} \sum_{s=\nu-2}^{t-\nu} \left[\frac{(b+2\nu-4-\sigma(s))^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}}(t-\nu+2)^{\underline{\nu-1}} - (t-\sigma(s))^{\underline{\nu-1}} \right] w(s) \\
&\qquad\qquad + \frac{1}{\Gamma(\nu)} \sum_{s=t-\nu+1}^{b+\nu-4} \left[\frac{(b+2\nu-4-\sigma(s))^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}}(t-\nu+2)^{\underline{\nu-1}} \right] w(s) \\
&= \sum_{s=\nu-2}^{b+\nu-4} G_2(t,s) w(s) = \sum_{s=\nu-2}^{b+\nu-2} G_2(t,s) w(s) \\
&= \sum_{s=\nu-2}^{b+\nu-2} G_2(t,s) \sum_{\tau=0}^{b} G_1(s,\tau) h(\tau) \\
&= \sum_{s=\nu-2}^{b+\nu-2} \sum_{\tau=0}^{b} G_2(t,s) G_1(s,\tau) h(\tau).
\end{aligned}
$$

So the theorem holds for $n=2$, and we have additional insight as to how the general case comes about.

We know finish proving the result by induction. Suppose the result holds for some $n \in \mathbb{N}$. We then consider the problem for $n+1$:

$$
\begin{cases}
(-1)^{n+1} \Delta_{\nu-2}^{\nu} \Delta_{2\nu-4}^{\nu} \cdots \Delta_{(n+1)(\nu-2)}^{\nu} y(t) = h(t), \qquad t \in \mathbb{N}_0^b, \ \ n \in \mathbb{N}, \\
\qquad y\left((n+1)(\nu-2)\right) = 0 = y\left(b+(n+1)(\nu-2)\right), \\
\quad \Delta_{(n+1-(i-1))(\nu-2)}^{\nu} \Delta_{(n+1-(i-2))(\nu-2)}^{\nu} \cdots \Delta_{(n+1)(\nu-2)}^{\nu} y\left((n+1-i)(\nu-2)\right) = 0, \\
\quad \Delta_{(n+1-(i-1))(\nu-2)}^{\nu} \Delta_{(n+1-(i-2))(\nu-2)}^{\nu} \cdots \Delta_{(n+1)(\nu-2)}^{\nu} y\left(b+(n+1-i)(\nu-2)\right) = 0,
\end{cases}
$$

for $i = 1, 2, 3, \ldots, n$.

Let $w(t) = -\Delta^{\nu}_{(n+1)(\nu-2)}y(t)$. Then we may consider the problem

$$\begin{cases} (-1)^n \Delta^{\nu}_{\nu-2}\Delta^{\nu}_{2\nu-4}\cdots\Delta^{\nu}_{n(\nu-2)}w(t) = h(t), \qquad t \in \mathbb{N}_0^b, \ \ n \in \mathbb{N}, \\ \qquad\qquad w(n(\nu-2) = 0 = w(b+n(\nu-2)) \\ \Delta^{\nu}_{(n+1-(i-1))(\nu-2)}\Delta^{\nu}_{(n+1-(i-2))(\nu-2)}\cdots\Delta^{\nu}_{n(\nu-2)}w\left((n+1-i)(\nu-2)\right) = 0, \\ \Delta^{\nu}_{(n+1-(i-1))(\nu-2)}\Delta^{\nu}_{(n+1-(i-2))(\nu-2)}\cdots\Delta^{\nu}_{n(\nu-2)}w\left(b+(n+1-i)(\nu-2)\right) = 0, \end{cases}$$

for $i = 2, 3, 4, \ldots, n$, which is equivalent to

$$\begin{cases} (-1)^n \Delta^{\nu}_{\nu-2}\Delta^{\nu}_{2\nu-4}\cdots\Delta^{\nu}_{n(\nu-2)}w(t) = h(t), \qquad t \in \mathbb{N}_0^b, \ \ n \in \mathbb{N}, \\ \qquad\qquad w(n(\nu-2) = 0 = w(b+n(\nu-2)) \\ \quad \Delta^{\nu}_{(n-(i-1))(\nu-2)}\Delta^{\nu}_{(n-(i-2))(\nu-2)}\cdots\Delta^{\nu}_{n(\nu-2)}w\left((n-i)(\nu-2)\right) = 0, \\ \quad \Delta^{\nu}_{(n-(i-1))(\nu-2)}\Delta^{\nu}_{(n-(i-2))(\nu-2)}\cdots\Delta^{\nu}_{n(\nu-2)}w\left(b+(n-i)(\nu-2)\right) = 0, \end{cases}$$

for $i = 1, 2, 3, \ldots, n-1$.

By assumption, we then have that

$$\begin{aligned} w(t) &= -\Delta^{\nu}_{(n+1)(\nu-2)}y(t) \\ &= \sum_{\tau_n=(n-1)(\nu-2)}^{b+(n-1)(\nu-2)} \sum_{\tau_{n-1}=(n-2)(\nu-2)}^{b+(n-2)(\nu-2)} \cdots \sum_{\tau_1=0}^{b} G_n(t,\tau_n)G_{n-1}(\tau_n,\tau_{n-1})\cdots G_1(\tau_2,\tau_1)h(\tau_1). \end{aligned}$$

Noting still that we have $y\left((n+1)(\nu-2)\right) = 0$ and $y\left(b+(n+1)(\nu-2)\right) = 0$ as boundary conditions, all that remains to be shown is that
$y(t) = \sum_{s=n(\nu-2)}^{b+n(\nu-2)} G_{n+1}(t,s)w(s)$.

As earlier, and from the fact that $w(t) = \Delta^{\nu}_{(n+1)(\nu-2)} y(t)$, we have

$$
\begin{aligned}
y(t) &= \alpha_0 (t-a)^{\underline{\nu-2}} + \alpha_1 (t-a)^{\underline{\nu-1}} - \Delta_a^{-\nu} w(t) \\
&= \alpha_0 (t - n(\nu-2))^{\underline{\nu-2}} + \alpha_1 (t - n(\nu-2))^{\underline{\nu-1}} - \Delta^{-\nu}_{n(\nu-2)} w(t),
\end{aligned}
$$

where $t \in \mathbb{N}_{(n+1)(\nu-2)}$.

Note now that

$$
\begin{aligned}
\Delta^{-\nu}_{n(\nu-2)} w((n+1)(\nu-2)) &= \frac{1}{\Gamma(\nu)} \sum_{s=n(\nu-2)}^{(n+1)(\nu-2)-\nu} ((n+1)(\nu-2) - \sigma(s))^{\underline{\nu-1}} w(s) \\
&= \frac{1}{\Gamma(\nu)} \sum_{s=n\nu-2n}^{n\nu-2n-2} ((n+1)(\nu-2) - \sigma(s))^{\underline{\nu-1}} w(s) \\
&= 0,
\end{aligned}
$$

again using the convention on sums. Using the first boundary condition, we have

$$
\begin{aligned}
0 &= y((n+1)(\nu-2)) \\
&= \alpha_0 ((n+1)(\nu-2) - n(\nu-2))^{\underline{\nu-2}} + \alpha_1 ((n+1)(\nu-2) - n(\nu-2))^{\underline{\nu-1}} \\
&\qquad - \Delta^{-\nu}_{n(\nu-2)} w((n+1)(\nu-2)) \\
&= \alpha_0 (\nu-2)^{\underline{\nu-2}} + \alpha_1 (\nu-2)^{\underline{\nu-1}} = \alpha_0 (\nu-2)^{\underline{\nu-2}} \\
&\implies \alpha_0 = 0.
\end{aligned}
$$

Using the second boundary condition, we have

$$\begin{aligned}
0 &= y((n+1)(\nu-2)+b) \\
&= \alpha_1((n+1)(\nu-2)+b-n(\nu-2))^{\underline{\nu-1}} - \Delta^{-\nu}_{n(\nu-2)} w((n+1)(\nu-2)+b) \\
&= \alpha_1(b+\nu-2)^{\underline{\nu-1}} - \Delta^{-\nu}_{n(\nu-2)} w((n+1)(\nu-2)+b) \\
\implies \alpha_1 &= \frac{\Delta^{-\nu}_{n(\nu-2)} w((n+1)(\nu-2)+b)}{(b+\nu-2)^{\underline{\nu-1}}}.
\end{aligned}$$

Now, since $b+(n+1)(\nu-2)$ is the maximum t-value considered on our domain, this implies that $t-\nu+1 \leq b+(n+1)(\nu-2)-\nu+1 = b+n(\nu-2)-1$ for all t that we are considering. Now when $s = b+n(\nu-2)-1$, note that

$$\begin{aligned}
(b-\sigma(s)+(n+1)(\nu-2))^{\underline{\nu-1}} &= (b-b-n(\nu-2)+(n+1)(\nu-2))^{\underline{\nu-1}} \\
&= (\nu-2)^{\underline{\nu-1}} = 0,
\end{aligned}$$

and, similarly, when $s = b+n(\nu-2)$, we have

$$(b-\sigma(s)+(n+1)(\nu-2))^{\underline{\nu-1}} = (\nu-3)^{\underline{\nu-1}} = 0.$$

So when $s = b+n(\nu-2)-1, b+n(\nu-2)$ we have $s \geq t-\nu+1$, which implies $G_{n+1}(t,s) = 0$ for these s-values for any t-value in consideration here.

So now we have

$$
\begin{aligned}
y(t) &= \frac{\Delta^{-\nu}_{n(\nu-2)} w((n+1)(\nu-2)+b)}{(b+\nu-2)^{\underline{\nu-1}}}(t-n(\nu-2))^{\underline{\nu-1}} - \Delta^{-\nu}_{n(\nu-2)} w(t) \\
&= \frac{(t-n(\nu-2))^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}} \frac{1}{\Gamma(\nu)} \sum_{s=n(\nu-2)}^{(n+1)(\nu-2)+b-\nu} ((n+1)(\nu-2)+b-\sigma(s))^{\underline{\nu-1}} w(s) \\
&\qquad - \frac{1}{\Gamma(\nu)} \sum_{s=n(\nu-2)}^{t-\nu} (t-\sigma(s))^{\underline{\nu-1}} w(s) \\
&= \frac{1}{\Gamma(\nu)} \sum_{s=n(\nu-2)}^{t-\nu} \left[u_{n+1}(t,s) - x(t,s)\right] w(s) + \frac{1}{\Gamma(\nu)} \sum_{s=t-\nu+1}^{(n+1)(\nu-2)+b-\nu} u_{n+1}(t,s) w(s) \\
&= \sum_{s=n(\nu-2)}^{(n+1)(\nu-2)+b-\nu} G_{n+1}(t,s) w(s) = \sum_{s=n(\nu-2)}^{b+n(\nu-2)-2} G_{n+1}(t,s) w(s) \\
&= \sum_{s=n(\nu-2)}^{b+n(\nu-2)} G_{n+1}(t,s) w(s).
\end{aligned}
$$

□

Remark 2.5. Note that $G_j(j(\nu-2), s) = 0$ and $G_j(b + j(\nu-2), s) = 0$ for all s such that (t, s) is in the domain of G_j:

$$
\begin{aligned}
G_j(j(\nu-2), s) &= \frac{(b-\sigma(s)+j(\nu-2))^{\underline{\nu-1}}}{\Gamma(\nu)(b+\nu-2)^{\underline{\nu-1}}}(j(\nu-2)-(j-1)(\nu-2))^{\underline{\nu-1}} \\
&= \frac{(b-\sigma(s)+j(\nu-2))^{\underline{\nu-1}}}{\Gamma(\nu)(b+\nu-2)^{\underline{\nu-1}}}(\nu-2)^{\underline{\nu-1}} \\
&= \frac{(b-\sigma(s)+j(\nu-2))^{\underline{\nu-1}}}{\Gamma(\nu)(b+\nu-2)^{\underline{\nu-1}}} \frac{\Gamma(\nu-1)}{\Gamma(0)} = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
G_j(b+&j(\nu-2),s)\\
&=\frac{1}{\Gamma(\nu)}\left[\frac{(b-\sigma(s)+j(\nu-2))^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}}(b+j(\nu-2)-(j-1)(\nu-2))^{\underline{\nu-1}}\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\left.-(b+j(\nu-2)-\sigma(s))^{\underline{\nu-1}}\right]\\
&=\frac{1}{\Gamma(\nu)}\left[\frac{(b-\sigma(s)+j(\nu-2))^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}}(b+\nu-2)^{\underline{\nu-1}}-(b+j(\nu-2)-\sigma(s))^{\underline{\nu-1}}\right]\\
&=\frac{1}{\Gamma(\nu)}\left[(b-\sigma(s)+j(\nu-2))^{\underline{\nu-1}}-(b+j(\nu-2)-\sigma(s))^{\underline{\nu-1}}\right]=0.
\end{aligned}
$$

2.3 Properties of the Green's Function

In this section, we highlight some of the important properties of the Green's function which help us prove further results.

Theorem 2.6. *For $G_j(t,s)$ defined above, we have $G_j(t,s)\geq 0$ on its domain for all $j\in\mathbb{N}$.*

Proof. First, to add some insight into our method, let us look at the case when $j=1$.

Note that when $s=b-1,b$, we have $G_1(t,s)=0$ from the previous proof. Thus if $0\leq t-\nu+1\leq s\leq b-2$, then

$$
(b-\sigma(s)+\nu-2)^{\underline{\nu-1}}=(b-(s+1)+\nu-2)^{\underline{\nu-1}}=\frac{\Gamma(b-s+\nu-2)}{\Gamma(b-s-1)}\geq 0,
$$

since both $b-s+\nu-2\geq\nu>1$ and $b-s-1\geq 1$. Also, since $t-\nu+1\geq 0$, then $t^{\underline{\nu-1}}=\frac{\Gamma(t+1)}{\Gamma(t-\nu+2)}\geq 0$. Therefore, when $0\leq t-\nu+1\leq s\leq b$, we have $G_1(t,s)\geq 0$.

We only need now to consider the case when $0\leq s\leq t-\nu\leq b-2$. We wish to show

$$\begin{aligned}&\frac{(b+\nu-2-\sigma(s))^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}}t^{\underline{\nu-1}}-(t-\sigma(s))^{\underline{\nu-1}}\geq 0\\ \iff &\frac{(b+\nu-2-\sigma(s))^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}}\frac{t^{\underline{\nu-1}}}{(t-\sigma(s))^{\underline{\nu-1}}}\geq 1.\end{aligned}$$

Thus, we consider, keeping in mind that $t \leq b+\nu-2$ and $s \leq t-\nu$,

$$\begin{aligned}&\frac{(b+\nu-2-\sigma(s))^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}}\frac{t^{\underline{\nu-1}}}{(t-\sigma(s))^{\underline{\nu-1}}}\\
&=\frac{(b+\nu-s-3)^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}}\frac{t^{\underline{\nu-1}}}{(t-s-1)^{\underline{\nu-1}}}\\
&=\frac{\Gamma(b+\nu-s-2)}{\Gamma(b-s-1)}\frac{\Gamma(b)}{\Gamma(b+\nu-1)}\frac{\Gamma(t+1)}{\Gamma(t-\nu+2)}\frac{\Gamma(t-\nu-s+1)}{\Gamma(t-s)}\\
&=\frac{\Gamma(b)}{\Gamma(b-s-1)}\frac{\Gamma(b+\nu-s-2)}{\Gamma(b+\nu-1)}\frac{\Gamma(t+1)}{\Gamma(t-s)}\frac{\Gamma(t-\nu-s+1)}{\Gamma(t-\nu+2)}\\
&=\frac{(b-1)(b-2)\cdots(b-s-1)\Gamma(b-s-1)}{\Gamma(b-s-1)}\\
&\qquad\cdot\frac{\Gamma(b+\nu-s-2)}{(b+\nu-2)(b+\nu-3)\cdots(b+\nu-s-2)\Gamma(b+\nu-s-2)}\\
&\qquad\cdot\frac{(t)(t-1)\cdots(t-s)\Gamma(t-s)}{\Gamma(t-s)}\\
&\qquad\cdot\frac{\Gamma(t-\nu-s+1)}{(t-\nu+1)(t-\nu)\cdots(t-\nu-s+1)\Gamma(t-\nu-s+1)}\\
&=\frac{(b-1)(b-2)\cdots(b-s-1)}{(b+\nu-2)(b+\nu-3)\cdots(b+\nu-s-2)}\frac{(t)(t-1)\cdots(t-s)}{(t-\nu+1)(t-\nu)\cdots(t-\nu-s+1)}\\
&=: A.\end{aligned}$$

We wish to show $A \geq 1$. To this end, consider y as some function of ν defined as $y_n(\nu) := \frac{(b+\nu-2-n)(t-\nu+1-n)}{(b-1-n)(t-n)}$, where $n \in \mathbb{N}_0^s$. Then $\lim_{\nu\to 1^+} y_n(\nu) = \frac{(b-1-n)(t-n)}{(b-1-n)(t-n)} = 1$, and

$$y_n'(\nu) = \frac{-b-2\nu+2+n+t+1-n}{(b-1-n)(t-n)} = \frac{t-(b+\nu-2)-\nu+1}{(b-1-n)(t-n)} \leq 0,$$

since $n \in \mathbb{N}_0^s$ and $s \leq t-\nu \leq b-2$ (and noting $\nu \in (1,2]$). Therefore, y_n is decreasing for $\nu \in (1,2]$, which implies $\frac{1}{y_n(\nu)} = \frac{(b-1-n)(t-n)}{(b+\nu-2-n)(t-\nu+1-n)}$ is increasing for $\nu \in (1,2]$ and $\lim_{\nu \to 1^+} \frac{(b-1-n)(t-n)}{(b+\nu-2-n)(t-\nu+1-n)} = 1$. Then $A = \prod_{n=0}^{s} \frac{1}{y_n(\nu)} \geq 1$ since every factor in the finite product is greater than or equal to 1. So

$$\frac{(b+\nu-2-\sigma(s))^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}} t^{\underline{\nu-1}} - (t-\sigma(s))^{\underline{\nu-1}} \geq 0,$$

and, therefore, $G_1(t,s) \geq 0$ on its domain.

Now let us look at the case for arbitrary $j \in \mathbb{N}$. From the previous proof, when $s = b+(j-1)(\nu-2)-1, b+(j-1)(\nu-2)$, we have $G_j(t,s) = 0$, and when $(j-1)(\nu-2) \leq t-\nu+1 \leq s \leq b+(j-1)(\nu-2)-2$, we have

$$(b-\sigma(s)+j(\nu-2))^{\underline{\nu-1}} = \frac{\Gamma(b-s+j(\nu-2))}{\Gamma(b-s+j(\nu-2)-\nu+1)} \geq 0,$$

since both $b-s+j(\nu-2) \geq b-(b-2+(j-1)(\nu-2))+j(\nu-2) = \nu > 1$ and $b-s+j(\nu-2)-\nu+1 \geq b-(b-2+(j-1)(\nu-2))+j(\nu-2)-\nu+1 = 1$. So when $(j-1)(\nu-2) \leq t-\nu+1 \leq s \leq b+(j-1)(\nu-2)$, we have $G_j(t,s) \geq 0$.

We only need now to consider the case when $(t,s) \in S_j$, or, in other words, when $(j-1)(\nu-2) \leq s \leq t-\nu \leq b-2+(j-1)(\nu-2)$. We wish to show

$$\begin{aligned} &\frac{(b-\sigma(s)+j(\nu-2))^{\underline{\nu-1}}}{(b-\nu-2)^{\underline{\nu-1}}}(t-(j-1)(\nu-2))^{\underline{\nu-1}} - (t-\sigma(s))^{\underline{\nu-1}} \geq 0 \\ \iff &\frac{(b-\sigma(s)+j(\nu-2))^{\underline{\nu-1}}}{(b-\nu-2)^{\underline{\nu-1}}} \frac{(t-(j-1)(\nu-2))^{\underline{\nu-1}}}{(t-\sigma(s))^{\underline{\nu-1}}} \geq 1. \end{aligned}$$

Therefore, consider

$$
\begin{aligned}
&\frac{(b-\sigma(s)+j(\nu-2))^{\underline{\nu-1}}}{(b-\nu-2)^{\underline{\nu-1}}}\frac{(t-(j-1)(\nu-2))^{\underline{\nu-1}}}{(t-\sigma(s))^{\underline{\nu-1}}}\\
&=\frac{(b-s-1+j(\nu-2))^{\underline{\nu-1}}}{(b-\nu-2)^{\underline{\nu-1}}}\frac{(t-(j-1)(\nu-2))^{\underline{\nu-1}}}{(t-s-1)^{\underline{\nu-1}}}\\
&=\frac{\Gamma(b-s+j(\nu-2))}{\Gamma(b-s+j(\nu-2)-\nu+1)}\frac{\Gamma(b)}{\Gamma(b+\nu-1)}\\
&\qquad\qquad\qquad\cdot\frac{\Gamma(t-(j-1)(\nu-2)+1)}{\Gamma(t-(j-1)(\nu-2)-\nu+2)}\frac{\Gamma(t-s-\nu+1)}{\Gamma(t-s)}\\
&=\frac{\Gamma(b)}{\Gamma(b-s+j(\nu-2)-\nu+1)}\frac{\Gamma(b-s+j(\nu-2))}{\Gamma(b+\nu-1)}\\
&\qquad\qquad\qquad\cdot\frac{\Gamma(t-(j-1)(\nu-2)+1)}{\Gamma(t-s)}\frac{\Gamma(t-s-\nu+1)}{\Gamma(t-(j-1)(\nu-2)-\nu+2)}\\
&=\frac{\Gamma(b)}{\Gamma(b-s+(j-1)(\nu-2)-1)}\frac{\Gamma(b-s+(j-1)(\nu-2)+\nu-2)}{\Gamma(b+\nu-1)}\\
&\qquad\qquad\qquad\cdot\frac{\Gamma(t-(j-1)(\nu-2)+1)}{\Gamma(t-s)}\frac{\Gamma(t-s-\nu+1)}{\Gamma(t-(j-1)(\nu-2)-\nu+2)}\\
&=\frac{\Gamma(b)}{\Gamma(b-k-1)}\frac{\Gamma(b+\nu-2-k)}{\Gamma(b+\nu-1)}\frac{\Gamma(t-s+k+1)}{\Gamma(t-s)}\frac{\Gamma(t-s-\nu+1)}{\Gamma(t-s-\nu+k+2)},
\end{aligned}
$$

where $k=s-(j-1)(\nu-2)$. Note $k\in\mathbb{N}_0$ for all s here.

Keeping in mind that $t\leq b-2+(j-1)(\nu-2)+\nu=b+j(\nu-2)$ and $s\leq t-\nu$, we have

$$
\begin{aligned}
&\frac{\Gamma(b)}{\Gamma(b-k-1)}\frac{\Gamma(b+\nu-2-k)}{\Gamma(b+\nu-1)}\frac{\Gamma(t-s+k+1)}{\Gamma(t-s)}\frac{\Gamma(t-s-\nu+1)}{\Gamma(t-s-\nu+k+2)}\\
&=\frac{(b-1)(b-2)\cdots(b-k-1)\Gamma(b-k-1)}{\Gamma(b-k-1)}\\
&\qquad\cdot\frac{\Gamma(b+\nu-2-k)}{(b+\nu-2)(b+\nu-3)\cdots(b+\nu-2-k)\Gamma(b+\nu-2-k)}\\
&\qquad\cdot\frac{(t-s+k)(t-s+k-1)\cdots(t-s)\Gamma(t-s)}{\Gamma(t-s)}\\
&\qquad\cdot\frac{\Gamma(t-s-\nu+1)}{(t-s-\nu+k+1)(t-s-\nu+k)\cdots(t-s-\nu+1)\Gamma(t-s-\nu+1)}\\
&=\frac{(b-1)(b-2)\cdots(b-k-1)}{(b+\nu-2)(b+\nu-3)\cdots(b+\nu-2-k)}\\
&\qquad\cdot\frac{(t-s+k)(t-s+k-1)\cdots(t-s)}{(t-s-\nu+k+1)(t-s-\nu+k)\cdots(t-s-\nu+1)}\\
&=: A.
\end{aligned}
$$

Again, we wish to show $A \geq 1$. As before, let us define $y_n(\nu) := \frac{(b+\nu-2-n)(t-s-\nu+k+1-n)}{(b-1-n)(t-s+k-n)}$ for $n \in \mathbb{N}_0^k$. Then $\lim_{\nu\to 1^+} y(\nu) = \frac{(b-1-n)(t-s+k-n)}{(b-1-n)(t-s+k-n)} = 1$, and

$$
y_n'(\nu) = \frac{-b-2\nu+2+n+t-s+k+1-n}{(b-1-n)(t-s+k-n)} = \frac{t-(b+\nu-2+s-k)-\nu+1}{(b-1-n)(t-s+k-n)} \leq 0,
$$

since $n \in \mathbb{N}_0^k$ and $s \leq t-\nu \leq b-2+(j-1)(\nu-2) = b-2+s-k$ (and still noting $\nu \in (1,2]$). We can also note that both factors of the denominator in the above inequality are positive since

$$\begin{aligned} b-1-n &\geq b-1-k \\ &= b-1-s+(j-1)(\nu-2) \\ &\geq b-1-(b-2+(j-1)(\nu-2))+(j-1)(\nu-2)=1, \end{aligned}$$

and $t > s$ and $k \geq n$. Thus, for $\nu \in (1,2]$, y_n is decreasing, which implies both that $\frac{1}{y_n(\nu)} = \frac{(b-1-n)(t-s+k-n)}{(b+\nu-2-n)(t-s-\nu+k+1-n)}$ is increasing and $\lim_{\nu\to 1^+} \frac{(b-1-n)(t-s+k-n)}{(b+\nu-2-n)(t-s-\nu+k+1-n)} = 1$. Then $A = \prod_{n=0}^{k} \frac{1}{y_n(\nu)} \geq 1$ since every factor in the finite product is greater than or equal to 1. Also, we may note that all factors are positive as $t-s \geq t-s-\nu+1 \geq 0$ (since $s \leq t-\nu$) and $b+\nu-2-k \geq b-k-1 = b-s+(j-1)(\nu-2)-1 \geq 0$ (since $s \leq b-2+(j-1)(\nu-2)$). So

$$\frac{(b-\sigma(s)+j(\nu-2))^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}}(t-(j-1)(\nu-2))^{\underline{\nu-1}} - (t-\sigma(s))^{\underline{\nu-1}} \geq 0,$$

and, therefore, $G_j(t,s) \geq 0$ on its domain. Since $j \in \mathbb{N}$ was arbitrary, the result holds for all $j \in \mathbb{N}$. □

We can note that in previous results, we have shown that $G_j(t,s) = 0$ when we have $t = j(\nu-2), b+j(\nu-2)$ or when $s = b+(j-1)(\nu-2)-1, b+j(\nu-2)$. We now show that G_j is positive everywhere else on its domain, and we will also find the maximum of G_j on its domain.

Theorem 2.7. *For each $s \in \mathbb{N}_{(j-1)(\nu-2)}^{b+(j-1)(\nu-2)-2}$ we have $G_j(t,s)$ is strictly increasing for $t \in \mathbb{N}_{j(\nu-2)}^{s+\nu-1}$ and strictly decreasing for $t \in \mathbb{N}_{s+\nu}^{b+j(\nu-2)-1}$, and*

$$\max_{t \in \mathbb{N}_{j(\nu-2)}^{b+j(\nu-2)}} G_j(t,s) = G_j(s+\nu-1,s).$$

Proof. We will assume $b \geq 2$ to keep the domains from being trivial. Let $t-\nu+1 \leq s$. We will show $\Delta_t G_j(t,s) > 0$. Now, $\Gamma(b-s+j(\nu-2)) > 0$ since $\Gamma(x) > 0$ on $(0,\infty)$ and

$$b-s+j(\nu-2) \geq b-(b+(j-1)(\nu-2)-2)+j(\nu-2) = \nu \in (1,2].$$

Also, $b-s-1+(j-1)(\nu-2) \geq b-(b+(j-1)(\nu-2)-2)-1+(j-1)(\nu-2) = 1$, so $\Gamma(b-s-1+(j-1)(\nu-2)) > 0$.

Let $C := \frac{(b-\sigma(s)+j(\nu-2))^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}}$. Then note

$$\begin{aligned} C &= \frac{\Gamma(b-s-1+j(\nu-2)+1)}{\Gamma(b-s-1+j(\nu-2)+1-\nu+1)} \frac{\Gamma(b+\nu-1-\nu+1)}{\Gamma(b+\nu-1)} \\ &= \frac{\Gamma(b-s+j(\nu-2))}{\Gamma(b-s-1+(j-1)(\nu-2))} \frac{\Gamma(b)}{\Gamma(b+\nu-1)} \\ &> 0. \end{aligned}$$

We may now consider

$$\begin{aligned} \Gamma(\nu)\Delta_t G_j(t,s) &= \Delta_t \left[\frac{(b-\sigma(s)+j(\nu-2))^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}} (t-(j-1)(\nu-2))^{\underline{\nu-1}} \right] \\ &= \Delta_t \left[C(t-(j-1)(\nu-2))^{\underline{\nu-1}} \right] \\ &= C(\nu-1)(t-(j-1)(\nu-2))^{\underline{\nu-2}} > 0 \end{aligned}$$

since $C, \nu-1 > 0$ and

$$\begin{aligned} (t-(j-1)(\nu-2))^{\underline{\nu-2}} &= \frac{\Gamma(t-(j-1)(\nu-2)+1)}{\Gamma(t-(j-1)(\nu-2)+1-(\nu-2))} \\ &= \frac{\Gamma(t-(j-1)(\nu-2)+1)}{\Gamma(t-j(\nu-2)+1)} \\ &> 0, \end{aligned}$$

keeping in mind that

$$\begin{aligned}
&t \geq j(\nu-2) \\
&\implies t-(\nu-2) \geq (j-1)(\nu-2) \\
&\implies t-(j-1)(\nu-2) \geq \nu-2 \\
&\implies t-(j-1)(\nu-2)+1 \geq \nu-1 > 0.
\end{aligned}$$

Therefore, $\Delta_t G_j(t,s) > 0$ for $t-\nu+1 \leq s$.

