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SYSTEMS OF NONLINEAR WAVE EQUATIONS WITH DAMPING AND SUPERCRITICAL SOURCES

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We consider the local and global well-posedness of the coupled nonlinear wave equations

$$\begin{aligned} u_{tt} - \Delta u + g_1(u_t) &= f_1(u,v) \\ v_{tt} - \Delta v + g_2(v_t) &= f_2(u,v), \end{aligned}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with a nonlinear Robin boundary condition on u and a zero boundary conditions on v. The nonlinearities $f_1(u,v)$ and $f_2(u,v)$ are with supercritical exponents representing strong sources, while $g_1(u_t)$ and $g_2(v_t)$ act as damping. It is well-known that the presence of a nonlinear boundary source causes significant difficulties since the linear Neumann problem for the single wave equation is not, in general, well-posed in the finite-energy space $H^1(\Omega) \times L^2(\partial\Omega)$ with boundary data from $L^2(\partial\Omega)$ (due to the failure of the uniform Lopatinskii condition). Additional challenges stem from the fact that the sources considered in this dissertation are *non-dissipative* and are not locally Lipschitz from $H^1(\Omega)$ into $L^2(\Omega)$ or $L^2(\partial\Omega)$. By employing nonlinear semigroups and the theory of monotone operators, we obtain several results on the existence of local and global weak solutions, and uniqueness of weak solutions. Moreover, we prove that such unique solutions depend continuously on the initial data. Under some restrictions on the parameters, we also prove that every weak solution to our system blows up in finite time, provided the initial energy is negative and the sources are more dominant than the damping in the system.

Additional results are obtained via careful analysis involving the Nehari Manifold. Specifically, we prove the existence of a unique global weak solution with initial data coming from the "good" part of the potential well. For such a global solution, we prove that the total energy of the system decays exponentially or algebraically, depending on the behavior of the dissipation in the system near the origin. Moreover, we prove a blow up result for weak solutions with *nonnegative initial energy*. Finally, we establish important generalization of classical results by H. Brézis in 1972 on convex integrals on Sobolev spaces. These results allowed us to overcome a major technical difficulty that faced us in the proof of the local existence of weak solutions.

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Chapter 1

Introduction

1.1 The Model

In this thesis, we study a system of coupled nonlinear wave equations which features two competing forces. One force is damping and the other is a strong source. Of central interest is the relationship of the source and damping terms to the behavior of solutions.

In order to simplify the exposition, we restrict our analysis to the physically more relevant case when $\Omega \subset \mathbb{R}^3$. Our results extend very easily to bounded domains in $\mathbb{R}^n$, by accounting for the corresponding Sobolev imbeddings, and accordingly adjusting the conditions imposed on the parameters. Therefore, throughout the paper we assume that Ω is bounded, open, and connected non-empty set in $\mathbb{R}^3$ with a smooth boundary $\Gamma = \partial\Omega$.

We study the local and global well-posedness of the following initial-boundary value problem:

$$\begin{cases} u_{tt} - \Delta u + g_1(u_t) = f_1(u,v) & \text{in } \Omega \times (0,T), \\ v_{tt} - \Delta v + g_2(v_t) = f_2(u,v) & \text{in } \Omega \times (0,T), \\ \partial_\nu u + u + g(u_t) = h(u) & \text{on } \Gamma \times (0,T), \\ v = 0 & \text{on } \Gamma \times (0,T), \\ u(0) = u_0 \in H^1(\Omega), u_t(0) = u_1 \in L^2(\Omega), \\ v(0) = v_0 \in H^1_0(\Omega), v_t(0) = v_1 \in L^2(\Omega), \end{cases} \tag{1.1.1}$$

where the nonlinearities $f_1(u,v)$, $f_2(u,v)$ and $h(u)$ are supercritical interior and boundary sources, and the damping functions g_1, g_2 and g are arbitrary continuous monotone increasing graphs vanishing at the origin.

The source-damping interaction in (1.1.1) encompasses a broad class of problems in quantum field theory and certain mechanical applications (Jörgens [25] and Segal [45]). A related model to (1.1.1) is the Reissner-Mindlin plate equations (see for instance, Ch. 3 in [28]), which consist of three coupled PDE's: a wave equations and two wave-like equations, where each equations is influenced by nonlinear damping and source terms. It is worth noting that *non-dissipative* "energy-building" sources, especially those on the boundary, arise when one considers a wave equation being coupled with other types of dynamics, such as structure-acoustic or fluid-structure interaction models (Lasiecka [32]). In light of these applications we are mainly interested in higher-order nonlinearities, as described in following assumption.

Assumption 1.1.1.

- ***Damping:*** *g_1, g_2 and g are continuous and monotone increasing functions with $g_1(0) = g_2(0) = g(0) = 0$. In addition, the following growth conditions at infinity hold: there exist positive constants a_j and b_j, $j = 1, 2, 3$, such that, for $|s| \geq 1$,*

$$\begin{aligned} a_1|s|^{m+1} &\leq g_1(s)s \leq b_1|s|^{m+1}, \ \ \textit{with } m \geq 1, \\ a_2|s|^{r+1} &\leq g_2(s)s \leq b_2|s|^{r+1}, \ \ \textit{with } r \geq 1, \\ a_3|s|^{q+1} &\leq g(s)s \leq b_3|s|^{q+1}, \ \ \textit{with } q \geq 1. \end{aligned}$$

- ***Interior sources:*** *$f_j(u, v) \in C^1(\mathbb{R}^2)$ such that*

$$|\nabla f_j(u, v)| \leq C(|u|^{p-1} + |v|^{p-1} + 1), \ \ j = 1, 2, \ \ \textit{with } 1 \leq p < 6.$$

- ***Boundary source:*** *$h \in C^1(\mathbb{R})$ such that*

$$|h'(s)| \leq C(|s|^{k-1} + 1), \ \ \textit{with } 1 \leq k < 4.$$

- ***Parameters:*** $\max\{p\frac{m+1}{m}, p\frac{r+1}{r}\} < 6$, $\ k\frac{q+1}{q} < 4$.

Let us note here that in view of the Sobolev imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ (in 3D), each of the Nemytski operators $f_j(u, v)$ is locally Lipschitz continuous from $H^1(\Omega) \times H^1(\Omega)$ into $L^2(\Omega)$ for the values $1 \leq p \leq 3$. Hence, when the exponent of the sources p lies in $1 \leq p < 3$, we call the source **sub-critical**, and **critical**, if $p = 3$. For the values $3 < p \leq 5$ the source is called **supercritical**, and in this case the operator $f_j(u, v)$ is **not** locally Lipschitz continuous from $H^1(\Omega) \times H^1(\Omega)$ into $L^2(\Omega)$.

However, for $3 < p \leq 5$, the potential energy induced by the source is well defined in the finite energy space. When $5 < p < 6$ the source is called **super-supercritical**. In this case, the potential energy may not be defined in the finite energy space and the problem itself is no longer within the framework of potential well theory (see for instance [3, 34, 35, 50, 51]).

A benchmark system, which is a special case of (1.1.1), is the following well-known polynomially damped system studied extensively in the literature (see for instance [2, 3, 39, 40]):

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{m-1}u_t = f_1(u,v) & \text{in } \Omega \times (0,T), \\ v_{tt} - \Delta v + |v_t|^{r-1}v_t = f_2(u,v) & \text{in } \Omega \times (0,T), \end{cases} \tag{1.1.2}$$

where the sources f_1, f_2 are very specific. Namely, $f_1(u,v) = \partial_u F(u,v)$ and $f_2(u,v) = \partial_v F(u,v)$, where $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is the C^1-function given by:

$$F(u,v) = a|u+v|^{p+1} + 2b|uv|^{\frac{p+1}{2}}, \tag{1.1.3}$$

where $p \geq 3$, $a > 1$ and $b > 0$. Systems of nonlinear wave equations such as (1.1.2) go back to Reed [42] who proposed a similar system in three space dimensions but without the presence of damping. Indeed, recently in [2] and later in [3] the authors studied system (1.1.2) with Dirichlét boundary conditions on both u and v where the exponent of the source was restricted to be *critical* ($p = 3$ in 3D). We note here that the functions f_1 and f_2 in (1.1.2) satisfy Assumption 1.1.1, even for the values $3 \leq p < 6$, and so our work extends and refines the results in [2], on one hand by allowing a larger class of sources (other than those in (1.1.2)) and having a larger range of exponents of sources, $p > 3$. On the other hand, system (1.1.1) has a Robin boundary condition which also features nonlinear damping and a source term. In particular, the Robin boundary condition, in combination with the interior damping, creates serious technical difficulties in the analysis (for more details, see Subsection 2.1.1).

In studying systems such as (1.1.2) or the more general system (1.1.1), several difficulties arise due to the coupling. On one hand, establishing blow up results for systems of wave equations (not just global nonexistence results which don't require local solvability) is known to be more subtle than the scalar case. Additional challenges stem from the fact that in many physical systems, such as (1.1.2), the sources are not necessarily C^2-functions, even when $3 < p \leq 5$. In such a case, uniqueness of solutions becomes problematic, and this particular issue will be addressed in this thesis.

It is important to note that the mixture of Robin and Dirichlét boundary conditions in the system (1.1.1) is not essential to the methods used in this paper nor to our results. Indeed, similar existence, uniqueness and blow-up results can be easily obtained if instead one imposes Robin boundary conditions on both u and v.

In recent years, wave equations under the influence of nonlinear damping and sources have generated considerable interest. If the sources are at most critical, i.e., $p \leq 3$ and $k \leq 2$, many authors have successfully studied such equations by using Galerkin approximations or standard fixed point theorems (see for example [2, 3, 4, 20, 36, 39, 40, 41]). Also, for other related work on hyperbolic problems, we refer the reader to [16, 18, 24, 27, 30, 31, 33, 38, 47, 49] and the references therein. However, only few papers [7, 10, 11, 12] have dealt with supercritical sources, i.e., when $p > 3$ and $k > 2$.

In this thesis we use the powerful theory of monotone operators and nonlinear semigroups (Kato's Theorem [6, 46]) to study system (1.1.1). Our strategy is similar to the one used by Bociu [7] and our proofs draw substantially from important ideas in [7, 10, 11, 12] and in [17]. However, we were faced with the following technical issue: in the operator theoretic formulation of (1.1.1), although the operators induced by interior and boundary damping terms are individually maximal monotone from $H^1(\Omega)$ into $(H^1(\Omega))'$, it was crucial to verify their sum is maximal monotone. Since neither of these two operators has the whole space $H^1(\Omega)$ as its domain, as the exponents m, r, and q of damping are arbitrary large, then checking the domain condition (see Theorem 1.5 (p.54) [6]), to assure maximal monotonicity of their sum, becomes infeasible. In order to overcome this difficulty, we construct a convex functional whose subdifferential can represent the sum of interior and boundary damping, which yields the desired maximal monotonicity. Some details can be found in Subsections 1.3.4 and 2.1.1.

1.2 Notation

The following notations will be used throughout the thesis:

$$\|u\|_s = \|u\|_{L^s(\Omega)}\,,\ |u|_s = \|u\|_{L^s(\Gamma)}\,,\ \|u\|_{1,\Omega} = \|u\|_{H^1(\Omega)}\,;$$
$$(u,v)_\Omega = (u,v)_{L^2(\Omega)},\ (u,v)_\Gamma = (u,v)_{L^2(\Gamma)},\ (u,v)_{1,\Omega} = (u,v)_{H^1(\Omega)};$$
$$\tilde{m} = \frac{m+1}{m},\ \tilde{r} = \frac{r+1}{r},\ \tilde{q} = \frac{q+1}{q}.$$

As usual, we denote the standard duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$ by $\langle\cdot,\cdot\rangle$. We also use the notation γu to denote the *trace* of u on Γ and we write $\frac{d}{dt}(\gamma u(t))$ as γu_t. In addition, the following Sobolev imbeddings will be used frequently, and sometimes without mention:

$$\begin{cases} H^{1-\epsilon}(\Omega) \hookrightarrow L^{\frac{6}{1+2\epsilon}}(\Omega), \text{ for } \epsilon \in [0,1], \\ H^{1-\epsilon}(\Omega) \hookrightarrow H^{\frac{1}{2}-\epsilon}(\Gamma) \hookrightarrow L^{\frac{4}{1+2\epsilon}}(\Gamma), \text{ for } \epsilon \in [0,\frac{1}{2}]. \end{cases} \tag{1.2.1}$$

We also remind the reader with the following interpolation inequality:

$$\|u\|^2_{H^\theta(\Omega)} \le \epsilon\,\|u\|^2_{1,\Omega} + C(\epsilon,\theta)\,\|u\|_2^2\,, \tag{1.2.2}$$

for all $0 \le \theta < 1$ and $\epsilon > 0$. We finally note that $(\|\nabla u\|_2^2 + |\gamma u|_2^2)^{1/2}$ is an equivalent norm to the standard $H^1(\Omega)$ norm. This fact follows from a Poincaré-Wirtinger type of inequality:

$$\|u\|_2^2 \le c_0(\|\nabla u\|_2^2 + |\gamma u|_2^2) \text{ for all } u \in H^1(\Omega). \tag{1.2.3}$$

Thus, throughout the thesis we put,

$$\|u\|^2_{1,\Omega} = \|\nabla u\|_2^2 + |\gamma u|_2^2 \text{ and } (u,v)_{1,\Omega} = (\nabla u, \nabla v)_\Omega + (\gamma u, \gamma v)_\Gamma, \tag{1.2.4}$$

for u, $v \in H^1(\Omega)$.

1.3 Main Results

In order to state our main result we begin by giving the definition of a weak solution to (1.1.1).

Definition 1.3.1. A pair of functions (u,v) is said to be a *weak solution* of (1.1.1) on $[0,T]$ if

- $u \in C([0,T];H^1(\Omega))$, $v \in C([0,T];H^1_0(\Omega))$, $u_t \in C([0,T];L^2(\Omega)) \cap L^{m+1}(\Omega \times (0,T))$, $\gamma u_t \in L^{q+1}(\Gamma \times (0,T))$, $v_t \in C([0,T];L^2(\Omega)) \cap L^{r+1}(\Omega \times (0,T))$;

- $(u(0),v(0)) = (u_0,v_0) \in H^1(\Omega) \times H^1_0(\Omega)$, $(u_t(0),v_t(0)) = (u_1,v_1) \in L^2(\Omega) \times L^2(\Omega)$;

- For all $t \in [0,T]$, u and v verify the following identities:

$$\begin{aligned}(u_t(t),&\phi(t))_\Omega - (u_t(0),\phi(0))_\Omega + \int_0^t [-(u_t(\tau),\phi_t(\tau))_\Omega + (u(\tau),\phi(\tau))_{1,\Omega}]d\tau \\ &+ \int_0^t \int_\Omega g_1(u_t(\tau))\phi(\tau)dxd\tau + \int_0^t \int_\Gamma g(\gamma u_t(\tau))\gamma\phi(\tau)d\Gamma d\tau \\ &= \int_0^t \int_\Omega f_1(u(\tau),v(\tau))\phi(\tau)dxd\tau + \int_0^t \int_\Gamma h(\gamma u(\tau))\gamma\phi(\tau)d\Gamma d\tau, \end{aligned} \quad (1.3.1)$$

$$\begin{aligned}(v_t(t),&\psi(t))_\Omega - (v_t(0),\psi(0))_\Omega + \int_0^t [-(v_t(\tau),\psi_t(\tau))_\Omega + (v(\tau),\psi(\tau))_{1,\Omega}]d\tau \\ &+ \int_0^t \int_\Omega g_2(v_t(\tau))\psi(\tau)dxd\tau = \int_0^t \int_\Omega f_2(u(\tau),v(\tau))\psi(\tau)dxd\tau, \end{aligned} \quad (1.3.2)$$

for all test functions satisfying:
$\phi \in C([0,T];H^1(\Omega)) \cap L^{m+1}(\Omega \times (0,T))$ such that $\gamma\phi \in L^{q+1}(\Gamma \times (0,T))$ with $\phi_t \in L^1([0,T];L^2(\Omega))$ and $\psi \in C([0,T];H_0^1(\Omega)) \cap L^{r+1}(\Omega \times (0,T))$ such that $\psi_t \in L^1([0,T];L^2(\Omega))$.

1.3.1 Existence and uniqueness

Our first theorem establishes the existence of a local weak solution to (1.1.1). Specifically, we have the following result.

Theorem 1.3.2 (Local weak solutions). *Assume the validity of Assumption 1.1.1, then there exists a local weak solution (u,v) to (1.1.1) defined on $[0,T_0]$ for some $T_0 > 0$ depending on the initial energy $\mathscr{E}(0)$, where*

$$\mathscr{E}(t) = \frac{1}{2}(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 + \|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2). \quad (1.3.3)$$

In addition, the following energy identity holds for all $t \in [0,T_0]$:

$$\begin{aligned}\mathscr{E}(t) &+ \int_0^t \int_\Omega [g_1(u_t)u_t + g_2(v_t)v_t]\,dxd\tau + \int_0^t \int_\Gamma g(\gamma u_t)\gamma u_t d\Gamma d\tau \\ &= \mathscr{E}(0) + \int_0^t \int_\Omega [f_1(u,v)u_t + f_2(u,v)v_t]\,dxd\tau + \int_0^t \int_\Gamma h(\gamma u)\gamma u_t d\Gamma d\tau. \end{aligned} \quad (1.3.4)$$

In order to state the next theorem, we need additional assumptions on the sources and the boundary damping.

Assumption 1.3.3.

(a) *For $p > 3$, there exists a function $F(u,v) \in C^3(\mathbb{R}^2)$ such that $f_1(u,v) = F_u(u,v)$, $f_2(u,v) = F_v(u,v)$ and $|D^\alpha F(u,v)| \leq C(|u|^{p-2} + |v|^{p-2} + 1)$, for all multi-indices $|\alpha| = 3$ and all $u, v \in \mathbb{R}$.*

(b) *For $k \geq 2$, $h \in C^2(\mathbb{R})$ such that $|h''(s)| \leq C(|s|^{k-2} + 1)$, for all $s \in \mathbb{R}$.*

(c) *For $k < 2$, there exists $m_g > 0$ such that $(g(s_1) - g(s_2))(s_1 - s_2) \geq m_g|s_1 - s_2|^2$, for all s_1, $s_2 \in \mathbb{R}$.*

Theorem 1.3.4 (Uniqueness of weak solutions–Part 1). *In addition to Assumptions 1.1.1 and 1.3.3, we further assume that u_0, $v_0 \in L^{\frac{3(p-1)}{2}}(\Omega)$ and $\gamma u_0 \in L^{2(k-1)}(\Gamma)$. Then weak solutions of (1.1.1) are unique.*

Remark 1.3.5. The additional assumptions on the initial data in Theorem 1.3.4 are redundant if $p \leq 5$ and $k \leq 3$, due to the imbeddings (1.2.1). Also, it is often the case that the interior sources f_1 and f_2 fail to satisfy Assumption 1.3.3(a), as in system (1.1.2) for the values $3 < p \leq 5$. To ensure uniqueness of weak solutions in such a case, we require the exponents m and r of the interior damping to be sufficiently large. More precisely, the following result resolves this issue.

Theorem 1.3.6 (Uniqueness of weak solutions–Part 2). *Under Assumption 1.1.1 and Assumption 1.3.3(b)(c), we additionally assume that u_0, $v_0 \in L^{3(p-1)}(\Omega)$, $\gamma u_0 \in L^{2(k-1)}(\Gamma)$, and m, $r \geq 3p - 4$ if $p > 3$. Then weak solutions of (1.1.1) are unique.*

Our next theorem states that weak solutions furnished by Theorem 1.3.2 are global solutions provided the exponents of damping are more dominant than the exponents of the corresponding sources.

Theorem 1.3.7 (Global weak solutions). *In addition to Assumption 1.1.1, further assume $u_0, v_0 \in L^{p+1}(\Omega)$ and $\gamma u_0 \in L^{k+1}(\Gamma)$. If $p \leq \min\{m, r\}$ and $k \leq q$, then the said solution (u,v) in Theorem 1.3.2 is a global weak solution and T_0 can be taken arbitrarily large.*

Our next result states that the weak solution of (1.1.1) depends continuously on the initial data.

Theorem 1.3.8 (Continuous dependence on initial data). *Assume the validity of Assumptions 1.1.1 and 1.3.3 and an initial data $U_0 = (u_0, v_0, u_1, v_1) \in X$, where X is given by $X := (H^1(\Omega) \cap L^{\frac{3(p-1)}{2}}(\Omega)) \times (H^1_0(\Omega) \cap L^{\frac{3(p-1)}{2}}(\Omega)) \times L^2(\Omega) \times L^2(\Omega)$, such that $\gamma u_0 \in L^{2(k-1)}(\Gamma)$. If $U_0^n = (u_0^n, u_1^n, v_0^n, v_1^n)$ is a sequence of initial data such that, as $n \longrightarrow \infty$,*

$$U_0^n \longrightarrow U_0 \ \text{ in } \ X \ \text{ and } \ \gamma u_0^n \longrightarrow \gamma u_0 \ \text{ in } \ L^{2(k-1)}(\Gamma),$$

then, the corresponding weak solutions (u^n, v^n) and (u, v) of (1.1.1) satisfy:

$$(u^n, v^n, u_t^n, v_t^n) \longrightarrow (u, v, u_t, v_t) \ \text{ in } \ C([0,T]; H), \ \text{ as } \ n \longrightarrow \infty,$$

where $H := H^1(\Omega) \times H^1_0(\Omega) \times L^2(\Omega) \times L^2(\Omega)$.

Remark 1.3.9. If $p \leq 5$, then the spaces X and H in the Theorem 1.3.8 are identical since $H^1(\Omega) \hookrightarrow L^6(\Omega)$. In addition, if $k \leq 3$, then the assumption $\gamma u_0^n \longrightarrow \gamma u_0$ in $L^{2(k-1)}(\Gamma)$ is redundant since $u_0^n \longrightarrow u_0$ in $H^1(\Omega)$ implies $\gamma u_0^n \longrightarrow \gamma u_0$ in $L^4(\Gamma)$.

1.3.2 Blow-up of weak solutions

In order to state our blow up results, we need additional assumptions on interior and boundary sources and initial data.

Assumption 1.3.10.

- *There exists a function $F \in C^2(\mathbb{R}^2)$ such that $f_1(u,v) = \partial_u F(u,v)$ and $f_2(u,v) = \partial_v F(u,v)$, $(u,v) \in \mathbb{R}^2$. Moreover, there exist $c_0 > 0$ and $c_1 > 2$ such that $F(u,v) \geq c_0(|u|^{p+1} + |v|^{p+1})$ and $uf_1(u,v) + vf_2(u,v) \geq c_1 F(u,v)$, for all $(u,v) \in \mathbb{R}^2$.*
- *There exist $c_2 > 0$ and $c_3 > 2$ such that $H(s) \geq c_2|s|^{k+1}$ and $h(s)s \geq c_3 H(s)$, for all $s \in \mathbb{R}$, where $H(s) = \int_0^s h(\tau)d\tau$.*
- *The initial energy $E(0) < 0$, where the total energy $E(t)$ is given by:*

$$\begin{aligned} E(t) =& \frac{1}{2}(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 + \|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2) \\ &- \int_\Omega F(u(t), v(t))dx - \int_\Gamma H(\gamma u(t))d\Gamma. \end{aligned} \tag{1.3.5}$$

Remark 1.3.11. It is important to note here that our restrictions on interior and boundary sources in Assumption 1.3.10 are natural and quite reasonable. For instance, the function F given in (1.1.3) satisfies Assumption 1.3.10. Indeed, a quick calculations show that there exists a constant $c_0 > 0$ such that $F(u,v) \geq c_0(|u|^{p+1} + |v|^{p+1})$, provided b is chosen large enough. Moreover, it is easy to compute and find that $uf_1(u,v) + vf_2(u,v) = (p+1)F(u,v)$. Since the blow-up theorems below require $p > m \geq 1$, then $p+1 > 2$, and so, the assumption $c_1 > 2$ is reasonable. A simple example of a boundary source term that satisfies Assumption 1.3.10 is $h(s) = |s|^{k-1}s$. In this case, $H(s) = \frac{1}{k+1}|s|^{k+1}$, and so, $h(s)s = (k+1)H(s)$. Again, the statement of Theorem 1.3.12 requires $k > q \geq 1$, implies that $k+1 > 2$. Thus, the restriction $c_3 > 2$ in Assumption 1.3.3 is also reasonable.

Our first blow-up result shows that if the interior and boundary sources are more dominant than their corresponding damping terms, and the initial energy is negative, then every weak solution of (1.1.1) blows up in finite time. In addition, we obtain an upper bound for the life span of solutions.

Theorem 1.3.12 (**Blow-up of solutions-Part 1**). *Assume the validity of Assumptions 1.1.1 and 1.3.10. If* $p > \max\{m, r\}$ *and* $k > q$*, then any weak solution* (u,v) *of (1.1.1) blows up in finite time. More precisely,* $\|u(t)\|_{1,\Omega} + \|v(t)\|_{1,\Omega} \to \infty$ *as* $t \to T^-$*, for some* $0 < T < \infty$.

Our second result shows that all solutions of (1.1.1) blows up in finite time, provided $E(0) < 0$, and the interior sources dominate both interior and boundary damping, without any restriction on the boundary source.

Theorem 1.3.13 (**Blow-up of solutions-Part 2**). *Assume the validity of Assumptions 1.1.1 and 1.3.10. If* $p > \max\{m, r, 2q-1\}$*, then any weak solution* (u,v) *of (1.1.1) blows up in finite time. Specifically,* $\|u(t)\|_{1,\Omega} + \|v(t)\|_{1,\Omega} \to \infty$ *as* $t \to T^-$*, for some* $0 < T < \infty$.

Remark 1.3.14. Although the existence and uniqueness results in Theorems 1.3.2 and 1.3.4 hold for sources that are super-supercritical (i.e., $p < 6$ and $k < 4$), however the assumptions in Theorem 1.3.12 and 1.3.13 force the restrictions $p < 5$ and $k < 3$. To see this, we note that both theorems require $p > m$, and by Assumption 1.1.1, it follows that, $6 > p(1 + \frac{1}{m}) > p(1 + \frac{1}{p}) = p + 1$, which implies $p < 5$. By the same observation, we conclude $k < 3$ in Theorem 1.3.12. Although $k > q$ is not required by Theorem 1.3.13, we still must have $k < 3$. Indeed, since $2q - 1 < p < 5$, then $q < 3$. Whence, by Assumption 1.1.1, we have $4 > k(1 + \frac{1}{q}) > \frac{4}{3}k$, and so, $k < 3$.

1.3.3 Decay of energy

This subsection is devoted to present our results of global existence of potential well solutions, uniform decay rates of energy, and blow up of solutions with non-negative initial energy. Comparing with the results of [3] for system (1.1.2) with $p = 3$, our results extend and refine the results of [3] in the following sense: (i) System (1.1.1) is more general than (1.1.2) with supercritical sources and subject to a nonlinear Robin boundary condition. (ii) The global existence and energy decay results in [3] are obtained only when the exponents of the damping functions are restricted to the case $m,\ r \leq 5$. Here, we allow $m,\ r$ to be larger than 5, provided we impose additional assumptions on the regularity of weak solutions. (iii) In addition to the standard case $p > \max\{m, r\}$ and $k > q$ for our blow up result, we consider another scenario in which the interior source is more dominant than both feedback mappings in the interior and on the boundary. Specifically, we prove a blow up result in the case $p > \max\{m, r, 2q-1\}$, and without the additional assumption $k > q$. Although this kind of blow up result has been established for solutions with *negative* initial energy [10, 22], to our knowledge, our result is new for wave equations with *non-negative* initial energy.

We begin by briefly pointing out the connection of problem (1.1.1) to some important aspects of the theory of elliptic equations. In order to do so, we need to impose additional assumptions on the interior sources f_1, f_2 and boundary source h.

Assumption 1.3.15.

- *There exists a nonnegative function* $F(u,v) \in C^1(\mathbb{R}^2)$ *such that* $\partial_u F(u,v) = f_1(u,v)$, $\partial_v F(u,v) = f_2(u,v)$, *and* F *is homogeneous of order* $p+1$, *i.e.,* $F(\lambda u, \lambda v) = \lambda^{p+1} F(u,v)$, *for all* $\lambda > 0$, $(u,v) \in \mathbb{R}^2$.

- *There exists a nonnegative function* $H(s) \in C^1(\mathbb{R})$ *such that* $H'(s) = h(s)$, *and* H *is homogeneous of order* $k+1$, *i.e.,* $H(\lambda s) = \lambda^{k+1} H(s)$, *for all* $\lambda > 0$, $s \in \mathbb{R}$.

Remark 1.3.16. We note that the special function $F(u,v)$ defined in (1.1.3) satisfies Assumption 1.3.15, provided $p \geq 3$. However, there is a large class of functions that satisfy Assumption 1.3.15. For instance, functions of the form (with an appropriate range of values for p, s and σ):

$$\mathcal{F}(u,v) = a|u|^{p+1} + b|v|^{p+1} + \alpha |u|^s |v|^{p+1-s} + \beta(|u|^\sigma + |v|^\sigma)^{\frac{p+1}{\sigma}},$$

satisfy Assumption 1.3.15. Moreover, since F and H are homogeneous, then the Euler homogeneous function theorem gives the following useful identities:

$$f_1(u,v)u + f_2(u,v)v = (p+1)F(u,v) \text{ and } h(s)s = (k+1)H(s). \tag{1.3.6}$$

Finally, we note that the assumptions $|\nabla f_j(u,v)| \le C(|u|^{p-1}+|v|^{p-1}+1)$, $j=1,2$ and $|h'(s)| \le C(|s|^{k-1}+1)$ (as required by Assumption 1.1.1), imply that there exists a constant $M>0$ such that $F(u,v) \le M(|u|^{p+1}+|v|^{p+1}+1)$ and $H(s) \le M(|s|^{k+1}+1)$, for all $u,v,s \in \mathbb{R}$. Therefore, by the homogeneity of F and H, we must have

$$F(u,v) \le M(|u|^{p+1}+|v|^{p+1}) \text{ and } H(s) \le M|s|^{k+1}. \tag{1.3.7}$$

Now we put $X := H^1(\Omega) \times H_0^1(\Omega)$, and define the functional $J : X \to \mathbb{R}$ by:

$$J(u,v) := \frac{1}{2}(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2) - \int_\Omega F(u,v)dx - \int_\Gamma H(\gamma u)d\Gamma, \tag{1.3.8}$$

where $J(u,v)$ represents the *potential energy* of the system. Therefore the total energy can be written as:

$$E(t) = \frac{1}{2}(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2) + J(u(t),v(t)). \tag{1.3.9}$$

In addition, simple calculations shows that the Fréchet derivative of J at $(u,v) \in X$ is given by:

$$\begin{aligned}\langle J'(u,v),(\phi,\psi)\rangle = &\int_\Omega \nabla u \cdot \nabla \phi dx + \int_\Gamma \gamma u \gamma \phi d\Gamma + \int_\Omega \nabla v \cdot \nabla \psi dx \\ &- \int_\Omega [f_1(u,v)\phi + f_2(u,v)\psi]dx - \int_\Gamma h(\gamma u)\gamma\phi d\Gamma,\end{aligned} \tag{1.3.10}$$

for all $(\phi,\psi) \in X$.

Associated to the functional J is the well-known *Nehari manifold*, namely

$$\mathcal{N} := \{(u,v) \in X \backslash \{(0,0)\} : \langle J'(u,v),(u,v)\rangle = 0\}. \tag{1.3.11}$$

It follows from (1.3.10) and (1.3.6) that the Nehari manifold can be put as:

$$\begin{aligned}\mathcal{N} = \Big\{&(u,v) \in X \backslash \{(0,0)\} : \\ &\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2 = (p+1)\int_\Omega F(u,v)dx + (k+1)\int_\Gamma H(\gamma u)d\Gamma\Big\}.\end{aligned} \tag{1.3.12}$$

In order to introduce the potential well, we first prove the following lemma.

Lemma 1.3.17. *In addition to Assumptions 1.1.1 and 1.3.15, further assume that* $1 < p \le 5$ *and* $1 < k \le 3$*. Then*

$$d := \inf_{(u,v)\in\mathcal{N}} J(u,v) > 0. \tag{1.3.13}$$

Proof. Fix $(u,v) \in \mathcal{N}$. Then, it follows from (1.3.8) and (1.3.12) that

$$J(u,v) \geq \left(\frac{1}{2} - \frac{1}{c}\right)(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2). \tag{1.3.14}$$

where $c := \min\{p+1, k+1\} > 2$. Since $(u,v) \in \mathcal{N}$, then the bounds (1.3.7) yield

$$\begin{aligned}\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2 &\leq C_{p,k}\Big(\int_\Omega (|u|^{p+1} + |v|^{p+1})dx + \int_\Gamma |\gamma u|^{k+1} d\Gamma\Big) \\ &\leq C\Big(\|u\|_{1,\Omega}^{p+1} + \|v\|_{1,\Omega}^{p+1} + \|u\|_{1,\Omega}^{k+1}\Big).\end{aligned} \tag{1.3.15}$$

Thus,

$$\|(u,v)\|_X^2 \leq C(\|(u,v)\|_X^{p+1} + \|(u,v)\|_X^{k+1}),$$

and since $(u,v) \neq (0,0)$, we have

$$1 \leq C(\|(u,v)\|_X^{p-1} + \|(u,v)\|_X^{k-1}).$$

It follows that $\|(u,v)\|_X \geq s_1 > 0$ where s_1 is the unique positive solution of the equation $C(s^{p-1} + s^{k-1}) = 1$, where p, $k > 1$. Then, by (1.3.14), we arrive at

$$J(u,v) \geq \left(\frac{1}{2} - \frac{1}{c}\right) s_1^2$$

for all $(u,v) \in \mathcal{N}$. Thus, (1.3.13) follows. $\square$

As in [3], we introduce the following sets:

$$\begin{aligned}\mathcal{W} &:= \{(u,v) \in X : J(u,v) < d\}, \\ \mathcal{W}_1 &:= \{(u,v) \in \mathcal{W} : \|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2 > (p+1)\int_\Omega F(u,v)dx + (k+1)\int_\Gamma H(\gamma u)d\Gamma\} \\ &\qquad \cup \{(0,0)\}, \\ \mathcal{W}_2 &:= \{(u,v) \in \mathcal{W} : \|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2 < (p+1)\int_\Omega F(u,v)dx + (k+1)\int_\Gamma H(\gamma u)d\Gamma\}.\end{aligned}$$

Clearly, $\mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset$, and $\mathcal{W}_1 \cup \mathcal{W}_2 = \mathcal{W}$. In addition, we refer to $\mathcal{W}$ as the *potential well* and d as the *depth* of the well. The set $\mathcal{W}_1$ is regarded as the "good" part of the well, as we will show that every weak solution exists globally in time, provided the initial data are taken from $\mathcal{W}_1$ and the initial energy is under the level d. On the other hand, if the initial data are taken from $\mathcal{W}_2$ and the sources dominate the damping, we will prove a blow up result for weak solutions with nonnegative initial energy.

The following result establishes the existence of a global weak solution to (1.1.1), provided the initial data come from $\mathcal{W}_1$ and the initial energy is less than d, and without imposing the conditions $p \leq \min\{m, r\}$, $k \leq q$, as required by Theorem 1.3.7.

In order to state our first result, we recall the quadratic energy $\mathscr{E}(t)$ and the total energy $E(t)$ as defined in (1.3.3) and (1.3.5), respectively.

Theorem 1.3.18 (Global solutions). *In addition to Assumptions 1.1.1 and 1.3.15, further assume $(u_0, v_0) \in \mathcal{W}_1$ and $E(0) < d$. If $1 < p \leq 5$ and $1 < k \leq 3$, then the weak solution (u, v) of (1.1.1) is a global solution. Furthermore, we have:*

- $(u(t), v(t)) \in \mathcal{W}_1$,
- $$\mathscr{E}(t) < d\left(\frac{c}{c-2}\right), \tag{1.3.16}$$
- $$\left(1 - \frac{2}{c}\right)\mathscr{E}(t) \leq E(t) \leq \mathscr{E}(t), \tag{1.3.17}$$

for all $t \geq 0$, where $c = \min\{p+1, k+1\} > 2$.

Since the weak solution furnished by Theorem 1.3.18 is a global solution and the total energy $E(t)$ remains positive for all $t \geq 0$, we may study the uniform decay rates of the energy. Specifically, we will show that if the initial data come from a closed subset of $\mathcal{W}_1$, then the energy $E(t)$ decays either exponentially or algebraically, depending on the behaviors of the functions g_1, g_2 and g near the origin.

In order to state our result on the energy decay, we need some preparations. Define the function

$$\mathcal{G}(s) := \frac{1}{2}s^2 - MR_1 s^{p+1} - MR_2 s^{k+1}, \tag{1.3.18}$$

where the constant $M > 0$ is as given in (1.3.7) and

$$R_1 := \sup_{u \in H^1(\Omega)\setminus\{0\}} \frac{\|u\|_{p+1}^{p+1}}{\|u\|_{1,\Omega}^{p+1}}, \quad R_2 := \sup_{u \in H^1(\Omega)\setminus\{0\}} \frac{|\gamma u|_{k+1}^{k+1}}{\|u\|_{1,\Omega}^{k+1}}. \tag{1.3.19}$$

Since $p \leq 5$ and $k \leq 3$, by Sobolev Imbedding Theorem, we know $0 < R_1$, $R_2 < \infty$.

A straightforward calculation shows that $\mathcal{G}'(s)$ has a unique positive zero, say at $s_0 > 0$, and

$$\sup_{s\in[0,\infty)} \mathcal{G}(s) = \mathcal{G}(s_0).$$

Thus, we define the set

$$\tilde{\mathcal{W}}_1 := \{(u,v) \in X : \|(u,v)\|_X < s_0,\ J(u,v) < \mathcal{G}(s_0)\}. \tag{1.3.20}$$

We will show in Proposition 4.2.3 that $\mathcal{G}(s_0) \leq d$ and $\tilde{\mathcal{W}}_1 \subset \mathcal{W}_1$.

Furthermore, for each fixed small value $\delta > 0$, we define a closed subset of $\tilde{\mathcal{W}}_1$, namely

$$\tilde{\mathcal{W}}_1^\delta := \{(u,v) \in X : \|(u,v)\|_X \leq s_0 - \delta,\ J(u,v) \leq \mathcal{G}(s_0 - \delta)\}. \tag{1.3.21}$$

Indeed, we will show in Proposition 4.2.4 that $\tilde{\mathcal{W}}_1^\delta$ is invariant under the dynamics, if the initial energy satisfies $E(0) \leq \mathcal{G}(s_0 - \delta)$.

The following theorem addresses the uniform decay rates of energy. In the standard case m, $r \leq 5$, $q \leq 3$, we don't impose any additional assumptions on the weak solutions furnished by Theorem 1.3.18. However, if any of the exponents of damping is *large*, then we need additional assumptions on the regularity of weak solutions. More precisely, we have the following result.

Theorem 1.3.19 (**Uniform decay rates).** *In addition to Assumptions 1.1.1 and 1.3.15, further assume: $1 < p < 5$, $1 < k < 3$, $u_0 \in L^{m+1}(\Omega)$, $v_0 \in L^{r+1}(\Omega)$, $\gamma u_0 \in L^{q+1}(\Gamma)$, $(u_0, v_0) \in \tilde{\mathcal{W}}_1^\delta$, and $E(0) < \mathcal{G}(s_0 - \delta)$ for some $\delta > 0$. In addition, assume $u \in L^\infty(\mathbb{R}^+; L^{\frac{3}{2}(m-1)}(\Omega))$ if $m > 5$, $v \in L^\infty(\mathbb{R}^+; L^{\frac{3}{2}(r-1)}(\Omega))$ if $r > 5$, and $\gamma u \in L^\infty(\mathbb{R}^+; L^{2(q-1)}(\Gamma))$ if $q > 3$, where (u,v) is the global solution of (1.1.1) furnished by Theorem 1.3.18.*

- *If g_1, g_2, and g are linearly bounded near the origin, then the total energy $E(t)$ decays exponentially:*

$$E(t) \leq \tilde{C}E(0)e^{-wt}, \quad \text{for all } t \geq 0, \tag{1.3.22}$$

where $\tilde{C}$ and w are positive constants.

- *If at least one of the feedback mappings g_1, g_2 and g is not linearly bounded near the origin, then $E(t)$ decays algebraically:*

$$E(t) \leq C(E(0))(1+t)^{-\beta}, \quad \text{for all } t \geq 0, \tag{1.3.23}$$

where $\beta > 0$ (specified in (4.2.19)) depends on the growth rates of g_1, g_2 and g near the origin.

Our final result in this section addresses the blow up of potential well solutions with *non-negative* initial energy. It is important to note that the blow up results in Theorems 1.3.12 and 1.3.13 deal with the case of *negative* initial energy for general weak solutions (not necessarily potential well solutions).

Theorem 1.3.20 (Blow-up of potential well solutions). *In addition to Assumptions 1.1.1 and 1.3.15, further assume for* <u>*all*</u> $s \in \mathbb{R}$,

$$\begin{aligned} &a_1|s|^{m+1} \leq g_1(s)s \leq b_1|s|^{m+1}, \text{ where } m \geq 1, \\ &a_2|s|^{r+1} \leq g_2(s)s \leq b_2|s|^{r+1}, \text{ where } r \geq 1, \\ &a_3|s|^{q+1} \leq g(s)s \leq b_3|s|^{q+1}, \text{ where } q \geq 1. \end{aligned} \tag{1.3.24}$$

In addition, we suppose $F(u,v) \geq \alpha_0(|u|^{p+1}+|v|^{p+1})$*, for some* $\alpha_0 > 0$*, and* $H(s) > 0$*, for all* $s \neq 0$*. If* $1 < p \leq 5$*,* $1 < k \leq 3$*,* $(u_0, v_0) \in \mathcal{W}_2$*,* $0 \leq E(0) < \rho d$*, where*

$$\rho := \frac{\min\left\{\frac{p+1}{p-1}, \frac{k+1}{k-1}\right\}}{\max\left\{\frac{p+1}{p-1}, \frac{k+1}{k-1}\right\}} \leq 1, \tag{1.3.25}$$

then, the weak solution (u,v) *of (1.1.1) (as furnished by Theorem 1.3.2) blows up in finite time; provided either*

- $p > \max\{m, r\}$ *and* $k > q$,

  or

- $p > \max\{m, r, 2q-1\}$.

Remark 1.3.21. The blow up result in Theorem 1.3.20 relies on the blow up results in Theorems 1.3.12 and 1.3.13 for negative initial energy. Therefore, as Theorems 1.3.12 and 1.3.13, we conclude from Theorem 1.3.20 that

$$\|u(t)\|_{1,\Omega} + \|v(t)\|_{1,\Omega} \to \infty,$$

as $t \to T^-$, for some $0 < T < \infty$.

1.3.4 Convex integrals on Sobolev spaces

In this subsection we introduce some abstract results which are essential for establishing the local existence of weak solutions to our system (1.1.1).

Let j_0, $j_1 : \mathbb{R} \to [0, +\infty)$ be convex functions vanishing at 0. Note that, since j_0 and j_1 are convex functions and finite everywhere, then they are continuous on $\mathbb{R}$. Let $\gamma : H^1(\Omega) \to H^{1/2}(\Gamma)$ denote the trace map, and define the functional $J : H^1(\Omega) \to [0, +\infty]$ by

$$J(u) = \int_\Omega j_0(u)dx + \int_\Gamma j_1(\gamma u)d\Gamma. \tag{1.3.26}$$

Clearly, J is convex and lower semicontinuous with its domain given by

$$D(J) = \{u \in H^1(\Omega) : j_0(u) \in L^1(\Omega) \quad \text{and} \quad j_1(\gamma u) \in L^1(\Gamma)\}. \tag{1.3.27}$$

As usual, $D(\partial J)$ represents the set of all functions $u \in H^1(\Omega)$ for which $\partial J(u)$ is nonempty. It is well known that $D(\partial J)$ is a dense subset of $D(J)$. The convex conjugate of J is defined by

$$J^*(T) = \sup\{\langle T, u\rangle - J(u) : u \in D(J)\} \text{ for } T \in (H^1(\Omega))', \tag{1.3.28}$$

where, here and later, $(H^1(\Omega))'$ denotes the dual space of $H^1(\Omega)$. Similarly, the convex conjugate of j_k, $k = 0, 1$; is given by

$$j_k^*(x) = \sup\{xy - j_k(y) : y \in \mathbb{R}\}, \quad x \in \mathbb{R}. \tag{1.3.29}$$

H. Brézis [14] studied the convex functional $J_0(u) = \int_\Omega j_0(u)dx$ on $H_0^1(\Omega)$ and characterized its conjugate J_0^* and its subdifferential ∂J_0. The main Theorems presented here generalize the results in [14] to the functional J. The strategy of the proof is conceptually similar to the one by Brézis, however our conclusions cannot be directly derived from the work in [14], and necessitate a number of nontrivial technical auxiliary results.

Our main findings are stated in the following theorems.

Theorem 1.3.22. *Suppose $T \in (H^1(\Omega))'$ such that $J^*(T) < +\infty$. Then T is a signed Radon measure on $\overline{\Omega}$ and there exist $T_a \in L^1(\Omega)$ and $T_{\Gamma,a} \in L^1(\Gamma)$ such that*

$$\langle T, v\rangle = \int_\Omega T_a v dx + \int_\Gamma T_{\Gamma,a}\gamma v d\Gamma, \quad \textit{for all} \quad v \in C(\overline{\Omega}). \tag{1.3.30}$$

Moreover,

$$J^*(T) = \int_\Omega j_0^*(T_a)dx + \int_\Gamma j_1^*(T_{\Gamma,a})d\Gamma.$$

Theorem 1.3.23. *Let $u \in H^1(\Omega)$. If $T \in (H^1(\Omega))'$ such that $T \in \partial J(u)$, then T is a signed Radon measure on $\overline{\Omega}$ and there exist $T_a \in L^1(\Omega)$, $T_{\Gamma,a} \in L^1(\Gamma)$ such that T satisfies (1.3.30). Moreover, T, T_a, $T_{\Gamma,a}$ verify the following:*

- $T_a \in \partial j_0(u)$ *a.e. in* Ω *and* $T_{\Gamma,a} \in \partial j_1(\gamma u)$ *a.e. on* Γ, (1.3.31)
- $T_a u \in L^1(\Omega)$ *and* $T_{\Gamma,a}\gamma u \in L^1(\Gamma)$, (1.3.32)
- $\langle T, u\rangle = \int_\Omega T_a u dx + \int_\Gamma T_{\Gamma,a}\gamma u d\Gamma.$ (1.3.33)

Conversely, if $T \in (H^1(\Omega))'$ such that there exist $T_a \in L^1(\Omega)$, $T_{\Gamma,a} \in L^1(\Gamma)$ satisfying (1.3.30) and (1.3.31), then $T \in \partial J(u)$.

Assume for the moment that Theorem 1.3.23 has been proven. Define the functionals J_0 and $J_1 : H^1(\Omega) \to [0, +\infty]$ by

$$J_0(u) = \int_\Omega j_0(u)dx \text{ and } J_1(u) = \int_\Gamma j_1(\gamma u)d\Gamma.$$

Then, following corollary is an immediate consequence of Theorem 1.3.23.

Corollary 1.3.24. *Let $u \in H^1(\Omega)$. Then,*

- *if $j_1 = 0$ (i.e., $J = J_0$), then*

$$\partial J_0(u) = \{T \in (H^1(\Omega))' \cap L^1(\Omega) : T \in \partial j_0(u) \text{ a.e. in } \Omega\}. \tag{1.3.34}$$

- *if $j_0 = 0$ (i.e., $J = J_1$), then*

$$\begin{aligned}\partial J_1(u) = \{T \in (H^1(\Omega))' : T = \gamma^* T_\Gamma, \text{ where } T_\Gamma \in H^{-\frac{1}{2}}(\Gamma) \cap L^1(\Gamma) \\ \text{such that } T_\Gamma \in \partial j_1(\gamma u) \text{ a.e. on } \Gamma\}.\end{aligned} \tag{1.3.35}$$

Proof. The first statement of the Corollary is clear from Theorem 1.3.23. As for the second statement, first assume that $T \in (H^1(\Omega))'$ such that $T = \gamma^* T_\Gamma$ where $T_\Gamma \in H^{-\frac{1}{2}}(\Gamma) \cap L^1(\Gamma)$ with $T_\Gamma \in \partial j_1(\gamma u)$ a.e. on Γ. Note for all $w \in C^1(\overline{\Omega})$,

$$\langle T, w\rangle = \langle \gamma^* T_\Gamma, w\rangle = \langle T_\Gamma, \gamma w\rangle = \int_\Gamma T_\Gamma \gamma w d\Gamma.$$

Let $v \in C(\overline{\Omega})$, then there exists a sequence $w_n \in C^1(\overline{\Omega})$ such that $w_n \to v$ in $C(\overline{\Omega})$. Then, it follows easily from the Lebesgue Dominated Convergence Theorem that we may extend T to a bounded linear functional on $C(\overline{\Omega})$ via

$$\langle T, v\rangle = \lim_{n\to\infty} \langle T, w_n\rangle = \lim_{n\to\infty} \int_\Gamma T_\Gamma \gamma w_n d\Gamma = \int_\Gamma T_\Gamma \gamma v d\Gamma.$$

Therefore, by Theorem 1.3.23, with $j_0 = 0$, we obtain $T \in \partial J_1(u)$.

Conversely, if $T \in (H^1(\Omega))'$ such that $T \in \partial J_1(u)$, then by Theorem 1.3.23, with $j_0 = 0$, T is a Radon measure on $\overline{\Omega}$ and there exists $T_\Gamma \in L^1(\Gamma)$ such that $T_\Gamma \in \partial j_1(\gamma u)$ and

$$\langle T, v\rangle = \int_\Gamma T_\Gamma \gamma v d\Gamma \ \text{ for all } \ v \in C(\overline{\Omega}). \tag{1.3.36}$$

Since $T_\Gamma \in L^1(\Gamma)$, we have $T_\Gamma \in (C(\Gamma))'$ such that $\langle T_\Gamma, \phi\rangle = \int_\Gamma T_\Gamma \phi d\Gamma$, for all $\phi \in C(\Gamma)$. Note, for any $\psi \in H^{\frac{1}{2}}(\Gamma)$, there exists a sequence $\phi_n \in C^1(\Gamma)$ such that $\phi_n \to \psi$ in $H^{\frac{1}{2}}(\Gamma)$. Since $\gamma : H^1(\Omega) \to H^{\frac{1}{2}}(\Gamma)$ is surjective and has a continuous linear right inverse γ^{-1}, then clearly $|\langle T, \gamma^{-1}\psi\rangle| \le \|T\|\, \|\gamma^{-1}\psi\|_{H^1(\Omega)} < \infty$, for all $\psi \in H^{\frac{1}{2}}(\Gamma)$. Therefore, we can extend T_Γ to a bounded linear functional on $H^{\frac{1}{2}}(\Gamma)$ as follows:

$$\langle T_\Gamma, \psi\rangle = \lim_{n\to\infty} \langle T_\Gamma, \phi_n\rangle = \lim_{n\to\infty} \int_\Gamma T_\Gamma \phi_n d\Gamma = \lim_{n\to\infty} \langle T, \gamma^{-1}\phi_n\rangle = \langle T, \gamma^{-1}\psi\rangle,$$

for all $\psi \in H^{\frac{1}{2}}(\Gamma)$, where we have used (1.3.36). Hence, $T_\Gamma \in H^{-\frac{1}{2}}(\Gamma)$ such that $\langle T_\Gamma, \gamma v\rangle = \int_\Gamma T_\Gamma \gamma v d\Gamma = \langle T, v\rangle$ for all $v \in C^1(\overline{\Omega})$. Since $C^1(\overline{\Omega})$ is dense in $H^1(\Omega)$, we obtain $\langle T_\Gamma, \gamma v\rangle = \langle T, v\rangle$ for all $v \in H^1(\Omega)$, i.e., $T = \gamma^* T_\Gamma$. $\square$

Chapter 2

Existence and Uniqueness

2.1 Local Existence

This section is devoted to prove the existence statement in Theorem 1.3.2, which will be carried out in the following five sub-sections.

2.1.1 Operator theoretic formulation

Our first goal is to put problem (1.1.1) in an operator theoretic form. In order to do so, we introduce the Robin Laplacian Δ_R: $\mathcal{D}(\Delta_R) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$ where $\Delta_R = -\Delta u$ with its domain $\mathcal{D}(\Delta_R) = \{u \in H^2(\Omega) : \partial_\nu u + u = 0 \text{ on } \Gamma\}$. We note here that the Robin Laplacian can be extended to a continuous operator $\Delta_R : H^1(\Omega) \longrightarrow (H^1(\Omega))'$ by:

$$\langle \Delta_R u, v \rangle = (\nabla u, \nabla v)_\Omega + (\gamma u, \gamma v)_\Gamma = (u, v)_{1,\Omega} \tag{2.1.1}$$

for all $u, v \in H^1(\Omega)$.

We also define the Robin map $R : H^s(\Gamma) \longrightarrow H^{s+\frac{3}{2}}(\Omega)$ as follows:

$$q = Rp \Longleftrightarrow q \text{ is a weak solution for } \begin{cases} \Delta q = 0 & \text{in } \Omega \\ \partial_\nu q + q = p & \text{on } \Gamma. \end{cases} \tag{2.1.2}$$

Hence, for $p \in L^2(\Gamma)$, we know from (2.1.2) that

$$(Rp, \phi)_{1,\Omega} = (p, \gamma\phi)_\Gamma \quad \text{for all } \phi \in H^1(\Omega). \tag{2.1.3}$$

Combining (2.1.1) and (2.1.3) gives the following useful identity:

$$\langle \Delta_R Rp, \phi \rangle = (Rp, \phi)_{1,\Omega} = (p, \gamma\phi)_\Gamma, \tag{2.1.4}$$

for all $p \in L^2(\Gamma)$ and $\phi \in H^1(\Omega)$.