Now let $s \leq t-\nu$. We will show that $\Delta_t G_j(t,s) < 0$. We want to show

$$\begin{aligned}
\Gamma(\nu)\Delta_t G_j(t,s) &= C(\nu-1)(t-(j-1)(\nu-2))^{\underline{\nu-2}} - (\nu-1)(t-\sigma(s))^{\underline{\nu-2}} < 0 \\
&\iff C(t-(j-1)(\nu-2))^{\underline{\nu-2}} < (t-s-1)^{\underline{\nu-2}} \\
&\iff C\frac{(t-(j-1)(\nu-2))^{\underline{\nu-2}}}{(t-s-1)^{\underline{\nu-2}}} < 1.
\end{aligned}$$

Note that the inequality's direction is preserved in the last step since

$$\begin{aligned}
&t \geq s+\nu \\
&\implies t \geq s+1 \\
&\implies \frac{\Gamma(t-s)}{\Gamma(t-s-\nu+2)} = (t-s-1)^{\underline{\nu-2}} > 0.
\end{aligned}$$

We can see the inequality above holds by the following argument:

$$\begin{aligned}
&C\frac{(t-(j-1)(\nu-2))^{\underline{\nu-2}}}{(t-s-1)^{\underline{\nu-2}}}\\
&\quad=\frac{(b-s-1+j(\nu-2))^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}}\frac{(t-(j-1)(\nu-2))^{\underline{\nu-2}}}{(t-s-1)^{\underline{\nu-2}}}\\
&\quad=\frac{\Gamma(b-s+j(\nu-2))\Gamma(b+\nu-1-\nu+1)\Gamma(t-(j-1)(\nu-2)+1)\Gamma(t-s-\nu+2)}{\Gamma(b-s+j(\nu-2)-\nu+1)\Gamma(b+\nu-1)\Gamma(t-(j-1)(\nu-2)+1-\nu+2)\Gamma(t-s)}\\
&\quad=\frac{\Gamma(b-s+j(\nu-2))}{\Gamma(b-s+(j-1)(\nu-2)-1)}\frac{\Gamma(b)}{\Gamma(b+\nu-1)}\\
&\qquad\qquad\cdot\frac{\Gamma(t-(j-1)(\nu-2)+1)}{\Gamma(t-j(\nu-2)+1)}\frac{\Gamma(t-s-\nu+2)}{\Gamma(t-s)}.
\end{aligned}$$

Now let $k := s-(j-1)(\nu-2)$, and note that $k \in \mathbb{N}_0$. So we have

$$\begin{aligned}
&C\frac{(t-(j-1)(\nu-2))^{\underline{\nu-2}}}{(t-s-1)^{\underline{\nu-2}}}\\
&\quad=\frac{\Gamma(b-k+\nu-2)}{\Gamma(b-k-1)}\frac{\Gamma(b)}{\Gamma(b+\nu-1)}\frac{\Gamma(t+k-s+1)}{\Gamma(t+k-s-\nu+3)}\frac{\Gamma(t-s-\nu+2)}{\Gamma(t-s)}\\
&\quad=\frac{[\Gamma(b-k+\nu-2)](b-1)(b-2)\cdots(b-k-1)\Gamma(b-k-1)}{[\Gamma(b-k-1)](b+\nu-2)(b+\nu-3)\cdots(b+\nu-k-2)\Gamma(b+\nu-k-2)}\\
&\qquad\qquad\cdot\frac{(t-s+k)(t-s+k-1)\cdots(t-s)\Gamma(t-s)}{(t-s-\nu+k+2)(t-s-\nu+k+1)\cdots(t-s-\nu+2)\Gamma(t-s-\nu+2)}\\
&\qquad\qquad\cdot\frac{[\Gamma(t-s-\nu+2)]}{[\Gamma(t-s)]}\\
&\quad=\frac{(b-1)(b-2)\cdots(b-k-1)}{(b+\nu-2)(b+\nu-3)\cdots(b+\nu-k-2)}\\
&\qquad\qquad\cdot\frac{(t-s+k)(t-s+k-1)\cdots(t-s)}{(t-s-\nu+k+2)(t-s-\nu+k+1)\cdots(t-s-\nu+2)}\\
&\quad<1.
\end{aligned}$$

The final inequality above holds since we have $b-1 < b+\nu-2 \iff 1 < \nu$ and $t-s+k \leq t-s-\nu+k+2 \iff 0 \leq 2-\nu$. Thus, each fraction in the product is composed of $k+1$ factors in both the numerator and denominator such that each factor in the numerator of the first fraction can be shown to be less than a distinct factor in the denominator, and each factor in the numerator of the second fraction can be shown to be less than or equal to a distinct factor in the denominator. Therefore, $G_j(t,s)$ is strictly increasing for $t-\nu+1 \leq s$ and strictly decreasing for $s \leq t-\nu$.

Now, this means the maximum of $G_j(t,s)$ must be either at $t = s+\nu$ or $t = s+\nu-1$. In the following, we will see that the maximum actually occurs at $t = s+\nu-1$:

$$
\begin{aligned}
&\Gamma(\nu)(G_j(s+\nu-1,s) - G_j(s+\nu,s)) \\
&\quad = C((s+\nu-1-(j-1)(\nu-2))^{\underline{\nu-1}} - (s+\nu-(j-1)(\nu-2))^{\underline{\nu-1}}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad + (s+\nu-s-1)^{\underline{\nu-1}} \\
&\quad = C((s+\nu-1-(j-1)(\nu-2))^{\underline{\nu-1}} - (s+\nu-(j-1)(\nu-2))^{\underline{\nu-1}}) + \Gamma(\nu) \\
&\quad = C\left(\frac{\Gamma(s+\nu-(j-1)(\nu-2))}{\Gamma(s+1-(j-1)(\nu-2))} - \frac{\Gamma(s+\nu-(j-1)(\nu-2)+1)}{\Gamma(s+2-(j-1)(\nu-2))}\right) + \Gamma(\nu) \\
&\quad = C\Bigg(\frac{(s+1-(j-1)(\nu-2))\Gamma(s+\nu-(j-1)(\nu-2))}{\Gamma(s+2-(j-1)(\nu-2))} \\
&\qquad\qquad\qquad\qquad - \frac{\Gamma(s+\nu-(j-1)(\nu-2)+1)}{\Gamma(s+2-(j-1)(\nu-2))}\Bigg) + \Gamma(\nu) \\
&\quad = \frac{C}{\Gamma(s+2-(j-1)(\nu-2))}\big[(s+1-(j-1)(\nu-2))\Gamma(s+\nu-(j-1)(\nu-2)) \\
&\qquad\qquad\qquad - (s+\nu-(j-1)(\nu-2))\Gamma(s+\nu-(j-1)(\nu-2))\big] + \Gamma(\nu) \\
&\quad = C\frac{\Gamma(s+\nu-(j-1)(\nu-2))}{\Gamma(s+2-(j-1)(\nu-2))}\big[s+1-(j-1)(\nu-2) \\
&\qquad\qquad\qquad\qquad - (s+\nu-(j-1)(\nu-2)\big] + \Gamma(\nu) \\
&\quad = C\frac{\Gamma(s+\nu-(j-1)(\nu-2))}{\Gamma(s+2-(j-1)(\nu-2))}(1-\nu) + \Gamma(\nu) \\
&\quad = C\frac{\Gamma(k+\nu)}{\Gamma(k+2)}(1-\nu) + \Gamma(\nu) \\
&\quad > 0,
\end{aligned}
$$

which is equivalent to

$$\begin{aligned}
&\Gamma(\nu) > \frac{C\Gamma(k+\nu)}{\Gamma(k+2)}(\nu-1)\\
&\iff \frac{\Gamma(\nu)\Gamma(k+2)}{C\Gamma(k+\nu)(\nu-1)} > 1\\
&\iff \frac{\Gamma(\nu-1)\Gamma(k+2)}{C\Gamma(k+\nu)} > 1\\
&\iff \frac{\Gamma(\nu-1)\Gamma(k+2)}{\Gamma(k+\nu)}\frac{\Gamma(b-s+(j-1)(\nu-2)-1)\Gamma(b+\nu-1)}{\Gamma(b-s+j(\nu-2)\Gamma(b)}\\
&= \frac{\Gamma(\nu-1)\Gamma(k+2)}{\Gamma(k+\nu)}\frac{\Gamma(b-k-1)\Gamma(b+\nu-1)}{\Gamma(b-k+\nu-2)\Gamma(b)}\\
&= \frac{\Gamma(k+2)\Gamma(\nu-1)}{(k+\nu-1)(k+\nu-2)\cdots(\nu)(\nu-1)\Gamma(\nu-1)}\\
&\qquad \cdot \frac{[\Gamma(b-k-1)](b+\nu-2)(b+\nu-3)\cdots(b+\nu-k-2)\Gamma(b+\nu-k-2)}{[\Gamma(b-k+\nu-2)](b-1)(b-2)\cdots(b-k-1)\Gamma(b-k-1)}\\
&= \frac{(k+1)(k)\cdots(1)}{(k+\nu-1)(k+\nu-2)\cdots(\nu-1)}\frac{(b+\nu-2)(b+\nu-3)\cdots(b+\nu-k-2)}{(b-1)(b-2)\cdots(b-k-1)}\\
&> 1,
\end{aligned}$$

since the first fraction in the product is greater than 1 if $\nu < 2$ and the second fraction is greater than 1 if $\nu > 1$. Therefore, for each $s \in \mathbb{N}_{(j-1)(\nu-2)}^{b+(j-1)(\nu-2)}$,

$$\max_{t\in\mathbb{N}_{j(\nu-2)}^{b+j(\nu-2)}} G_j(t,s) = G_j(s+\nu-1,s).$$

□

To help condense and notationally simplify some future expressions, we make the following definition.

Definition 2.8. Let

$$\mathcal{G}_n(t,\tau_n) := G_n(t,\tau_n) \sum_{\tau_{n-1}=(n-2)(\nu-2)}^{b+(n-2)(\nu-2)} \sum_{\tau_{n-2}=(n-3)(\nu-2)}^{b+(n-3)(\nu-2)} \cdots \sum_{\tau_1=0}^{b} G_{n-1}(\tau_n,\tau_{n-1}) G_{n-2}(\tau_{n-1},\tau_{n-2}) \cdots G_1(\tau_2,\tau_1).$$

Corollary 2.9. *For any* $\tau_n \in \mathbb{N}_{(n-1)(\nu-2)}^{b+(n-1)(\nu-2)}$

$$\max_{t\in\mathbb{N}_{n(\nu-2)}^{b+n(\nu-2)}} \mathcal{G}_n(t,\tau_n) = \mathcal{G}_n(\tau_n+\nu-1,\tau_n)$$
$$= G_n(\tau_n+\nu-1,\tau_n) \sum_{\tau_{n-1}=(n-2)(\nu-2)}^{b+(n-2)(\nu-2)} G_{n-1}(\tau_n,\tau_{n-1}) \cdots \sum_{\tau_1=0}^{b} G_1(\tau_2,\tau_1).$$

Proof. For each $s \in \mathbb{N}_{(j-1)(\nu-2)}^{b+(j-1)(\nu-2)}$, we have, from Theorem 2.7,

$$\max_{t\in\mathbb{N}_{j(\nu-2)}^{b+j(\nu-2)}} G_j(t,s) = G_j(s+\nu-1,s),$$

and from Theorem 2.6, $G_j(t,s) \geq 0$ on its domain. So for all $t \in \mathbb{N}_{n(\nu-2)}^{b+n(\nu-2)}$, we have

$$\begin{aligned}
&\mathcal{G}_n(t,\tau_n)\\
&= G_n(t,\tau_n) \sum_{\tau_{n-1}=(n-2)(\nu-2)}^{b+(n-2)(\nu-2)} G_{n-1}(\tau_n,\tau_{n-1}) \sum_{\tau_{n-2}=(n-3)(\nu-2)}^{b+(n-3)(\nu-2)} G_{n-2}(\tau_{n-1},\tau_{n-2})\\
&\qquad \cdots \sum_{\tau_1=0}^{b} G_1(\tau_2,\tau_1)\\
&= G_n(t,\tau_n) \sum_{\tau_{n-1}=(n-2)(\nu-2)}^{b+(n-2)(\nu-2)} G_{n-1}(\tau_n,\tau_{n-1}) \sum_{\tau_{n-2}=(n-3)(\nu-2)}^{b+(n-3)(\nu-2)} \mathcal{G}_{n-2}(\tau_{n-1},\tau_{n-2})\\
&\leq G_n(\tau_n+\nu-1,\tau_n) \sum_{\tau_{n-1}=(n-2)(\nu-2)}^{b+(n-2)(\nu-2)} G_{n-1}(\tau_n,\tau_{n-1}) \sum_{\tau_{n-2}=(n-3)(\nu-2)}^{b+(n-3)(\nu-2)} \mathcal{G}_{n-2}(\tau_{n-1},\tau_{n-2}),
\end{aligned}$$

and

$$\mathcal{G}_n(\tau_n+\nu-1,\tau_n)$$
$$= G_n(\tau_n+\nu-1,\tau_n)\sum_{\tau_{n-1}=(n-2)(\nu-2)}^{b+(n-2)(\nu-2)} G_{n-1}(\tau_n,\tau_{n-1})\sum_{\tau_{n-2}=(n-3)(\nu-2)}^{b+(n-3)(\nu-2)} \mathcal{G}_{n-2}(\tau_{n-1},\tau_{n-2}).$$

Thus, we have our result. □

2.4 Existence and Uniqueness Theorems

We can find results related to the existence of solutions of fractional differential equations in [13], [14], [15], [16], and [22]. Here, we will discuss the existence and uniqueness of positive solutions of nonlinear fractional difference equations.

While nearly all of the results in this work would be either trivial or undefined for $b = 0$ or $b = 1$, perhaps it should be said that we are really only considering b-values that one could use to gather interesting or well-defined results, i.e., those values of $b \in \mathbb{N}_2$. In anticipation of our existence and uniqueness results, let us define the following domain:

Definition 2.10. For $j \in \mathbb{N}_0$,

$$\mathcal{D}_j := [b/4 + j(\nu-2), 3b/4 + j(\nu-2)] \cap \mathbb{N}_{j(\nu-2)},$$

unless $b = 2$, in which case let $\mathcal{D}_j := \{j(\nu-2)\}$.

Lemma 2.11. *There exists $\gamma \in (0,1)$ such that for any τ_n*

$$\min_{t\in\mathcal{D}_n} \mathcal{G}_n(t,\tau_n) \geq \gamma\left(\max_{t\in\mathbb{N}_{n(\nu-2)}^{b+n(\nu-2)}} \mathcal{G}_n(t,\tau_n)\right) = \gamma\mathcal{G}_n(\tau_n+\nu-1,\tau_n).$$

Proof. For any $t \in \mathcal{D}_n$, a set of a finite number of points, we have

$$\frac{\mathcal{G}_n(t,\tau_n)}{\max\limits_{t\in\mathbb{N}_{n(\nu-2)}^{b+n(\nu-2)}} \mathcal{G}_n(t,\tau_n)} = \frac{\mathcal{G}_n(t,\tau_n)}{\mathcal{G}_n(\tau_n+\nu-1,\tau_n)} \in (0,1],$$

since $\max_{t\in\mathbb{N}_{n(\nu-2)}^{b+n(\nu-2)}} \mathcal{G}_n(t,\tau_n) \geq \mathcal{G}_n(t,\tau_n)$ for any $t \in \mathbb{N}_{n(\nu-2)}^{b+n(\nu-2)}$ and $\mathcal{G}_n(t,\tau_n) \neq 0$ for $t \in \mathcal{D}_n \subseteq \mathbb{N}_{j(\nu-2)+1}^{b+j(\nu-2)-1}$ as a result of Theorem 2.7. Since t (and τ_n) comes from a domain with a finite number of points, we can find γ such that

$$0 < \gamma < \min_{t\in\mathcal{D}_n} \frac{\mathcal{G}_n(t,\tau_n)}{\mathcal{G}_n(\tau_n+\nu-1,\tau_n)} \leq 1.$$

Therefore, we have $\gamma \in (0,1)$ such that

$$\min_{t\in\mathcal{D}_n} \mathcal{G}_n(t,\tau_n) \geq \gamma \left(\max_{t\in\mathbb{N}_{n(\nu-2)}^{b+n(\nu-2)}} \mathcal{G}_n(t,\tau_n) \right) = \gamma\mathcal{G}_n(\tau_n+\nu-1,\tau_n).$$

$\square$

We consider a fractional boundary value problem of the form

$$\begin{cases} (-1)^n \Delta_{\nu-2}^{\nu}\Delta_{2\nu-4}^{\nu}\cdots\Delta_{n(\nu-2)}^{\nu} y(t) = f(t, y(t+n(\nu-2))), \qquad t\in\mathbb{N}_0^b,\ n\in\mathbb{N}, \\ y\left(n(\nu-2)\right) = 0 = y\left(b+n(\nu-2)\right), \\ \Delta_{(n-(i-1))(\nu-2)}^{\nu}\Delta_{(n-(i-2))(\nu-2)}^{\nu}\cdots\Delta_{(n-1)(\nu-2)}^{\nu}\Delta_{n(\nu-2)}^{\nu} y\left((n-i)(\nu-2)\right) = 0, \\ \Delta_{(n-(i-1))(\nu-2)}^{\nu}\Delta_{(n-(i-2))(\nu-2)}^{\nu}\cdots\Delta_{(n-1)(\nu-2)}^{\nu}\Delta_{n(\nu-2)}^{\nu} y\left(b+(n-i)(\nu-2)\right) = 0, \end{cases} \tag{2.4.1}$$

where $i = 1,2,3,\ldots,n-1$, and $f : \mathbb{N}_0^b \times \mathbb{R} \to \mathbb{R}$ (and, still, $\nu \in (1,2]$).

We can note that y solves this fractional boundary value problem if and only if y

is a fixed point of the operator $T : \mathcal{B} \to \mathcal{B}$ defined by

$$\begin{aligned} Ty &:= \sum_{\tau_n=(n-1)(\nu-2)}^{b+(n-1)(\nu-2)} \mathcal{G}_n(t,\tau_n) f(\tau_1, y(\tau_1 + n(\nu-2))) \\ &= \sum_{\tau_n=(n-1)(\nu-2)}^{b+(n-1)(\nu-2)} G_n(t,\tau_n) \sum_{\tau_{n-1}=(n-2)(\nu-2)}^{b+(n-2)(\nu-2)} G_{n-1}(\tau_n,\tau_{n-1}) \cdots \\ &\qquad\qquad \sum_{\tau_1=0}^{b} G_1(\tau_2,\tau_1) f(\tau_1, y(\tau_1 + n(\nu-2))), \end{aligned}$$

and where

$$\mathcal{B} := \{ y : \mathbb{N}_{n(\nu-2)}^{b+n(\nu-2)} \to \mathbb{R} \,|\, \text{the boundary conditions of (2.4.1) hold} \} \tag{2.4.2}$$

along with the supremum norm, $\|\cdot\|$, which, as in [27], is a Banach space. Let us define the following constants (again, where $b \geq 2$) which will appear in the next proof:

$$\begin{aligned} \eta &:= \left(\sum_{\tau_n=(n-1)(\nu-2)}^{b+(n-1)(\nu-2)} \mathcal{G}_n(\tau_n + \nu - 1, \tau_n) \right)^{-1}, \\ \lambda &:= \left(\sum_{\tau_n=(n-1)(\nu-2)}^{b+(n-1)(\nu-2)} G_n(n(\nu-2)+1,\tau_n) \sum_{\tau_{n-1}=(n-2)(\nu-2)}^{b+(n-2)(\nu-2)} G_{n-1}(\tau_n,\tau_{n-1}) \cdots \right. \\ &\qquad\qquad \left. \sum_{\tau_2=\nu-2}^{b+\nu-2} G_2(\tau_3,\tau_2) \sum_{\tau_1 \in \mathcal{D}_0} G_1(\tau_2,\tau_1) \right)^{-1}. \end{aligned}$$

Since G is nonzero and positive at least at some points in a nontrivial domain, both η and λ will be positive real numbers. Also, consider two conditions regarding f that will be used in the next theorem:

(C1) There exists a number $r > 0$ such that $f(t,y) \leq \eta r$ whenever $0 \leq y \leq r$.

(C2) There exists a number $r > 0$ such that $f(t,y) \geq \lambda r$ whenever $t \in \mathcal{D}_0$ and $\gamma r \leq y \leq r$, where γ is as in Lemma 2.11.

Remark 2.12. We may note that in what follows, we will be supposing that the conditions above hold for different r-values. A function f may satisfy (C1) for $r = r_1$, and f might also satisfy (C2) at $r = r_2$ such that $r_1 < \gamma r_2$. Thus, (C1) indicates that f is bounded above on one region while (C2) indicates that f is bounded below on a second disjoint region. Thus, there are easily functions f which satisfy the above conditions at distinct values of r. Also, it is important to note that a positive solution, as referred to below, may take on the value of 0 but only at the endpoints.

Theorem 2.13. *Suppose there exist positive and distinct r_1 and r_2 such that (C1) holds at $r = r_1$ and (C2) holds at $r = r_2$. Suppose also that $f(t, y) \geq 0$ and continuous. Then the fractional boundary value problem (2.4.1) has at least one positive solution, y_0, such that $\|y_0\|$ lies between r_1 and r_2.*

Proof. Without loss of generality, suppose $0 < r_1 < r_2$. We will now consider the set $\mathcal{K} := \{y \in \mathcal{B} \,|\, y(t) \geq 0, \min_{t\in\mathcal{D}_n} y(t) \geq \gamma\|y\|\} \subseteq \mathcal{B}$, where γ is as in Lemma 2.11. Note that $\mathcal{K}$ is a cone: given $y \in \mathcal{K}$, any positive scalar multiple of y is also in $\mathcal{K}$, and, since for $y \in \mathcal{K}$ we have $y(t) \geq 0$, if $-y \in \mathcal{K}$, then $y \equiv 0$. Now whenever $y \in \mathcal{K}$, we have $(Ty)(t) \geq 0$, and

$$\begin{aligned}
\min_{t\in\mathcal{D}_n}(Ty)(t) &= \min_{t\in\mathcal{D}_n} \sum_{\tau_n=(n-1)(\nu-2)}^{b+(n-1)(\nu-2)} \mathcal{G}_n(t,\tau_n) f(\tau_1, y(\tau_1 + n(\nu-2))) \\
&\geq \gamma \left(\sum_{\tau_n=(n-1)(\nu-2)}^{b+(n-1)(\nu-2)} \mathcal{G}_n(\tau_n + \nu - 1, \tau_n) f(\tau_1, y(\tau_1 + n(\nu-2))) \right) \\
&= \gamma \left(\max_{t\in\mathbb{N}_{n(\nu-2)}^{b+n(\nu-2)}} \sum_{\tau_n=(n-1)(\nu-2)}^{b+(n-1)(\nu-2)} \mathcal{G}_n(t,\tau_n) f(\tau_1, y(\tau_1 + n(\nu-2))) \right) \\
&= \gamma\|Ty\|,
\end{aligned}$$

i.e., $Ty \in \mathcal{K}$. So $T : \mathcal{K} \to \mathcal{K}$. We can also note that T is a completely continuous operator.

Now let $\Omega_1 := \{y \in \mathcal{B} : \|y\| < r_1\}$. For $y \in \partial\Omega_1$, we have $\|y\| = r_1$; therefore, condition (C1) holds for all $y \in \partial\Omega_1$. Thus, for $y \in \mathcal{K} \cap \partial\Omega_1$, we have

$$\begin{aligned}
\|Ty\| &= \max_{t \in \mathbb{N}_{n(\nu-2)}^{b+n(\nu-2)}} \sum_{\tau_n=(n-1)(\nu-2)}^{b+(n-1)(\nu-2)} \mathcal{G}_n(t, \tau_n) f(\tau_1, y(\tau_1 + n(\nu-2))) \\
&\leq \sum_{\tau_n=(n-1)(\nu-2)}^{b+(n-1)(\nu-2)} \mathcal{G}_n(\tau_n + \nu - 1, \tau_n) f(\tau_1, y(\tau_1 + n(\nu-2))) \\
&\leq \eta r_1 \sum_{\tau_n=(n-1)(\nu-2)}^{b+(n-1)(\nu-2)} \mathcal{G}_n(\tau_n + \nu - 1, \tau_n) \\
&= r_1 \\
&= \|y\|.
\end{aligned}$$

Therefore, $\|Ty\| \leq \|y\|$ whenever $y \in \mathcal{K} \cap \partial\Omega_1$, which implies that T is a cone compression on $\mathcal{K} \cap \partial\Omega_1$.

Now let $\Omega_2 := \{y \in \mathcal{B} : \|y\| < r_2\}$. For $y \in \partial\Omega_2$, we have $\|y\| = r_2$; therefore, condition (C2) holds for all $y \in \partial\Omega_2$. Thus, for $y \in \mathcal{K} \cap \partial\Omega_2$, we have

$$
\begin{aligned}
&\|Ty\| \\
&\quad \geq (Ty)\,(n(\nu-2)+1) \\
&\quad = \sum_{\tau_n=(n-1)(\nu-2)}^{b+(n-1)(\nu-2)} \mathcal{G}_n\,(n(\nu-2)+1,\tau_n)\, f(\tau_1, y(\tau_1+n(\nu-2))) \\
&\quad = \sum_{\tau_n=(n-1)(\nu-2)}^{b+(n-1)(\nu-2)} G_n(n(\nu-2)+1,\tau_n) \sum_{\tau_{n-1}=(n-2)(\nu-2)}^{b+(n-2)(\nu-2)} G_{n-1}(\tau_n,\tau_{n-1}) \cdots \\
&\qquad\qquad \sum_{\tau_1=0}^{b} G_1(\tau_2,\tau_1) f(\tau_1, y(\tau_1+n(\nu-2))) \\
&\quad \geq \sum_{\tau_n=(n-1)(\nu-2)}^{b+(n-1)(\nu-2)} G_n(n(\nu-2)+1,\tau_n) \sum_{\tau_{n-1}=(n-2)(\nu-2)}^{b+(n-2)(\nu-2)} G_{n-1}(\tau_n,\tau_{n-1}) \cdots \\
&\qquad\qquad \sum_{\tau_1\in\mathcal{D}_0} G_1(\tau_2,\tau_1) f(\tau_1, y(\tau_1+n(\nu-2))) \\
&\quad \geq \lambda r_2 \sum_{\tau_n=(n-1)(\nu-2)}^{b+(n-1)(\nu-2)} G_n(n(\nu-2)+1,\tau_n) \sum_{\tau_{n-1}=(n-2)(\nu-2)}^{b+(n-2)(\nu-2)} G_{n-1}(\tau_n,\tau_{n-1}) \cdots \\
&\qquad\qquad \sum_{\tau_1\in\mathcal{D}_0} G_1(\tau_2,\tau_1) \\
&\quad = r_2 \\
&\quad = \|y\|.
\end{aligned}
$$

Therefore, $\|Ty\| \geq \|y\|$ whenever $y \in \mathcal{K} \cap \partial\Omega_2$, which implies that T is a cone expansion on $\mathcal{K} \cap \partial\Omega_2$. So now, by Theorem 2.2 we have that T has a fixed point, which implies that our fractional boundary value problem has a positive solution y_0 such that $r_1 \leq \|y_0\| \leq r_2$. □

Now we introduce a Lemma that will help show uniqueness under a Lipschitz condition.

Lemma 2.14. *For $\mathcal{G}_n(t,\tau_n)$ defined previously, we have*

$$\max_{t\in\mathbb{N}_{n(\nu-2)}^{b+n(\nu-2)}} \sum_{\tau_n=(n-1)(\nu-2)}^{b+(n-1)(\nu-2)} \mathcal{G}_n(t,\tau_n) \leq \left[\frac{(b+n-2)^{\underline{n}}\Gamma(b+\nu)}{b\Gamma(\nu+1)}\right]^n.$$

Proof. We have

$$\begin{aligned}
G_j(\tau_j+\nu-1,\tau_j) &= \frac{1}{\Gamma(\nu)}\frac{(b-\sigma(\tau_j)+j(\nu-2))^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}}(\tau_j+\nu-1-(j-1)(\nu-2))^{\underline{\nu-1}}\\
&= \frac{1}{\Gamma(\nu)}\frac{(b-\tau_j-1+j(\nu-2))^{\underline{\nu-1}}}{(b+\nu-2)^{\underline{\nu-1}}}(\tau_j+\nu-1-(j-1)(\nu-2))^{\underline{\nu-1}}\\
&= \frac{1}{\Gamma(\nu)}\frac{\Gamma(b-\tau_j+j(\nu-2))}{\Gamma(b-\tau_j+j(\nu-2)-\nu+1)}\frac{\Gamma(b)}{\Gamma(b+\nu-1)}(\tau_j+\nu-1-(j-1)(\nu-2))^{\underline{\nu-1}}.
\end{aligned}$$

Now $\tau_j \in \mathbb{N}_{(j-1)(\nu-2)}^{b+(j-1)(\nu-2)}$, so when $\tau_j = b+(j-1)(\nu-2)-1$,

$$b-\tau_j+j(\nu-2)-\nu+1 = b-b-(j-1)(\nu-2)+1+j(\nu-2)-\nu+1 = 0,$$

and when $\tau_j = b+(j-1)(\nu-2)$,

$$b-\tau_j+j(\nu-2)-\nu+1 = b-(b+(j-1)(\nu-2))+j(\nu-2)-\nu+1 = -1.$$

Thus, for these two values of τ_j

$$\frac{\Gamma(b-\tau_j+j(\nu-2))}{\Gamma(b-\tau_j+j(\nu-2)-\nu+1)} = 0,$$

noting that $b-\tau_j+j(\nu-2)$ will not be an integer except in the case that $\nu=2$, in which case our work would be simplified from the beginning. Also note that when

$\tau_j = b + (j-1)(\nu-2) - 2$, we have

$$\frac{\Gamma(b-\tau_j+j(\nu-2))}{\Gamma(b-\tau_j+j(\nu-2)-\nu+1)} = \frac{\Gamma(\nu)}{\Gamma(1)} = \Gamma(\nu) \leq 1.$$

Now for $\tau_j \in \mathbb{N}_{(j-1)(\nu-2)}^{b+(j-1)(\nu-2)-3}$ we have

$$b-\tau_j+j(\nu-2) \geq b-(b+(j-1)(\nu-2)-3)+j(\nu-2) = \nu+1 > 2,$$

and

$$\begin{aligned}
b-\tau_j+j(\nu-2) &\leq b-\tau_j \\
&\leq b-(j-1)(\nu-2) \\
&= b+(j-1)(2-\nu) \\
&\leq b+(j-1)(1) = b+j-1,
\end{aligned}$$

while

$$\begin{aligned}
b-\tau_j+j(\nu-2)-\nu+1 &\geq b-(b+(j-1)(\nu-2)-3)-\nu+1 \\
&= -(j-1)(\nu-2)-\nu+3+1 \\
&= -j(\nu-2)+2 \geq 2.
\end{aligned}$$

Thus, for $\tau_j \in \mathbb{N}_{(j-1)(\nu-2)}^{b+(j-1)(\nu-2)-3}$, we have

$$\frac{\Gamma(b-\tau_j+j(\nu-2))}{\Gamma(b-\tau_j+j(\nu-2)-\nu+1)} \leq \frac{\Gamma(b-\tau_j)}{\Gamma(2)} \leq \Gamma(b+j-1) = (b+j-2)!,$$

so for all $\tau_j \in \mathbb{N}_{(j-1)(\nu-2)}^{b+(j-1)(\nu-2)}$,

$$\frac{\Gamma(b-\tau_j+j(\nu-2))}{\Gamma(b-\tau_j+j(\nu-2)-\nu+1)} \leq (b+j-2)!.$$

Therefore,

$$\begin{aligned}
&G_j(\tau_j+\nu-1,\tau_j)\\
&\quad= \frac{1}{\Gamma(\nu)} \frac{\Gamma(b-\tau_j+j(\nu-2))}{\Gamma(b-\tau_j+j(\nu-2)-\nu+1)} \frac{\Gamma(b)}{\Gamma(b+\nu-1)} (\tau_j+\nu-1-(j-1)(\nu-2))^{\underline{\nu-1}}\\
&\quad\leq \frac{1}{\Gamma(\nu)} \frac{(b+j-2)!\Gamma(b)}{\Gamma(b+\nu-1)} (\tau_j+\nu-1-(j-1)(\nu-2))^{\underline{\nu-1}}\\
&\quad\leq \frac{1}{\Gamma(\nu)} \frac{(b+j-2)!\Gamma(b)}{\Gamma(b-1)} (\tau_j+\nu-1-(j-1)(\nu-2))^{\underline{\nu-1}}\\
&\quad= \frac{1}{\Gamma(\nu)} \frac{(b+j-2)!(b-1)!}{(b-2)!} (\tau_j+\nu-1-(j-1)(\nu-2))^{\underline{\nu-1}}\\
&\quad= \frac{(b-1)(b+j-2)!}{\Gamma(\nu)} (\tau_j+\nu-1-(j-1)(\nu-2))^{\underline{\nu-1}}.
\end{aligned}$$

Now

$$
\begin{aligned}
&\max_{t\in\mathbb{N}_{j(\nu-2)}^{b+j(\nu-2)}} \sum_{\tau_j=(j-1)(\nu-2)}^{b+(j-1)(\nu-2)} G_j(t,\tau_j) \\
&\quad \le \sum_{\tau_j=(j-1)(\nu-2)}^{b+(j-1)(\nu-2)} \frac{(b-1)(b+j-2)!}{\Gamma(\nu)}(\tau_j+\nu-1-(j-1)(\nu-2))^{\underline{\nu-1}} \\
&\quad = \frac{(b-1)(b+j-2)!}{\Gamma(\nu)}\cdot\frac{1}{\nu}(\tau_j+\nu-1-(j-1)(\nu-2))^{\underline{\nu}}\Big|_{\tau_j=(j-1)(\nu-2)}^{b+(j-1)(\nu-2)+1} \\
&\quad = \frac{(b-1)(b+j-2)!}{\nu\Gamma(\nu)}[(b+\nu)^{\underline{\nu}}-(\nu-1)^{\underline{\nu}}] \\
&\quad = \frac{(b-1)(b+j-2)!}{\Gamma(\nu+1)}(b+\nu)^{\underline{\nu}} \\
&\quad = \frac{(b-1)(b+j-2)!\Gamma(b+\nu+1)}{\Gamma(\nu+1)\Gamma(b+1)} \\
&\quad = \frac{(b+j-2)!\Gamma(b+\nu+1)}{\Gamma(\nu+1)b(b-2)!} \\
&\quad \le \frac{(b+n-2)!\Gamma(b+\nu)}{b\Gamma(\nu+1)(b-2)!} \\
&\quad = \frac{(b+n-2)^{\underline{n}}\Gamma(b+\nu)}{b\Gamma(\nu+1)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\sum_{\tau_n=(n-1)(\nu-2)}^{b+(n-1)(\nu-2)} \mathcal{G}_n(t,\tau_n) &= \sum_{\tau_n=(n-1)(\nu-2)}^{b+(n-1)(\nu-2)} G_n(t,\tau_n) \sum_{\tau_{n-1}=(n-2)(\nu-2)}^{b+(n-2)(\nu-2)} G_{n-1}(\tau_n,\tau_{n-1}) \\
&\qquad \cdots \sum_{\tau_1=0}^{b} G_1(\tau_2,\tau_1) \\
&\le \left[\frac{(b+n-2)^{\underline{n}}\Gamma(b+\nu)}{b\Gamma(\nu+1)}\right]^n,
\end{aligned}
$$

giving us our result. $\square$

Here we prove a uniqueness theorem when f satisfies a Lipschitz condition.

Theorem 2.15. *Suppose $f(t,y)$ satisfies a Lipschitz condition in y with Lipschitz constant α, i.e., $|f(t,y_2) - f(t,y_1)| \le \alpha|y_2 - y_1|$ for all $(t,y_1),(t,y_2)$. Then if $\left[\frac{(b+n-2)^{\underline{n}}\Gamma(b+\nu)}{b\Gamma(\nu+1)}\right]^n < \frac{1}{\alpha}$, the fractional BVP (2.4.1) has a unique solution.*

Proof. Let $y_1, y_2 \in \mathcal{B}$, where $\mathcal{B}$ is the Banach space from (2.4.2). Then

$$\begin{aligned}
&\|Ty_2 - Ty_1\| \\
&\le \max_{t \in \mathbb{N}_{n(\nu-2)}^{b+n(\nu-2)}} \sum_{\tau_n=(n-1)(\nu-2)}^{b+(n-1)(\nu-2)} \left|\mathcal{G}_n(t,\tau_n)\right| \big|f(\tau_1, y_2(\tau_1 + n(\nu-2))) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - f(\tau_1, y_1(\tau_1 + n(\nu-2)))\big| \\
&\le \alpha \sum_{\tau_n=(n-1)(\nu-2)}^{b+(n-1)(\nu-2)} \mathcal{G}_n(\tau_n + \nu - 1, \tau_n)\big|y_2(\tau_1 + n(\nu-2)) - y_1(\tau_1 + n(\nu-2))\big| \\
&\le \alpha\|y_2 - y_1\| \sum_{\tau_n=(n-1)(\nu-2)}^{b+(n-1)(\nu-2)} \mathcal{G}_n(\tau_n + \nu - 1, \tau_n) \\
&\le \alpha\|y_2 - y_1\| \left[\frac{(b+n-2)^{\underline{n}}\Gamma(b+\nu)}{b\Gamma(\nu+1)}\right]^n,
\end{aligned}$$

which implies, by Theorem 2.3 (the Banach Contraction Theorem), we have a unique solution since $\alpha\left[\frac{(b+n-2)^{\underline{n}}\Gamma(b+\nu)}{b\Gamma(\nu+1)}\right]^n < 1$. □

Example 2.16. In the case $n = 2$, $\nu = 1.3$, and $\alpha = 0.01$, if f in 2.4.1 is Lipschitz continuous with Lipschitz constant α, then Theorem 2.15 guarantees we will have a unique solution if

$$\left(\frac{b(b-1)\Gamma(b+1.3)}{b\Gamma(2.3)}\right)^2 < 100.$$

Solving for b (numerically) implies that $b_{max} \approx 4.011$, where b_{max} is the largest value of b such that the hypotheses of Theorem 2.15 are satisfied. If instead we have $\alpha = 0.001$, then $b_{max} \approx 5.182$.

Chapter 3

Discrete q-Calculus

This chapter is largely the result of joint collaboration to produce a paper that incorporates and extends some results regarding calculus on a q-time scale [12]. In this chapter, we introduce the q-calculus, highlight definitions and properties of important functions and operators on a q-time scale, and solve initial value problems.

3.1 Preliminaries

The functions that we are considering are defined on sets of the form

$$aq^{\mathbb{N}_0} := \left\{a, aq, aq^2, \ldots\right\},$$

where $a, q \in \mathbb{R}$, $a > 0$ and $q > 1$. We will also consider sets of the form

$$aq^{\mathbb{N}_0^n} := \left\{a, aq, aq^2, \ldots, aq^n\right\},$$

where $n \in \mathbb{N}$.

Definition 3.1. We define the forward jump operator σ by

$$\sigma(t) := qt$$

for $t \in aq^{\mathbb{N}_0^{n-1}}$.

3.2 The q-Difference and q-Integral

The derivative is the rate of change from one point to the next. For our domain, $aq^{\mathbb{N}_0}$, the horizontal change is $\sigma(t) - t$, and the vertical change is $f(\sigma(t)) - f(t)$. Thus, we have the following definition.

Definition 3.2. Let $f : aq^{\mathbb{N}_0^n} \to \mathbb{R}$. We define the q-difference Δ_q by

$$\Delta_q f(t) := \frac{f(\sigma(t)) - f(t)}{\mu(t)}$$

where $\mu(t) = \sigma(t) - t = t(q-1)$ and $t \in aq^{\mathbb{N}_0^{n-1}}$. We also define $\Delta_q^n f(t)$, $n = 1, 2, 3, ...$ recursively by $\Delta_q(\Delta_q^{n-1} f(t))$ and Δ_q^0 to be the identity operator.

Remark 3.3. We could suppress the subscript on Δ_q since throughout this chapter, the difference represented by Δ will always be the q-difference, but we will leave it there as other works have shown the subscript throughout (both in places where Δ would and would not be ambiguous).

Theorem 3.4. *Assume $f, g : aq^{\mathbb{N}_0^n} \to \mathbb{R}$ and $\alpha \in \mathbb{R}$. Then for $t \in aq^{\mathbb{N}_0^{n-1}}$*

(i) $\Delta_q \alpha = 0$*;*

(ii) $\Delta_q \alpha f(t) = \alpha \Delta_q f(t)$*;*

(iii) $\Delta_q (f(t) + g(t)) = \Delta_q f(t) + \Delta_q g(t)$*;*

(iv) $\Delta_q(f(t)g(t)) = f(\sigma(t))\Delta_q g(t) + \Delta_q(f(t))g(t)$;

(v) $\Delta_q \frac{f(t)}{g(t)} = \frac{g(t)\Delta_q f(t) - f(t)\Delta_q g(t)}{g(\sigma(t))g(t)}$,

where in (v) we assume that $g(t) \neq 0$ *for* $t \in aq^{\mathbb{N}_0^n}$.

Proof. The proofs of (i), (ii), and (iii) follow easily from the definition of the q-difference as

$$\Delta_q \alpha = \frac{\alpha - \alpha}{\mu(t)} = 0,$$

$$\Delta_q \alpha f(t) = \frac{\alpha f(\sigma(t)) - \alpha f(t)}{\mu(t)} = \alpha \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \alpha \Delta_q f(t),$$

and

$$\begin{aligned}
\Delta_q(f(t) + g(t)) &= \frac{(f(\sigma(t)) + g(\sigma(t))) - (f(t) + g(t))}{\mu(t)} \\
&= \frac{(f(\sigma(t)) - f(t)) + (g(\sigma(t)) - g(t))}{\mu(t)} \\
&= \frac{f(\sigma(t)) - f(t)}{\mu(t)} + \frac{g(\sigma(t)) - g(t)}{t(q-1)} \\
&= \Delta_q f(t) + \Delta_q g(t).
\end{aligned}$$

To see (iv), consider

$$\begin{aligned}
\Delta_q(f(t)g(t)) &= \frac{f(\sigma(t))g(\sigma(t)) - f(t)g(t)}{\mu(t)} \\
&= \frac{f(\sigma(t))g(\sigma(t)) - f(\sigma(t))g(t) + f(\sigma(t))g(t) - f(t)g(t)}{\mu(t)} \\
&= \frac{f(\sigma(t))[g(\sigma(t)) - g(t)] + g(t)[f(\sigma(t)) - f(t)]}{\mu(t)} \\
&= f(\sigma(t))\Delta_q g(t) + g(t)\Delta_q f(t).
\end{aligned}$$

The proof of (v) employs a similar trick to that of (iv), for

$$\begin{aligned}
\Delta_q \frac{f(t)}{g(t)} &= \frac{\frac{f(\sigma(t))}{g(\sigma(t))} - \frac{f(t)}{g(t)}}{\mu(t)} \\
&= \frac{f(\sigma(t))g(t) - f(t)g(\sigma(t))}{\mu(t)g(\sigma(t))(g(t))} \\
&= \frac{f(\sigma(t))g(t) + f(t)g(t) - f(t)g(t) - f(t)g(\sigma(t))}{\mu(t)g(\sigma(t))g(t)} \\
&= \frac{g(t)[f(\sigma(t)) - f(t)]}{\mu(t)g(\sigma(t))g(t)} - \frac{f(t)[g(\sigma(t)) - g(t)]}{\mu(t)g(\sigma(t))g(t)} \\
&= \frac{g(t)\Delta_q f(t) - f(t)\Delta_q g(t)}{g(\sigma(t))g(t)}.
\end{aligned}$$

□

Definition 3.5. For $n \in \mathbb{R}$, we define $[n]_q$ by

$$[n]_q := \frac{q^n - 1}{q - 1}.$$

Note that when $n \in \mathbb{N}$

$$[n]_q := 1 + q + q^2 + \ldots + q^{n-1}.$$

Also, we define $[n]_q!$ by

$$[n]_q! := [n]_q[n-1]_q \ldots [2]_q[1]_q, \quad [0]_q! := 1.$$

Next we define the q-falling function. While we do not extend the definition to fractional falling powers for $q \in (1, \infty)$ here, one can do this similarly to previous work in this thesis with respect to the time scale $\mathbb{N}_a$ while using a Gamma function as presented in [26], [33], or [40].

Definition 3.6. On the time scale $aq^{\mathbb{N}_0}$, and for $n \in \mathbb{N}$, the q-falling function, $(\beta-\alpha)^{\underline{n}}$, is defined to be

$$(\beta - \alpha)^{\underline{n}} := \prod_{k=0}^{n-1}(\beta - \alpha q^k), \quad (\beta - \alpha)^{\underline{0}} := 1,$$

for $\beta, \alpha \in \mathbb{R}$.

Remark 3.7. Though the definition above only requires $\beta, \alpha \in \mathbb{R}$, in application, we generally have $\beta, \alpha \in aq^{N_0}$.

Remark 3.8. We may note that earlier on $\mathbb{N}_a$, we defined

$$(t-c)^{\underline{n}} = (t-c)(t-c-1)(t-c-2)\cdots(t-c-n+1).$$

We may view this definition as either

(i) $(t-c)^{\underline{n}} = \prod_{k=0}^{n-1}(t - \sigma^k(c))$, or as

(ii) $(t-c)^{\underline{n}} = \prod_{k=0}^{n-1}\rho^k(t-c)$.

The definition above takes the view of case (i). It tends to imply that $t, c \in aq^{N_0}$ and is consistent with [33]; it is the definition that will be used in what follows in this chapter. The view of case (ii) tends to imply $t - c \in aq^{N_0}$; it is consistent with the definition of the falling function on a mixed time scale, as seen in [25] and in the following chapter. While this difference in views of definition is not of utmost importance to this work overall, one might find that there are could be significant differences in results by taking one view over the other.

Remark 3.9. When looking at the domain $aq^{\mathbb{N}_0}$, there is interest in results for the case when $q \in (0, 1)$ [6]. On a domain of this type, one can define a fractional q-falling function. To motivate the definition of the q-falling function where n is not a positive integer, note the following:

$$
\begin{aligned}
(\beta-\alpha)^{\underline{n}} = \prod_{k=0}^{n-1}(\beta-\alpha q^k) &= \beta^n \prod_{k=0}^{n-1}\left(1-\frac{\alpha}{\beta}q^k\right) \\
&= \beta^n \frac{\prod_{k=0}^{n-1}(1-\frac{\alpha}{\beta}q^k)\prod_{r=n}^{\infty}(1-\frac{\alpha}{\beta}q^r)}{\prod_{r=n}^{\infty}(1-\frac{\alpha}{\beta}q^r)} \\
&= \beta^n \frac{\prod_{k=0}^{\infty}(1-\frac{\alpha}{\beta}q^k)}{\prod_{k=0}^{\infty}(1-\frac{\alpha}{\beta}q^{k+n})}.
\end{aligned}
$$

Using this motivation, we define the fractional q-falling function for

$q \in (0,1)$.

Definition 3.10. With respect to $aq^{\mathbb{N}_0}$, the fractional q-falling function is defined as

$$
(\beta-\alpha)^{\underline{\nu}} := \beta^{\nu}\frac{\prod_{k=0}^{\infty}(1-\frac{\alpha}{\beta}q^k)}{\prod_{k=0}^{\infty}(1-\frac{\alpha}{\beta}q^{k+\nu})}
$$

for $\beta, \alpha, \nu \in \mathbb{R}$.

Remark 3.11. Note that in the above definition, both products can be shown to converge here for $q \in (0,1)$: we have that $\prod_{k=0}^{\infty}\left(1+\left(-\frac{\alpha}{\beta}q^k\right)\right)$ converges if and only if $\sum_{k=0}^{\infty}\left(-\frac{\alpha}{\beta}q^k\right)$ converges, and $\sum_{k=0}^{\infty}\left(-\frac{\alpha}{\beta}q^k\right) = -\frac{\alpha}{\beta}\sum_{k=0}^{\infty}q^k$ converges since it is a geometric series with ratio $q \in (0,1)$.

The next two theorems highlight some results for such q.

Theorem 3.12. *The following are properties of the q-falling function for $\nu, \gamma, \alpha, \beta \in \mathbb{R}$ and $q \in (0,1)$:*

(i) $(\beta-\alpha)^{\underline{\nu+\gamma}} = (\beta-\alpha)^{\underline{\nu}}(\beta-q^{\nu}\alpha)^{\underline{\gamma}}$;

(ii) $(\gamma\beta-\gamma\alpha)^{\underline{\nu}} = \gamma^{\nu}(\beta-\alpha)^{\underline{\nu}}$;

(iii) For $t, s \in aq^{\mathbb{N}_0}$ such that $t \geq s$ and $\nu \notin \mathbb{N}_0$, $(t-s)^{\underline{\nu}} = 0$.

Proof. (i) Given ν, γ, α, and β as above,

$$\begin{aligned}
(\beta-\alpha)^{\underline{\nu+\gamma}} &= \beta^{\nu+\gamma}\frac{\prod_{k=0}^{\infty}(1-\frac{\alpha}{\beta}q^k)}{\prod_{k=0}^{\infty}(1-\frac{\alpha}{\beta}q^{k+\nu+\gamma})} \\
&= \beta^{\nu+\gamma}\frac{\prod_{k=0}^{\infty}(1-\frac{\alpha}{\beta}q^k)}{\prod_{k=0}^{\infty}(1-\frac{\alpha}{\beta}q^{k+\nu+\gamma})}\frac{\prod_{k=0}^{\infty}(1-\frac{\alpha q^\nu}{\beta}q^k)}{\prod_{k=0}^{\infty}(1-\frac{\alpha}{\beta}q^{k+\nu})} \\
&= \beta^{\nu}\frac{\prod_{k=0}^{\infty}(1-\frac{\alpha}{\beta}q^k)}{\prod_{k=0}^{\infty}(1-\frac{\alpha}{\beta}q^{k+\nu})}\beta^{\gamma}\frac{\prod_{k=0}^{\infty}(1-\frac{\alpha q^\nu}{\beta}q^k)}{\prod_{k=0}^{\infty}(1-\frac{\alpha}{\beta}q^{k+\nu+\gamma})} \\
&= (\beta-\alpha)^{\underline{\nu}}(\beta-q^\nu\alpha)^{\underline{\gamma}}.
\end{aligned}$$

(ii) Given ν, γ, α, and β as above,

$$\begin{aligned}
(\gamma\beta-\gamma\alpha)^{\underline{\nu}} &= (\gamma\beta)^{\nu}\frac{\prod_{k=0}^{\infty}(1-\frac{\alpha\gamma}{\beta\gamma}q^k)}{\prod_{k=0}^{\infty}(1-\frac{\alpha\gamma}{\beta\gamma}q^{k+\nu})} \\
&= \gamma^{\nu}\beta^{\nu}\frac{\prod_{k=0}^{\infty}(1-\frac{\alpha}{\beta}q^k)}{\prod_{k=0}^{\infty}(1-\frac{\alpha}{\beta}q^{k+\nu})} \\
&= \gamma^{\nu}(\beta-\alpha)^{\underline{\nu}}.
\end{aligned}$$

(iii) Let $t = aq^n$ and $s = aq^m$ such that $n \geq m$, and let $\nu \notin \mathbb{N}_0$.

$$\begin{aligned}
(t-s)^{\underline{\nu}} &= (aq^n - aq^m)^{\underline{\nu}} \\
&= (aq^n)^{\nu}\frac{\prod_{k=0}^{\infty}(1-\frac{aq^m}{aq^n}q^k)}{\prod_{k=0}^{\infty}(1-\frac{aq^m}{aq^n}q^{k+\nu})} \\
&= (aq^n)^{\nu}\frac{\prod_{k=0}^{\infty}(1-q^{m-n}q^k)}{\prod_{k=0}^{\infty}(1-q^{m-n}q^{k+\nu})} \\
&= (aq^n)^{\nu}(1-q^{m-n}q^{n-m})\frac{\prod_{k=0}^{n-m-1}(1-q^{m-n}q^k)\prod_{k=n-m+1}^{\infty}(1-q^{m-n}q^k)}{\prod_{k=0}^{\infty}(1-q^{m-n}q^{k+\nu})} \\
&= 0
\end{aligned}$$

□

Remark 3.13. Regarding part (iii) of the previous theorem, if $\nu \in \mathbb{N}_0$, then for $t = aq^n$ and $s = aq^m$ such that $n \geq m$ the result still holds if $\nu \geq n - m + 1$ for any $q \geq 0$.

Theorem 3.14. *For $t \in aq^{\mathbb{N}_0}$ and $\alpha, \nu \in \mathbb{R}, n \in \mathbb{N}$ and $q \in (0,1)$, the following equalities hold.*

(i) $\Delta_q(t-\alpha)^{\underline{\nu}} = q^{\nu-1}[\nu]_{1/q}(\sigma(t)-\alpha)^{\underline{\nu-1}}$;

(ii) $\Delta_q(\alpha-t)^{\underline{\nu}} = -q^{\nu-1}[\nu]_{1/q}(\alpha-t)^{\underline{\nu-1}}$.

Proof. (i) By direct calculation,

$$\begin{aligned}
\Delta_q(t-\alpha)^{\underline{\nu}} &= \Delta_q t^\nu \frac{\prod_{k=0}^\infty(1-\frac{\alpha}{t}q^k)}{\prod_{k=0}^\infty(1-\frac{\alpha}{t}q^{k+\nu})} \\
&= \frac{\left(\frac{t}{q}\right)^\nu \frac{\prod_{k=0}^\infty(1-\frac{\alpha}{t}q^{k+1})}{\prod_{k=0}^\infty(1-\frac{\alpha}{t}q^{k+\nu+1})} - t^\nu \frac{\prod_{k=0}^\infty(1-\frac{\alpha}{t}q^k)}{\prod_{k=0}^\infty(1-\frac{\alpha}{t}q^{k+\nu})}}{\mu(t)} \\
&= \frac{\left(\frac{t}{q}\right)^\nu \frac{\prod_{k=0}^\infty(1-\frac{\alpha}{t}q^{k+1})}{\prod_{k=0}^\infty(1-\frac{\alpha}{t}q^{k+\nu+1})}\frac{(1-\frac{\alpha}{t}q^\nu)}{(1-\frac{\alpha}{t}q^\nu)} - t^\nu \frac{\prod_{k=0}^\infty(1-\frac{\alpha}{t}q^k)}{\prod_{k=0}^\infty(1-\frac{\alpha}{t}q^{k+\nu})}}{\mu(t)} \\
&= t^\nu \frac{\prod_{k=0}^\infty\left(1-\frac{\alpha}{t}q^{k+1}\right)}{\prod_{k=0}^\infty\left(1-\frac{\alpha}{t}q^{k+\nu}\right)}\left[\frac{\left(\frac{1}{q}\right)^\nu\left(1-\frac{\alpha}{t}q^\nu\right)-\left(1-\frac{\alpha}{t}\right)}{t\left(\frac{1}{q}-1\right)}\right] \\
&= t^{\nu-1} \frac{\prod_{k=0}^\infty\left(1-\frac{\alpha}{t}q^{k+1}\right)}{\prod_{k=0}^\infty\left(1-\frac{\alpha}{t}q^{k+\nu}\right)}\left[\frac{\left(\frac{1}{q}\right)^\nu-\frac{\alpha}{t}-1+\frac{\alpha}{t}}{\frac{1}{q}-1}\right] \\
&= q^{\nu-1}\left(\frac{t}{q}\right)^{\nu-1} \frac{\prod_{k=0}^\infty\left(1-\frac{\alpha}{t}q^{k+1}\right)}{\prod_{k=0}^\infty\left(1-\frac{\alpha}{t}q^{k+\nu}\right)}\left[\frac{\frac{1}{q^\nu}-1}{\frac{1}{q}-1}\right] \\
&= q^{\nu-1}[\nu]_{1/q}(\sigma(t)-\alpha)^{\underline{\nu-1}}.
\end{aligned}$$

(ii) Again, by direct calculation,

$$\begin{aligned}
\Delta_q(\alpha - t)^{\underline{\nu}} &= \Delta_q \alpha^{\nu} \frac{\prod_{k=0}^{\infty}(1-\frac{t}{\alpha}q^k)}{\prod_{k=0}^{\infty}(1-\frac{t}{\alpha}q^{k+\nu})} \\
&= \frac{\alpha^{\nu}\frac{\prod_{k=0}^{\infty}(1-\frac{t}{\alpha}q^{k-1})}{\prod_{k=0}^{\infty}(1-\frac{t}{\alpha}q^{k+\nu-1})} - \alpha^{\nu}\frac{\prod_{k=0}^{\infty}(1-\frac{t}{\alpha}q^{k})}{\prod_{k=0}^{\infty}(1-\frac{t}{\alpha}q^{k+\nu})}}{\mu(t)} \\
&= \frac{\alpha^{\nu}\frac{\prod_{k=0}^{\infty}(1-\frac{t}{\alpha}q^{k-1})}{\prod_{k=0}^{\infty}(1-\frac{t}{\alpha}q^{k+\nu-1})} - \alpha^{\nu}\frac{\prod_{k=0}^{\infty}(1-\frac{t}{\alpha}q^{k})}{\prod_{k=0}^{\infty}(1-\frac{t}{\alpha}q^{k+\nu})}\frac{(1-\frac{t}{\alpha}q^{\nu-1})}{(1-\frac{t}{\alpha}q^{\nu-1})}}{\mu(t)} \\
&= \alpha^{\nu}\frac{\prod_{k=0}^{\infty}(1-\frac{t}{\alpha}q^{k})}{\prod_{k=0}^{\infty}(1-\frac{t}{\alpha}q^{k+\nu-1})}\left[\frac{(1-\frac{t}{\alpha}q^{-1})-(1-\frac{t}{\alpha}q^{\nu-1})}{t\left(\frac{1}{q}-1\right)}\right] \\
&= \alpha^{\nu}\frac{\prod_{k=0}^{\infty}(1-\frac{t}{\alpha}q^{k})}{\prod_{k=0}^{\infty}(1-\frac{t}{\alpha}q^{k+\nu-1})} \cdot \frac{t}{\alpha}\left[\frac{q^{\nu-1}(1-\frac{1}{q^{\nu}})}{t(\frac{1}{q}-1)}\right] \\
&= \alpha^{\nu-1}\frac{\prod_{k=0}^{\infty}(1-\frac{t}{\alpha}q^{k})}{\prod_{k=0}^{\infty}(1-\frac{t}{\alpha}q^{k+\nu-1})}(-q^{\nu-1})[\nu]_{1/q} \\
&= -q^{\nu-1}[\nu]_{1/q}(\alpha - t)^{\underline{\nu-1}}
\end{aligned}$$

□

Definition 3.15. The n^{th} Taylor monomial, $h_n(t,\alpha)$, is defined as

$$h_n(t,\alpha) := \frac{(t-\alpha)^{\underline{n}}}{[n]_q!}$$

for $t \in aq^{\mathbb{N}_0}, \alpha \in \mathbb{R}$, and $n \in \mathbb{N}$.

Remark 3.16. We also define the Taylor monomials for a mixed time scale in the following chapter, of which $aq^{\mathbb{N}_a}$ is a particular example. For other examples of Taylor monomials calculated for different time scales, see [31].

Theorem 3.17. *For $t \in aq^{\mathbb{N}_0}$ and $\alpha, \nu \in \mathbb{R}, n \in \mathbb{N}$,*

$$\Delta_q h_n(t,\alpha) = h_{n-1}(t,\alpha).$$

Proof. Given t, α, ν, and n as in the statement of the theorem,

$$\begin{aligned}
\Delta_q h_n(t,\alpha) &= \Delta_q \frac{(t-\alpha)^{\underline{n}}}{[n]_q!} \\
&= [n]_q \frac{(t-\alpha)^{\underline{n-1}}}{[n]_q!} \\
&= \frac{(t-\alpha)^{\underline{n-1}}}{[n-1]_q!} \\
&= h_{n-1}(t,\alpha).
\end{aligned}$$

□

Remark 3.18. With regard to what follows, especially the next definition, any sum or product should be understood to only consider elements of our domain, $aq^{\mathbb{N}_0}$. For example,

$$\sum_{s=a}^{t} f(s) = \sum_{s \in aq^{\mathbb{N}_a^{\log_q \frac{t}{a}}}} f(s) = f(a) + f(aq) + \cdots + f(t/q) + f(t).$$

At times, however, the operators will function normally. How the operator functions will be clear from context. By observing the expression within the operator, one can know how the operator itself is to be handled. If we let $t = aq^n$, then one can consider the sum above given in the equivalent form

$$\sum_{k=0}^{n} f(aq^k) = f(a) + f(aq) + \cdots + f(aq^{n-1}) + f(aq^n).$$

One could also make use of an index function as seen in the next chapter.

Definition 3.19. Let $f : aq^{\mathbb{N}_0} \to \mathbb{R}$ and $c, t \in aq^{\mathbb{N}_0}$. We define the integral by

$$\int_c^t f(s)\,\Delta_q s := \begin{cases} \sum_{s=c}^{t/q} f(s)\mu(s), & t > c \\ 0, & t \leq c \end{cases}.$$

Note that the integral is simply a left-hand Riemann sum.

The following theorem gives some properties of the integral.

Theorem 3.20. *Assume $f, g : aq^{\mathbb{N}_0^n} \to \mathbb{R}$ and $b, c, d \in aq^{\mathbb{N}_0^{n-1}}$, with $b < c < d$. Then, for $\alpha \in \mathbb{R}$,*

(i) $\int_b^c \alpha f(t)\,\Delta_q t = \alpha \int_b^c f(t)\,\Delta_q t$;

(ii) $\int_b^c (f(t) + g(t))\,\Delta_q t = \int_b^c f(t)\,\Delta_q t + \int_b^c g(t)\,\Delta_q t$;

(iii) $\int_b^b f(t)\,\Delta_q t = 0$;

(iv) $\int_b^d f(t)\,\Delta_q t = \int_b^c f(t)\,\Delta_q t + \int_c^d f(t)\,\Delta_q t$;

(v) $|\int_b^c f(t)\,\Delta_q t| \leq \int_b^c |f(t)|\,\Delta_q t$;

(vi) *if* $f(t) \geq g(t)$ *for* $t \in \{b, bq, \ldots, \frac{c}{q}\}$, *then* $\int_b^c f(t)\,\Delta_q t \geq \int_b^c g(t)\,\Delta_q t$;

(vii) *if* $F(t) := \int_b^t f(s)\,\Delta_q s$, *then* $\Delta_q F(t) = f(t), t \in \{b, bq, \ldots, n\}$.