By using the operators introduced above, we can put (1.1.1) in the following form:

$$\begin{cases} u_{tt} + \Delta_R(u - Rh(\gamma u) + Rg(\gamma u_t)) + g_1(u_t) = f_1(u,v) \\ v_{tt} - \Delta v + g_2(v_t) = f_2(u,v) \\ u(0) = u_0 \in H^1(\Omega), u_t(0) = u_1 \in L^2(\Omega) \\ v(0) = v_0 \in H_0^1(\Omega), v_t(0) = v_1 \in L^2(\Omega). \end{cases} \tag{2.1.5}$$

It is important to point out here that in (2.1.5), we can show $\mathcal{S}_1 := \Delta_R Rg(\gamma u_t)$ and $\mathcal{S}_2 := g(u_t)$ are both maximal monotone from $H^1(\Omega)$ into $(H^1(\Omega))'$. However, in order to show that $\mathcal{S}_1 + \mathcal{S}_2$ is also maximal monotone, one needs to check the validity of domain condition: $(\text{int } \mathcal{D}(\mathcal{S}_1)) \cap \mathcal{D}(\mathcal{S}_2) \neq \emptyset$. The fact that the exponents of the interior and boundary damping, m and q, are allowed to be arbitrary large makes it infeasible to verify the above domain condition.

In order to overcome this difficulty, we shall introduce a maximal monotone operator $\mathcal{S}$ representing the sum of interior and boundary damping. To do so, we first define the functional $J : H^1(\Omega) \longrightarrow [0, +\infty]$ by

$$J(u) = \int_\Omega j_1(u)dx + \int_\Gamma j(\gamma u)d\Gamma. \tag{2.1.6}$$

where j_1 and $j : \mathbb{R} \longrightarrow [0, +\infty)$ are convex functions defined by:

$$j_1(s) = \int_0^s g_1(\tau)d\tau \text{ and } j(s) = \int_0^s g(\tau)d\tau. \tag{2.1.7}$$

Clearly, J is convex and lower semicontinuous. The subdifferential of J, $\partial J : H^1(\Omega) \longrightarrow (H^1(\Omega))'$ is defined by,

$$\partial J(u) = \{u^* \in (H^1(\Omega))' : J(u) + \langle u^*, v - u\rangle \leq J(v) \text{ for all } v \in H^1(\Omega)\}. \tag{2.1.8}$$

The domain $\mathcal{D}(\partial J)$ represents the set of all functions $u \in H^1(\Omega)$ for which $\partial J(u)$ is nonempty.

By Theorem 1.3.23, we know that, for any $u \in \mathcal{D}(\partial J)$, $\partial J(u)$ is a singleton, and thus we may define the operator $\mathcal{S} : \mathcal{D}(\mathcal{S}) = \mathcal{D}(\partial J) \subset H^1(\Omega) \longrightarrow (H^1(\Omega))'$ such that

$$\partial J(u) = \{\mathcal{S}(u)\}. \tag{2.1.9}$$

It is well known that any subdifferential is maximal monotone, thus $\mathcal{S} : \mathcal{D}(\mathcal{S}) \subset H^1(\Omega) \longrightarrow (H^1(\Omega))'$ is a maximal monotone operator. Moreover, by Theorem 1.3.23, we also know that, for all $u \in \mathcal{D}(\mathcal{S})$, we have $g_1(u) \in L^1(\Omega)$, $g_1(u)u \in L^1(\Omega)$, $g(\gamma u) \in L^1(\Gamma)$ and $g(\gamma u)\gamma u \in L^1(\Gamma)$. In addition,

$$\langle \mathcal{S}(u), u\rangle = \int_\Omega g_1(u)u dx + \int_\Gamma g(\gamma u)\gamma u d\Gamma, \tag{2.1.10}$$

and

$$\langle \mathcal{S}(u), v\rangle = \int_\Omega g_1(u)v dx + \int_\Gamma g(\gamma u)\gamma v d\Gamma \text{ for all } v \in C(\overline{\Omega}). \tag{2.1.11}$$

It follows that for all $u \in \mathcal{D}(S)$,

$$\langle \mathcal{S}(u), v\rangle = \int_\Omega g_1(u)v dx + \int_\Gamma g(\gamma u)\gamma v d\Gamma \text{ for all } v \in H^1(\Omega) \cap L^\infty(\Omega). \tag{2.1.12}$$

In fact, if $v \in H^1(\Omega) \cap L^\infty(\Omega)$, then there exists $v_n \in C(\overline{\Omega})$ such that $v_n \to v$ in $H^1(\Omega)$ and a.e. in Ω with $|v_n| \leq M$ in Ω for some $M > 0$. By (2.1.11) and the Lebesgue Dominated Convergence Theorem, we obtain (2.1.12).

By using the operator $\mathcal{S}$ we may rewrite (2.1.5) as

$$\begin{cases} u_{tt} + \Delta_R(u - Rh(\gamma u)) + \mathcal{S}(u_t) = f_1(u,v), \\ v_{tt} - \Delta v + g_2(v_t) = f_2(u,v), \\ u(0) = u_0 \in H^1(\Omega), u_t(0) = u_1 \in L^2(\Omega), \\ v(0) = v_0 \in H_0^1(\Omega), v_t(0) = v_1 \in L^2(\Omega). \end{cases} \tag{2.1.13}$$

It is important to note here that $\mathcal{S}(u_t)$ represents the sum of the interior damping $g(u_t)$ and the boundary damping $\Delta_R Rg(\gamma u_t)$. However, $\mathcal{D}(\mathcal{S})$ is not necessarily the same as the domain of the operator $\Delta_R Rg(\gamma\cdot) + g(\cdot) : H^1(\Omega) \longrightarrow (H^1(\Omega))'$. Therefore, systems (2.1.5) and (2.1.13) are not exactly equivalent. Nonetheless, we shall see that if (u,v) is a strong solution for (2.1.13), then (u,v) must be a weak solution for (1.1.1) in the sense of Definition 1.3.1. So, instead of studying (1.1.1) directly, we show system (2.1.13) has a unique strong solution first.

Let $H = H^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ and define the nonlinear operator

$$\mathscr{A} : \mathcal{D}(\mathscr{A}) \subset H \longrightarrow H$$

by

$$\mathscr{A}\begin{bmatrix} u \\ v \\ y \\ z \end{bmatrix}^{tr} = \begin{bmatrix} -y \\ -z \\ \Delta_R(u - Rh(\gamma u)) + \mathcal{S}(y) - f_1(u,v) \\ -\Delta v + g_2(z) - f_2(u,v) \end{bmatrix}^{tr}, \tag{2.1.14}$$

where

$$\begin{aligned}\mathcal{D}(\mathscr{A}) = \Big\{ & (u,v,y,z) \in \Big(H^1(\Omega)\times H_0^1(\Omega)\Big)^2 : \\ & \Delta_R(u - Rh(\gamma u)) + \mathcal{S}(y) - f_1(u,v) \in L^2(\Omega), \quad y \in \mathcal{D}(\mathcal{S}), \\ & -\Delta v + g_2(z) - f_2(u,v) \in L^2(\Omega), \quad g_2(z) \in H^{-1}(\Omega)\cap L^1(\Omega)\Big\}.\end{aligned}$$

Put $U = (u, v, u_t, v_t)$. Then the system (2.1.13) is equivalent to

$$U_t + \mathscr{A}U = 0 \text{ with } U(0) = (u_0, v_0, u_1, v_1) \in H. \tag{2.1.15}$$

2.1.2 Globally Lipschitz sources

First, we deal with the case where the boundary damping is assumed strongly monotone and the sources are globally Lipschitz. In this case, we have the following lemma.

Lemma 2.1.1. *Assume that,*

- g_1, g_2 *and* g *are continuous and monotone increasing functions with* $g_1(0) = g_2(0) = g(0) = 0$. *Moreover, the following strong monotonicity condition is imposed on* g*:*
  there exists $m_g > 0$ *such that* $(g(s_1) - g(s_2))(s_1 - s_2) \geq m_g|s_1 - s_2|^2$.
- $f_1,\ f_2 : H^1(\Omega)\times H_0^1(\Omega) \longrightarrow L^2(\Omega)$ *are globally Lipschitz.*
- $h\circ\gamma : H^1(\Omega) \longrightarrow L^2(\Gamma)$ *is globally Lipschitz.*

Then, system (2.1.15) has a unique global strong solution $U \in W^{1,\infty}(0,T;H)$ *for arbitrary* $T > 0$*; provided the datum* $U_0 \in \mathcal{D}(\mathscr{A})$*.*

Proof. In order to prove Lemma 2.1.1 it suffices to show that the operator $\mathscr{A} + \omega I$ is m-accretive for some positive ω. We say an operator $\mathscr{A} : \mathcal{D}(\mathscr{A}) \subset H \longrightarrow H$ is *accretive* if $(\mathscr{A}x_1 - \mathscr{A}x_2, x_1 - x_2)_H \geq 0$, for all $x_1, x_2 \in \mathcal{D}(\mathscr{A})$, and it is *m-accretive* if, in addition, $\mathscr{A} + I$ maps $\mathcal{D}(\mathscr{A})$ onto H. In fact, by Kato's Theorem (see [46] for instance), if $\mathscr{A} + \omega I$ is m-accretive for some positive ω, then for each $U_0 \in \mathcal{D}(\mathscr{A})$ there is a unique strong solution U of (2.1.15), i.e., $U \in W^{1,\infty}(0,T;H)$ such that $U(0) = U_0$, $U(t) \in \mathcal{D}(\mathscr{A})$ for all $t \in [0,T]$, and equation (2.1.15) is satisfied a.e. $[0,T]$, where $T > 0$ is arbitrary.

Step 1: Proof for $\mathscr{A}+\omega I$ is accretive for some positive ω. Let $U=[u,v,y,z]$, $\hat{U}=[\hat{u},\hat{v},\hat{y},\hat{z}]\in\mathcal{D}(\mathscr{A})$. We aim to find $\omega>0$ such that

$$((\mathscr{A}+\omega I)U-(\mathscr{A}+\omega I)\hat{U},\, U-\hat{U})_H\geq 0.$$

By straightforward calculations, we obtain

$$\begin{aligned}
&((\mathscr{A}+\omega I)U-(\mathscr{A}+\omega I)\hat{U},U-\hat{U})_H=(\mathscr{A}(U)-\mathscr{A}(\hat{U}),U-\hat{U})_H+\omega|U-\hat{U}|_H^2\\
&=-(y-\hat{y},u-\hat{u})_{1,\Omega}-(z-\hat{z},v-\hat{v})_{1,\Omega}+\langle\Delta_R(u-\hat{u}),y-\hat{y}\rangle\\
&\quad-\langle\Delta_R R(h(\gamma u)-h(\gamma\hat{u})),y-\hat{y}\rangle+\langle\mathcal{S}(y)-\mathcal{S}(\hat{y}),y-\hat{y}\rangle\\
&\quad-(f_1(u,v)-f_1(\hat{u},\hat{v}),y-\hat{y})_\Omega-\langle\Delta(v-\hat{v}),z-\hat{z}\rangle\\
&\quad+\langle g_2(z)-g_2(\hat{z}),z-\hat{z}\rangle-(f_2(u,v)-f_2(\hat{u},\hat{v}),z-\hat{z})_\Omega\\
&\quad+\omega(\|u-\hat{u}\|_{1,\Omega}^2+\|v-\hat{v}\|_{1,\Omega}^2+\|y-\hat{y}\|_2^2+\|z-\hat{z}\|_2^2).
\end{aligned}\tag{2.1.16}$$

Notice

$$-\langle\Delta(v-\hat{v}),z-\hat{z}\rangle=(\nabla(v-\hat{v}),\nabla(z-\hat{z}))_\Omega=(v-\hat{v},z-\hat{z})_{1,\Omega}.\tag{2.1.17}$$

Moreover, since $g_2(y)-g_2(\hat{y})\in H^{-1}(\Omega)\cap L^1(\Omega)$ and $z-\hat{z}\in H_0^1(\Omega)$ satisfying $(g_2(z(x))-g_2(\hat{z}(x)))(z(x)-\hat{z}(x))\geq 0$, for all $x\in\Omega$, then by Lemma 2.2 (p.89) in [6], we have $(g_2(z)-g_2(\hat{z}))(z-\hat{z})\in L^1(\Omega)$ and

$$\langle g_2(z)-g_2(\hat{z}),z-\hat{z}\rangle=\int_\Omega(g_2(z)-g_2(\hat{z}))(z-\hat{z})dx\geq 0.\tag{2.1.18}$$

Now we show

$$\begin{aligned}
&\langle\mathcal{S}(y)-\mathcal{S}(\hat{y}),y-\hat{y}\rangle\\
&\geq\int_\Omega(g_1(y)-g_1(y))(y-\hat{y})dx+\int_\Gamma(g(\gamma y)-g(\gamma\hat{y}))(\gamma y-\gamma\hat{y})d\Gamma.
\end{aligned}\tag{2.1.19}$$

Since $y-\hat{y}\in H^1(\Omega)$, if we set

$$w_n=\begin{cases} n & \text{if } y-\hat{y}\geq n\\ y-\hat{y} & \text{if } |y-\hat{y}|\leq n\\ -n & \text{if } y-\hat{y}\leq -n,\end{cases}\tag{2.1.20}$$

then $w_n\in H^1(\Omega)\cap L^\infty(\Omega)$. So by (2.1.12) one has

$$\langle\mathcal{S}(y)-\mathcal{S}(\hat{y}),w_n\rangle=\int_\Omega(g_1(y)-g_1(\hat{y}))w_ndx+\int_\Gamma(g(\gamma y)-g(\gamma\hat{y}))\gamma w_nd\Gamma.\tag{2.1.21}$$

Moreover, by (2.1.20) we know w_n and $y-\hat{y}$ have the same sign, then since g_1 is monotone increasing, one has $(g_1(y)-g_1(\hat{y}))w_n \geq 0$ a.e. in Ω. Therefore, by Fatou's Lemma, we obtain

$$\liminf_{n\to\infty} \int_\Omega (g_1(y)-g_1(\hat{y}))w_n dx \geq \int_\Omega (g_1(y)-g_1(\hat{y}))(y-\hat{y})dx. \tag{2.1.22}$$

Likewise, we have

$$\liminf_{n\to\infty} \int_\Gamma (g(\gamma y)-g(\gamma\hat{y}))\gamma w_n d\Gamma \geq \int_\Omega (g(\gamma y)-g(\gamma\hat{y}))(\gamma y-\gamma\hat{y})d\Gamma. \tag{2.1.23}$$

Since $w_n \to y-\hat{y}$ in $H^1(\Omega)$, by taking the lower limit on both sides of (2.1.21) and using (2.1.22)-(2.1.23), we conclude that the inequality (2.1.19) holds.

By using (2.1.1), (2.1.4), (2.1.17), (2.1.18) and (2.1.19), we obtain from (2.1.16) that

$$\begin{aligned}
&((\mathscr{A}+\omega I)U-(\mathscr{A}+\omega I)\hat{U}, U-\hat{U})_H \\
&\geq (g(\gamma y)-g(\gamma\hat{y}), \gamma y-\gamma\hat{y})_\Gamma - (h(\gamma u)-h(\gamma\hat{u}), \gamma y-\gamma\hat{y})_\Gamma \\
&\quad - (f_1(u,v)-f_1(\hat{u},\hat{v}), y-\hat{y})_\Omega - (f_2(u,v)-f_2(\hat{u},\hat{v}), z-\hat{z})_\Omega \\
&\quad + \omega(\|u-\hat{u}\|_{1,\Omega}^2 + \|v-\hat{v}\|_{1,\Omega}^2 + \|y-\hat{y}\|_2^2 + \|z-\hat{z}\|_2^2).
\end{aligned} \tag{2.1.24}$$

Let $V = H^1(\Omega)\times H_0^1(\Omega)$ and recall the assumption that f_1, f_2 and h are globally Lipschitz continuous with Lipshcitz constant L_{f_1}, L_{f_2}, and L_h; respectively. Let $L = \max\{L_{f_1}, L_{f_2}, L_h\}$. Therefore, by employing the strong monotonicity condition on g and Young's inequality, we have

$$\begin{aligned}
&(g(\gamma y)-g(\gamma\hat{y}), \gamma y-\gamma\hat{y})_\Gamma - (h(\gamma u)-h(\gamma\hat{u}), \gamma y-\gamma\hat{y})_\Gamma \\
&\quad - (f_1(u,v)-f_1(\hat{u},\hat{v}), y-\hat{y})_\Omega - (f_2(u,v)-f_2(\hat{u},\hat{v}), z-\hat{z})_\Omega \\
&\geq m_g|\gamma y-\gamma\hat{y}|_2^2 - L\left\|u-\hat{u}\right\|_{1,\Omega}|\gamma y-\gamma\hat{y}|_2 - L\left\|(u-\hat{u}, v-\hat{v})\right\|_V \left\|y-\hat{y}\right\|_2 \\
&\quad - L\left\|(u-\hat{u}, v-\hat{v})\right\|_V \left\|z-\hat{z}\right\|_2 \\
&\geq m_g|\gamma y-\gamma\hat{y}|_2^2 - \frac{L^2}{4\epsilon}\|u-\hat{u}\|_{1,\Omega}^2 - \epsilon|\gamma y-\gamma\hat{y}|_2^2 - \frac{L}{2}(\|u-\hat{u}\|_{1,\Omega}^2 + \|v-\hat{v}\|_{1,\Omega}^2) \\
&\quad - \frac{L}{2}\|y-\hat{y}\|_2^2 - \frac{L}{2}(\|u-\hat{u}\|_{1,\Omega}^2 + \|v-\hat{v}\|_{1,\Omega}^2) - \frac{L}{2}\|z-\hat{z}\|_2^2.
\end{aligned} \tag{2.1.25}$$

Combining (2.1.24) and (2.1.25) leads to

$$
\begin{aligned}
&((\mathscr{A}+\omega I)U-(\mathscr{A}+\omega I)\hat{U},U-\hat{U})_H\\
&\geq (m_g-\epsilon)|\gamma y-\gamma\hat{y}|_2^2+(\omega-\frac{L^2}{4\epsilon}-L)\|u-\hat{u}\|_{1,\Omega}^2\\
&\quad+(\omega-L)\|v-\hat{v}\|_{1,\Omega}^2+(\omega-\frac{L}{2})\|y-\hat{y}\|_2^2+(\omega-\frac{L}{2})\|z-\hat{z}\|_2^2.
\end{aligned}
$$

Therefore, by choosing $\epsilon < m_g$ and $\omega > \frac{L^2}{4\epsilon}+L$, then $\mathscr{A}+\omega I$ is accretive.

Step 2: Proof for $\mathscr{A}+\lambda I$ is m-accretive, for some $\lambda>0$. To this end, it suffices to show that the range of $\mathscr{A}+\lambda I$ is all of H, for some $\lambda>0$.

Let $(a,b,c,d)\in H$. We have to show that there exists $(u,v,y,z)\in\mathcal{D}(\mathscr{A})$ such that $(\mathscr{A}+\lambda I)(u,v,y,z)=(a,b,c,d)$, for some $\lambda>0$, i.e.,

$$
\begin{cases}
-y+\lambda u=a\\
-z+\lambda v=b\\
\Delta_R(u-Rh(\gamma u))+\mathcal{S}(y)-f_1(u,v)+\lambda y=c\\
-\Delta v+g_2(z)-f_2(u,v)+\lambda z=d.
\end{cases}
\tag{2.1.26}
$$

Note, (2.1.26) is equivalent to

$$
\begin{cases}
\frac{1}{\lambda}\Delta_R(y)-\Delta_R Rh\left(\gamma\frac{a+y}{\lambda}\right)+\mathcal{S}(y)-f_1\left(\frac{a+y}{\lambda},\frac{b+z}{\lambda}\right)+\lambda y=c-\frac{1}{\lambda}\Delta_R(a)\\
-\frac{1}{\lambda}\Delta z+g_2(z)-f_2\left(\frac{a+y}{\lambda},\frac{b+z}{\lambda}\right)+\lambda z=d+\frac{1}{\lambda}\Delta b.
\end{cases}
\tag{2.1.27}
$$

Recall that $V=H^1(\Omega)\times H_0^1(\Omega)$ and notice that the right hand side of (2.1.27) belongs to V'. Thus, we define the operator $\mathscr{B}:\mathcal{D}(\mathscr{B})\subset V\longrightarrow V'$ by:

$$
\mathscr{B}\begin{bmatrix} y\\ z\end{bmatrix}^{tr}=\begin{bmatrix}\frac{1}{\lambda}\Delta_R(y)-\Delta_R Rh\left(\gamma\frac{a+y}{\lambda}\right)+\mathcal{S}(y)-f_1\left(\frac{a+y}{\lambda},\frac{b+z}{\lambda}\right)+\lambda y\\ -\frac{1}{\lambda}\Delta z+g_2(z)-f_2\left(\frac{a+y}{\lambda},\frac{b+z}{\lambda}\right)+\lambda z\end{bmatrix}^{tr}
$$

where $\mathcal{D}(\mathscr{B})=\{(y,z)\in V : y\in\mathcal{D}(\mathcal{S}),\, g_2(z)\in H^{-1}(\Omega)\cap L^1(\Omega)\}$. Therefore, the issue reduces to proving that $\mathscr{B}:\mathcal{D}(\mathscr{B})\subset V\longrightarrow V'$ is surjective. By Corollary 1.2 (p.45) in [6], it is enough to show that $\mathscr{B}$ is maximal monotone and coercive.

We split $\mathscr{B}$ as two operators:

$$
\mathscr{B}_1\begin{bmatrix} y\\ z\end{bmatrix}^{tr}=\begin{bmatrix}\frac{1}{\lambda}\Delta_R(y)-\Delta_R Rh\left(\gamma\frac{a+y}{\lambda}\right)-f_1\left(\frac{a+y}{\lambda},\frac{b+z}{\lambda}\right)+\lambda y\\ -\frac{1}{\lambda}\Delta z-f_2\left(\frac{a+y}{\lambda},\frac{b+z}{\lambda}\right)+\lambda z\end{bmatrix}^{tr},
$$

and

$$\mathscr{B}_2 \begin{bmatrix} y \\ z \end{bmatrix}^{tr} = \begin{bmatrix} \mathcal{S}(y) \\ g_2(z) \end{bmatrix}^{tr}.$$

$\mathscr{B}_1$ **is maximal monotone and coercive:** First we note $\mathcal{D}(\mathscr{B}_1) = V$. To see $\mathscr{B}_1 : V \longrightarrow V'$ is monotone, we let $Y = (y, z) \in V$ and $\hat{Y} = (\hat{y}, \hat{z}) \in V$. By straightforward calculations, we obtain

$$\begin{aligned}
&\langle \mathscr{B}_1 Y - \mathscr{B}_1 \hat{Y}, Y - \hat{Y}\rangle \\
&= \frac{1}{\lambda}\langle \Delta_R(y-\hat{y}), y-\hat{y}\rangle - \left\langle \Delta_R R\left(h\left(\gamma\frac{a+y}{\lambda}\right) - h\left(\gamma\frac{a+\hat{y}}{\lambda}\right)\right), y-\hat{y}\right\rangle \\
&- \left(f_1\left(\frac{a+y}{\lambda}, \frac{b+z}{\lambda}\right) - f_1\left(\frac{a+\hat{y}}{\lambda}, \frac{b+\hat{z}}{\lambda}\right), y-\hat{y}\right)_\Omega \\
&+ \lambda\|y-\hat{y}\|_2^2 - \frac{1}{\lambda}\langle \Delta(z-\hat{z}), z-\hat{z}\rangle \\
&- \left(f_2\left(\frac{a+y}{\lambda}, \frac{b+z}{\lambda}\right) - f_2\left(\frac{a+\hat{y}}{\lambda}, \frac{b+\hat{z}}{\lambda}\right), z-\hat{z}\right)_\Omega + \lambda\|z-\hat{z}\|_2^2.
\end{aligned}$$

By (2.1.1) and (2.1.4) we have,

$$\begin{aligned}
&\langle \mathscr{B}_1 Y - \mathscr{B}_1 \hat{Y}, Y - \hat{Y}\rangle \\
&= \frac{1}{\lambda}(y-\hat{y}, y-\hat{y})_{1,\Omega} - \left(h\left(\gamma\frac{a+y}{\lambda}\right) - h\left(\gamma\frac{a+\hat{y}}{\lambda}\right), \gamma y - \gamma\hat{y}\right)_\Gamma \\
&- \left(f_1\left(\frac{a+y}{\lambda}, \frac{b+z}{\lambda}\right) - f_1\left(\frac{a+\hat{y}}{\lambda}, \frac{b+\hat{z}}{\lambda}\right), y-\hat{y}\right)_\Omega \\
&+ \lambda\|y-\hat{y}\|_2^2 + \frac{1}{\lambda}(z-\hat{z}, z-\hat{z})_{1,\Omega} \\
&- \left(f_2\left(\frac{a+y}{\lambda}, \frac{b+z}{\lambda}\right) - f_2\left(\frac{a+\hat{y}}{\lambda}, \frac{b+\hat{z}}{\lambda}\right), z-\hat{z}\right)_\Omega + \lambda\|z-\hat{z}\|_2^2.
\end{aligned}$$

Since f_1, f_2, h are Lipschitz continuous with Lipschitz constant L,

$$\begin{aligned}
\langle \mathscr{B}_1 Y - \mathscr{B}_1 \hat{Y}, Y - \hat{Y}\rangle &\geq \frac{1}{\lambda}\|y-\hat{y}\|_{1,\Omega}^2 - \frac{L}{\lambda}\|y-\hat{y}\|_{1,\Omega}|\gamma y - \gamma\hat{y}|_2 \\
&- \frac{L}{\lambda}\|(y-\hat{y}, z-\hat{z})\|_V\|y-\hat{y}\|_2 + \lambda\|y-\hat{y}\|_2^2 + \frac{1}{\lambda}\|z-\hat{z}\|_{1,\Omega}^2 \\
&- \frac{L}{\lambda}\|(y-\hat{y}, z-\hat{z})\|_V\|z-\hat{z}\|_2 + \lambda\|z-\hat{z}\|_2^2.
\end{aligned}$$

Applying Young's inequality yields,

$$
\begin{aligned}
\langle \mathscr{B}_1 Y - \mathscr{B}_1 \hat{Y}, Y - \hat{Y} \rangle &\geq \frac{1}{\lambda}\|y-\hat{y}\|_{1,\Omega}^2 - \frac{L^2}{4\eta\lambda}\|y-\hat{y}\|_{1,\Omega}^2 - \frac{\eta}{\lambda}|\gamma y - \gamma\hat{y}|_2^2 \\
&- \frac{L^2}{4\eta\lambda}(\|y-\hat{y}\|_{1,\Omega}^2 + \|z-\hat{z}\|_{1,\Omega}^2) - \frac{\eta}{\lambda}\|y-\hat{y}\|_2^2 + \lambda\|y-\hat{y}\|_2^2 + \frac{1}{\lambda}\|z-\hat{z}\|_{1,\Omega}^2 \\
&- \frac{L^2}{4\eta\lambda}(\|y-\hat{y}\|_{1,\Omega}^2 + \|z-\hat{z}\|_{1,\Omega}^2) - \frac{\eta}{\lambda}\|z-\hat{z}\|_2^2 + \lambda\|z-\hat{z}\|_2^2 \\
&\geq \left(\frac{1}{\lambda} - \frac{3L^2}{4\eta\lambda}\right)\|y-\hat{y}\|_{1,\Omega}^2 - \frac{\eta}{\lambda}|\gamma y - \gamma\hat{y}|_2^2 \\
&+ \left(\frac{1}{\lambda} - \frac{2L^2}{4\eta\lambda}\right)\|z-\hat{z}\|_{1,\Omega}^2 + \left(\lambda - \frac{\eta}{\lambda}\right)(\|y-\hat{y}\|_2^2 + \|z-\hat{z}\|_2^2).
\end{aligned}
$$

By using the imbedding $H^{\frac{1}{2}}(\Omega) \hookrightarrow L^2(\Gamma)$ and the interpolation inequality (1.2.2), we obtain,

$$
|\gamma u|_2^2 \leq C\|u\|_{H^{\frac{1}{2}}(\Omega)}^2 \leq \delta\|u\|_{1,\Omega}^2 + C_\delta\|u\|_2^2,
$$

for all $u \in H^1(\Omega)$, where $\delta > 0$. It follows that,

$$
|\gamma y - \gamma\hat{y}|_2^2 \leq \delta\|y-\hat{y}\|_{1,\Omega}^2 + C_\delta\|y-\hat{y}\|_2^2.
$$

Thus,

$$
\begin{aligned}
\langle \mathscr{B}_1 Y - \mathscr{B}_1 \hat{Y}, Y - \hat{Y} \rangle &\geq \left(\frac{1}{2\lambda} - \frac{3L^2}{4\eta\lambda} - \frac{\eta\delta}{\lambda}\right)\|y-\hat{y}\|_{1,\Omega}^2 \\
&+ (\lambda - \frac{\eta+\eta C_\delta}{\lambda})\|y-\hat{y}\|_2^2 + (\lambda - \frac{\eta}{\lambda})\|z-\hat{z}\|_2^2 + \left(\frac{1}{2\lambda} - \frac{2L^2}{4\eta\lambda}\right)\|z-\hat{z}\|_{1,\Omega}^2 \\
&+ \frac{1}{2\lambda}(\|y-\hat{y}\|_{1,\Omega}^2 + \|z-\hat{z}\|_{1,\Omega}^2).
\end{aligned}
$$

Note that the sign of

$$
\frac{1}{2\lambda} - \frac{3L^2}{4\eta\lambda} - \frac{\eta\delta}{\lambda} = \frac{2 - 3L^2/\eta - 4\eta\delta}{4\lambda},
$$

does not depend on the value of λ. So, we let $\eta > 3L^2$ and choose $\delta > 0$ sufficiently small so that $4\eta\delta < 1$. In addition, we select λ sufficiently large such that $\lambda^2 > \eta + \eta C_\delta$. Therefore,

$$
\langle \mathscr{B}_1 Y - \mathscr{B}_1 \hat{Y}, Y - \hat{Y} \rangle \geq \frac{1}{2\lambda}(\|y-\hat{y}\|_{1,\Omega}^2 + \|z-\hat{z}\|_{1,\Omega}^2) = \frac{1}{2\lambda}\|Y-\hat{Y}\|_V^2,
$$

proving that $\mathscr{B}_1$ is strongly monotone. It is easy to see that strong monotonicity implies coercivity of $\mathscr{B}_1$.

Next, we show that $\mathscr{B}_1$ is continuous. Clearly, $\Delta_R : H^1(\Omega) \longrightarrow (H^1(\Omega))'$ and $\Delta : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ are continuous. Moreover, if we set

$$\tilde{f}_j(y,z) := f_j\left(\frac{a+y}{\lambda}, \frac{b+z}{\lambda}\right), \quad j = 1, 2,$$

then, since $f_1, f_2 : V \longrightarrow L^2(\Omega)$ are globally Lipschitz, it is clear that the mappings $\tilde{f}_1 : V \longrightarrow (H^1(\Omega))'$ and $\tilde{f}_2 : V \longrightarrow H^{-1}(\Omega)$ are also Lipschitz continuous.

To see the mapping

$$\tilde{h}(y) := \Delta_R R h\left(\gamma \frac{a+y}{\lambda}\right)$$

is Lipschitz continuous form $H^1(\Omega)$ into $(H^1(\Omega))'$, we use (2.1.4) and the assumption that $h \circ \gamma : H^1(\Omega) \longrightarrow L^2(\Gamma)$ is globally Lipschitz continuous. Indeed,

$$\begin{aligned}\left\|\tilde{h}(y) - \tilde{h}(\hat{y})\right\|_{(H^1(\Omega))'} &= \sup_{\|\varphi\|_{1,\Omega}=1} \left(h\left(\gamma\frac{a+y}{\lambda}\right) - h\left(\gamma\frac{a+\hat{y}}{\lambda}\right), \gamma\varphi\right)_\Gamma \\ &\le C\left|h\left(\gamma\frac{a+y}{\lambda}\right) - h\left(\gamma\frac{a+\hat{y}}{\lambda}\right)\right|_2 \le \frac{CL}{\lambda}\|y-\hat{y}\|_{1,\Omega}.\end{aligned}$$

It follows that $\mathscr{B}_1 : V \longrightarrow V'$ is continuous and along with the monotonicity of $\mathscr{B}_1$, we conclude that $\mathscr{B}_1$ is maximal monotone.

$\mathscr{B}_2$ is maximal monotone: First we note $\mathcal{D}(\mathscr{B}_2) = \mathcal{D}(\mathscr{B}) = \{(y,z) \in V : y \in \mathcal{D}(\mathcal{S}),\, g_2(z) \in H^{-1}(\Omega) \cap L^1(\Omega)\}$. Remember in Subsection 2.1.1 we have already known $\mathcal{S} : \mathcal{D}(\mathcal{S}) \subset H^1(\Omega) \longrightarrow (H^1(\Omega))'$ is maximal monotone. In order to study the operator $g_2(z)$, we define the functional $J_2 : H_0^1(\Omega) \longrightarrow [0,\infty]$ by

$$J_2(z) = \int_\Omega j_2(z(x))dx$$

where $j_2 : \mathbb{R} \longrightarrow [0,+\infty)$ is a convex function defined by

$$j_2(s) = \int_0^s g_2(\tau)d\tau.$$

Clearly J_2 is proper, convex and lower semi-continuous. Moreover, by Corollary 1.3.24 we know that $\partial J_2 : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ is described by

$$\partial J_2(z) = \{\mu \in H^{-1}(\Omega) \cap L^1(\Omega) : \mu = g_2(z) \text{ a.e. in } \Omega\}. \tag{2.1.28}$$

That is to say, $\mathcal{D}(\partial J_2) = \{z \in H_0^1(\Omega) : g_2(z) \in H^{-1}(\Omega) \cap L^1(\Omega)\}$ and for all $z \in \mathcal{D}(\partial J_2)$, $\partial J_2(z)$ is a singleton such that $\partial J_2(z) = \{g_2(z)\}$. Since any subdifferential is maximal monotone, we obtain the maximal monotonicity of the operator $g_2(\cdot) : \mathcal{D}(\partial J_2) \subset H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$. Hence, by Proposition 2.6.1 in the Appendix, it follows that $\mathscr{B}_2 : \mathcal{D}(\mathscr{B}_2) \subset V \longrightarrow V'$ is maximal monotone. Now, Since $\mathscr{B}_1$ and $\mathscr{B}_2$ are both maximal monotone and $\mathcal{D}(\mathscr{B}_1) = V$, we conclude that $\mathscr{B} = \mathscr{B}_1 + \mathscr{B}_2$ is maximal monotone.

Finally, since $\mathscr{B}_2$ is monotone and $\mathscr{B}_2 0 = 0$, it follows that $\langle \mathscr{B}_2 Y, Y\rangle \geq 0$ for all $Y \in \mathcal{D}(\mathcal{S})$, and along with the fact $\mathscr{B}_1$ is coercive, we obtain $\mathscr{B} = \mathscr{B}_1 + \mathscr{B}_2$ is coercive as well. Then, the surjectivity of $\mathscr{B}$ follows immediately by Corollary 1.2 (p.45) in [6]. Thus, we proved the existence of (y,z) in $\mathcal{D}(\mathscr{B}) \subset V = H^1(\Omega) \times H_0^1(\Omega)$ such that (y,z) satisfies (2.1.27). So by (2.1.26), $(u,v) = (\frac{y+a}{\lambda}, \frac{z+b}{\lambda}) \in H^1(\Omega) \times H_0^1(\Omega)$. In addition, one can easily see that $(u,v,y,z) \in \mathcal{D}(\mathscr{A})$. Indeed, we have $\Delta_R(u - Rh(\gamma u)) + \mathcal{S}(y) - f_1(u,v) = -\lambda y + c \in L^2(\Omega)$ and $-\Delta v + g_2(z) - f_2(u,v) = -\lambda z + d \in L^2(\Omega)$. Thus, the proof of maximal accretivity is completed and so is the proof of Lemma 2.1.1. $\square$

2.1.3 Locally Lipschitz sources

In this subsection, we loosen the restrictions on sources and allow f_1, f_2 and h to be locally Lipschitz continuous.

Lemma 2.1.2. *For $m, r, q \geq 1$, we assume that:*

- *g_1, g_2 and g are continuous and monotone increasing functions with $g_1(0) = g_2(0) = g(0) = 0$. In addition, the following growth conditions hold: there exist positive constants a_j, $j = 1,2,3$, such that $g_1(s)s \geq a_1|s|^{m+1}$, $g_2(s)s \geq a_2|s|^{r+1}$ and $g(s)s \geq a_3|s|^{q+1}$ for $|s| \geq 1$. Moreover, there exists $m_g > 0$ such that $(g(s_1) - g(s_2))(s_1 - s_2) \geq m_g|s_1 - s_2|^2$.*
- *f_1, $f_2 : H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^2(\Omega)$ are locally Lipschitz continuous.*
- *$h \circ \gamma : H^1(\Omega) \longrightarrow L^2(\Gamma)$ is locally Lipschitz continuous.*

Then, system (2.1.15) has a unique local strong solution $U \in W^{1,\infty}(0,T_0;H)$ for some $T_0 > 0$; provided the initial datum $U_0 \in \mathcal{D}(\mathscr{A})$.

Proof. As in [12, 17], we use standard truncation of the sources. Recall $V = H^1(\Omega) \times H_0^1(\Omega)$ and define

$$f_1^K(u,v) = \begin{cases} f_1(u,v) & \text{if } \|(u,v)\|_V \leq K \\ f_1\Big(\frac{Ku}{\|(u,v)\|_V}, \frac{Kv}{\|(u,v)\|_V}\Big) & \text{if } \|(u,v)\|_V > K, \end{cases}$$

$$f_2^K(u,v) = \begin{cases} f_2(u,v) & \text{if } \|(u,v)\|_V \leq K \\ f_2\Big(\frac{Ku}{\|(u,v)\|_V}, \frac{Kv}{\|(u,v)\|_V}\Big) & \text{if } \|(u,v)\|_V > K, \end{cases}$$

$$h^K(u) = \begin{cases} h(\gamma u) & \text{if } \|u\|_{1,\Omega} \leq K \\ h\Big(\gamma \frac{Ku}{\|u\|_{1,\Omega}}\Big) & \text{if } \|u\|_{1,\Omega} > K, \end{cases}$$

where K is a positive constant such that $K^2 \geq 4\mathscr{E}(0)+1$, where the quadratic energy $\mathscr{E}(t)$ is given by $\mathscr{E}(t) = \frac{1}{2}\Big(\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2 + \|u_t(t)\|_2^2 + \|v_t(t)\|_2^2\Big)$.

With the truncated sources above, we consider the following (K) problem:

$$(K) \begin{cases} u_{tt} + \Delta_R(u - Rh^K(u)) + \mathcal{S}(u_t) = f_1^K(u,v) & \text{in } \Omega \times (0,\infty) \\ v_{tt} - \Delta v + g_2(v_t) = f_2^K(u,v) & \text{in } \Omega \times (0,\infty) \\ u(x,0) = u_0(x) \in H^1(\Omega), u_t(x,0) = u_1(x) \in H^1(\Omega) \\ v(x,0) = v_0(x) \in H_0^1(\Omega), v_t(x,0) = v_1(x) \in H_0^1(\Omega). \end{cases}$$

We note here that for each such K, the operators f_1^K, $f_2^K : H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^2(\Omega)$ and $h^K : H^1(\Omega) \longrightarrow L^2(\Gamma)$ are globally Lipschitz continuous (see [17]). Therefore, by Lemma 2.1.1, the (K) problem has a unique global strong solution $U_K \in W^{1,\infty}(0,T;H)$ for any $T > 0$ provided the initial datum $U_0 \in \mathcal{D}(\mathscr{A})$.

In what follows, we shall express $(u_K(t), v_K(t))$ as $(u(t), v(t))$. Since $u_t \in \mathcal{D}(\mathcal{S}) \subset H^1(\Omega)$ and $v_t \in H_0^1(\Omega)$ such that $g(v_t) \in H^{-1}(\Omega) \cap L^1(\Omega)$, then by (2.1.10) and Lemma 2.2 (p.89) in [6], we may use the multiplier u_t and v_t on the (K) problem and obtain the following energy identity:

$$\begin{aligned} &\mathscr{E}(t) + \int_0^t \int_\Omega \Big(g_1(u_t)u_t + g_2(v_t)v_t\Big) dx d\tau + \int_0^t \int_\Gamma g(\gamma u_t)\gamma u_t d\Gamma d\tau \\ &= \mathscr{E}(0) + \int_0^t \int_\Omega (f_1^K(u,v)u_t + f_2^K(u,v)v_t) dx d\tau + \int_0^t \int_\Gamma h^K(u)\gamma u_t d\Gamma d\tau. \end{aligned} \tag{2.1.29}$$

In addition, since m, r, $q \geq 1$, we know $\tilde{m} = \frac{m+1}{m}$, $\tilde{r} = \frac{r+1}{r}$, $\tilde{q} = \frac{q+1}{q} \leq 2$. Hence, by our assumptions on the sources, it follows that f_1: $H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^{\tilde{m}}(\Omega)$, f_2: $H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^{\tilde{r}}(\Omega)$, and $h \circ \gamma$: $H^1(\Omega) \longrightarrow L^{\tilde{q}}(\Gamma)$ are all locally Lipschitz with Lipschitz constant $L_{f_1}(K)$, $L_{f_2}(K)$, $L_h(K)$, respectively, on the ball $\{(u,v) \in V : \|(u,v)\|_V \leq K$. Put

$$L_K = \max\{L_{f_1}(K), L_{f_2}(K), L_h(K)\}.$$

By using similar calculations as in [17], we deduce $f_1^K : H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^{\tilde{m}}(\Omega)$, $f_2^K : H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^{\tilde{r}}(\Omega)$ and $h^K : H^1(\Omega) \longrightarrow L^{\tilde{q}}(\Gamma)$ are globally Lipschitz with Lipschitz constant L_K.

We now estimate the terms due to the sources in the energy identity (2.1.29). By using Hölder's and Young's inequalities, we have

$$\begin{aligned}
&\int_0^t \int_\Omega f_1^K(u,v) u_t dx d\tau \leq \int_0^t \left\| f_1^K(u,v) \right\|_{\tilde{m}} \|u_t\|_{m+1} d\tau \\
&\leq \epsilon \int_0^t \|u_t\|_{m+1}^{m+1} d\tau + C_\epsilon \int_0^t \left\| f_1^K(u,v) \right\|_{\tilde{m}}^{\tilde{m}} d\tau \\
&\leq \epsilon \int_0^t \|u_t\|_{m+1}^{m+1} d\tau + C_\epsilon \int_0^t \left(\left\| f_1^K(u,v) - f_1^K(0,0) \right\|_{\tilde{m}}^{\tilde{m}} d + \left\| f_1^K(0,0) \right\|_{\tilde{m}}^{\tilde{m}} \right) d\tau \\
&\leq \epsilon \int_0^t \|u_t\|_{m+1}^{m+1} d\tau + C_\epsilon L_K^{\tilde{m}} \int_0^t (\|u\|_{1,\Omega}^{\tilde{m}} + \|v\|_{1,\Omega}^{\tilde{m}}) d\tau + C_\epsilon t |f_1(0,0)|^{\tilde{m}} |\Omega|. \qquad (2.1.30)
\end{aligned}$$

Likewise, we deduce

$$\begin{aligned}
&\int_0^t \int_\Omega f_2^K(u,v) v_t dx d\tau \\
&\leq \epsilon \int_0^t \|v_t\|_{r+1}^{r+1} d\tau + C_\epsilon L_K^{\tilde{r}} \int_0^t (\|u\|_{1,\Omega}^{\tilde{r}} + \|v\|_{1,\Omega}^{\tilde{r}}) d\tau + C_\epsilon t |f_2(0,0)|^{\tilde{r}} |\Omega|, \qquad (2.1.31)
\end{aligned}$$

and

$$\int_0^t \int_\Gamma h^K(u) \gamma u_t d\Gamma d\tau \leq \epsilon \int_0^t |\gamma u_t|_{q+1}^{q+1} d\tau + C_\epsilon L_K^{\tilde{q}} \int_0^t \|u\|_{1,\Omega}^{\tilde{q}} d\tau + C_\epsilon t |h(0)|^{\tilde{q}} |\Gamma|. \qquad (2.1.32)$$

If we set $\alpha := \min\{a_1, a_2, a_3\}$, then by the assumptions on damping, it follows that

$$g_1(s)s \geq \alpha(|s|^{m+1} - 1), \;\; g_2(s)s \geq \alpha(|s|^{r+1} - 1), \;\; g(s)s \geq \alpha(|s|^{q+1} - 1) \qquad (2.1.33)$$

for all $s \in \mathbb{R}$. Therefore,

$$\begin{cases}
\int_0^t \int_\Omega g_1(u_t) u_t dx d\tau \geq \alpha \int_0^t \|u_t\|_{m+1}^{m+1} d\tau - \alpha t |\Omega|, \\
\int_0^t \int_\Omega g_2(v_t) v_t dx d\tau \geq \alpha \int_0^t \|v_t\|_{r+1}^{r+1} d\tau - \alpha t |\Omega|, \\
\int_0^t \int_\Gamma g(\gamma u_t) \gamma u_t d\Gamma d\tau \geq \alpha \int_0^t |\gamma u_t|_{q+1}^{q+1} d\tau - \alpha t |\Gamma|.
\end{cases} \qquad (2.1.34)$$

By combining (2.1.30)-(2.1.34) in the energy identity (2.1.29), one has

$$\begin{aligned}
&\mathscr{E}(t) + \alpha \int_0^t (\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1}) d\tau - \alpha t(2|\Omega| + |\Gamma|) \\
&\quad \leq \mathscr{E}(0) + \epsilon \int_0^t (\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1}) d\tau \\
&\quad + C_\epsilon L_K^{\tilde{m}} \int_0^t (\|u\|_{1,\Omega}^{\tilde{m}} + \|v\|_{1,\Omega}^{\tilde{m}}) d\tau + C_\epsilon L_K^{\tilde{r}} \int_0^t (\|u\|_{1,\Omega}^{\tilde{r}} + \|v\|_{1,\Omega}^{\tilde{r}}) d\tau \\
&\quad + C_\epsilon L_K^{\tilde{q}} \int_0^t \|u\|_{1,\Omega}^{\tilde{q}} \, d\tau + C_\epsilon t(|f_1(0,0)|^{\tilde{m}}|\Omega| + |f_2(0,0)|^{\tilde{r}}|\Omega| + |h(0)|^{\tilde{q}}|\Gamma|). \quad (2.1.35)
\end{aligned}$$

If $\epsilon \leq \alpha$, then (2.1.35) implies

$$\begin{aligned}
\mathscr{E}(t) \leq &\mathscr{E}(0) + C_\epsilon L_K^{\tilde{m}} \int_0^t (\|u\|_{1,\Omega}^{\tilde{m}} + \|v\|_{1,\Omega}^{\tilde{m}}) d\tau \\
&+ C_\epsilon L_K^{\tilde{r}} \int_0^t (\|u\|_{1,\Omega}^{\tilde{r}} + \|v\|_{1,\Omega}^{\tilde{r}}) d\tau + C_\epsilon L_K^{\tilde{q}} \int_0^t \|u\|_{1,\Omega}^{\tilde{q}} \, d\tau \\
&+ C_\epsilon t(|f_1(0,0)|^{\tilde{m}}|\Omega| + |f_2(0,0)|^{\tilde{r}}|\Omega| + |h(0)|^{\tilde{q}}|\Gamma|) + \alpha t(2|\Omega| + |\Gamma|). \quad (2.1.36)
\end{aligned}$$

Since $\tilde{m}, \tilde{r}, \tilde{q} \leq 2$, then by Young's inequality,

$$\int_0^t (\|u\|_{1,\Omega}^{\tilde{m}} + \|v\|_{1,\Omega}^{\tilde{m}}) d\tau \leq \int_0^t (\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2 + \tilde{C}) d\tau \leq 2 \int_0^t \mathscr{E}(\tau) d\tau + \tilde{C}t,$$

$$\int_0^t (\|u\|_{1,\Omega}^{\tilde{r}} + \|v\|_{1,\Omega}^{\tilde{r}}) d\tau \leq 2 \int_0^t \mathscr{E}(\tau) d\tau + \tilde{C}t,$$

$$\int_0^t \|u\|_{1,\Omega}^{\tilde{q}} \, d\tau \leq 2 \int_0^t \mathscr{E}(\tau) d\tau + \tilde{C}t,$$

where $\tilde{C}$ is positive constant that depends on m, r and q. Therefore, if we set $C(L_K) = 2C_\epsilon(L_K^{\tilde{m}} + L_K^{\tilde{r}} + L_K^{\tilde{q}})$ and $C_0 = C_\epsilon(|f_1(0,0)|^{\tilde{m}}|\Omega| + |f_2(0,0)|^{\tilde{r}}|\Omega| + |h(0)|^{\tilde{q}}|\Gamma|) + \alpha(2|\Omega| + |\Gamma|) + 3\tilde{C}$, then it follows from (2.1.36) that

$$\mathscr{E}(t) \leq (\mathscr{E}(0) + C_0 T_0) + C(L_K) \int_0^t \mathscr{E}(\tau) d\tau, \text{ for all } t \in [0, T_0],$$

where T_0 will be chosen below. By Gronwall's inequality, one has

$$\mathscr{E}(t) \leq (\mathscr{E}(0) + C_0 T_0) e^{C(L_K)t} \text{ for all } t \in [0, T_0]. \quad (2.1.37)$$

We select

$$T_0 = \min\left\{\frac{1}{4C_0}, \frac{1}{C(L_K)}\log 2\right\}, \tag{2.1.38}$$

and recall our assumption that $K^2 \geq 4\mathscr{E}(0) + 1$. Then, it follows from (2.1.37) that

$$\mathscr{E}(t) \leq 2(\mathscr{E}(0) + 1/4) \leq K^2/2, \tag{2.1.39}$$

for all $t \in [0, T_0]$. This implies that $\|(u(t), v(t))\|_V \leq K$, for all $t \in [0, T_0]$, and therefore, $f_1^K(u, v) = f_1(u, v)$, $f_2^K(u, v) = f_2(u, v)$ and $h^K(u) = h(\gamma u)$ on the time interval $[0, T_0]$. Because of the uniqueness of solutions for the (K) problem, the solution to the truncated problem (K) coincides with the solution to the system (2.1.13) for $t \in [0, T_0]$, completing the proof of Lemma 2.1.2. $\square$

Remark 2.1.3. In Lemma 2.1.2, the local existence time T_0 depends on L_K, which is the local Lipschitz constant of: $f_1 : H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^{\frac{m+1}{m}}(\Omega)$, $f_2 : H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^{\frac{r+1}{r}}(\Omega)$ and $h(\gamma u) : H^1(\Omega) \longrightarrow L^{\frac{q+1}{q}}(\Gamma)$. The advantage of this result is that T_0 does not depends on the locally Lipschitz constants for the mapping $f_1, f_2 : H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^2(\Omega)$ and $h(\gamma u) : H^1(\Omega) \longrightarrow L^2(\Gamma)$. This fact is critical for the remaining parts of the proof of the local existence statement in Theorem 1.3.2.

2.1.4 Lipschitz approximations of the sources

This subsection is devoted for constructing Lipschitz approximations of the sources. The following propositions are needed.

Proposition 2.1.4. *Assume $1 \leq p < 6$, $m, r \geq 1$, $p\frac{m+1}{m} \leq \frac{6}{1+2\epsilon}$, and $p\frac{r+1}{r} \leq \frac{6}{1+2\epsilon}$, for some $\epsilon > 0$. Further assume that f_1, $f_2 \in C^1(\mathbb{R}^2)$ such that*

$$|\nabla f_j(u, v)| \leq C(|u|^{p-1} + |v|^{p-1} + 1), \tag{2.1.40}$$

for $j = 1, 2$ and all $u, v \in \mathbb{R}$. Then, $f_j : H^{1-\epsilon}(\Omega) \times H_0^{1-\epsilon}(\Omega) \longrightarrow L^\sigma(\Omega)$ is locally Lipschitz continuous, $j = 1, 2$, where $\sigma = \frac{m+1}{m}$ or $\sigma = \frac{r+1}{r}$.

Remark 2.1.5. Since $H^1(\Omega) \hookrightarrow H^{1-\epsilon}(\Omega)$, then it follows from Proposition 2.1.4 that each f_j is locally Lipschitz from $H^1(\Omega) \times H_0^1(\Omega)$ into $L^{\frac{m+1}{m}}(\Omega)$ or $L^{\frac{r+1}{r}}(\Omega)$. In particular, if $1 \leq p \leq 3$, then it is easy to verify that each f_j is locally Lipschitz from $H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^2(\Omega)$.