Proof. Recall that the integral is defined to be a sum. Thus, properties (i)-(vi) regarding the integral hold since the corresponding properties for sums hold.

To prove (vii), consider

$$\Delta_q F(t) = \frac{\sum_{s=b}^{t} f(s)\mu(s) - \sum_{s=b}^{t/q} f(s)\mu(s)}{\mu(t)} = \frac{f(t)\mu(t)}{\mu(t)} = f(t).$$

□

Using the product rules proved in Theorem 2.2, we can prove the following integration by parts theorem.

Theorem 3.21. (Integration By Parts) *Given two functions* $u, v : aq^{\mathbb{N}_0} \to \mathbb{R}$ *and* $b, c \in aq^{\mathbb{N}_0}, b < c$, *we have the following integration by parts formulas:*

(i) $\int_b^c u(t)\Delta_q v(t)\,\Delta_q t = u(t)v(t)\Big|_b^c - \int_b^c v(\sigma(t))\Delta_q u(t)\,\Delta_q t$;

(ii) $\int_b^c u(\sigma(t))\Delta_q v(t)\,\Delta_q t = u(t)v(t)\Big|_b^c - \int_b^c v(t)\Delta_q u(t)\,\Delta_q t$.

Proof.

(i) Assume $u, v : aq^{\mathbb{N}_0} \to \mathbb{R}$ and $b, c \in aq^{\mathbb{N}_0}, b < c$. Using the product rule we can obtain

$$\Delta_q(u(t)v(t)) = \Delta_q(u(t))v(\sigma(t)) + u(t)\Delta_q v(t)$$
$$\implies u(t)\Delta_q v(t) = \Delta_q(u(t)v(t)) - \Delta_q(u(t))v(\sigma(t)).$$

Integrating both sides we obtain the following:

$$\int_b^c u(t)\Delta_q v(t)\,\Delta_q t = \int_b^c \Delta_q(u(t)v(t))\,\Delta_q t - \int_b^c \Delta_q u(t)v(\sigma(t))\,\Delta_q t$$
$$\implies \int_b^c u(t)\Delta_q v(t)\,\Delta_q t = u(t)v(t)\Big|_b^c - \int_b^c v(\sigma(t))\Delta_q u(t)\,\Delta_q t.$$

(ii) Assume $u, v : aq^{\mathbb{N}_0} \to \mathbb{R}$ and $b, c \in aq^{\mathbb{N}_0}, b < c$. Using a variation of the product rule we can obtain

$$\Delta_q(u(t)v(t)) = \Delta_q(u(t))v(t) + u(\sigma(t))\Delta_q v(t)$$
$$\Longrightarrow u(\sigma(t))\Delta_q v(t) = \Delta_q(u(t)v(t)) - \Delta_q(u(t))v(t).$$

Integrating both sides we obtain the following:

$$\int_b^c u(\sigma(t))\Delta_q v(t)\,\Delta_q t = \int_b^c \Delta_q(u(t)v(t))\,\Delta_q t - \int_b^c \Delta_q u(t)v(t)\,\Delta_q t$$
$$\Longrightarrow \int_b^c u(\sigma(t))\Delta_q v(t)\,\Delta_q t = u(t)v(t)\Big|_b^c - \int_b^c v(t)\Delta_q u(t)\,\Delta_q t.$$

□

Definition 3.22. Assume $f : aq^{\mathbb{N}_0^n} \to \mathbb{R}$. F(t) is an antidifference of $f(t)$ on $aq^{\mathbb{N}_0^n}$ provided

$$\Delta_q F(t) = f(t), \qquad t \in aq^{\mathbb{N}_0^{n-1}}.$$

Theorem 3.23. *If $f : aq^{\mathbb{N}_0^n} \to \mathbb{R}$ and $G(t)$ is an antidifference of $f(t)$ on $aq^{\mathbb{N}_0^n}$, then $F(t) = G(t) + C$ is a general antidifference of $f(t)$.*

Proof. Assume $G(t)$ is an antidifference of $f(t)$ on $aq^{\mathbb{N}_0^n}$.
Let $F(t) = G(t) + C$, where C is constant and $t \in aq^{\mathbb{N}_0^n}$. Then

$$\Delta_q F(t) = \Delta_q(G(t) + C) = \Delta_q G(t) + \Delta_q C = \Delta_q G(t) + 0 = f(t), \qquad t \in aq^{\mathbb{N}_0^{n-1}}$$

and so $F(t)$ is an antidifference of $f(t)$ on $aq^{\mathbb{N}_0^n}$. Conversely, assume $F(t)$ is an antidifference of $f(t)$ on $aq^{\mathbb{N}_0^n}$. Then

$$\Delta_q(F(t) - G(t)) = \Delta_q F(t) - \Delta_q G(t) = f(t) - f(t) = 0$$

for $t \in aq^{\mathbb{N}_0^{n-1}}$. This implies $F(t) - G(t) = C$, for $t \in aq^{\mathbb{N}_0^n}$. Hence

$$F(t) = G(t) + C, \qquad t \in aq^{\mathbb{N}_0^n}.$$

$\square$

Theorem 3.24. (Fundemental Theorem of q-Calculus)
Assume $f : aq^{\mathbb{N}_0^n} \to \mathbb{R}$ and $F(t)$ is any antidifference of $f(t)$ on $aq^{\mathbb{N}_0^n}$. Then

$$\int_a^t f(s)\,\Delta_q s = \int_a^t \Delta_q F(s)\,\Delta_q s = F(s)\Big|_a^t.$$

Proof. Assume $F(t)$ is any antidifference of $f(t)$ on $aq^{\mathbb{N}_0^n}$. Let

$$G(t) := \int_a^t f(s)\,\Delta_q s, \qquad t \in aq^{\mathbb{N}_0^n}.$$

By Theorem 2.13 (vii), $G(t)$ is an antidifference of $f(t)$. Hence, by the previous theorem, $F(t) = G(t) + C$, where C is a constant. Then

$$\begin{aligned} F(s)\Big|_a^t &= F(t) - F(a) \\ &= (G(t) + C) - (G(a) + C)) \\ &= G(t) - G(a). \end{aligned}$$

By Theorem 2.13 (iii), $G(a) = 0$

$$\Longrightarrow F(s)\Big|_a^t = \int_a^t f(t)\,\Delta_q t.$$

$\square$

3.3 The q-Exponential

Recall from traditional calculus that $x(t) = e^{pt}$ is the unique solution of the following initial value problem

$$\begin{cases} x' = px, \\ x(0) = 1. \end{cases}$$

We will define our exponential function in this manner by finding the solution to the following initial value problem

$$\begin{cases} \Delta_q x(t) = p(t)x(t), \\ x(a) = 1. \end{cases}$$

To see what the solution of the initial value problem will be, we will generate a pattern recursively. We start at $t = a$ to find $x(aq)$:

$$\Delta_q x(a) = \frac{x(aq) - x(a)}{\mu(a)} = p(a)x(a) = p(a)$$

$$\Longrightarrow x(aq) = 1 + \mu(a)p(a).$$

Similarly, we find $x(aq^2)$:

$$\Delta_q x(aq) = \frac{x(aq^2) - x(aq)}{\mu(aq)} = p(aq)x(aq) = p(aq)(1 + \mu(a)p(a)),$$

which implies

$$\begin{aligned} x(aq^2) &= 1 + \mu(a)p(a) + \mu(aq)p(aq)(1 + \mu(a)p(a)) \\ &= (1 + \mu(a)p(a))(1 + \mu(aq)p(aq)). \end{aligned}$$

Solving for $x(aq^3)$ we obtain

$$x(aq^3) = (1 + \mu(a)p(a))(1 + \mu(aq)p(aq))(1 + \mu(aq^2)p(aq^2)).$$

Continuing inductively, the solution of the initial value problem can be written as

$$\prod_{s=a}^{t/q}(1 + \mu(s)p(s)).$$

This analysis leads to the definition of the exponential function on our domain. Later, it will be shown this does satisfy the above initial value problem.

Definition 3.25. The q-exponential function $e_p(t, a)$ is defined to be

$$e_p(t, a) := \prod_{s=a}^{t/q}(1 + \mu(s)p(s)).$$

Remark 3.26. If $c > d$ in $\prod_{s=c}^{d} f(s)$, then we consider this an empty product. In other words, $\prod_{s=c}^{d} f(s) = 1$.

Remark 3.27. It is worth noting that the definition above differs from the two q-exponential function definitions given in [33]. For examples of exponential functions on other time scales, see [20].

In order to develop analogues of certain familiar laws of exponents, we can define circle operators which, in the discrete q-calculus, will behave similarly to the related operators in real calculus.

Definition 3.28. We define $\oplus$ by

$$(p \oplus r)(t) := p(t) + r(t) + \mu(t)p(t)r(t).$$

Theorem 3.29. *$e_p(t,a)e_r(t,a) = e_{p\oplus r}(t,a)$.*

Proof.

$$\begin{aligned}
e_p(t,a)e_r(t,a) &= \prod_{s=a}^{t/q}[1+\mu(s)p(s)]\prod_{\ell=a}^{t/q}[1+\mu(\ell)r(\ell)] \\
&= \prod_{s=a}^{t/q}[(1+\mu(s)p(s))(1+\mu(s)r(s))] \\
&= \prod_{s=a}^{t/q}[1+\mu(s)r(s)+\mu(s)p(s)+\mu^2(s)p(s)r(s)] \\
&= \prod_{s=a}^{t/q}[1+\mu(s)[p(s)+r(s)+\mu(s)p(s)r(s)]] \\
&= \prod_{s=a}^{t/q}[1+\mu(s)[p(s)\oplus r(s)]] \\
&= e_{p\oplus r}(t,a).
\end{aligned}$$

□

Definition 3.30. We define the set of regressive functions, $\mathcal{R}_q$, by

$$\mathcal{R}_q := \left\{p : aq^{N_0} \to \mathbb{C} \,|\, 1+\mu(t)p(t) \neq 0 \ \forall t\right\}.$$

We also define the set of regressive constant functions in the following way.

$$\mathcal{R}_q^c := \mathbb{C} \cap \mathcal{R}_q = \{\alpha \in \mathbb{C} : \alpha \neq -\frac{1}{\mu(t)} \ \forall t\},$$

where $\mathbb{C}$ is the set of all complex constant functions.

Theorem 3.31. *$\mathcal{R}_q$, $\oplus$ is an Abelian group.*

Proof. Let $p(t), \ell(t), r(t) \in \mathcal{R}_q$ throughout. First we check for commutativity:

$$\begin{aligned}
p(t) \oplus r(t) &= p(t) + r(t) + \mu(t)p(t)r(t) \\
&= r(t) + p(t) + \mu(t)r(t)p(t) \\
&= r(t) \oplus p(t).
\end{aligned}$$

Next we check for associativity:

$$\begin{aligned}
(p(t) \oplus \ell(t)) \oplus r(t) &= (p(t) + \ell(t) + \mu(t)p(t)\ell(t)) \oplus r(t) \\
&= (p(t) + \ell(t) + \mu(t)p(t)\ell(t)) + r(t) \\
&\qquad + \mu(t)[p(t) + \ell(t) + \mu(t)p(t)\ell(t)]r(t) \\
&= p(t) + \ell(t) + r(t) + \mu(t)p(t)\ell(t) \\
&\qquad + \mu(t)p(t)r(t) + \mu(t)\ell(t)r(t) + \mu^2(t)p(t)\ell(t)r(t) \\
&= p(t) + \ell(t) + r(t) + \mu(t)\ell(t)r(t) \\
&\qquad + p(t)\ell(t)\mu(t) + p(t)r(t)\mu(t) + \mu^2(t)p(t)\ell(t)r(t) \\
&= p(t) + [\ell(t) + r(t) + \mu(t)\ell(t)r(t)] \\
&\qquad + p(t)[\ell(t) + r(t) + \mu(t)\ell(t)r(t)]\mu(t) \\
&= p(t) \oplus (\ell(t) \oplus r(t)).
\end{aligned}$$

To check for closure, keep in mind that $1 + \mu(t)p(t) \neq 0$ and $1 + \mu(t)\ell(t) \neq 0$ since $p(t), \ell(t) \in \mathcal{R}_q$. We want to show that $1 + \mu(t)[p(t) \oplus \ell(t)] \neq 0$:

$$\begin{aligned}
1 + \mu(t)[p(t) \oplus \ell(t)] &= 1 + \mu(t)[p(t) + \ell(t) + \mu(t)p(t)\ell(t)] \\
&= 1 + \mu(t)p(t) + \mu(t)\ell(t) + \mu^2(t)p(t)\ell(t) \\
&= (1 + \mu(t)p(t))(1 + \mu(t)\ell(t)) \\
&\neq 0 \\
&\implies p(t) \oplus \ell(t) \in \mathcal{R}_q.
\end{aligned}$$

Now we show that the zero function, 0, is the identity element in $\mathcal{R}_q$: we see that $0 \in \mathcal{R}_q$ since $1 + \mu(t)(0) = 1 \neq 0$, and

$$0 \oplus p(t) = p(t) \oplus 0 = 0 + p(t) + \mu(t)(0)p(t) = p(t).$$

To show that every element in $\mathcal{R}_q$ has an additive inverse in $\mathcal{R}_q$, let $\tilde{p}(t) := \frac{-p(t)}{1+\mu(t)p(t)}$. We have that

$$1 + \mu(t)\tilde{p}(t) = 1 + \frac{-\mu(t)p(t)}{1 + \mu(t)p(t)} = \frac{1}{1 + \mu(t)p(t)} \neq 0.$$

Thus $\tilde{p}(t) \in \mathcal{R}_q$, and

$$\begin{aligned}
p(t) \oplus \tilde{p}(t) &= p(t) \oplus \frac{-p(t)}{1 + \mu(t)p(t)} \\
&= p(t) + \frac{-p(t)}{1 + \mu(t)p(t)} + \frac{-\mu(t)p^2(t)}{1 + \mu(t)p(t)} \\
&= \frac{p(t)(1 + \mu(t)p(t)) - p(t) - \mu(t)p^2(t)}{1 + \mu(t)p(t)} \\
&= 0,
\end{aligned}$$

showing that $\tilde{p}(t)$ is the additive inverse of $p(t)$. Thus, we have that $\mathcal{R}_q$, $\oplus$ is an Abelian group. $\square$

Definition 3.32. We define $\ominus$ by

$$\ominus p(t) = \frac{-p(t)}{1+\mu(t)p(t)},$$

the additive inverse of $p(t)$ in $\mathcal{R}_q$. As a binary operator, we define $\ominus$ by

$$\begin{aligned} p(t) \ominus \ell(t) &:= p(t) \oplus [\ominus \ell(t)] \\ &= p(t) - \frac{\ell(t)}{1+\mu(t)\ell(t)} - \frac{\mu(t)p(t)\ell(t)}{1+\mu(t)\ell(t)} \\ &= \frac{p(t)-\ell(t)}{1+\mu(t)\ell(t)}. \end{aligned}$$

Theorem 3.33. *Assume $p(t), \ell(t) \in \mathcal{R}_q$ and $t, s \in aq^{\mathbb{N}_0}$. Then*

(i) $e_0(t,a) = 1$ *and* $e_p(t,t) = 1$*;*

(ii) $e_p(t,a) \neq 0$*, for any* $t \in aq^{\mathbb{N}_0}$*;*

(iii) *if* $1+\mu(t)p(t) > 0$*, then* $e_p(t,a) > 0$*;*

(iv) $e_p(\sigma(t),a) = [1+\mu(t)p(t)]e_p(t,a)$;

(v) $\Delta_q e_p(t,a) = pe_p(t,a)$;

(vi) $e_p(t,s)e_p(s,a) = e_p(t,a)$;

(vii) $e_{\ominus p}(t,a) = \frac{1}{e_p(t,a)}$;

(viii) $\frac{e_p(t,a)}{e_\ell(t,a)} = e_{p\ominus\ell}(t,a)$;

(ix) *for* p*,* $\ell \in \mathcal{R}_q^c$ *and* $|p| < |\ell|$, $\lim_{t\to\infty} e_{p\ominus\ell}(t,a) = 0$.

Proof.

(i) $e_0(t,a) = \prod_{s=a}^{t/q}(1+0) = 1$, and $e_p(t,t) = \prod_{s=t}^{t/q}(1+\mu(s)p(s)) = 1$ by our convention on products.

(ii) By way of contradiction, assume that there is a $t \in aq^{\mathbb{N}_0}$ such that $e_p(t,a) = 0$. Since, by definition, $e_p(t,a) = \prod_{s=a}^{t/q}(1+\mu(s)p(s))$, then there must be an s such that $1+\mu(s)p(s) = 0$. However, this contradicts the fact that $p(t) \in \mathcal{R}_q$. Thus we have that $e_p(t,a) \neq 0$.

(iii) Assume that $1+\mu(t)p(t) > 0$ for all t. Then, $\prod_{s=a}^{t/q}(1+\mu(s)p(s)) > 0$. Therefore, we conclude that $e_p(t,a) > 0$.

(iv) By direct calculation,

$$\begin{aligned}
e_p(\sigma(t),a) &= \prod_{s=a}^{t}(1+\mu(s)p(s)) \\
&= [1+\mu(t)p(t)]\prod_{s=a}^{t/q}(1+\mu(s)p(s)) \\
&= [1+\mu(t)p(t)]e_p(t,a).
\end{aligned}$$

(v) By definition of the delta difference and property (iv) above,

$$\begin{aligned}
\Delta_q e_p(t,a) &= \frac{e_p(\sigma(t),a) - e_p(t,a)}{\mu(t)} \\
&= \frac{(1+\mu(t)p(t))e_p(t,a) - e_p(t,a)}{\mu(t))} \\
&= \frac{e_p(t,a)(1+p(t)\mu(t)-1)}{\mu(t)} \\
&= p(t)e_p(t,a).
\end{aligned}$$

(vi) By definition of our exponential function,

$$
\begin{aligned}
e_p(t,s)e_p(s,a) &= \prod_{r=a}^{s/q}(1+\mu(r)p(r))\prod_{k=s}^{t/q}(1+\mu(k)p(k)) \\
&= \prod_{r=a}^{t/q}(1+\mu(r)p(r)) \\
&= e_p(t,a).
\end{aligned}
$$

(vii) By definition of $\ominus$,

$$
\begin{aligned}
e_{\ominus p}(t,a) &= \prod_{s=a}^{t/q}\left(1+\frac{-p(s)}{1+\mu(s)p(s)}\mu(s)\right) \\
&= \prod_{s=a}^{t/q}\frac{1}{1+\mu(s)p(s)} \\
&= \frac{1}{\prod_{s=a}^{t/q}(1+\mu(s)p(s))} \\
&= \frac{1}{e_p(t,a)}.
\end{aligned}
$$

(viii) By direct calculation,

$$\begin{aligned}
\frac{e_p(t,a)}{e_\ell(t,a)} &= \frac{\prod_{s=a}^{t/q}(1+\mu(s)p(s))}{\prod_{s=a}^{t/q}(1+\mu(s)\ell(s))} \\
&= \prod_{s=a}^{t/q}\frac{(1+\mu(s)p(s))}{(1+\mu(s)\ell(s))} \\
&= \prod_{s=a}^{t/q}\frac{(1+\mu(s)p(s))+\mu(s)\ell(s)-\mu(s)\ell(s)}{1+\mu(s)\ell(s)} \\
&= \prod_{s=a}^{t/q}[1+\frac{\mu(s)p(s)-\mu(s)\ell(s)}{1+\mu(s)\ell(s)}] \\
&= \prod_{s=a}^{t/q}[1+\mu(s)\frac{p(s)-\ell(s)}{1+\mu(s)\ell(s)}] \\
&= \prod_{s=a}^{t/q}[1+\mu(s)(p(s)\ominus\ell(s))] \\
&= e_{p\ominus\ell}(t,a).
\end{aligned}$$

(ix) Assume p, $\ell \in \mathcal{R}_q^c$ such that $|p| < |\ell|$. Then

$$\lim_{t\to\infty}|e_{p\ominus\ell}(t,a)| = \lim_{t\to\infty}\Big|\frac{\prod_{s=a}^{t/q}(1+\mu(s)p)}{\prod_{s=a}^{t/q}(1+\mu(s)\ell)}\Big| = \prod_{s=a}^{\infty}\Big|\frac{(1+\mu(s)p)}{(1+\mu(s)\ell)}\Big|.$$

Consider the limit

$$\lim_{t\to\infty}\Big|\frac{(1+\mu(t)p)}{(1+\mu(t)\ell)}\Big| = \frac{|p|}{|\ell|} < 1.$$

We can then assert that there is a t_0 such that for all $t \geq t_0$

$$\Big|\frac{(1+\mu(t)p)}{(1+\mu(t)\ell)}\Big| \leq \delta_0,$$

for some constant δ_0 such that $\frac{|p|}{|\ell|} \leq \delta_0 < 1$.

$$\begin{aligned}
\Longrightarrow 0 &\leq \lim_{t\to\infty} |e_{p\ominus\ell}(t,a)| \\
&= \prod_{s=a}^{t_0/q} \Big|\frac{(1+\mu(s)p)}{(1+\mu(s)\ell)}\Big| \prod_{r=t_0}^{\infty} \Big|\frac{(1+\mu(r)p)}{(1+\mu(r)\ell)}\Big| \\
&\leq \prod_{s=a}^{t_0/q} \Big|\frac{(1+\mu(s)p)}{(1+\mu(s)\ell)}\Big| \prod_{r=t_0}^{\infty} \delta_0 \\
&= 0.
\end{aligned}$$

$\square$

To define a circle dot multiplication, we consider using circle plus addition, $\oplus$, multiple times on an element from our set. From this, one can see a pattern that motivates the definition for circle dot multiplication.

$$\begin{aligned}
\overbrace{p \oplus p \oplus \cdots \oplus p}^{\text{n terms}} &= \sum_{k=1}^{n} \binom{n}{k} \mu^{k-1}(t) p^k(t) \\
&= \frac{1}{\mu(t)} \sum_{k=1}^{n} \binom{n}{k} \mu^{k}(t) p^k(t) \\
&= \frac{\sum_{k=0}^{n} \binom{n}{k} \mu^{k}(t) p^k(t) - 1}{\mu(t)} \\
&= \frac{(1+\mu(t)p(t))^n - 1}{\mu(t)}.
\end{aligned}$$

Definition 3.34. We define a circle dot multiplication, $\odot$, by

$$\alpha \odot p := \frac{(1+\mu(t)p(t))^{\alpha} - 1}{\mu(t)},$$

for $\alpha \in \mathbb{R}$.

Theorem 3.35. *If $\alpha \in \mathbb{R}$ and $p(t) \in \mathcal{R}_q$, then*

$$e_p^{\alpha}(t,a) = e_{\alpha \odot p}(t,a).$$

Proof. Assume that $\alpha \in \mathbb{R}$ and $p(t) \in \mathcal{R}_q$.

$$\begin{aligned} e_p^{\alpha}(t,a) &= \left(\prod_{s=a}^{t/q} (1+\mu(s)p(s)) \right)^{\alpha} \\ &= \prod_{s=a}^{t/q} (1+\mu(s)p(s))^{\alpha} \\ &= \prod_{s=a}^{t/q} \left(1+\mu(s) \frac{(1+\mu(s)p(s))^{\alpha}-1}{\mu(s)} \right) \\ &= \prod_{s=a}^{t/q} (1+\mu(s)(\alpha \odot p)) \\ &= e_{\alpha \odot p}(t,a). \end{aligned}$$

□

Lemma 3.36. *If $p(t), \ell(t) \in \mathcal{R}_q$ and $e_p(t,a) = e_\ell(t,a)$, then $p(t) = \ell(t)$.*

Proof. We assume that $p(t), \ell(t) \in \mathcal{R}_q$ and $e_p(t,a) = e_\ell(t,a)$. Thus, we have that $\Delta_q e_p(t,a) = \Delta_q e_\ell(t,a)$, which implies that $p(t)e_p(t,a) = \ell(t)e_q(t,a)$. Dividing by $e_p(t,a) = e_\ell(t,a)$, we get that $p(t) = \ell(t)$. □

Definition 3.37. The set of positively regressive functions, $\mathcal{R}_q^+$, is defined by

$$\mathcal{R}_q^+ := \{p(t) : 1+\mu(t)p(t) > 0\}.$$

Notice that $\mathcal{R}_q^+$ is a sub-group of $\mathcal{R}_q$. The details of this proof are left to the reader.

Theorem 3.38. *The set $\mathcal{R}_q^+$ with $\oplus$ and $\odot$ is a vector space.*

Proof. Since we have already proved that $\mathcal{R}_q^+$ with $\oplus$ is an Abelian group, it only remains to show the following, where $\alpha, \beta \in \mathbb{R}$ and $p(t) \in \mathcal{R}_q^+$.

First, we show associativity of scalar multiplication.

$$\begin{aligned} e_{\alpha\odot(\beta\odot p)}(t,a) &= [e_{\beta\odot p}(t,a)]^{\alpha} \\ &= [e_p(t,a)^{\beta}]^{\alpha} \\ &= [e_p(t,a)]^{\alpha\beta} \\ &= e_{\alpha\beta\odot p}(t,a). \end{aligned}$$

Therefore, by the previous lemma, $\alpha \odot (\beta \odot p) = (\alpha\beta) \odot p$.

Next, we show the distributivity of scalar sums.

$$\begin{aligned} e_{(r+s)\odot p}(t,a) &= [e_p(t,a)]^{r+s} \\ &= [e_p(t,a)]^{r}[e_p(t,a)]^{s} \\ &= e_{r\odot p}(t,a)e_{s\odot p}(t,a) \\ &= e_{(r\odot p)\oplus(s\odot p)}(t,a). \end{aligned}$$

Therefore, by the previous lemma, $(r+s) \odot p(t) = (r \odot p(t)) \oplus (s \odot p(t))$.

Now, we show a distributive property involving $\oplus$. Let $r \in \mathbb{R}$ and $p(t), \ell(t) \in \mathcal{R}_q$.

$$\begin{aligned} e_{r\odot(p\oplus\ell)}(t,a) &= [e_{p\oplus\ell}(t,a)]^r \\ &= [e_p(t,a)e_\ell(t,a)]^r \\ &= [e_p(t,a)]^r[e_q(t,a)]^r \\ &= e_{r\odot p}(t,a)e_{r\odot\ell}(t,a) \\ &= e_{(r\odot p)\oplus(r\odot\ell)}(t,a). \end{aligned}$$

Therefore, by the previous lemma,

$$r \odot (p(t) \oplus \ell(t)) = (r \odot p(t)) \oplus (r \odot \ell(t)).$$

Finally, we show that we have a scalar multiplicative identity.

$$1 \odot p = \frac{1 + \mu(t)p(t) - 1}{\mu(t)} = \frac{\mu(t)p(t)}{\mu(t)} = p(t).$$

Thus, we have that the constant function 1 is our multiplicative identity. □

Definition 3.39. For $\pm p(t) \in \mathcal{R}_q$,the generalized hyperbolic sine and cosine functions are defined as follows:

$$\begin{aligned} \cosh_p(t,a) &:= \frac{e_p(t,a) + e_{-p}(t,a)}{2}, \\ \sinh_p(t,a) &:= \frac{e_p(t,a) - e_{-p}(t,a)}{2}. \end{aligned}$$

Following these definitions, we arrive at the following theorem concerning some properties of the generalized hyperbolic sine and cosine functions on $aq^{\mathbb{N}_0}$.

Theorem 3.40. *Assume $\pm p(t) \in \mathcal{R}_q, t \in aq^{\mathbb{N}_0}$. Then* *(i)* $\cosh_p^2(t,a) - \sinh_p^2(t,a) = \prod_{s=a}^{t/q}(1-\mu^2(s)p^2(s))$;

(ii) $\Delta_q \cosh_p(t,a) = p(t)\sinh_p(t,a)$;

(iii) $\Delta_q \sinh_p(t,a) = p(t)\cosh_p(t,a)$;

Proof. We can see that (i) holds by direct calculation:

$$\begin{aligned}
\cosh_p^2(t,a) - \sinh_p^2(t,a) &= \frac{(e_p(t,a)+e_{-p}(t,a))^2 - (e_p(t,a)-e_{-p}(t,a))^2}{4} \\
&= e_p(t,a)e_{-p}(t,a) \\
&= e_{p\oplus -p}(t,a) \\
&= e_{-\mu p^2}(t,a) \\
&= \prod_{s=a}^{t/q}[1+\mu(s)(-\mu(s)p^2(s))] \\
&= \prod_{s=a}^{t/q}[1-\mu^2(s)p^2(s)].
\end{aligned}$$

Similarly, (ii) holds by direct calculation:

$$\begin{aligned}
\Delta_q \cosh_p(t,a) &= \frac{1}{2}\Delta_q e_p(t,a) + \frac{1}{2}\Delta_q e_{-p}(t,a) \\
&= \frac{1}{2}p(t)e_p(t,a) - \frac{1}{2}p(t)e_{-p}(t,a) \\
&= p(t)\frac{e_p(t,a)-e_{-p}(t,a)}{2} \\
&= p(t)\sinh_p(t,a).
\end{aligned}$$

The proof of (iii) is similar. □

Next, we define the generalized sine and cosine functions.

Definition 3.41. For $\pm ip(t) \in \mathcal{R}_q, t \in aq^{\mathbb{N}_0}$,

$$\cos_p(t,a) = \frac{e_{ip}(t,a) + e_{-ip}(t,a)}{2},$$
$$\sin_p(t,a) = \frac{e_{ip}(t,a) - e_{-ip}(t,a)}{2i}.$$

The following theorem relates the generalized hyperbolic trigonometric functions and the generalized trigonometric functions.

Theorem 3.42. *Assume* $\pm p(t) \in \mathcal{R}_q, t \in aq^{\mathbb{N}_0}$. *Then*

(i) $\sin_{ip}(t,a) = i\sinh_p(t,a)$;

(ii) $\cos_{ip}(t,a) = \cosh_p(t,a,)$.

Proof.

(i) By definition,

$$\begin{aligned}\sin_{ip}(t,a) &= \frac{1}{2i}(e_{i^2p}(t,a) - e_{-i^2p}(t,a)) \\ &= \frac{i(e_p(t,a) - e_{-p}(t,a))}{2} \\ &= i\sinh_p(t,a).\end{aligned}$$

(ii) Again by definition,

$$\begin{aligned}\cos_{ip}(t,a) &= \frac{e_{i^2p}(t,a) + e_{-i^2p}(t,a)}{2} \\ &= \frac{e_{-p}(t,a) + e_p(t,a)}{2} \\ &= \cosh_p(t,a).\end{aligned}$$

□

The following theorem gives various properties of the generalized sine and cosine functions.

Theorem 3.43. *Assume $\mu(t)p(t) \neq \pm i$. Then, for $t \in aq^{\mathbb{N}_0}$,*

(i) $\cos_p^2(t,a) + \sin_p^2(t,a) = \prod_{s=a}^{t/q}(1+\mu^2(s)p^2(s))$;

(ii) $\Delta_q \cos_p(t,a) = -p(t)\sin_p(t,a)$;

(iii) $\Delta_q \sin_p(t,a) = p(t)\cos_p(t,a)$;

Proof. By direct calculation, (i) holds:

$$\begin{aligned}
\cos_p^2(t,a) + \sin_p^2(t,a) &= \frac{(e_{ip}(t,a)+e_{-ip}(t,a))^2 - (e_{ip}(t,a)-e_{-ip}(t,a))^2}{4} \\
&= e_{ip}(t,a)e_{-ip}(t,a) \\
&= e_{ip\oplus -ip}(t,a) \\
&= e_{\mu p^2}(t,a) \\
&= \prod_{s=a}^{t/q}[1+\mu(s)(\mu(s)p^2(s))] \\
&= \prod_{s=a}^{t/q}[1+\mu^2(s)p^2(s)].
\end{aligned}$$

Again, by direct calculation, (ii) holds:

$$\begin{aligned}
\Delta_q \cos_p(t,a) &= \frac{1}{2}\Delta_q e_{ip}(t,a) + \frac{1}{2}\Delta_q e_{-ip}(t,a) \\
&= \frac{1}{2} ip(t)e_p(t,a) - \frac{1}{2} ip(t)e_{-p}(t,a) \\
&= -p(t)\frac{e_{ip}(t,a) - e_{-ip}(t,a)}{2i} \\
&= -p(t)\sin_p(t,a).
\end{aligned}$$

The proof of (iii) is similar. $\square$

3.4 The q-Laplace Transform

Recall that the traditional Laplace transform is

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

We define the q-Laplace transform in a similar manner using our definition of the exponential.

Definition 3.44. Assume $f : aq^{\mathbb{N}_0} \to \mathbb{R}$ and $t_0 \in aq^{\mathbb{N}_0}$. Then the q-Laplace transform of f is defined by

$$\mathcal{L}_{t_0}\{f\}(s) = F_{t_0}(s) := \int_{t_0}^\infty e_{\ominus s}(\sigma(t), a) f(t)\, \Delta_q t$$

for $s \in \mathbb{C} \setminus \left\{ -\dfrac{1}{\mu(aq^n)} : n \in \mathbb{N}_0 \right\}$ such that this improper integral converges.

If we suppose $t_0 = aq^m$ for some $m \in \mathbb{N}_0$, we can also write the q-Laplace transform as a sum, using the definition of the integral

$$\mathcal{L}_{t_0}\{f\}(s) = \sum_{n=m}^\infty \frac{f(aq^n)\mu(aq^n)}{\prod_{k=0}^n (1 + \mu(aq^k)s)}.$$

For more on whole-ordered and fractional-ordered Laplace transforms on discrete domains, see [7] and [19]. To help identify functions whose Laplace transforms exist, we introduce the next definition.

Definition 3.45. A function $f : aq^{\mathbb{N}_0} \to \mathbb{R}$ is of exponential order $r > 0, r \in \mathbb{R}$, if for some constant $A > 0$

$$|f(aq^n)| \leq A(\mu(a) q^{\frac{n-1}{2}})^n r^n$$

for all sufficiently large $n \in \mathbb{N}_0$.

Theorem 3.46. (*Existence of q-Laplace Tranform) If $f : aq^{\mathbb{N}_0} \to \mathbb{R}$ is of exponential order $r > 0$, then $\mathcal{L}_a \{f\}(s)$ exists for $|s| > r$.*

Proof. Assume $f(t)$ is of exponential order r. Then there is a constant $A > 0$ and a $t_0 = aq^N \in aq^{\mathbb{N}_0}$ such that $|f(t)| \leq A(\mu(a)q^{\frac{n-1}{2}})^n r^n$ for all $t \in aq^{\mathbb{N}_N}$. We now show that

$$\mathcal{L}_a \{f\}(s) = \int_a^{\infty} e_{\ominus s}(\sigma(t), a) f(t) \Delta_q t = \sum_{n=0}^{\infty} \frac{f(aq^n)\mu(aq^n)}{\prod_{k=0}^{n}(1 + \mu(aq^k)s)}$$

converges for $|s| > r$. Since

$$\sum_{n=N}^{\infty} \frac{f(aq^n)\mu(aq^n)}{\prod_{k=0}^{n}(1 + \mu(aq^k)s)} \leq \sum_{n=N}^{\infty} \frac{A(\mu(a)q^{\frac{n-1}{2}})^n r^n \mu(aq^n)}{\prod_{k=0}^{n}|1 + \mu(aq^k)s|},$$

and noting $\mu(aq^{n+1}) = aq^{n+1}(q-1) = \mu(a)q^{n+1}$, consider

$$\begin{aligned}
\lim_{n\to\infty} \frac{\dfrac{A(\mu(a)q^{\frac{n}{2}})^{n+1} r^{n+1} \mu(aq^{n+1})}{\prod_{k=0}^{n+1}|1 + \mu(aq^k)s|}}{\dfrac{A(\mu(a)q^{\frac{n-1}{2}})^n r^n \mu(aq^n)}{\prod_{k=0}^{n}|1 + \mu(aq^k)s|}} &= \mu(a) r \lim_{n\to\infty} \frac{q^{n+1}}{|1 + \mu(a)q^{n+1}s|} \\
&= \mu(a) r \lim_{n\to\infty} \frac{1}{|\frac{1}{q^{n+1}} + \mu(a)s|} \\
&= \frac{\mu(a) r}{\mu(a)|s|} = \frac{r}{|s|}.
\end{aligned}$$

Therefore, by the ratio test, for $|s| > r$,

$$\sum_{n=N}^{\infty} \frac{A(\mu(a)q^{\frac{n-1}{2}})^n r^n \mu(aq^n)}{\prod_{k=0}^{n}|1 + \mu(aq^k)s|}$$

converges. It follows that $\mathcal{L}_a \{f\}(s)$ converges absolutely for $|s| > r$. □

Remark 3.47. Notice from above that for $t = aq^n$, $\sum_{n=0}^{\infty} \frac{\mu(aq^n)f(aq^n)}{\prod_{k=0}^{n}(1+\mu(aq^k)s)}$ converges for $|s| > r$, which implies that $\sum_{n=0}^{\infty} e_{\ominus s}(\sigma(aq^n), a) f(aq^n)\mu(aq^n)$ converges for $|s| > r$. Therefore,

$$\lim_{t\to\infty} e_{\ominus s}(\sigma(t), a) f(t)\mu(t) = 0$$

by the n^{th} term test. Since $\mu(t) = t(q-1) > 1$ for large t, we have that

$$\lim_{t\to\infty} e_{\ominus s}(\sigma(t), a) f(t) = 0.$$

Theorem 3.48. (*Linearity*) *Assume $f, g : aq^{\mathbb{N}_0} \to \mathbb{R}$ are of exponential order $r > 0$. Then for $|s| > r$ and $\alpha, \beta \in \mathbb{C}$,*

$$\mathcal{L}_a \{\alpha f + \beta g\}(s) = \alpha \mathcal{L}_a \{f\}(s) + \beta \mathcal{L}_a \{g\}(s).$$

Proof. Let $f, g : aq^{\mathbb{N}_0} \to \mathbb{R}$ be of exponential order $r > 0$. Then for $|s| > r$ and $\alpha, \beta \in \mathbb{C}$ we have

$$\begin{aligned}
\mathcal{L}_a \{\alpha f + \beta g\}(s) &= \int_a^\infty e_{\ominus s}(\sigma(t), a)(\alpha f + \beta g)\,\Delta_q t \\
&= \int_a^\infty \big(e_{\ominus s}(\sigma(t), a)\alpha f + e_{\ominus s}(\sigma(t), a)\beta g\big)\,\Delta_q t \\
&= \int_a^\infty e_{\ominus s}(\sigma(t), a)\alpha f\,\Delta_q t + \int_a^\infty e_{\ominus s}(\sigma(t), a)\beta g\,\Delta_q t \\
&= \alpha \int_a^\infty e_{\ominus s}(\sigma(t), a) f\,\Delta_q t + \beta \int_a^\infty e_{\ominus s}(\sigma(t), a) g\,\Delta_q t \\
&= \alpha \mathcal{L}_a \{f\}(s) + \beta \mathcal{L}_a \{g\}(s).
\end{aligned}$$

□

Remark 3.49. Clearly $\mathcal{L}_a\{0\}(s) = 0$, and, according to [18], if $\mathcal{L}_a\{f\}(s) = 0$, then

$f \equiv 0$. Therefore, by the linearity of the q-Laplace transform, we can show that the q-Laplace transform is unique. To see this, suppose we have functions f, g such that $\mathcal{L}_a\{f\}(s) = \mathcal{L}_a\{g\}(s)$. Then

$$\begin{aligned} 0 &= \mathcal{L}_a\{f\}(s) - \mathcal{L}_a\{g\}(s) = \mathcal{L}_a\{f-g\}(s) = \mathcal{L}_a\{0\}(s) \\ &\implies f \equiv g. \end{aligned}$$

Theorem 3.50. *Let $m \in \mathbb{N}_0$ be given, and suppose $f : aq^{N_0} \to \mathbb{R}$ is of exponential order $r > 0$. Then for $|s| > r$,*

$$\mathcal{L}_{aq^m}\{f\}(s) = \mathcal{L}_a\{f\}(s) - \sum_{n=0}^{m-1} \frac{\mu(aq^n)f(aq^n)}{\prod_{k=0}^{n}(1+\mu(aq^k)s)}.$$

Proof. Assume $m \in \mathbb{N}_0$ and f is of exponential order $r > 0$. Then for $|s| > r$,

$$\begin{aligned} \mathcal{L}_{aq^m}\{f\}(s) &= \int_{aq^m}^{\infty} e_{\ominus s}(\sigma(t), a) f(t) \Delta_q t \\ &= \sum_{n=m}^{\infty} \frac{\mu(aq^n)f(aq^n)}{\prod_{k=0}^{n}(1+\mu(aq^k)s)} \\ &= \sum_{n=0}^{\infty} \frac{\mu(aq^n)f(aq^n)}{\prod_{k=0}^{n}(1+\mu(aq^k)s)} - \sum_{n=0}^{m-1} \frac{\mu(aq^n)f(aq^n)}{\prod_{k=0}^{n}(1+\mu(aq^k)s)} \\ &= \mathcal{L}_a\{f\}(s) - \sum_{n=0}^{m-1} \frac{\mu(aq^n)f(aq^n)}{\prod_{k=0}^{n}(1+\mu(aq^k)s)}. \end{aligned}$$

□

Theorem 3.51. *The function $f(t) = e_p(t,a)$ for $p \in \mathcal{R}_q^c$ is of exponential order $r = |p| + \varepsilon$ for all $\varepsilon > 0$.*

Proof. For $t = aq^n, n \in \mathbb{N}_0$, we have

$$|e_p(t,a)| = |e_p(aq^n,a)| = \left|\prod_{k=0}^{n-1}(1+\mu(aq^k)p)\right| = \prod_{k=0}^{n-1}|1+\mu(aq^k)p|.$$

By applying the triangle inequality to each term in the product we obtain

$$|e_p(t,a)| \leq \prod_{k=0}^{n-1}(1+\mu(aq^k)|p|).$$

Let $\varepsilon > 0$ be given. Then there exists N such that for all $n \geq N$ we have $\mu(aq^n)\varepsilon > 1$, which implies

$$\mu(aq^N)(|p|+\varepsilon) \geq \mu(aq^k)|p| + 1$$

for all $k \leq N$, and

$$\mu(aq^k)(|p|+\varepsilon) \geq \mu(aq^k)|p| + 1$$

for all $k \in \mathbb{N}_N$. So, for n sufficiently large, we have

$$\begin{aligned}
|e_p(t,a)| &\leq \prod_{k=0}^{n-1}(1+\mu(aq^k)|p|) = \prod_{k=0}^{N}(1+\mu(aq^k)|p|)\prod_{s=N+1}^{n-1}(1+\mu(aq^s)|p|)\\
&< (\mu(aq^N)(|p|+\varepsilon))^{N+1}\prod_{s=N+1}^{n-1}\mu(aq^s)(|p|+\varepsilon)\\
&< (a(q-1)q^N(|p|+\varepsilon))^{N+1}(a(q-1)(|p|+\varepsilon))^{n-N-1}q^{\frac{n(n-1)}{2}}q^{-\frac{N(N+1)}{2}}\\
&= a^n(q-1)^n(|p|+\varepsilon)^n q^{\frac{N(N+1)}{2}}q^{\frac{n(n-1)}{2}}\\
&= q^{\frac{N(N+1)}{2}}(\mu(a)q^{\frac{n-1}{2}})^n(|p|+\varepsilon)^n.
\end{aligned}$$

Since this holds for an arbitrary $\varepsilon > 0$, it holds for any $\varepsilon > 0$. Since N is a finite number, we have the exponential order of $e_p(t,a)$ as $r = |p| + \varepsilon$ for all $\varepsilon > 0$. $\square$

Theorem 3.52. *For $p \in \mathcal{R}_q^c$ and $|s| > |p|$,*

$$\mathcal{L}_a \{e_p(t,a)\}(s) = \frac{1}{s-p}$$

Proof. We previously defined the q-exponential function $e_p(t,a)$ to be

$$e_p(t,a) = \prod_{s=a}^{t/q}(1+\mu(s)p(s)).$$

Let $\varepsilon > 0$ be given. Since, for any $\varepsilon > 0$, $e_p(t,a)$ is of exponential order $|p| + \varepsilon$, we have for $|s| > |p| + \varepsilon$

$$\begin{aligned}
\mathcal{L}_a\{e_p(t,a)\}(s) &= \int_a^\infty e_{\ominus s}(\sigma(t),a)e_p(t,a)\Delta_q t \\
&= \int_a^\infty \frac{e_p(t,a)}{\prod_{r=a}^{t}(1+\mu(r)s)}\,\Delta_q t \\
&= \int_a^\infty \frac{e_p(t,a)}{(1+\mu(t)s)\prod_{r=a}^{t/q}(1+\mu(r)s)}\,\Delta_q t \\
&= \int_a^\infty \frac{e_{\ominus s}(t,a)e_p(t,a)}{1+\mu(t)s}\,\Delta_q t \\
&= \int_a^\infty \frac{e_{p\ominus s}(t,a)}{1+\mu(t)s}\,\Delta_q t \\
&= \frac{1}{p-s}\int_a^\infty (p\ominus s)\ e_{p\ominus s}(t,a)\,\Delta_q t \\
&= \frac{1}{p-s}\int_a^\infty \Delta_q e_{p\ominus s}(t,a)\,\Delta_q t \\
&= \frac{1}{p-s} e_{p\ominus s}(t,a)\Big|_a^\infty \\
&= \frac{1}{p-s}(0-1) \\
&= \frac{1}{s-p}.
\end{aligned}$$

Hence, for $|s| > |p| + \varepsilon$,

$$\mathcal{L}_a\{e_p(t,a)\}(s) = \frac{1}{s-p}.$$

Since $\varepsilon > 0$ is arbitrary, then this holds for all $|s| > |p|$. $\square$

Remark 3.53. Since $e_p(t,a) \equiv 1$ when $p = 0$, the q-Laplace transform of a constant function follows from the above theorem and the linearity of the q-Laplace transform:

$$\mathcal{L}_a\left\{c\right\}(s) = c\mathcal{L}_a\{e_0(t,a)\} = \frac{c}{s},$$

where $|s| > 0$.

Theorem 3.54. *The function $f(t) = h_m(t,a)$ for $m \in \mathbb{N}_0$ is of exponential order $r = \varepsilon$ for all $\varepsilon > 0$.*

Proof. For $t = aq^n \in aq^{\mathbb{N}_0}$,

$$|h_m(t,a)| = \frac{(t-a)^{\underline{m}}}{[m]_q!} \leq (t-a)^{\underline{m}} \leq t^m = (aq^n)^m = a^m(q^m)^n$$

For any fixed $\delta > 0$ and any constant $\alpha > 1$

$$\lim_{n\to\infty} \alpha^{\frac{n^2}{2}}\delta^n = \infty$$

which implies that there exists N such that for all $n > N$ we have

$$\alpha^{n^2}\delta^n > 1.$$

Let $\varepsilon > 0$ be given and take $\delta = (aq^{-\frac{1}{2}}(q-1)\varepsilon q^{-m})$ and $\alpha = q$. Then for all $n > N$

such that the above inequality holds, we have

$$
\begin{aligned}
&q^{\frac{n^2}{2}}(aq^{-\frac{1}{2}}(q-1)\varepsilon q^{-m})^n = \mu^n(a)q^{\frac{(n-1)(n)}{2}}(\varepsilon q^{-m})^n > 1 \\
&\quad \Longrightarrow |t^m| < a^m(q^m)^n\mu^n(a)q^{\frac{(n-1)(n)}{2}}\varepsilon^n(q^{-m})^n = a^m(\mu(a)q^{\frac{(n-1)}{2}})^n\varepsilon^n.
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary, then for all $\varepsilon > 0$, $h_m(t,a)$ is of exponential order $r = \varepsilon$ with $A = a^m$. □

Theorem 3.55. *For* $|s| > 0$

$$
\mathcal{L}_a\{h_n(t,a)\}(s) = \frac{1}{s^{n+1}}.
$$

Proof. For the n^{th} order Taylor monomial, $h_n(t,a)$, for $|s| > \varepsilon$ for all $\varepsilon > 0$ we have, using integration by parts,

$$
\begin{aligned}
\mathcal{L}_a\{h_n(t,a)\}(s) &= \int_a^\infty h_n(t,a)e_{\ominus s}(\sigma(t),a)\Delta_q t \\
&= \frac{1}{\ominus s}h_n(t,a)e_{\ominus s}(\sigma(t),a)\Big|_{t=a}^{\infty} - \int_a^\infty \frac{h_{n-1}(t,a)e_{\ominus s}(\sigma(\sigma(t)),a)}{\ominus s}\Delta_q t \\
&= 0 \; - \int_a^\infty h_{n-1}(t,a)\left(\frac{1}{-s}\frac{1+\mu(\sigma(t))s}{\prod_{r=a}^{\sigma(t)}(1+\mu(r)s)}\right)\Delta_q t \\
&= -\int_a^\infty h_{n-1}(t,a)\left(\frac{1}{-s}\frac{1}{\prod_{r=a}^{t}(1+\mu(r)s)}\right)\Delta_q t \\
&= \frac{1}{s}\int_a^\infty h_{n-1}(t,a)e_{\ominus s}(\sigma(t),a)\,\Delta_q t,
\end{aligned}
$$

where the 0 in the third line of the equation results from an earlier remark. Repeating the integration steps above we can obtain for any $\varepsilon > 0$ and $|s| > \varepsilon$

$$\begin{aligned}\mathcal{L}_a\{h_n(t,a)\}(s) &= \frac{1}{s^n}\int_a^\infty e_{\ominus s}(\sigma(t),a)\,\Delta_q t \\ &= \frac{1}{s^n}\mathcal{L}_a\left\{1\right\}(s) \\ &= \left(\frac{1}{s^n}\right)\left(\frac{1}{s}\right) \\ &= \frac{1}{s^{n+1}}.\end{aligned}$$

Since ε is an arbitrarily small constant greater than 0, we can thus say that this holds for $|s| > 0$. □

Lemma 3.56. *For $t \in aq^{\mathbb{N}_0}$ and $p \in \mathcal{R}_q$,*

$$|e_p(t,a) \pm e_{-p}(t,a)| \leq 2e_{|p|}(t,a).$$

Proof. Since

$$|e_p(t,a) \pm e_{-p}(t,a)| \leq |e_p(t,a)| + |e_{-p}(t,a)|.$$

We can apply the triangle inequality to each exponential in a similar way done in the proof of Theorem 3.51 ($|e_p(t,a)| \leq e_{|p|}(t,a)$) to obtain

$$|e_p(t,a) \pm e_{-p}(t,a)| \leq e_{|p|}(t,a) + e_{|p|}(t,a) = 2e_{|p|}(t,a).$$

□

Theorem 3.57. *For $t \in aq^{\mathbb{N}_0}$, the exponential order of the following functions is $|p| + \varepsilon$ for any $\varepsilon > 0$:*

(i) $\cosh_p(t,a)$*;*

(ii) $\sinh_p(t,a)$*;*

(iii) $\cos_p(t,a)$*;*

(iv) $\sin_p(t,a)$.

Proof. (i) Consider

$$|\cosh_p(t,a)| = \frac{|e_p(t,a) + e_{-p}(t,a)|}{2}.$$

By the previous lemma,

$$|\cosh_p(t,a)| \leq \frac{2e_{|p|}(t,a)}{2} = e_{|p|}(t,a)$$

By steps similar to those in Theorem 4.7, we can conclude

$$|\cosh_p(t,a)| \leq e_{|p|}(t,a) \leq A(\mu(a)q^{\frac{n-1}{2}})^n(|p|+\varepsilon)^n$$

for any fixed $\varepsilon > 0$ and A defined in the same way as in the proof of Theorem 4.7. Therefore, $\cosh_p(t,a)$ is of exponential order $|p|+\varepsilon$.

(ii) Similar to that above,

$$|\sinh_p(t,a)| = \frac{|e_p(t,a) - e_{-p}(t,a)|}{2}.$$

By the previous lemma,

$$|\sinh_p(t,a)| \leq \frac{2e_{|p|}(t,a)}{2} = e_{|p|}(t,a)$$

By steps similar to those in Theorem 4.7, we can conclude

$$|\sinh_p(t,a)| \leq e_{|p|}(t,a) \leq A(\mu(a)q^{\frac{n-1}{2}})^n(|p|+\varepsilon)^n$$

for any fixed $\varepsilon > 0$ and A defined in the same way as in the proof of Theorem 4.7 . Therefore, $\sinh_p(t,a)$ is of exponential order $|p| + \varepsilon$.

(iii) Using an earlier theorem

$$|\cos_p(t,a)| = |\cosh_{ip}(t,a)|.$$

By part (i) of this theorem, $\cos_p(t,a)$ must be of exponential order $|ip| + \varepsilon = |p| + \varepsilon$ for any fixed $\varepsilon > 0$.

(iv) Also using an earlier theorem

$$|\sin_p(t,a)| = \left|\frac{\sinh_{ip}(t,a)}{i}\right| = |\sinh_{ip}(t,a)|.$$

By part (ii) of this theorem, we know that $\sin_p(t,a)$ must be of exponential order $|ip| + \varepsilon = |p| + \varepsilon$ for any fixed $\varepsilon > 0$.

$\square$

Theorem 3.58. *For $t \in aq^{\mathbb{N}_0}$ and $|s| > |p|$,*

(i) $\mathcal{L}_a\{\cosh_p(t,a)\}(s) = \frac{s}{s^2-p^2}$, *for* $\pm p(t) \in \mathcal{R}_q^c$;

(ii) $\mathcal{L}_a\{\sinh_p(t,a)\}(s) = \frac{p}{s^2-p^2}$, *for* $\pm p(t) \in \mathcal{R}_q^c$;

(iii) $\mathcal{L}_a\{\cos_p(t,a)\}(s) = \frac{s}{s^2+p^2}$, *for* $\pm ip(t) \in \mathcal{R}_q^c$;

(iv) $\mathcal{L}_a\{\sin_p(t,a)\}(s) = \frac{p}{s^2+p^2}$, *for* $\pm ip(t) \in \mathcal{R}_q^c$.

Proof.

(i) For $|s| > |p|$,

$$\begin{aligned}
\mathcal{L}_a\{\cosh_p(t,a)\}(s) &= \mathcal{L}_a\Big\{\frac{e_p(t,a)+e_{-p}(t,a)}{2}\Big\}(s) \\
&= \frac{1}{2}\Big(\mathcal{L}_a\{e_p(t,a)\}(s)+\mathcal{L}_a\{e_{-p}(t,a)\}(s)\Big) \\
&= \frac{1}{2}\Big(\frac{1}{s-p}+\frac{1}{s-(-p)}\Big) \\
&= \frac{2s}{2(s-p)(s+p)} \\
&= \frac{s}{s^2-p^2}.
\end{aligned}$$

(ii) For $|s| > |p|$,

$$\begin{aligned}
\mathcal{L}_a\{\sinh_p(t,a)\}(s) &= \mathcal{L}_a\Big\{\frac{e_p(t,a)-e_{-p}(t,a)}{2}\Big\}(s) \\
&= \frac{1}{2}\Big(\mathcal{L}_a\{e_p(t,a)\}(s)-\mathcal{L}_a\{e_{-p}(t,a)\}(s)\Big) \\
&= \frac{1}{2}\Big(\frac{1}{s-p}-\frac{1}{s-(-p)}\Big) \\
&= \frac{2p}{2(s-p)(s+p)} \\
&= \frac{p}{s^2-p^2}.
\end{aligned}$$

(iii) For $|s| > |p|$,

$$\begin{aligned}
\mathcal{L}_a\{\cos_p(t,a)\}(s) &= \mathcal{L}_a\left\{\frac{e_{ip}(t,a)+e_{-ip}(t,a)}{2}\right\}(s) \\
&= \frac{1}{2}\big(\mathcal{L}_a\{e_{ip}(t,a)\}(s) + \mathcal{L}_a\{e_{-ip}(t,a)\}(s)\ \big) \\
&= \frac{1}{2}\left(\Big(\frac{1}{s-ip}\Big) + \Big(\frac{1}{s-(-ip)}\Big)\right) \\
&= \frac{1}{2}\Big(\frac{1}{s-ip} + \frac{1}{s+ip}\Big) \\
&= \frac{2s}{2(s^2-i^2p^2)} \\
&= \frac{s}{s^2+p^2}.
\end{aligned}$$

(iv) For $|s| > |p|$,

$$\begin{aligned}
\mathcal{L}_a\{\sin_p(t,a)\}(s) &= \mathcal{L}_a\Big\{\frac{e_{ip}(t,a)-e_{-ip}(t,a)}{2i}\Big\}(s) \\
&= \frac{1}{2i}\Big(\mathcal{L}_a\{e_{ip}(t,a)\}(s) - \mathcal{L}_a\{e_{-ip}(t,a)\}(s)\Big) \\
&= \frac{1}{2i}\Big(\frac{1}{(s-ip)} - \frac{1}{(s-(-ip))}\Big) \\
&= \frac{1}{2i}\Big(\frac{2ip}{s^2+p^2}\Big) \\
&= \frac{p}{s^2+p^2}.
\end{aligned}$$

□

Lemma 3.59. *If $f : aq^{\mathbb{N}_0} \to \mathbb{R}$ and $f(t)$ is of exponential order $r > 0$, then $\Delta_q f$ is also of exponential order $r > 0$.*

Proof. Since f is of exponential order $r > 0$, then we know for all sufficiently large n, $|f(aq^n)| \leq |A(\mu(a)q^{\frac{n-1}{2}})^n r^n \mu(aq^n)|$ for some constant A. Thus, for sufficiently large

$n \in \mathbb{N}$, we have

$$\begin{aligned}
|\Delta_q f(aq^n)| &= \left|\frac{f(aq^{n+1}) - f(aq^n)}{\mu(aq^n)}\right| = \left|\frac{f(aq^{n+1}) - f(aq^n)}{aq^{n+1} - aq^n}\right| = \left|\frac{f(aq^{n+1}) - f(aq^n)}{aq^n(q-1)}\right| \\
&\leq \frac{|f(aq^{n+1})| + |f(aq^n)|}{aq^n(q-1)} \\
&\leq \frac{A(\mu(a)q^{\frac{n}{2}})^{n+1}r^{n+1} + A(\mu(a)q^{\frac{n-1}{2}})^n r^n}{aq^n(q-1)} \\
&= \frac{A(\mu(a))^{n+1}q^{\frac{n(n+1)}{2}}r^{n+1} + A(\mu(a))^n q^{\frac{(n-1)n}{2}}r^n}{aq^n(q-1)} \\
&= \frac{A(\mu(a))^n q^{\frac{(n-1)n}{2}}r^n[\mu(a)q^n r + 1]}{aq^n(q-1)} \\
&= A(\mu(a)q^{\frac{n-1}{2}})^n r^n \left[\frac{\mu(a)r}{a(q-1)} + \frac{1}{aq^n(q-1)}\right] \\
&\leq A(\mu(a)q^{\frac{n-1}{2}})^n r^n \left[\frac{\mu(a)r}{a(q-1)} + 1\right] \\
&= B(\mu(a)q^{\frac{n-1}{2}})^n r^n,
\end{aligned}$$

where $B := A\left[\frac{\mu(a)r}{a(q-1)} + 1\right]$. So by definition, $\Delta_q f$ is of exponential order $r > 0$. □

Remark 3.60. By inductively applying the previous lemma, one can conclude that if f is of exponential order $r > 0$, then $\Delta_q^n f$ is of exponential order $r > 0$ for all $n \in \mathbb{N}$.

Theorem 3.61. *If $f : aq^{\mathbb{N}_0} \to \mathbb{R}$ and $f(t)$ is of exponential order $r > 0$, then for $|s| > r$*

$$\mathcal{L}_a\{\Delta_q^n f\}(s) = s^n F_a(s) - \sum_{k=0}^{n-1} s^{n-1-k}\Delta_q^k f(a),$$

where $n \in \mathbb{N}$.

Proof. If $n = 1$ then we get the following for $|s| > r$.

$$\mathcal{L}_a\{\Delta_q^1 f\}(s) = \int_a^\infty e_{\ominus s}(\sigma(t), a)\Delta_q f(t)\,\Delta_q t.$$

Using integration by parts (ii) (and an earlier remark) we can rewrite the integral as

$$\begin{aligned}\mathcal{L}_a\{\Delta_q^1 f\}(s) &= e_{\ominus s}(t,a)f(t)\Big|_a^\infty - \int_a^\infty \ominus s e_{\ominus s}(t,a)f(t)\,\Delta_q t\\ &= -f(a) + s\int_a^\infty e_{\ominus s}(\sigma(t),a)f(t)\,\Delta_q t\\ &= sF_a(s) - f(a).\end{aligned}$$

Now assume that the theorem is true for some $k \in \mathbb{N}$. By the previous lemma, note that $\Delta_q^k f$ is of exponential order $r > 0$. So consider

$$\begin{aligned}\mathcal{L}_a\{\Delta_q^{k+1} f\}(s) &= \mathcal{L}_a\{\Delta_q \Delta_q^k f\}(s)\\ &= s\mathcal{L}_a\{\Delta_q^k f\}(s) - \Delta_q^k f(a)\\ &= s\Big(s^k F_a(s) - \sum_{r=0}^{k-1} s^{k-1-r}\Delta_q^r f(a)\Big) - \Delta_q^k f(a)\\ &= s^{k+1}F_a(s) - \sum_{r=0}^{k-1} s^{k-r}\Delta_q^r f(a) - \Delta_q^k f(a)\\ &= s^{k+1}F_a(s) - \sum_{r=0}^{k} s^{k-r}\Delta_q^r f(a).\end{aligned}$$

Therefore, inductively, the theorem holds for any $n \in \mathbb{N}$. $\square$

Theorem 3.62. *Assume $\alpha, \beta \in \mathcal{R}_q^c$, $\pm\frac{\beta}{1+\alpha\mu(t)} \in \mathcal{R}_q$. Then, for $|s| > \max\{|\alpha+\beta|, |\alpha-\beta|\}$,*

(i) $\mathcal{L}_a\Big\{e_\alpha(t,a)\cosh_{\frac{\beta}{1+\alpha\mu(t)}}(t,a)\Big\}(s) = \frac{s-\alpha}{(s-\alpha)^2-\beta^2}$;

(ii) $\mathcal{L}_a\Big\{e_\alpha(t,a)\sinh_{\frac{\beta}{1+\alpha\mu(t)}}(t,a)\Big\}(s) = \frac{\beta}{(s-\alpha)^2-\beta^2}$.