Proof. It is enough to prove that $f_1 : H^{1-\epsilon}(\Omega) \times H_0^{1-\epsilon}(\Omega) \longrightarrow L^{\tilde{m}}(\Omega)$ is locally Lipschitz continuous, where $\tilde{m} = \frac{m+1}{m}$. Let $(u,v), (\hat{u},\hat{v}) \in \tilde{V} := H^{1-\epsilon}(\Omega) \times H_0^{1-\epsilon}(\Omega)$ such that $\|(u,v)\|_{\tilde{V}}, \|(\hat{u},\hat{v})\|_{\tilde{V}} \leq R$, where $R > 0$. By (2.1.40) and the mean value theorem, we have

$$\begin{aligned} &|f_1(u,v) - f_1(\hat{u},\hat{v})| \\ &\leq C\Big(|u-\hat{u}| + |v-\hat{v}|\Big)\Big(|u|^{p-1} + |\hat{u}|^{p-1} + |v|^{p-1} + |\hat{v}|^{p-1} + 1\Big). \end{aligned} \tag{2.1.41}$$

Therefore,

$$\begin{aligned} \|f_1(u,v) - f_1(\hat{u},\hat{v})\|_{\tilde{m}}^{\tilde{m}} &= \int_\Omega |f_1(u,v) - f_1(\hat{u},\hat{v})|^{\tilde{m}} dx \\ \leq C \int_\Omega &\left(|u-\hat{u}|^{\tilde{m}} + |v-\hat{v}|^{\tilde{m}}\right) \\ &\left(|u|^{(p-1)\tilde{m}} + |v|^{(p-1)\tilde{m}} + |\hat{u}|^{(p-1)\tilde{m}} + |\hat{v}|^{(p-1)\tilde{m}} + 1\right) dx. \end{aligned} \tag{2.1.42}$$

All terms in (2.1.42) are estimated in the same manner. In particular, for a typical term in (2.1.42), we estimate it by Hölder's inequality and the Sobolev imbedding $H^{1-\epsilon}(\Omega) \hookrightarrow L^{\frac{6}{1+2\epsilon}}(\Omega)$ together with the assumption $p\tilde{m} \leq \frac{6}{1+2\epsilon}$ and $\|u\|_{H^{1-\epsilon}(\Omega)} \leq R$. For instance,

$$\begin{aligned} \int_\Omega |u-\hat{u}|^{\tilde{m}} |u|^{(p-1)\tilde{m}} dx &\leq \left(\int_\Omega |u-\hat{u}|^{p\tilde{m}} dx\right)^{\frac{1}{p}} \left(\int_\Omega |u|^{p\tilde{m}} dx\right)^{\frac{p-1}{p}} \\ &\leq C \|u-\hat{u}\|_{H^{1-\epsilon}(\Omega)}^{\tilde{m}} \|u\|_{H^{1-\epsilon}(\Omega)}^{(p-1)\tilde{m}} \leq CR^{(p-1)\tilde{m}} \|u-\hat{u}\|_{H^{1-\epsilon}(\Omega)}^{\tilde{m}}. \end{aligned}$$

Hence, we obtain

$$\|f_1(u,v) - f_1(\hat{u},\hat{v})\|_{\tilde{m}} \leq C(R) \|(u-\hat{u}, v-\hat{v})\|_{H^{1-\epsilon}(\Omega) \times H_0^{1-\epsilon}(\Omega)},$$

completing the proof. □

Recall that for the values $3 < p < 6$, the source $f_1(u,v)$ and $f_2(u,v)$ are not locally Lipschitz continuous from $H^1(\Omega) \times H_0^1(\Omega)$ into $L^2(\Omega)$. So, in order to apply Lemma 2.1.2 to prove Theorem 1.3.2, we shall construct Lipschitz approximations of the sources f_1 and f_2. In particular, we shall use smooth cutoff functions $\eta_n \in C_0^\infty(\mathbb{R}^2)$, similar to those used in [37], such that each η_n satisfies: $0 \leq \eta_n \leq 1$; $\eta_n(u,v) = 1$ if $|(u,v)| \leq n$; $\eta_n(u,v) = 0$ if $|(u,v)| \geq 2n$; and $|\nabla \eta_n(u,v)| \leq C/n$. Put

$$f_j^n(u,v) = f_j(u,v)\eta_n(u,v), \quad u,v \in \mathbb{R}, \quad j = 1,2, \quad n \in \mathbb{N}, \tag{2.1.43}$$

where f_1 and f_2 satisfy Assumption 1.1.1. The following proposition summarizes important properties of f_1^n and f_2^n.

Proposition 2.1.6. *For each $j = 1, 2,\ \ n \in \mathbb{N}$, then function f_j^n, defined in (2.1.43), satisfies:*

- *$f_j^n(u, v) : H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^2(\Omega)$ is globally Lipschitz continuous with Lipschitz constant depending on n.*
- *There exists $\epsilon > 0$ such that $f_j^n : H^{1-\epsilon}(\Omega) \times H_0^{1-\epsilon}(\Omega) \longrightarrow L^\sigma(\Omega)$ is locally Lipschitz continuous where the local Lipschitz constant is independent of n, and where $\sigma = \frac{m+1}{m}$ or $\sigma = \frac{r+1}{r}$.*

Proof. It is enough to prove the proposition for the function f_1^n. Let $(u, v), (\hat{u}, \hat{v}) \in H^1(\Omega) \times H_0^1(\Omega)$ and put

$$\begin{aligned}
\Omega_1 &= \{x \in \Omega : |(u(x), v(x))| < 2n, |(\hat{u}(x), \hat{v}(x))| < 2n\},\\
\Omega_2 &= \{x \in \Omega : |(u(x), v(x))| < 2n, |(\hat{u}(x), \hat{v}(x))| \geq 2n\},\\
\Omega_3 &= \{x \in \Omega : |(u(x), v(x))| \geq 2n, |(\hat{u}(x), \hat{v}(x))| < 2n\}. \qquad (2.1.44)
\end{aligned}$$

By the definition of η, it is clear that $f_1^n(u, v) = f_1^n(\hat{u}, \hat{v}) = 0$ if $|(u, v)| \geq 2n$ and $|(\hat{u}, \hat{v})| \geq 2n$. Therefore, by (2.1.43) we have

$$\|f_1^n(u, v) - f_1^n(\hat{u}, \hat{v})\|_2^2 = I_1 + I_2 + I_3, \qquad (2.1.45)$$

where $I_j = \int_{\Omega_j} |f_1(u, v)\eta_n(u, v) - f_1(\hat{u}, \hat{v})\eta_n(\hat{u}, \hat{v})|^2 dx$, $j = 1, 2, 3$.

Notice

$$\begin{aligned}
I_1 \leq\ & 2\int_{\Omega_1} |f_1(u, v)|^2 |\eta_n(u, v) - \eta_n(\hat{u}, \hat{v})|^2 dx\\
&+ 2\int_{\Omega_1} |\eta_n(\hat{u}, \hat{v})|^2 |f_1(u, v) - f_1(\hat{u}, \hat{v})|^2 dx. \qquad (2.1.46)
\end{aligned}$$

Since $|\nabla f_1(u, v)| \leq C(|u|^{p-1} + |v|^{p-1} + 1)$, we have

$$|f_1(u, v)| \leq C(|u|^p + |v|^p + 1) \qquad (2.1.47)$$

and along with the fact $|u|, |v| \leq 2n$ in Ω_1 and $|\nabla\eta_n| \leq C/n$, we obtain

$$\begin{aligned}
&\int_{\Omega_1} |f_1(u, v)|^2 |\eta_n(u, v) - \eta_n(\hat{u}, \hat{v})|^2 dx\\
&\leq C\int_{\Omega_1} (|u|^p + |v|^p + 1)^2 |\nabla\eta_n(\xi_1, \xi_2)|^2 |(u - \hat{u}, v - \hat{v})|^2 dx\\
&\leq Cn^{2p-2}\int_{\Omega_1} (|u - \hat{u}|^2 + |v - \hat{v}|^2) dx. \qquad (2.1.48)
\end{aligned}$$

Moreover, since $|\eta_n| \leq 1$ and $|u|, |\hat{u}|, |v|, |\hat{v}| \leq 2n$ in Ω_1, then by (2.1.41) we deduce

$$\begin{aligned}
&\int_{\Omega_1} |\eta_n(\hat{u},\hat{v})|^2 |f_1(u,v) - f_1(\hat{u},\hat{v})|^2 dx \\
&\leq C \int_{\Omega_1} (|u-\hat{u}|^2 + |v-\hat{v}|^2) \left(|u|^{p-1} + |v|^{p-1} + |\hat{u}|^{p-1} + |\hat{v}|^{p-1} + 1\right)^2 dx \\
&\leq C n^{2p-2} \int_{\Omega_1} (|u-\hat{u}|^2 + |v-\hat{v}|^2) dx. \qquad (2.1.49)
\end{aligned}$$

Therefore, it follows from (2.1.46), (2.1.48) and (2.1.49) that

$$I_1 \leq C(n) \int_{\Omega_1} (|u-\hat{u}|^2 + |v-\hat{v}|^2) dx,$$

where $C(n) = Cn^{2p-2}$. To estimate I_2, we note $\eta_n(\hat{u},\hat{v}) = 0$ in Ω_2. Then similar argument as in (2.1.48) yields

$$I_2 = \int_{\Omega_2} |f_1(u,v)|^2 |\eta_n(u,v) - \eta_n(\hat{u},\hat{v})|^2 dx \leq C(n) \int_{\Omega_2} (|u-\hat{u}|^2 + |v-\hat{v}|^2) dx,$$

where $C(n)$ is as in (2.1.49). By reversing the roles of (u,v) and $(\hat{u},\hat{v})$, one also obtains $I_3 \leq C(n) \int_{\Omega_3} (|u-\hat{u}|^2 + |v-\hat{v}|^2) dx$. Thus it follows that

$$\begin{aligned}
\|f_1^n(u,v) - f_1^n(\hat{u},\hat{v})\|_2^2 &\leq C(n)(\|u-\hat{u}\|_2^2 + \|v-\hat{v}\|_2^2) \\
&\leq C(n) \|(u-\hat{u}, v-\hat{v})\|^2_{H^1(\Omega)\times H^1_0(\Omega)},
\end{aligned}$$

where $C(n) = Cn^{2p-2}$, which completes the proof of the first statement of the proposition.

To prove the second statement we recall Assumption 1.1.1, in particular, $p\frac{m+1}{m} < 6$. Then, there exists $\epsilon > 0$ such that $p\frac{m+1}{m} \leq \frac{6}{1+2\epsilon}$. Let $(u,v), (\hat{u},\hat{v}) \in \tilde{V} := H^{1-\epsilon}(\Omega) \times H_0^{1-\epsilon}(\Omega)$ such that $\|(u,v)\|_{\tilde{V}}, \|(\hat{u},\hat{v})\|_{\tilde{V}} \leq R$, where $R > 0$, and recall the notation $\tilde{m} = \frac{m+1}{m}$. Then,

$$\|f_1^n(u,v) - f_1^n(\hat{u},\hat{v})\|_{\tilde{m}}^{\tilde{m}} = P_1 + P_2 + P_3 \qquad (2.1.50)$$

where

$$P_j = \int_{\Omega_j} |f_1(u,v)\eta_n(u,v) - f_1(\hat{u},\hat{v})\eta_n(\hat{u},\hat{v})|^{\tilde{m}} dx, \quad j = 1,2,3,$$

and each Ω_j is as defined in (2.1.44). Since $|\eta_n| \leq 1$, one has

$$\begin{aligned} P_1 &\leq C\int_{\Omega_1} |f_1(u,v)|^{\tilde{m}} |\eta_n(u,v) - \eta_n(\hat{u},\hat{v})|^{\tilde{m}} dx \\ &\quad + C\int_{\Omega_1} |\eta_n(\hat{u},\hat{v})|^{\tilde{m}} |f_1(u,v) - f_1(\hat{u},\hat{v})|^{\tilde{m}} dx \\ &\leq C\int_{\Omega_1} |f_1(u,v)|^{\tilde{m}} |\eta_n(u,v) - \eta_n(\hat{u},\hat{v})|^{\tilde{m}} dx + C\left\| f_1(u,v) - f_1(\hat{u},\hat{v})\right\|_{\tilde{m}}^{\tilde{m}}. \end{aligned} \tag{2.1.51}$$

By (2.1.47) and the mean value theorem, we obtain

$$\begin{aligned} &\int_{\Omega_1} |f_1(u,v)|^{\tilde{m}} |\eta_n(u,v) - \eta_n(\hat{u},\hat{v})|^{\tilde{m}} dx \\ &\leq C\int_{\Omega_1} (|u|^p + |v|^p + 1)^{\tilde{m}} |\nabla\eta_n(\xi_1,\xi_2)|^{\tilde{m}} |(u-\hat{u}, v-\hat{v})|^{\tilde{m}} dx \\ &\leq C\int_{\Omega_1} (|u|^{(p-1)\tilde{m}} + |v|^{(p-1)\tilde{m}} + 1)(|u-\hat{u}|^{\tilde{m}} + |v-\hat{v}|^{\tilde{m}}) dx, \end{aligned} \tag{2.1.52}$$

where we have used the facts $|u|, |v| \leq 2n$ in Ω_1 and $|\nabla\eta_n| \leq C/n$.

All terms in (2.1.52) are estimated in the same manner. By using Hölder's inequality, the Sobolev imbedding $H^{1-\epsilon}(\Omega) \hookrightarrow L^{\frac{6}{1+2\epsilon}}(\Omega)$ together with the assumption $p\tilde{m} \leq \frac{6}{1+2\epsilon}$ and $\|u\|_{H^{1-\epsilon}(\Omega)} \leq R$, we obtain

$$\begin{aligned} &\int_{\Omega_1} |u|^{(p-1)\tilde{m}} |u-\hat{u}|^{\tilde{m}} dx \leq \left(\int_{\Omega_1} |u|^{p\tilde{m}} dx\right)^{\frac{p-1}{p}} \left(\int_{\Omega_1} |u-\hat{u}|^{p\tilde{m}}\right)^{\frac{1}{p}} \\ &\leq C\left\|u\right\|_{H^{1-\epsilon}(\Omega)}^{(p-1)\tilde{m}} \left\|u-\hat{u}\right\|_{H^{1-\epsilon}(\Omega)}^{\tilde{m}} \leq CR^{(p-1)\tilde{m}} \left\|u-\hat{u}\right\|_{H^{1-\epsilon}(\Omega)}^{\tilde{m}}. \end{aligned} \tag{2.1.53}$$

Therefore, it is easy to see that

$$\int_{\Omega_1} |f_1(u,v)|^{\tilde{m}} |\eta_n(u,v) - \eta_n(\hat{u},\hat{v})|^{\tilde{m}} dx \leq C(R) \left\|(u-\hat{u}, v-\hat{v})\right\|_{\tilde{V}}^{\tilde{m}}. \tag{2.1.54}$$

By Proposition 2.1.4, we know $f_1 : \tilde{V} = H^{1-\epsilon}(\Omega) \times H_0^{1-\epsilon}(\Omega) \longrightarrow L^{\tilde{m}}(\Omega)$ is locally Lipschitz. Therefore, it follows from (2.1.51) and (2.1.54) that

$$P_1 \leq C(R) \left\|(u-\hat{u}, v-\hat{v})\right\|_{\tilde{V}}^{\tilde{m}}.$$

To estimate P_2, we use $\eta_n(\hat{u},\hat{v})=0$ in Ω_2 and adopt the same computation in (2.1.52)-(2.1.54). Thus, we deduce

$$P_2=\int_{\Omega_2}|f_1(u,v)|^{\tilde{m}}|\eta_n(u,v)-\eta_n(\hat{u},\hat{v})|^{\tilde{m}}dx\leq C(R)\left\|(u-\hat{u},v-\hat{v})\right\|_{\tilde{V}}^{\tilde{m}}.$$

Likewise, $P_3\leq C(R)\left\|(u-\hat{u},v-\hat{v})\right\|_{\tilde{V}}^{\tilde{m}}$. Therefore, by (2.1.50) we have

$$\|f_1^n(u,v)-f_1^n(\hat{u},\hat{v})\|_{\tilde{m}}^{\tilde{m}}\leq C(R)\left\|(u-\hat{u},v-\hat{v})\right\|_{\tilde{V}}^{\tilde{m}},$$

where the local Lipschitz constant $C(R)$ is independent of n. This completes the proof of the proposition. □

The following proposition deals with the boundary source h.

Proposition 2.1.7. *Assume $1\leq k<4$, $q\geq 1$ and $k\frac{q+1}{q}\leq\frac{4}{1+2\epsilon}$, for some $\epsilon>0$. If $h\in C^1(\mathbb{R})$ such that $|h'(s)|\leq C(|s|^{k-1}+1)$, then $h\circ\gamma$ is Locally Lipschitz: $H^{1-\epsilon}(\Omega)\longrightarrow L^{\frac{q+1}{q}}(\Gamma)$.*

Proof. The proof is very similar to the proof of Proposition 2.1.4 and it is omitted. □

Remark 2.1.8. Since $H^1(\Omega)\hookrightarrow H^{1-\epsilon}(\Omega)$, then by Proposition 2.1.7, we know $h\circ\gamma$ is locally Lipschitz from $H^1(\Omega)$ into $L^{\frac{q+1}{q}}(\Gamma)$. In particular, if $1\leq k\leq 2$, we can directly verify $h\circ\gamma$ is locally Lipschitz from $H^1(\Omega)$ into $L^2(\Gamma)$.

We note here that if $2<k<4$, then $h\circ\gamma$ is not locally Lipschitz continuous from $H^1(\Omega)$ into $L^2(\Gamma)$. As we have done for the interior sources, we shall construct Lipschitz approximations for the boundary source h. Let $\zeta_n\in C_0^\infty(\mathbb{R})$ be a cutoff function such that $0\leq\zeta_n\leq 1$; $\zeta_n(s)=1$ if $|s|\leq n$; $\zeta_n(s)=0$ if $|s|\geq 2n$; and $|\zeta_n'(s)|\leq C/n$. Put

$$h^n(s)=h(s)\zeta_n(s),\quad s\in\mathbb{R},\quad n\in\mathbb{N},\tag{2.1.55}$$

where h satisfies Assumption 1.1.1. The following proposition summarizes some important properties of h^n.

Proposition 2.1.9. *For each $n\in\mathbb{N}$, the function h^n defined in (2.1.55) has the following properties:*

- *$h^n\circ\gamma:H^1(\Omega)\longrightarrow L^2(\Gamma)$ is globally Lipschitz continuous with Lipschitz constant depending on n.*
- *There exists $\epsilon>0$ such that $h^n\circ\gamma:H^{1-\epsilon}(\Omega)\longrightarrow L^{\frac{q+1}{q}}(\Gamma)$ is locally Lipschitz continuous where the local Lipschitz constant does not depend on n.*

Proof. The proof is similar to the proof of Proposition 2.1.6 and it is omitted. □

2.1.5 Approximate solutions and passage to the limit

We complete the proof of the local existence statement in Theorem 1.3.2 in the following four steps.

Step 1: Approximate system. Recall that in Lemma 2.1.2, the boundary damping g is assumed *strongly* monotone. However, in Assumption 1.1.1, we only impose the monotonicity condition on g. To remedy this, we approximate the boundary damping with:

$$g^n(s) = g(s) + \frac{1}{n}s, \quad n \in \mathbb{N}. \tag{2.1.56}$$

Note that, g^n is strongly monotone with the constant $m_g = \frac{1}{n} > 0$, since g is monotone increasing. Indeed, for all $s_1, s_2 \in \mathbb{R}$,

$$(g^n(s_1) - g^n(s_2))(s_1 - s_2) = (g(s_1) - g(s_2))(s_1 - s_2) + \frac{1}{n}|s_1 - s_2|^2 \geq \frac{1}{n}|s_1 - s_2|^2.$$

Corresponding to g^n, we define the operator $\mathcal{S}^n$ as follows: replace g with g^n in (2.1.7) to define the functional J^n like J in (2.1.6), and then similar to (2.1.9), we define the operator $\mathcal{S}^n : \mathcal{D}(\mathcal{S}^n) = \mathcal{D}(\partial J^n) \subset H^1(\Omega) \longrightarrow (H^1(\Omega))'$ such that $\partial J^n(u) = \{\mathcal{S}^n(u)\}$. As in (2.1.10) and (2.1.11), we have for all $u \in \mathcal{D}(\mathcal{S}^n)$,

$$\langle \mathcal{S}^n(u), u \rangle = \int_\Omega g_1(u)u dx + \int_\Gamma g^n(\gamma u)\gamma u d\Gamma \tag{2.1.57}$$

and

$$\langle \mathcal{S}^n(u), v \rangle = \int_\Omega g_1(u)v dx + \int_\Gamma g^n(\gamma u)\gamma v d\Gamma \text{ for all } v \in C(\overline{\Omega}). \tag{2.1.58}$$

Recall $H = H^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$, and the approximate sources f_1^n, f_2^n, h^n which were introduced in (2.1.43) and (2.1.55). Now, we define the nonlinear operator $\mathscr{A}^n : \mathcal{D}(\mathscr{A}^n) \subset H \longrightarrow H$ by:

$$\mathscr{A}^n \begin{bmatrix} u \\ v \\ y \\ z \end{bmatrix}^{tr} = \begin{bmatrix} -y \\ -z \\ \Delta_R(u - Rh^n(\gamma u)) + \mathcal{S}^n(y) - f_1^n(u,v) \\ -\Delta v + g_2(z) - f_2^n(u,v) \end{bmatrix}^{tr}, \tag{2.1.59}$$

where $\mathcal{D}(\mathscr{A}^n) = \Big\{(u,v,y,z) \in \Big(H^1(\Omega) \times H^1_0(\Omega)\Big)^2 : \Delta_R(u - Rh^n(\gamma u)) + \mathcal{S}^n(y) - f_1^n(u,v) \in L^2(\Omega),\ y \in \mathcal{D}(\mathcal{S}^n),\ -\Delta v + g_2(z) - f_2^n(u,v) \in L^2(\Omega),\ g_2(z) \in H^{-1}(\Omega) \cap L^1(\Omega)\Big\}$.

Clearly, the space of test functions $\mathscr{D}(\Omega)^4 \subset \mathcal{D}(\mathscr{A}^n)$, and since $\mathscr{D}(\Omega)^4$ is dense in H, for each $U_0 = (u_0, v_0, u_1, v_1) \in H$ there exists a sequence of functions $U_0^n = (u_0^n, v_0^n, u_1^n, v_1^n) \in \mathscr{D}(\Omega)^4$ such that $U_0^n \longrightarrow U_0$ in H.

Put $U = (u, v, u_t, v_t)$ and consider the approximate system:

$$U_t + \mathscr{A}^n U = 0 \text{ with } U(0) = (u_0^n, v_0^n, u_1^n, v_1^n) \in \mathscr{D}(\Omega)^4. \tag{2.1.60}$$

Step 2: Approximate solutions. Since g^n, f_1^n, f_2^n and h^n satisfy the assumptions of Lemma 2.1.2, then for each n, the approximate problem (2.1.60) has a strong local solution $U^n = (u^n, v^n, u_t^n, v_t^n) \in W^{1,\infty}(0, T_0; H)$ such that $U^n(t) \in \mathcal{D}(\mathscr{A}^n)$ for $t \in [0, T_0]$. It is important to note here that T_0 is totally independent of n. In fact, by (2.1.38), T_0 does not depend on the strong monotonicity constant $m_g = \frac{1}{n}$, and although T_0 depends on the local Lipschitz constants of the mappings $f_1^n : H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^{\tilde{m}}(\Omega)$, $f_2^n : H^1(\Omega) \times H_0^1(\Omega) \longrightarrow L^{\tilde{r}}(\Omega)$ and $h^n \circ \gamma : H^1(\Omega) \longrightarrow L^{\tilde{q}}(\Gamma)$, it is fortunate that these Lipschitz constants are independent of n, thanks to Propositions 2.1.6 and 2.1.9. Also, recall that T_0 depends on K which itself depends on the initial data, and since $U_0^n \to U_0$ in H, we can choose K sufficiently large such that K is uniform for all n. Thus, we will only emphasize the dependence of T_0 on K.

Now, by (2.1.39), we know $\mathscr{E}^n(t) \leq K^2/2$ for all $t \in [0, T_0]$, which implies that,

$$\|U^n(t)\|_H^2 = \|u^n(t)\|_{1,\Omega}^2 + \|v^n(t)\|_{1,\Omega}^2 + \|u_t^n(t)\|_2^2 + \|v_t^n(t)\|_2^2 \leq K^2, \tag{2.1.61}$$

for all $t \in [0, T_0]$. In addition, by letting $0 < \epsilon \leq \alpha/2$ in (2.1.35) and by the fact $\tilde{m}, \tilde{q}, \tilde{r} \leq 2$ and the bound (2.1.61), we deduce that,

$$\int_0^{T_0} \|u_t^n\|_{m+1}^{m+1} dt + \int_0^{T_0} \|v_t^n\|_{r+1}^{r+1} dt + \int_0^{T_0} |\gamma u_t^n|_{q+1}^{q+1} dt < C(K), \tag{2.1.62}$$

for some constant $C(K) > 0$. Since $|g_1(s)| \leq b_1|s|^m$ for $|s| \geq 1$ and g_1 is increasing with $g_1(0) = 0$, then $|g_1(s)| \leq b_1(|s|^m + 1)$ for all $s \in \mathbb{R}$. Hence, it follows from (2.1.62) that

$$\int_0^{T_0} \int_\Omega |g_1(u_t^n)|^{\tilde{m}} dx dt \leq b_1^{\tilde{m}} \int_0^{T_0} \int_\Omega (|u_t^n|^{m+1} + 1) dx dt < C(K). \tag{2.1.63}$$

Similarly, one has

$$\int_0^{T_0}\int_\Omega |g_2(v_t^n)|^{\tilde r}dxdt < C(K) \text{ and } \int_0^{T_0}\int_\Omega |g^n(\gamma u_t^n)|^{\tilde q}dxdt < C(K). \tag{2.1.64}$$

Next, we shall prove the following statement: If $w \in H^1(\Omega) \cap L^{m+1}(\Omega)$ with $\gamma w \in L^{q+1}(\Gamma)$, then

$$\langle \mathcal{S}^n(u_t^n), w\rangle = \int_\Omega g_1(u_t^n)wdx + \int_\Gamma g^n(\gamma u_t^n)\gamma w d\Gamma, \text{ a.e. } [0,T_0]. \tag{2.1.65}$$

Indeed, by Lemma 5.1.1 in Chapter 5, there exists a sequence $\{w_k\} \subset H^2(\Omega)$ such that $w_k \longrightarrow w$ in $H^1(\Omega)$, $|w_k|^{m+1} \longrightarrow |w|^{m+1}$ in $L^1(\Omega)$ and $|\gamma w_k|^{q+1} \longrightarrow |\gamma w|^{q+1}$ in $L^1(\Gamma)$. By the Generalized Dominated Convergence Theorem, we conclude, on a subsequence labeled the same as $\{w_k\}$,

$$w_k \longrightarrow w \text{ in } L^{m+1}(\Omega) \text{ and } \gamma w_k \longrightarrow \gamma w \text{ in } L^{q+1}(\Gamma). \tag{2.1.66}$$

Since $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$ (in 3D), and the fact that $u_t^n \in \mathcal{D}(\mathcal{S}^n)$, then it follows from (2.1.58) that,

$$\langle \mathcal{S}^n(u_t^n), w_k\rangle = \int_\Omega g_1(u_t^n)w_k dx + \int_\Gamma g^n(\gamma u_t^n)\gamma w_k d\Gamma. \tag{2.1.67}$$

From (2.1.63) and (2.1.64) we note that $\|g_1(u_t^n)\|_{\tilde m}$ and $|g^n(\gamma u_t^n)|_{\tilde q} < \infty$, a.e. $[0,T_0]$. Therefore, by using (2.1.66), we can pass to the limit in (2.1.67) as $k \longrightarrow \infty$ to obtain (2.1.65) as claimed.

Recall that $U^n = (u^n, v^n, u_t^n, v_t^n) \in \mathcal{D}(\mathscr{A}^n)$ is a strong solution of (2.1.60). If ϕ and ψ satisfy the conditions imposed on test functions in Definition 1.3.1, then by (2.1.63)-(2.1.65), we can test the approximate system (2.1.60) against ϕ and ψ to obtain

$$\begin{aligned}
&(u_t^n(t),\phi(t))_\Omega - (u_1^n,\phi(0))_\Omega - \int_0^t (u_t^n,\phi_t)_\Omega d\tau + \int_0^t (u^n,\phi)_{1,\Omega}d\tau \\
&\quad + \int_0^t\int_\Omega g_1(u_t^n)\phi dxd\tau + \int_0^t\int_\Gamma g(\gamma u_t^n)\gamma\phi d\Gamma d\tau + \frac{1}{n}\int_0^t\int_\Gamma \gamma u_t^n\gamma\phi d\Gamma d\tau \\
&= \int_0^t\int_\Omega f_1^n(u^n,v^n)\phi dxd\tau + \int_0^t\int_\Gamma h^n(\gamma u^n)\gamma\phi d\Gamma d\tau,
\end{aligned} \tag{2.1.68}$$

and

$$(v_t^n(t),\psi(t))_\Omega-(v_1^n,\psi(0))_\Omega-\int_0^t(v_t^n,\psi_t)_\Omega d\tau+\int_0^t(v^n,\psi)_{1,\Omega}d\tau$$
$$+\int_0^t\int_\Omega g_2(v_t^n)\psi dxd\tau=\int_0^t\int_\Omega f_2^n(u^n,v^n)\psi dxd\tau \tag{2.1.69}$$

for all $t\in[0,T_0]$.

Step 3: Passage to the limit. We aim to prove that there exists a subsequence of $\{U^n\}$, labeled again as $\{U^n\}$, that converges to a solution of the original problem (1.1.1). In what follows, we focus on passing to the limit in (2.1.68) only, since passing to the limit in (2.1.69) is similar and in fact it is simpler.

First, we note that (2.1.61) shows $\{U^n\}$ is bounded in $L^\infty(0,T_0;H)$. So, by Alaoglu's Theorem, there exists a subsequence, labeled by $\{U^n\}$, such that

$$U^n\longrightarrow U \text{ weakly}^* \text{ in } L^\infty(0,T_0;H). \tag{2.1.70}$$

Also, by (2.1.61), we know $\{u^n\}$ is bounded in $L^\infty(0,T_0;H^1(\Omega))$, and so, $\{u^n\}$ is bounded in $L^s(0,T_0;H^1(\Omega))$ and for any $s>1$. In addition, by (2.1.62), we know $\{u_t^n\}$ is bounded in $L^{m+1}(\Omega\times(0,T_0))$, and since $m\geq 1$, we see that $\{u_t^n\}$ is also bounded in $L^{\tilde{m}}(\Omega\times(0,T_0))=L^{\tilde{m}}(0,T_0;L^{\tilde{m}}(\Omega))$. We note here that for sufficiently small $\epsilon>0$, the imbedding $H^1(\Omega)\hookrightarrow H^{1-\epsilon}(\Omega)$ is compact, and $H^{1-\epsilon}(\Omega)\hookrightarrow L^{\tilde{m}}(\Omega)$ (since $\tilde{m}\leq 2$). If $s>1$ is fixed, then by Aubin's Compactness Theorem, there exists a subsequence such that

$$u^n\longrightarrow u \text{ strongly in } L^s(0,T_0;H^{1-\epsilon}(\Omega)), \tag{2.1.71}$$

Similarly, we deduce that there exists a subsequence such that

$$v^n\longrightarrow v \text{ strongly in } L^s(0,T_0;H^{1-\epsilon}(\Omega)). \tag{2.1.72}$$

Now, fix $t\in[0,T_0]$. Since $\phi\in C([0,t];H^1(\Omega))$ and $\phi_t\in L^1(0,t;L^2(\Omega))$, then by (2.1.70), we obtain

$$\lim_{n\longrightarrow\infty}\int_0^t(u^n,\phi)_{1,\Omega}dxd\tau=\int_0^t(u,\phi)_{1,\Omega}dxd\tau \tag{2.1.73}$$

and

$$\lim_{n\longrightarrow\infty}\int_0^t(u_t^n,\phi_t)_\Omega dxd\tau=\int_0^t(u_t,\phi_t)_\Omega dxd\tau. \tag{2.1.74}$$

In addition, since $\tilde{q} \leq 2 \leq q+1$ and $\gamma\phi \in L^{q+1}(\Gamma \times (0,t))$, then $\gamma\phi \in L^{\tilde{q}}(\Gamma \times (0,t))$, and along with (2.1.62), one has

$$\left|\frac{1}{n}\int_0^t\int_\Gamma \gamma u_t^n \gamma\phi d\Gamma d\tau\right| \leq \frac{1}{n}\left(\int_0^t |\gamma u_t^n|_{q+1}^{q+1} d\tau\right)^{\frac{1}{q+1}}\left(\int_0^t |\gamma\phi|_{\tilde{q}}^{\tilde{q}} dt\right)^{\frac{q}{q+1}} \longrightarrow 0. \quad (2.1.75)$$

Moreover, by (2.1.63)-(2.1.64), on a subsequence,

$$\begin{cases} g_1(u_t^n) \longrightarrow g_1^* \text{ weakly in } L^{\tilde{m}}(\Omega \times (0,t)), \\ g(\gamma u_t^n) \longrightarrow g^* \text{ weakly in } L^{\tilde{q}}(\Gamma \times (0,t)), \end{cases} \quad (2.1.76)$$

for some $g_1^* \in L^{\tilde{m}}(\Omega \times (0,t))$ and some $g^* \in L^{\tilde{q}}(\Gamma \times (0,t))$. Our goal is to show that $g_1^* = g_1(u_t)$ and $g^* = g(\gamma u_t)$. In order to do so, we consider two solutions to the approximate problem (2.1.60), U^n and U^j. For sake of simplifying the notation, put $\tilde{u} = u^n - u^j$. Since U^n, $U^j \in W^{1,\infty}(0,T_0;H)$ and $U^n(t)$, $U^j(t) \in \mathcal{D}(\mathscr{A}^n)$, then $\tilde{u}_t \in W^{1,\infty}(0,T_0;L^2(\Omega))$ and $\tilde{u}_t(t) \in H^1(\Omega)$. Moreover, by (2.1.62) we know $\tilde{u}_t \in L^{m+1}(\Omega \times (0,T_0))$ and $\gamma\tilde{u}_t \in L^{q+1}(\Gamma \times (0,T_0))$. Hence, we may consider the difference of the approximate problems corresponding to the parameters n and j, and then use the multiplier $\tilde{u}_t$ on the first equation. By performing integration by parts in the first equation, one has the following energy identity:

$$\begin{aligned}
&\frac{1}{2}\left(\|\tilde{u}_t(t)\|_2^2 + \|\tilde{u}(t)\|_{1,\Omega}^2\right) + \int_0^t\int_\Omega (g_1(u_t^n) - g_1(u_t^j))\tilde{u}_t dx d\tau \\
&\quad + \int_0^t\int_\Gamma (g(\gamma u_t^n) - g(\gamma u_t^j))\gamma\tilde{u}_t d\Gamma d\tau + \int_0^t\int_\Gamma \left(\frac{1}{n}\gamma u_t^n - \frac{1}{j}\gamma u_t^j\right)\gamma\tilde{u}_t d\Gamma d\tau \\
&= \frac{1}{2}\left(\|\tilde{u}_t(0)\|_2^2 + \|\tilde{u}(0)\|_{1,\Omega}^2\right) + \int_0^t\int_\Omega (f_1^n(u^n,v^n) - f_1^j(u^j,v^j))\tilde{u}_t dx d\tau \\
&\quad + \int_0^t\int_\Gamma (h^n(\gamma u^n) - h^j(\gamma u^j))\gamma\tilde{u}_t d\Gamma d\tau, \qquad (2.1.77)
\end{aligned}$$

where we have used (2.1.65). It follows from (2.1.77) that,

$$\begin{aligned}
&\frac{1}{2}\left(\|\tilde{u}_t(t)\|_2^2+\|\tilde{u}(t)\|_{1,\Omega}^2\right)+\int_0^t\int_\Omega(g_1(u_t^n)-g_1(u_t^j))\tilde{u}_t dxd\tau\\
&\quad+\int_0^t\int_\Gamma(g(\gamma u_t^n)-g(\gamma u_t^j))\gamma\tilde{u}_t d\Gamma d\tau\\
&\le\frac{1}{2}\left(\|\tilde{u}_t(0)\|_2^2+\|\tilde{u}(0)\|_{1,\Omega}^2\right)+2\left(\frac{1}{n}+\frac{1}{j}\right)\int_0^t\int_\Gamma\left(|\gamma u_t^n|^2+|\gamma u_t^j|^2\right)d\Gamma d\tau\\
&\quad+\int_0^t\int_\Omega|f_1^n(u^n,v^n)-f_1^j(u^j,v^j)||\tilde{u}_t|dxd\tau\\
&\quad+\int_0^t\int_\Gamma|h^n(\gamma u^n)-h^j(\gamma u^j)||\gamma\tilde{u}_t|d\Gamma d\tau. \qquad (2.1.78)
\end{aligned}$$

We will show that each term on the right hand side of (2.1.78) converges to 0 as $n,j\longrightarrow\infty$. First, since $\lim_{n\longrightarrow 0}\|u_0^n-u_0\|_{1,\Omega}=0$ and $\lim_{n\longrightarrow 0}\|u_1^n-u_1\|_2=0$, we obtain

$$\begin{aligned}
\lim_{n,j\longrightarrow 0}\|\tilde{u}(0)\|_{1,\Omega}&=\lim_{n,j\longrightarrow 0}\left\|u_0^n-u_0^j\right\|_{1,\Omega}=0,\\
\lim_{n,j\longrightarrow 0}\|\tilde{u}_t(0)\|_2&=\lim_{n,j\longrightarrow 0}\left\|u_1^n-u_1^j\right\|_2=0. \qquad (2.1.79)
\end{aligned}$$

By (2.1.62), we know $\int_0^t|\gamma u_t^n|_{q+1}^{q+1}d\tau<C(K)$ for all $n\in\mathbb{N}$. Since $q\ge 1$, it is easy to see $\int_0^t|\gamma u_t^n|_2^2 d\tau$ is also uniformly bounded in n. Thus,

$$\lim_{n,j\longrightarrow\infty}\left(\frac{1}{n}+\frac{1}{j}\right)\int_0^t\int_\Gamma\left(|\gamma u_t^n|^2+|\gamma u_t^j|^2\right)d\Gamma d\tau=0. \qquad (2.1.80)$$

Next we look at the third term on the right hand side of (2.1.78). We have,

$$\begin{aligned}
&\int_0^t\int_\Omega|f_1^n(u^n,v^n)-f_1^j(u^j,v^j)||\tilde{u}_t|dxd\tau\\
&\le\int_0^t\int_\Omega|f_1^n(u^n,v^n)-f_1^n(u,v)||\tilde{u}_t|dxd\tau+\int_0^t\int_\Omega|f_1^n(u,v)-f_1(u,v)||\tilde{u}_t|dxd\tau\\
&\quad+\int_0^t\int_\Omega|f_1(u,v)-f_1^j(u,v)||\tilde{u}_t|dxd\tau\\
&\quad+\int_0^t\int_\Omega|f_1^j(u,v)-f_1^j(u^j,v^j)||\tilde{u}_t|dxd\tau \qquad (2.1.81)
\end{aligned}$$

We now estimate each term on the right-hand side of (2.1.81) as follows. Recall, by Proposition 2.1.6, $f_1^n : H^{1-\epsilon}(\Omega) \times H_0^{1-\epsilon}(\Omega) \longrightarrow L^{\tilde{m}}(\Omega)$ is locally Lipschitz where the local Lipschitz constant is independent of n. By using Hölder's inequality, we obtain

$$\begin{aligned}
&\int_0^t \int_\Omega |f_1^n(u^n, v^n) - f_1^n(u,v)||\tilde{u}_t| dx d\tau \\
&\qquad \leq \left(\int_0^t \int_\Omega |f_1^n(u^n, v^n) - f_1^n(u,v)|^{\tilde{m}} dx d\tau \right)^{\frac{m}{m+1}} \left(\int_0^t \int_\Omega |u_t|^{m+1} dx d\tau \right)^{\frac{1}{m+1}} \\
&\qquad \leq C(K) \left(\int_0^t (\|u^n - u\|_{H^{1-\epsilon}(\Omega)}^{\tilde{m}} + \|v^n - v\|_{H^{1-\epsilon}(\Omega)}^{\tilde{m}}) d\tau \right)^{\frac{m}{m+1}} \longrightarrow 0, \qquad (2.1.82)
\end{aligned}$$

as $n \longrightarrow \infty$, where we have used the convergence (2.1.71)-(2.1.72) and the uniform bound in (2.1.62) .

To handle the second term on the right-hand side of (2.1.81), we shall show

$$f_1^n(u,v) \longrightarrow f_1(u,v) \text{ in } L^{\tilde{m}}(\Omega \times (0, T_0)). \qquad (2.1.83)$$

Indeed, by (2.1.70), we know $U \in L^\infty(0, T_0; H)$, thus $u \in L^\infty(0, T_0; H^1(\Omega))$ and $v \in L^\infty(0, T_0; H_0^1(\Omega))$. In addition, by (2.1.43), the definition of f_1^n, we have

$$\|f_1^n(u,v) - f_1(u,v)\|_{L^{\tilde{m}}(\Omega \times (0,T_0))}^{\tilde{m}} = \int_0^{T_0} \int_\Omega (|f_1(u,v)||\eta_n(u,v) - 1|)^{\tilde{m}} \, dx dt. \qquad (2.1.84)$$

Since $\eta_n(u,v) \leq 1$, it follows $(|f_1(u,v)||\eta_n(u,v) - 1|)^{\tilde{m}} \leq 2^{\tilde{m}} |f_1(u,v)|^{\tilde{m}}$. To see $|f_1(u,v)|^{\tilde{m}} \in L^1(\Omega \times (0, T_0))$, we use the assumptions $|f_1(u,v)| \leq C(|u|^p + |v|^p + 1)$ and $p\tilde{m} < 6$ along with the imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$. Indeed,

$$\begin{aligned}
\int_0^{T_0} \int_\Omega |f_1(u,v)|^{\tilde{m}} dx dt &\leq C \int_0^{T_0} \int_\Omega (|u|^{p\tilde{m}} + |v|^{p\tilde{m}} + 1) dx dt \\
&\leq C \int_0^{T_0} (\|u\|_{H^1(\Omega)}^{p\tilde{m}} + \|v\|_{H_0^1(\Omega)}^{p\tilde{m}} + |\Omega|) dt < \infty.
\end{aligned}$$

Clearly, $\eta_n(u(x), v(x)) \longrightarrow 1$ a.e. on Ω. By applying the Lebesgue Dominated Convergence Theorem on (2.1.84), (2.1.83) follows, as desired. Now, by using Hölder's inequality and the limit (2.1.83), one has

$$\begin{aligned}
&\int_0^t \int_\Omega |f_1^n(u,v) - f_1(u,v)||\tilde{u}_t| dx d\tau \\
&\leq \left(\int_0^t \int_\Omega |f_1^n(u,v) - f_1(u,v)|^{\tilde{m}} dx d\tau \right)^{\frac{m}{m+1}} \left(\int_0^t \int_\Omega |\tilde{u}_t|^{m+1} dx d\tau \right)^{\frac{1}{m+1}} \longrightarrow 0,
\end{aligned}$$
(2.1.85)

as $n \longrightarrow \infty$, where we have used the uniform bound in (2.1.62).

Combining (2.1.82) and (2.1.85) in (2.1.81) gives us the desired result

$$\lim_{n,j\longrightarrow\infty} \int_0^t \int_\Omega |f_1^n(u^n, v^n) - f_1^j(u^j, v^j)||\tilde{u}_t| dx d\tau = 0. \tag{2.1.86}$$

Next we show,

$$\lim_{n,j\longrightarrow\infty} \int_0^t \int_\Gamma |h^n(\gamma u^n) - h^j(\gamma u^j)||\gamma\tilde{u}_t| d\Gamma d\tau = 0. \tag{2.1.87}$$

To see this, we write

$$\begin{aligned}
&\int_0^t \int_\Gamma |h^n(\gamma u^n) - h^j(\gamma u^j)||\gamma\tilde{u}_t| d\Gamma d\tau \\
&\le \int_0^t \int_\Gamma |h^n(\gamma u^n) - h^n(\gamma u)||\gamma\tilde{u}_t| d\Gamma d\tau + \int_0^t \int_\Gamma |h^n(\gamma u) - h(\gamma u)||\gamma\tilde{u}_t| d\Gamma d\tau \\
&+ \int_0^t \int_\Gamma |h(\gamma u) - h^j(\gamma u)||\gamma\tilde{u}_t| d\Gamma d\tau + \int_0^t \int_\Gamma |h^j(\gamma u) - h^j(\gamma u^j)||\gamma\tilde{u}_t| d\Gamma d\tau. \qquad (2.1.88)
\end{aligned}$$

By Proposition 2.1.9, $h^n \circ \gamma : H^{1-\epsilon}(\Omega) \longrightarrow L^{\tilde{q}}(\Gamma)$ is locally Lipschitz where the local Lipschitz constant is independent of n. Therefore, by Hölder's inequality

$$\begin{aligned}
&\int_0^t \int_\Gamma |h^n(\gamma u^n) - h^n(\gamma u)||\gamma\tilde{u}_t| d\Gamma d\tau \\
&\qquad \le \left(\int_0^t \int_\Gamma |h^n(\gamma u^n) - h^n(\gamma u)|^{\tilde{q}} d\Gamma d\tau\right)^{\frac{q}{q+1}} \left(\int_0^t \int_\Gamma |\gamma\tilde{u}_t|^{q+1} d\Gamma d\tau\right)^{\frac{1}{q+1}} \\
&\qquad \le C(K) \left(\int_0^t \|u^n - u\|^{\tilde{q}}_{H^{1-\epsilon}(\Omega)}\, d\tau\right)^{\frac{q}{q+1}} \longrightarrow 0, \text{ as } n \longrightarrow \infty, \qquad (2.1.89)
\end{aligned}$$

where we have used the convergence (2.1.71) and the uniform bound in (2.1.62).

Since $u \in L^\infty(0, T; H^1(\Omega))$, then similar to (2.1.83), we may deduce that,

$$h^n(\gamma u) \longrightarrow h(\gamma u) \text{ in } L^{\tilde{q}}(\Omega \times (0, T_0)).$$

Again, by using the uniform bound in (2.1.62), we obtain,

$$\begin{aligned}
&\int_0^t \int_\Gamma |h^n(\gamma u) - h(\gamma u)||\gamma\tilde{u}_t| d\Gamma d\tau \\
&\quad \le \left(\int_0^t \int_\Gamma |h^n(\gamma u) - h(\gamma u)|^{\tilde{q}} d\Gamma d\tau\right)^{\frac{q}{q+1}} \left(\int_0^t \int_\Gamma |\gamma\tilde{u}_t|^{q+1} d\Gamma d\tau\right)^{\frac{1}{q+1}} \longrightarrow 0, \quad (2.1.90)
\end{aligned}$$

as $n \longrightarrow \infty$. By combining the estimates (2.1.88)-(2.1.90), then (2.1.87) follows as claimed.

Now, by using the fact that g_1 and g are monotone increasing and using (2.1.79)-(2.1.80),(2.1.86)-(2.1.87), we can take limit as $n, j \longrightarrow \infty$ in (2.1.78) to deduce

$$\lim_{n,j\longrightarrow\infty} \int_0^t \int_\Omega (g_1(u_t^n) - g_1(u_t^j))(u_t^n - u_t^j)dxd\tau = 0, \tag{2.1.91}$$

$$\lim_{n,j\longrightarrow\infty} \int_0^t \int_\Gamma (g(\gamma u_t^n) - g(\gamma u_t^j))(\gamma u_t^n - \gamma u_t^j)d\Gamma d\tau = 0. \tag{2.1.92}$$

In addition, it follows from (2.1.62) that, on a relabeled subsequence, $u_t^n \longrightarrow u_t$ weakly in $L^{m+1}(\Omega \times (0, T_0))$. Therefore, Lemma 1.3 (p.49) [6] along with (2.1.76) and (2.1.91) assert that $g_1^* = g_1(u_t)$; provided we show that

$$g_1 : L^{m+1}(\Omega \times (0,t)) \longrightarrow L^{\tilde{m}}(\Omega \times (0,t))$$

is maximal monotone. Indeed, since g_1 is monotone increasing, it is easy to see g_1 is a monotone operator. Thus, we need to verify that g_1 is hemi-continuous, i.e., we have to show that

$$\lim_{\lambda\longrightarrow\infty} \int_0^t \int_\Omega g_1(u + \lambda v)wdxd\tau = \int_0^t \int_\Omega g_1(u)wdxd\tau, \tag{2.1.93}$$

for all $u, v, w \in L^{m+1}(\Omega \times (0,t))$.

Indeed, since g_1 is continuous, then $g_1(u + \lambda v)w \longrightarrow g_1(u)w$ point-wise as $\lambda \longrightarrow 0$. Moreover, since $|g_1(s)| \le \beta(|s|^m + 1)$ for all $s \in \mathbb{R}$, we know if $|\lambda| \le 1$, then $|g_1(u+\lambda v)w| \le \beta(|u+\lambda v|^m + 1)|w| \le C(|u|^m|w| + |v|^m|w| + |w|) \in L^1(\Omega \times (0,t))$, by Hölder's inequality. Thus, (2.1.93) follows from the Lebesgue Dominated Convergence Theorem. Hence, g_1 is maximal monotone and we conclude that that $g_1^* = g_1(u_t)$, i.e.,

$$g_1(u_t^n) \longrightarrow g_1(u_t) \ \text{ weakly in } \ L^{\tilde{m}}(\Omega \times (0,t)). \tag{2.1.94}$$

In a similar way, one can show that $g^* = g(\gamma u_t)$, that is

$$g(\gamma u_t^n) \longrightarrow g(\gamma u_t) \ \text{ weakly in } \ L^{\tilde{q}}(\Gamma \times (0,t)). \tag{2.1.95}$$

It remains to show that

$$\lim_{n\longrightarrow\infty} \int_0^t \int_\Omega f_1^n(u^n, v^n)\phi dxd\tau = \int_0^t \int_\Omega f_1(u,v)\phi dxd\tau. \tag{2.1.96}$$

To prove (2.1.96), we write

$$\begin{aligned}
&\left|\int_0^t\int_\Omega (f_1^n(u^n,v^n)-f_1(u,v))\phi dxd\tau\right| \\
&\le \int_0^t\int_\Omega |f_1^n(u^n,v^n)-f_1^n(u,v)||\phi|dxd\tau+\int_0^t\int_\Omega |f_1^n(u,v)-f_1(u,v)||\phi|dxd\tau.
\end{aligned} \tag{2.1.97}$$

Since $\phi\in L^{m+1}(\Omega\times(0,t))$, then by replacing $\tilde{u}_t$ with ϕ in (2.1.82), we deduce

$$\lim_{n\longrightarrow\infty}\int_0^t\int_\Omega |f_1^n(u^n,v^n)-f_1^n(u,v)||\phi|dxd\tau=0. \tag{2.1.98}$$

In addition, (2.1.83) yields

$$\lim_{n\to\infty}\int_0^t\int_\Omega |f_1^n(u,v)-f_1(u,v)||\phi|dxd\tau=0. \tag{2.1.99}$$

Hence, (2.1.96) is verified.

In a similar manner, one can deduce

$$\lim_{n\longrightarrow\infty}\int_0^t\int_\Gamma h^n(\gamma u^n)\gamma\phi d\Gamma d\tau=\int_0^t\int_\Gamma h(\gamma u)\gamma\phi d\Gamma d\tau. \tag{2.1.100}$$

Finally, by using (2.1.70)-(2.1.75), (2.1.94)-(2.1.96) and (2.1.100) we can pass to the limit in (2.1.68) to obtain (1.3.1). In a similar way, we can work on (2.1.69) term by term to pass to the limit and obtain (1.3.2) .

Step 4: Completion of the proof. Since $t\in[0,T_0]$ and g, g_1 are monotone increasing on $\mathbb{R}$, then (2.1.78) implies

$$\begin{aligned}
&\frac{1}{2}\left(\|\tilde{u}_t(t)\|_2^2+\|\tilde{u}(t)\|_{1,\Omega}^2\right) \\
&\le \frac{1}{2}\left(\|\tilde{u}_t(0)\|_2^2+\|\tilde{u}(0)\|_{1,\Omega}^2\right)+2\left(\frac{1}{n}+\frac{1}{m}\right)\int_0^{T_0}\int_\Gamma(|\gamma u_t^n|^2+|\gamma u_t^j|^2)d\Gamma d\tau \\
&\quad+\int_0^{T_0}\int_\Omega |f_1^n(u^n,v^n)-f_1^j(u^j,v^j)||\tilde{u}_t|dxd\tau \\
&\quad+\int_0^{T_0}\int_\Gamma |h^n(\gamma u^n)-h^j(\gamma u^j)||\gamma\tilde{u}_t|d\Gamma d\tau.
\end{aligned} \tag{2.1.101}$$

By (2.1.79)-(2.1.80) and (2.1.86)-(2.1.87), we know the right hand side of (2.1.101) converges to 0 as $n, j \longrightarrow \infty$, so

$$\lim_{n,j\longrightarrow\infty} \left\|u^n(t) - u^j(t)\right\|_{1,\Omega} = \lim_{n,j\longrightarrow\infty} \|\tilde{u}(t)\|_{1,\Omega} = 0 \text{ uniformly in } t \in [0, T_0];$$
$$\lim_{n,j\longrightarrow\infty} \left\|u_t^n(t) - u_t^j(t)\right\|_2 = \lim_{n,j\longrightarrow\infty} \|\tilde{u}_t(t)\|_2 = 0 \text{ uniformly in } t \in [0, T_0].$$

Hence

$$\begin{aligned} &u^n(t) \longrightarrow u(t) \text{ in } H^1(\Omega) \text{ uniformly on } [0, T_0];\\ &u_t^n(t) \longrightarrow u_t(t) \text{ in } L^2(\Omega) \text{ uniformly on } [0, T_0]. \end{aligned} \tag{2.1.102}$$

Since $u^n \in W^{1,\infty}([0, T_0]; H^1(\Omega))$ and $u_t^n \in W^{1,\infty}([0, T_0]; L^2(\Omega))$, by (2.1.102), we conclude

$$u \in C([0, T_0]; H^1(\Omega)) \text{ and } u_t \in C([0, T_0]; L^2(\Omega)).$$

Moreover, (2.1.102) shows $u^n(0) \longrightarrow u(0)$ in $H^1(\Omega)$. Since $u^n(0) = u_0^n \longrightarrow u_0$ in $H^1(\Omega)$, then the initial condition $u(0) = u_0$ holds. Also, since $u_t^n(0) \longrightarrow u_t(0)$ in $L^2(\Omega)$ and $u_t^n(0) = u_1^n \longrightarrow u_1$ in $L^2(\Omega)$, we obtain $u_t(0) = u_1$. Similarly, we may deduce v, v_t satisfy the required regularity and the imposed initial conditions, as stated in Definition 1.3.1. This completes the proof of the local existence statement in Theorem 1.3.2.