Proof.

(i) For α and β as above,

$$\begin{aligned}
&\mathcal{L}_a\Big\{e_\alpha(t,a)\cosh_{\frac{\beta}{1+\alpha\mu(t)}}(t,a)\Big\}(s)\\
&\qquad= \mathcal{L}_a\Big\{e_\alpha(t,a)\Big(\frac{1}{2}\Big(e_{\frac{\beta}{1+\alpha\mu(t)}}(t,a)+e_{\frac{-\beta}{1+\alpha\mu(t)}}(t,a)\Big)\Big)\Big\}(s)\\
&\qquad= \frac{1}{2}\mathcal{L}_a\Big\{e_\alpha(t,a)e_{\frac{\beta}{1+\alpha\mu(t)}}(t,a)\Big\}+\frac{1}{2}\mathcal{L}_a\Big\{e_\alpha(t,a)e_{\frac{-\beta}{1+\alpha\mu(t)}}(t,a)\Big\}\\
&\qquad= \frac{1}{2}\mathcal{L}_a\Big\{e_{\alpha\oplus\frac{\beta}{1+\alpha\mu(t)}}(t,a)\Big\}+\frac{1}{2}\mathcal{L}_a\Big\{e_{\alpha\oplus\frac{-\beta}{1+\alpha\mu(t)}}(t,a)\Big\}\\
&\qquad= \frac{1}{2}\mathcal{L}_a\Big\{e_{\alpha+\beta}(t,a)\Big\}+\frac{1}{2}\mathcal{L}_a\Big\{e_{\alpha-\beta}(t,a)\Big\}.
\end{aligned}$$

For $|s|>\max\{|\alpha+\beta|,|\alpha-\beta|\}$,

$$\begin{aligned}
\mathcal{L}_a\Big\{e_\alpha(t,a)\cosh_{\frac{\beta}{1+\alpha\mu(t)}}(t,a)\Big\}(s) &= \frac{1}{2}\Big(\frac{1}{s-(\alpha+\beta)}+\frac{1}{s-(\alpha-\beta)}\Big)\\
&= \frac{1}{2}\Big(\frac{1}{(s-\alpha)-\beta}+\frac{1}{(s-\alpha)+\beta}\Big)\\
&= \frac{1}{2}\Big(\frac{2(s-\alpha)}{(s-\alpha)^2-\beta^2}\Big)\\
&= \frac{s-\alpha}{(s-\alpha)^2-\beta^2}.
\end{aligned}$$

(ii) For α and β as above,

$$\begin{aligned}
&\mathcal{L}_a\Big\{e_\alpha(t,a)\sinh_{\frac{\beta}{1+\alpha\mu(t)}}(t,a)\Big\}(s)\\
&\qquad= \mathcal{L}_a\Big\{e_\alpha(t,a)\Big(\frac{1}{2}\Big(e_{\frac{\beta}{1+\alpha\mu(t)}}(t,a)-e_{\frac{-\beta}{1+\alpha\mu(t)}}(t,a)\Big)\Big)\Big\}(s)\\
&\qquad= \frac{1}{2}\mathcal{L}_a\Big\{e_\alpha(t,a)e_{\frac{\beta}{1+\alpha\mu(t)}}(t,a)\Big\}-\frac{1}{2}\mathcal{L}_a\Big\{e_{\alpha)}(t,a)e_{\frac{-\beta}{1+\alpha\mu(t)}}(t,a)\Big\}\\
&\qquad= \frac{1}{2}\mathcal{L}_a\Big\{e_{\alpha\oplus\frac{\beta}{1+\alpha\mu(t)}}(t,a)\Big\}-\frac{1}{2}\mathcal{L}_a\Big\{e_{\alpha\oplus\frac{-\beta}{1+\alpha\mu(t)}}(t,a)\Big\}\\
&\qquad= \frac{1}{2}\mathcal{L}_a\Big\{e_{\alpha+\beta}(t,a)\Big\}-\frac{1}{2}\mathcal{L}_a\Big\{e_{\alpha-\beta}(t,a)\Big\}
\end{aligned}$$

For $|s| > \max\{|\alpha+\beta|, |\alpha-\beta|\}$,

$$\begin{aligned}
\mathcal{L}_a\Big\{e_\alpha(t,a)\sinh_{\frac{\beta}{1+\alpha\mu(t)}}(t,a)\Big\}(s) &= \frac{1}{2}\Big(\frac{1}{s-(\alpha+\beta)} - \frac{1}{s-(\alpha-\beta)}\Big) \\
&= \frac{1}{2}\Big(\frac{1}{(s-\alpha)-\beta} - \frac{1}{(s-\alpha)+\beta}\Big) \\
&= \frac{1}{2}\Big(\frac{2\beta}{(s-\alpha)^2-\beta^2}\Big) \\
&= \frac{\beta}{(s-\alpha)^2-\beta^2}.
\end{aligned}$$

□

Similarly to Theorem 3.62, one can prove the following theorem.

Theorem 3.63. *Assume $\alpha(t) \in \mathcal{R}_q$, $\pm\frac{\beta}{1+\alpha\mu(t)} \neq i$. Then, for $|s| > \max\{|\alpha+i\beta|, |\alpha-i\beta|\}$,*

(i) $\mathcal{L}_a\Big\{e_\alpha(t,a)\cos_{\frac{\beta}{1+\alpha\mu(t)}}(t,a)\Big\}(s) = \frac{s-\alpha}{(s-\alpha)^2+\beta^2}$;

(ii) $\mathcal{L}_a\Big\{e_\alpha(t,a)\sin_{\frac{\beta}{1+\alpha\mu(t)}}(t,a)\Big\}(s) = \frac{\beta}{(s-\alpha)^2+\beta^2}$.

Proof.

(i) For α and β as above,

$$\begin{aligned}
&\mathcal{L}_a\Big\{e_\alpha(t,a)\cos_{\frac{\beta}{1+\alpha\mu(t)}}(t,a)\Big\}(s) \\
&\quad= \mathcal{L}_a\Big\{e_\alpha(t,a)\Big(\frac{1}{2}\Big(e_{\frac{i\beta}{1+\alpha\mu(t)}}(t,a) + e_{\frac{-i\beta}{1+\alpha\mu(t)}}(t,a)\Big)\Big)\Big\}(s) \\
&\quad= \frac{1}{2}\mathcal{L}_a\Big\{e_\alpha(t,a)e_{\frac{i\beta}{1+\alpha\mu(t)}}(t,a)\Big\} + \frac{1}{2}\mathcal{L}_a\Big\{e_\alpha(t,a)e_{\frac{-i\beta}{1+\alpha\mu(t)}}(t,a)\Big\} \\
&\quad= \frac{1}{2}\mathcal{L}_a\Big\{e_{\alpha\oplus\frac{i\beta}{1+\alpha\mu(t)}}(t,a)\Big\} + \frac{1}{2}\mathcal{L}_a\Big\{e_{\alpha\oplus\frac{-i\beta}{1+\alpha\mu(t)}}(t,a)\Big\} \\
&\quad= \frac{1}{2}\mathcal{L}_a\Big\{e_{\alpha+i\beta}(t,a)\Big\} + \frac{1}{2}\mathcal{L}_a\Big\{e_{\alpha-i\beta}(t,a)\Big\}.
\end{aligned}$$

For $|s| > \max\{|\alpha + i\beta|, |\alpha - i\beta|\}$,

$$\begin{aligned}\mathcal{L}_a\Big\{e_\alpha(t,a)\cos_{\frac{\beta}{1+\alpha\mu(t)}}(t,a)\Big\}(s) &= \frac{1}{2}\Big(\frac{1}{s-(\alpha+i\beta)} + \frac{1}{s-(\alpha-i\beta)}\Big)\\ &= \frac{1}{2}\Big(\frac{1}{(s-\alpha)-i\beta} + \frac{1}{(s-\alpha)+i\beta}\Big)\\ &= \frac{1}{2}\Big(\frac{2(s-\alpha)}{(s-\alpha)^2+\beta^2}\Big)\\ &= \frac{s-\alpha}{(s-\alpha)^2+\beta^2}.\end{aligned}$$

(ii) For α and β as above,

$$\begin{aligned}&\mathcal{L}_a\Big\{e_\alpha(t,a)\sin_{\frac{\beta}{1+\alpha\mu(t)}}(t,a)\Big\}(s)\\ &\quad= \mathcal{L}_a\Big\{e_\alpha(t,a)\Big(\frac{1}{2i}\Big(e_{\frac{i\beta}{1+\alpha\mu(t)}}(t,a) - e_{\frac{-i\beta}{1+\alpha\mu(t)}}(t,a)\Big)\Big)\Big\}(s)\\ &\quad= \frac{1}{2i}\mathcal{L}_a\Big\{e_\alpha(t,a)e_{\frac{i\beta}{1+\alpha\mu(t)}}(t,a)\Big\} - \frac{1}{2i}\mathcal{L}_a\Big\{e_{\alpha)}(t,a)e_{\frac{-i\beta}{1+\alpha\mu(t)}}(t,a)\Big\}\\ &\quad= \frac{1}{2i}\mathcal{L}_a\Big\{e_{\alpha\oplus\frac{i\beta}{1+\alpha\mu(t)}}(t,a)\Big\} - \frac{1}{2i}\mathcal{L}_a\Big\{e_{\alpha\oplus\frac{-i\beta}{1+\alpha\mu(t)}}(t,a)\Big\}\\ &\quad= \frac{1}{2i}\mathcal{L}_a\Big\{e_{\alpha+i\beta}(t,a)\Big\} - \frac{1}{2i}\mathcal{L}_a\Big\{e_{\alpha-i\beta}(t,a)\Big\}.\end{aligned}$$

For $|s| > \max\{|\alpha + i\beta|, |\alpha - i\beta|\}$,

$$\begin{aligned}\mathcal{L}_a\Big\{e_\alpha(t,a)\sin_{\frac{\beta}{1+\alpha\mu(t)}}(t,a)\Big\}(s) &= \frac{1}{2i}\Big(\frac{1}{s-(\alpha+i\beta)} - \frac{1}{s-(\alpha-i\beta)}\Big)\\ &= \frac{1}{2i}\Big(\frac{1}{(s-\alpha)-i\beta} - \frac{1}{(s-\alpha)+i\beta}\Big)\\ &= \frac{1}{2i}\Big(\frac{2i\beta}{(s-\alpha)^2+\beta^2}\Big)\\ &= \frac{\beta}{(s-\alpha)^2+\beta^2}.\end{aligned}$$

$\square$

Next, we provide an example of solving an initial value problem using the q-Laplace transform.

Example 3.64. Use the q-Laplace transform to solve the following initial value problem:

$$\begin{cases} \Delta_q^2 f(t) - 2\Delta_q f(t) - 8f(t) = 0 \\ \Delta_q f(a) = 0, \quad f(a) = -3/2. \end{cases}$$

Taking the Laplace transform of both sides, we have

$$\Big(s^2 F_a(s) - \Delta_q f(a) - sf(a)\Big) - 2\Big(sF_a(s) - f(a)\Big) - 8F_a(s) = 0.$$

Plugging in the initial conditions, we have

$$\begin{aligned} s^2F_a(s) + \frac{3s}{2} - 2sF_a(s) - 3 - 8F_a(s) = 0 \\ &\Longrightarrow (s^2 - 2s - 8)F_a(s) = 3 - \frac{3s}{2} \\ &\Longrightarrow F_a(s) = \frac{3 - \frac{3s}{2}}{s^2 - 2s - 8} = \frac{-1/2}{s-4} - \frac{1}{s+2} \\ &\Longrightarrow f(t) = -\frac{1}{2}e_4(t,a) - e_{-2}(t,a). \end{aligned}$$

Remark 3.65. Note that in the second to last line above, we could have split the fraction as follows to find a different, but equivalent, form of the solution.

$$\begin{aligned} F_a(s) &= \frac{3 - \frac{3s}{2}}{s^2 - 2s - 8} = -\frac{3}{2}\left(\frac{s-1}{(s-1)^2 - 9}\right) + \frac{1}{2}\left(\frac{3}{(s-1)^2 - 9}\right) \\ \Longrightarrow f(t) &= -\frac{3}{2}e_1(t,a)\cosh_{\frac{3}{1+\mu(t)}}(t,a) + \frac{1}{2}e_1(t,a)\sinh_{\frac{3}{1+\mu(t)}}(t,a). \end{aligned}$$

The following Leibniz formula will be useful for later theorems.

Lemma 3.66. *(Leibniz formula) Assume* $t \in aq^{\mathbb{N}_n}$*,* $n \in \mathbb{Z}$*, and*

$f : aq^{\mathbb{N}_n} \times aq^{\mathbb{N}_0} \to \mathbb{R}$. *Then*

$$\Delta_q \left(\sum_{s=a}^{tq^{-n}} f(t,s)\mu(s) \right) = \sum_{s=a}^{tq^{-n}} \Delta_q f(t,s)\mu(s) + f(tq, tq^{-n+1})q^{-n+1}.$$

Proof.

$$\begin{aligned}
\Delta_q \left(\sum_{s=a}^{tq^{-n}} f(t,s)\mu(s) \right) &= \frac{\sum_{s=a}^{tq^{-n+1}} f(tq,s)\mu(s) - \sum_{s=a}^{tq^{-n}} f(t,s)\mu(s)}{\mu(t)} \\
&= \frac{\sum_{s=a}^{tq^{-n}} [(f(tq,s) - f(t,s))\mu(s)]}{\mu(t)} + \frac{f(tq, tq^{-n+1})\mu(tq^{-n+1})}{\mu(t)} \\
&= \sum_{s=a}^{tq^{-n}} \Delta_q f(t,s)\mu(s) + f(tq, tq^{-n+1})q^{-n+1}.
\end{aligned}$$

□

We will now use the Leibniz formula to prove the following theorem.

Theorem 3.67. *(Variation of Constants Formula) Assume* $n \geq 1$ *is an integer. Then the solution to the initial value problem*

$$\begin{cases} \Delta_q^n y(t) = f(t) \\ \Delta_q^i y(a) = 0, \quad i = 0, 1, \dots, n-1. \end{cases}$$

is given by

$$y(t) = \int_a^t h_{n-1}(t, \sigma(s)) f(s) \, \Delta_q s.$$

Proof. First note that if $n = 1$, then

$$y(t) = \int_a^t h_0(t, \sigma(s)) f(s)\, \Delta_q s$$
$$= \sum_{s=a}^{t/q} f(s)\mu(s).$$

Then $y(a) = 0$ and

$$\Delta_q y(t) = \frac{\sum_{s=a}^{t} f(s)\mu(s) - \sum_{s=a}^{t/q} f(s)\mu(s)}{\mu(t)}$$
$$= \frac{f(t)\mu(t)}{\mu(t)} = f(t).$$

So the result holds for $n = 1$. If now $n > 1$,

$$y(t) = \int_a^t h_{n-1}(t, \sigma(s)) f(s)\, \Delta_q s$$
$$= \sum_{s=a}^{t/q} h_{n-1}(t, \sigma(s)) f(s)\mu(s).$$

Taking the q-difference of this summation and applying the Leibniz formula, we have

$$\Delta_q y(t) = \sum_{s=a}^{t/q} \Delta_q h_{n-1}(t, \sigma(s)) f(s)\mu(s) + h_{n-1}(tq, tq) f(t)\mu(t)$$
$$= \sum_{s=a}^{t/q} \Delta_q h_{n-1}(t, \sigma(s)) f(s)\mu(s).$$

Continuing inductively, we find for $i = 0, 1, 2, \ldots, n-1$,

$$\Delta_q^i y(t) = \sum_{s=a}^{t/q} \Delta_q^i h_{n-1}(t, \sigma(s)) f(s)\mu(s),$$

which implies

$$\begin{aligned}\Delta_q^i y(a) &= \sum_{s=a}^{a/q} \Delta_q^i h_{n-1}(a,\sigma(s)) f(s)\mu(s) \\ &= \int_a^a \Delta_q^i h_{n-1}(a,\sigma(s)) f(s)\, \Delta_q s \\ &= 0\end{aligned}$$

by definition of the integral. Thus, the initial conditions hold.

From above, we have for $i = 0, 1, 2, \ldots, n-1$,

$$\Delta_q^i y(t) = \sum_{s=a}^{t/q} \Delta_q^i h_{n-1}(t,\sigma(s)) f(s)\mu(s),$$

which implies

$$\begin{aligned}\Delta_q^{n-1} y(t) &= \sum_{s=a}^{t/q} \Delta_q^{n-1} h_{n-1}(t,\sigma(s)) f(s)\mu(s) \\ &= \sum_{s=a}^{t/q} f(s)\mu(s),\end{aligned}$$

and

$$\Delta_q \sum_{s=a}^{t/q} f(s)\mu(s) = f(t).$$

Thus,

$$\Delta_q^n y(t) = \sum_{s=a}^{t/q} 0 + f(t) = f(t).$$

□

Example 3.68. Use the variation of constants formula to solve the initial value problem

$$\begin{cases} \Delta_q^2 y(t) = e_p(t,1) \\ y(1) = \Delta_q y(1) = 0. \end{cases}$$

From the Variation of Constants formula, the solution of this initial value problem is given by

$$y(t) = \int_1^t h_1(t,\sigma(s)) e_p(s,1)\,\Delta_q s.$$

Integrating by parts, we have

$$\begin{aligned}
y(t) &= \int_1^t h_1(t,\sigma(s)) e_p(s,1)\,\Delta_q s \\
&= \frac{1}{p}\left[h_1(t,s)e_p(s,1)\Big|_{s=1}^{t}\right] - \frac{1}{p}\int_1^t e_p(s,1)\Delta_q h_1(t,s)\,\Delta_q s \\
&= \frac{1}{p}(h_1(t,t)e_p(t,1) - h_1(t,1)e_p(s,s)) - \frac{1}{p}\int_1^t e_p(s,1)\,\Delta_q s \\
&= \frac{1}{p}\Big(- h_1(t,1)\Big) + \frac{1}{p^2}\left[e_p(s,1)\Big|_{s=1}^{t}\right] \\
&= -\frac{1}{p}h_1(t,1) + \frac{1}{p^2}e_p(t,1) - \frac{1}{p^2}e_p(1,1) \\
&= -\frac{1}{p}h_1(t,1) + \frac{1}{p^2}e_p(t,1) - \frac{1}{p^2}.
\end{aligned}$$

We can verify the initial conditions:

$$\begin{aligned}
y(1) &= -\frac{1}{p}h_1(1,1) + \frac{1}{p^2}e_p(1,1) - \frac{1}{p^2} \\
&= 0 + \frac{1}{p^2} - \frac{1}{p^2} \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\Delta_q y(1) &= \Delta_q\Big(-\frac{1}{p}h_1(t,1) + \frac{1}{p^2}e_p(t,1) - \frac{1}{p^2}\Big)\Big|_{t=1} \\
&= \left(-\frac{1}{p} + \frac{1}{p}e_p(t,1) - 0\right)\Big|_{t=1} \\
&= \frac{1}{p}e_p(1,1) - \frac{1}{p} \\
&= \frac{1}{p} - \frac{1}{p} \\
&= 0.
\end{aligned}$$

Thus, the initial conditions hold, and we can also verify that we have a solution by finding $\Delta^2 y(t)$:

$$\begin{aligned}
\Delta_q^2 y(t) &= \Delta_q^2\Big(-\frac{1}{p}h_1(t,1) + \frac{1}{p^2}e_p(t,1) - \frac{1}{p^2}\Big) \\
&= \Delta_q\Big(-\frac{1}{p} + \frac{1}{p}e_p(t,1) - 0\Big) \\
&= e_p(t,1).
\end{aligned}$$

Thus,

$$y(t) = -\frac{1}{p}h_1(t,1) + \frac{1}{p^2}e_p(t,1) - \frac{1}{p^2}$$

is the solution to the given initial value problem.

Chapter 4

Green's Functions on Mixed Time Scales with Lidstone Boundary Conditions

This chapter focuses on finding a Green's function and its properties for a boundary value problem on a mixed time scale, i.e., a time scale whose elements may be thought of as being recursively defined according to a linear function. A mixed time scale "mixes" together our previous two experiences with time scales in this thesis. Whereas earlier, our forward jumps in a time scale were determined either by strictly adding or strictly multiplying, our forward jumps here will be determined by a combination of both multiplying and adding.

4.1 Preliminaries

In this section, we introduce some fundamental concepts and properties of a mixed time scale. Rather than determining a formula for the jump operators after one has defined a particular time scale, we might think of mixed time scales as being recursively defined through the following jump operators.

Definition 4.1. For $a, b \in \mathbb{R}$ with $a \geq 1, b \geq 0$, and $a + b > 1$, we define the forward jump operator by

$$\sigma_{a,b}(t) := at + b.$$

Throughout this chapter, and anytime there is no ambiguity, we can refer to $\sigma_{a,b}$ simply as σ. Additionally, any reference to a and b will refer to these constants from the definition of σ. As any case in which $a = 1$ results in a time scale that is simply an analogue of the traditional discrete time scale $\mathbb{N}_a$, we will generally be interested here in the case when $a > 1$. The q-time scale discussed in the previous chapter is simply a particular mixed time scale in which a here is equivalent to q and $b = 0$.

Definition 4.2. The n^{th} forward jump operator is recursively defined for $n \in \mathbb{N}_0$ as

$$\sigma^n(t) := \begin{cases} \sigma(\sigma^{n-1}(t)), & n \geq 1, \\ 1, & n = 0. \end{cases}$$

We can also similarly define a backward jump operator as follows.

Definition 4.3. Given $\sigma = \sigma_{a,b}$, we can define the backward jump operator as

$$\rho_{a,b}(t) = \rho(t) := \frac{t - b}{a},$$

and, for $n \in \mathbb{N}_0$,

$$\rho^n(t) := \begin{cases} \rho(\rho^{n-1}(t)), & n \geq 1, \\ 1, & n = 0, \end{cases}$$

where the subscript on ρ will be suppressed in this chapter as there is no ambiguity.

We now define our time scale $\mathbb{T}_\alpha$ based on our jump operators.

Definition 4.4. Given $\alpha \in \mathbb{R}$ such that $\alpha > \frac{b}{1-a}$ (where $a > 1$), we define the mixed time scale

$$\mathbb{T}_\alpha := \{\alpha, \sigma(\alpha), \sigma^2(\alpha), \ldots\}.$$

Remark 4.5. In [25], for $a > 1$ and $\alpha > \frac{b}{1-a}$,

$$\mathbb{T}_\alpha := \{\ldots, \rho^2(\alpha), \rho(\alpha), \alpha, \sigma(\alpha), \sigma^2(\alpha), \ldots\}.$$

Note that this is not technically a time scale. This can be remedied by defining

$$\mathbb{T}_\alpha := \{\ldots, \rho^2(\alpha), \rho(\alpha), \alpha, \sigma(\alpha), \sigma^2(\alpha), \ldots\} \cup \left\{\frac{b}{1-a}\right\}$$

or by considering Definition 4.4 above. This is a minor point since in any uses of the time scale that follow, we only consider $\mathbb{T}_\alpha$ as in Definition 4.4, though one could easily consider

$$\mathbb{T}_\alpha := \{\rho^k(\alpha), \ldots, \rho^2(\alpha), \rho(\alpha), \alpha, \sigma(\alpha), \sigma^2(\alpha), \ldots\},$$

for some $k \in \mathbb{N}_0$.

We may note that $\mathbb{T}_\alpha$ in either the definition or remark above is bounded below by $\frac{b}{1-a}$ (though in the remark, we are actually dealing with the infimum instead of just a lower bound), a proof of which may be found in [25]. This property does not hold if $a = 1$. Additionally, if $\alpha = \frac{b}{1-a}$, then $\mathbb{T}_\alpha = \{\alpha\}$, and if $\alpha < \frac{b}{1-a}$, then σ as defined earlier would function as a backward jump operator (and ρ as the forward jump operator) while $\mathbb{T}_\alpha$ will be bounded above by $\frac{b}{1-a}$. For simplicity, we will consider $\alpha \geq 0 > \frac{b}{1-a}$.

Definition 4.6. For $c, d \in \mathbb{T}_\alpha$ such that $d \geq c$, we define

$$\mathbb{T}_{[c,d]} := \mathbb{T}_\alpha \cap [c,d] = \{c, \sigma(c), \sigma^2(c), \ldots, \rho(d), d\}.$$

We define $\mathbb{T}_{(c,d)}, \mathbb{T}_{(c,d]}$, and $\mathbb{T}_{[c,d)}$ similarly. Additionally, we may use the notation $\mathbb{T}_\alpha^d$ where

$$\mathbb{T}_\alpha^d := \mathbb{T}_{[\alpha,d]}.$$

Definition 4.7. As before, we define a graininess function (or, as in [25], a forward distance operator)

$$\mu(t) := \sigma(t) - t = (at+b) - t = (a-1)t + b.$$

Below are some properties of μ, as found in [25].

Theorem 4.8. *For $t \in \mathbb{T}_\alpha$ and $n \in \mathbb{N}_0$, the following hold:*

(i) $\mu(t) > 0$*;*

(ii) $\mu(\sigma^n(t)) = a^n \mu(t)$*;*

(iii) $\mu(\rho^n(t)) = a^{-n} \mu(t)$*.*

We now define an index function, whose value gives the number of forward jumps in $\mathbb{T}_\alpha$ between two elements of the time scale.

Definition 4.9. We define the index function $K : \mathbb{T}_\alpha \times \mathbb{T}_\alpha \to \mathbb{Z}$ by

$$K(t,s) := \log_a \left(\frac{\mu(t)}{\mu(s)} \right).$$

For simplicity, we will use the notation $K(t) := K(t, \alpha)$.

As presented in [25], some properties of the index function are below.

Theorem 4.10. *For $t, s, r \in \mathbb{T}_\alpha$ such that $t = \sigma^k(s)$, the following hold:*

(i) $K(t,t) = 0$;

(ii) $K(t,s) = k$;

(iii) $K(s,t) = -K(t,s)$;

(iv) $K(t,s) = K(t,r) + K(r,s)$.

We define the difference operator here exactly as before. Thus, all relevant properties from before hold since they are simply based on the following definition.

Definition 4.11. For $f : \mathbb{T}_\alpha \to \mathbb{R}$, the forward difference operator (or delta difference) is defined as

$$\Delta_{a,b} f(t) = \Delta f(t) := \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

Here, again, we will suppress the subscript on Δ throughout this chapter.

We also define a definite integral on $\mathbb{T}_\alpha$. As before, it is essentially a left-hand Riemann sum associated with $\mathbb{T}_c^d \subseteq \mathbb{T}_\alpha$.

Definition 4.12. For $f : \mathbb{T}_\alpha \to \mathbb{R}$ and $c, d \in \mathbb{T}_\alpha$,

$$\int_c^d f(t)\,\Delta t := \begin{cases} \displaystyle\sum_{j=0}^{K(d,c)-1} f(\sigma^j(c))\mu(\sigma^j(c)), & c < d \\ 0, & c \geq d. \end{cases}$$

Remark 4.13. As we are considering a time scale $\mathbb{T}_\alpha$ that can be considered a generalization of the time scale $aq_0^{\mathbb{N}}$ from the previous chapter, we can note that the properties in Theorem 3.20 similarly hold for this definition of a definite integral.

As in the previous two chapters, it is important to have functions that are factorial in nature with respect to our time scale. We also define bracket and brace functions.

Definition 4.14. For $n \in \mathbb{N}$ and $t \in \mathbb{T}_\alpha$, we define the rising function by

$$t^{\overline{n}} := \prod_{j=0}^{n-1} \sigma^j(t).$$

Additionally, we define

$$t^{\overline{0}} := 1.$$

We also define

$$t^{\underline{n}} := \prod_{j=0}^{n-1} \rho^j(t), \quad t^{\underline{0}} := 1.$$

Definition 4.15. For $n \in \mathbb{Z}$, we define the a-bracket function by

$$[n]_a := \left(\frac{a^n - 1}{a - 1}\right)$$

and the a-brace function by

$$\{n\}_a := \left(\frac{a^n - 1}{(a-1)a^{n-1}}\right).$$

Remark 4.16. We may note from the definitions above that $[0]_a = 0$, $[1]_a = 1$, $\{0\}_a = 0$, and $\{1\}_a = 1$. Also, in [25], we can see that the following hold:

(i) $\Delta t^{\overline{n}} = [n]_a (\sigma(t))^{\overline{n-1}}$;

(ii) $\Delta t^{\underline{n}} = \{n\}_a t^{\underline{n-1}}$.

In anticipation of defining Taylor monomials, we also define the following factorial functions.

Definition 4.17. For $n \in \mathbb{N}_0$ we recursively define the a-bracket factorial and a-brace factorial functions, respectively, by

$$[0]_a! := 1, \quad [n]_a! := [n]_a \cdot [n-1]_a!,$$

and

$$\{0\}_a! := 1, \quad \{n\}_a! := \{n\}_a \cdot \{n-1\}_a!.$$

We now define the Taylor monomials on $\mathbb{T}_\alpha$.

Definition 4.18. For $s \in \mathbb{T}_\alpha$, we define the Taylor monomials $h(\cdot, s) : \mathbb{T}_\alpha \to \mathbb{R}$ by

$$h_n(t,s) := \sum_{i=0}^{n} \frac{(-1)^i s^{\bar{i}} t^{\underline{n-i}}}{[i]_a!\{n-i\}_a!}.$$

From [25] we have the following two theorems regarding some properties and an application of Taylor monomials.

Theorem 4.19. *For $n \in \mathbb{N}$ and $t, s \in \mathbb{T}_\alpha$ we have*

(i) $h_0(t,s) = 1$;

(ii) $h_n(s,s) = 0$;

(iii) $\Delta h_n(t,s) = h_{n-1}(t,s)$;

(iv) $h_1(t,s) = t - s$;

(v) $h_n(t,s) = \prod_{i=1}^{n} \frac{t-\sigma^{i-1}(s)}{[i]_a}$.

Theorem 4.20. *(Taylor's Formula) Let $f : \mathbb{T}_\alpha \to \mathbb{R}$ and $s \in \mathbb{T}_\alpha$. Then for any $n \in \mathbb{N}_0$ we have for all $t \in \mathbb{T}_\alpha$*

$$f(t) = p_n(t,s) + R_n(t,s),$$

where $p_n(\cdot, s) : \mathbb{T}_\alpha \to \mathbb{R}$ *is given by*

$$p_n(t, s) := \sum_{k=0}^{n} \Delta^k f(s) h_k(t, s),$$

and $R_n(\cdot, s) : \mathbb{T}_\alpha \to \mathbb{R}$ *is given by*

$$R_n(t, s) := \int_s^t h_n(t, \sigma(\tau)) \Delta^{n+1} f(\tau) \, \Delta\tau.$$

4.2 Green's Function on a Mixed Time Scale

In this section, we will find a Green's function on a mixed time scale where $a > 1$ and investigate some of its properties. We will first provide a theorem regarding the existence of nontrivial solutions for a certain boundary value problem. Many of the results in this section can be viewed as analogues to Green's function results in [34]. Some other results on mixed time scales can also be found in [24].

Theorem 4.21. *Given that* $y : \mathbb{T}_\alpha \to \mathbb{R}$*,* $\beta \in \mathbb{T}_\alpha$*, and* $A, B, C, D \in \mathbb{R}$*, the homogeneous boundary value problem*

$$\begin{cases} -\Delta^2 y(t) = 0, \quad t \in \mathbb{T}_\alpha \\ Ay(\alpha) - B\Delta y(\alpha) = 0 \\ Cy(\beta) + D\Delta y(\beta) = 0 \end{cases}$$

has only the trivial solution if and only if

$$\gamma := AC(\beta - \alpha) + BC + AD \neq 0.$$

Proof. Given that $-\Delta^2 y(t) = -\Delta\Delta y(t) = 0$, we have

$$\Delta y(t) = c_0 h_0(t,\alpha) = c_0.$$

Summing both sides from α to $t-1$ gives

$$\sum_{\tau=\alpha}^{t-1} \Delta y(t) = \sum_{\tau=\alpha}^{t-1} c_0$$
$$\implies y(t) - y(\alpha) = c_0(t-\alpha)$$
$$\implies y(t) = y(\alpha) + c_0(t-\alpha)$$

and letting $y(\alpha) = c_1$ gives

$$y(t) = c_1 + c_0(t-\alpha).$$

Using our boundary conditions, we have

$$Ay(\alpha) - B\Delta y(\alpha) = Ac_1 - Bc_0 = 0,$$

and

$$Cy(\beta) + D\Delta y(\beta) = C(c_1 + c_0(\beta - \alpha) + Dc_0 = 0.$$

Thus, we have the following sytem of equations

$$c_0(-B) + c_1 A = 0,$$
$$c_0(C\beta - C\alpha + D) + c_1 C = 0,$$

which has only the trivial solution if and only if the determinant

$$-\gamma = \begin{vmatrix} -B & A \\ C\beta - C\alpha + D & C \end{vmatrix} \neq 0.$$

From above, we can see

$$\begin{aligned} \gamma &= -[(-B)C - A(C\beta - C - C\alpha + D)] \\ &= BC + AC\beta - AC\alpha + AD \\ &= AC(\beta - \alpha) + BC + AD. \end{aligned}$$

□

Lemma 4.22. *For $y : \mathbb{T}_\alpha \to \mathbb{R}$, $\beta \in \mathbb{T}_\alpha$, and $A_1, A_2 \in \mathbb{R}$, the boundary value problem*

$$\begin{cases} -\Delta^2 y(t) = 0, \quad t \in \mathbb{T}_\alpha, \\ y(\alpha) = A_1, \quad y(\beta) = A_2 \end{cases}$$

has the solution

$$y(t) = A_1 + \frac{A_2 - A_1}{\beta - \alpha}(t - \alpha).$$

Proof. The general solution to the difference equation is

$$y(t) = c_1 + c_0(t - \alpha).$$

Using the first boundary condition,

$$y(\alpha) = c_1 + 0 = c_1,$$

and, using the second boundary condition,

$$y(\beta) = A_1 + c_0(\beta - \alpha) = A_2$$
$$\implies c_0 = \frac{A_2 - A_1}{\beta - \alpha}.$$

So then

$$y(t) = A_1 + \frac{A_2 - A_1}{\beta - \alpha}(t - \alpha).$$

□

Remark 4.23. We might notice that the solution to the boundary value problem above is essentially just a line connecting the points (α, A_1) and (β, A_2) (though, of course, the solution $y(t)$ only exists at $t \in \mathbb{T}_\alpha$).

Theorem 4.24. *For $y, f : \mathbb{T}_\alpha \to \mathbb{R}$ and $\beta \in \mathbb{T}_\alpha$ such that $\beta > \alpha$, the boundary value problem*

$$\begin{cases} -\Delta^2 y(t) = f(t), \quad t \in \mathbb{T}_\alpha \\ y(\alpha) = 0 = y(\beta) \end{cases} \tag{4.2.1}$$

has solution

$$y(t) = \int_\alpha^\beta G(t,\tau) f(\tau)\,\Delta\tau = \sum_{j=0}^{K(\beta)-1} G(t, \sigma^j(\alpha)) f(\sigma^j(\alpha)) \mu(\sigma^j(\alpha)),$$

where $G : \mathbb{T}_\alpha^\beta \times \mathbb{T}_\alpha^{\rho(\beta)} \to \mathbb{R}$ is defined by

$$G(t,\tau) := \begin{cases} \dfrac{h_1(\beta, \sigma(\tau))}{h_1(\beta, \alpha)} h_1(t, \alpha) - h_1(t, \sigma(\tau)), & 0 \le K(\tau) \le K(t) - 1 \le K(\beta) - 1, \\ \dfrac{h_1(\beta, \sigma(\tau))}{h_1(\beta, \alpha)} h_1(t, \alpha), & 0 \le K(t) \le K(\tau) \le K(\beta) - 1. \end{cases}$$

Proof. We may note that the related homogeneous boundary value problem, i.e., the case when $f(t) \equiv 0$, does not have only the trivial solution as γ (defined in Theorem 4.21) is not equal to 0:

$$\gamma = AC(\beta - \alpha) + BC + AD = (\beta - \alpha) \neq 0.$$

By an application of Taylor's theorem from [31] the general solution to (4.2.1) is given by

$$\begin{aligned} y(t) &= c_0 h_0(t, \alpha) + c_1 h_t(t, \alpha) - \int_\alpha^t h_1(t, \sigma(\tau)) f(\tau)\, \Delta\tau \\ &= c_0 + c_1 h_1(t, \alpha) - \sum_{j=0}^{K(t)-1} h_1(t, \sigma(\sigma^j(\alpha))) f(\sigma^j(\alpha)) \mu(\sigma^j(\alpha)) \end{aligned}$$

Using the first boundary condition,

$$\begin{aligned} y(\alpha) &= c_0 + c_1 h_1(\alpha, \alpha) - \int_\alpha^\alpha h_1(t, \sigma(\tau)) f(\tau)\, \Delta\tau \\ &= c_0 + 0 - 0 = 0 \\ &\implies c_0 = 0, \end{aligned}$$

and using the second boundary condition,

$$\begin{aligned} y(\beta) &= c_1 h_1(\beta, \alpha) - \sum_{j=0}^{K(\beta)-1} h_1(\beta, \sigma^{j+1}(\alpha)) f(\sigma^j(\alpha)) \mu(\sigma^j(\alpha)) = 0 \\ \implies c_1 &= \frac{\sum_{j=0}^{K(\beta)-1} h_2(\beta, \sigma^{j+1}(\alpha)) f(\sigma^j(\alpha)) \mu(\sigma^j(\alpha))}{h_1(\beta, \alpha)}. \end{aligned}$$

Thus,

$$
\begin{aligned}
y(t) &= \frac{\sum_{j=0}^{K(\beta)-1} h_1(\beta, \sigma^{j+1}(\alpha)) f(\sigma^j(\alpha))\mu(\sigma^j(\alpha))}{h_1(\beta,\alpha)} h_1(t,\alpha) \\
&\quad - \sum_{j=0}^{K(\beta)-1} h_1(t, \sigma^{j+1}(\alpha)) f(\sigma^j(\alpha))\mu(\sigma^j(\alpha)) \\
&= \mu(\sigma^j(\alpha)) \Bigg[\sum_{j=0}^{K(t)-1} \left(\frac{h_1(\beta, \sigma^{j+1}(\alpha))}{h_1(\beta,\alpha)} h_1(t,\alpha) - h_1(t, \sigma^{j+1}(\alpha)) \right) \\
&\quad + \sum_{j=K(t)}^{K(\beta)-1} \frac{h_1(\beta, \sigma^{j+1}(\alpha))}{h_1(\beta,\alpha)} h_1(t,\alpha) \Bigg] f(\sigma^j(\alpha)) \\
&= \sum_{j=0}^{K(\beta)-1} G(t, \sigma^j(\alpha)) f(\sigma^j(\alpha)) \mu(\sigma^j(\alpha)) \\
&= \int_\alpha^\beta G(t,\tau) f(\tau)\, \Delta\tau,
\end{aligned}
$$

for $G(t,\tau)$ defined in the statement of the theorem. $\square$

Remark 4.25. We can also simplify the Green's function above as

$$
\begin{aligned}
G(t,\tau) &:= \begin{cases} \dfrac{h_1(\beta,\sigma(\tau))}{h_1(\beta,\alpha)} h_1(t,\alpha) - h_1(t,\sigma(\tau)), & 0 \le K(\tau) \le K(t)-1 \le K(\beta)-1, \\ \dfrac{h_1(\beta,\sigma(\tau))}{h_1(\beta,\alpha)} h_1(t,\alpha), & 0 \le K(t) \le K(\tau) \le K(\beta)-1, \end{cases} \\
&= \begin{cases} \dfrac{\beta-\sigma(\tau)}{\beta-\alpha}(t-\alpha) - (t-\sigma(\tau)), & 0 \le K(\tau) \le K(t)-1 \le K(\beta)-1, \\ \dfrac{\beta-\sigma(\tau)}{\beta-\alpha}(t-\alpha), & 0 \le K(t) \le K(\tau) \le K(\beta)-1, \end{cases} \\
&= \begin{cases} \dfrac{(\beta-\alpha)(t-\sigma(\tau))}{\beta-\alpha}, & 0 \le K(\tau) \le K(t)-1 \le K(\beta)-1, \\ \dfrac{(\beta-\sigma(\tau))(t-\alpha)}{\beta-\alpha}, & 0 \le K(t) \le K(\tau) \le K(\beta)-1. \end{cases}
\end{aligned}
$$

This is symmetric in t and $\sigma(\tau)$ and closely resembles

$$\begin{cases} \dfrac{(b-s)(t-a)}{b-a}, & a \le t \le s \le b, \\ \dfrac{(b-t)(s-a)}{b-a}, & a \le s \le t \le b, \end{cases}$$

which is the Green's function for the conjugate boundary value problem on $[a,b] \subset \mathbb{R}$

$$\begin{cases} -y''(t) = 0, \\ y(a) = 0 = y(b), \end{cases}$$

as can be seen in [35].

Theorem 4.26. *For $G(t,\tau)$ defined in Theorem 4.24, $G(t,\tau) \ge 0$ on its domain and*

$$\max_{t \in \mathbb{T}_\alpha^\beta} G(t,\tau) = G(\sigma(\tau),\tau).$$

Proof. First, note that

$$G(\alpha,\tau) = \frac{h_1(\beta,\sigma(\tau))}{h_1(\beta,\alpha)} h_1(\alpha,\alpha) = \frac{h_1(\beta,\sigma(\tau))}{h_1(\beta,\alpha)}(\alpha-\alpha) = 0,$$

and

$$G(\beta,\tau) = \frac{h_1(\beta,\sigma(\tau))}{h_1(\beta,\alpha)} h_1(\beta,\alpha) - h_1(\beta,\sigma(\tau)) = 0.$$

To prove our result, we will show that $\Delta G(t,\tau) \ge 0$ for $t < \tau$, $\Delta G(t,\tau) \le 0$ for $\tau < t$, and $G(\sigma(\tau),\tau) \ge G(\tau,\tau)$ (which then shows $\Delta G(t,\tau) \ge 0$ for $t \le \tau$). First consider the domain $0 \leq K(t) < K(\tau) \leq K(\beta) - 1$:

$$\begin{aligned}\Delta G(t,\tau) &= \frac{h_1(\beta,\sigma(\tau))}{h_1(\beta,\alpha)}\Delta h_1(t,\alpha)\\ &= \frac{\beta-\sigma(\tau)}{\beta-\alpha}h_0(t,\alpha)\\ &= \frac{\beta-\sigma(\tau)}{\beta-\alpha}\\ &\geq 0.\end{aligned}$$

Now consider the domain $0 \leq K(\tau) \leq K(t) - 1 \leq K(\beta) - 1$:

$$\begin{aligned}\Delta G(t,\tau) &= \frac{h_1(\beta,\sigma(\tau))}{h_1(\beta,\alpha)}\Delta h_1(t,\alpha) - \Delta h_1(t,\sigma(\tau))\\ &= \frac{\beta-\sigma(\tau)}{\beta-\alpha}h_0(t,\alpha) - h_0(t,\sigma(\tau))\\ &= \frac{\beta-\sigma(\tau)}{\beta-\alpha} - 1\\ &\leq 0,\end{aligned}$$

as $\beta - \sigma(\tau) \leq \beta - \alpha$. Since G is increasing for $t < \tau$ and decreasing for $\tau < t$, we need to see which is larger: $G(\sigma(\tau),\tau)$ or $G(\tau,\tau)$. We can observe

$$\begin{aligned}G(\sigma(\tau),\tau) - G(\tau,\tau) &= \frac{\beta-\sigma(\tau)}{\beta-\alpha}(\sigma(\tau)-\alpha) - (\sigma(\tau)-\sigma(\tau)) - \frac{\beta-\sigma(\tau)}{\beta-\alpha}(\tau-\alpha)\\ &= \frac{\beta-\sigma(\tau)}{\beta-\alpha}[\sigma(\tau)-\alpha-\tau+\alpha]\\ &= \frac{\beta-\sigma(\tau)}{\beta-\alpha}(\sigma(\tau)-\tau)\\ &\geq 0,\end{aligned}$$

which implies that $\max_{t\in\mathbb{T}_\alpha^\beta} G(t,\tau) = G(\sigma(\tau),\tau)$. Also, since $\Delta G(t,\tau) \geq 0$ for $t \in \mathbb{T}_{[\alpha,\tau]}$, $\Delta G(t,\tau) \leq 0$ for $t \in \mathbb{T}_{(\tau,\beta)}$, and $G(\alpha,\tau) = 0 = G(\beta,\tau)$, we have $G(t,\tau) \geq 0$

on its domain. □

Remark 4.27. Note that in the above proof, we have $\Delta G(t,\tau) > 0$ for $t \leq \tau < \rho(\beta)$, $\Delta G(t,\tau) < 0$ for $\alpha < \tau < t$.

Corollary 4.28. *For $G(t,\tau)$ defined in Theorem 4.24, we have*

$$-\Delta^2 G(t,\tau) = \frac{\delta_{t\tau}}{\mu(\tau)},$$

on its domain, where $\delta_{t\tau}$ is the Kronecker delta, i.e., $\delta_{t\tau} = 1$ for $t = \tau$ and $\delta_{t\tau} = 0$ for $t \neq \tau$.

Proof. We will show this by direct computation. First, for $t \leq \tau$,

$$\begin{aligned}
-\Delta^2 G(t,\tau) &= -\Delta\Delta G(t,\tau) \\
&= -\Delta\left(\frac{h_1(\beta,\sigma(\tau))}{h_1(\beta,\alpha)} h_0(t,\alpha)\right) \\
&= -\Delta\left(\frac{h_1(\beta,\sigma(\tau))}{h_1(\beta,\alpha)}\right) \\
&= 0.
\end{aligned}$$

Now, for $t > \tau$,

$$\begin{aligned}
-\Delta^2 G(t,\tau) &= -\Delta\Delta G(t,\tau) \\
&= -\Delta\left(\frac{h_1(\beta,\sigma(\tau))}{h_1(\beta,\alpha)} h_0(t,\alpha) - h_0(t,\sigma(\tau))\right) \\
&= -\Delta\left(\frac{h_1(\beta,\sigma(\tau))}{h_1(\beta,\alpha)} - 1\right) \\
&= 0.
\end{aligned}$$

Finally now for $t = \tau$,

$$
\begin{aligned}
&-\Delta^2 G(t,\tau)\\
&= -\Delta\Delta G(t,\tau)\\
&= -\Delta\left(\frac{G(\sigma(t),\tau) - G(t,\tau)}{\mu(t)}\right)\\
&= -\frac{\frac{G(\sigma^2(t),\tau)-G(\sigma(t),\tau)}{\mu(\sigma(t))} - \frac{G(\sigma(t),\tau)-G(t,\tau)}{\mu(t)}}{\mu(t)}\\
&= -\frac{1}{\mu(t)}\left[\frac{\frac{\beta-\sigma(\tau)}{\beta-\alpha}(\sigma^2(t)-\alpha) - (\sigma^2(t)-\sigma(\tau)) - \frac{\beta-\sigma(\tau)}{\beta-\alpha}(\sigma(t)-\alpha) + (\sigma(t)-\sigma(\tau))}{\mu(\sigma(t))}\right.\\
&\qquad\left. - \frac{\frac{\beta-\sigma(\tau)}{\beta-\alpha}(\sigma(t)-\alpha) - (\sigma(t)-\sigma(\tau)) - \frac{\beta-\sigma(\tau)}{\beta-\alpha}(t-\alpha)}{\mu(t)}\right]\\
&= -\frac{1}{\mu(\tau)}\left[\frac{\frac{\beta-\sigma(\tau)}{\beta-\alpha}(\sigma^2(\tau)-\alpha-\sigma(\tau)+\alpha) - \sigma^2(\tau) + \sigma(\tau) + \sigma(\tau) - \sigma(\tau)}{\mu(\sigma(\tau))}\right.\\
&\qquad\left. - \frac{\frac{\beta-\sigma(\tau)}{\beta-\alpha}(\sigma(\tau)-\alpha-\tau+\alpha) - \sigma(\tau) + \sigma(\tau)}{\mu(\tau)}\right.\\
&= -\frac{1}{\mu(\tau)}\left[\frac{\frac{\beta-\sigma(\tau)}{\beta-\alpha}\mu(\sigma(\tau)) - \mu(\sigma(\tau))}{\mu(\sigma(\tau))} - \frac{\frac{\beta-\sigma(\tau)}{\beta-\alpha}\mu(\tau)}{\mu(\tau)}\right]\\
&= -\frac{1}{\mu(\tau)}\left[\frac{\beta-\sigma(\tau)}{\beta-\alpha} - 1 - \frac{\beta-\sigma(\tau)}{\beta-\alpha}\right]\\
&= \frac{1}{\mu(\tau)}.
\end{aligned}
$$

Therefore,

$$-\Delta^2 G(t,\tau) = \frac{\delta_{t\tau}}{\mu(\tau)}.$$

□

Remark 4.29. As a result of the above corollary, we can verify the solution to the difference equation of Theorem 4.24 (where $i = K(t)$):

$$\begin{aligned}
-\Delta^2 y(t) &= \sum_{j=0}^{K(\beta)-1} (-\Delta^2 G(t, \sigma^j(\alpha))) f(\sigma^j(\alpha)) \mu(\sigma^j(\alpha)) \\
&= \frac{1}{\mu(\sigma^i(\alpha))} f(\sigma^i(\alpha)) \mu(\sigma^i(\alpha)) \\
&= f(\sigma^i(\alpha)) \\
&= f(t).
\end{aligned}$$

Corollary 4.30. *As defined in Theorem 4.24, $G(t,\tau)$ is the unique function defined on $\mathbb{T}_\alpha^\beta \times \mathbb{T}_\alpha^{\rho(\beta)}$ such that $G(\alpha,\tau) = 0 = G(\beta,\tau)$ and $-\Delta^2 G(t,\tau) = \frac{\delta_{t\tau}}{\mu(\tau)}$.*

Proof. If there exists another such function, say $H(t,\tau)$, then fix $\tau \in \mathbb{T}_\alpha^{\rho(\beta)}$ and let

$$y(t) := G(t,\tau) - H(t,\tau).$$

Then

$$\begin{aligned}
-\Delta^2 y(t) = -\Delta^2 G(t,\tau) + \Delta^2 H(t,\tau) = 0 \\
\implies y(t) = c_0 + c_1 h_1(t,\alpha).
\end{aligned}$$

But then

$$y(\alpha) = G(\alpha,\tau) - H(\alpha,\tau) = 0,$$

and

$$y(\beta) = G(\beta,\tau) - H(\beta,\tau) = 0,$$

so

$$y(t) = G(t,\tau) - H(t,\tau) \equiv 0.$$

As $\tau \in \mathbb{T}_\alpha^{\rho(\beta)}$ was arbitrary, we have $G(t,\tau) \equiv H(t,\tau)$ on $\mathbb{T}_\alpha^\beta \times \mathbb{T}_\alpha^{\rho(\beta)}$. $\square$

Theorem 4.31. *The unique solution of*

$$\begin{cases} -\Delta^2 y(t) = f(t), \quad t \in \mathbb{T}_\alpha, \\ y(\alpha) = A_1, \quad y(\beta) = A_2 \end{cases}$$

is given by

$$y(t) = u(t) + \int_\alpha^\beta G(t,\tau) f(\tau)\,\Delta\tau = u(t) + \sum_{j=0}^{K(\beta)-1} G(t,\sigma^j(\alpha)) f(\sigma^j(\alpha)) \mu(\sigma^j(\alpha)),$$

where $u(t)$ solves

$$\begin{cases} -\Delta^2 y(t) = 0, \quad t \in \mathbb{T}_\alpha, \\ y(\alpha) = A_1, \quad y(\beta) = A_2 \end{cases}$$

and $G(t,\tau)$ is the Green's function as defined in Theorem 4.24.

Proof. Let us first verify that the solution satisfies the initial conditions. At $t = \alpha$ we have

$$\begin{aligned} y(\alpha) &= u(\alpha) + \int_\alpha^\beta G(\alpha,\tau) f(\tau)\,\Delta\tau \\ &= u(\alpha) + \int_\alpha^\beta 0 \cdot f(\tau)\,\Delta\tau \\ &= u(\alpha) = A_1, \end{aligned}$$

and, similarly, at $t = \beta$, we have

$$y(\beta) = u(\beta) + \int_\alpha^\beta G(\beta, \tau) f(\tau)\,\Delta\tau = A_2.$$

Now

$$\begin{aligned} -\Delta^2 y(t) &= -\Delta^2\Big[u(t) + \int_\alpha^\beta G(t,\tau) f(\tau)\,\Delta\tau\Big] \\ &= -\Delta^2 u(t) - \Delta^2 \int_\alpha^\beta G(t,\tau) f(\tau)\,\Delta\tau \\ &= 0 + f(t) = f(t). \end{aligned}$$

Since $G(t, \tau)$ is the unique Green's function with the properties from Corollary 4.30, we have our result. □

We now prove a comparison theorem for solutions of boundary value problems like that in Theorem 4.31.

Theorem 4.32. *If $u(t)$ and $v(t)$ satisfy*

$$\begin{cases} \Delta^2 u(t) \le \Delta^2 v(t), \quad t \in \mathbb{T}_\alpha^\beta, \\ u(\alpha) \ge v(\alpha), \\ u(\beta) \ge v(\beta), \end{cases}$$

then $u(t) \ge v(t)$ on $\mathbb{T}_\alpha^\beta$.

Proof. Let $w(t) := u(t) - v(t)$. Then for $t \in \mathbb{T}_\alpha^\beta$

$$f(t) := -\Delta^2 w(t) = -\Delta^2 u(t) + \Delta^2 v(t) \ge 0.$$

If $A_1 := u(\alpha) - v(\alpha) \geq 0$ and $A_2 := u(\beta) - v(\beta) \geq 0$, then $w(t)$ solves the boundary value problem

$$\begin{cases} -\Delta^2 w(t) = f(t), \\ w(\alpha) = A_1, \quad w(\beta) = A_2. \end{cases}$$

Thus, by Theorem 4.31

$$w(t) = y(t) + \int_{\alpha}^{\beta} G(t,\tau) f(\tau)\, \Delta\tau,$$

where $G(t,\tau)$ is the Green's function defined in Theorem 4.24 and $y(t)$ is the solution of

$$\begin{cases} -\Delta^2 y(t) = 0, \quad t \in \mathbb{T}_{\alpha}, \\ y(\alpha) = A_1, \quad y(\beta) = A_2. \end{cases}$$

Since $-\Delta^2 y(t) = 0$ has solution

$$y(t) = c_0 + c_1 h_t(t,\alpha) = c_0 + c_1(t-\alpha),$$

and both $y(\alpha), y(\beta) \geq 0$, then we have $y(t) \geq 0$. By Theorem 4.26 $G(t,\tau) \geq 0$, and, thus, we have

$$w(t) = y(t) + \int_{\alpha}^{\beta} G(t,\tau) f(\tau)\, \Delta\tau \geq 0.$$

□

4.3 Even-Ordered Boundary Value Problems with Even-Ordered Boundary Conditions

We will now focus our attention on problems similar to those from Chapter 2, of which the second-order boundary value problems in Section 4.2 are a special case. A minor difference here is that we will make the problem a little more general by allowing the boundary conditions to be nonzero. Note that in the iterated integrals that follow, many Green's functions will only take into account a domain of $\mathbb{T}_\alpha^{\rho(\beta)} \times \mathbb{T}_\alpha^{\rho(\beta)}$, but this only neglects domain elements where the Green's function's value is zero.

Theorem 4.33. *For $y : \mathbb{T}_\alpha^{\sigma^{2n}(\beta)} \to \mathbb{R}$, $f : \mathbb{T}_\alpha^\beta \to \mathbb{R}$, and $n \in \mathbb{N}$, the boundary value problem*

$$\begin{cases} (-1)^n \Delta^{2n} y(t) = f(t), \\ (-1)^k \Delta^{2k} y(\alpha) = A_k, \\ (-1)^k \Delta^{2k} y(\beta) = B_k, \end{cases}$$

where $k = 0, 1, 2, \ldots, n-1$, has the unique solution

$$y(t) = u_0(t) + \sum_{k=1}^{n} \int_\alpha^\beta \int_\alpha^\beta \cdots \int_\alpha^\beta G(t, \tau_n) G(\tau_n, \tau_{n-1}) \cdots$$
$$G(\tau_{n-k+2}, \tau_{n-k+1}) u_k(\tau_{n-k+1}) \, \Delta\tau_{n-k+1} \, \Delta\tau_{n-k+2} \cdots \Delta\tau_n,$$

where $u_n(t) \equiv f(t)$ and $u_k(t)$ is the solution to

$$\begin{cases} -\Delta^2 y(t) = 0, \\ y(\alpha) = A_k, \quad y(\beta) = B_k. \end{cases}$$

Proof. By Theorem 4.31, we already have the result for the case $n = 1$. Before we do the inductive step, let us see what happens in the case $n = 2$, which results in the boundary value problem

$$\begin{cases} \Delta^4 y(t) = f(t), \\ y(\alpha) = A_0, \quad y(\beta) = B_0, \\ -\Delta^2 y(\alpha) = A_1, \quad -\Delta^2 y(\beta) = B_1. \end{cases}$$

Let $w(t) := -\Delta^2 y(t)$. We may then turn our attention to the problem

$$\begin{cases} -\Delta^2 w(t) = f(t), \\ w(\alpha) = A_1, \quad w(\beta) = B_1, \end{cases}$$

which, from the case $n = 1$ has solution

$$w(t) = u_1(t) + \int_\alpha^\beta G(t, \tau_1) f(\tau_1)\, \Delta\tau_1.$$

We also have

$$\begin{cases} -\Delta^2 y(t) = w(t) = u_1(t) + \displaystyle\int_\alpha^\beta G(t, \tau_1) w(\tau_1)\, \Delta\tau_1 \\ y(\alpha) = A_0, \quad y(\beta) = B_0, \end{cases}$$

which has solution

$$\begin{aligned} y(t) &= u_0(t) + \int_\alpha^\beta G(t, \tau_2) w(\tau_2)\, \Delta\tau_2 \\ &= u_0(t) + \int_\alpha^\beta G(t, \tau_2) \big[u_1(\tau_2) + \int_\alpha^\beta G(\tau_2, \tau_1) f(\tau_1) \Delta\tau_1\big]\, \Delta\tau_2 \\ &= u_0(t) + \int_\alpha^\beta \Big[G(t, \tau_2) u_1(\tau_2) + G(t, \tau_2) \int_\alpha^\beta G(\tau_2, \tau_1) f(\tau_1)\, \Delta\tau_1\Big]\, \Delta\tau_2 \\ &= u_0(t) + \int_\alpha^\beta G(t, \tau_2) u_1(\tau_2) \Delta\tau_2 + \int_\alpha^\beta \int_\alpha^\beta G(t, \tau_2) G(\tau_2, \tau_1) f(\tau_1)\, \Delta\tau_1\, \Delta\tau_2 \\ &= u_0(t) + \int_\alpha^\beta G(t, \tau_2) u_1(\tau_2) \Delta\tau_2 + \int_\alpha^\beta \int_\alpha^\beta G(t, \tau_2) G(\tau_2, \tau_1) u_2(\tau_1)\, \Delta\tau_1\, \Delta\tau_2, \end{aligned}$$

thus showing the result holds when $n = 2$.