2.2 Energy Identity

This section is devoted to derive the energy identity (1.3.4) in Theorem 1.3.2. One is tempted to test (1.3.1) with u_t and (1.3.2) with v_t, and carry out standard calculations to obtain energy identity. However, this procedure is only *formal*, since u_t and v_t are not regular enough and cannot be used as test functions in (1.3.1) and (1.3.2). In order to overcome this difficulty we shall use the difference quotients $D_h u$ and $D_h v$ and their well-known properties (see [26] and also [40, 43] for more details).

2.2.1 Properties of the difference quotient

Let X be a Banach space. For any function $u \in C([0,T];X)$ and $h > 0$, we define the *symmetric difference quotient* by:

$$D_h u(t) = \frac{u_e(t+h) - u_e(t-h)}{2h}, \tag{2.2.1}$$

where $u_e(t)$ denotes the extension of $u(t)$ to $\mathbb{R}$ given by:

$$u_e(t) = \begin{cases} u(0) \text{ for } t \leq 0, \\ u(t) \text{ for } t \in (0,T), \\ u(T) \text{ for } t \geq T. \end{cases} \tag{2.2.2}$$

The results in the following proposition have been established by Koch and Lasiecka in [26].

Proposition 2.2.1 ([26])**.** *Let $u \in C([0,T];X)$ where X is a Hilbert space with inner product $(\cdot,\cdot)_X$. Then,*

$$\lim_{h \longrightarrow 0} \int_0^T (u, D_h u)_X dt = \frac{1}{2}\left(\|u(T)\|_X^2 - \|u(0)\|_X^2\right). \tag{2.2.3}$$

If, in addition, $u_t \in C([0,T];X)$, then

$$\int_0^T (u_t, (D_h u)_t)_X dt = 0, \text{ for each } h > 0, \tag{2.2.4}$$

and, as $h \longrightarrow 0$,

$$D_h u(t) \longrightarrow u_t(t) \text{ weakly in } X, \text{ for every } t \in (0,T), \tag{2.2.5}$$

$$D_h u(0) \longrightarrow \frac{1}{2} u_t(0) \text{ and } D_h u(T) \longrightarrow \frac{1}{2} u_t(T) \text{ weakly in } X. \tag{2.2.6}$$

The following proposition is essential for the proof of the energy identity (1.3.4).

Proposition 2.2.2. *Let X and Y be Banach spaces. Assume $u \in C([0,T];Y)$ and $u_t \in L^1(0,T;Y) \cap L^p(0,T;X)$, where $1 \leq p < \infty$. Then $D_h u \in L^p(0,T;X)$ and $\|D_h u\|_{L^p(0,T;X)} \leq \|u_t\|_{L^p(0,T;X)}$. Moreover, $D_h u \longrightarrow u_t$ in $L^p(0,T;X)$, as $h \longrightarrow 0$.*

Proof. Throughout the proof, we write u_t as u'. Since $u \in C([0,T];Y)$, then by (2.2.2) $u_e \in C([-h,T+h];Y)$. Also note that,

$$u_e'(t) = u'(t) \text{ for } t \in (0,T) \text{ and } u_e'(t) = 0 \text{ for } t \in (-h,0) \cup (T,T+h), \tag{2.2.7}$$

and along with the assumption $u' \in L^1(0,T;Y)$, one has $u_e' \in L^1(-h,T+h;Y)$. Since u_e and $u_e' \in L^1(-h,T+h;Y)$, we conclude (for instance, see Lemma 1.1, page 250 in [48])

$$D_h u(t) = \frac{u_e(t+h) - u_e(t-h)}{2h} = \frac{1}{2h}\int_{t-h}^{t+h} u_e'(s)ds, \text{ a.e. } t \in [0,T]. \tag{2.2.8}$$

By using Jensen's inequality, it follows that

$$\|D_h u(t)\|_X^p \leq \frac{1}{2h}\int_{t-h}^{t+h} \|u_e'(s)\|_X^p \, ds, \text{ a.e. } t \in [0,T]. \tag{2.2.9}$$

By integrating both sides of (2.2.9) over $[0,T]$ and by using Tonelli's Theorem, one has

$$\begin{aligned} \int_0^T \|D_h u(t)\|_X^p \, dt &\leq \frac{1}{2h}\int_0^T \int_{t-h}^{t+h} \|u_e'(s)\|_X^p \, dsdt = \frac{1}{2h}\int_0^T \int_{-h}^{h} \|u_e'(s+t)\|_X^p \, dsdt \\ &= \frac{1}{2h}\int_{-h}^{h} \int_0^T \|u_e'(s+t)\|_X^p \, dtds = \frac{1}{2h}\int_{-h}^{h} \int_s^{T+s} \|u_e'(t)\|_X^p \, dtds. \end{aligned} \tag{2.2.10}$$

We split the last integral in (2.2.10) as the sum of two integrals, and by recalling (2.2.7), we deduce

$$\begin{aligned} \int_0^T \|D_h u(t)\|_X^p \, dt &\leq \frac{1}{2h}\int_{-h}^{0} \int_s^{T+s} \|u_e'(t)\|_X^p \, dtds + \frac{1}{2h}\int_0^{h} \int_s^{T+s} \|u_e'(t)\|_X^p \, dtds \\ &= \frac{1}{2h}\int_{-h}^{0} \int_0^{T+s} \|u'(t)\|_X^p \, dtds + \frac{1}{2h}\int_0^{h} \int_s^{T} \|u'(t)\|_X^p \, dsdt \\ &\leq \frac{1}{2h}\int_{-h}^{0} \int_0^{T} \|u'(t)\|_X^p \, dtds + \frac{1}{2h}\int_0^{h} \int_0^{T} \|u'(t)\|_X^p \, dtds \\ &= \frac{1}{2h}\int_{-h}^{h} \int_0^{T} \|u'(t)\|_X^p \, dtds = \int_0^T \|u'(t)\|_X^p \, dt. \end{aligned}$$

Thus,

$$\|D_h u\|_{L^p(0,T;X)} \leq \|u'\|_{L^p(0,T;X)}, \tag{2.2.11}$$

as desired.

It remains to show: $D_h u \longrightarrow u'$ in $L^p(0,T;X)$, as $h \longrightarrow 0$.

Let $\epsilon > 0$ be given. By Lemma 2.6.2 in the Appendices, $C_0((0,T);X)$ is dense in $L^p(0,T;X)$, and since $u' \in L^p(0,T;X)$, there exists $\phi \in C_0((0,T);X)$ such that $\|u' - \phi\|_{L^p(0,T;X)} \leq \epsilon/3$. Note that (2.2.8) yields,

$$D_h u(t) - u'(t) = \frac{1}{2h}\int_{t-h}^{t+h} (u_e'(s) - u'(t))ds, \text{ a.e. } t \in [0,T].$$

In particular,

$$\begin{aligned}
&\|D_h u(t) - u'(t)\|_X^p \leq \frac{1}{2h}\int_{t-h}^{t+h} \|u_e'(s) - u'(t)\|_X^p \, ds \\
&\leq \frac{1}{2h}\int_{t-h}^{t+h} \Big(\|u_e'(s) - \phi(s)\|_X + \|\phi(s) - \phi(t)\|_X + \|\phi(t) - u'(t)\|_X \Big)^p ds \\
&\leq \frac{3^{p-1}}{2h}\int_{t-h}^{t+h} \|u_e'(s) - \phi(s)\|_X^p \, ds + \frac{3^{p-1}}{2h}\int_{t-h}^{t+h} \|\phi(s) - \phi(t)\|_X^p \, ds \\
&\quad + 3^{p-1}\|\phi(t) - u'(t)\|_X^p, \qquad (2.2.12)
\end{aligned}$$

where we have used Jensen's inequality. Now, integrating both sides of (2.2.12) over $[0,T]$ to obtain,

$$\int_0^T \|D_h u(t) - u'(t)\|_X^p \, dt \leq I_1 + I_2 + I_3 \qquad (2.2.13)$$

where

$$\begin{aligned}
I_1 &= \frac{3^{p-1}}{2h}\int_0^T \int_{t-h}^{t+h} \|u_e'(s) - \phi(s)\|_X^p \, dsdt, \\
I_2 &= \frac{3^{p-1}}{2h}\int_0^T \int_{t-h}^{t+h} \|\phi(s) - \phi(t)\|_X^p \, dsdt, \\
I_3 &= 3^{p-1}\|\phi(t) - u'(t)\|_{L^p(0,T;X)}^p.
\end{aligned}$$

Since $\|u' - \phi\|_{L^p(0,T;X)} \leq \epsilon/3$, then

$$I_3 \leq 3^{p-1}\frac{\epsilon^p}{3^p} = \frac{\epsilon^p}{3}. \qquad (2.2.14)$$

In addition, since $\phi \in C_0((0,T);X)$, then $\phi : \mathbb{R} \longrightarrow X$ is uniformly continuous. Thus, there exists $\delta > 0$ (say $\delta < T$) such that $\|\phi(s) - \phi(t)\|_X < \frac{\epsilon}{3T^{1/p}}$ whenever $|s-t| < \delta$. So, if $0 < h < \frac{\delta}{2}$, then one has

$$I_2 \leq \frac{3^{p-1}}{2h} \int_0^T \int_{t-h}^{t+h} \left(\frac{\epsilon}{3T^{1/p}} \right)^p dsdt = \frac{\epsilon^p}{3}. \tag{2.2.15}$$

As for I_1, we change variables and use Tonelli's theorem as follows:

$$\begin{aligned} I_1 &= \frac{3^{p-1}}{2h} \int_0^T \int_{-h}^{h} \|u_e'(s+t) - \phi(s+t)\|_X^p \, dsdt \\ &= \frac{3^{p-1}}{2h} \int_{-h}^{h} \int_s^{T+s} \|u_e'(t) - \phi(t)\|_X^p \, dtds. \end{aligned} \tag{2.2.16}$$

Now, split I_1 into two integrals and recall (2.2.7) to obtain (for sufficiently small h),

$$\begin{aligned} I_1 &= \frac{3^{p-1}}{2h} \left(\int_{-h}^{0} \int_0^{T+s} \|u'(t) - \phi(t)\|_X^p \, dtds + \int_0^h \int_s^T \|u'(t) - \phi(t)\|_X^p \, dtds \right) \\ &\leq \frac{3^{p-1}}{2h} \int_{-h}^{h} \int_0^T \|u'(t) - \phi(t)\|_X^p \, dtds = 3^{p-1} \int_0^T \|u'(t) - \phi(t)\|_X^p \, dt \\ &= 3^{p-1} \|u' - \phi\|_{L^p(0,T;X)}^p \leq 3^{p-1} \cdot \frac{\epsilon^p}{3^p} = \frac{\epsilon^p}{3}. \end{aligned} \tag{2.2.17}$$

Therefore, if if $0 < h < \frac{\delta}{2}$, then it follows from (2.2.14), (2.2.15), (2.2.17), and (2.2.13) that

$$\|D_h u - u'\|_{L^p(0,T;X)}^p \leq \epsilon^p,$$

completing the proof. □

2.2.2 Proof of the energy identity

Throughout the proof, we fix $t \in [0, T_0]$ and let (u, v) be a weak solution of system (1.1.1) in the sense of Definition 1.3.1. Recall the regularity of u and v, in particular, $u_t \in C([0,t]; L^2(\Omega))$ and $u_t \in L^{m+1}(\Omega \times (0,t)) = L^{m+1}(0,t; L^{m+1}(\Omega))$. We can define the difference quotient $D_h u(\tau)$ on $[0,t]$ as (2.2.1), i.e., $D_h u(\tau) = \frac{1}{2h}[u_e(\tau+h) - u_e(\tau - h)]$, where $u_e(\tau)$ extends $u(\tau)$ from $[0,t]$ to $\mathbb{R}$ as in (2.2.2):

$$u_e(\tau) = \begin{cases} u(0) & \text{for } \tau \leq 0, \\ u(\tau) & \text{for } \tau \in (0,t), \\ u(t) & \text{for } \tau \geq t. \end{cases}$$

By Proposition 2.2.2, with $X = L^{m+1}(\Omega)$ and $Y = L^2(\Omega)$, we have

$$D_h u \in L^{m+1}(\Omega \times (0,t)) \text{ and } D_h u \longrightarrow u_t \text{ in } L^{m+1}(\Omega \times (0,t)). \tag{2.2.18}$$

Similar argument yields,

$$D_h v \in L^{r+1}(\Omega \times (0,t)) \text{ and } D_h v \longrightarrow v_t \text{ in } L^{r+1}(\Omega \times (0,t)). \tag{2.2.19}$$

Recall the notation γu_t stands for $(\gamma u)_t$, and since $u \in C([0,t];H^1(\Omega))$, then $\gamma u \in C([0,t];L^2(\Gamma))$. Moreover, we know $(\gamma u)_t = \gamma u_t \in L^{q+1}(\Gamma\times(0,t)) = L^{q+1}(0,t;L^{q+1}(\Gamma))$, so $(\gamma u)_t \in L^2(\Gamma \times (0,t)) = L^2(0,t;L^2(\Gamma))$. So, by Proposition 2.2.2 with $X = L^{q+1}(\Gamma)$ and $Y = L^2(\Gamma)$, one has

$$\begin{aligned}&\gamma D_h u = D_h(\gamma u) \in L^{q+1}(\Gamma \times (0,t)) \text{ and}\\ &\gamma D_h u = D_h(\gamma u) \longrightarrow (\gamma u)_t = \gamma u_t \text{ in } L^{q+1}(\Gamma \times (0,t)).\end{aligned} \tag{2.2.20}$$

Moreover, since $u \in C([0,t];H^1(\Omega))$ and $v \in C([0,t];H_0^1(\Omega))$, then

$$D_h u \in C([0,t];H^1(\Omega)) \text{ and } D_h v \in C([0,t];H_0^1(\Omega)). \tag{2.2.21}$$

We now show

$$(D_h u)_t \in L^1(0,t;L^2(\Omega)) \text{ and } (D_h v)_t \in L^1(0,t;L^2(\Omega)). \tag{2.2.22}$$

Indeed, for $0 < h < \frac{t}{2}$, we note that

$$(D_h u)_t(\tau) = \begin{cases} \frac{1}{2h}[u_t(\tau+h) - u_t(\tau-h)], & \text{if } h < \tau < t-h,\\ -\frac{1}{2h}u_t(\tau-h), & \text{if } t-h < \tau < t,\\ \frac{1}{2h}u_t(\tau+h), & \text{if } 0 < \tau < h,\end{cases}$$

and since $u_t \in C([0,t];L^2(\Omega))$, we conclude $(D_h u)_t \in L^1(0,t;L^2(\Omega))$. Similarly, $(D_h v)_t \in L^1(0,t;L^2(\Omega))$.

Thus, (2.2.18)-(2.2.22) show that $D_h u$ and $D_h v$ satisfy the required regularity conditions to be suitable test functions in Definition 1.3.1. Therefore, by taking $\phi = D_h u$ in (1.3.1) and $\psi = D_h v$ in (1.3.2), we obtain

$$\begin{aligned}(u_t(t), D_h u(t))_\Omega - (u_t(0), D_h u(0))_\Omega - \int_0^t (u_t, (D_h u)_t)_\Omega d\tau + \int_0^t (u, D_h u)_{1,\Omega} d\tau\\ + \int_0^t \int_\Omega g_1(u_t) D_h u dx d\tau + \int_0^t \int_\Gamma g(\gamma u_t)\gamma D_h u d\Gamma d\tau\\ = \int_0^t \int_\Omega f_1(u,v) D_h u dx d\tau + \int_0^t \int_\Gamma h(\gamma u)\gamma D_h u d\Gamma d\tau,\end{aligned} \tag{2.2.23}$$

and

$$(v_t(t), D_hv(t))_\Omega - (v_t(0), D_hv(0))_\Omega - \int_0^t (v_t, (D_hv)_t)_\Omega d\tau + \int_0^t (v, D_hv)_{1,\Omega} d\tau$$
$$+ \int_0^t \int_\Omega g_2(v_t) D_hv dx d\tau = \int_0^t \int_\Omega f_2(u,v) D_hv dx d\tau. \tag{2.2.24}$$

We will pass to the limit as $h \longrightarrow 0$ in (2.2.23) only, since passing to the limit in (2.2.24) can be handled in the same way.

Since u, $u_t \in C([0,t]; L^2(\Omega))$, then (2.2.6) shows

$$D_hu(0) \longrightarrow \frac{1}{2}u_t(0) \text{ and } D_hu(t) \longrightarrow \frac{1}{2}u_t(t) \text{ weakly in } L^2(\Omega).$$

It follows that

$$\lim_{h\longrightarrow 0}(u_t(0), D_hu(0))_\Omega = \frac{1}{2}\left\|u_t(0)\right\|_2^2,$$
$$\lim_{h\longrightarrow 0}(u_t(t), D_hu(t))_\Omega = \frac{1}{2}\left\|u_t(t)\right\|_2^2. \tag{2.2.25}$$

Also, by (2.2.4)

$$\int_0^t (u_t, (D_hu)_t)_\Omega d\tau = 0. \tag{2.2.26}$$

In addition, since $u \in C([0,t]; H^1(\Omega))$, then (2.2.3) yields

$$\lim_{h\longrightarrow 0}\int_0^t (u, D_hu)_{1,\Omega} d\tau = \frac{1}{2}\left(\left\|u(t)\right\|_{1,\Omega}^2 - \left\|u(0)\right\|_{1,\Omega}^2\right). \tag{2.2.27}$$

Since $u_t \in L^{m+1}(\Omega \times (0,t))$ and $|g_1(s)| \le b_1|s|^m$ whenever $|s| \ge 1$, then clearly $g_1(u_t) \in L^{\tilde{m}}(\Omega \times (0,t))$, where $\tilde{m} = \frac{m+1}{m}$. Hence, by (2.2.18)

$$\lim_{h\longrightarrow 0}\int_0^t \int_\Omega g_1(u_t) D_hu dx d\tau = \int_0^t \int_\Omega g_1(u_t) u_t dx d\tau. \tag{2.2.28}$$

Similarly, since $g(\gamma u_t) \in L^{\tilde{q}}(\Gamma \times (0,t))$, then (2.2.20) implies

$$\lim_{h\longrightarrow 0}\int_0^t \int_\Gamma g(\gamma u_t)\gamma D_hu d\Gamma d\tau = \int_0^t \int_\Gamma g(\gamma u_t)\gamma u_t dx d\tau. \tag{2.2.29}$$

In order to handle the interior source, we note that since $u \in C([0,t];H^1(\Omega))$ and $v \in C([0,t];H_0^1(\Omega))$, then there exists $M_0 > 0$ such that $\|u(\tau)\|_6$, $\|v(\tau)\|_6 \leq M_0$ for all $\tau \in [0,t]$. Also, since $|f_1(u,v)| \leq C(|u|^p + |v|^p + 1)$, then

$$\int_\Omega |f_1(u(\tau),v(\tau))|^{\frac{6}{p}} dx \leq C \int_\Omega (|u(\tau)|^6 + |v(\tau)|^6 + 1) dx \leq C(M_0),$$

for all $\tau \in [0,t]$. Hence, $f_1(u,v) \in L^\infty(0,t;L^{\frac{6}{p}}(\Omega))$, and so, $f_1(u,v) \in L^{\frac{6}{p}}(\Omega \times (0,t))$. Since $\frac{6}{p} > \tilde{m}$, then $f_1(u,v) \in L^{\tilde{m}}(\Omega \times (0,t))$. Therefore, it follows from (2.2.18) that

$$\lim_{h \longrightarrow 0} \int_0^t \int_\Omega f_1(u,v) D_h u dx d\tau = \int_0^t \int_\Omega f_1(u,v) u_t dx d\tau. \tag{2.2.30}$$

Finally, we consider the boundary source. Again, since $u \in C([0,t];H^1(\Omega))$ and $H^1(\Omega) \hookrightarrow L^4(\Gamma)$, then there exists $M_1 > 0$ such that $|\gamma u(\tau)|_4 \leq M_1$ for all $\tau \in [0,t]$. By recalling the assumption $|h(\gamma u)| \leq C(|\gamma u|^k + 1)$, then

$$\int_\Gamma |h(\gamma u(\tau))|^{\frac{4}{k}} dx \leq C \int_\Gamma (|\gamma u(\tau)|^4 + 1) d\Gamma \leq C(M_1)$$

for all $\tau \in [0,t]$. Hence, $h(\gamma u) \in L^\infty(0,t;L^{\frac{4}{k}}(\Gamma))$, and in particular, $h(\gamma u) \in L^{\frac{4}{k}}(\Gamma \times (0,t))$. Since $\frac{4}{k} > \tilde{q}$, we conclude $h(\gamma u) \in L^{\tilde{q}}(\Gamma \times (0,t))$. Therefore, (2.2.20) yields

$$\lim_{h \longrightarrow 0} \int_0^t \int_\Gamma h(\gamma u) \gamma D_h u d\Gamma d\tau = \int_0^t \int_\Gamma h(\gamma u) \gamma u_t d\Gamma d\tau. \tag{2.2.31}$$

By combining (2.2.25)-(2.2.31), we can pass to the limit as $h \longrightarrow 0$ in (2.2.23) to obtain

$$\begin{aligned} &\frac{1}{2}(\|u_t(t)\|_2^2 + \|u(t)\|_{1,\Omega}^2) + \int_0^t \int_\Omega g_1(u_t) u_t dx d\tau + \int_0^t \int_\Gamma g(\gamma u_t) \gamma u_t d\Gamma d\tau \\ &= \frac{1}{2}(\|u_t(0)\|_2^2 + \|u(0)\|_{1,\Omega}^2) + \int_0^t \int_\Omega f_1(u,v) u_t dx d\tau + \int_0^t \int_\Gamma h(\gamma u) \gamma u_t d\Gamma d\tau. \end{aligned} \tag{2.2.32}$$

Similarly, we can also pass to the limit as $h \longrightarrow 0$ in (2.2.24) and obtain

$$\begin{aligned} &\frac{1}{2}(\|v_t(t)\|_2^2 + \|v(t)\|_{1,\Omega}^2) + \int_0^t \int_\Omega g_2(v_t) v_t dx d\tau \\ &= \frac{1}{2}(\|v_t(0)\|_2^2 + \|v(0)\|_{1,\Omega}^2) + \int_0^t \int_\Omega f_2(u,v) v_t dx d\tau. \end{aligned} \tag{2.2.33}$$

By adding (2.2.32) to (2.2.33), then the energy identity (1.3.4) follows.

2.3 Uniqueness of Weak Solutions

The uniqueness results of Theorem 1.3.4 and Theorem 1.3.6 will be justified in the following two subsections.

2.3.1 Proof of Theorem 1.3.4.

The proof of Theorem 1.3.4 will be carried out in the following four steps.

Step 1: Let (u, v) and $(\hat{u}, \hat{v})$ be two weak solutions on $[0, T]$ in the sense of Definition 1.3.1 satisfying the same initial conditions. Put $y = u - \hat{u}$ and $z = v - \hat{v}$. The energy corresponding to (y, z) is given by:

$$\tilde{\mathscr{E}}(t) = \frac{1}{2}(\|y(t)\|_{1,\Omega}^2 + \|z(t)\|_{1,\Omega}^2 + \|y_t(t)\|_2^2 + \|z_t(t)\|_2^2) \tag{2.3.1}$$

for all $t \in [0, T]$. We aim to show that $\tilde{\mathscr{E}}(t) = 0$, and thus $y(t) = 0$ and $z(t) = 0$ for all $t \in [0, T]$.

By the regularity imposed on weak solutions in Definition 1.3.1, there exists a constant $R > 0$ such that

$$\begin{cases} \|u(t)\|_{1,\Omega}, \|\hat{u}(t)\|_{1,\Omega}, \|v(t)\|_{1,\Omega}, \|\hat{v}(t)\|_{1,\Omega} \leq R, \\ \|u_t(t)\|_2, \|\hat{u}_t(t)\|_2, \|v_t(t)\|_2, \|\hat{v}_t(t)\|_2 \leq R, \\ \int_0^T \|u_t\|_{m+1}^{m+1} dt, \int_0^T \|\hat{u}_t\|_{m+1}^{m+1} dt, \int_0^T |\gamma u_t|_{q+1}^{q+1} dt, \int_0^T |\gamma \hat{u}_t|_{q+1}^{q+1} dt \leq R, \\ \int_0^T \|v_t\|_{r+1}^{r+1} dt, \int_0^T \|\hat{v}_t\|_{r+1}^{r+1} dt \leq R \end{cases} \tag{2.3.2}$$

for all $t \in [0, T]$. Since $y(0) = y_t(0) = z(0) = z_t(0) = 0$, then by Definition 1.3.1, y and z satisfy:

$$\begin{aligned} (y_t(t), \phi(t))_\Omega &- \int_0^t (y_t, \phi_t)_\Omega d\tau + \int_0^t (y, \phi)_{1,\Omega} d\tau \\ &+ \int_0^t \int_\Omega (g_1(u_t) - g_1(\hat{u}_t))\phi dx d\tau + \int_0^t \int_\Gamma (g(\gamma u_t) - g(\gamma \hat{u}_t))\gamma\phi d\Gamma d\tau \\ &= \int_0^t \int_\Omega (f_1(u, v) - f_1(\hat{u}, \hat{v}))\phi dx d\tau + \int_0^t \int_\Gamma (h(\gamma u) - h(\gamma \hat{u}))\gamma\phi d\Gamma d\tau, \end{aligned} \tag{2.3.3}$$

and

$$(z_t(t), \psi(t))_\Omega - \int_0^t (z_t, \psi_t)_\Omega d\tau + \int_0^t (z, \psi)_{1,\Omega} d\tau + \int_0^t \int_\Omega (g_2(v_t) - g_2(\hat{v}_t))\psi dx d\tau$$
$$= \int_0^t \int_\Omega (f_2(u,v) - f_2(\hat{u},\hat{v}))\psi dx d\tau, \tag{2.3.4}$$

for all $t \in [0,T]$ and for all test functions ϕ and ψ as described in Definition 1.3.1.

Let $\phi(\tau) = D_h y(\tau)$ in (2.3.3) and $\psi(\tau) = D_h z(\tau)$ in (2.3.4) for $\tau \in [0,t]$ where the difference quotients $D_h y$ and $D_h z$ are defined in (2.2.1). Using a similar argument as in obtaining the energy identity (1.3.4), we can pass to the limit as $h \longrightarrow 0$ and deduce

$$\frac{1}{2}(\|y(t)\|_{1,\Omega}^2 + \|y_t(t)\|_2^2) + \int_0^t \int_\Omega (g_1(u_t) - g_1(\hat{u}_t)) y_t dx d\tau$$
$$+ \int_0^t \int_\Gamma (g(\gamma u_t) - g(\gamma \hat{u}_t))\gamma y_t d\Gamma d\tau$$
$$= \int_0^t \int_\Omega (f_1(u,v) - f_1(\hat{u},\hat{v})) y_t dx d\tau + \int_0^t \int_\Gamma (h(\gamma u) - h(\gamma \hat{u}))\gamma y_t d\Gamma d\tau \tag{2.3.5}$$

and

$$\frac{1}{2}(\|z(t)\|_{1,\Omega}^2 + \|z_t(t)\|_2^2) + \int_0^t \int_\Omega (g_2(v_t) - g_2(\hat{v}_t)) z_t dx d\tau$$
$$= \int_0^t \int_\Omega (f_2(u,v) - f_2(\hat{u},\hat{v})) z_t dx d\tau. \tag{2.3.6}$$

Adding (2.3.5) and (2.3.6) and employing the monotonicity properties of g_1, g_2 yield

$$\tilde{\mathscr{E}}(t) \leq \int_0^t \int_\Omega (f_1(u,v) - f_1(\hat{u},\hat{v})) y_t dx d\tau + \int_0^t \int_\Omega (f_2(u,v) - f_2(\hat{u},\hat{v})) z_t dx d\tau$$
$$+ \int_0^t \int_\Gamma (h(\gamma u) - h(\gamma \hat{u}))\gamma y_t d\Gamma d\tau - \int_0^t \int_\Gamma (g(\gamma u_t) - g(\gamma \hat{u}_t))\gamma y_t d\Gamma, \tag{2.3.7}$$

for all $t \in [0,T]$ where $\tilde{\mathscr{E}}(t)$ is defined in (2.3.1).

We will estimate each term on the right hand side of (2.3.7).

Step 2: "Estimate for the terms due to the interior sources."

Put

$$R_f = \int_0^t \int_\Omega (f_1(u,v) - f_1(\hat{u},\hat{v}))y_t dx d\tau + \int_0^t \int_\Omega (f_2(u,v) - f_2(\hat{u},\hat{v}))z_t dx d\tau. \quad (2.3.8)$$

First we note that, if $1 \leq p \leq 3$, then by Remark 2.1.5 we know f_1 and f_2 are both locally Lipschitz from $H^1(\Omega) \times H_0^1(\Omega)$ into $L^2(\Omega)$. In this case, the estimate for R_f is straightforward. By using Hölder's inequality, we have

$$\begin{aligned}
&\int_0^t \int_\Omega (f_1(u,v) - f_1(\hat{u},\hat{v}))y_t dx d\tau \\
&\leq \left(\int_0^t \int_\Omega |f_1(u,v) - f_1(\hat{u},\hat{v})|^2 dx d\tau\right)^{1/2} \left(\int_0^t \int_\Omega |y_t|^2 dx d\tau\right)^{1/2} \\
&\leq C(R) \left(\int_0^t (\|y\|_{1,\Omega}^2 + \|z\|_{1,\Omega}^2) d\tau\right)^{1/2} \left(\int_0^t \|y_t\|_2^2 \, d\tau\right)^{1/2} \leq C(R) \int_0^t \tilde{\mathscr{E}}(\tau) d\tau. \quad (2.3.9)
\end{aligned}$$

Likewise, $\int_0^t \int_\Omega (f_2(u,v) - f_2(\hat{u},\hat{v}))z_t dx d\tau \leq C(R) \int_0^t \tilde{\mathscr{E}}(\tau) d\tau$. Therefore, for $1 \leq p \leq 3$, we have the following estimate for R_f:

$$R_f \leq C(R) \int_0^t \tilde{\mathscr{E}}(\tau) d\tau. \quad (2.3.10)$$

For the case $3 < p < 6$, f_1 and f_2 are not locally Lipschitz from $H^1(\Omega) \times H_0^1(\Omega)$ into $L^2(\Omega)$, and therefore the computation in (2.3.9) does not work. To overcome this difficulty, we shall use a clever idea by Bociu and Lasiecka [11, 12] which involves integration by parts. In order to do so, we require f_1 and f_2 to be C^2-functions. More precisely, we impose the following assumption: there exists $F \in C^3(\mathbb{R}^2)$ such that $f_1(u,v) = \partial_u F(u,v)$, $f_2(u,v) = \partial_v F(u,v)$ and $|D^\alpha F(u,v)| \leq C(|u|^{p-2} + |v|^{p-2} + 1)$ for all α such that $|\alpha| = 3$. It follows from this assumption that $f_j \in C^2(\mathbb{R}^2)$, $j = 1, 2$, and

$$\begin{cases}
\left|D^\beta f_j(u,v)\right| \leq C(|u|^{p-2} + |v|^{p-2} + 1), \ \text{ for all } \ |\beta| = 2; \\
|\nabla f_j(u,v)| \leq C(|u|^{p-1} + |v|^{p-1} + 1) \ \text{ and } \ |f_j(u,v)| \leq C(|u|^p + |v|^p + 1); \\
|\nabla f_j(u,v) - \nabla f_j(\hat{u},\hat{v})| \leq C(|u|^{p-2} + |\hat{u}|^{p-2} + |v|^{p-2} + |\hat{v}|^{p-2} + 1)(|y| + |z|); \\
|f_j(u,v) - f_j(\hat{u},\hat{v})| \leq C(|u|^{p-1} + |\hat{u}|^{p-1} + |v|^{p-1} + |\hat{v}|^{p-1} + 1)(|y| + |z|)
\end{cases} \quad (2.3.11)$$

where $y = u - \hat{u}$ and $z = v - \hat{v}$.

Now, we evaluate R_f in the case $3 < p < 6$. By integration by parts in time and by recalling $y(0) = 0$, one has

$$
\begin{aligned}
&\int_0^t \int_\Omega [f_1(u,v) - f_1(\hat{u},\hat{v})] y_t dx d\tau = \int_\Omega [f_1(u(t),v(t)) - f_1(\hat{u}(t),\hat{v}(t))] y(t) dx \\
&- \int_\Omega \int_0^t \left[\nabla f_1(u,v) \cdot \begin{pmatrix} u_t \\ v_t \end{pmatrix} - \nabla f_1(\hat{u},\hat{v}) \cdot \begin{pmatrix} \hat{u}_t \\ \hat{v}_t \end{pmatrix} \right] y d\tau dx \\
&= \int_\Omega [f_1(u(t),v(t)) - f_1(\hat{u}(t),\hat{v}(t))] y(t) dx - \int_\Omega \int_0^t \nabla f_1(u,v) \cdot \begin{pmatrix} y_t \\ z_t \end{pmatrix} y d\tau dx \\
&- \int_\Omega \int_0^t [\nabla f_1(u,v) - \nabla f_1(\hat{u},\hat{v})] \cdot \begin{pmatrix} \hat{u}_t \\ \hat{v}_t \end{pmatrix} y d\tau dx. \qquad (2.3.12)
\end{aligned}
$$

As (2.3.12), we have a similar expression for $\int_0^t \int_\Omega [f_2(u,v) - f_2(\hat{u},\hat{v})] z_t dx d\tau$. Therefore, we deduce

$$
R_f = P_1 + P_2 + P_3 + P_4 + P_5 \qquad (2.3.13)
$$

where,

$$
\begin{cases}
P_1 = \int_\Omega [f_1(u(t),v(t)) - f_1(\hat{u}(t),\hat{v}(t))] y(t) dx \\
P_2 = \int_\Omega [f_2(u(t),v(t)) - f_2(\hat{u}(t),\hat{v}(t))] z(t) dx \\
P_3 = \int_\Omega \int_0^t [\nabla f_1(u,v) - \nabla f_1(\hat{u},\hat{v})] \cdot \begin{pmatrix} \hat{u}_t \\ \hat{v}_t \end{pmatrix} y d\tau dx \\
P_4 = \int_\Omega \int_0^t [\nabla f_2(u,v) - \nabla f_2(\hat{u},\hat{v})] \cdot \begin{pmatrix} \hat{u}_t \\ \hat{v}_t \end{pmatrix} z) d\tau dx \\
P_5 = \int_\Omega \int_0^t \left(\nabla f_1(u,v) \cdot \begin{pmatrix} y_t \\ z_t \end{pmatrix} y + \nabla f_2(u,v) \cdot \begin{pmatrix} y_t \\ z_t \end{pmatrix} z \right) d\tau dx.
\end{cases}
$$

By using (2.3.11) and Young's inequality, we obtain

$$
\begin{aligned}
&|P_1 + P_2| \\
&\leq C \int_\Omega (|u(t)|^{p-1} + |\hat{u}(t)|^{p-1} + |v(t)|^{p-1} + |\hat{v}(t)|^{p-1} + 1)(y^2(t) + z^2(t)) dx, \qquad (2.3.14) \\
&|P_3 + P_4| \\
&\leq C \int_\Omega \int_0^t (|u|^{p-2} + |\hat{u}|^{p-2} + |v|^{p-2} + |\hat{v}|^{p-2} + 1)(y^2 + z^2)(|\hat{u}_t| + |\hat{v}_t|) d\tau dx. \qquad (2.3.15)
\end{aligned}
$$

As for P_5, we integrate by parts one more time and use the assumption $f_1(u,v) = \partial_u F(u,v)$ and $f_2(u,v) = \partial_v F(u,v)$. Indeed,

$$\begin{aligned}
P_5 &= \int_\Omega \int_0^t \Big(\partial_u f_1(u,v) y_t y + \partial_v f_1(u,v) z_t y \Big) d\tau dx \\
&\qquad\qquad + \int_\Omega \int_0^t \Big(\partial_u f_2(u,v) y_t z + \partial_v f_2(u,v) z_t z \Big) d\tau dx \\
&= \int_\Omega \int_0^t \Big(\frac{1}{2} \partial_u f_1(u,v)(y^2)_t + \partial^2_{uv} F(u,v)(yz)_t + \frac{1}{2} \partial_v f_2(u,v)(z^2)_t \Big) d\tau dx \\
&= \int_\Omega \Big(\frac{1}{2} \partial_u f_1(u(t),v(t)) y(t)^2 + \partial^2_{uv} F(u(t),v(t)) y(t) z(t) \Big) dx \\
&\qquad\qquad\qquad + \int_\Omega \frac{1}{2} \partial_v f_2(u(t),v(t)) z(t)^2) dx \\
&+ \int_\Omega \int_0^t \Big(\frac{1}{2} \nabla \partial_u f_1(u,v) y^2 + \nabla \partial^2_{uv} F(u,v) yz \Big) \cdot \begin{pmatrix} u_t \\ v_t \end{pmatrix} d\tau dx \\
&\qquad\qquad\qquad + \int_\Omega \int_0^t \frac{1}{2} \nabla \partial_v f_2(u,v) z^2 \cdot \begin{pmatrix} u_t \\ v_t \end{pmatrix} d\tau dx. \qquad (2.3.16)
\end{aligned}$$

By employing (2.3.11) and Young's inequality, we deduce

$$\begin{aligned}
P_5 &\le C \int_\Omega (|u(t)|^{p-1} + |v(t)|^{p-1} + 1)(|y(t)|^2 + |z(t)|^2) dx \\
&\quad + C \int_\Omega \int_0^t (|u|^{p-2} + |v|^{p-2} + 1)(y^2 + z^2)(|u_t| + |v_t|) d\tau dx. \qquad (2.3.17)
\end{aligned}$$

It follows from (2.3.14), (2.3.15), (2.3.17), and (2.3.13) that

$$\begin{aligned}
R_f &\le C \int_\Omega (|y(t)|^2 + |z(t)|^2) dx + C \int_0^t \int_\Omega (y^2 + z^2)(|u_t| + |v_t| + |\hat{u}_t| + |\hat{v}_t|) dx d\tau \\
&+ C \int_0^t \int_\Omega (|u|^{p-2} + |\hat{u}|^{p-2} + |v|^{p-2} + |\hat{v}|^{p-2})(y^2 + z^2)(|u_t| + |v_t| + |\hat{u}_t| + |\hat{v}_t|) dx d\tau \\
&+ C \int_\Omega (|u(t)|^{p-1} + |\hat{u}(t)|^{p-1} + |v(t)|^{p-1} + |\hat{v}(t)|^{p-1})(|y(t)|^2 + |z(t)|^2) dx. \qquad (2.3.18)
\end{aligned}$$

Now, we estimate the terms on the right-hand side of (2.3.18) as follows.

1. Estimate for

$$I_1 = \int_\Omega (|y(t)|^2 + |z(t)|^2) dx.$$

Since y, $y_t \in C([0,T];L^2(\Omega))$ and $y(0)=0$, we obtain

$$\int_\Omega |y(t)|^2 dx = \int_\Omega \left| \int_0^t y_t(\tau) d\tau \right|^2 dx \leq t \int_0^t \|y_t(\tau)\|_2^2 \, d\tau \leq 2T \int_0^t \tilde{\mathscr{E}}(\tau) d\tau. \quad (2.3.19)$$

Likewise, $\int_\Omega |z(t)|^2 dx \leq 2T \int_0^t \tilde{\mathscr{E}}(\tau) d\tau$. Therefore,

$$I_1 \leq 4T \int_0^t \tilde{\mathscr{E}}(\tau) d\tau. \quad (2.3.20)$$

2. Estimate for

$$I_2 = \int_0^t \int_\Omega (y^2 + z^2)(|u_t| + |v_t| + |\hat{u}_t| + |\hat{v}_t|) dx d\tau.$$

A typical term in I_2 is estimated as follows. By using Hölder's inequality and the imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we have

$$\begin{aligned} \int_0^t \int_\Omega y^2 |u_t| dx d\tau &\leq \int_0^t \|y\|_6^2 \, \|u_t\|_{3/2} \, d\tau \\ &\leq C \int_0^t \|y\|_{1,\Omega}^2 \, \|u_t\|_2 \, d\tau \leq C(R) \int_0^t \tilde{\mathscr{E}}(\tau) d\tau, \end{aligned} \quad (2.3.21)$$

where we have used the fact $\|u_t(t)\|_2 \leq R$ for all $t \in [0,T]$ (see (2.3.2)). Therefore,

$$I_2 \leq C(R) \int_0^t \tilde{\mathscr{E}}(\tau) d\tau. \quad (2.3.22)$$

3. Estimate for

$$I_3 = \int_0^t \int_\Omega (|u|^{p-2} + |\hat{u}|^{p-2} + |v|^{p-2} + |\hat{v}|^{p-2})(y^2 + z^2)(|u_t| + |v_t| + |\hat{u}_t| + |\hat{v}_t|) dx d\tau.$$

A typical term in I_3 is estimated as follows. Recall the assumption $p\frac{m+1}{m} < 6$ which implies $\frac{6}{6-p} < m+1$. Thus, by using Hölder's inequality and (2.3.2), one has

$$\begin{aligned} &\int_0^t \int_\Omega |u|^{p-2} y^2 |u_t| dx d\tau \leq \int_0^t \|u\|_6^{p-2} \, \|y\|_6^2 \, \|u_t\|_{\frac{6}{6-p}} \, d\tau \\ &\leq C \int_0^t \|u\|_{1,\Omega}^{p-2} \, \|y\|_{1,\Omega}^2 \, \|u_t\|_{m+1} \, d\tau \leq C(R) \int_0^t \tilde{\mathscr{E}}(\tau) \, \|u_t\|_{m+1} \, d\tau. \end{aligned} \quad (2.3.23)$$

Therefore,

$$I_3 \leq C(R) \int_0^t \tilde{\mathscr{E}}(\tau) \Big(\|u_t\|_{m+1} + \|v_t\|_{r+1} + \|\hat{u}_t\|_{m+1} + \|\hat{v}_t\|_{r+1} \Big) d\tau. \tag{2.3.24}$$

4. Estimate for

$$I_4 = \int_\Omega (|u(t)|^{p-1} + |\hat{u}(t)|^{p-1} + |v(t)|^{p-1} + |\hat{v}(t)|^{p-1})(|y(t)|^2 + |z(t)|^2) dx.$$

Estimating I_4 is quite involved. We focus on the typical term $\int_\Omega |u(t)|^{p-1}|y(t)|^2 dx$ in the following two cases for the exponent $p \in (3, 6)$.

Case 1: $3 < p < 5$. In this case, we have

$$\int_\Omega |u(t)|^{p-1}|y(t)|^2 dx \leq \int_\Omega |y(t)|^2 dx + \int_{\{x\in\Omega:|u(t)|>1\}} |u(t)|^{p-1}|y(t)|^2 dx. \tag{2.3.25}$$

The first term on the right-hand side of (2.3.25) has been already estimated in (2.3.19). For the second term, we notice if $0 < \epsilon < 5 - p$, then $|u(t)|^{p-1} \leq |u(t)|^{4-\epsilon}$, since $|u(t)| > 1$. Again, by using Hölder's inequality, (2.3.2), (1.2.1), and (1.2.2), it follows that

$$\begin{aligned} \int_{\{x\in\Omega:|u(t)|>1\}} |u(t)|^{p-1}|y(t)|^2 dx &\leq \int_\Omega |u(t)|^{4-\epsilon}|y(t)|^2 dx \leq \|u(t)\|_6^{4-\epsilon} \|y(t)\|^2_{\frac{6}{1+\epsilon/2}} \\ &\leq C \|u(t)\|_{1,\Omega}^{4-\epsilon} \|y(t)\|^2_{H^{1-\epsilon/4}(\Omega)} \\ &= C(R) \Big(\epsilon \|y(t)\|_{1,\Omega}^2 + C_\epsilon \|y(t)\|_2^2 \Big). \end{aligned} \tag{2.3.26}$$

By using (2.3.19) and (2.3.26), then from (2.3.25) it follows that

$$\int_\Omega |u(t)|^{p-1}|y(t)|^2 dx \leq C(R) \left(\epsilon \tilde{\mathscr{E}}(t) + C_\epsilon T \int_0^t \tilde{\mathscr{E}}(\tau) d\tau \right), \tag{2.3.27}$$

in the case $3 < p < 5$ and where $0 < \epsilon < 5 - p$.

Case 2: $5 \leq p < 6$. In this case, the assumption $p\frac{m+1}{m} < 6$ implies $m > 5$. Recall that in Theorem 1.3.4 we required a higher regularity of initial data u_0, v_0, namely,

$u_0, v_0 \in L^{\frac{3}{2}(p-1)}(\Omega)$. By density of $C_0(\Omega)$ in $L^{\frac{3}{2}(p-1)}(\Omega)$, then for any $\epsilon > 0$, there exists $\phi \in C_0(\Omega)$ such that $\|u_0 - \phi\|_{\frac{3}{2}(p-1)} < \epsilon^{\frac{1}{p-1}}$.

Now,

$$\begin{aligned}\int_\Omega |u(t)|^{p-1}|y(t)|^2 dx \leq C\int_\Omega |u(t)-u_0|^{p-1}|y(t)|^2 dx + C\int_\Omega |u_0-\phi|^{p-1}|y(t)|^2 dx \\ + C\int_\Omega |\phi|^{p-1}|y(t)|^2 dx. \qquad (2.3.28)\end{aligned}$$

Since $p < \frac{6m}{m+1}$ and $m > 5$, then $\frac{3(p-1)}{2(m+1)} < 1$. So, by using Hölder's inequality and the bound $\int_0^T \|u_t\|_{m+1}^{m+1} dt \leq R$, one has

$$\begin{aligned}\int_\Omega |u(t)-u_0|^{p-1}|y(t)|^2 dx &\leq \left(\int_\Omega |u(t)-u(0)|^{\frac{3(p-1)}{2}} dx\right)^{2/3} \|y(t)\|_6^2 \\ &\leq C\left(\int_\Omega \left|\int_0^t u_t(\tau)d\tau\right|^{\frac{3(p-1)}{2}} dx\right)^{2/3} \|y(t)\|_{1,\Omega}^2 \\ &\leq C\left[\int_\Omega \left(\int_0^t |u_t|^{m+1} d\tau\right)^{\frac{3(p-1)}{2(m+1)}} dx\right]^{2/3} T^{\frac{m(p-1)}{m+1}} \tilde{\mathscr{E}}(t) \\ &\leq C(R) T^{\frac{m(p-1)}{m+1}} \tilde{\mathscr{E}}(t), \qquad (2.3.29)\end{aligned}$$

where we have used the important fact that $\frac{3(p-1)}{2(m+1)} < 1$.

The second term on the right hand side of (2.3.28) is easily estimated as follows:

$$\int_\Omega |u_0-\phi|^{p-1}|y(t)|^2 dx \leq \|u_0-\phi\|_{\frac{3(p-1)}{2}}^{p-1} \|y(t)\|_6^2 \leq C\epsilon\tilde{\mathscr{E}}(t). \qquad (2.3.30)$$

Since $\phi \in C_0(\Omega)$ then $|\phi(x)| \leq C(\epsilon)$, for all $x \in \Omega$. So, by (2.3.19), the last term on the right hand side of (2.3.28) is estimated as follows:

$$\int_\Omega |\phi|^{p-1}|y(t)|^2 dx \leq C(\epsilon)\int_\Omega |y(t)|^2 dx \leq C(\epsilon, T)\int_0^t \tilde{\mathscr{E}}(\tau)d\tau. \qquad (2.3.31)$$

By combining (2.3.29)-(2.3.31) then (2.3.28) yields

$$\int_\Omega |u(t)|^{p-1}|y(t)|^2 dx \leq C(R)\left(T^{\frac{m(p-1)}{m+1}} + \epsilon\right)\tilde{\mathscr{E}}(t) + C(\epsilon, T)\int_0^t \tilde{\mathscr{E}}(\tau)d\tau, \qquad (2.3.32)$$

in the case $5 \leq p < 6$.

By combining the estimates in (2.3.27) and (2.3.32), then for the case $3 < p < 6$, one has

$$\int_\Omega |u(t)|^{p-1}|y(t)|^2 dx \leq C(R)\left(T^{\frac{m(p-1)}{m+1}} + \epsilon\right)\tilde{\mathscr{E}}(t) + C(\epsilon, R, T)\int_0^t \tilde{\mathscr{E}}(\tau)d\tau, \quad (2.3.33)$$

where $\epsilon > 0$ such that $\epsilon < 5 - p$, if $3 < p < 5$.

The other terms in I_4 can be estimated in the same way, and we have

$$I_4 \leq C(R)\left(T^{\frac{m(p-1)}{m+1}} + T^{\frac{r(p-1)}{r+1}} + \epsilon\right)\tilde{\mathscr{E}}(t) + C(\epsilon, R, T)\int_0^t \tilde{\mathscr{E}}(\tau)d\tau. \quad (2.3.34)$$

Finally, by combining the estimates (2.3.20), (2.3.22), (2.3.24) and (2.3.34) back into (2.3.18), we obtain for $3 < p < 6$:

$$\begin{aligned} R_f &\leq C(R)\left(T^{\frac{m(p-1)}{m+1}} + T^{\frac{r(p-1)}{r+1}} + \epsilon\right)\tilde{\mathscr{E}}(t) \\ &+ C(\epsilon, R, T)\int_0^t \tilde{\mathscr{E}}(\tau)(\|u_t\|_{m+1} + \|v_t\|_{r+1} + \|\hat{u}_t\|_{m+1} + \|\hat{v}_t\|_{r+1} + 1)d\tau, \end{aligned} \quad (2.3.35)$$

where $\epsilon > 0$ is sufficiently small. According to (2.3.10), estimate (2.3.35) also holds for $1 \leq p \leq 3$, i.e., (2.3.35) holds for all $1 \leq p < 6$.

Step 3: Estimate for

$$R_h = \int_0^t \int_\Gamma (h(\gamma u) - h(\gamma \hat{u}))\gamma y_t d\Gamma d\tau.$$

First, we consider the subcritical case: $1 \leq k < 2$. Although, in this case, h is locally Lipschitz from $H^1(\Omega)$ into $L^2(\Gamma)$, we cannot estimate R_h by using the same method as we have done for R_f. More precisely, an estimate as in (2.3.9) won't work for R_h, because the energy $\tilde{\mathscr{E}}$ does not control the boundary trace γy_t.

In order to overcome this difficulty, we shall take advantage of the boundary damping term: $\int_0^t \int_\Gamma (g(\gamma u_t) - g(\gamma \hat{u}_t))\gamma y_t d\Gamma d\tau$. It is here where the strong monotonicity condition imposed on g in Assumption 1.3.3 is critical. Namely, the assumption that: there exists $m_g > 0$ such that $(g(s_1) - g(s_2))(s_1 - s_2) \geq m_g |s_1 - s_2|^2$. Now, by recalling $y = u - \hat{u}$, we have

$$\int_0^t \int_\Gamma (g(\gamma u_t) - g(\gamma \hat{u}_t))\gamma y_t d\Gamma d\tau \geq m_g \int_0^t \int_\Gamma |\gamma y_t|^2 d\Gamma d\tau. \quad (2.3.36)$$

To estimate R_h, we employ Hölder's inequality followed by Young's inequality, and the fact that h is locally Lipschitz from $H^1(\Omega)$ into $L^2(\Gamma)$ when $1 \le k < 2$ (see Remark 2.1.8). Thus,

$$\begin{aligned}
R_h &\le \left(\int_0^t \int_\Gamma |h(\gamma u) - h(\gamma \hat{u})|^2 d\Gamma d\tau\right)^{\frac{1}{2}} \left(\int_0^t \int_\Gamma |\gamma y_t|^2 d\Gamma d\tau\right)^{\frac{1}{2}} \\
&\le C(R) \left(\int_0^t \|y\|_{1,\Omega}^2 \, d\tau\right)^{\frac{1}{2}} \left(\int_0^t \int_\Gamma |\gamma y_t|^2 d\Gamma d\tau\right)^{\frac{1}{2}} \\
&\le \epsilon \int_0^t \int_\Gamma |\gamma y_t|^2 d\Gamma d\tau + C(R,\epsilon) \int_0^t \tilde{\mathscr{E}}(\tau) d\tau. \qquad (2.3.37)
\end{aligned}$$

Therefore, if we choose $\epsilon \le m_g$, then by (2.3.36) and (2.3.37), we obtain for $1 \le k < 2$:

$$R_h - \int_0^t \int_\Gamma (g(\gamma u_t) - g(\gamma \hat{u}_t))\gamma y_t \le C(R,\epsilon) \int_0^t \tilde{\mathscr{E}}(\tau) d\tau. \qquad (2.3.38)$$

Next, we consider the case $2 \le k < 4$. In this case, we need the extra assumption $h \in C^2(\mathbb{R})$ such that $h''(s) \le C(|s|^{k-2} + 1)$, which implies:

$$\begin{cases}
|h'(s)| \le C(|s|^{k-1} + 1), \quad |h(s)| \le C(|s|^k + 1), \\
|h'(u) - h'(\hat{u})| \le C(|u|^{k-2} + |\hat{u}|^{k-2} + 1)|y|, \\
|h(u) - h(\hat{u})| \le C(|u|^{k-1} + |\hat{u}|^{k-1} + 1)|y|,
\end{cases} \qquad (2.3.39)$$

where $y = u - \hat{u}$.

To evaluate R_h, integrate by parts twice with respect to time, employ (2.3.39) and the fact $y(0) = 0$, to obtain

$$\begin{aligned}
R_h &\le \left|\int_\Gamma [h(\gamma u(t)) - h(\gamma \hat{u}(t))]\gamma y(t) d\Gamma\right| + \left|\int_0^t \int_\Gamma [h'(\gamma u)\gamma u_t - h'(\gamma \hat{u})\gamma \hat{u}_t]\gamma y d\Gamma d\tau\right| \\
&\le \left|\int_\Gamma [h(\gamma u(t)) - h(\gamma \hat{u}(t))]\gamma y(t) d\Gamma\right| + \left|\int_0^t \int_\Gamma [h'(\gamma u) - h'(\gamma \hat{u})]\gamma \hat{u}_t \gamma y d\Gamma d\tau\right| \\
&+ \frac{1}{2}\left|\int_\Gamma h'(\gamma u(t))(\gamma y(t))^2 d\Gamma\right| + \frac{1}{2}\left|\int_0^t \int_\Gamma h''(\gamma u)\gamma u_t (\gamma y)^2 d\Gamma d\tau\right| \\
&\le I_5 + I_6 + I_7 + I_8 \qquad (2.3.40)
\end{aligned}$$

where,

$$
\begin{aligned}
I_5 &= C\int_\Gamma |\gamma y(t)|^2 d\Gamma;\\
I_6 &= C\int_0^t\int_\Gamma (|\gamma u_t| + |\gamma \hat{u}_t|)|\gamma y|^2 d\Gamma d\tau;\\
I_7 &= C\int_0^t\int_\Gamma (|\gamma u|^{k-2} + |\gamma \hat{u}|^{k-2})(|\gamma u_t| + |\gamma \hat{u}_t|)|\gamma y|^2 d\Gamma d\tau;\\
I_8 &= C\int_\Gamma (|\gamma u(t)|^{k-1} + |\gamma \hat{u}(t)|^{k-1})|\gamma y(t)|^2 d\Gamma.
\end{aligned}
$$

Since $y(t) \in H^1(\Omega)$, then I_5 is estimated easily as follows:

$$
\begin{aligned}
I_5 = |\gamma y(t)|_2^2 \leq C\,\|y(t)\|^2_{H^{\frac{1}{2}}(\Omega)} &\leq \epsilon\,\|y(t)\|^2_{1,\Omega} + C_\epsilon\,\|y(t)\|_2^2\\
&\leq 2\epsilon\tilde{\mathscr{E}}(t) + C_\epsilon T\int_0^t \tilde{\mathscr{E}}(\tau)d\tau, \qquad (2.3.41)
\end{aligned}
$$

where we have used (1.2.1), (1.2.2) and (2.3.19).

Since $q \geq 1$ and $H^1(\Omega) \hookrightarrow L^4(\Gamma)$, then I_6 is estimated by:

$$
I_6 \leq C\int_0^t (|\gamma u_t|_2 + |\gamma \hat{u}_t|_2)|\gamma y|_4^2 d\tau \leq C\int_0^t (|\gamma u_t|_{q+1} + |\gamma \hat{u}_t|_{q+1})\tilde{\mathscr{E}}(\tau)d\tau. \qquad (2.3.42)
$$

In I_7 we focus on the typical term: $\int_0^t\int_\Gamma |\gamma u|^{k-2}|\gamma u_t||\gamma y|^2 d\Gamma d\tau$. Notice, the assumption $k\frac{q+1}{q} < 4$ implies $\frac{4}{4-k} < q+1$. Therefore,

$$
\begin{aligned}
&\int_0^t\int_\Gamma |\gamma u|^{k-2}|\gamma u_t||\gamma y|^2 d\Gamma d\tau \leq \int_0^t |\gamma u|_4^{k-2}|\gamma u_t|_{\frac{4}{4-k}}|\gamma y|_4^2 d\tau\\
&\leq C\int_0^t \|u\|^{k-2}_{1,\Omega}\,|\gamma u_t|_{q+1}\,\|y\|^2_{1,\Omega}\,d\tau \leq C(R)\int_0^t |\gamma u_t|_{q+1}\tilde{\mathscr{E}}(\tau)d\tau, \qquad (2.3.43)
\end{aligned}
$$

where we have used (2.3.2). The other terms in I_7 can be estimated in the same manner, thus

$$
I_7 \leq C(R)\int_0^t (|\gamma u_t|_{q+1} + |\gamma \hat{u}_t|_{q+1})\tilde{\mathscr{E}}(\tau)d\tau. \qquad (2.3.44)
$$

Finally, we estimate I_8 by focusing on the typical term: $\int_\Gamma |\gamma u(t)|^{k-1}|\gamma y(t)|^2 d\Gamma$. We consider the following two cases for the exponent $k \in [2,4)$.

Case 1: $2 \le k < 3$. First, we note that

$$\int_\Gamma |\gamma u(t)|^{k-1}|\gamma y(t)|^2 d\Gamma \le \int_\Gamma |\gamma y(t)|^2 d\Gamma + \int_{\{x\in\Gamma:|\gamma u(t)|>1\}} |\gamma u(t)|^{k-1}|\gamma y(t)|^2 d\Gamma. \tag{2.3.45}$$

The first term on the right-hand side of (2.3.45) has been estimated in (2.3.41). As for the second term, we choose $0 < \epsilon < 3-k$, and so, $k-1 < 2-\epsilon$. By using Hölder's inequality, (1.2.1) and (1.2.2), we obtain

$$\begin{aligned}\int_{\{x\in\Gamma:|\gamma u(t)|>1\}} |\gamma u(t)|^{k-1}|\gamma y(t)|^2 d\Gamma &\le \int_\Gamma |\gamma u(t)|^{2-\epsilon}|\gamma y(t)|^2 d\Gamma \\ &\le |\gamma u(t)|_4^{2-\epsilon}|\gamma y(t)|^2_{\frac{4}{1+\epsilon/2}} \\ &\le C\,\|u(t)\|_{1,\Omega}^{2-\epsilon}\,\|y(t)\|^2_{H^{1-\epsilon/4}(\Omega)} \\ &\le C(R)(\epsilon\,\|y(t)\|^2_{1,\Omega} + C_\epsilon\,\|y(t)\|_2^2).\end{aligned} \tag{2.3.46}$$

Therefore, by using the estimates (2.3.46), (2.3.41) and (2.3.19), we obtain from (2.3.45) that

$$\int_\Gamma |\gamma u(t)|^{k-1}|\gamma y(t)|^2 d\Gamma \le C(R)\left(\epsilon\tilde{\mathscr{E}}(t) + C_\epsilon T\int_0^t \tilde{\mathscr{E}}(\tau)d\tau\right) \tag{2.3.47}$$

for the case $2 < k < 3$, and where $0 < \epsilon < 3-k$.

Case 2: $3 \le k < 4$. Observe that, in this case, the assumption $k\frac{q+1}{q} < 4$ implies $q > 3$. Also, recall that in Theorem 1.3.4 we required the extra assumption: $\gamma u_0 \in L^{2(k-1)}(\Gamma)$.

By density of $C(\Gamma)$ in $L^{2(k-1)}(\Gamma)$, for any $\epsilon > 0$, there exists $\psi \in C(\Gamma)$ such that $|\gamma u_0 - \psi|_{2(k-1)} \le \epsilon^{\frac{1}{k-1}}$. Note that,

$$\begin{aligned}\int_\Gamma |\gamma u(t)|^{k-1}|\gamma y(t)|^2 d\Gamma &\le C\int_\Gamma |\gamma u(t) - \gamma u_0|^{k-1}|\gamma y(t)|^2 d\Gamma \\ &\quad + C\int_\Gamma |\gamma u_0 - \psi|^{k-1}|\gamma y(t)|^2 d\Gamma + C\int_\Gamma |\psi|^{k-1}|\gamma y(t)|^2 d\Gamma.\end{aligned} \tag{2.3.48}$$

Since $k < \frac{4q}{q+1}$ and $q > 3$, then $\frac{2(k-1)}{q+1} < 1$. Therefore, by using (2.3.2), we have

$$\int_\Gamma |\gamma u(t) - \gamma u_0|^{k-1} |\gamma y(t)|^2 d\Gamma \leq \left(\int_\Gamma \left| \int_0^t \gamma u_t(\tau) d\tau \right|^{2(k-1)} d\Gamma \right)^{\frac{1}{2}} \left(\int_\Gamma |\gamma y(t)|^4 d\Gamma \right)^{\frac{1}{2}}$$

$$\leq C \Big(\int_\Gamma \left| \int_0^t |\gamma u_t(\tau)|^{q+1} d\tau \right|^{\frac{2(k-1)}{q+1}} d\Gamma \Big)^{\frac{1}{2}} T^{\frac{q(k-1)}{q+1}} \|y(t)\|^2_{1,\Omega} \leq C(R) T^{\frac{q(k-1)}{q+1}} \tilde{\mathscr{E}}(t). \quad (2.3.49)$$

The second term on the right-hand side of (2.3.48) is estimated by:

$$\int_\Gamma |\gamma u_0 - \psi|^{k-1} |\gamma y(t)|^2 d\Gamma \leq |\gamma u_0 - \psi|^{k-1}_{2(k-1)} |\gamma y(t)|^2_4 \leq C\epsilon \|y(t)\|^2_{1,\Omega} \leq C\epsilon \tilde{\mathscr{E}}(t). \quad (2.3.50)$$

Finally, we estimate the last term on the right-hand side of (2.3.48). Since $\psi \in C(\Gamma)$, then $|\psi(x)| \leq C(\epsilon)$, for all $x \in \Gamma$. It follows from (2.3.41) that,

$$\begin{aligned} \int_\Gamma |\psi|^{k-1} |\gamma y(t)|^2 d\Gamma &\leq C(\epsilon) \int_\Gamma |\gamma y(t)|^2 d\Gamma \\ &\leq \epsilon C(\epsilon) \tilde{\mathscr{E}}(t) + C(\epsilon, T) \int_0^t \tilde{\mathscr{E}}(\tau) d\tau. \end{aligned} \quad (2.3.51)$$

Now, (2.3.49)-(2.3.51) and (2.3.48) yield

$$\int_\Gamma |\gamma u(t)|^{k-1} |\gamma y(t)|^2 d\Gamma \leq C(R,\epsilon)(T^{\frac{q(k-1)}{q+1}} + \epsilon) \tilde{\mathscr{E}}(t) + C(\epsilon, T) \int_0^t \tilde{\mathscr{E}}(\tau) d\tau, \quad (2.3.52)$$

for the case $3 \leq k < 4$. It is easy to see that the other term in I_8 has the same estimate as (2.3.47) and (2.3.52). So, we may conclude that for $2 < k < 4$ and sufficiently small $\epsilon > 0$:

$$I_8 \leq C(R,\epsilon)(T^{\frac{q(k-1)}{q+1}} + \epsilon) \tilde{\mathscr{E}}(t) + C(\epsilon, R, T) \int_0^t \tilde{\mathscr{E}}(\tau) d\tau. \quad (2.3.53)$$

Combine (2.3.41), (2.3.42), (2.3.44) and (2.3.53) back into (2.3.40) to obtain the following estimate for R_h in the case $2 \leq k < 4$:

$$\begin{aligned} R_h &\leq C(R,\epsilon)(T^{\frac{q(k-1)}{q+1}} + \epsilon) \tilde{\mathscr{E}}(t) \\ &\quad + C(\epsilon, R, T) \int_0^t (|\gamma u_t|_{q+1} + |\gamma \hat{u}_t|_{q+1} + 1) \tilde{\mathscr{E}}(\tau) d\tau, \end{aligned} \quad (2.3.54)$$

where $\epsilon > 0$ is sufficiently small.

Step 4: Completion of the proof.

By the estimates (2.3.35), (2.3.38), (2.3.54) and employing the monotonicity property of g, we obtain from (2.3.7) that

$$\begin{aligned}\tilde{\mathscr{E}}(t) \leq C(R)\left(T^{\frac{m(p-1)}{m+1}} + T^{\frac{r(p-1)}{r+1}} + T^{\frac{q(k-1)}{q+1}} + \epsilon\right)\tilde{\mathscr{E}}(t) \\ + C(\epsilon, R, T)\int_0^t \tilde{\mathscr{E}}(\tau)\Big(\|u_t\|_{m+1} + \|v_t\|_{r+1} + \|\hat{u}_t\|_{m+1} + \|\hat{v}_t\|_{r+1} \\ + |\gamma u_t|_{q+1} + |\gamma \hat{u}_t|_{q+1} + 1\Big)d\tau,\end{aligned}$$

for all $t \in [0, T]$. Choose ϵ and T small enough so that

$$C(R)\left(T^{\frac{m(p-1)}{m+1}} + T^{\frac{r(p-1)}{r+1}} + T^{\frac{q(k-1)}{q+1}} + \epsilon\right) < 1.$$

By applying Gronwall's inequality with an L^1-kernel, it follows that $\tilde{\mathscr{E}}(t) = 0$ on $[0, T]$. Hence, $y(t) = z(t) = 0$ on $[0, T]$. Finally we note that, it is sufficient to consider a small time interval $[0, T]$, since this process can be reiterated. The proof of Theorem 1.3.4 is now complete.

2.3.2 Proof of Theorem 1.3.6.

We begin by pointing out that the only difference between Theorem 1.3.6 and Theorem 1.3.4 is that Assumption 1.3.3 (a) is not imposed in Theorem 1.3.6. Thus, the proof of Theorem 1.3.6 is essentially the same as Theorem 1.3.4, with the exception of the estimate for R_f in (2.3.8). So, we focus on estimating R_f in the case $p > 3$ and the interior sources f_1, f_2 are not necessarily C^2-functions. With this scenario in place, the method of integration by parts twice fails. To handle this difficulty, recall the additional restriction on parameters and the initial data in Theorem 1.3.6, namely, $m, r \geq 3p - 4$, if $p > 3$, and u_0, $v_0 \in L^{3(p-1)}(\Omega)$.

Now, since $|\nabla f_1(u, v)| \leq C(|u|^{p-1} + |v|^{p-1} + 1)$, then by the mean value theorem,

$$|f_1(u, v) - f_1(\hat{u}, \hat{v})| \leq C(|u|^{p-1} + |\hat{u}|^{p-1} + |v|^{p-1} + |\hat{v}|^{p-1} + 1)(|y| + |z|) \tag{2.3.55}$$

where $y = u - \hat{u}$ and $z = v - \hat{v}$. Thus,

$$\int_0^t \int_\Omega (f_1(u, v) - f_1(\hat{u}, \hat{v}))y_t dx d\tau \leq I_1 + I_2 \tag{2.3.56}$$

where

$$I_1 = C\int_0^t \int_\Omega (|y|+|z|)|y_t|dxd\tau;$$
$$I_2 = C\int_0^t \int_\Omega (|u|^{p-1}+|\hat{u}|^{p-1}+|v|^{p-1}+|\hat{v}|^{p-1})(|y|+|z|)|y_t|dxd\tau.$$

The estimate for I_1 is straightforward. Invoking Hölder's inequality yields,

$$I_1 \le C\int_0^t (\|y\|_6+\|z\|_6)\,\|y_t\|_2\,d\tau \le C\int_0^t \tilde{\mathscr{E}}(\tau)^{\frac{1}{2}}\tilde{\mathscr{E}}(\tau)^{\frac{1}{2}}d\tau = C\int_0^t \tilde{\mathscr{E}}(\tau)d\tau. \quad (2.3.57)$$

A typical term in I_2 is estimated as follows:

$$\begin{aligned}&\int_0^t \int_\Omega |u|^{p-1}|y||y_t|dxd\tau\\ &\le C\int_0^t \int_\Omega |u-u_0|^{p-1}|y||y_t|dxd\tau + C\int_0^t\int_\Omega |u_0|^{p-1}|y||y_t|dxd\tau.\end{aligned} \quad (2.3.58)$$

By Hölder's inequality,

$$\begin{aligned}&\int_0^t \int_\Omega |u-u_0|^{p-1}|y||y_t|dxd\tau\\ &\le \int_0^t \left(\int_\Omega |u(\tau)-u_0|^{3(p-1)}dx\right)^{\frac{1}{3}}\left(\int_\Omega |y(\tau)|^6dx\right)^{\frac{1}{6}}\left(\int_\Omega |y_t(\tau)|^2dx\right)^{\frac{1}{2}}d\tau.\end{aligned} \quad (2.3.59)$$

Since u, $u_t \in C([0,T];L^2(\Omega))$, we can write

$$\begin{aligned}\int_\Omega |u(\tau)-u_0|^{3(p-1)}dx &= \int_\Omega \left|\int_0^\tau u_t(s)ds\right|^{3(p-1)}dx\\ &\le C(T)\int_\Omega \left(\int_0^\tau |u_t(s)|^{m+1}ds\right)^{\frac{3(p-1)}{m+1}}dx.\end{aligned} \quad (2.3.60)$$

Since $m \ge 3p-4$, then $\frac{3(p-1)}{m+1} \le 1$. Therefore, by using Hölder's inequality and (2.3.2), it follows that

$$\int_\Omega |u(\tau)-u_0|^{3(p-1)}dx \le C(T)\left(\int_\Omega\int_0^\tau |u_t(s)|^{m+1}dsdx\right)^{\frac{3(p-1)}{m+1}} \le C(R,T). \quad (2.3.61)$$

So, (2.3.61) and (2.3.59) yield

$$\int_0^t\int_\Omega |u-u_0|^{p-1}|y||y_t|dxd\tau \le C(R,T)\int_0^t \|y(\tau)\|_6\,\|y_t(\tau)\|_2\,d\tau$$
$$\le C(R,T)\int_0^t \tilde{\mathscr{E}}(\tau)^{\frac{1}{2}}\tilde{\mathscr{E}}(\tau)^{\frac{1}{2}}d\tau = C(R,T)\int_0^t \tilde{\mathscr{E}}(\tau)d\tau. \tag{2.3.62}$$

By recalling the assumption $u_0 \in L^{3(p-1)}(\Omega)$, then the second term on the right hand side of (2.3.58) is estimated by:

$$\int_0^t\int_\Omega |u_0|^{p-1}|y||y_t|dxd\tau \le \int_0^t \|u_0\|_{3(p-1)}^{p-1}\|y(\tau)\|_6\,\|y_t(\tau)\|_2\,d\tau$$
$$\le C\,\|u_0\|_{3(p-1)}^{p-1}\int_0^t \tilde{\mathscr{E}}(\tau)d\tau. \tag{2.3.63}$$

Combining (2.3.62) and (2.3.63) back into (2.3.58) yields

$$\int_0^t\int_\Omega |u|^{p-1}|y||y_t|dxd\tau \le C\Big(R,T,\|u_0\|_{3(p-1)}\Big)\int_0^t \tilde{\mathscr{E}}(\tau)d\tau. \tag{2.3.64}$$

The other terms in I_2 are estimated in the same manner, and one has

$$I_2 \le C\Big(R,T,\|u_0\|_{3(p-1)},\|v_0\|_{3(p-1)}\Big)\int_0^t \tilde{\mathscr{E}}(\tau)d\tau. \tag{2.3.65}$$

Hence, (2.3.57), (2.3.65), and (2.3.56) yield

$$\int_0^t\int_\Omega (f_1(u,v)-f_1(\hat{u},\hat{v}))y_t dxd\tau$$
$$\le C\Big(R,T,\|u_0\|_{3(p-1)},\|v_0\|_{3(p-1)}\Big)\int_0^t \tilde{\mathscr{E}}(\tau)d\tau. \tag{2.3.66}$$

It is clear that $\int_0^t\int_\Omega (f_2(u,v)-f_2(\hat{u},\hat{v}))z_t dxd\tau$ has the same estimate as in (2.3.66). Finally, we may use the same argument as Step 3 and Step 4 in the proof of Theorem 1.3.4 and complete the proof of Theorem 1.3.6.

2.4 Global Existence

This section is devoted to prove the existence of global solutions (Theorem 1.3.7). Here, we apply a standard continuation procedure for ODE's to conclude that either the weak solution (u,v) is global or there exists $0<T<\infty$ such that $\limsup_{t\longrightarrow T^-} E_1(t) = +\infty$ where $E_1(t)$ is the modified energy defined by

$$\begin{aligned}E_1(t) :=& \frac{1}{2}(\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2 + \|u_t(t)\|_2^2 + \|v_t(t)\|_2^2) \\ &+ \frac{1}{p+1}(\|u(t)\|_{p+1}^{p+1} + \|v(t)\|_{p+1}^{p+1}) + \frac{1}{k+1}|\gamma u(t)|_{k+1}^{k+1}.\end{aligned} \tag{2.4.1}$$

We aim to show that the latter cannot happen under the assumptions of Theorem 1.3.7. Indeed, this assertion is contained in the following proposition.

Proposition 2.4.1. *Let (u,v) be a weak solution of (1.1.1) on $[0,T_0]$ as furnished by Theorem 1.3.2. Assume u_0, $v_0 \in L^{p+1}(\Omega)$, if $p>5$, and $\gamma u_0 \in L^{k+1}(\Gamma)$, if $k>3$. We have:*

- *If $p \le \min\{m,r\}$ and $k \le q$, then for all $t\in[0,T_0]$, (u,v) satisfies*

$$E_1(t) + \int_0^t \left(\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1}\right) d\tau \le C(T_0, E_1(0)), \tag{2.4.2}$$

  where $T_0>0$ is being arbitrary.

- *If $p>\min\{m,r\}$ or $k>q$, then the bound in (2.4.2) holds for $0\le t<T'$, for some $T'>0$ depending on $E_1(0)$ and T_0.*

Proof. With the modified energy as given in (2.4.1), the energy identity (1.3.4) yields,

$$\begin{aligned}&E_1(t) + \int_0^t\int_\Omega [g_1(u_t)u_t + g_2(v_t)v_t]\,dxd\tau + \int_0^t\int_\Gamma g(\gamma u_t)\gamma u_t d\Gamma d\tau \\ &= E_1(0) + \int_0^t\int_\Omega [f_1(u,v)u_t + f_2(u,v)v_t]\,dxd\tau + \int_0^t\int_\Gamma h(\gamma u)\gamma u_t d\Gamma d\tau \\ &\quad + \frac{1}{p+1}\int_\Omega (|u(t)|^{p+1} - |u(0)|^{p+1} + |v(t)|^{p+1} - |v(0)|^{p+1})dx \\ &\quad + \frac{1}{k+1}\int_\Gamma (|\gamma u(t)|^{k+1} - |\gamma u(0)|^{k+1})d\Gamma \\ &= E_1(0) + \int_0^t\int_\Omega [f_1(u,v)u_t + f_2(u,v)v_t]\,dxd\tau + \int_0^t\int_\Gamma h(\gamma u)\gamma u_t d\Gamma d\tau \\ &\quad + \int_0^t\int_\Omega (|u|^{p-1}uu_t + |v|^{p-1}vv_t)dxd\tau + \int_0^t\int_\Gamma |\gamma u|^{k-1}\gamma u\gamma u_t d\Gamma d\tau.\end{aligned} \tag{2.4.3}$$

To estimate the source terms on the right-hand side of (2.4.3), we recall the assumptions: $|h(s)| \leq C(|s|^k + 1)$, $|f_j(u,v)| \leq C(|u|^p + |v|^p + 1)$, $j = 1, 2$. So, by employing Hölder's and Young's inequalities, we find

$$\begin{aligned}
&\left|\int_0^t \int_\Omega f_1(u,v) u_t dx d\tau\right| \leq C \int_0^t \int_\Omega (|u|^p + |v|^p + 1)|u_t| dx d\tau \\
&\leq C \int_0^t \|u_t\|_{p+1} \left(\|u\|_{p+1}^p + \|v\|_{p+1}^p + |\Omega|^{\frac{p}{p+1}}\right) d\tau \\
&\leq \epsilon \int_0^t \|u_t\|_{p+1}^{p+1} d\tau + C_\epsilon \int_0^t \left(\|u\|_{p+1}^{p+1} + \|v\|_{p+1}^{p+1} + |\Omega|\right) d\tau \\
&\leq \epsilon \int_0^t \|u_t\|_{p+1}^{p+1} d\tau + C_\epsilon \int_0^t E_1(\tau) d\tau + C_\epsilon T_0 |\Omega|. \qquad (2.4.4)
\end{aligned}$$

Similarly, we deduce

$$\left|\int_0^t \int_\Omega f_2(u,v) v_t dx d\tau\right| \leq \epsilon \int_0^t \|v_t\|_{p+1}^{p+1} d\tau + C_\epsilon \int_0^t E_1(\tau) d\tau + C_\epsilon T_0 |\Omega|, \qquad (2.4.5)$$

and

$$\left|\int_0^t \int_\Gamma h(\gamma u) \gamma u_t\right| \leq \epsilon \int_0^t |\gamma u_t|_{k+1}^{k+1} d\tau + C_\epsilon \int_0^t E_1(\tau) d\tau + C_\epsilon T_0 |\Gamma|. \qquad (2.4.6)$$

By adopting similar estimates as in (2.4.4), we obtain

$$\begin{aligned}
&\left|\int_0^t \int_\Omega (|u|^{p-1} u u_t + |v|^{p-1} v v_t) dx d\tau + \int_0^t \int_\Gamma |\gamma u|^{k-1} \gamma u \gamma u_t d\Gamma d\tau\right| \\
&\leq \int_0^t \int_\Omega (|u|^p |u_t| + |v|^p |v_t|) dx d\tau + \int_0^t \int_\Gamma |\gamma u|^k |\gamma u_t| d\Gamma d\tau \\
&\leq \epsilon \int_0^t (\|u_t\|_{p+1}^{p+1} + \|v_t\|_{p+1}^{p+1} + |\gamma u_t|_{k+1}^{k+1}) d\tau + C_\epsilon \int_0^t E_1(\tau) d\tau. \qquad (2.4.7)
\end{aligned}$$

By recalling (2.1.34), one has

$$\begin{aligned}
&\int_0^t \int_\Omega \left[g_1(u_t) u_t + g_2(v_t) v_t\right] dx d\tau + \int_0^t \int_\Gamma g(\gamma u_t) \gamma u_t d\Gamma d\tau \\
&\qquad \geq \alpha \int_0^t (\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1}) d\tau - \alpha T_0 (2|\Omega| + |\Gamma|). \qquad (2.4.8)
\end{aligned}$$

Now, if $p \leq \min\{m, r\}$ and $k \leq q$, it follows from (2.4.4)-(2.4.8) and the energy identity (2.4.3) that, for $t \in [0, T_0]$,

$$\begin{aligned} &E_1(t) + \alpha \int_0^t (\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1}) d\tau \\ &\leq E_1(0) + \epsilon \int_0^t (\|u_t\|_{p+1}^{p+1} + \|v_t\|_{p+1}^{p+1} + |\gamma u_t|_{k+1}^{k+1}) d\tau + C_\epsilon \int_0^t E_1(\tau) d\tau + C_{T_0,\epsilon} \\ &\leq E_1(0) + \epsilon \int_0^t (\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1}) d\tau + C_\epsilon \int_0^t E_1(\tau) d\tau + C_{T_0,\epsilon}, \end{aligned} \tag{2.4.9}$$

where we have used Hölder's and Young's inequalities in the last line of (2.4.9). By choosing $0 < \epsilon \leq \alpha/2$, then (2.4.9) yields

$$\begin{aligned} E_1(t) + \frac{\alpha}{2} \int_0^t (\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1}) d\tau \\ \leq C_\epsilon \int_0^t E_1(\tau) d\tau + E_1(0) + C_{T_0,\epsilon}. \end{aligned} \tag{2.4.10}$$

In particular,

$$E_1(t) \leq C_\epsilon \int_0^t E_1(\tau) d\tau + E_1(0) + C_{T_0,\epsilon}. \tag{2.4.11}$$

By Gronwall's inequality, we conclude that

$$E_1(t) \leq (E_1(0) + C_{T_0,\epsilon}) e^{C_\epsilon T_0} \text{ for } t \in [0, T_0], \tag{2.4.12}$$

where $T_0 > 0$ is arbitrary, and by combining (2.4.10) and (2.4.12), the desired result in (2.4.2) follows.

Now, if $p > \min\{m, r\}$ or $k > q$, then we slightly modify estimate (2.4.4) by using different Hölder's conjugates. Specifically, we apply Hölder's inequality with $m+1$ and $\tilde{m} = \frac{m+1}{m}$ followed by Young's inequality to obtain

$$\begin{aligned} &\left| \int_0^t \int_\Omega f_1(u, v) u_t dx d\tau \right| \leq C \int_0^t \int_\Omega (|u|^p + |v|^p + 1)|u_t| dx d\tau \\ &\leq C \int_0^t \|u_t\|_{m+1} \left(\|u\|_{p\tilde{m}}^p + \|v\|_{p\tilde{m}}^p + |\Omega|^{1/\tilde{m}} \right) d\tau \\ &\leq \epsilon \int_0^t \|u_t\|_{m+1}^{m+1} d\tau + C_\epsilon \int_0^t \left(\|u\|_{p\tilde{m}}^{p\tilde{m}} + \|v\|_{p\tilde{m}}^{p\tilde{m}} + |\Omega| \right) d\tau. \end{aligned} \tag{2.4.13}$$

Since $p\tilde{m} < 6$ and $H^1(\Omega) \hookrightarrow Ł^6(\Omega)$, Then

$$\begin{aligned}\left|\int_0^t \int_\Omega f_1(u,v)u_t dx d\tau\right| &\leq \epsilon \int_0^t \|u_t\|_{m+1}^{m+1} d\tau + C_\epsilon \int_0^t \left(\|u\|_{1,\Omega}^{p\tilde{m}} + \|v\|_{1,\Omega}^{p\tilde{m}} + |\Omega|\right) d\tau \\ &\leq \epsilon \int_0^t \|u_t\|_{m+1}^{m+1} d\tau + C_\epsilon \int_0^t E_1(\tau)^{\frac{p\tilde{m}}{2}} d\tau + C_\epsilon T_0 |\Omega|. \quad (2.4.14)\end{aligned}$$

Likewise, we may deduce

$$\left|\int_0^t \int_\Omega f_2(u,v)v_t dx d\tau\right| \leq \epsilon \int_0^t \|v_t\|_{r+1}^{r+1} d\tau + C_\epsilon \int_0^t E_1(\tau)^{\frac{p\tilde{r}}{2}} d\tau + C_\epsilon T_0 |\Omega| \quad (2.4.15)$$

and

$$\left|\int_0^t \int_\Gamma h(\gamma u)\gamma u_t\right| \leq \epsilon \int_0^t |\gamma u_t|_{q+1}^{q+1} d\tau + C_\epsilon \int_0^t E_1(\tau)^{\frac{k\tilde{q}}{2}} d\tau + C_\epsilon T_0 |\Gamma|. \quad (2.4.16)$$

In addition, by employing similar estimates as in (2.4.13)-(2.4.14), we have

$$\begin{aligned}&\Big| \int_0^t \int_\Omega (|u|^{p-1}uu_t + |v|^{p-1}vv_t) dx d\tau + \int_0^t \int_\Gamma |\gamma u|^{k-1}\gamma u \gamma u_t d\Gamma d\tau \Big| \\ &\leq \int_0^t \int_\Omega (|u|^p |u_t| + |v|^p |v_t|) dx d\tau + \int_0^t \int_\Gamma |\gamma u|^k |\gamma u_t| d\Gamma d\tau \\ &\leq \epsilon \int_0^t (\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1}) d\tau \\ &+ C_\epsilon \int_0^t (E_1(\tau)^{\frac{p\tilde{m}}{2}} + E_1(\tau)^{\frac{p\tilde{r}}{2}} + E_1(\tau)^{\frac{k\tilde{q}}{2}}) d\tau. \quad (2.4.17)\end{aligned}$$

By using (2.4.14)-(2.4.17) along with (2.4.8), we obtain from the energy identity (2.4.3) that

$$\begin{aligned}&E_1(t) + \alpha \int_0^t (\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1}) d\tau \\ &\leq E_1(0) + \epsilon \int_0^t (\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1}) d\tau + C_\epsilon \int_0^t E_1(\tau)^\sigma d\tau + C_{T_0,\epsilon} \quad (2.4.18)\end{aligned}$$

where $\sigma = \max\{\frac{p\tilde{m}}{2}, \frac{p\tilde{r}}{2}, \frac{k\tilde{q}}{2}\} > 1$. Notice, the assumption $p > \min\{m, r\}$ or $k > q$, implies that $\sigma > 1$. By choosing $0 < \epsilon \leq \alpha/2$, then it follows that

$$\begin{aligned}&E_1(t) + \frac{\alpha}{2} \int_0^t (\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1}) d\tau \\ &\leq C_\epsilon \int_0^t E_1(\tau)^\sigma d\tau + E_1(0) + C_{T_0,\epsilon} \text{ for } t \in [0, T_0]. \quad (2.4.19)\end{aligned}$$

In particular,

$$E_1(t) \leq C_\epsilon \int_0^t E_1(\tau)^\sigma d\tau + E_1(0) + C_{T_0,\epsilon} \ \text{ for } \ t \in [0, T_0]. \tag{2.4.20}$$

By using a standard comparison theorem (see [29] for instance), then (2.4.20) yields that $E_1(t) \leq z(t)$, where $z(t) = [(E_1(0) + C_{T_0,\epsilon})^{1-\sigma} - C_\epsilon(\sigma - 1)t]^{-\frac{1}{\sigma-1}}$ is the solution of the Volterra integral equation

$$z(t) = C_\epsilon \int_0^t z(s)^\sigma ds + E_1(0) + C_{T_0,\epsilon}.$$

Since $\sigma > 1$, then clearly $z(t)$ blows up at the finite time $T_1 = \frac{1}{C_\epsilon(\sigma-1)}(E_1(0)+C_{T_0,\epsilon})^{1-\sigma}$, i.e., $z(t) \longrightarrow \infty$, as $t \longrightarrow T_1^-$. Note that T_1 depends on the initial energy $E_1(0)$ and the original existence time T_0. Nonetheless, if we choose $T' = \min\{T_0, \frac{1}{2}T_1\}$, then

$$E_1(t) \leq z(t) \leq C_0 := [(E_1(0) + C_{T_0,\epsilon})^{1-\sigma} - C_\epsilon(\sigma - 1)T']^{-\frac{1}{\sigma-1}}, \tag{2.4.21}$$

for all $t \in [0, T']$. Finally, we may combine (2.4.19) and (2.4.21) to obtain

$$E_1(t) + \frac{\alpha}{2} \int_0^t (\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1}) d\tau \leq C_\epsilon T' C_0^\sigma + E_1(0) + C_{T_0,\epsilon}, \tag{2.4.22}$$

for all $t \in [0, T']$, which completes the proof of the proposition. □

2.5 Continuous Dependence on Initial Data

In this section, we provide the proof of Theorem 1.3.8. The strategy here is to adopt the same argument as in the proof of Theorem 1.3.4 and use the bounds of Proposition 2.4.1.

Proof. Let $U_0 = (u_0, v_0, u_1, v_1) \in X$, where

$$X = \left(H^1(\Omega) \cap L^{\frac{3(p-1)}{2}}(\Omega)\right) \times \left(H_0^1(\Omega) \cap L^{\frac{3(p-1)}{2}}(\Omega)\right) \times L^2(\Omega) \times L^2(\Omega)$$

such that $\gamma u_0 \in L^{2(k-1)}(\Gamma)$. Assume that $\{U_0^n = (u_0^n, u_1^n, v_0^n, v_1^n)\}$ is a sequence of initial data that satisfies:

$$U_0^n \longrightarrow U_0 \ \ in \ \ X \ \text{ and } \gamma u_0^n \longrightarrow \gamma u_0 \ \ in \ \ L^{2(k-1)}(\Gamma), \ \text{as } n \longrightarrow \infty. \tag{2.5.1}$$

Notice that in Remark 1.3.9, we have pointed out that if $p \leq 5$, then the space X is identical to $H = H^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$, and if $k \leq 3$, then the assumption $\gamma u_0^n \longrightarrow \gamma u_0$ in $L^{2(k-1)}(\Gamma)$ is redundant.

Let $\{(u^n, v^n)\}$ and (u, v) be the unique weak solutions to (1.1.1) defined on $[0, T_0]$ in the sense of Definition 1.3.1, corresponding to the initial data $\{U_0^n\}$ and $\{U_0\}$, respectively. First, we show that the local existence time T_0 can be taken independent of $n \in \mathbb{N}$. To see this, we recall that the local existence time provided by Theorem 1.3.2 depends on the initial energy $E(0)$. In addition, since $U_0^n \longrightarrow U_0$ in X, then $u_0^n \longrightarrow u_0$, $v_0^n \longrightarrow v_0$ in $L^{p+1}(\Omega)$, if $p > 5$; and $\gamma u_0^n \longrightarrow \gamma u_0$ in $L^{k+1}(\Gamma)$, if $k > 3$. Hence, we may assume $E_1^n(0) \leq E_1(0) + 1$, for all $n \in \mathbb{N}$, where $E_1(t)$ is defined in (2.4.1) and $E_1^n(t)$ is defined by

$$E_1^n(t) := E^n(t) + \frac{1}{p+1}(\|u^n(t)\|_{p+1}^{p+1} + \|v^n(t)\|_{p+1}^{p+1}) + \frac{1}{k+1}|\gamma u^n(t)|_{k+1}^{k+1}$$

where $E^n(t) = \frac{1}{2}(\|u^n(t)\|_{1,\Omega}^2 + \|v^n(t)\|_{1,\Omega}^2 + \|u_t^n(t)\|_2^2 + \|v_t^n(t)\|_2^2)$. Therefore, we can choose K, as in (2.1.38), sufficiently large, say $K^2 \geq 4E_1(0) + 5$, then the local existence time T_0 for the solutions $\{(u^n, v^n)\}$ and (u, v) can be chosen independent of $n \in \mathbb{N}$. Moreover, in view of (2.4.2), T_0 can be taken arbitrarily large in the case when $p \leq \min\{m, r\}$ and $k \leq q$. However, in the case when $p > \min\{m, r\}$ or $k > q$, we select the local existence time to be $T = T'$ where T' is given in Proposition 2.4.1 (which is also uniform in n). In either case, it follows from (2.4.2) that there exists $R > 0$ such that, for all $n \in \mathbb{N}$ and all $t \in [0, T]$,

$$\begin{cases} E_1(t) + \int_0^t (\|u_t\|_{m+1}^{m+1} + \|v_t\|_{r+1}^{r+1} + |\gamma u_t|_{q+1}^{q+1}) d\tau \leq R, \\ E_1^n(t) + \int_0^t (\|u_t^n\|_{m+1}^{m+1} + \|v_t^n\|_{r+1}^{r+1} + |\gamma u_t^n|_{q+1}^{q+1}) d\tau \leq R, \end{cases} \tag{2.5.2}$$

where T can be arbitrarily large if $p \leq \min\{m, r\}$ and $k \leq q$, or T is sufficiently small if $p > \min\{m, r\}$ or $k > q$.

Now, put $y^n(t) = u(t) - u^n(t)$, $z^n(t) = v(t) - v^n(t)$, and

$$\tilde{\mathscr{E}}_n(t) = \frac{1}{2}(\|y^n(t)\|_{1,\Omega}^2 + \|z^n(t)\|_{1,\Omega}^2 + \|y_t^n(t)\|_2^2 + \|z_t^n(t)\|_2^2), \tag{2.5.3}$$

for $t \in [0, T]$. We aim to show $\tilde{\mathscr{E}}_n(t) \longrightarrow 0$ uniformly on $[0, T]$, for sufficiently small T.

We begin by following the proof of Theorem 1.3.4, where here u, v, u^n, v^n, y^n, z^n, $\tilde{\mathscr{E}}_n$ replace u, v, $\hat{u}$, $\hat{v}$, y, z, $\tilde{\mathscr{E}}$ in the proof of Theorem 1.3.4; respectively. However, due to having non-zero initial data, $y^n(0) = u_0 - u_0^n$ and $z^n(0) = v_0 - v_0^n$, we have to take care of the additional terms resulting from integration by parts.

First, as in (2.3.7), accounting for the non-zero initial data, we obtain the energy inequality

$$\tilde{\mathscr{E}}_n(t) \leq \tilde{\mathscr{E}}_n(0) + R_f^n + R_h^n, \tag{2.5.4}$$

where

$$R_f^n = \int_0^t \int_\Omega (f_1(u,v) - f_1(u^n, v^n)) y_t^n dx d\tau + \int_0^t \int_\Omega (f_2(u,v) - f_2(u^n, v^n)) z_t^n dx d\tau$$

and

$$R_h^n = \int_0^t \int_\Gamma (h(\gamma u) - h(\gamma u^n)) \gamma y_t^n d\Gamma d\tau.$$

As in Step 2 in the proof of Theorem 1.3.4, the estimate for R_f^n when $1 \leq p \leq 3$ is straightforward. Indeed, following (2.3.9)-(2.3.10), we find

$$R_f^n \leq C(R) \int_0^t \tilde{\mathscr{E}}_n(\tau) d\tau. \tag{2.5.5}$$

If $3 < p < 5$, we utilize Assumption 1.3.3 and integration by parts in (2.3.12)-(2.3.13) yields the additional terms:

$$Q_1 = \left| \int_\Omega (f_1(u_0, v_0) - f_1(u_0^n, v_0^n))\, y^n(0) dx \right| + \left| \int_\Omega (f_2(u_0, v_0) - f_2(u_0^n, v_0^n))\, z^n(0) dx \right|,$$

which must be added to the right-hand side of (2.3.13). Another place where we pick up additional non-zero terms is in (2.3.16), namely the terms:

$$Q_2 = \left| \int_\Omega \left(\frac{1}{2} \partial_u f_1(u_0, v_0) |y^n(0)|^2 + \partial_{uv}^2 F(u_0, v_0) y^n(0) z^n(0) + \frac{1}{2} \partial_v f_2(u_0, v_0) |z^n(0)|^2 \right) dx \right|$$

must be added to the right-hand side of (2.3.16).

By using (2.3.11), we deduce

$$\begin{aligned} &Q_1 + Q_2 \\ &\quad \leq C \int_\Omega (|u_0|^{p-1} + |u_0^n|^{p-1} + |v_0|^{p-1} + |v_0^n|^{p-1} + 1)(|y^n(0)|^2 + |z^n(0)|^2) dx. \end{aligned} \tag{2.5.6}$$

A typical term on the right-hand side of (2.5.6) is estimated in the following manner. By using Hölder's inequality and (2.5.2), we have

$$\int_\Omega |u_0^n|^{p-1}|y^n(0)|^2dx \le \|u_0^n\|^{p-1}_{\frac{3(p-1)}{2}} \|y^n(0)\|_6^2 \le C(R)\,\|y^n(0)\|^2_{1,\Omega} \le C(R)\tilde{\mathscr{E}}_n(0). \quad (2.5.7)$$

Thus,

$$Q_1 + Q_2 \le C(R)\tilde{\mathscr{E}}_n(0). \quad (2.5.8)$$

The non-zero initial data, $y^n(0) \neq 0$ and $z^n(0) \neq 0$, also changes the estimates in (2.3.19)-(2.3.20). Indeed,

$$\begin{aligned}\int_\Omega |y^n(t)|^2dx &= \int_\Omega \left|y^n(0) + \int_0^t y_t^n(\tau)d\tau\right|^2 dx \\ &\le 2\int_\Omega |y^n(0)|^2dx + 2\int_\Omega \left|\int_0^t y_t^n(\tau)d\tau\right|^2 dx \\ &\le C\Big(\,\|y^n(0)\|^2_{1,\Omega} + t\int_0^t \|y_t^n(\tau)\|_2^2\,d\tau\Big) \\ &\le C\left(\tilde{\mathscr{E}}_n(0) + T\int_0^t \tilde{\mathscr{E}}_n(\tau)d\tau\right). \end{aligned} \quad (2.5.9)$$

Also, since the integral $\int_\Omega |z^n(t)|^2dx$ can be estimated as in (2.5.9), we conclude

$$\int_\Omega \left(|y^n(t)|^2 + |z^n(t)|^2\right)dx \le C\left(\tilde{\mathscr{E}}_n(0) + T\int_0^t \tilde{\mathscr{E}}_n(\tau)d\tau\right). \quad (2.5.10)$$

Another place where one must exercise caution in estimating the typical term: $\int_\Omega |u^n(t)|^{p-1}|y^n(t)|^2dx$. As in the proof of Theorem 1.3.4, we consider two cases: $3 < p < 5$ and $5 \le p < 6$.

If $3 < p < 5$, then by using (2.3.25), (2.3.26) and (2.5.9), we obtain for $0 < \epsilon < 5 - p$:

$$\int_\Omega |u^n(t)|^{p-1}|y^n(t)|^2dx \le 2\epsilon\tilde{\mathscr{E}}(t) + C(\epsilon,R)\tilde{\mathscr{E}}_n(0) + C(\epsilon,R,T)\int_0^t \tilde{\mathscr{E}}(\tau)d\tau. \quad (2.5.11)$$

If $5 \le p < 6$, the non-zero initial data make the computations more involved than (2.3.28)-(2.3.32). Recall the choice of $\phi \in C_0(\Omega)$ such that $\|u_0 - \phi\|_{\frac{3(p-1)}{2}} \le \epsilon^{\frac{1}{p-1}}$

where the value of $\epsilon > 0$ will be chosen later. Then, we have

$$\int_\Omega |u^n(t)|^{p-1}|y^n(t)|^2dx \leq C\Big(\int_\Omega |u^n(t)-u_0^n|^{p-1}|y^n(t)|^2dx + \int_\Omega |u_0^n-u_0|^{p-1}|y^n(t)|^2dx + \int_\Omega |u_0-\phi|^{p-1}|y^n(t)|^2dx + \int_\Omega |\phi|^{p-1}|y^n(t)|^2dx\Big). \tag{2.5.12}$$

As in (2.3.29), we deduce that

$$\int_\Omega |u^n(t)-u_0^n|^{p-1}|y^n(t)|^2dx \leq C(R)T^{\frac{m(p-1)}{m+1}}\tilde{\mathscr{E}}_n(t). \tag{2.5.13}$$

Also, by using Hölder's inequality and the embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we obtain

$$\int_\Omega |u_0^n-u_0|^{p-1}|y^n(t)|^2dx \leq \|u_0^n-u_0\|^{p-1}_{\frac{3(p-1)}{2}}\|y^n(t)\|_6^2 \leq \epsilon\tilde{\mathscr{E}}_n(t), \tag{2.5.14}$$

for all sufficiently large n, since $u_0^n \longrightarrow u_0$ in $L^{\frac{3(p-1)}{2}}(\Omega)$. Moreover, from (2.3.30), we know

$$\int_\Omega |u_0-\phi|^{p-1}|y^n(t)|^2dx \leq C\epsilon\tilde{\mathscr{E}}_n(t). \tag{2.5.15}$$

As for the last term on the right-hand side of (2.5.12), we refer to (2.3.31) and (2.5.9), and we have

$$\int_\Omega |\phi|^{p-1}|y^n(t)|^2dx \leq C(\epsilon)\int_\Omega |y^n(t)|^2dx \leq C(\epsilon)\left(\tilde{\mathscr{E}}_n(0) + T\int_0^t \tilde{\mathscr{E}}_n(\tau)d\tau\right). \tag{2.5.16}$$

Thus, for the case $5 \leq p < 6$, it follows from (2.5.13)-(2.5.16), and (2.5.12) that

$$\int_\Omega |u^n(t)|^{p-1}|y^n(t)|^2dx \leq C(\epsilon)\tilde{\mathscr{E}}_n(0) + C(R)\Big(T^{\frac{m(p-1)}{m+1}}+\epsilon\Big)\tilde{\mathscr{E}}_n(t) + C(\epsilon)T\int_0^t \tilde{\mathscr{E}}_n(\tau)d\tau. \tag{2.5.17}$$

By combining the two cases (2.5.11) and (2.5.17), we have for $3 < p < 6$:

$$\int_\Omega |u^n(t)|^{p-1}|y^n(t)|^2dx \leq C(\epsilon,R)\tilde{\mathscr{E}}_n(0) + C(R)\Big(T^{\frac{m(p-1)}{m+1}}+\epsilon\Big)\tilde{\mathscr{E}}_n(t) + C(\epsilon,R,T)\int_0^t \tilde{\mathscr{E}}_n(\tau)d\tau. \tag{2.5.18}$$

Now, by looking at (2.5.8), (2.5.10) and (2.5.18), we notice that the non-zero initial data $y^n(0)$ and $z^n(0)$ also contribute the additional term $C(\epsilon, R)\tilde{\mathscr{E}}_n(0)$, which should be added to the right-hand side of R_f, and so, for $3 < p < 6$ we have:

$$\begin{aligned} R_f^n &\leq C(\epsilon, R)\tilde{\mathscr{E}}_n(0) + C(R)\Big(T^{\frac{m(p-1)}{m+1}} + T^{\frac{r(p-1)}{r+1}} + \epsilon\Big)\tilde{\mathscr{E}}_n(t) \\ &+ C(\epsilon, R, T)\int_0^t \tilde{\mathscr{E}}_n(\tau)\Big(\, \|u_t\|_{m+1} + \|v_t\|_{r+1} + \|u_t^n\|_{m+1} + \|v_t^n\|_{r+1} + 1\Big)d\tau, \end{aligned} \tag{2.5.19}$$

for all sufficiently large n, and $\epsilon > 0$ is sufficiently small, and according to (2.5.5), the estimate (2.5.19) also holds for the case $1 \leq p \leq 3$.

By using the similar approach (which is omitted) we can estimate R_h^n in (2.5.4) as well. Finally, from (2.5.4), we conclude

$$\begin{aligned} \tilde{\mathscr{E}}_n(t) &\leq \tilde{\mathscr{E}}_n(0) + R_f^n + R_h^n \\ &\leq C(\epsilon, R)\tilde{\mathscr{E}}_n(0) + C(R)\left(T^{\frac{m(p-1)}{m+1}} + T^{\frac{r(p-1)}{r+1}} + T^{\frac{q(k-1)}{q+1}} + \epsilon\right)\tilde{\mathscr{E}}_n(t) \\ &+ C(\epsilon, R, T)\int_0^t \tilde{\mathscr{E}}_n(\tau)\Big(\, \|u_t\|_{m+1} + \|v_t\|_{r+1} + \|u_t^n\|_{m+1} + \|v_t^n\|_{r+1} \\ &\qquad\qquad + |u_t|_{q+1} + |u_t^n|_{q+1} + 1\Big)d\tau. \end{aligned}$$

Again, we can choose ϵ and T small enough so that

$$C(R)\left(T^{\frac{m(p-1)}{m+1}} + T^{\frac{r(p-1)}{r+1}} + T^{\frac{q(k-1)}{q+1}} + \epsilon\right) < 1.$$

By Gronwall's inequality, we obtain

$$\begin{aligned} \tilde{\mathscr{E}}_n(t) \leq C(\epsilon, R, T)\tilde{\mathscr{E}}_n(0)\exp\Big[\int_0^t \Big(\, &\|u_t\|_{m+1} + \|v_t\|_{r+1} \\ &+ \|u_t^n\|_{m+1} + \|v_t^n\|_{r+1} + |u_t|_{q+1} + |u_t^n|_{q+1} + 1\Big)d\tau\Big], \end{aligned} \tag{2.5.20}$$

and so, by (2.5.2), we have

$$\tilde{\mathscr{E}}_n(t) \leq C(\epsilon, R, T)\tilde{\mathscr{E}}_n(0), \tag{2.5.21}$$

for all sufficiently large n. Hence, $\tilde{\mathscr{E}}_n(t) \longrightarrow 0$ uniformly on $[0, T]$. This concludes the proof of Theorem 1.3.8. $\square$

2.6 Appendix

Proposition 2.6.1. *Let X and Y be Banach spaces and assume $A_1 : \mathcal{D}(A_1) \subset X \longrightarrow X^*$, $A_2 : \mathcal{D}(A_2) \subset Y \longrightarrow Y^*$ are single-valued maximal monotone operators. Then, the operator $A : \mathcal{D}(A_1) \times \mathcal{D}(A_2) \subset X \times Y \longrightarrow X^* \times Y^*$ defined by $A\begin{pmatrix} x \\ y \end{pmatrix}^{tr} = \begin{pmatrix} A_1(x) \\ A_2(y) \end{pmatrix}^{tr}$ is also maximal monotone.*

Proof. The fact that A is monotone is trivial. In order to show that A is maximal monotone, assume $(x_0, y_0) \in X \times Y$ and $(x_0^*, y_0^*) \in X^* \times Y^*$ such that

$$\langle x - x_0, A_1(x) - x_0^* \rangle + \langle y - y_0, A_2(y) - y_0^* \rangle \geq 0, \tag{2.6.1}$$

for all $(x, y) \in \mathcal{D}(A_1) \times \mathcal{D}(A_2)$.

If $x_0 \in \mathcal{D}(A_1)$, then by taking $x = x_0$ in (2.6.1) and using the maximal monotonicity of A_2, we obtain $y_0 \in \mathcal{D}(A_2)$ and $y_0^* = A_2(y_0)$, and then we can put $y = y_0$ in (2.6.1) and conclude from the maximal monotonicity of A_1 that $x_0^* = A_1(x_0)$. Similarly, if $y_0 \in \mathcal{D}(A_2)$, then it follows that $x_0 \in \mathcal{D}(A_1)$, $x_0^* = A_1(x_0)$ and $y_0^* = A_2(y_0)$.

Now, if $x_0 \notin \mathcal{D}(A_1)$ and $y_0 \notin \mathcal{D}(A_2)$, then since A_1 and A_2 are both maximal monotone, there exist $x_1 \in \mathcal{D}(A_1)$, $y_1 \in \mathcal{D}(A_2)$ such that $\langle x_1 - x_0, A_1(x_1) - x_0^* \rangle < 0$ and $\langle y_1 - y_0, A_2(y_1) - y_0^* \rangle < 0$. Therefore, $\langle x_1 - x_0, A_1(x_1) - x_0^* \rangle + \langle y_1 - y_0, A_2(y_1) - y_0^* \rangle < 0$, which contradicts (2.6.1).

Therefore, we must have $x_0 \in \mathcal{D}(A_1)$, $y_0 \in \mathcal{D}(A_2)$, with $x_0^* = A_1(x_0)$ and $y_0^* = A_2(y_0)$. Thus, A is maximal monotone. □

Lemma 2.6.2. *Let X be a Banach space and $1 \leq p < \infty$. Then, $C_0((0,T); X)$ is dense in $L^p(0,T;X)$, where $C_0((0,T); X)$ denotes the space of continuous functions $u : (0,T) \longrightarrow X$ with compact support in $(0,T)$.*

Remark 2.6.3. The result is well-known if $X = \mathbb{R}^n$. Although for a general Banach space X such a result is expected, we couldn't find a reference for it in the literature. Thus, we provide a proof for it.