Now suppose the result holds for some $n \in \mathbb{N}$. We will then consider the boundary value problem

$$\begin{cases} (-1)^{n+1}\Delta^{2(n+1)}y(t) = f(t), \\ (-1)^{2i}\Delta^{2i}y(\alpha) = A_i, \\ (-1)^{2i}\Delta^{2i}y(\beta) = B_i, \end{cases}$$

where $i = 0, 1, 2, \ldots, n$. Let $w(t) := (-1)^{2n}y(t)$. We may then consider the boundary value problem

$$\begin{cases} (-1)^{n}\Delta^{2n}y(t) = w(t), \\ (-1)^{2i}\Delta^{2i}y(\alpha) = A_i, \\ (-1)^{2i}\Delta^{2i}y(\beta) = B_i, \end{cases}$$

where $i = 0, 1, 2, \ldots, n-1$, which, by the inductive step, has solution

$$\begin{aligned} y(t) = u_0(t) + \sum_{k=1}^{n-1} \int_\alpha^\beta \int_\alpha^\beta \cdots \int_\alpha^\beta G(t,\tau_n)G(\tau_n,\tau_{n-1})\cdots \\ G(\tau_{n-k+2},\tau_{n-k+1})u_k(\tau_{n-k+1})\,\Delta\tau_{n-k+1}\,\Delta\tau_{n-k+2}\cdots\Delta\tau_n \\ + \int_\alpha^\beta \int_\alpha^\beta \cdots \int_\alpha^\beta G(t,\tau_n)G(\tau_n,\tau_{n-1})\cdots G(\tau_2,\tau_1)w(\tau_1)\,\Delta\tau_1\,\Delta\tau_2\cdots\Delta\tau_n \\ = u_0(t) + \sum_{k=1}^{n-1} \int_\alpha^\beta \int_\alpha^\beta \cdots \int_\alpha^\beta G(t,\tau_n)G(\tau_n,\tau_{n-1})\cdots \\ G(\tau_{n-k+2},\tau_{n-k+1})u_k(\tau_{n-k+1})\,\Delta\tau_{n-k+1}\,\Delta\tau_{n-k+2}\cdots\Delta\tau_n \\ + \int_\alpha^\beta \int_\alpha^\beta \cdots \int_\alpha^\beta G(t,\tau_{n+1})G(\tau_{n+1},\tau_n)\cdots G(\tau_3,\tau_2)w(\tau_2)\,\Delta\tau_2\,\Delta\tau_3\cdots\Delta\tau_{n+1} \\ = u_0(t) + \sum_{k=1}^{n-1} \int_\alpha^\beta \int_\alpha^\beta \cdots \int_\alpha^\beta G(t,\tau_{n+1})G(\tau_{n+1},\tau_n)\cdots \\ G(\tau_{n-k+3},\tau_{n-k+2})u_k(\tau_{n-k+2})\,\Delta\tau_{n-k+2}\,\Delta\tau_{n-k+3}\cdots\Delta\tau_{n+1} \\ + \int_\alpha^\beta \int_\alpha^\beta \cdots \int_\alpha^\beta G(t,\tau_{n+1})G(\tau_{n+1},\tau_n)\cdots G(\tau_3,\tau_2)w(\tau_2)\,\Delta\tau_2\,\Delta\tau_3\cdots\Delta\tau_{n+1}. \end{aligned}$$

Now, as

$$\begin{cases} -\Delta^2 w(t) = f(t), \\ w(\alpha) = A_n, \quad w(\beta) = B_n, \end{cases}$$

we have

$$\begin{aligned} w(t) &= u_n(t) + \int_\alpha^\beta G(t,\tau_1) f(\tau_1)\,\Delta\tau_1 \\ &= u_n(t) + \int_\alpha^\beta G(t,\tau_1) u_{n+1}(\tau_1)\,\Delta\tau_1, \end{aligned}$$

where $u_{n+1} \equiv f$. Therefore,

$$
\begin{aligned}
&y(t) \\
&= u_0(t) + \sum_{k=1}^{n-1} \int_\alpha^\beta \int_\alpha^\beta \cdots \int_\alpha^\beta G(t,\tau_{n+1}) G(\tau_{n+1},\tau_n) \cdots \\
&\qquad\qquad G(\tau_{n-k+3},\tau_{n-k+2}) u_k(\tau_{n-k+2}) \, \Delta\tau_{n-k+2} \, \Delta\tau_{n-k+3} \cdots \Delta\tau_{n+1} \\
&\quad + \int_\alpha^\beta \int_\alpha^\beta \cdots \int_\alpha^\beta G(t,\tau_{n+1}) G(\tau_{n+1},\tau_n) \cdots G(\tau_3,\tau_2) w(\tau_2) \, \Delta\tau_2 \, \Delta\tau_3 \cdots \Delta\tau_{n+1} \\
&= u_0(t) + \sum_{k=1}^{n-1} \int_\alpha^\beta \int_\alpha^\beta \cdots \int_\alpha^\beta G(t,\tau_{n+1}) G(\tau_{n+1},\tau_n) \cdots \\
&\qquad\qquad G(\tau_{n-k+3},\tau_{n-k+2}) u_k(\tau_{n-k+2}) \, \Delta\tau_{n-k+2} \, \Delta\tau_{n-k+3} \cdots \Delta\tau_{n+1} \\
&\quad + \int_\alpha^\beta \int_\alpha^\beta \cdots \int_\alpha^\beta G(t,\tau_{n+1}) G(\tau_{n+1},\tau_n) \cdots \\
&\qquad\qquad G(\tau_3,\tau_2) \big[u_n(\tau_2) + \int_\alpha^\beta G(\tau_2,\tau_1) u_{n+1}(\tau_1) \, \Delta\tau_1 \big] \, \Delta\tau_2 \, \Delta\tau_3 \cdots \Delta\tau_{n+1} \\
&= u_0(t) + \sum_{k=1}^{n-1} \int_\alpha^\beta \int_\alpha^\beta \cdots \int_\alpha^\beta G(t,\tau_{n+1}) G(\tau_{n+1},\tau_n) \cdots \\
&\qquad\qquad G(\tau_{n-k+3},\tau_{n-k+2}) u_k(\tau_{n-k+2}) \, \Delta\tau_{n-k+2} \, \Delta\tau_{n-k+3} \cdots \Delta\tau_{n+1} \\
&\quad + \int_\alpha^\beta \int_\alpha^\beta \cdots \int_\alpha^\beta G(t,\tau_{n+1}) G(\tau_{n+1},\tau_n) \cdots G(\tau_3,\tau_2) u_n(\tau_2) \, \Delta\tau_2 \, \Delta\tau_3 \cdots \Delta\tau_{n+1} \\
&\quad + \int_\alpha^\beta \int_\alpha^\beta \cdots \int_\alpha^\beta G(t,\tau_{n+1}) G(\tau_{n+1},\tau_n) \cdots \\
&\qquad\qquad G(\tau_3,\tau_2) \int_\alpha^\beta G(\tau_2,\tau_1) u_{n+1}(\tau_1) \, \Delta\tau_1 \, \Delta\tau_2 \, \Delta\tau_3 \cdots \Delta\tau_{n+1} \\
&= u_0(t) + \sum_{k=1}^{n+1} \int_\alpha^\beta \int_\alpha^\beta \cdots \int_\alpha^\beta G(t,\tau_{n+1}) G(\tau_{n+1},\tau_n) \cdots \\
&\qquad\qquad G(\tau_{n-k+2},\tau_{n-k+1}) u_k(\tau_{n-k+1}) \, \Delta\tau_{n-k+1} \, \Delta\tau_{n-k+2} \cdots \Delta\tau_{n+1},
\end{aligned}
$$

which gives the result. □

Remark 4.34. If, as in Chapter 2, we focus on Lidstone boundary conditions, we have that, according to Theorem 4.33,

$$\begin{cases} (-1)^n \Delta^{2n} y(t) = f(t), \\ \qquad \Delta^{2k} y(\alpha) = 0, \\ \qquad \Delta^{2k} y(\beta) = 0, \end{cases}$$

where $k = 0, 1, 2, \ldots, n-1$, has solution

$$y(t) = \int_\alpha^\beta \int_\alpha^\beta \cdots \int_\alpha^\beta G(t, \tau_n) G(\tau_n, \tau_{n-1}) \cdots G(\tau_2, \tau_1) f(\tau_1)\, \Delta\tau_1\, \Delta\tau_2 \cdots \Delta\tau_n,$$

since $u_k(t) \equiv 0$ for all $k = 0, 1, 2, \ldots, n-1$.

As in Chapter 2, we may find it convenient to make the following definition.

Definition 4.35. Let

$$\begin{aligned} \mathcal{G}_n(t, \tau_n) := G(t, \tau_n) \sum_{j_{n-1}=0}^{K(\beta)-1} G(\tau_n, \sigma^{j_{n-1}}(\alpha)) \sum_{j_{n-2}=0}^{K(\beta)-1} G(\sigma^{j_{n-1}}(\alpha), \sigma^{j_{n-2}}(\alpha)) \cdots \\ \sum_{j_1=0}^{K(\beta)-1} G(\sigma^{j_2}(\alpha), \sigma^{j_1}(\alpha)) \mu(\sigma^{j_1}(\alpha)) \mu(\sigma^{j_2}(\alpha)) \cdots \mu(\sigma^{j_{n-1}}(\alpha)). \end{aligned}$$

Remark 4.36. We can note that

$$\mathcal{G}_n(t, \tau_n) = G(t, \tau_n) \int_\alpha^\beta G(\tau_n, \tau_{n-1}) \int_\alpha^\beta G(\tau_{n-1}, \tau_{n-2}) \cdots \int_\alpha^\beta G(\tau_2, \tau_1)\, \Delta\tau_1\, \Delta\tau_2 \cdots \Delta\tau_{n-1},$$

but in the results that follow, we will typically use the summation form from the definition above.

Corollary 4.37. *For any* $\tau_n \in \mathbb{T}_\alpha^{\rho(\beta)}$

$$\max_{t\in\mathbb{T}_\alpha^\beta} \mathcal{G}_n(t,\tau_n) = \mathcal{G}_n(\sigma(\tau_n),\tau_n).$$

Proof. For each $\tau \in \mathbb{T}_\alpha^{\rho(\beta)}$, we have, from Theorem 4.26

$$\max_{t\in\mathbb{T}_\alpha^\beta} G(t,\tau) = G(\sigma(\tau),\tau),$$

and $G(t,\tau) \geq 0$ on its domain. So for all $t \in \mathbb{T}_\alpha^\beta$, we have

$$\begin{aligned}
&\mathcal{G}_n(t,\tau_n) \\
&= G(t,\tau_n) \sum_{j_{n-1}=0}^{K(\beta)-1} G(\tau_n, \sigma^{j_{n-1}}(\alpha)) \sum_{j_{n-2}=0}^{K(\beta)-1} G(\sigma^{j_{n-1}}(\alpha), \sigma^{j_{n-2}}(\alpha)) \cdots \\
&\qquad\qquad \sum_{j_1=0}^{K(\beta)-1} G(\sigma^{j_2}(\alpha), \sigma^{j_1}(\alpha))\mu(\sigma^{j_1}(\alpha))\mu(\sigma^{j_2}(\alpha)) \cdots \mu(\sigma^{j_{n-1}}(\alpha)) \\
&= G(t,\tau_n) \sum_{j_{n-1}=0}^{K(\beta)-1} \mathcal{G}_{n-1}(\tau_n, \sigma^{j_{n-1}}(\alpha))\mu(\sigma^{j_{n-1}}(\alpha)) \\
&\leq G(\sigma(\tau_n),\tau_n) \sum_{j_{n-1}=0}^{K(\beta)-1} \mathcal{G}_{n-1}(\tau_n, \sigma^{j_{n-1}}(\alpha))\mu(\sigma^{j_{n-1}}(\alpha)) \\
&= \mathcal{G}_n(\sigma(\tau_n),\tau_n),
\end{aligned}$$

and

$$\mathcal{G}_n(\sigma(\tau_n),\tau_n) = G(\sigma(\tau_n),\tau_n) \sum_{j_{n-1}=0}^{K(\beta)-1} \mathcal{G}_{n-1}(\tau_n, \sigma^{j_{n-1}}(\alpha))\mu(\sigma^{j_{n-1}}(\alpha)) = \mathcal{G}_n(\sigma(\tau_n),\tau_n).$$

Thus, we have our result. $\square$

4.4 Existence and Uniqueness Theorems

Similar to earlier work in this thesis, we will define the following domain and present a lemma which will be used to help us provide some existence and uniqueness results.

Definition 4.38. Given the domain $\mathbb{T}_\alpha^\beta$, let

$$\mathcal{D} := [\sigma^{\lceil K(\beta)/4 \rceil}(\alpha), \sigma^{\lfloor 3K(\beta)/4 \rfloor}(\alpha)] \cap \mathbb{T}_\alpha.$$

Remark 4.39. Note that we have $\varnothing \neq \mathcal{D} \subsetneq \mathbb{T}_\alpha^\beta$ if $K(\beta) \geq 2$.

Lemma 4.40. *Given that $\mathcal{D} \neq \varnothing$, there exists $\gamma \in (0,1)$ such that for any $\tau_n \in \mathbb{T}_\alpha^{\rho(\beta)}$,*

$$\min_{t \in \mathcal{D}} \mathcal{G}_n(t, \tau_n) \geq \gamma \left(\max_{t \in \mathbb{T}_\alpha^\beta} \mathcal{G}_n(t, \tau_n) \right) = \gamma \mathcal{G}_n(\sigma(\tau_n), \tau_n).$$

Proof. Note that this is trivially true if τ_n is chosen such that $\mathcal{G}_n(t, \tau_n) = 0$. Otherwise, for any $t \in \mathcal{D}$, a set of finite points, we have

$$\frac{\mathcal{G}_n(t, \tau_n)}{\max_{t \in \mathbb{T}_\alpha^\beta} \mathcal{G}_n(t, \tau_n)} = \frac{\mathcal{G}_n(t, \tau_n)}{\mathcal{G}_n(\sigma(\tau_n), \tau_n)} \in (0, 1],$$

since $\max_{t \in \mathbb{T}_\alpha^\beta} \mathcal{G}_n(t, \tau_n) \geq \mathcal{G}_n(t, \tau_n)$ for any $t \in \mathbb{T}_\alpha^\beta$ and $\mathcal{G}_n(t, \tau_n) \neq 0$ for $t \in \mathcal{D}$ as a result of Remark 4.27. Since t (and τ_n) comes from a domain with a finite number of points, we can find γ such that

$$0 < \gamma < \min_{t \in \mathcal{D}} \frac{\mathcal{G}_n(t, \tau_n)}{\mathcal{G}_n(\sigma(\tau_n), \tau_n)} \leq 1.$$

Therefore, we have $\gamma \in (0,1)$ such that

$$\min_{t \in \mathcal{D}} \mathcal{G}_n(t, \tau_n) \geq \gamma \left(\max_{t \in \mathbb{T}_\alpha^\beta} \mathcal{G}_n(t, \tau_n) \right) = \gamma \mathcal{G}_n(\sigma(\tau_n), \tau_n).$$

□

Throughout this section, we will consider a problem of the form

$$\begin{cases} (-1)^n \Delta^{2n} y(t) = f(t, y(t)), \quad t \in \mathbb{T}_\alpha^\beta, n \in \mathbb{N} \\ \qquad \Delta^{2i} y(\alpha) = 0, \\ \qquad \Delta^{2i} y(\beta) = 0, \end{cases} \tag{4.4.1}$$

where $i = 0, 1, 2, \ldots, n-1$, and $f : \mathbb{T}_\alpha^\beta \times \mathbb{R} \to \mathbb{R}$.

We can note that y solves this boundary value problem if and only if y is a fixed point of the operator $T : \mathcal{B} \to \mathcal{B}$ defined by

$$Ty := \sum_{j_n=0}^{K(\beta)-1} \mathcal{G}_n(t, \sigma^{j_n}(\alpha)) f(\sigma^{j_1}(\alpha), y(\sigma^{j_1}(\alpha)))\mu(\sigma^{j_n}(\alpha),$$

and where $\mathcal{B}$ is the Banach space

$$\mathcal{B} := \{y : \mathbb{T}_\alpha \to \mathbb{R} \,\big|\, \text{the boundary conditions of (4.4.1) hold}\}, \tag{4.4.2}$$

equipped with the supremum norm $\|\cdot\|$.

As in Chapter 2, we will define two constants:

$$\eta := \left(\sum_{j_n=0}^{K(\beta)-1} \mathcal{G}_n(\sigma^{j_n+1}(\alpha), \sigma^{j_n}(\alpha))\mu(\sigma^{j_n}(\alpha)) \right)^{-1}$$

$$\lambda := \Bigg(\sum_{j_n=0}^{K(\beta)-1} G(\sigma(\alpha), \sigma^{j_n}(\alpha)) \sum_{j_{n-1}=0}^{K(\beta)-1} G(\sigma^{j_n}(\alpha), \sigma^{j_{n-1}}(\alpha)) \cdots$$
$$\sum_{j_2=0}^{K(\beta)-1} G(\sigma^{j_3}(\alpha), \sigma^{j_2}(\alpha)) \sum_{j_1=\lceil K(\beta)/4 \rceil}^{\lfloor 3K(\beta)/4 \rfloor} G(\sigma^{j_2}(\alpha), \sigma^{j_1}(\alpha))\mu(\sigma^{j_1}(\alpha))\mu(\sigma^{j_2}(\alpha)) \cdots$$
$$\mu(\sigma^{j_n}(\alpha)) \Bigg)^{-1}$$

As in Chapter 2, since G is nonzero and positive at least at some points in a nontrivial domain, both η and λ will be positive real numbers. Also, let us consider two conditions regarding f that will be used in the following theorem:

(C1) There exists a number $r > 0$ such that $f(t, y) \leq \eta r$ whenever $0 \leq y \leq r$.

(C2) There exists a number $r > 0$ such that $f(t, y) \geq \lambda r$ whenever $t \in \mathcal{D}$ and $\gamma r \leq y \leq r$, where γ is as in Lemma 4.40.

Remark 4.41. As in Chapter 2, there would be many such functions f that satisfy the above conditions since f is free to satisfy (C1) and (C2) at distinct values of r. Also note a characterization of positive solutions mentioned in Remark 2.12.

Theorem 4.42. *Suppose there exist positive and distinct r_1 and r_2 such that (C1) holds at $r = r_1$ and (C2) holds at $r = r_2$. Suppose also that $f(t, y) \geq 0$ and continuous. Then the boundary value problem (4.4.1) has at least one positive solution y_0 such that $\|y_0\|$ lies between r_1 and r_2.*

Proof. Without loss of generality, suppose $0 < r_1 < r_2$. We will consider the cone $\mathcal{K} := \{y \in \mathcal{B} \,|\, y(t) \geq 0, \min_{t \in \mathcal{D}} y(t) \geq \gamma \|y\|\} \subseteq \mathcal{B}$. Now whenever $y \in \mathcal{K}$, we have

$(Ty)(t) \geq 0$, and

$$\begin{aligned}
\min_{t\in\mathcal{D}}(Ty)(t) &= \min_{t\in\mathcal{D}} \sum_{j_n=0}^{K(\beta)-1} \mathcal{G}_n(t,\sigma^{j_n}(\alpha))f(\sigma^{j_1}(\alpha),y(\sigma^{j_1}(\alpha)))\mu(\sigma^{j_n}(\alpha)) \\
&\geq \gamma \left(\sum_{j_n=0}^{K(\beta)-1} \mathcal{G}_n(\sigma^{j_n+1}(\alpha),\sigma^{j_n}(\alpha))f(\sigma^{j_1}(\alpha),y(\sigma^{j_1}(\alpha)))\mu(\sigma^{j_n}(\alpha)) \right) \\
&= \gamma \left(\max_{t\in\mathbb{T}_\alpha^\beta} \mathcal{G}_n(t,\sigma^{j_n}(\alpha))f(\sigma^{j_1}(\alpha),y(\sigma^{j_1}(\alpha)))\mu(\sigma^{j_n}(\alpha)) \right) \\
&= \gamma\|Ty\|,
\end{aligned}$$

i.e., $Ty \in \mathcal{K}$. So $T : \mathcal{K} \to \mathcal{K}$. We can also note that T is a completely continuous operator.

Now let $\Omega_1 := \{y \in \mathcal{K} : \|y\| < r_1\}$. For $y \in \partial\Omega_1$, we have $\|y\| = r_1$, so condition (C1) applies for all $y \in \partial\Omega_1$. Thus, for $y \in \mathcal{K} \cap \partial\Omega_1$, we have

$$\begin{aligned}
\|Ty\| &= \max_{t\in\mathbb{T}_\alpha^\beta} \sum_{j_n=0}^{K(\beta)-1} \mathcal{G}_n(t,\sigma^{j_n}(\alpha))f(\sigma^{j_1}(\alpha),y(\sigma^{j_1}(\alpha)))\mu(\sigma^{j_n}(\alpha)) \\
&\leq \sum_{j_n=0}^{K(\beta)-1} \mathcal{G}_n(\sigma^{j_n+1}(\alpha),\sigma^{j_n}(\alpha))f(\sigma^{j_1}(\alpha),y(\sigma^{j_1}(\alpha)))\mu(\sigma^{j_n}(\alpha)) \\
&\leq \eta r_1 \sum_{j_n=0}^{K(\beta)-1} \mathcal{G}_n(\sigma^{j_n+1}(\alpha),\sigma^{j_n}(\alpha))\mu(\sigma^{j_n}(\alpha)) \\
&= r_1 \\
&= \|y\|.
\end{aligned}$$

Therefore, $\|Ty\| \leq \|y\|$ whenever $y \in \mathcal{K} \cap \partial\Omega_1$, which implies that T is a cone compression on $\mathcal{K} \cap \partial\Omega_1$.

Now let $\Omega_2 := \{y \in \mathcal{K} : \|y\| < r_2\}$. For $y \in \partial\Omega_2$, we have $\|y\| = r_2$, so condition

(C2) applies for all $y \in \partial\Omega_2$. Thus, for $y \in \mathcal{K} \cap \partial\Omega_2$, we have

$$
\begin{aligned}
&\|Ty\| \\
&\quad \geq (Ty)(\sigma(\alpha)) \\
&\quad = \sum_{j_n=0}^{K(\beta)-1} \mathcal{G}_n(\sigma(\alpha), \sigma^{j_n}(\alpha)) f(\sigma^{j_1}(\alpha), y(\sigma^{j_1}(\alpha)))\mu(\sigma^{j_n}(\alpha)) \\
&\quad = \sum_{j_n=0}^{K(\beta)-1} G(\sigma(\alpha), \sigma^{j_n}(\alpha)) \sum_{j_{n-1}=0}^{K(\beta)-1} G(\sigma^{j_n}(\alpha), \sigma^{j_{n-1}}(\alpha)) \cdots \\
&\qquad \sum_{j_1=0}^{K(\beta)-1} G(\sigma^{j_2}(\alpha), \sigma^{j_1}(\alpha)) f(\sigma^{j_1}(\alpha), y(\sigma^{j_1}(\alpha)))\mu(\sigma^{j_1}(\alpha))\mu(\sigma^{j_2}(\alpha)) \cdots \mu(\sigma^{j_n}(\alpha)) \\
&\quad \geq \sum_{j_n=0}^{K(\beta)-1} G(\sigma(\alpha), \sigma^{j_n}(\alpha)) \sum_{j_{n-1}=0}^{K(\beta)-1} G(\sigma^{j_n}(\alpha), \sigma^{j_{n-1}}(\alpha)) \cdots \\
&\qquad \sum_{j_1=\lceil K(\beta)/4 \rceil}^{\lfloor 3K(\beta)/4 \rfloor} G(\sigma^{j_2}(\alpha), \sigma^{j_1}(\alpha)) f(\sigma^{j_1}(\alpha), y(\sigma^{j_1}(\alpha)))\mu(\sigma^{j_1}(\alpha))\mu(\sigma^{j_2}(\alpha)) \cdots \mu(\sigma^{j_n}(\alpha)) \\
&\quad \geq \lambda r_2 \sum_{j_n=0}^{K(\beta)-1} G(\sigma(\alpha), \sigma^{j_n}(\alpha)) \sum_{j_{n-1}=0}^{K(\beta)-1} G(\sigma^{j_n}(\alpha), \sigma^{j_{n-1}}(\alpha)) \cdots \\
&\qquad \sum_{j_1=\lceil K(\beta)/4 \rceil}^{\lfloor 3K(\beta)/4 \rfloor} G(\sigma^{j_2}(\alpha), \sigma^{j_1}(\alpha))\mu(\sigma^{j_1}(\alpha))\mu(\sigma^{j_2}(\alpha)) \cdots \mu(\sigma^{j_n}(\alpha)) \\
&\quad = r_2 \\
&\quad = \|y\|.
\end{aligned}
$$

Therefore, $\|Ty\| \geq \|y\|$ whenever $y \in \mathcal{K} \cap \partial\Omega_2$, which implies that T is a cone expansion on $\mathcal{K} \cap \partial\Omega_2$. So now, by Theorem 2.2 we have that T has a fixed point, which implies that our boundary value problem has a positive solution y_0 such that $r_1 \leq \|y_0\| \leq r_2$. □

Now we introduce a Lemma that will help show uniqueness under a Lipschitz condition.

Lemma 4.43. *For $\mathcal{G}_n(t,\tau_n)$ defined previously, we have*

$$\max_{t\in\mathbb{T}_\alpha^\beta} \sum_{j_n=0}^{K(\beta)-1} \mathcal{G}_n(t,\tau_n) \leq \left[\frac{(\beta-\alpha)^2}{4}\right]^n.$$

Proof. We know $\max_{t\in\mathbb{T}_\alpha^\beta} G(t,\tau) = G(\sigma(\tau),\tau)$, and

$$\begin{aligned}
G(\sigma(\tau),\tau) &= \frac{\beta-\sigma(\tau)}{\beta-\alpha}(\sigma(\tau)-\alpha) - (\sigma(\tau)-\sigma(\tau)) \\
&= \frac{\beta-\sigma(\tau)}{\beta-\alpha}(\sigma(\tau)-\alpha) \\
&= \frac{\beta-a\tau-b}{\beta-\alpha}(a\tau+b-\alpha).
\end{aligned}$$

Now

$$\begin{aligned}
\frac{d}{d\tau}\left[\frac{\beta-a\tau-b}{\beta-\alpha}(a\tau+b-\alpha)\right] &= \frac{-a}{\beta-\alpha}(a\tau+b-\alpha) + \frac{\beta-a\tau-b}{\beta-\alpha}(a) \\
&= \frac{1}{\beta-\alpha}[-a^2\tau-ab+a\alpha+a\beta-a^2\tau-ab] \\
&= \frac{1}{\beta-\alpha}[-2a^2\tau-2ab+a\alpha+a\beta] \\
&= 0 \\
&\iff \tau = \frac{a\alpha+a\beta-2ab}{2a^2} = \frac{\alpha+\beta-2b}{2a},
\end{aligned}$$

and

$$\frac{d^2}{d\tau 2}\left[\frac{\beta-a\tau-b}{\beta-\alpha}(a\tau+b-\alpha)\right] = \frac{1}{\beta-\alpha}[-2a^2] < 0,$$

which implies

$$\begin{aligned}
G(\sigma(\tau),\tau) &\leq G\left(\sigma\left(\frac{\alpha+\beta-2b}{2a}\right),\frac{\alpha+\beta-2b}{2a}\right)\\
&= \frac{\beta - a\left(\frac{\alpha+\beta-2b}{2a}\right)+b}{\beta-\alpha}\left(a\left(\frac{\alpha+\beta-2b}{2a}\right)+b-\alpha\right)\\
&= \frac{\beta-\frac{1}{2}(\alpha+\beta-2b)+b}{\beta-\alpha}\left(\frac{1}{2}(\alpha+\beta-2b)+b-\alpha\right)\\
&= \frac{\frac{1}{2}(\beta-\alpha)}{\beta-\alpha}\left(\frac{1}{2}(\beta-\alpha)\right)\\
&= \frac{\beta-\alpha}{4}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{j_i=0}^{K(\beta)-1} G(\sigma^{j_i+1}(\alpha),\sigma^{j_i}(\alpha))\mu(\sigma^{j_i}(\alpha))\\
&\qquad\leq \sum_{j_i=0}^{K(\beta)-1}\frac{\beta-\alpha}{4}\mu(\sigma^{j_i}(\alpha))\\
&\qquad= \frac{\beta-\alpha}{4}[(\sigma(\alpha)-\alpha)+(\sigma^2(\alpha)-\sigma(\alpha))+\cdots+(\sigma^{K(\beta)}(\alpha)-\sigma^{K(\beta)-1}(\alpha)]\\
&\qquad= \frac{(\beta-\alpha)^2}{4}
\end{aligned}$$

for $i=1,2,3,\ldots,n$.

Therefore,

$$\begin{aligned}
&\max_{t\in\mathbb{T}_\alpha^\beta} \sum_{j_n=0}^{K(\beta)-1} \mathcal{G}_n(t,\tau_n)\mu(\sigma^{j_n}(\alpha)) \\
&= \max_{t\in\mathbb{T}_\alpha^\beta} \sum_{j_n=0}^{K(\beta)-1} G(t,\sigma^{j_n}(\alpha))\mu(\sigma^{j_n}(\alpha)) \sum_{j_{n-1}=0}^{K(\beta)-1} G(\sigma^{j_n}(\alpha),\sigma^{j_{n-1}}(\alpha))\mu(\sigma^{j_{n-1}}(\alpha))\cdots \\
&\qquad\qquad \sum_{j_1=0}^{K(\beta)-1} G(\sigma^{j_2}(\alpha),\sigma^{j_1}(\alpha))\mu(\sigma^{j_1}(\alpha)) \\
&\le \left[\frac{(\beta-\alpha)^2}{4}\right]\left[\frac{(\beta-\alpha)^2}{4}\right]\cdots\left[\frac{(\beta-\alpha)^2}{4}\right] \\
&= \left[\frac{(\beta-\alpha)^2}{4}\right]^n .
\end{aligned}$$

□

Finally, we prove a uniqueness theorem when f satisfies a Lipschitz condition.

Theorem 4.44. *Suppose $f(t,y)$ satisfies a Lipschitz condition in y with Lipschitz constant ξ, i.e., $|f(t,y_2)-f(t,y_1)| \le \xi|y_2-y_1|$ for all $(t,y_1),(t,y_2)$. Then if we have $\left[\frac{(\beta-\alpha)^2}{4}\right]^n < \frac{1}{\xi}$, the BVP (4.4.1) has a unique solution.*

Proof. Let $y_1, y_2 \in \mathcal{B}$, where $\mathcal{B}$ is the Banach space from (4.4.2). Then

$$\begin{aligned}
\|Ty_2 - Ty_1\| &\le \max_{t\in\mathbb{T}_\alpha^\beta} \sum_{j_n=0}^{K(\beta)-1} |\mathcal{G}_n(t,\sigma^{j_n}(\alpha)||f(\sigma^{j_1}(\alpha),y_2(\sigma^{j_1}(\alpha))) - f(\sigma^{j_1}(\alpha),y_1(\sigma^{j_1}(\alpha)))| \\
&\le \xi \sum_{j_n=0}^{K(\beta)-1} \mathcal{G}_n(\sigma^{j_n+1}(\alpha),\sigma^{j_n}(\alpha))|y_2(\sigma^{j_1}(\alpha)) - y_1(\sigma^{j_1}(\alpha))| \\
&\le \xi\|y_2-y_1\| \sum_{j_n=0}^{K(\beta)-1} \mathcal{G}_n(\sigma^{j_n+1}(\alpha),\sigma^{j_n}(\alpha)) \\
&\le \xi\|y_2-y_1\| \left[\frac{(\beta-\alpha)^2}{4}\right]^n,
\end{aligned}$$

which implies, by Theorem 2.3 (the Banach Contraction Theorem), since $\xi\left[\frac{(\beta-\alpha)^2}{4}\right]^n < 1$, we have a unique solution. $\square$

Example 4.45. If we consider the time scale $\mathbb{T}_\alpha = \mathbb{T}_0$ such that $\sigma(t) = 2t+1$, i.e.,

$$\mathbb{T}_0 := \{0, 1, 3, 7, 15, 31, 63, \ldots\},$$

then by the preceding theorem, if we let $\xi = \frac{1}{250}$ and $n = 2$, the boundary value problem (4.4.1) has a unique solution if

$$\left[\frac{(\beta-0)^2}{4}\right]^2 < 250$$
$$\Longrightarrow \beta^4 < 4000$$
$$\Longrightarrow \beta_{max} = 7,$$

where β_{max} is the largest value of $\beta \in \mathbb{T}_0$ such that the hypotheses of Theorem 4.44 are satisfied since $7^4 = 2401$ whereas $15^4 = 50625$.

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