Proof. Let $u \in L^p(0,T;X)$, $\epsilon > 0$ be given. By the definition of $L^p(0,T;X)$, there exists a simple function ϕ with values in X such that

$$\int_0^T \|\phi(t) - u(t)\|_X^p \, dt < \epsilon^p. \tag{2.6.2}$$

Say $\phi(t) = \sum_{j=1}^{n} x_j \chi_{E_j}(t)$, where $x_j \in X$ are distinct, each $x_j \neq 0$, and $E_j \subset (0,T)$ are Lebesgue measurable such that $E_j \cap E_k = \emptyset$, for all $j \neq k$.

By a standard result in analysis, for each E_j, there exists a finite disjoint sequence of open segments $\{I_{j,k}\}_{k=1}^{m_j}$ such that

$$m\left(E_j \triangle \bigcup_{k=1}^{m_j} I_{j,k}\right) < \left(\frac{\epsilon}{2n\left\|x_j\right\|_X}\right)^p \text{ for } j = 1, 2, ..., n, \tag{2.6.3}$$

where m denotes the Lebesgue measure, and $E \triangle F$ is the symmetric difference of the sets E and F. In particular, we have

$$m\left(\left(E_j \triangle \bigcup_{k=1}^{m_j} I_{j,k}\right) \cap [0,T]\right) < \left(\frac{\epsilon}{2n\left\|x_j\right\|_X}\right)^p \text{ for } j = 1, 2, ..., n.$$

Let us note that $\left(E_j \triangle \bigcup_{k=1}^{m_j} I_{j,k}\right) \cap [0,T] = E_j \triangle \left(\bigcup_{k=1}^{m_j} I_{j,k} \cap [0,T]\right)$. So, we may assume, without loss of generality, that each $I_{j,k} \subset [0,T]$.

Now, if E, $F \subset [0,T]$ are Lebesgue measurable, then

$$\begin{aligned}
&\int_0^T |\chi_E(t) - \chi_F(t)|^p dt \\
&= \int_{E\setminus F} |\chi_E(t) - \chi_F(t)|^p dt + + \int_{F\setminus E} |\chi_E(t) - \chi_F(t)|^p dt + \int_{E\cap F} |\chi_E(t) - \chi_F(t)|^p dt \\
&= \int_{E\setminus F} \chi_E(t) dt + \int_{F\setminus E} \chi_F(t) dt = m(E \triangle F).
\end{aligned} \tag{2.6.4}$$

Therefore, by (2.6.4) and (2.6.3),

$$\|x_j\|_X^p \int_0^T |\chi_{E_j}(t) - \chi_{\bigcup_{k=1}^{m_j} I_{j,k}}(t)|^p dt = \|x_j\|_X^p \, m\left(E_j \triangle \bigcup_{k=1}^{m_j} I_{j,k}\right) < \left(\frac{\epsilon}{2n}\right)^p. \tag{2.6.5}$$

Since $I_{j,k} \subset [0,T]$, we can select $\delta_{j,k}$ such that $0 < \delta_{j,k} < \frac{1}{4}(b_{j,k} - a_{j,k})$ where $I_{j,k} = (a_{j,k}, b_{j,k})$. Choose $\delta > 0$ such that

$$\delta < \min\left\{\delta_{j,k}, \ \frac{1}{8(2n)^{p-1} \sum_{j=1}^{n} (\|x_j\|_X^p \, m_j)} \epsilon^p : \ k = 1, \cdots, m_j; \ j = 1, \cdots, n\right\}. \tag{2.6.6}$$

Now we define the functions $g_{j,k} \in C_0((0,T);\mathbb{R})$ such that $g_{j,k}(t) = 1$ on $[a_{j,k} + 2\delta, b_{j,k} - 2\delta]$, $g_{j,k}(t)$ is linear on $[a_{j,k} + \delta, b_{j,k} + 2\delta] \cup [b_{j,k} - 2\delta, b_{j,k} - \delta]$, and $g_{j,k}(t) = 0$

outside $[a_{j,k}+\delta, b_{j,k}-\delta]$. Let us notice that

$$\int_0^T \left|\sum_{k=1}^{m_j}(\chi_{I_{j,k}}(t) - g_{j,k}(t))\right|^p dt \le \int_0^T \left(\sum_{k=1}^{m_j}(\chi_{(a_{j,k},a_{j,k}+2\delta)}(t) + \chi_{(b_{j,k}-2\delta,b_{j,k})}(t))\right)^p dt$$
$$= \int_0^T \sum_{k=1}^{m_j}\Big(\chi_{(a_{j,k},a_{j,k}+2\delta)}(t) + \chi_{(b_{j,k}-2\delta,b_{j,k})}(t)\Big)dt = \sum_{k=1}^{m_j} 4\delta = 4m_j\delta. \tag{2.6.7}$$

Finally, we define $g(t) = \sum_{j=1}^{n} x_j \sum_{k=1}^{m_j} g_{j,k}(t)$. Clearly, $g \in C_0((0,T);X)$. Then, (2.6.2) yields

$$\|u-g\|_{L^p(0,T;X)} \le \|u-\phi\|_{L^p(0,T;X)} + \|\phi - g\|_{L^p(0,T;X)} < \epsilon + \|\phi - g\|_{L^p(0,T;X)}. \tag{2.6.8}$$

For $t \in (0,T)$, we note that

$$\|\phi(t)-g(t)\|_X = \left\|\sum_{j=1}^{n}\left(x_j\chi_{E_j}(t) - x_j\sum_{k=1}^{m_j} g_{j,k}(t)\right)\right\|_X$$
$$= \left\|\sum_{j=1}^{n}\left(x_j\chi_{E_j}(t) - x_j\sum_{j=1}^{m_j}\chi_{I_{j,k}}(t) + x_j\sum_{j=1}^{m_j}\chi_{I_{j,k}}(t) - x_j\sum_{k=1}^{m_j} g_{j,k}(t)\right)\right\|_X$$
$$\le \sum_{j=1}^{n}\|x_j\|_X\left|\chi_{E_j}(t) - \chi_{\bigcup_{k=1}^{m_j} I_{j,k}}(t)\right| + \sum_{j=1}^{n}\|x_j\|_X\sum_{k=1}^{m_j}\Big|\chi_{I_{j,k}}(t) - g_{j,k}(t)\Big|.$$

So, by Jensen's inequality and (2.6.5)-(2.6.7), we have

$$\int_0^T \|\phi(t)-g(t)\|_X^p\, dt \le (2n)^{p-1}\sum_{j=1}^{n}\|x_j\|^p\int_0^T |\chi_{E_j}(t) - \chi_{\bigcup_{k=1}^{m_j} I_{j,k}}(t)|^p dt$$
$$+ (2n)^{p-1}\sum_{j=1}^{n}\|x_j\|^p\int_0^T\left(\sum_{k=1}^{m_j}|\chi_{I_{j,k}}(t) - g_{j,k}(t)|\right)^p dt$$
$$< (2n)^{p-1}\sum_{j=1}^{n}\left(\frac{\epsilon}{2n}\right)^p + (2n)^{p-1}\sum_{j=1}^{n}\|x_j\|_X^p\, 4m_j\delta < \frac{1}{2}\epsilon^p + \frac{1}{2}\epsilon^p = \epsilon^p. \tag{2.6.9}$$

Combining (2.6.9) with (2.6.8) yields $\|u-g\|_{L^p(0,T;X)} < 2\epsilon$. □

Chapter 3

Blow-up of Weak Solutions

This chapter is devoted to prove our blow-up results: Theorems 1.3.12 and 1.3.13. These results are inspired by the work of [10, 20, 34] for their treatment of a single wave equation. Although the basic calculus in the proofs draw from ideas in [2, 20, 34] and also from the recent results in [10], our proofs had to be significantly adjusted to accommodate the coupling in the system (1.1.1).

3.1 Proof of Theorem 1.3.12

Proof. Let (u, v) be a weak solution to (1.1.1) in the sense of Definition 1.3.1. Throughout the proof, we assume the validity of Assumptions 1.1.1 and 1.3.10, $p > \max\{m, r\}$ and $k > q$. We define the life span T of such a solution (u, v) to be the supremum of all $T^* > 0$ such that (u, v) is a solution to (1.1.1) in the sense of Definition (1.3.1) on $[0, T^*]$. Our goal is to show that T is necessarily finite, and obtain an upper bound for T.

As in [2, 10], for $t \in [0, T)$, we define:

$$\begin{aligned} &G(t) = -E(t), \\ &N(t) = \|u(t)\|_2^2 + \|v(t)\|_2^2, \\ &S(t) = \int_\Omega F(u(t), v(t))dx + \int_\Gamma H(\gamma u(t))d\Gamma. \end{aligned}$$

It follows that,

$$G(t) = -\frac{1}{2}(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 + \|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2) + S(t), \tag{3.1.1}$$

and

$$N'(t) = 2\int_\Omega [u(t)u_t(t) + v(t)v_t(t)]dx. \tag{3.1.2}$$

Moreover, by the assumptions $H(s) \geq c_2|s|^{k+1}$ and $F(u,v) \geq c_0(|u|^{p+1} + |v|^{p+1})$, one has

$$S(t) \geq c_0(\|u(t)\|_{p+1}^{p+1} + \|v(t)\|_{p+1}^{p+1}) + c_2|\gamma u(t)|_{k+1}^{k+1}. \tag{3.1.3}$$

Let

$$0 < \alpha < \min\Big\{\frac{1}{m+1} - \frac{1}{p+1}, \frac{1}{r+1} - \frac{1}{p+1}, \frac{1}{q+1} - \frac{1}{k+1}, \frac{p-1}{2(p+1)}\Big\}. \tag{3.1.4}$$

In particular, $\alpha < \frac{1}{2}$. To simplify the notation, we introduce the following constants:

$$\begin{aligned} &K_1 = b_1|\Omega|^{\frac{p-m}{(p+1)(m+1)}} c_0^{-\frac{1}{p+1}}, \quad K_2 = b_2|\Omega|^{\frac{p-r}{(p+1)(r+1)}} c_0^{-\frac{1}{p+1}}, \quad K_3 = b_3|\Gamma|^{\frac{k-q}{(k+1)(q+1)}} c_2^{-\frac{1}{k+1}}, \\ &\delta_1 = \frac{\lambda}{6}G(0)^{\frac{1}{m+1}-\frac{1}{p+1}}, \quad \delta_2 = \frac{\lambda}{6}G(0)^{\frac{1}{r+1}-\frac{1}{p+1}}, \quad \delta_3 = \frac{\lambda}{6}G(0)^{\frac{1}{q+1}-\frac{1}{k+1}} \end{aligned} \tag{3.1.5}$$

where $\lambda = \min\{c_1 - 2, c_3 - 2\} > 0$, and $|\Omega|, |\Gamma|$ denote the Lebesgue measures of Ω and Γ.

Note that the energy identity (1.3.4) is equivalent to

$$G(t) = G(0) + \int_0^t \int_\Omega [g_1(u_t)u_t + g_2(v_t)v_t]dxd\tau + \int_0^t \int_\Gamma g(\gamma u_t)\gamma u_t d\Gamma d\tau.$$

So, by Assumption 1.1.1 and the regularity of the solution (u, v), we conclude that $G(t)$ is absolutely continuous and

$$\begin{aligned} G'(t) &= \int_\Omega [g_1(u_t(t))u_t(t) + g_2(v_t(t))v_t(t)]dx + \int_\Gamma g(\gamma u_t(t))\gamma u_t(t)d\Gamma \\ &\geq a_1\|u_t(t)\|_{m+1}^{m+1} + a_2\|v_t(t)\|_{r+1}^{r+1} + a_3|\gamma u_t(t)|_{q+1}^{q+1} \geq 0, \text{ a.e. } [0,T). \end{aligned} \tag{3.1.6}$$

Thus, $G(t)$ is non-decreasing. Since $G(0) = -E(0) > 0$, then it follows that

$$0 < G(0) \leq G(t) \leq S(t) \text{ for } 0 \leq t < T. \tag{3.1.7}$$

Now, put

$$Y(t) = G(t)^{1-\alpha} + \epsilon N'(t). \tag{3.1.8}$$

where $0 < \epsilon \le G(0)$. Later in the proof we further adjust the requirements on ϵ. We shall show that

$$Y'(t) = (1-\alpha)G(t)^{-\alpha}G'(t) + \epsilon N''(t), \tag{3.1.9}$$

where

$$\begin{aligned} N''(t) &= 2\Big(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 \Big) - 2\Big(\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2 \Big) \\ &\quad - 2\int_\Omega (g_1(u_t)u + g_2(v_t)v)dx - 2\int_\Gamma g(\gamma u_t)\gamma u d\Gamma \\ &\quad + 2\int_\Omega (f_1(u,v)u + f_2(u,v)v)dx + 2\int_\Gamma h(\gamma u)\gamma u d\Gamma, \text{ a.e. } [0,T). \end{aligned} \tag{3.1.10}$$

In order to obtain (3.1.10), we first verify $u \in L^{m+1}(\Omega \times (0,t))$ for all $t \in [0,T)$. Indeed, since both u and $u_t \in C([0,t]; L^2(\Omega))$, we can write

$$\begin{aligned} \int_0^t \int_\Omega |u|^{m+1} dx d\tau &= \int_0^t \int_\Omega \left| \int_0^\tau u_t(s)ds + u_0 \right|^{m+1} dx d\tau \\ &\le 2^m \left[\int_0^t \int_\Omega \left| \int_0^\tau u_t(s)ds \right|^{m+1} dx d\tau + \int_0^t \int_\Omega |u_0|^{m+1} dx d\tau \right] \\ &\le 2^m \left[t^m \int_0^t \int_\Omega \int_0^t |u_t(s)|^{m+1} ds dx d\tau + t \|u_0\|_{m+1}^{m+1} \right] \\ &\le 2^m \Big(t^{m+1} \|u_t\|_{L^{m+1}(\Omega\times(0,t))}^{m+1} + t\|u_0\|_{m+1}^{m+1} \Big) < \infty, \end{aligned} \tag{3.1.11}$$

for all $t \in [0,T)$, where we have used the regularity enjoyed by u, namely, the fact $u_t \in L^{m+1}(\Omega \times (0,t))$, and the assumption $u_0 \in H^1(\Omega) \hookrightarrow L^{m+1}(\Omega)$ since $m < p < 5$, as stated in Remark 1.3.14. Hence, $u \in L^{m+1}(\Omega \times (0,t))$ for all $t \in [0,T)$. Likewise, one can show that $v \in L^{r+1}(\Omega \times (0,t))$ for all $t \in [0,T)$. Moreover, by similar estimates as in (3.1.11), we deduce

$$\|\gamma u\|_{L^{q+1}(\Gamma\times(0,t))}^{q+1} \le 2^q \Big(t^{q+1} \|\gamma u_t\|_{L^{q+1}(\Gamma\times(0,t))}^{q+1} + t|\gamma u_0|_{q+1}^{q+1} \Big) < \infty.$$

Thus, $\gamma u \in L^{q+1}(\Gamma \times (0,t))$, for all $t \in [0,T)$.

The above shows that u and v enjoy, respectively, the regularity restrictions imposed on the test functions ϕ and ψ, as stated in Definition 1.3.1. Therefore, we can replace ϕ by u in (1.3.1) and ψ by v in (1.3.2), and by (3.1.2), we obtain

$$\begin{aligned}\frac{1}{2}N'(t) &= \int_\Omega (u_1u_0 + v_1v_0)dx + \int_0^t\int_\Omega (|u_t|^2 + |v_t|^2)dxd\tau - \int_0^t (\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2)d\tau \\ &- \int_0^t\int_\Omega (g_1(u_t)u + g_2(v_t)v)dxd\tau - \int_0^t\int_\Gamma g(\gamma u_t)\gamma u d\Gamma d\tau \\ &+ \int_0^t\int_\Omega (f_1(u,v)u + f_2(u,v)v)dxd\tau + \int_0^t\int_\Gamma h(\gamma u)\gamma u d\Gamma d\tau, \text{ a.e. } [0,T). \end{aligned} \tag{3.1.12}$$

By Assumption 1.1.1, $|\nabla f_j(u,v)| \leq C(|u|^{p-1} + |v|^{p-1} + 1)$, and so, by the Mean Value Theorem, one has $|f_j(u,v)| \leq C(|u|^p + |v|^p + 1)$, $j = 1, 2$. Thus, by using Young's and Hölder's inequality, we have

$$\begin{aligned}\int_0^t \left|\int_\Omega (f_1(u,v)u + f_2(u,v)v)dx\right| d\tau &\leq C\int_0^t\int_\Omega (|u|^p + |v|^p + 1)(|u| + |v|)dxd\tau \\ &\leq C_T \int_0^t\int_\Omega (|u|^{p+1} + |v|^{p+1})dxdt < \infty, \end{aligned} \tag{3.1.13}$$

for all $t \in [0,T)$, where we have used the fact $u \in C([0,t]; H^1(\Omega))$, the imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and the restriction $p < 5$, as mentioned in Remark 1.3.14.

In addition, by using the regularity of the solution (u,v) and the assumptions on the parameters, we infer

$$\begin{aligned}\int_0^t \left|\int_\Omega (g_1(u_t)u + g_2(v_t)v)dx\right| d\tau &+ \int_0^t \left|\int_\Gamma g(\gamma u_t)\gamma u d\Gamma\right| d\tau \\ &+ \int_0^t \left|\int_\Gamma h(\gamma u)\gamma u d\Gamma\right| d\tau < \infty, \end{aligned} \tag{3.1.14}$$

for all $t \in [0,T)$. Hence, it follows from (3.1.13)-(3.1.14), (3.1.12), and the regularity of (u,v) that $N'(t)$ is absolutely continuous, and thus (4.3.7) follows.

Now, note that (3.1.1) yields

$$\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2 = -(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2) + 2S(t) - 2G(t). \tag{3.1.15}$$

So, by (3.1.9), (3.1.10), (3.1.15) and the assumptions $uf_1(u,v) + vf_2(u,v) \geq c_1F(u,v)$,

$h(s)s \geq c_3 H(s)$, we deduce

$$
\begin{aligned}
&Y'(t) \geq (1-\alpha)G(t)^{-\alpha}G'(t) + 4\epsilon\Big(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 \Big) + 4\epsilon G(t) \\
&+ 2\epsilon(c_1 - 2)\int_\Omega F(u(t),v(t))dx + 2\epsilon(c_3-2)\int_\Gamma H(\gamma u(t))d\Gamma \\
&- 2\epsilon\int_\Omega g_1(u_t(t))u(t)dx - 2\epsilon\int_\Omega g_2(v_t(t))v(t)dx - 2\epsilon\int_\Gamma g(\gamma u_t(t))\gamma u(t)d\Gamma. \qquad (3.1.16)
\end{aligned}
$$

We begin by estimating the last three terms on the right hand side of (3.1.16). First, by using the assumption $g_1(s)s \leq b_1|s|^{m+1}$, Hölder's inequality, the fact $p > m$, and the inequality (3.1.3), we have

$$
\begin{aligned}
&\left|\int_\Omega g_1(u_t(t))u(t)dx\right| \leq b_1\int_\Omega |u(t)||u_t(t)|^m dx \leq b_1 \|u(t)\|_{m+1}\|u_t(t)\|_{m+1}^m \\
&\leq b_1|\Omega|^{\frac{p-m}{(p+1)(m+1)}}\|u(t)\|_{p+1}\|u_t(t)\|_{m+1}^m \leq K_1 S(t)^{\frac{1}{p+1}}\|u_t(t)\|_{m+1}^m \qquad (3.1.17)
\end{aligned}
$$

where K_1 is defined in (3.1.5). Observe, the definition of α implies $\frac{1}{p+1} - \frac{1}{m+1} + \alpha < 0$. Therefore, by using (3.1.6)-(3.1.7), Young's inequality, and recalling the definition of $\delta_1, \delta_2, \delta_3$ in (3.1.5), we obtain from (3.1.17) that

$$
\begin{aligned}
&\Big|\int_\Omega g_1(u_t(t))u(t)dx\Big| \leq K_1 S(t)^{\frac{1}{p+1}-\frac{1}{m+1}} S(t)^{\frac{1}{m+1}}\|u_t(t)\|_{m+1}^m \\
&\leq G(t)^{\frac{1}{p+1}-\frac{1}{m+1}}(\delta_1 S(t) + C_{\delta_1}K_1^{\frac{m+1}{m}}\|u_t(t)\|_{m+1}^{m+1}) \\
&\leq \delta_1 G(t)^{\frac{1}{p+1}-\frac{1}{m+1}}S(t) + C_{\delta_1}K_1^{\frac{m+1}{m}}a_1^{-1}G'(t)G(t)^{-\alpha}G(t)^{\frac{1}{p+1}-\frac{1}{m+1}+\alpha} \\
&\leq \delta_1 G(0)^{\frac{1}{p+1}-\frac{1}{m+1}}S(t) + C_{\delta_1}K_1^{\frac{m+1}{m}}a_1^{-1}G'(t)G(t)^{-\alpha}G(0)^{\frac{1}{p+1}-\frac{1}{m+1}+\alpha}. \qquad (3.1.18)
\end{aligned}
$$

By repeating the estimates (3.1.17)-(3.1.18), replacing $u(t)$ by $v(t)$ and m by r, we deduce

$$
\begin{aligned}
&\Big|\int_\Omega g_2(v_t(t))v(t)dx\Big| \\
&\leq \delta_2 G(0)^{\frac{1}{p+1}-\frac{1}{r+1}}S(t) + C_{\delta_2}K_2^{\frac{r+1}{r}}a_2^{-1}G'(t)G(t)^{-\alpha}G(0)^{\frac{1}{p+1}-\frac{1}{r+1}+\alpha}. \qquad (3.1.19)
\end{aligned}
$$

Likewise, by replacing $u(t)$ by $\gamma u(t)$, Ω by Γ, p by k, m by q in (3.1.17)-(3.1.18), we obtain

$$
\begin{aligned}
&\Big|\int_\Gamma g(\gamma u_t(t))\gamma u(t)d\Gamma\Big| \\
&\leq \delta_3 G(0)^{\frac{1}{k+1}-\frac{1}{q+1}}S(t) + C_{\delta_3}K_3^{\frac{q+1}{q}}a_3^{-1}G'(t)G(t)^{-\alpha}G(0)^{\frac{1}{k+1}-\frac{1}{q+1}+\alpha}. \qquad (3.1.20)
\end{aligned}
$$

Now, since $0<\alpha<\frac{1}{2}$, we may choose $0<\epsilon<1$ small enough such that

$$\begin{aligned} L := & 1-\alpha-2\epsilon\Big(C_{\delta_1}K_1^{\frac{m+1}{m}}a_1^{-1}G(0)^{\frac{1}{p+1}-\frac{1}{m+1}+\alpha} \\ & + C_{\delta_2}K_2^{\frac{r+1}{r}}a_2^{-1}G(0)^{\frac{1}{p+1}-\frac{1}{r+1}+\alpha}+C_{\delta_3}K_3^{\frac{q+1}{q}}a_3^{-1}G(0)^{\frac{1}{k+1}-\frac{1}{q+1}+\alpha}\Big)\geq 0. \end{aligned} \tag{3.1.21}$$

In addition, since $\lambda=\min\{c_1-2,c_3-2\}$, then

$$(c_1-2)\int_\Omega F(u(t),v(t))dx+(c_3-2)\int_\Gamma H(\gamma u(t))d\Gamma\geq \lambda S(t). \tag{3.1.22}$$

Hence, by inserting (3.1.18)-(3.1.20) into (3.1.16) and using (3.1.21), (3.1.22) and (3.1.5), we conclude

$$\begin{aligned} Y'(t) &\geq LG(t)^{-\alpha}G'(t)+4\epsilon(\|u_t(t)\|_2^2+\|v_t(t)\|_2^2)+4\epsilon G(t)+\lambda\epsilon S(t) \\ &\geq 4\epsilon\Big(\|u_t(t)\|_2^2+\|v_t(t)\|_2^2+G(t)\Big)+\lambda\epsilon S(t). \end{aligned} \tag{3.1.23}$$

In particular, the inequality (3.1.23) shows that $Y(t)$ is increasing on $[0,T)$, with

$$Y(t)=G(t)^{1-\alpha}+\epsilon N'(t)\geq G(0)^{1-\alpha}+\epsilon N'(0). \tag{3.1.24}$$

If $N'(0)\geq 0$, then no further condition on ϵ is needed. However, if $N'(0)<0$, then we further adjust ϵ so that $0<\epsilon\leq -\frac{G(0)^{1-\alpha}}{2N'(0)}$. In any case, one has

$$Y(t)\geq \frac{1}{2}G(0)^{1-\alpha}>0 \text{ for } t\in[0,T). \tag{3.1.25}$$

Finally, we show that

$$Y'(t)\geq C\epsilon^{1+\sigma}Y(t)^\eta \text{ for } t\in[0,T), \tag{3.1.26}$$

where

$$1<\eta=\frac{1}{1-\alpha}<2,\ \ \sigma=1-\frac{2}{(1-2\alpha)(p+1)}>0,$$

and $C>0$ is a generic constant independent of ϵ. Notice that $\sigma>0$ follows from the assumption $\alpha<\frac{p-1}{2(p+1)}$.

Now, if $N'(t)\leq 0$ for some $t\in[0,T)$, then for such value of t we have

$$Y(t)^\eta=[G(t)^{1-\alpha}+\epsilon N'(t)]^\eta\leq G(t) \tag{3.1.27}$$

and in this case, (3.1.23) and (3.1.27) yield

$$Y'(t) \geq 4\epsilon G(t) \geq 4\epsilon^{1+\sigma} G(t) \geq 4\epsilon^{1+\sigma} Y(t)^{\eta}.$$

Hence, (3.1.26) holds for all $t \in [0, T)$ for which $N'(t) \leq 0$. However, if $t \in [0, T)$ is such that $N'(t) > 0$, then showing the validity of (3.1.26) requires a little more effort. First, we note that $Y(t) = G(t)^{1-\alpha} + \epsilon N'(t) \leq G(t)^{1-\alpha} + N'(t)$, and so

$$Y(t)^{\eta} \leq 2^{\eta-1}[G(t) + N'(t)^{\eta}]. \tag{3.1.28}$$

We estimate $N'(t)^{\eta}$ as follows. By using Hölder's and Young's inequality and noting that $1 < \eta < 2$, we obtain from (3.1.2) that

$$\begin{aligned} N'(t)^{\eta} &\leq 2^{\eta}\Big(\|u_t(t)\|_2 \|u(t)\|_2 + \|v_t(t)\|_2 \|v(t)\|_2 \Big)^{\eta} \\ &\leq C\Big(\|u_t(t)\|_2^{\eta} \|u(t)\|_{p+1}^{\eta} + \|v_t(t)\|_2^{\eta} \|v(t)\|_{p+1}^{\eta} \Big) \\ &\leq C\Big(\|u_t(t)\|_2^2 + \|u(t)\|_{p+1}^{\frac{2\eta}{2-\eta}} + \|v_t(t)\|_2^2 + \|v(t)\|_{p+1}^{\frac{2\eta}{2-\eta}} \Big). \end{aligned} \tag{3.1.29}$$

Since $\eta = \frac{1}{1-\alpha}$ and $\sigma > 0$, it is easy to see that

$$\frac{2\eta}{(2-\eta)(p+1)} - 1 = \frac{2}{(1-2\alpha)(p+1)} - 1 = -\sigma < 0. \tag{3.1.30}$$

Therefore, by (3.1.3), (3.1.30), (3.1.7), and by recalling $\epsilon \leq G(0)$, we have

$$\begin{aligned} \|u(t)\|_{p+1}^{\frac{2\eta}{2-\eta}} &= (\|u(t)\|_{p+1}^{p+1})^{\frac{2\eta}{(2-\eta)(p+1)}} \leq CS(t)^{\frac{2\eta}{(2-\eta)(p+1)}} \\ &\leq CS(t)^{\frac{2\eta}{(2-\eta)(p+1)}-1} S(t) \leq CG(0)^{-\sigma} S(t) \leq C\epsilon^{-\sigma} S(t). \end{aligned} \tag{3.1.31}$$

Similarly,

$$\|v(t)\|_{p+1}^{\frac{2\eta}{2-\eta}} \leq C\epsilon^{-\sigma} S(t). \tag{3.1.32}$$

By (3.1.29) and (3.1.31)-(3.1.32) and noting $\epsilon^{-\sigma} > 1$, we obtain

$$\begin{aligned} N'(t)^{\eta} &\leq C\Big(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 + \epsilon^{-\sigma} S(t)\Big) \\ &\leq C\epsilon^{-\sigma}\Big(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 + S(t)\Big). \end{aligned} \tag{3.1.33}$$

Finally, the estimates (3.1.23), (3.1.33) and (3.1.28) allow us to conclude that

$$\begin{aligned} Y'(t) &\geq C\epsilon[G(t) + \|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 + S(t)] \geq C\epsilon[G(t) + \epsilon^\sigma N'(t)^\eta] \\ &\geq C\epsilon^{1+\sigma}[G(t) + N'(t)^\eta] \geq C\epsilon^{1+\sigma} Y(t)^\eta \end{aligned}$$

for all values of $t \in [0, T)$ for which $N'(t) > 0$. Hence, (3.1.26) is valid. By simple calculations, it follows from (3.1.25)-(3.1.26) that T is necessarily finite and

$$T < C\epsilon^{-(1+\sigma)} Y(0)^{-\frac{\alpha}{1-\alpha}} \leq C\epsilon^{-(1+\sigma)} G(0)^{-\alpha}. \tag{3.1.34}$$

As a result,

$$Y(t) = G(t)^{1-\alpha} + \epsilon N'(t) \to \infty \quad \text{as} \quad t \to T^-. \tag{3.1.35}$$

It remains to show $\|u(t)\|_{1,\Omega} + \|v(t)\|_{1,\Omega} \to \infty$ as $t \to T^-$. Indeed, by the definition of $Y(t)$ and the first inequality in (3.1.33), one has

$$\begin{aligned} Y(t)^\eta &\leq 2^{\eta-1}[G(t) + \epsilon^\eta N'(t)^\eta] \\ &\leq 2^{\eta-1}\left[G(t) + \epsilon^\eta C\left(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 + \epsilon^{-\sigma} S(t)\right)\right]. \end{aligned} \tag{3.1.36}$$

By recalling (3.1.1), and by further adjusting ϵ so that $-\frac{1}{2} + \epsilon^\eta C \leq 0$, then (3.1.36) implies

$$Y(t)^\eta \leq 2^{\eta-1}[S(t) + C\epsilon^{\eta-\sigma} S(t)]. \tag{3.1.37}$$

However, by using the assumptions on the sources and employing Hölder's inequality, we have

$$\begin{aligned} S(t) &= \int_\Omega F(u(t), v(t))dx + \int_\Gamma H(\gamma u(t))d\Gamma \\ &\leq \frac{1}{c_1}\int_\Omega \Big[u(t) f_1(u(t), v(t)) + v(t) f_2(u(t), v(t))\Big] dx + \frac{1}{c_3}\int_\Gamma h(\gamma u(t))\gamma u(t) d\Gamma \\ &\leq C\left(\|u(t)\|_{p+1}^{p+1} + \|v(t)\|_{p+1}^{p+1} + |\gamma u(t)|_{k+1}^{k+1}\right) \\ &\leq C\left(\|u(t)\|_{1,\Omega}^{p+1} + \|v(t)\|_{1,\Omega}^{p+1} + \|u(t)\|_{1,\Omega}^{k+1}\right), \end{aligned} \tag{3.1.38}$$

where we have used the fact $p < 5$ and $k < 3$, as mentioned in Remark 1.3.14. Consequently, by combining (3.1.37) and (3.1.38) one has

$$Y(t)^\eta \leq C\left(\|u(t)\|_{1,\Omega}^{p+1} + \|v(t)\|_{1,\Omega}^{p+1} + \|u(t)\|_{1,\Omega}^{k+1}\right),$$

and along with (3.1.35), we conclude $\|u(t)\|_{1,\Omega} + \|v(t)\|_{1,\Omega} \to \infty$ as $t \to T^-$. This completes the proof of Theorem 1.3.12. $\square$

3.2 Proof of Theorem 1.3.13

The proof of Theorem 1.3.13 goes along the same lines as the proof of Theorem 1.3.12; except for the estimate of the last term on the right hand side of (3.1.16). Here, we shall utilize the following trace and interpolation theorems:

- Trace theorem (see [1] for instance):

$$|\gamma u|_{q+1} \leq C \|u\|_{W^{s,q+1}(\Omega)}, \quad \text{where } s > \frac{1}{q+1}. \tag{3.2.1}$$

- Interpolation theorem (see [52]):

$$W^{1-\theta,r}(\Omega) = [H^1(\Omega), L^{p+1}(\Omega)]_\theta, \tag{3.2.2}$$

  where $r = \frac{2(p+1)}{(1-\theta)(p+1)+2\theta}$, $\theta \in [0,1]$, and as usual $[\cdot,\cdot]_\theta$ denotes the interpolation bracket.

We select θ such that

$$1 - \theta = \frac{1}{\beta(q+1)} > \frac{1}{q+1} \text{ for some } \frac{1}{q+1} < \beta < 1. \tag{3.2.3}$$

Additionally, we require that

$$r = \frac{2(p+1)}{(1-\theta)(p+1)+2\theta} \geq q+1. \tag{3.2.4}$$

Note $p > q$ since by assumption $p > 2q - 1 = q + (q-1) \geq q$. So, inserting (3.2.3) into (3.2.4) yields the following restriction on β:

$$\beta \geq \frac{p-1}{2(p-q)} > 0. \tag{3.2.5}$$

However, since $q \geq 1$, and by assumption, $p > 2q-1$, it follows that $1 > \frac{p-1}{2(p-q)} \geq \frac{1}{q+1}$. Thus, it is enough to impose the following restriction on β:

$$\frac{p-1}{2(p-q)} \leq \beta < 1. \tag{3.2.6}$$

Now, we turn our attention to the proof of Theorem 1.3.13.

Proof. Under the above restrictions on the parameters, we first show that

$$|\gamma u|_{q+1} \leq C_1(\|u\|_{1,\Omega}^{\frac{2\beta}{q+1}} + \|u\|_{p+1}^{\frac{(p+1)\beta}{q+1}}), \tag{3.2.7}$$

for some β satisfying (3.2.6), where C_1 is a generic constant.

In order to prove (3.2.7), we use (3.2.1)-(3.2.4) and Young's inequality to obtain

$$\begin{aligned} |\gamma u|_{q+1} &\leq C\,\|u\|_{W^{1-\theta,q+1}(\Omega)} \leq C\,\|u\|_{W^{1-\theta,r}(\Omega)} \leq C\,\|u\|_{1,\Omega}^{1-\theta}\,\|u\|_{p+1}^{\theta} \\ &= C\,\|u\|_{1,\Omega}^{\frac{1}{\beta(q+1)}}\,\|u\|_{p+1}^{1-\frac{1}{\beta(q+1)}} \leq C_1(\|u\|_{1,\Omega}^{\frac{2\beta}{q+1}} + \|u\|_{p+1}^{\frac{2\beta^2(q+1)-2\beta}{(2\beta^2-1)(q+1)}}). \end{aligned} \tag{3.2.8}$$

By comparing (3.2.7) and (3.2.8), it suffice to show that there exists β satisfying (3.2.6) such that $\frac{2\beta^2(q+1)-2\beta}{(2\beta^2-1)(q+1)} = \frac{(p+1)\beta}{q+1}$. We note that the latter is equivalent to $2(p+1)\beta^2 - 2(q+1)\beta - (p-1) = 0$. By the assumption $2q < p+1$, the positive root of the above quadratic equation satisfies:

$$\beta := \frac{2(q+1) + \sqrt{4(q+1)^2 + 8(p^2-1)}}{4(p+1)} < \frac{(p+3) + \sqrt{(3p+1)^2}}{4(p+1)} = 1. \tag{3.2.9}$$

Additionally, we must show that

$$\frac{2(q+1) + \sqrt{4(q+1)^2 + 8(p^2-1)}}{4(p+1)} \geq \frac{p-1}{2(p-q)}, \tag{3.2.10}$$

as required by (3.2.6). Indeed, by routine calculations, it is easy to see that (3.2.10) is equivalent to

$$(p-1)(p+1)^2(p-2q+1) \geq 0. \tag{3.2.11}$$

Obviously, (3.2.11) is valid since $p \geq 1$ and $p > 2q-1$. Hence, (3.2.7) verified.

Now, we turn our attention to estimating the last term on the right hand side of (3.1.16). First, we note that (3.1.15) yields

$$\|u(t)\|_{1,\Omega}^2 \leq 2S(t). \tag{3.2.12}$$

By Hölder's inequality and the estimates (3.2.7), (3.2.12) and (3.1.3), we obtain

$$\begin{aligned} \Big|\int_\Gamma g(\gamma u_t(t))\gamma u(t)d\Gamma\Big| &\leq b_3\int_\Gamma |\gamma u(t)||\gamma u_t(t)|^q d\Gamma \leq b_3|\gamma u(t)|_{q+1}|\gamma u_t(t)|_{q+1}^q \\ &\leq b_3C_1\left(\|u\|_{1,\Omega}^{\frac{2\beta}{q+1}} + \|u\|_{p+1}^{\frac{(p+1)\beta}{q+1}}\right)|\gamma u_t(t)|_{q+1}^q \\ &\leq b_3C_1\left(2^{\frac{\beta}{q+1}}S(t)^{\frac{\beta}{q+1}} + c_0^{-\frac{\beta}{q+1}}S(t)^{\frac{\beta}{q+1}}\right)|\gamma u_t(t)|_{q+1}^q \\ &\leq K_4S(t)^{\frac{\beta}{q+1}}|\gamma u_t(t)|_{q+1}^q \end{aligned} \tag{3.2.13}$$

where $K_4 = b_3 C_1 \cdot \max\{2^{\frac{\beta}{q+1}}, c_0^{-\frac{\beta}{q+1}}\}$. In addition to the restriction on α in (3.1.4), we further require $\alpha < \frac{1-\beta}{q+1}$, so $\frac{\beta-1}{q+1} + \alpha < 0$. Thus, by using (3.1.6)-(3.1.7) and Young's inequality, we can continue the estimate in (3.2.13) as follows.

$$\begin{aligned}
\Big| \int_\Gamma g(\gamma u_t(t))\gamma u(t) d\Gamma \Big| &\leq K_4 S(t)^{\frac{\beta-1}{q+1}} S(t)^{\frac{1}{q+1}} |\gamma u_t(t)|_{q+1}^q \\
&\leq G(t)^{\frac{\beta-1}{q+1}} \left(\delta_4 S(t) + C_{\delta_4} K_4^{\frac{q+1}{q}} |\gamma u_t(t)|_{q+1}^{q+1} \right) \\
&\leq \delta_4 G(t)^{\frac{\beta-1}{q+1}} S(t) + C_{\delta_4} K_4^{\frac{q+1}{q}} a_3^{-1} G'(t) G(t)^{-\alpha} G(t)^{\frac{\beta-1}{q+1}+\alpha} \\
&\leq \delta_4 G(0)^{\frac{\beta-1}{q+1}} S(t) + C_{\delta_4} K_4^{\frac{q+1}{q}} a_3^{-1} G'(t) G(t)^{-\alpha} G(0)^{\frac{\beta-1}{q+1}+\alpha} \qquad (3.2.14)
\end{aligned}$$

where $\delta_4 = \frac{\lambda}{6} G(0)^{\frac{1-\beta}{q+1}}$.

Now, instead of estimate (3.1.20) we use (3.2.14), and instead of (3.1.21) in Theorem 1.3.12, we choose $0 < \epsilon < 1$ small enough so that

$$\begin{aligned}
L_1 = 1 - \alpha - 2\epsilon \Big(& C_{\delta_1} K_1^{\frac{m+1}{m}} a_1^{-1} G(0)^{\frac{1}{p+1} - \frac{1}{m+1} + \alpha} + C_{\delta_2} K_2^{\frac{r+1}{r}} a_2^{-1} G(0)^{\frac{1}{p+1} - \frac{1}{r+1} + \alpha} \\
& + C_{\delta_4} K_4^{\frac{q+1}{q}} a_3^{-1} G(0)^{\frac{\beta-1}{q+1}+\alpha} \Big) \geq 0.
\end{aligned}$$

After replacing L with L_1 in (3.1.23), the rest of the proof continues exactly as in the proof of Theorem 1.3.12. $\square$

Chapter 4

Decay of Energy

The main goal of the present chapter is to establish global existence of potential well solutions, uniform decay rates of energy, and blow up of solutions with non-negative initial energy. Our strategy for the blow up results in this proof follows the general framework of [3] and [13]. However, our proofs had to be significantly adjusted to accommodate the coupling in the system (1.1.1) and the new case $p > \max\{m, r, 2q-1\}$. For the decay of energy, we follow the roadmap paper by Lasiecka and Tataru [30] and its refined versions in [3, 13, 33, 49] which involve comparing the energy of the system to a suitable ordinary differential equation.

4.1 Global Solutions

This section is devoted to the proof of Theorem 1.3.18.

Proof. The argument will be carried out in two steps.

Step 1. We first show the invariance of $\mathcal{W}_1$ under the dynamics, i.e., $(u(t), v(t)) \in \mathcal{W}_1$ for all $t \in [0, T)$, where $[0, T)$ is the maximal interval of existence.

Notice the energy identity (1.3.4) is equivalent to

$$E(t) + \int_0^t \int_\Omega [g_1(u_t)u_t + g_2(v_t)v_t]dxd\tau + \int_0^t \int_\Gamma g(\gamma u_t)\gamma u_t d\Gamma d\tau = E(0). \tag{4.1.1}$$

Since g_1, g_2 and g are all monotone increasing, then it follows from the regularity of the solutions (u, v) that

$$E'(t) = -\int_\Omega [g_1(u_t)u_t + g_2(v_t)v_t]dx - \int_\Gamma g(\gamma u_t)\gamma u_t d\Gamma \leq 0. \tag{4.1.2}$$

Thus,

$$J(u(t), v(t)) \leq E(t) \leq E(0) < d, \quad \text{for all} \quad t \in [0, T). \tag{4.1.3}$$

It follows that $(u(t), v(t)) \in \mathcal{W}$ for all $t \in [0, T)$.

To show that $(u(t), v(t)) \in \mathcal{W}_1$ on $[0, T)$, we proceed by contradiction. Assume that there exists $t_1 \in (0, T)$ such that $(u(t_1), v(t_1)) \notin \mathcal{W}_1$. Since $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$ and $\mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset$, then it must be the case that $(u(t_1), v(t_1)) \in \mathcal{W}_2$.

Let us show now that the function $t \mapsto \int_\Omega F(u(t), v(t))dx$ is continuous on $[0, T)$. Indeed, since $|\nabla f_j(u, v)| \leq C(|u|^{p-1} + |v|^{p-1} + 1)$, it follows that $|f_j(u, v)| \leq C(|u|^p + |v|^p + 1)$, $j = 1, 2$. By recalling that F is homogeneous of order $p+1$, one has $f_j(u, v)$ are homogeneous of order p, $j = 1, 2$. Therefore,

$$|f_j(u, v)| \leq C(|u|^p + |v|^p), \quad j = 1, 2. \tag{4.1.4}$$

Fix an arbitrary $t_0 \in [0, T)$. By the Mean Value Theorem and (4.1.4), we have

$$\begin{aligned}
&\int_\Omega |F(u(t), v(t)) - F(u(t_0), v(t_0))|dx \\
&\leq C \int_\Omega \Big(|u(t)|^p + |v(t)|^p + |u(t_0)|^p + |v(t_0)|^p\Big)\Big(|u(t) - u(t_0)| + |v(t) - v(t_0)|\Big)dx \\
&\leq C\Big(\|u(t)\|^p_{\frac{6}{5}p} + \|v(t)\|^p_{\frac{6}{5}p} + \|u(t_0)\|^p_{\frac{6}{5}p} + \|v(t_0)\|^p_{\frac{6}{5}p} \Big) \\
&\qquad\qquad\qquad \Big(\|u(t) - u(t_0)\|_6 + \|v(t) - v(t_0)\|_6 \Big).
\end{aligned} \tag{4.1.5}$$

Since $p \leq 5$, we know $\frac{6}{5}p \leq 6$, so by the imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and the regularity of the weak solution $(u, v) \in C([0, T); H^1(\Omega) \times H^1_0(\Omega))$, we obtain from (4.1.5) that

$$\lim_{t \to t_0} \int_\Omega |F(u(t), v(t)) - F(u(t_0), v(t_0))|dx = 0,$$

that is, $\int_\Omega F(u(t), v(t))dx$ is continuous on $[0, T)$.

Likewise, the function $t \mapsto \int_\Gamma H(\gamma u(t))d\Gamma$ is also continuous on $[0, T)$. Therefore, since $(u(0), v(0)) \in \mathcal{W}_1$ and $(u(t_1), v(t_1)) \in \mathcal{W}_2$, then it follows from the definition of $\mathcal{W}_1$ and $\mathcal{W}_2$ that there exists $s \in (0, t_1)$ such that

$$\|u(s)\|^2_{1,\Omega} + \|v(s)\|^2_{1,\Omega} = (p + 1) \int_\Omega F(u(s), v(s))dx + (k + 1) \int_\Gamma H(\gamma u(s))d\Gamma. \tag{4.1.6}$$

As a result, we may define t^* as the supremum of all $s \in (0, t_1)$ satisfying (4.1.6). Clearly, $t^* \in (0, t_1)$, t^* satisfies (4.1.6), and $(u(t), v(t)) \in \mathcal{W}_2$ for all $t \in (t^*, t_1]$.

We have two cases to consider:

Case 1: $(u(t^*), v(t^*)) \neq (0,0)$. In this case, since t^* satisfies (4.1.6), we see that $(u(t^*), v(t^*)) \in \mathcal{N}$, the Nehari manifold given in (1.3.12). Thus, by Lemma 1.3.17, it follows that $J(u(t^*), v(t^*)) \geq d$. Since $E(t) \geq J(u(t), v(t))$ for all $t \in [0,T)$, one has $E(t^*) \geq d$, which contradicts (4.1.3).

Case 2: $(u(t^*), v(t^*)) = (0,0)$. Since $(u(t), v(t)) \in \mathcal{W}_2$ for all $t \in (t^*, t_1]$, then by (1.3.7) and the definition of $\mathcal{W}_2$, we obtain

$$\begin{aligned}\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2 &< C(\|u(t)\|_{p+1}^{p+1} + \|v(t)\|_{p+1}^{p+1} + |\gamma u(t)|_{k+1}^{k+1}) \\ &\leq C(\|u(t)\|_{1,\Omega}^{p+1} + \|v(t)\|_{1,\Omega}^{p+1} + \|u(t)\|_{1,\Omega}^{k+1}), \quad \text{for all} \quad t \in (t^*, t_1].\end{aligned}$$

Therefore,

$$\|(u(t), v(t))\|_X^2 < C(\|(u(t), v(t))\|_X^{p+1} + \|(u(t), v(t))\|_X^{k+1}), \quad \text{for all} \quad t \in (t^*, t_1],$$

which yields,

$$1 < C(\|(u(t), v(t))\|_X^{p-1} + \|(u(t), v(t))\|_X^{k-1}), \quad \text{for all} \quad t \in (t^*, t_1].$$

It follows that $\|(u(t), v(t))\|_X > s_1$, for all $t \in (t^*, t_1]$, where $s_1 > 0$ is the unique positive solution of the equation $C(s^{p-1} + s^{k-1}) = 1$, where p, $k > 1$. Employing the continuity of the weak solution $(u(t), v(t))$, we obtain that

$$\|(u(t^*), v(t^*))\|_X \geq s_1 > 0,$$

which contradicts the assumption $(u(t^*), v(t^*)) = (0,0)$. Hence, $(u(t), v(t)) \in \mathcal{W}_1$ for all $t \in [0,T)$.

Step 2. We show the weak solution $(u(t), v(t))$ is global solution. By (4.1.3), we know $J(u(t), v(t)) < d$ for all $t \in [0,T)$, that is,

$$\frac{1}{2}(\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2) - \int_\Omega F(u(t), v(t))dx - \int_\Gamma H(\gamma u(t))d\Gamma < d, \quad \text{on} \quad [0,T). \tag{4.1.7}$$

Since $(u(t), v(t)) \in \mathcal{W}_1$ for all $t \in [0,T)$, one has

$$\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2 \geq c\left(\int_\Omega F(u(t), v(t))dx + \int_\Gamma H(\gamma u(t))d\Gamma\right), \quad \text{on} \quad [0,T), \tag{4.1.8}$$

where $c = \min\{p+1, k+1\} > 2$. Combining (4.1.7) and (4.1.8) yields

$$\int_\Omega F(u(t), v(t))dx + \int_\Gamma H(\gamma u(t))d\Gamma < \frac{2d}{c-2}, \quad \text{for all} \quad t \in [0,T). \tag{4.1.9}$$

By using the energy identity (4.1.1) and (4.1.9), we deduce

$$\begin{aligned} &\mathscr{E}(t) + \int_0^t \int_\Omega [g_1(u_t)u_t + g_2(v_t)v_t]dxd\tau + \int_0^t \int_\Gamma g(\gamma u_t)\gamma u_t d\Gamma d\tau \\ &= E(0) + \int_\Omega F(u(t), v(t))dx + \int_\Gamma H(\gamma u(t))d\Gamma \\ &< d + \frac{2d}{c-2} = d\frac{c}{c-2}, \quad \text{for all} \ \ t \in [0,T). \end{aligned} \tag{4.1.10}$$

By virtue of the monotonicity of g_1, g_2 and g, inequality (1.3.16) follows. Consequently, by a standard continuation argument we conclude that the weak solution $(u(t), v(t))$ is indeed a global solutions and it can be extended to $[0, \infty)$.

It remains to show inequality (1.3.17). Obviously $E(t) \leq \mathscr{E}(t)$ since $F(u,v)$ and $H(s)$ are non-negative functions. On the other hand, by (4.1.8) and the definition of $E(t)$, one has

$$E(t) \geq \frac{1}{2}(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2) + \left(\frac{1}{2} - \frac{1}{c}\right)(\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2) \geq \left(1 - \frac{2}{c}\right)\mathscr{E}(t).$$

Thus, the proof of Theorem 1.3.18 is now complete. □

4.2 Uniform Decay Rates of Energy

In this section we study the uniform decay rate of the energy for the global solution furnished by Theorem 1.3.18. More precisely, we shall prove Theorem 1.3.19.

Recall that the depth of the potential well d is defined in (1.3.13). The following lemma will be needed in the sequel.

Lemma 4.2.1. *Under the assumptions of Lemma 1.3.17, the depth of the potential well d coincides with the mountain pass level. Specifically,*

$$d = \inf_{(u,v) \in X \setminus \{(0,0)\}} \sup_{\lambda \geq 0} J(\lambda(u,v)). \tag{4.2.1}$$

Proof. Recall $X = H^1(\Omega) \times H_0^1(\Omega)$. Let $(u, v) \in X\backslash\{(0,0)\}$ be fixed. By recalling Assumption 1.3.15, it follows that,

$$J(\lambda(u,v)) = \frac{1}{2}\lambda^2(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2) - \lambda^{p+1}\int_\Omega F(u,v)dx - \lambda^{k+1}\int_\Gamma H(\gamma u)d\Gamma, \quad (4.2.2)$$

for $\lambda \geq 0$. Then,

$$\frac{d}{d\lambda}J(\lambda(u,v)) = \lambda\Big[(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2) - (p+1)\lambda^{p-1}\int_\Omega F(u,v)dx \\ - (k+1)\lambda^{k-1}\int_\Gamma H(\gamma u)d\Gamma\Big]. \quad (4.2.3)$$

Hence, the only critical point in $(0, \infty)$ for the mapping $\lambda \mapsto J(\lambda(u,v))$ is λ_0 which satisfies the equation:

$$(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2) = (p+1)\lambda_0^{p-1}\int_\Omega F(u,v)dx + (k+1)\lambda_0^{k-1}\int_\Gamma H(\gamma u)d\Gamma. \quad (4.2.4)$$

Moreover, it is easy to see that

$$\sup_{\lambda\geq 0} J(\lambda(u,v)) = J(\lambda_0(u,v)). \quad (4.2.5)$$

By the definition of $\mathcal{N}$ and noting (4.2.4), we conclude that $\lambda_0(u,v) \in \mathcal{N}$. As a result,

$$J(\lambda_0(u,v)) \geq \inf_{(y,z)\in\mathcal{N}} J(y,z) = d. \quad (4.2.6)$$

By combining (4.2.5) and (4.2.6), one has

$$\inf_{(u,v)\in X\backslash\{(0,0)\}} \sup_{\lambda\geq 0} J(\lambda(u,v)) \geq d. \quad (4.2.7)$$

On the other hand, for each fixed $(y,z) \in \mathcal{N}$, we find that (using (1.3.12) and (4.2.4)) the only critical point in $(0,\infty)$ of the mapping $\lambda \mapsto J(\lambda(y,z))$ is $\lambda_0 = 1$. Therefore, $\sup_{\lambda\geq 0} J(\lambda(y,z)) = J(y,z)$ for each $(y,z) \in \mathcal{N}$. Hence

$$\inf_{(u,v)\in X\backslash\{(0,0)\}} \sup_{\lambda\geq 0} J(\lambda(u,v)) \leq \inf_{(y,z)\in\mathcal{N}} \sup_{\lambda\geq 0} J(\lambda(y,z)) = \inf_{(y,z)\in\mathcal{N}} J(y,z) = d. \quad (4.2.8)$$

Combining (4.2.7) and (4.2.8) gives the desired result (4.2.1). $\square$

Now we introduce several functions. Let φ_j, $\varphi : [0,\infty) \to [0,\infty)$ be continuous, increasing, concave functions, vanishing at the origin, and such that

$$\varphi_j(g_j(s)s) \geq |g_j(s)|^2 + s^2 \quad \text{for} \quad |s| < 1, \quad j = 1, 2; \tag{4.2.9}$$

and

$$\varphi(g(s)s) \geq |g(s)|^2 \quad \text{for} \quad |s| < 1. \tag{4.2.10}$$

We also define the function $\Phi : [0,\infty) \to [0,\infty)$ by

$$\Phi(s) := \varphi_1(s) + \varphi_2(s) + \varphi(s) + s, \quad s \geq 0. \tag{4.2.11}$$

We note here that the concave functions φ_1, φ_2 and φ mentioned in (4.2.9)-(4.2.10) can always be constructed. To see this, recall the damping g_1, g_2 and g are monotone increasing functions passing through the origin. If g_1, g_2 and g are bounded above and below by linear or superlinear functions near the origin, i.e., for all $|s| < 1$,

$$c_1|s|^m \leq |g_1(s)| \leq c_2|s|^m,\ c_3|s|^r \leq |g_2(s)| \leq c_4|s|^r,\ c_5|s|^q \leq |g(s)| \leq c_6|s|^q, \tag{4.2.12}$$

where m, r, $q \geq 1$ and $c_j > 0$, $j = 1, ..., 6$, then we can select

$$\varphi_1(s) = c_1^{-\frac{2}{m+1}}(1+c_2^2)s^{\frac{2}{m+1}},\ \ \varphi_2(s) = c_3^{-\frac{2}{r+1}}(1+c_4^2)s^{\frac{2}{r+1}},\ \ \varphi = c_5^{-\frac{2}{q+1}}c_6^2 s^{\frac{2}{q+1}}. \tag{4.2.13}$$

It is straightforward to see the functions in (4.2.13) verify (4.2.9)-(4.2.10). To see this, consider φ_1 for example:

$$\begin{aligned}\varphi_1(g_1(s)s) &= c_1^{-\frac{2}{m+1}}(1+c_2^2)(g_1(s)s)^{\frac{2}{m+1}} \geq c_1^{-\frac{2}{m+1}}(1+c_2^2)(c_1|s|^{m+1})^{\frac{2}{m+1}} \\ &= (1+c_2^2)s^2 \geq s^2 + (c_2|s|^m)^2 \geq s^2 + |g_1(s)|^2, \text{ for all } |s| < 1.\end{aligned}$$

In particular, we note that, if g_1, g_2 and g are all linearly bounded near the origin, then (4.2.13) shows φ_1, φ_2 and φ are all linear functions.

However, if the damping are bounded by sublinear functions near the origin, namely, for all $|s| < 1$,

$$c_1|s|^{\theta_1} \leq |g_1(s)| \leq c_2|s|^{\theta_1},\ c_3|s|^{\theta_2} \leq |g_2(s)| \leq c_4|s|^{\theta_2},\ c_5|s|^{\theta} \leq |g(s)| \leq c_6|s|^{\theta}, \tag{4.2.14}$$

where $0 < \theta_1$, θ_2, $\theta < 1$ and $c_j > 0$, $j = 1, ..., 6$, then instead we can select

$$\varphi_1(s) = c_1^{-\frac{2\theta_1}{\theta_1+1}}(1+c_2^2)s^{\frac{2\theta_1}{\theta_1+1}},\ \ \varphi_2(s) = c_3^{-\frac{2\theta_2}{\theta_2+1}}(1+c_4^2)s^{\frac{2\theta_2}{\theta_2+1}},\ \ \varphi = c_5^{-\frac{2\theta}{\theta+1}}c_6^2 s^{\frac{2\theta}{\theta+1}}. \tag{4.2.15}$$

In sum, by (4.2.13) and (4.2.15), there exist constants C_1, C_2, $C_3 > 0$ such that

$$\varphi_1(s) = C_1 s^{z_1}, \quad \varphi_2(s) = C_2 s^{z_2}, \quad \varphi(s) = C_3 s^{z}, \tag{4.2.16}$$

where

$$z_1 := \frac{2}{m+1} \text{ or } \frac{2\theta_1}{\theta_1+1}, \quad z_2 := \frac{2}{r+1} \text{ or } \frac{2\theta_2}{\theta_2+1}, \quad z := \frac{2}{q+1} \text{ or } \frac{2\theta}{\theta+1} \tag{4.2.17}$$

depending on the growth rates of g_1, g_2 and g near the origin, which are specified in (4.2.12) and (4.2.14).

Now, we define

$$j := \max\left\{\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z}\right\}. \tag{4.2.18}$$

It is important to note that $j > 1$ if at least one of g_1, g_2 and g are not linearly bounded near the origin, and in this case we put

$$\beta := \frac{1}{j-1} > 0. \tag{4.2.19}$$

For the sake of simplifying the notations, we define

$$\mathbf{D}(t) := \int_0^t \int_\Omega [g_1(u_t)u_t + g_2(v_t)v_t] dx d\tau + \int_0^t \int_\Gamma g(\gamma u_t)\gamma u_t d\Gamma d\tau.$$

We note here that $\mathbf{D}(t) \geq 0$, by the monotonicity of g_1, g_2 and g, and the energy identity (4.1.1) can be written as

$$E(t) + \mathbf{D}(t) = E(0). \tag{4.2.20}$$

For the remainder of the proof of Theorem 1.3.19, we define

$$T_0 := \max\left\{1, \frac{1}{|\Omega|}, \frac{1}{|\Gamma|}, 8c_0\left(\frac{c}{c-2}\right)\right\} \tag{4.2.21}$$

where c_0 is the constant in the Poincaré-Wirtinger type of inequality (1.2.3), and $c = \min\{p+1, k+1\} > 2$.

4.2.1 Perturbed stabilization estimate

Proposition 4.2.2. *In addition to Assumptions 1.1.1 and 1.3.15, assume that* $1 < p < 5$, $1 < k < 3$, $u_0 \in L^{m+1}(\Omega)$, $v_0 \in L^{r+1}(\Omega)$, $\gamma u_0 \in L^{q+1}(\Gamma)$, $(u_0, v_0) \in \mathcal{W}_1$, *and* $E(0) < d$. *We further assume that* $u \in L^\infty(\mathbb{R}^+; L^{\frac{3}{2}(m-1)}(\Omega))$ *if* $m > 5$, $v \in L^\infty(\mathbb{R}^+; L^{\frac{3}{2}(r-1)}(\Omega))$ *if* $r > 5$, *and* $\gamma u \in L^\infty(\mathbb{R}^+; L^{2(q-1)}(\Gamma))$ *if* $q > 3$, *where* (u, v) *is the global solution of (1.1.1) furnished by Theorem 1.3.18. Then*

$$E(T) \leq \hat{C}\left[\Phi(\boldsymbol{D}(T)) + \int_0^T (\|u(t)\|_2^2 + \|v(t)\|_2^2)dt\right], \tag{4.2.22}$$

for all $T \geq T_0$, *where* T_0 *is defined in (4.2.21),* Φ *is given in (4.2.11), and* $\hat{C} > 0$ *is independent of* T.

Proof. Let $T \geq T_0$ be fixed. We begin by verifying $u \in L^{m+1}(\Omega \times (0,T))$ for all $T \in [0, \infty)$. Since both u and $u_t \in C([0,T]; L^2(\Omega))$, we can write

$$\begin{aligned}\int_0^T \int_\Omega |u|^{m+1} dxdt &= \int_0^T \int_\Omega \left|\int_0^t u_t(\tau)d\tau + u_0\right|^{m+1} dxdt \\ &\leq 2^m (T^{m+1} \|u_t\|_{L^{m+1}(\Omega\times(0,T))}^{m+1} + T\|u_0\|_{m+1}^{m+1}) < \infty,\end{aligned}$$

where we have used the regularity enjoyed by u, namely, $u_t \in L^{m+1}(\Omega \times (0,T))$, and the assumption $u_0 \in L^{m+1}(\Omega)$. Note, if $m \leq 5$, then $u_0 \in L^{m+1}(\Omega)$ is not an extra assumption since $u_0 \in H^1(\Omega) \hookrightarrow L^6(\Omega)$.

Similarly, we can show $v \in L^{r+1}(\Omega \times (0,T))$ and $\gamma u \in L^{q+1}(\Gamma \times (0,T))$. It follows that u and v enjoy, respectively, the regularity restrictions imposed on the test function ϕ and ψ, as stated in Definition 1.3.1. Consequently, we can replace ϕ by u in (1.3.1) and ψ by v in (1.3.2), and then the sum of two equations gives

$$\begin{aligned}&\left[\int_\Omega (u_t u + v_t v)dx\right]_0^T - \int_0^T (\|u_t\|_2^2 + \|v_t\|_2^2)dt + \int_0^T (\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2)dt \\ &\quad + \int_0^T \int_\Omega (g_1(u_t)u + g_2(v_t)v)dxdt + \int_0^T \int_\Gamma g(\gamma u_t)\gamma u d\Gamma dt \\ &\quad = \int_0^T \int_\Omega [f_1(u,v)u + f_2(u,v)v]dxdt + \int_0^T \int_\Gamma h(\gamma u)\gamma u d\Gamma dt.\end{aligned} \tag{4.2.23}$$

After a rearrangement of (4.2.23) and employing the identity (1.3.6), we obtain

$$
\begin{aligned}
2\int_0^T \mathscr{E}(t)dt =& 2\int_0^T (\|u_t\|_2^2 + \|v_t\|_2^2)dt - \left[\int_\Omega (u_t u + v_t v)dx\right]_0^T \\
& - \int_0^T \int_\Omega (g_1(u_t)u + g_2(v_t)v)dxdt - \int_0^T \int_\Gamma g(\gamma u_t)\gamma u d\Gamma dt \\
& + (p+1)\int_0^T \int_\Omega F(u,v)dxdt + (k+1)\int_0^T \int_\Gamma H(\gamma u)d\Gamma dt. \qquad (4.2.24)
\end{aligned}
$$

By recalling (1.3.7), one has

$$
\begin{aligned}
\int_0^T \mathscr{E}(t)dt \le& \int_0^T (\|u_t\|_2^2 + \|v_t\|_2^2)dt + \left|\left[\int_\Omega (u_t u + v_t v)dx\right]_0^T\right| \\
& + \left[\int_0^T \int_\Omega |g_1(u_t)u + g_2(v_t)v|dxdt + \int_0^T \int_\Gamma |g(\gamma u_t)\gamma u|d\Gamma dt\right] \\
& + C\int_0^T (\|u\|_{p+1}^{p+1} + \|v\|_{p+1}^{p+1} + |\gamma u|_{k+1}^{k+1})dt. \qquad (4.2.25)
\end{aligned}
$$

Now we start with estimating each term on the right-hand side of (4.2.25).

1. **Estimate for**

$$
\left|\left[\int_\Omega (u_t u + v_t v)dx\right]_0^T\right|.
$$

Notice

$$
\begin{aligned}
\left|\int_\Omega (u_t(t)u(t) + v_t(t)v(t))dx\right| &\le \|u_t(t)\|_2 \|u(t)\|_2 + \|v_t(t)\|_2 \|v(t)\|_2 \\
&\le \frac{1}{2}(\|u_t(t)\|_2^2 + \|u(t)\|_2^2 + \|v_t(t)\|_2^2 + \|v(t)\|_2^2) \le c_0\mathscr{E}(t), \quad \text{for all} \quad t \ge 0,
\end{aligned}
$$

where $c_0 > 0$ is the constant in the Poincaré-Wirtinger type of inequality (1.2.3). Thus, by (1.3.17) and (4.2.20), it follows that

$$
\begin{aligned}
\left|\left[\int_\Omega (u_t u + v_t v)dx\right]_0^T\right| &\le c_0(\mathscr{E}(T) + \mathscr{E}(0)) \le c_0\left(\frac{c}{c-2}\right)(E(T) + E(0)) \\
&\le c_0\left(\frac{c}{c-2}\right)\Big(2E(T) + \mathbf{D}(T)\Big). \qquad (4.2.26)
\end{aligned}
$$

2. **Estimate for**

$$\int_0^T (\|u\|_{p+1}^{p+1} + \|v\|_{p+1}^{p+1} + |\gamma u|_{k+1}^{k+1})dt.$$

Since $p < 5$, then by the Sobolev Imbedding Theorem, $H^{1-\delta}(\Omega) \hookrightarrow L^{p+1}(\Omega)$, for sufficiently small $\delta > 0$, and by using a standard interpolation, we obtain

$$\|u\|_{p+1} \le C\,\|u\|_{H^{1-\delta}(\Omega)} \le C\,\|u\|_{1,\Omega}^{1-\delta}\,\|u\|_2^{\delta}.$$

Applying Young's inequality yields

$$\|u\|_{p+1}^{p+1} \le C\,\|u\|_{1,\Omega}^{(1-\delta)(p+1)}\,\|u\|_2^{\delta(p+1)} \le \epsilon_0\,\|u\|_{1,\Omega}^{\frac{2(1-\delta)(p+1)}{2-\delta(p+1)}} + C_{\epsilon_0}\,\|u\|_2^2 \tag{4.2.27}$$

for all $\epsilon_0 > 0$, and where we have required $\delta < \frac{2}{p+1}$. By (1.3.17) and (4.1.3), one has

$$\|u\|_{1,\Omega}^2 \le 2\mathscr{E}(t) \le \left(\frac{2c}{c-2}\right)E(t) \le \left(\frac{2c}{c-2}\right)E(0). \tag{4.2.28}$$

Since $p > 1$ and $\delta < \frac{2}{p+1}$, then $\frac{2(1-\delta)(p+1)}{2-\delta(p+1)} > 2$, and thus combining (4.2.27) and (4.2.28) implies

$$\|u\|_{p+1}^{p+1} \le \epsilon_0 C(E(0))\,\|u\|_{1,\Omega}^2 + C_{\epsilon_0}\,\|u\|_2^2. \tag{4.2.29}$$

For each $\epsilon > 0$, if we choose $\epsilon_0 = \frac{\epsilon}{C(E(0))}$, then (4.2.29) gives

$$\|u\|_{p+1}^{p+1} \le \epsilon\,\|u\|_{1,\Omega}^2 + C(\epsilon, E(0))\,\|u\|_2^2. \tag{4.2.30}$$

Replacing u by v in (4.2.27)-(4.2.30) yields

$$\|v\|_{p+1}^{p+1} \le \epsilon\,\|v\|_{1,\Omega}^2 + C(\epsilon, E(0))\,\|v\|_2^2. \tag{4.2.31}$$

Also, since $k < 3$, then by the Sobolev Imbedding Theorem $|\gamma u|_{k+1} \le C\,\|u\|_{H^{1-\delta}(\Omega)}$, for sufficiently small $\delta > 0$. By employing similar estimates as in (4.2.27)-(4.2.30), we deduce

$$|\gamma u|_{k+1}^{k+1} \le \epsilon\,\|u\|_{1,\Omega}^2 + C(\epsilon, E(0))\,\|u\|_2^2. \tag{4.2.32}$$

A combination of the estimates (4.2.30)-(4.2.32) yields

$$\begin{aligned}&\int_0^T (\|u\|_{p+1}^{p+1} + \|v\|_{p+1}^{p+1} + |\gamma u|_{k+1}^{k+1})dt\\ &\le 4\epsilon\int_0^T \mathscr{E}(t)dt + C(\epsilon, E(0))\int_0^T (\|u\|_2^2 + \|v\|_2^2)dt.\end{aligned} \tag{4.2.33}$$

3. **Estimate for**
$$\int_0^T (\|u_t\|_2^2 + \|v_t\|_2^2) dt.$$

We introduce the sets:

$$\begin{aligned} A &:= \{(x,t) \in \Omega \times (0,T) : |u_t(x,t)| < 1\} \\ B &:= \{(x,t) \in \Omega \times (0,T) : |u_t(x,t)| \geq 1\}. \end{aligned}$$

By Assumption 1.1.1, we know $g_1(s)s \geq a_1|s|^{m+1} \geq a_1|s|^2$ for $|s| \geq 1$. Therefore, applying (4.2.9) and the fact φ_1 is concave and increasing implies,

$$\begin{aligned} \int_0^T \|u_t\|_2^2 \, dt &= \int_A |u_t|^2 dxdt + \int_B |u_t|^2 dxdt \\ &\leq \int_A \varphi_1(g_1(u_t)u_t) dxdt + \int_B g_1(u_t)u_t dxdt \\ &\leq T|\Omega|\varphi_1\left(\int_0^T \int_\Omega g_1(u_t)u_t dxdt\right) + \int_0^T \int_\Omega g_1(u_t)u_t dxdt, \end{aligned} \tag{4.2.34}$$

where we have used Jensen's inequality and our choice of T, namely $T|\Omega| \geq 1$. Likewise, one has

$$\int_0^T \|v_t\|_2^2 \, dt \leq T|\Omega|\varphi_2\left(\int_0^T \int_\Omega g_2(v_t)v_t dxdt\right) + \int_0^T \int_\Omega g_2(v_t)v_t dxdt. \tag{4.2.35}$$

4. **Estimate for**
$$\int_0^T \int_\Omega |g_1(u_t)u + g_2(v_t)v| dxdt + \int_0^T \int_\Gamma |g(\gamma u_t)\gamma u| d\Gamma dt.$$

<u>*Case 1*</u>: m, $r \leq 5$ and $q \leq 3$.

We will concentrate on evaluating $\int_0^T \int_\Omega |g_1(u_t)u| dxdt$. Notice

$$\begin{aligned} \int_0^T \int_\Omega |g_1(u_t)u| dxdt &= \int_A |g_1(u_t)u| dxdt + \int_B |g_1(u_t)u| dxdt \\ &\leq \left(\int_0^T \|u\|_2^2 \, dt\right)^{\frac{1}{2}} \left(\int_A |g_1(u_t)|^2 dxdt\right)^{\frac{1}{2}} + \int_B |g_1(u_t)u| dxdt \\ &\leq \epsilon \int_0^T \mathscr{E}(t) dt + C_\epsilon \int_A |g_1(u_t)|^2 dxdt + \int_B |g_1(u_t)u| dxdt \end{aligned} \tag{4.2.36}$$

where we have used Hölder's and Young's inequalities. By (4.2.9), Jensen's inequality and the fact $T|\Omega| \geq 1$, we have

$$\int_A |g_1(u_t)|^2 dxdt \leq \int_A \varphi_1(g_1(u_t)u_t)dxdt \leq T|\Omega|\varphi_1\left(\int_0^T \int_\Omega g_1(u_t)u_t dxdt\right). \quad (4.2.37)$$

Next, we estimate the last term on the right-hand side of (4.2.36). Since $m \leq 5$, then by Assumption 1.1.1, we know $|g_1(s)| \leq b_1|s|^m \leq b_1|s|^5$ for $|s| \geq 1$. Therefore, by Hölder's inequality, we deduce

$$\begin{aligned}
\int_B |g_1(u_t)u|dxdt &\leq \left(\int_B |u|^6 dxdt\right)^{\frac{1}{6}} \left(\int_B |g_1(u_t)|^{\frac{6}{5}} dxdt\right)^{\frac{5}{6}} \\
&\leq \left(\int_0^T \|u\|_6^6 \, dt\right)^{\frac{1}{6}} \left(\int_B |g_1(u_t)||g_1(u_t)|^{\frac{1}{5}} dxdt\right)^{\frac{5}{6}} \\
&\leq b_1^{\frac{1}{6}} \left(\int_0^T \|u\|_6^6 \, dt\right)^{\frac{1}{6}} \left(\int_B |g_1(u_t)||u_t| dxdt\right)^{\frac{5}{6}}. \quad (4.2.38)
\end{aligned}$$

By recalling inequality (1.3.16) which states $\mathscr{E}(t) \leq d\left(\frac{c}{c-2}\right)$, for all $t \geq 0$, we have

$$\int_0^T \|u\|_6^6 \, dt \leq C \int_0^T \|u\|_{1,\Omega}^6 \, dt \leq C \int_0^T \mathscr{E}(t)^3 dt \leq C \int_0^T \mathscr{E}(t)dt. \quad (4.2.39)$$

Combining (4.2.38) and (4.2.39) yields

$$\begin{aligned}
\int_B |g_1(u_t)u|dxdt &\leq C\left(\int_0^T \mathscr{E}(t)dt\right)^{\frac{1}{6}} \left(\int_0^T \int_\Omega g_1(u_t)u_t dxdt\right)^{\frac{5}{6}} \\
&\leq \epsilon \int_0^T \mathscr{E}(t)dt + C_\epsilon \int_0^T \int_\Omega g_1(u_t)u_t dxdt \quad (4.2.40)
\end{aligned}$$

where we have used Young's inequality.

By applying the estimates (4.2.37) and (4.2.40), we obtain from (4.2.36) that

$$\begin{aligned}
\int_0^T \int_\Omega |g_1(u_t)u|dxdt &\leq 2\epsilon \int_0^T \mathscr{E}(t)dt \\
&+ C_\epsilon T|\Omega|\varphi_1\left(\int_0^T \int_\Omega g_1(u_t)u_t dxdt\right) + C_\epsilon \int_0^T \int_\Omega g_1(u_t)u_t dxdt, \text{ if } m \leq 5.
\end{aligned} \quad (4.2.41)$$

Similarly,

$$\int_0^T \int_\Omega |g_2(v_t)v| dxdt \leq 2\epsilon \int_0^T \mathscr{E}(t)dt$$
$$+ C_\epsilon T|\Omega|\varphi_2 \left(\int_0^T \int_\Omega g_2(v_t)v_t dxdt \right) + C_\epsilon \int_0^T \int_\Omega g_2(v_t)v_t dxdt, \text{ if } r \leq 5. \quad (4.2.42)$$

Likewise, since $T|\Gamma| \geq 1$, we similarly derive

$$\int_0^T \int_\Gamma |g(\gamma u_t)\gamma u| d\Gamma dt \leq 2\epsilon \int_0^T \mathscr{E}(t)dt$$
$$+ C_\epsilon T|\Gamma|\varphi \left(\int_0^T \int_\Gamma g(\gamma u_t)\gamma u_t d\Gamma dt \right) + C_\epsilon \int_0^T \int_\Gamma g(\gamma u_t)\gamma u_t d\Gamma dt, \text{ if } q \leq 3. \quad (4.2.43)$$

Case 2: $\max\{m, r\} > 5$ or $q > 3$.

In this case, we impose the additional assumption $u \in L^\infty(\mathbb{R}^+; L^{\frac{3}{2}(m-1)}(\Omega))$ if $m > 5$, $v \in L^\infty(\mathbb{R}^+; L^{\frac{3}{2}(r-1)}(\Omega))$ if $r > 5$, and $\gamma u \in L^\infty(\mathbb{R}^+; L^{2(q-1)}(\Gamma))$ if $q > 3$.

We evaluate the last term on the right-hand side of (4.2.36) for the case $m > 5$. By Hölder's inequality, we have

$$\int_B |g_1(u_t)u| dxdt \leq \left[\int_B |g_1(u_t)|^{\frac{m+1}{m}} dxdt \right]^{\frac{m}{m+1}} \left[\int_B |u|^{m+1} dxdt \right]^{\frac{1}{m+1}}. \quad (4.2.44)$$

Since $|g_1(s)| \leq b_1|s|^m$ for all $|s| \geq 1$, one has

$$\int_B |g_1(u_t)|^{\frac{m+1}{m}} dxdt = \int_B |g_1(u_t)||g_1(u_t)|^{\frac{1}{m}} dxdt \leq b_1^{\frac{1}{m}} \int_B |g_1(u_t)||u_t| dxdt. \quad (4.2.45)$$

We evaluate the last term in (4.2.44) using Hölder's inequality:

$$\int_B |u|^{m+1} dxdt \leq \int_0^T \int_\Omega |u|^2 |u|^{m-1} dxdt \leq \int_0^T \|u\|_6^2 \|u\|_{\frac{3}{2}(m-1)}^{m-1} dt$$
$$\leq C \|u\|_{L^\infty(\mathbb{R}^+; L^{\frac{3}{2}(m-1)}(\Omega))}^{m-1} \int_0^T \mathscr{E}(t)dt. \quad (4.2.46)$$

Now, combining (4.2.44)-(4.2.46) yields

$$
\begin{aligned}
&\int_B |g_1(u_t)u|dxdt \\
&\leq C\,\|u\|_{L^\infty(\mathbb{R}^+;L^{\frac{3}{2}(m-1)}(\Omega))}^{\frac{m-1}{m+1}} \left(\int_0^T \mathscr{E}(t)dt\right)^{\frac{1}{m+1}} \left(\int_B |g_1(u_t)||u_t|dxdt\right)^{\frac{m}{m+1}} \\
&\leq \epsilon\,\|u\|_{L^\infty(\mathbb{R}^+;L^{\frac{3}{2}(m-1)}(\Omega))}^{m-1} \int_0^T \mathscr{E}(t)dt + C_\epsilon \int_0^T \int_\Omega g_1(u_t)u_t dxdt \qquad (4.2.47)
\end{aligned}
$$

where we have used Young's inequality.

By (4.2.36), (4.2.37) and (4.2.47), one has

$$
\begin{aligned}
&\int_0^T \int_\Omega |g_1(u_t)u|dxdt \leq \epsilon \left(1 + \|u\|_{L^\infty(\mathbb{R}^+;L^{\frac{3}{2}(m-1)}(\Omega))}^{m-1}\right) \int_0^T \mathscr{E}(t)dt \\
&+ C_\epsilon T|\Omega|\varphi_1 \left(\int_0^T \int_\Omega g_1(u_t)u_t dxdt\right) + C_\epsilon \int_0^T \int_\Omega g_1(u_t)u_t dxdt, \text{ if } m > 5. \qquad (4.2.48)
\end{aligned}
$$

Similarly, we can deduce

$$
\begin{aligned}
&\int_0^T \int_\Omega |g_2(v_t)v|dxdt \leq \epsilon \left(1 + \|v\|_{L^\infty(\mathbb{R}^+;L^{\frac{3}{2}(r-1)}(\Omega))}^{r-1}\right) \int_0^T \mathscr{E}(t)dt \\
&\quad + C_\epsilon T|\Omega|\varphi_2 \left(\int_0^T \int_\Omega g_2(v_t)v_t dxdt\right) + C_\epsilon \int_0^T \int_\Omega g_2(v_t)v_t dxdt, \text{ if } r > 5; \qquad (4.2.49)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^T \int_\Gamma |g(\gamma u_t)\gamma u|dxdt \leq \epsilon \left(1 + \|\gamma u\|_{L^\infty(\mathbb{R}^+;L^{2(q-1)}(\Gamma))}^{q-1}\right) \int_0^T \mathscr{E}(t)dt \\
&+ C_\epsilon T|\Gamma|\varphi \left(\int_0^T \int_\Gamma g(\gamma u_t)\gamma u_t d\Gamma dt\right) + C_\epsilon \int_0^T \int_\Gamma g(\gamma u_t)\gamma u_t d\Gamma dt, \text{ if } q > 3. \qquad (4.2.50)
\end{aligned}
$$

Now, if we combine the estimates (4.2.25), (4.2.26), (4.2.33)-(4.2.35), (4.2.41)-(4.2.43), (4.2.48)-(4.2.50), then by selecting ϵ sufficiently small and since $T \geq T_0 \geq 1$, we conclude

$$
\begin{aligned}
\frac{1}{2}\int_0^T \mathscr{E}(t)dt &\leq c_0 \left(\frac{c}{c-2}\right)(2E(T) + \mathbf{D}(T)) + C(\epsilon, E(0)) \int_0^T (\|u\|_2^2 + \|v\|_2^2)dt \\
&\quad + T \cdot C(\epsilon, |\Omega|, |\Gamma|)\Phi(\mathbf{D}(T)). \qquad (4.2.51)
\end{aligned}
$$

Since $\mathscr{E}(t) \geq E(t)$ for all $t \geq 0$ and $E(t)$ is non-increasing, one has

$$\int_0^T \mathscr{E}(t)dt \geq \int_0^T E(t)dt \geq TE(T). \tag{4.2.52}$$

Appealing to the fact $T \geq T_0 \geq 8c_0\left(\frac{c}{c-2}\right)$, then (4.2.51) and (4.2.52) yield

$$\begin{aligned}\frac{1}{4}TE(T) \leq c_0\left(\frac{c}{c-2}\right)\mathbf{D}(T) + C(\epsilon, E(0))\int_0^T(\|u\|_2^2 + \|v\|_2^2)dt \\ + T\cdot C(\epsilon, |\Omega|, |\Gamma|)\Phi(\mathbf{D}(T)).\end{aligned} \tag{4.2.53}$$

Since $T \geq 1$, dividing both sides of (4.2.53) by T yields

$$\begin{aligned}\frac{1}{4}E(T) \leq c_0\left(\frac{c}{c-2}\right)\mathbf{D}(T) + C(\epsilon, E(0))\int_0^T(\|u\|_2^2 + \|v\|_2^2)dt \\ + C(\epsilon, |\Omega|, |\Gamma|)\Phi(\mathbf{D}(T)).\end{aligned} \tag{4.2.54}$$

Finally, if we put $\hat{C} := 4[c_0\left(\frac{c}{c-2}\right) + C(\epsilon, |\Omega|, |\Gamma|) + C(\epsilon, E(0))]$, then (4.2.54) shows

$$E(T) \leq \hat{C}\left[\Phi(\mathbf{D}(T)) + \int_0^T(\|u(t)\|_2^2 + \|v(t)\|_2^2)dt\right] \tag{4.2.55}$$

for all $T \geq T_0 = \max\{1, \frac{1}{|\Omega|}, \frac{1}{|\Gamma|}, 8c_0\left(\frac{c}{c-2}\right)\}$. $\square$

4.2.2 Explicit approximation of the "good" part $\mathcal{W}_1$ of the potential well

In order to estimate the lower order terms $\int_0^T(\|u(t)\|_2^2 + \|v(t)\|_2^2)dt$ in (4.2.22), we shall construct an explicit subset $\tilde{\mathcal{W}}_1 \subset \mathcal{W}_1$, which approximates the "good" part of the well $\mathcal{W}_1$. By the definition of $J(u,v)$ in (1.3.8) and the bounds in (1.3.7), it follows that

$$J(u,v) \geq \frac{1}{2}(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2) - M(\|u\|_{p+1}^{p+1} + \|v\|_{p+1}^{p+1} + |\gamma u|_{k+1}^{k+1}).$$

By recalling the constants defined in (1.3.19), we have

$$\begin{aligned}J(u,v) &\geq \frac{1}{2}(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2) - MR_1(\|u\|_{1,\Omega}^{p+1} + \|v\|_{1,\Omega}^{p+1}) - MR_2\|u\|_{1,\Omega}^{k+1} \\ &\geq \frac{1}{2}\|(u,v)\|_X^2 - MR_1\|(u,v)\|_X^{p+1} - MR_2\|(u,v)\|_X^{k+1}\end{aligned} \tag{4.2.56}$$

where $X = H^1(\Omega) \times H_0^1(\Omega)$.

By recalling the function $\mathcal{G}(s)$ defined in (1.3.18), namely

$$\mathcal{G}(s) := \frac{1}{2}s^2 - MR_1 s^{p+1} - MR_2 s^{k+1},$$

then inequality (4.2.56) is equivalent to

$$J(u,v) \geq \mathcal{G}(\|(u,v)\|_X). \tag{4.2.57}$$

Since p, $k > 1$, then

$$\mathcal{G}'(s) = s\left(1 - MR_1(p+1)s^{p-1} - MR_2(k+1)s^{k-1}\right)$$

has only one positive zero at, say at $s_0 > 0$, where s_0 satisfies:

$$MR_1(p+1)s_0^{p-1} + MR_2(k+1)s_0^{k-1} = 1. \tag{4.2.58}$$

It is easy to verify that $\sup_{s\in[0,\infty)} \mathcal{G}(s) = \mathcal{G}(s_0) > 0$. Thus, we can define the following set as in (1.3.20):

$$\tilde{\mathcal{W}}_1 := \{(u,v) \in X : \|(u,v)\|_X < s_0,\ J(u,v) < \mathcal{G}(s_0)\}.$$

It is important to note $\tilde{\mathcal{W}}_1$ is not a trivial set. In fact, for any $(u,v) \in X$, there exists a scalar $\epsilon > 0$ such that $\epsilon(u,v) \in \tilde{\mathcal{W}}_1$. Moreover, we have the following result.

Proposition 4.2.3. *$\tilde{\mathcal{W}}_1$ is a subset of $\mathcal{W}_1$.*

Proof. We first show $\mathcal{G}(s_0) \leq d$. Fix $(u,v) \in X \setminus \{(0,0)\}$, then (4.2.57) yields $J(\lambda(u,v)) \geq \mathcal{G}(\lambda \|(u,v)\|_X)$ for all $\lambda \geq 0$. It follows that

$$\sup_{\lambda\geq 0} J(\lambda(u,v)) \geq \mathcal{G}(s_0).$$

Therefore, by Lemma 4.2.1, one has

$$d = \inf_{(u,v)\in X\setminus\{(0,0)\}} \sup_{\lambda\geq 0} J(\lambda(u,v)) \geq \mathcal{G}(s_0).$$

Moreover, for all $\|(u,v)\|_X < s_0$, by employing (1.3.7) and (1.3.19), we argue

$$\begin{aligned}
&(p+1)\int_\Omega F(u,v)dx + (k+1)\int_\Gamma H(\gamma u)d\Gamma \\
&\leq (p+1)MR_1(\|u\|_{1,\Omega}^{p+1} + \|v\|_{1,\Omega}^{p+1}) + (k+1)MR_2\|u\|_{1,\Omega}^{k+1} \\
&\leq \|(u,v)\|_X^2\left[(p+1)MR_1\|(u,v)\|_X^{p-1} + (k+1)MR_2\|(u,v)\|_X^{k-1}\right] \\
&< \|(u,v)\|_X^2\left[(p+1)MR_1 s_0^{p-1} + (k+1)MR_2 s_0^{k-1}\right] \\
&= \|(u,v)\|_X^2 = \|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2
\end{aligned}$$

where we have used (4.2.58). Therefore, by the definition of $\mathcal{W}_1$, it follows that $\tilde{\mathcal{W}}_1 \subset \mathcal{W}_1$. $\square$

For each fixed sufficiently small $\delta > 0$, we can define a closed subset of $\tilde{\mathcal{W}}_1$ as in (1.3.21), namely,

$$\tilde{\mathcal{W}}_1^\delta := \{(u,v) \in X : \|(u,v)\|_X \leq s_0 - \delta,\ J(u,v) \leq \mathcal{G}(s_0 - \delta)\},$$

and we show $\tilde{\mathcal{W}}_1^\delta$ is invariant under the dynamics.

Proposition 4.2.4. *Assume $\delta > 0$ is sufficiently small and $E(0) \leq \mathcal{G}(s_0 - \delta)$. If (u,v) is the global solution of (1.1.1) furnished by Theorem 1.3.18 and $(u_0, v_0) \in \tilde{\mathcal{W}}_1^\delta$, then $(u(t), v(t)) \in \tilde{\mathcal{W}}_1^\delta$ for all $t \geq 0$.*

Proof. By the fact $J(u(t), v(t)) \leq E(t) \leq E(0)$ and by assumption $E(0) \leq \mathcal{G}(s_0 - \delta)$, we obtain $J(u(t), v(t)) \leq \mathcal{G}(s_0 - \delta)$ for all $t \geq 0$. To show $\|(u(t), v(t))\|_X \leq s_0 - \delta$ for all $t \geq 0$, we argue by contradiction. Since $\|(u_0, v_0)\|_X \leq s_0 - \delta$ and $(u,v) \in C(\mathbb{R}^+; X)$, we can assume in contrary that there exists $t_1 > 0$ such that $\|(u(t_1), v(t_1))\|_X = s_0 - \delta + \epsilon$ for some $\epsilon \in (0, \delta)$. Therefore, by (4.2.57) we obtain that $J((u(t_1), v(t_1))) \geq \mathcal{G}(s_0 - \delta + \epsilon) > \mathcal{G}(s_0 - \delta)$ since $\mathcal{G}(t)$ is strictly increasing on $(0, s_0)$. However, this contradicts the fact that $J(u(t), v(t)) \leq \mathcal{G}(s_0 - \delta)$ for all $t \geq 0$. $\square$

4.2.3 Absorption of the lower order terms

Proposition 4.2.5. *In addition to Assumptions 1.1.1 and 1.3.15, further assume $(u_0, v_0) \in \tilde{\mathcal{W}}_1^\delta$ and $E(0) < \mathcal{G}(s_0 - \delta)$ for some $\delta > 0$. If $1 < p < 5$ and $1 < k < 3$, then the global solution (u,v) of the system (1.1.1) furnished by Theorem 1.3.18 satisfies the inequality*

$$\int_0^T (\|u(t)\|_2^2 + \|v(t)\|_2^2) dt \leq C_T \Phi(\boldsymbol{D}(T)) \tag{4.2.59}$$

for all $T \geq T_0$, where T_0 is specified in (4.2.21).

Proof. We follow the standard compactness-uniqueness approach and argue by contradiction.

Step 1: Limit problem from the contradiction hypothesis. Let us fix $T \geq T_0$. Suppose there is a sequence of initial data

$$\{u_0^n, v_0^n, u_1^n, v_1^n\} \subset \mathcal{W}_1^\delta \times (L^2(\Omega))^2$$

such that the corresponding weak solutions (u^n, v^n) verify

$$\lim_{n\to\infty} \frac{\Phi(\mathbf{D}_n(T))}{\int_0^T (\|u^n(t)\|_2^2 + \|v^n(t)\|_2^2)dt} = 0, \tag{4.2.60}$$

where

$$\mathbf{D}_n(T) := \int_0^T \int_\Omega [g_1(u_t^n)u_t^n + g_2(v_t^n)v_t^n]dxdt + \int_0^T \int_\Gamma g(\gamma u_t^n)\gamma u_t^n d\Gamma dt.$$

By the energy estimate (1.3.16), we have $\int_0^T (\|u^n(t)\|_2^2 + \|v^n(t)\|_2^2)dt \leq 2Td\left(\frac{c}{c-2}\right)$ for all $n \in \mathbb{N}$. Therefore, it follows from (4.2.60) that

$$\lim_{n\to\infty} \Phi(\mathbf{D}_n(T)) = 0. \tag{4.2.61}$$

By recalling (4.2.34)-(4.2.35) and (4.2.61), one has

$$\lim_{n\to\infty} \int_0^T (\|u_t^n\|_2^2 + \|v_t^n\|_2^2)dt = 0. \tag{4.2.62}$$

By Assumption 1.1.1, we know $a_1|s|^{m+1} \leq g_1(s)s \leq b_1|s|^{m+1}$ for all $|s| \geq 1$, and so

$$|g_1(s)|^{\frac{m+1}{m}} \leq b_1^{\frac{m+1}{m}}|s|^{m+1} \leq b_1^{\frac{m+1}{m}}\frac{1}{a_1}g_1(s)s, \qquad \text{for all} \quad |s| \geq 1. \tag{4.2.63}$$

In addition, since g_1 is increasing and vanishing at the origin, we know

$$|g_1(s)| \leq b_1, \qquad \text{for all} \quad |s| < 1. \tag{4.2.64}$$

If we define the sets

$$\begin{aligned} A_n &:= \{(x,t) \in \Omega \times (0,T) : |u_t^n(x,t)| < 1\} \\ B_n &:= \{(x,t) \in \Omega \times (0,T) : |u_t^n(x,t)| \geq 1\}, \end{aligned} \tag{4.2.65}$$

then (4.2.63) and (4.2.64) imply

$$\begin{aligned} \int_0^T \int_\Omega |g_1(u_t^n)|^{\frac{m+1}{m}} dxdt &= \int_{A_n} |g_1(u_t^n)|^{\frac{m+1}{m}} dxdt + \int_{B_n} |g_1(u_t^n)|^{\frac{m+1}{m}} dxdt \\ &\leq b_1^{\frac{m+1}{m}}|\Omega|T + b_1^{\frac{m+1}{m}}\frac{1}{a_1}\int_0^T \int_\Omega g_1(u_t^n)u_t^n dxdt. \end{aligned} \tag{4.2.66}$$

Since $\int_0^T\int_\Omega g_1(u_t^n)u_t^n dxdt \to 0$, as $n\to\infty$, (implied by (4.2.61)), then (4.2.66) shows

$$\sup_{n\in\mathbb{N}}\int_0^T\int_\Omega |g_1(u_t^n)|^{\frac{m+1}{m}}dxdt<\infty. \tag{4.2.67}$$

Note (4.2.62) implies, on a subsequence, $u_t^n\to 0$ a.e. in $\Omega\times(0,T)$. Thus, $g_1(u_t^n)\to 0$ a.e. in $\Omega\times(0,T)$. Consequently, by (4.2.67) and the fact $\frac{m+1}{m}>1$, we conclude,

$$g_1(u_t^n)\to 0 \quad \text{weakly in} \quad L^{\frac{m+1}{m}}(\Omega\times(0,T)). \tag{4.2.68}$$

Similarly, by following (4.2.63)-(4.2.67) step by step, we may deduce

$$\sup_{n\in\mathbb{N}}\int_0^T\int_\Gamma |g(\gamma u_t^n)|^{\frac{q+1}{q}}d\Gamma dt<\infty. \tag{4.2.69}$$

Notice (4.2.61) shows $\int_0^T\int_\Gamma g(\gamma u_t^n)\gamma u_t^n d\Gamma dt\to 0$ as $n\to\infty$. So on a subsequence $g(\gamma u_t^n)\gamma u_t^n\to 0$ a.e. in $\Gamma\times(0,T)$, and since g is increasing and vanishing at the origin, we see $g(\gamma u_t^n)\to 0$ a.e. in $\Gamma\times(0,T)$. Therefore, by (4.2.69), it follows that

$$g(\gamma u_t^n)\to 0 \quad \text{weakly in} \quad L^{\frac{q+1}{q}}(\Gamma\times(0,T)). \tag{4.2.70}$$

Now, notice (1.3.16) implies that the sequence of quadratic energy $\mathscr{E}_n(t):=\frac{1}{2}(\|u^n\|_{1,\Omega}^2+\|v^n\|_{1,\Omega}^2+\|u_t^n\|_2^2+\|v_t^n\|_2^2)$ is uniformly bounded on $[0,T]$. Therefore, $\{u^n,v^n,u_t^n,v_t^n\}$ is a bounded sequence in $L^\infty(0,T;H^1(\Omega)\times H_0^1(\Omega)\times L^2(\Omega)\times L^2(\Omega))$. So, on a subsequence, we have

$$\begin{aligned} &u^n\longrightarrow u \ \text{ weakly}^* \text{ in } \ L^\infty(0,T;H^1(\Omega)),\\ &v^n\longrightarrow v \ \text{ weakly}^* \text{ in } \ L^\infty(0,T;H_0^1(\Omega)). \end{aligned} \tag{4.2.71}$$

We note here that for any $0<\epsilon\le 1$, the imbedding $H^1(\Omega)\hookrightarrow H^{1-\epsilon}(\Omega)$ is compact, and $H^{1-\epsilon}(\Omega)\hookrightarrow L^2(\Omega)$. Thus, by Aubin's Compactness Theorem, for any $\alpha>1$, there exists a subsequence such that

$$\begin{aligned} &u^n\longrightarrow u \ \text{ strongly in } \ L^\alpha(0,T;H^{1-\epsilon}(\Omega)),\\ &v^n\longrightarrow v \ \text{ strongly in } \ L^\alpha(0,T;H_0^{1-\epsilon}(\Omega)). \end{aligned} \tag{4.2.72}$$

In addition, for any fixed $1\le s<6$, we know $H^{1-\epsilon}(\Omega)\hookrightarrow L^s(\Omega)$ for sufficiently small $\epsilon>0$. Hence, it follows from (4.2.72) that

$$u^n\longrightarrow u \ \text{ and } \ v^n\longrightarrow v \ \text{ strongly in } \ L^s(\Omega\times(0,T)), \tag{4.2.73}$$

for any $1 \leq s < 6$. Similarly, by (4.2.72), one also has

$$\gamma u^n \longrightarrow \gamma u \text{ strongly in } L^{s_0}(\Gamma \times (0,T)), \tag{4.2.74}$$

for any $s_0 < 4$. Consequently, on a subsequence,

$$\begin{aligned} &u^n \to u \text{ and } v^n \to v \text{ a.e. in } \Omega \times (0,T), \\ &\gamma u^n \to \gamma u \text{ a.e. in } \Gamma \times (0,T). \end{aligned} \tag{4.2.75}$$

Now let $t \in (0,T)$ be fixed. If $\phi \in C(\overline{\Omega \times (0,t)})$, then by (4.1.4), we have

$$|f_j(u^n, v^n)\phi| \leq C(|u^n|^p + |v^n|^p) \quad \text{in} \quad \Omega \times (0,t), \quad j = 1,2. \tag{4.2.76}$$

Since $p < 5$, using (4.2.73), (4.2.75)-(4.2.76) and the Generalized Dominated Convergence Theorem, we arrive at

$$\lim_{n\to\infty} \int_0^t \int_\Omega f_j(u^n, v^n)\phi dx d\tau = \int_0^t \int_\Omega f_j(u,v)\phi dx d\tau, \quad j = 1,2. \tag{4.2.77}$$

Similarly, applying (4.2.74)-(4.2.75), the assumption $k < 4$ and $|h(s)| \leq C|s|^k$, we may deduce

$$\lim_{n\to\infty} \int_0^t \int_\Gamma h(\gamma u^n)\gamma\phi d\Gamma d\tau = \int_0^t \int_\Gamma h(\gamma u)\gamma\phi d\Gamma d\tau. \tag{4.2.78}$$

If we select a test function $\phi \in C(\overline{\Omega \times (0,t)}) \cap C([0,t]; H^1(\Omega))$ such that $\phi(t) = \phi(0) = 0$ and $\phi_t \in L^2(\Omega \times (0,t))$, then (1.3.1) gives

$$\begin{aligned} &\int_0^t [-(u_t^n, \phi_t)_\Omega + (u^n, \phi)_{1,\Omega}] d\tau + \int_0^t \int_\Omega g_1(u_t^n)\phi dx d\tau + \int_0^t \int_\Gamma g(\gamma u_t^n)\gamma\phi d\Gamma d\tau \\ &= \int_0^t \int_\Omega f_1(u^n, v^n)\phi dx d\tau + \int_0^t \int_\Gamma h(\gamma u^n)\gamma\phi d\Gamma d\tau. \end{aligned} \tag{4.2.79}$$

By employing (4.2.62), (4.2.68), (4.2.70), (4.2.71), (4.2.77)-(4.2.78), we can pass to the limit in (4.2.79) to obtain

$$\int_0^t (u,\phi)_{1,\Omega} d\tau = \int_0^t \int_\Omega f_1(u,v)\phi dx d\tau + \int_0^t \int_\Gamma h(\gamma u)\gamma\phi d\Gamma d\tau. \tag{4.2.80}$$

Now we fix $\tilde{\phi} \in H^1(\Omega) \cap C(\overline{\Omega})$ and substitute $\phi(x,\tau) := \tau(t-\tau)\tilde{\phi}(x)$ into (4.2.80). Differentiating the result twice with respect to t yields

$$(u(t), \tilde{\phi})_{1,\Omega} = \int_\Omega f_1(u(t), v(t))\tilde{\phi}dx + \int_\Gamma h(\gamma u(t))\gamma\tilde{\phi}d\Gamma. \tag{4.2.81}$$

If we select a sequence $\tilde{\phi}_n \in H^1(\Omega) \cap C(\overline{\Omega})$ such that $\tilde{\phi}_n \to u(t)$ in $H^1(\Omega)$, for a fixed t, then $\tilde{\phi}_n \to u(t)$ in $L^6(\Omega)$. Now, since $|f_1(u,v)| \leq C(|u|^p + |v|^p)$ with $p < 5$, $|h(s)| \leq C|s|^k$ with $k < 3$, then by Hölder's inequality, we can pass to the limit as $n \to \infty$ in (4.2.81) (where $\tilde{\phi}$ is replaced by $\tilde{\phi}_n$), to obtain

$$\|u(t)\|_{1,\Omega}^2 = \int_\Omega f_1(u(t), v(t))u(t)dx + \int_\Gamma h(\gamma u(t))\gamma u(t)d\Gamma. \tag{4.2.82}$$

In addition, by repeating (4.2.79)-(4.2.82) for (1.3.2), we can derive

$$\|v(t)\|_{1,\Omega}^2 = \int_\Omega f_2(u(t), v(t))v(t)dx. \tag{4.2.83}$$

Adding (4.2.82) and (4.2.83) gives

$$\begin{aligned} \|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2 = & \int_\Omega (f_1(u(t), v(t))u(t) + f_2(u(t), v(t))v(t))dx \\ & + \int_\Gamma h(\gamma u(t))\gamma u(t)d\Gamma, \ \text{ for any } \ t \in (0,T). \end{aligned} \tag{4.2.84}$$

Next, we show $(u(t), v(t)) \in \tilde{\mathcal{W}}_1^\delta$ a.e. on $[0,T]$. Indeed, by (4.2.71)-(4.2.72) and referring to Proposition 2.9 in [39], we obtain, on a subsequence

$$\begin{aligned} u^n(t) &\longrightarrow u(t) \ \text{ weakly in } \ H^1(\Omega) \ \text{ a.e. } \ t \in [0,T]; \\ v^n(t) &\longrightarrow v(t) \ \text{ weakly in } \ H_0^1(\Omega) \ \text{ a.e. } \ t \in [0,T]. \end{aligned} \tag{4.2.85}$$

It follows that

$$\|u(t)\|_{1,\Omega} \leq \liminf_{n\to\infty} \|u^n(t)\|_{1,\Omega} \ \text{ and } \ \|v(t)\|_{1,\Omega} \leq \liminf_{n\to\infty} \|v^n(t)\|_{1,\Omega}, \tag{4.2.86}$$

for a.e. $t \in [0,T]$. Since the initial data $\{u_0^n, v_0^n\} \in \tilde{\mathcal{W}}_1^\delta$ and $E(0) < \mathcal{G}(s_0 - \delta)$, then Proposition 4.2.4 shows the corresponding global solutions $\{u^n(t), v^n(t)\} \in \tilde{\mathcal{W}}_1^\delta$ for all $t \geq 0$. Then, by the definition of $\tilde{\mathcal{W}}_1^\delta$ one knows $\|(u^n(t), v^n(t))\|_X \leq s_0 - \delta$, and

$J(u^n(t), v^n(t)) \leq \mathcal{G}(s_0-\delta)$ for all $t \geq 0$. Thus, (4.2.86) implies $\|(u(t), v(t))\|_X \leq s_0-\delta$ a.e. on $[0,T]$. In order to show $J(u(t), v(t)) \leq \mathcal{G}(s_0-\delta)$ a.e. on $[0,T]$, we note that

$$\begin{aligned}&\mathcal{G}(s_0-\delta) \geq J(u^n(t), v^n(t))\\ &\quad= \frac{1}{2}(\|u^n(t)\|_{1,\Omega} + \|v^n(t)\|_{1,\Omega}) - \int_\Omega F(u^n(t), v^n(t))dx - \int_\Gamma H(\gamma u^n(t))d\Gamma. \end{aligned} \tag{4.2.87}$$

Since the imbedding $H^1(\Omega) \to H^{1-\epsilon}(\Omega)$ is compact and $p<5$, $k<3$, we obtain from (4.2.85) that

$$\begin{aligned}&u^n(t) \longrightarrow u(t), \;\; v^n(t) \longrightarrow v(t) \;\text{ strongly in }\; L^{p+1}(\Omega), \;\text{ a.e. on }\; [0,T]\\ &\gamma u^n(t) \longrightarrow \gamma u(t) \;\text{ strongly in }\; L^{k+1}(\Gamma), \;\text{ a.e. on }\; [0,T].\end{aligned} \tag{4.2.88}$$

By (1.3.7), (4.2.88) and the Generalized Dominated Convergence Theorem, one has, on a subsequence

$$\begin{aligned}&\lim_{n\to\infty} \int_\Omega F(u^n(t), v^n(t))dx = \int_\Omega F(u(t), v(t))dx, \;\text{ a.e. on }\; [0,T],\\ &\lim_{n\to\infty} \int_\Gamma H(\gamma u^n(t))d\Gamma = \int_\Gamma H(\gamma u(t))d\Gamma, \;\text{ a.e. on }\; [0,T].\end{aligned} \tag{4.2.89}$$

Applying (4.2.86) and (4.2.89), we can take the limit inferior on both side of the inequality (4.2.87) to obtain

$$\mathcal{G}(s_0-\delta) \geq J(u(t), v(t)), \;\text{ a.e. on }\; [0,T].$$

Hence $(u(t), v(t)) \in \tilde{\mathcal{W}}_1^\delta \subset \mathcal{W}_1$ a.e. on $[0,T]$. Therefore, by the definition of $\mathcal{W}_1$ and (4.2.84), necessarily we have $(u(t), v(t)) = (0,0)$ a.e. on $[0,T]$. Therefore, (4.2.73) implies

$$u^n \longrightarrow 0 \quad\text{and}\quad v^n \longrightarrow 0 \quad\text{strongly in}\quad L^s(\Omega\times(0,T)), \;\text{ for any }\; s<6. \tag{4.2.90}$$

Step 2: Re-normalize the sequence $\{u^n, v^n\}$. We define

$$N_n := \left(\int_0^T (\|u^n\|_2^2 + \|v^n\|_2^2)dt\right)^{\frac{1}{2}}.$$

By (4.2.90), one has $u^n \longrightarrow 0$ and $v^n \longrightarrow 0$ in $L^2(\Omega\times(0,T))$, and so, $N_n \longrightarrow 0$ as $n\to\infty$. If we set

$$y^n := \frac{u^n}{N_n} \quad\text{and}\quad z^n := \frac{v^n}{N_n},$$

then clearly

$$\int_0^T (\|y^n\|_2^2 + \|z^n\|_2^2)dt = 1. \tag{4.2.91}$$

By the contradiction hypothesis (4.2.60), namely

$$\lim_{n\to\infty} \frac{\Phi(\mathbf{D}_n(T))}{N_n^2} = 0, \tag{4.2.92}$$

and along with (4.2.34)-(4.2.35), we obtain

$$\lim_{n\to\infty} \frac{\int_0^T (\|u_t^n\|_2^2 + \|v_t^n\|_2^2)dt}{N_n^2} = 0,$$

which is equivalent to

$$\lim_{n\to\infty} \int_0^T (\|y_t^n\|_2^2 + \|z_t^n\|_2^2)dt = 0. \tag{4.2.93}$$

We next show

$$\frac{g_1(u_t^n)}{N_n} \longrightarrow 0 \text{ strongly in } L^{\frac{m+1}{m}}(\Omega \times (0,T)). \tag{4.2.94}$$

Recall the definition of the sets A_n and B_n in (4.2.65). Since $N_n \longrightarrow 0$ as $n \to \infty$, we can let n be sufficiently large such that $N_n < 1$, then by using (4.2.9), (4.2.63), Hölder's and Jensen's inequalities, we deduce

$$\begin{aligned}
&\int_0^T \int_\Omega \left|\frac{g_1(u_t^n)}{N_n}\right|^{\frac{m+1}{m}} dxdt = \int_{A_n} \left|\frac{g_1(u_t^n)}{N_n}\right|^{\frac{m+1}{m}} dxdt + \int_{B_n} \left|\frac{g_1(u_t^n)}{N_n}\right|^{\frac{m+1}{m}} dxdt \\
&\le C(T,|\Omega|) \left(\int_{A_n} \left|\frac{g_1(u_t^n)}{N_n}\right|^2 dxdt\right)^{\frac{m+1}{2m}} + \frac{1}{N_n^2} \int_{B_n} |g_1(u_t^n)|^{\frac{m+1}{m}} dxdt \\
&\le C(T,|\Omega|) \left(\frac{1}{N_n^2} \int_{A_n} \varphi_1(g_1(u_t^n)u_t^n) dxdt\right)^{\frac{m+1}{2m}} + \frac{b_1^{\frac{m+1}{m}}}{a_1 N_n^2} \int_{B_n} g_1(u_t^n)u_t^n dxdt \\
&\le C(T,|\Omega|) \left(\frac{\Phi(\mathbf{D}_n(T))}{N_n^2}\right)^{\frac{m+1}{2m}} + \frac{b_1^{\frac{m+1}{m}}}{a_1} \frac{\Phi(\mathbf{D}_n(T))}{N_n^2} \longrightarrow 0 \text{ as } n \to \infty,
\end{aligned}$$

where we have used $T \geq T_0 \geq \frac{1}{|\Omega|}$ and (4.2.92). Thus our desired result (4.2.94) follows.

Likewise, we can prove

$$\frac{g(\gamma u_t^n)}{N_n} \longrightarrow 0 \text{ strongly in } L^{\frac{q+1}{q}}(\Gamma \times (0,T)). \tag{4.2.95}$$

Let E_n be the total energy corresponding to the solution (u^n, v^n). So (1.3.17) shows $E_n(t) \geq 0$ for all $t \geq 0$. Also by (4.2.22) and (4.2.91)-(4.2.92), we obtain $\lim_{n\to\infty} \frac{E_n(T)}{N_n^2} \leq \hat{C}$, which implies $\{\frac{E_n(T)}{N_n^2}\}$ is uniformly bounded. The energy identity (4.2.20) shows $E_n(T) + \mathbf{D}_n(T) = E_n(0)$, and thus $\{\frac{E_n(0)}{N_n^2}\}$ is also uniformly bounded. Moreover, since $E_n'(t) \leq 0$ for all $t \geq 0$, one has $\{\frac{E_n(t)}{N_n^2}\}$ is uniformly bounded on $[0,T]$, and along with the energy inequality (1.3.17), we conclude that the sequence

$$\Big\{\frac{\mathscr{E}_n(t)}{N_n^2} = \frac{1}{2}(\|y^n\|_{1,\Omega}^2 + \|z^n\|_{1,\Omega}^2 + \|y_t^n\|_2^2 + \|z_t^n\|_2^2)\Big\}$$

is uniformly bounded on $[0,T]$. Therefore, $\{y^n, z^n, y_t^n, z_t^n\}$ is a bounded sequence in $L^\infty(0,T; H^1(\Omega) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega))$. Therefore, on a subsequence,

$$\begin{aligned} &y^n \longrightarrow y \text{ weakly}^* \text{ in } L^\infty(0,T; H^1(\Omega)), \\ &z^n \longrightarrow z \text{ weakly}^* \text{ in } L^\infty(0,T; H_0^1(\Omega)). \end{aligned} \tag{4.2.96}$$

As in (4.2.72)-(4.2.75), we may deduce that, on subsequences

$$y^n \longrightarrow y \text{ and } z^n \longrightarrow z \text{ strongly in } L^s(\Omega \times (0,T)), \tag{4.2.97}$$

for any $s < 6$, and

$$\gamma y^n \longrightarrow \gamma y \text{ strongly in } L^{s_0}(\Gamma \times (0,T)), \tag{4.2.98}$$

for any $s_0 < 4$. Note (4.2.91) and (4.2.97) show that

$$\lim_{n\to\infty} \int_0^T (\|y^n\|_2^2 + \|z^n\|_2^2)dt = \int_0^T (\|y\|_2^2 + \|z\|_2^2)dt = 1. \tag{4.2.99}$$

However, by Hölder's inequality,

$$\begin{aligned} \int_0^T \int_\Omega |y^n||u^n|^{p-1}dxdt &\leq \Big(\int_0^T \int_\Omega |y^n|^5 dxdt\Big)^{\frac{1}{5}} \Big(\int_0^T \int_\Omega |u^n|^{\frac{5}{4}(p-1)}dxdt\Big)^{\frac{4}{5}} \\ &\longrightarrow \|y\|_{L^5(\Omega\times(0,T))} \cdot 0 = 0 \end{aligned} \tag{4.2.100}$$

where we have used (4.2.97), (4.2.90) and the fact $\frac{5}{4}(p-1) < 5$.

Similarly,

$$\lim_{n\to\infty}\int_0^T\int_\Omega |z^n||v^n|^{p-1}dxdt = 0 \quad\text{and}\quad \lim_{n\to\infty}\int_0^T\int_\Gamma |\gamma y^n||\gamma u^n|^{k-1}d\Gamma dt = 0. \tag{4.2.101}$$

Since $|f_j(u^n, v^n)| \le C(|u^n|^p + |v^n|^p)$, $j = 1, 2$, it follows that,

$$\int_0^t\int_\Omega \left|\frac{f_j(u^n, v^n)}{N_n}\phi\right| dxd\tau \le C\int_0^t\int_\Omega (|y^n||u^n|^{p-1} + |z^n||v^n|^{p-1})dxd\tau \longrightarrow 0, \tag{4.2.102}$$

for any $t \in (0, T)$, $\phi \in C(\overline{\Omega \times (0, t)})$, and where we have used (4.2.100)-(4.2.101). Likewise,

$$\int_0^t\int_\Gamma \left|\frac{h(\gamma u^n)}{N_n}\gamma\phi\right| d\Gamma d\tau \le C\int_0^t\int_\Omega |\gamma y^n||\gamma u^n|^{k-1}d\Gamma d\tau \longrightarrow 0. \tag{4.2.103}$$

Dividing both sides of (4.2.79) by N_n yields

$$\begin{aligned}&\int_0^t [-(y_t^n, \phi_t)_\Omega + (y^n, \phi)_{1,\Omega}]d\tau + \int_0^t\int_\Omega \frac{g_1(u_t^n)}{N_n}\phi dxd\tau + \int_0^t\int_\Gamma \frac{g(\gamma u_t^n)}{N_n}\gamma\phi d\Gamma d\tau\\ &= \int_0^t\int_\Omega \frac{f_1(u^n, v^n)}{N_n}\phi dxd\tau + \int_0^t\int_\Gamma \frac{h(\gamma u^n)}{N_n}\gamma\phi d\Gamma d\tau.\end{aligned} \tag{4.2.104}$$

where $\phi \in C(\overline{\Omega \times (0, t)}) \cap C([0, t]; H^1(\Omega))$ such that $\phi(t) = \phi(0) = 0$ and $\phi_t \in L^2(\Omega \times (0, t))$.

By using (4.2.93), (4.2.94)-(4.2.95), (4.2.96), and (4.2.102)-(4.2.103), we can pass to the limit in (4.2.104) to find

$$\int_0^t (y^n, \phi)_{1,\Omega}d\tau = 0, \text{ for all } t \in (0, T). \tag{4.2.105}$$

Now, fix an arbitrary $\tilde{\phi} \in H^1(\Omega) \cap C(\overline{\Omega})$ and substitute $\phi(x, \tau) = \tau(t - \tau)\tilde{\phi}(x)$ into (4.2.105). Differentiating the result twice yields

$$(y(t), \tilde{\phi})_{1,\Omega} = 0, \text{ for all } t \in (0, T), \tag{4.2.106}$$

which implies $y(t) = 0$ in $H^1(\Omega)$ for all $t \in (0, T)$. Similarly, we can show $z(t) = 0$ in $H_0^1(\Omega)$ for all $t \in (0, T)$. However, this contradicts the fact (4.2.99). Hence, the proof of Proposition 4.2.5 is complete. $\square$

Remark 4.2.6. We can iterate the estimate (4.2.59) on time intervals $[mT,(m+1)T]$, $m = 0, 1, 2, ...$, and obtain

$$\int_{mT}^{(m+1)T} (\|u(t)\|_2^2 + \|v(t)\|_2^2)dt \leq C_T \Phi(\mathbf{D}(T)), \;\; m = 0, 1, 2, ... \tag{4.2.107}$$

It is important to note, by the contradiction hypothesis made in the proof of Proposition 4.2.5, the constant C_T in (4.2.107) does not depend on m.

4.2.4 Proof of Theorem 1.3.19

We are now ready to prove Theorem 1.3.19: the uniform decay rates of energy.

Proof. Combining Propositions 4.2.2 and 4.2.5 yields $E(T) \leq \hat{C}(1 + C_T)\Phi(\mathbf{D}(T))$ for all $T \geq T_0$. If we set $\Phi_T = \hat{C}(1 + C_T)\Phi$, where C_T is as given in (4.2.59), then the energy identity (4.2.20) shows that

$$E(T) \leq \Phi_T(\mathbf{D}(T)) = \Phi_T(E(0) - E(T)),$$

which implies

$$E(T) + \Phi_T^{-1}(E(T)) \leq E(0).$$

By iterating the estimate on intervals $[mT, (m+1)T]$, $m = 0, 1, 2, ...$, we have

$$E((m+1)T) + \Phi_T^{-1}(E((m+1)T)) \leq E(mT), \;\; m = 0, 1, 2, ...$$

Therefore, by Lemma 3.3 in [30], one has

$$E(mT) \leq S(m) \text{ for all } m = 0, 1, 2, ... \tag{4.2.108}$$

where S is the solution the ODE:

$$S' + [I - (I + \Phi_T^{-1})^{-1}](S) = 0, \quad S(0) = E(0), \tag{4.2.109}$$

where I denotes the identity mapping. However, we note that

$$\begin{aligned} I - (I + \Phi_T^{-1})^{-1} &= (I + \Phi_T^{-1}) \circ (I + \Phi_T^{-1})^{-1} - (I + \Phi_T^{-1})^{-1} = \Phi_T^{-1} \circ (I + \Phi_T^{-1})^{-1} \\ &= \Phi_T^{-1} \circ (\Phi_T \circ \Phi_T^{-1} + \Phi_T^{-1})^{-1} = \Phi_T^{-1} \circ \Phi_T \circ (I + \Phi_T)^{-1} = (I + \Phi_T)^{-1}. \end{aligned}$$

It follows that the ODE (4.2.109) can be reduced to:

$$S' + (I + \Phi_T)^{-1}(S) = 0, \quad S(0) = E(0), \tag{4.2.110}$$

where (4.2.110) has a unique solutions defined on $[0, \infty)$. Since Φ_T is increasing passing through the origin, we have $(I + \Phi_T)^{-1}$ is also increasing and vanishing at zero. So if we write (4.2.110) in the form $S' = -(I + \Phi_T)^{-1}(S)$, then it follows that $S(t)$ is decreasing and $S(t) \to 0$ as $t \to \infty$.

For any $t > T$, there exists $m \in \mathbb{N}$ such that $t = mT + \delta$ with $0 \le \delta < T$, and so $m = \frac{t}{T} - \frac{\delta}{T} > \frac{t}{T} - 1$. By (4.2.108) and the fact $E(t)$ and $S(t)$ are decreasing, we obtain

$$E(t) = E(mT + \delta) \le E(mT) \le S(m) \le S\left(\frac{t}{T} - 1\right), \text{ for any } t > T. \tag{4.2.111}$$

If g_1, g_2, g are linearly bounded near the origin, then (4.2.13) shows that φ_1, φ_2, φ are linear, and it follows that Φ_T is linear, which implies $(I + \Phi_T)^{-1}$ is also linear. Therefore, the ODE (4.2.110) is of the form $S' + w_0 S = 0$, $S(0) = E(0)$ (for some positive constant w_0), whose solution is given by: $S(t) = E(0)e^{-w_0 t}$. Thus, from (4.2.111) we know

$$E(t) \le E(0)e^{-w_0\left(\frac{t}{T} - 1\right)} = (e^{w_0}E(0))e^{-\frac{w_0}{T}t}$$

for $t > T$. Consequently, if we set $w := \frac{w_0}{T}$ and choose $\tilde{C}$ sufficiently large, then we conclude

$$E(t) \le \tilde{C}E(0)e^{-wt}, \ \ t \ge 0,$$

which provides the exponential decay estimate (1.3.22).

If at least one of g_1, g_2 and g are not linearly bounded near the origin, then we can show the decay of $E(t)$ is algebraic. Indeed, by (4.2.16) we may choose $\varphi_1(s) = C_1 s^{z_1}$, $\varphi_2(s) = C_2 s^{z_2}$, $\varphi(s) = C_3 s^{z}$, where $0 < z_1, z_2, z \le 1$ are given in (4.2.17). Also recall that $j := \max\{\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z}\} > 1$, as defined in (4.2.18). Now, we study the function $(I + \Phi_T)^{-1}$. Notice, if $y = (I + \Phi_T)^{-1}(s)$ for $s \ge 0$, then $y \ge 0$. In addition,

$$\begin{aligned} s &= (I + \Phi_T)y = y + \hat{C}(1 + C_T)(\varphi_1(y) + \varphi_2(y) + \varphi(y) + y) \\ &\le C(\varphi_1(y) + \varphi_2(y) + \varphi(y) + y) \le C y^{\min\{z_1, z_2, z\}}, \text{ for all } 0 \le y \le 1. \end{aligned}$$

It follows that there exists $C_0 > 0$ such that $y \ge C_0 s^j$ for all $0 \le y \le 1$, i.e.,

$$(I + \Phi_T)^{-1}(s) \ge C_0 s^j \text{ provided } 0 \le (I + \Phi_T)^{-1}(s) \le 1. \tag{4.2.112}$$

Recall we have pointed out that $S(t)$ is decreasing to zero as $t \to \infty$, so $(I + \Phi_T)^{-1}(S(t))$ is also decreasing to zero as $t \to \infty$. Hence, there exists $t_0 \geq 0$ such that $(I + \Phi_T)^{-1}(S(t)) \leq 1$, whenever $t \geq t_0$. Therefore, (4.2.112) implies

$$S'(t) = -(I + \Phi_T)^{-1}(S(t)) \leq -C_0 S(t)^j \text{ if } t \geq t_0.$$

So, $S(t) \leq \hat{S}(t)$ for all $t \geq t_0$ where $\hat{S}$ is the solution of the ODE

$$\hat{S}'(t) = -C_0 \hat{S}(t)^j, \;\; \hat{S}(t_0) = S(t_0). \tag{4.2.113}$$

Since the solution of (4.2.113) is

$$\hat{S}(t) = [C_0(j-1)(t - t_0) + S(t_0)^{1-j}]^{-\frac{1}{j-1}} \text{ for all } t \geq t_0,$$

and along with (4.2.111), it follows that

$$E(t) \leq S\left(\frac{t}{T} - 1\right) \leq \hat{S}\left(\frac{t}{T} - 1\right) = \left[C_0(j-1)\left(\frac{t}{T} - 1 - t_0\right) + S(t_0)^{1-j}\right]^{-\frac{1}{j-1}}$$

for all $t \geq T(t_0 + 1)$. Since $S(t_0)$ depends on the initial energy $E(0)$, there exists a positive constant $C(E(0))$ depending on $E(0)$ such that

$$E(t) \leq C(E(0))(1+t)^{-\frac{1}{j-1}}, \text{ for all } t \geq 0,$$

where $j > 1$. Thus, the proof of Theorem 1.3.19 is complete. □

4.3 Blow-up of Potential Well Solutions

This section is devoted to prove the blow up result: Theorem 1.3.20. We begin by showing $\mathcal{W}_2$ is invariant under the dynamics of (1.1.1). More precisely, we have the following lemma.

Lemma 4.3.1. *In addition to Assumptions 1.1.1 and 1.3.15, further assume that* $(u_0, v_0) \in \mathcal{W}_2$ *and* $E(0) < d$. *If* $1 < p \leq 5$ *and* $1 < k \leq 3$, *then the weak solution* $(u(t), v(t)) \in \mathcal{W}_2$ *for all* $t \in [0, T)$, *and*

$$\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2 > 2 \min\left\{\frac{p+1}{p-1}, \frac{k+1}{k-1}\right\} d, \;\; \textit{for all } t \in [0, T), \tag{4.3.1}$$

where $[0, T)$ *is the maximal interval of existence.*

Proof. Since $E(0) < d$, we have shown in the proof of Theorem 1.3.18 that $(u(t), v(t)) \in \mathcal{W}$ for all $t \in [0, T)$. To show that $(u(t), v(t)) \in \mathcal{W}_2$ for all $t \in [0, T)$, we proceed by contradiction. Assume there exists $t_1 \in (0, T)$ such that $(u(t_1), v(t_1)) \notin \mathcal{W}_2$, then it must be $(u(t_1), v(t_1)) \in \mathcal{W}_1$. Recall that the weak solution $(u, v) \in C([0, T); H^1(\Omega) \times H^1_0(\Omega))$, and in the proof of Theorem 1.3.18 we have shown the continuity of the function

$$t \mapsto (p+1)\int_\Omega F(u(t), v(t))dt + (k+1)\int_\Gamma H(\gamma u(t))d\Gamma.$$

Since $(u(0), v(0)) \in \mathcal{W}_2$ and $(u(t_1), v(t_1)) \in \mathcal{W}_1$, it follows that there exists $s \in (0, t_1]$ such that

$$\|u(s)\|^2_{1,\Omega} + \|v(s)\|^2_{1,\Omega} = (p+1)\int_\Omega F(u(s), v(s))dx + (k+1)\int_\Gamma H(\gamma u(s))d\Gamma. \quad (4.3.2)$$

Now we define t^* as the infinimum of all $s \in (0, t_1]$ satisfying (4.3.2). By continuity, one has $t^* \in (0, t_1]$ satisfying (4.3.2), and $(u(t), v(t)) \in \mathcal{W}_2$ for all $t \in [0, t^*)$. Thus, we have two cases to consider.

Case 1: $(u(t^*), v(t^*)) \neq (0, 0)$. Since t^* satisfies (4.3.2), it follows $(u(t^*), v(t^*)) \in \mathcal{N}$, and by Lemma 2.1.1, we know $J(u(t^*), v(t^*)) \geq d$. Thus $E(t^*) \geq d$, contradicting $E(t) \leq E(0) < d$ for all $t \in [0, T)$.

Case 2: $(u(t^*), v(t^*)) = (0, 0)$. Since $(u(t), v(t)) \in \mathcal{W}_2$ for all $t \in [0, t^*)$, by utilizing a similar argument as in the proof of Theorem 1.3.18, we obtain $\|(u(t), v(t))\|_X > s_1$, for all $t \in [0, t^*)$, where $s_1 > 0$. By the continuity of the weak solution $(u(t), v(t))$, we obtain that $\|(u(t^*), v(t^*))\|_X \geq s_1 > 0$, contradicting the assumption $(u(t^*), v(t^*)) = (0, 0)$. It follows that $(u(t), v(t)) \in \mathcal{W}_2$ for all $t \in [0, T)$.

It remains to show inequality (4.3.1). Let $(u, v) \in \mathcal{W}_2$ be fixed. By recalling (4.2.4) in Lemma 4.2.1 which states that the only critical point in $(0, \infty)$ for the function $\lambda \mapsto J(\lambda(u, v))$ is $\lambda_0 > 0$, where λ_0 satisfies the equation

$$(\|u\|^2_{1,\Omega} + \|v\|^2_{1,\Omega}) = (p+1)\lambda_0^{p-1}\int_\Omega F(u, v)dx + (k+1)\lambda_0^{k-1}\int_\Gamma H(\gamma u)d\Gamma. \quad (4.3.3)$$

Since $(u, v) \in \mathcal{W}_2$, then $\lambda_0 < 1$. In addition, we recall the function $\lambda \mapsto J(\lambda(u, v))$ attains its absolute maximum over the positive axis at its critical point $\lambda = \lambda_0$. Thus, by Lemma 4.2.1 and (4.3.3), it follows that

$$
\begin{aligned}
d &\leq \sup_{\lambda \geq 0} J(\lambda(u,v)) = J(\lambda_0(u,v)) \\
&= \frac{1}{2}\lambda_0^2(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2) - \lambda_0^{p+1}\int_\Omega F(u,v)dx - \lambda_0^{k+1}\int_\Gamma H(\gamma u)d\Gamma \\
&\leq \lambda_0^2\left[\frac{1}{2}(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2) - \min\left\{\frac{1}{p+1}, \frac{1}{k+1}\right\}(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2)\right] \\
&= \frac{1}{2}\lambda_0^2 \max\left\{\frac{p-1}{p+1}, \frac{k-1}{k+1}\right\}(\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2).
\end{aligned}
$$

Since $\lambda_0 < 1$, one has

$$
\|u\|_{1,\Omega}^2 + \|v\|_{1,\Omega}^2 \geq \frac{2d}{\lambda_0^2}\min\left\{\frac{p+1}{p-1}, \frac{k+1}{k-1}\right\} > 2\min\left\{\frac{p+1}{p-1}, \frac{k+1}{k-1}\right\} d,
$$

completing the proof of Lemma 4.3.1. □

Now, we prove Theorem 1.3.20: the blow up of potential well solutions.

Proof. In order to show the maximal existence time T is finite, we argue by contradiction. Assume the weak solution $(u(t), v(t))$ can be extended to $[0, \infty)$, then Lemma 4.3.1 says $(u(t), v(t)) \in \mathcal{W}_2$ for all $t \in [0, \infty)$. Moreover, by the assumption $0 \leq E(0) < \rho d$, the energy $E(t)$ remains nonnegative:

$$
0 \leq E(t) \leq E(0) < \rho d \text{ for all } t \in [0, \infty). \tag{4.3.4}
$$

To see this, assume that $E(t_0) < 0$ for some $t_0 \in (0, \infty)$. Then, Theorems 1.3.12 and 1.3.13 assert that

$$
\|u(t)\|_{1,\Omega} + \|v(t)\|_{1,\Omega} \to \infty,
$$

as $t \to T^-$, for some $0 < T < \infty$, i.e., the weak solution $(u(t), v(t))$ must blow up in finite time, which contradicts our assumption.

Now, define

$$
\begin{aligned}
N(t) &:= \|u(t)\|_2^2 + \|v(t)\|_2^2, \\
S(t) &:= \int_\Omega F(u(t), v(t))dx + \int_\Gamma H(\gamma u(t))d\Gamma \geq 0.
\end{aligned}
$$

Since $u_t,\ v_t \in C([0,\infty); L^2(\Omega))$, it follows that

$$
N'(t) = 2\int_\Omega [u(t)u_t(t) + v(t)v_t(t)]dx. \tag{4.3.5}
$$

Recall in the proof of Proposition 4.2.2, we have verified u and v enjoy, respectively, the regularity restrictions imposed on the test function ϕ and ψ, as stated in Definition 1.3.1. Consequently, we can replace ϕ by u in (1.3.1) and ψ by v in (1.3.2), and sum the two equations to obtain:

$$\begin{aligned}\frac{1}{2}N'(t) &= \int_\Omega (u_1u_0 + v_1v_0)dx + \int_0^t \int_\Omega (|u_t|^2 + |v_t|^2)dxd\tau - \int_0^t (\|u\|^2_{1,\Omega} + \|v\|^2_{1,\Omega})d\tau \\ &- \int_0^t \int_\Omega (g_1(u_t)u + g_2(v_t)v)dxd\tau - \int_0^t \int_\Gamma g(\gamma u_t)\gamma u d\Gamma d\tau \\ &+ (p+1)\int_0^t \int_\Omega F(u,v)dxd\tau + (k+1)\int_0^t \int_\Gamma H(\gamma u)d\Gamma d\tau, \text{ a.e. } [0,\infty), \end{aligned} \tag{4.3.6}$$

where we have used (1.3.6). Since $p \le 5$ and $k \le 3$, then by Assumption 1.1.1, one can check that the RHS of (4.3.6) is absolutely continuous, and thus we can differentiate both sides of (4.3.6) to obtain

$$\begin{aligned}\frac{1}{2}N''(t) &= \Big(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 \Big) - \Big(\|u(t)\|^2_{1,\Omega} + \|v(t)\|^2_{1,\Omega} \Big) \\ &- \int_\Omega (g_1(u_t)u + g_2(v_t)v)dx - \int_\Gamma g(\gamma u_t)\gamma u d\Gamma \\ &+ (p+1)\int_\Omega F(u,v)dx + (k+1)\int_\Gamma H(\gamma u)d\Gamma, \text{ a.e. } [0,\infty). \end{aligned} \tag{4.3.7}$$

The assumption $|g_1(s)| \le b_1|s|^m$ for all $s \in \mathbb{R}$ implies

$$\begin{aligned}\left| \int_\Omega g_1(u_t(t))u(t)dx \right| &\le b_1 \int_\Omega |u_t(t)|^m |u(t)|dx \\ &\le C\|u(t)\|_{m+1} \|u_t(t)\|^m_{m+1} \\ &\le C\|u(t)\|_{p+1} \|u_t(t)\|^m_{m+1}, \end{aligned} \tag{4.3.8}$$

where we have used Hölder's inequality and the assumption $p > m$. In addition, the assumption $F(u,v) \ge \alpha_0(|u|^{p+1} + |v|^{p+1})$ for some $\alpha_0 > 0$ yields

$$\|u(t)\|^{p+1}_{p+1} + \|v(t)\|^{p+1}_{p+1} \le \frac{1}{\alpha_0}\int_\Omega F(u(t),v(t))dx \le \frac{1}{\alpha_0}S(t). \tag{4.3.9}$$

It follows from (4.3.8)-(4.3.9) that

$$\left| \int_\Omega g_1(u_t(t))u(t)dx \right| \le CS(t)^{\frac{1}{p+1}} \|u_t(t)\|^m_{m+1} \le \epsilon S(t)^{\frac{m+1}{p+1}} + C_\epsilon \|u_t(t)\|^{m+1}_{m+1}, \tag{4.3.10}$$

where we have used Young's inequality.

Since $p > r$, we may similarly deduce

$$\left|\int_\Omega g_2(v_t(t))v(t)dx\right| \le \epsilon S(t)^{\frac{r+1}{p+1}} + C_\epsilon \|v_t(t)\|_{r+1}^{r+1}. \tag{4.3.11}$$

In order to estimate $|\int_\Gamma g(\gamma u_t(t))\gamma u(t) d\Gamma|$, depending on different assumptions on parameters, there are two cases to consider: either $k > q$ or $p > 2q-1$.

Case 1: $k > q$. In this case, the estimate is straightforward. As in (4.3.8), we have

$$\left|\int_\Gamma g(\gamma u_t(t))\gamma u(t)dx\right| \le C|\gamma u(t)|_{k+1}|\gamma u_t(t)|_{q+1}^q. \tag{4.3.12}$$

Since $H(s)$ is homogeneous of order $k+1$ and $H(s) > 0$ for all $s \in \mathbb{R}$, then $H(s) \ge \min\{H(1), H(-1)\}|s|^{k+1}$, where $H(1)$, $H(-1) > 0$. Thus,

$$\int_\Gamma |\gamma u(t)|^{k+1} d\Gamma \le C\int_\Gamma H(\gamma u(t))d\Gamma \le CS(t). \tag{4.3.13}$$

It follows from (4.3.12)-(4.3.13), Young's inequality, and the assumption $k > q$ that

$$\left|\int_\Gamma g(\gamma u_t(t))\gamma u(t)dx\right| \le CS(t)^{\frac{1}{k+1}}|\gamma u_t(t)|_{q+1}^q \le \epsilon S(t)^{\frac{q+1}{k+1}} + C_\epsilon|\gamma u_t(t)|_{q+1}^{q+1}. \tag{4.3.14}$$

Case 2: $p > 2q-1$. We shall employ a useful inequality which was shown in the proof of Theorem 1.3.13, namely,

$$|\gamma u|_{q+1} \le C\left(\|u\|_{1,\Omega}^{\frac{2\beta}{q+1}} + \|u\|_{p+1}^{\frac{(p+1)\beta}{q+1}}\right), \tag{4.3.15}$$

where $\frac{p-1}{2(p-q)} \le \beta < 1$.

In addition, since $(u(t), v(t)) \in \mathcal{W}_2$ for all $t \ge 0$, one has

$$\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2 \le \max\{p+1, k+1\}S(t), \text{ for all } t \ge 0. \tag{4.3.16}$$

Now we apply (4.3.15) and the assumption $|g(s)| \le b_3|s|^q$ to obtain

$$\begin{aligned}\left|\int_\Gamma g(\gamma u_t(t))\gamma u(t)d\Gamma\right| &\le b_3\int_\Gamma |\gamma u(t)||\gamma u_t(t)|^q d\Gamma \le b_3|\gamma u(t)|_{q+1}|\gamma u_t(t)|_{q+1}^q \\ &\le C\left(\|u\|_{1,\Omega}^{\frac{2\beta}{q+1}} + \|u\|_{p+1}^{\frac{(p+1)\beta}{q+1}}\right)|\gamma u_t(t)|_{q+1}^q \\ &\le CS(t)^{\frac{\beta}{q+1}}|\gamma u_t(t)|_{q+1}^q \le \epsilon S(t)^\beta + C_\epsilon|\gamma u_t(t)|_{q+1}^{q+1}. \end{aligned} \tag{4.3.17}$$

where we have used (4.3.16), (4.3.9) and Young's inequality.

Combining (4.3.7), (4.3.10)-(4.3.11), (4.3.14) and (4.3.17) yields

$$\begin{aligned}
&\frac{1}{2}N''(t) + C_\epsilon \left(\|u_t(t)\|_{m+1}^{m+1} + \|v_t(t)\|_{r+1}^{r+1} + |\gamma u_t(t)|_{q+1}^{q+1}\right) \\
&\geq \Big(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2 \Big) - \Big(\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2 \Big) \\
&\quad - \epsilon \Big(S(t)^{\frac{m+1}{p+1}} + S(t)^{\frac{r+1}{p+1}} + S(t)^{j_0} \Big) \\
&\quad + (p+1) \int_\Omega F(u,v)dx + (k+1) \int_\Gamma H(\gamma u) d\Gamma, \;\; \text{a.e. } t \in [0,\infty), \qquad (4.3.18)
\end{aligned}$$

where

$$j_0 := \begin{cases} \frac{q+1}{k+1}, & \text{if } k > q, \\ \beta, & \text{if } p > 2q-1. \end{cases}$$

Since $\beta < 1$, it follows $j_0 < 1$.

Rearranging the terms in the definition (1.3.5) of the total energy $E(t)$ gives

$$\begin{aligned}
-\Big(\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2\Big) &= \left(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2\right) - 2\int_\Omega F(u(t),v(t))dx \\
&\quad - 2\int_\Gamma H(\gamma u(t)) d\Gamma - 2E(t). \qquad (4.3.19)
\end{aligned}$$

It follows from (4.3.18)-(4.3.19) that

$$\begin{aligned}
&\frac{1}{2}N''(t) + C_\epsilon \left(\|u_t(t)\|_{m+1}^{m+1} + \|v_t(t)\|_{r+1}^{r+1} + |\gamma u_t(t)|_{q+1}^{q+1}\right) \\
&\geq (p-1)\int_\Omega F(u(t),v(t))dx + (k-1)\int_\Gamma H(\gamma u(t)) d\Gamma \\
&\quad - 2E(t) - \epsilon \Big(S(t)^{\frac{m+1}{p+1}} + S(t)^{\frac{r+1}{p+1}} + S(t)^{j_0} \Big), \;\; \text{a.e. } t \in [0,\infty). \qquad (4.3.20)
\end{aligned}$$

Since $(u(t),v(t)) \in \mathcal{W}_2$ for all $t \in [0,\infty)$, then by Lemma 4.3.1, we deduce

$$\begin{aligned}
&(p-1)\int_\Omega F(u(t),v(t))dx + (k-1)\int_\Gamma H(\gamma u(t)) d\Gamma \\
&> \min\left\{\frac{p-1}{p+1}, \frac{k-1}{k+1}\right\} (\|u(t)\|_{1,\Omega}^2 + \|v(t)\|_{1,\Omega}^2) \\
&> 2\min\left\{\frac{p-1}{p+1}, \frac{k-1}{k+1}\right\} \cdot \min\left\{\frac{p+1}{p-1}, \frac{k+1}{k-1}\right\} d = 2\rho d, \qquad (4.3.21)
\end{aligned}$$

for all $t \in [0, \infty)$, where $\rho \le 1$ is defined in (1.3.25).

Note (4.3.4) implies there exists $\delta > 0$ such that

$$0 \le E(t) \le E(0) \le (1-\delta)\rho d \text{ for all } t \in [0,\infty). \tag{4.3.22}$$

Combining (4.3.20)-(4.3.22) yields

$$\begin{aligned}
&\frac{1}{2}N''(t) + C_\epsilon \left(\|u_t(t)\|_{m+1}^{m+1} + \|v_t(t)\|_{r+1}^{r+1} + |\gamma u_t(t)|_{q+1}^{q+1}\right) \\
&> \delta\Big[(p-1)\int_\Omega F(u(t),v(t))dx + (k-1)\int_\Gamma H(\gamma u(t))d\Gamma\Big] + 2(1-\delta)\rho d \\
&\quad - 2E(t) - \epsilon\left(S(t)^{\frac{m+1}{p+1}} + S(t)^{\frac{r+1}{p+1}} + S(t)^{j_0}\right) \\
&\ge \delta\Big[(p-1)\int_\Omega F(u(t),v(t))dx + (k-1)\int_\Gamma H(\gamma u(t))d\Gamma\Big] \\
&\quad - \epsilon\left(S(t)^{\frac{m+1}{p+1}} + S(t)^{\frac{r+1}{p+1}} + S(t)^{j_0}\right), \text{ a.e. } t \in [0,\infty).
\end{aligned} \tag{4.3.23}$$

Now, we consider two cases: $S(t) > 1$ and $S(t) \le 1$.

If $S(t) > 1$, then since $p > \max\{m, r\}$ and $j_0 < 1$, one has $S(t)^{\frac{m+1}{p+1}} + S(t)^{\frac{r+1}{p+1}} + S(t)^{j_0} \le 3S(t)$. In this case, we choose $0 < \epsilon \le \frac{1}{6}\delta \min\{p-1, k-1\}$, and thus, (4.3.23) and the definition of $S(t)$ imply

$$\begin{aligned}
&\frac{1}{2}N''(t) + C_\epsilon \left(\|u_t(t)\|_{m+1}^{m+1} + \|v_t(t)\|_{r+1}^{r+1} + |\gamma u_t(t)|_{q+1}^{q+1}\right) \\
&\ge \delta\Big[(p-1)\int_\Omega F(u(t),v(t))dx + (k-1)\int_\Gamma H(\gamma u(t))d\Gamma\Big] - 3\epsilon S(t) \\
&\ge \frac{1}{2}\delta\Big[(p-1)\int_\Omega F(u(t),v(t))dx + (k-1)\int_\Gamma H(\gamma u(t))d\Gamma\Big] > \delta\rho d,
\end{aligned} \tag{4.3.24}$$

for a.e. $t \in [0, \infty)$, where the inequality (4.3.21) has been used.

If $S(t) \le 1$, then $S(t)^{\frac{m+1}{p+1}} + S(t)^{\frac{r+1}{p+1}} + S(t)^{j_0} \le 3$. In this case, we choose $0 < \epsilon \le \frac{1}{3}\delta\rho d$. Thus, it follows from (4.3.23) and (4.3.21) that

$$\begin{aligned}
&\frac{1}{2}N''(t) + C_\epsilon \left(\|u_t(t)\|_{m+1}^{m+1} + \|v_t(t)\|_{r+1}^{r+1} + |\gamma u_t(t)|_{q+1}^{q+1}\right) \\
&\ge \delta\Big[(p-1)\int_\Omega F(u(t),v(t))dx + (k-1)\int_\Gamma H(\gamma u(t))d\Gamma\Big] - 3\epsilon \\
&> 2\delta\rho d - 3\epsilon \ge \delta\rho d, \text{ a.e. } t \in [0,\infty).
\end{aligned} \tag{4.3.25}$$

Therefore, if we choose $\epsilon \leq \frac{1}{6}\delta \min\{p-1, k-1, 2\rho d\}$, then it follows from (4.3.24)-(4.3.25) that

$$N''(t) + 2C_\epsilon \left(\|u_t(t)\|_{m+1}^{m+1} + \|v_t(t)\|_{r+1}^{r+1} + |\gamma u_t(t)|_{q+1}^{q+1} \right) > 2\delta\rho d, \text{ a.e. } t \in [0, \infty). \tag{4.3.26}$$

Integrating (4.3.26) yields

$$N'(t) - N'(0) + 2C_\epsilon \int_0^t (\|u_t(\tau)\|_{m+1}^{m+1} + \|v_t(\tau)\|_{r+1}^{r+1} + |\gamma u_t(\tau)|_{q+1}^{q+1}) d\tau \geq (2\delta\rho d)t, \tag{4.3.27}$$

for all $t \in [0, \infty)$.

By the restrictions on damping in (1.3.24), one has

$$\begin{aligned}
&\int_0^t \left(\|u_t(\tau)\|_{m+1}^{m+1} + \|v_t(\tau)\|_{r+1}^{r+1} + |\gamma u_t(\tau)|_{q+1}^{q+1} \right) d\tau \\
&\leq C \left(\int_0^t \int_\Omega (g_1(u_t)u_t + g_2(v_t)v_t) dx d\tau + \int_0^t \int_\Gamma g(\gamma u_t)\gamma u_t d\Gamma d\tau \right) \\
&= C(E(0) - E(t)) < C\rho d \leq Cd, \text{ for all } t \in [0, \infty),
\end{aligned} \tag{4.3.28}$$

where we have used the energy identity (4.1.1) and the energy estimate (4.3.4).

A combination of (4.3.27) and (4.3.28) yields

$$N'(t) \geq (2\delta\rho d)t + N'(0) - C(\epsilon)d, \text{ for all } t \in [0, \infty). \tag{4.3.29}$$

Integrating (4.3.29) yields

$$N(t) \geq (\delta\rho d)t^2 + [N'(0) - C(\epsilon)d]t + N(0), \text{ for all } t \in [0, \infty). \tag{4.3.30}$$

It is important to note here (4.3.30) asserts $N(t)$ has a *quadratic* growth rate as $t \to \infty$.

On the other hand, we can estimate $N(t)$ directly as follows. Note,

$$\begin{aligned}
\|u(t)\|_2^2 &= \int_\Omega \left| u_0 + \int_0^t u_t(\tau) d\tau \right|^2 dx \\
&\leq 2\|u_0\|_2^2 + 2t \left(\int_0^t \int_\Omega |u_t(\tau)|^2 dx d\tau \right) \\
&\leq 2\|u_0\|_2^2 + Ct^{1+\frac{m-1}{m+1}} \left(\int_0^t \int_\Omega |u_t(\tau)|^{m+1} dx d\tau \right)^{\frac{2}{m+1}} \\
&\leq 2\|u_0\|_2^2 + Cd^{\frac{2}{m+1}} t^{\frac{2m}{m+1}}, \text{ for all } t \in [0, \infty)
\end{aligned}$$

where we have used (4.3.28). Likewise,

$$\|v(t)\|_2^2 \leq 2\|v_0\|_2^2 + Cd^{\frac{2}{r+1}} t^{\frac{2r}{r+1}}, \text{ for all } t \in [0,\infty).$$

It follows that

$$N(t) \leq 2(\|u_0\|_2^2 + \|v_0\|_2^2) + C(d^{\frac{2}{m+1}} t^{\frac{2m}{m+1}} + d^{\frac{2}{r+1}} t^{\frac{2r}{r+1}}), \text{ for all } t \in [0,\infty). \quad (4.3.31)$$

Since $\frac{2m}{m+1} < 2$ and $\frac{2r}{r+1} < 2$, then (4.3.31) contradicts the quadratic growth of $N(t)$, as $t \to \infty$. Therefore, we conclude that weak solution $(u(t), v(t))$ cannot be extended to $[0,\infty)$, and thus it must be the case that there exists $t_0 \in (0,\infty)$ such that $E(t_0) < 0$. Hence, the proof of Theorem 1.3.20 is complete. $\square$

Chapter 5

Convex Integrals on Sobolev Spaces

5.1 Approximation Results

In order to prove Theorems 1.3.22 and 1.3.23, we shall need several approximation lemmas. Throughout, $C_0(\Omega)$ denotes the space of continuous functions with compact support in Ω.

Lemma 5.1.1. *If $u \in D(J)$, then there exists a sequence $v_n \in H^2(\Omega)$ such that $v_n \to u$ in $H^1(\Omega)$, $j_0(v_n) \to j_0(u)$ in $L^1(\Omega)$ and $j_1(\gamma v_n) \to j_1(\gamma u)$ in $L^1(\Gamma)$.*

Proof. We consider the functional $\varphi : L^2(\Omega) \to [0, +\infty]$ defined by

$$\varphi(v) = \int_\Omega \left(\frac{1}{2}|\nabla v|^2 + j_0(v) \right) dx + \int_\Gamma j_1(\gamma v) d\Gamma, \tag{5.1.1}$$

if $v \in H^1(\Omega)$, $j_0(v) \in L^1(\Omega)$, $j_1(\gamma v) \in L^1(\Gamma)$; otherwise $\varphi(v) = +\infty$. Clearly, the functional φ is convex and lower semicontinuous. By Corollary 13 in [15, p 115] it follows that, $\partial\varphi : L^2(\Omega) \to L^2(\Omega)$ is given by

$$\partial\varphi(v) = \{w \in L^2(\Omega) : w + \Delta v \in \partial j_0(v) \text{ a.e. in } \Omega\}$$

with its domain

$$D(\partial\varphi) = \{v \in H^2(\Omega) : -\frac{\partial v}{\partial \nu} \in \partial j_1(v) \text{ a.e. on } \Gamma\}.$$

Next, fix $u \in D(J) \subset H^1(\Omega)$ and put:

$$v_n = \left(I + \frac{1}{n}\partial\varphi\right)^{-1} u. \tag{5.1.2}$$

Since $\partial\varphi$ is maximal monotone then $(I + \frac{1}{n}\partial\varphi) : D(\partial\varphi) \subset L^2(\Omega) \to L^2(\Omega)$ is one-to-one, onto, and $v_n \to u$ in $L^2(\Omega)$. Also notice that, $v_n \in D(\partial\phi) \subset H^2(\Omega)$.

Let us first show that,

$$\lim_{n\to\infty} \varphi(v_n) = \varphi(u). \tag{5.1.3}$$

To see this, note that (5.1.2) implies $\frac{1}{n}\partial\varphi(v_n) = u - v_n$. So, by the definition of subdifferential, we have

$$\frac{1}{n}\left\|\partial\varphi(v_n)\right\|^2_{L^2(\Omega)} = \left(\partial\varphi(v_n), \frac{1}{n}\partial\varphi(v_n)\right) = (\partial\varphi(v_n), u - v_n) \leq \varphi(u) - \varphi(v_n).$$

Consequently $\varphi(v_n) \leq \varphi(u)$. Since φ is lower semicontinuous and $v_n \to u$ in $L^2(\Omega)$, we have $\liminf_{n\to\infty} \varphi(v_n) \geq \varphi(u)$, and so (5.1.3) holds.

Our next step is to show that

$$v_n \to u \text{ strongly in } H^1(\Omega). \tag{5.1.4}$$

Indeed,

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla(v_n - u)|^2 dx &= \frac{1}{2}\int_\Omega |\nabla v_n|^2 dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega \nabla v_n \cdot \nabla u dx \\ &= \varphi(v_n) - \varphi(u) - \int_\Omega j_0(v_n)dx - \int_\Gamma j_1(\gamma v_n)d\Gamma \\ &+ \int_\Omega j_0(u)dx + \int_\Gamma j_1(\gamma u)d\Gamma - \int_\Omega \nabla(v_n - u)\cdot\nabla u dx. \end{aligned} \tag{5.1.5}$$

The fact that $u \in D(J)$ (whence $\varphi(u) < +\infty$), the definition of φ (5.1.1), and the convergence result (5.1.3) imply that $\{\|\nabla v_n\|_{L^2(\Omega)}\}$ is bounded. Also, since v_n is bounded in $L^2(\Omega)$, we infer that $\{v_n\}$ is bounded in $H^1(\Omega)$ and so, on a subsequence labeled by $\{v_n\}$, we have

$$v_n \to u \text{ weakly in } H^1(\Omega). \tag{5.1.6}$$

Now, since the embedding $H^1(\Omega) \hookrightarrow H^{1-\epsilon}(\Omega)$ is compact for $0 < \epsilon < 1$, then on a subsequence, $v_n \to u$ strongly in $H^{1-\epsilon}(\Omega)$ (for sufficiently small $\epsilon > 0$) and therefore $\gamma v_n \to \gamma u$ in $L^2(\Gamma)$.

By extracting a subsequence, still labeled by $\{v_n\}$, one has $v_n \to u$ a.e. in Ω and $\gamma v_n \to \gamma u$ a.e. on Γ. Then, Fatou's lemma gives us

$$\liminf_{n\to\infty} \left(\int_\Omega j_0(v_n)dx + \int_\Gamma j_1(\gamma v_n)d\Gamma \right) \geq \int_\Omega j_0(u)dx + \int_\Gamma j_1(\gamma u)d\Gamma. \tag{5.1.7}$$

Combining (5.1.7), (5.1.3) and (5.1.6), then from (5.1.5) we obtain

$$\liminf_{n\to\infty} \int_\Omega |\nabla(v_n - u)|^2 dx \leq 0.$$

Therefore, on a subsequence one has

$$\lim_{n\to\infty} \int_\Omega |\nabla(v_n - u)|^2 dx = 0. \tag{5.1.8}$$

Since $v_n \to u$ in $L^2(\Omega)$, then (5.1.4) follows. Moreover, by using (5.1.3), (5.1.6) and (5.1.8), then (5.1.5) yields

$$\lim_{n\to\infty} \left(\int_\Omega j_0(v_n)dx + \int_\Gamma j_1(\gamma v_n)d\Gamma \right) = \int_\Omega j_0(u)dx + \int_\Gamma j_1(\gamma u)d\Gamma. \tag{5.1.9}$$

However, by Fatou's lemma,

$$\liminf_{n\to\infty} \int_\Omega j_0(v_n)dx \geq \int_\Omega j_0(u)dx \ \text{ and } \ \liminf_{n\to\infty} \int_\Gamma j_1(\gamma v_n)d\Gamma \geq \int_\Gamma j_1(\gamma u)d\Gamma. \tag{5.1.10}$$

Hence, it follows from (5.1.9)-(5.1.10) (by extracting a further subsequence) that

$$\lim_{n\to\infty} \int_\Omega j_0(v_n)dx = \int_\Omega j_0(u)dx \ \text{ and } \ \lim_{n\to\infty} \int_\Gamma j_1(\gamma v_n)d\Gamma = \int_\Gamma j_1(\gamma u)d\Gamma,$$

which completes the proof of Lemma 5.1.1. $\square$

Lemma 5.1.2. *Let $K \subset \mathbb{R}^2$ be a convex closed set containing the origin. Then*

$$\left\{(u,v) \in \left[C_0(\Omega) \cap W^{1,\infty}(\Omega)\right]^2 : (u(x), v(x)) \in K, \quad \text{for all } x \in \Omega\right\}$$

is dense in

$$\{(u,v) \in L^1(\Omega) \times L^1(\Omega) : (u(x), v(x)) \in K, \ \text{a.e. } x \in \Omega\}.$$

Proof. Let u, $v \in L^1(\Omega)$ such that $(u(x), v(x)) \in K$ for a.e. $x \in \Omega$. Since $C_0^1(\Omega)$ is dense in $L^1(\Omega)$, there exist $\tilde{u}$, $\tilde{v} \in C_0^1(\Omega)$ such that

$$\|u - \tilde{u}\|_{L^1(\Omega)} < \epsilon \text{ and } \|v - \tilde{v}\|_{L^1(\Omega)} < \epsilon. \tag{5.1.11}$$

Let $P : \mathbb{R}^2 \to K \subset \mathbb{R}^2$ be the projection onto the convex closed set K. Put $(\hat{u}(x), \hat{v}(x)) = P(\tilde{u}(x), \tilde{v}(x))$ for all $x \in \Omega$. Since P is a (non-strict) contraction on $\mathbb{R}^2$, then for any x_1, $x_2 \in \Omega$, we have

$$\begin{aligned}
&|(\hat{u}(x_1), \hat{v}(x_1)) - (\hat{u}(x_2), \hat{v}(x_2))| \leq |(\tilde{u}(x_1), \tilde{v}(x_1)) - (\tilde{u}(x_2), \tilde{v}(x_2))| \\
&\leq |\tilde{u}(x_1) - \tilde{u}(x_2)| + |\tilde{v}(x_1) - \tilde{v}(x_2)| \leq C|x_1 - x_2|,
\end{aligned}$$

where in the last inequality we used the fact $\tilde{u}$, $\tilde{v} \in C_0^1(\Omega)$. Therefore,

$$|\hat{u}(x_1) - \hat{u}(x_2)| \leq C|x_1 - x_2| \text{ and } |\hat{v}(x_1) - \hat{v}(x_2)| \leq C|x_1 - x_2| \text{ for any } x_1, x_2 \in \Omega.$$

That is, $\hat{u}$ and $\hat{v}$ are both Lipschitz continuous on Ω, which is equivalent to $\hat{u}$, $\hat{v} \in W^{1,\infty}(\Omega)$. Moreover, since K contains the origin, one has $P(0,0) = (0,0)$, and therefore $\hat{u}$ and $\hat{v}$ both have compact supports in Ω. Also note,

$$|(u(x), v(x)) - (\hat{u}(x), \hat{v}(x))| \leq |(u(x), v(x)) - (\tilde{u}(x), \tilde{v}(x))| \text{ a.e. } x \in \Omega,$$

and so, (5.1.11) yields

$$\|u - \hat{u}\|_{L^1(\Omega)} < 2\epsilon \text{ and } \|v - \hat{v}\|_{L^1(\Omega)} < 2\epsilon,$$

which completes the proof. □

Proposition 5.1.3. *Let $j : \mathbb{R} \to [0, \infty)$ be a convex function with $j(0) = 0$. If $u \in L^1(\Omega)$, then*

$$\int_\Omega j^*(u)dx = \sup\left\{\int_\Omega (uv - j(v))dx : v \in C_0(\Omega) \cap W^{1,\infty}(\Omega)\right\}.$$

Proof. Since $u \in L^1(\Omega)$ and $j^{**} = j$ on $\mathbb{R}$, then by identity (1) in [14] we obtain

$$\int_\Omega j^*(u)dx = \sup\left\{\int_\Omega (uv - j(v))dx : v \in L^\infty(\Omega)\right\}. \tag{5.1.12}$$

So, if we put

$$\theta = \sup\left\{\int_\Omega (uv - j(v))dx : v \in C_0(\Omega) \cap W^{1,\infty}(\Omega)\right\},$$

then $\theta \leq \int_\Omega j^*(u)dx$.

Let $\epsilon > 0$ be given. Then, from (5.1.12) there exists $v_0 \in L^\infty(\Omega)$, such that

$$\int_\Omega (uv_0 - j(v_0))dx \geq \int_\Omega j^*(u)dx - \epsilon. \tag{5.1.13}$$

Now, put

$$h(r) = \begin{cases} j(r) & \text{if } |r| \leq \|v_0\|_{L^\infty(\Omega)}, \\ +\infty & \text{if } |r| > \|v_0\|_{L^\infty(\Omega)}, \end{cases} \tag{5.1.14}$$

and consider the set $K = \{(r, \rho) \in \mathbb{R}^2 : \rho \geq h(r)\}$. Note, K is the epigraph of h, and since h is convex, lower semicontinuous and $h(0) = 0$, then K is convex, closed and contains the origin. Since $(v_0(x), h(v_0(x))) \in K$ for all $x \in \Omega$, we may apply Lemma 5.1.2 to $(v_0, h(v_0)) \in L^1(\Omega) \times L^1(\Omega)$ to obtain sequences $\{v_n\}$, $\{\alpha_n\} \subset C_0(\Omega) \cap W^{1,\infty}(\Omega)$ such that,

$$v_n \to v_0, \qquad \alpha_n \to h(v_0) \text{ in } L^1(\Omega), \tag{5.1.15}$$

and $\alpha_n \geq h(v_n)$ in Ω. It follows (5.1.14) that, $\|v_n\|_{L^\infty(\Omega)} \leq \|v_0\|_{L^\infty(\Omega)}$. In addition, $\alpha_n \to j(v_0)$ in $L^1(\Omega)$ and $\alpha_n \geq j(v_n)$ in Ω.

After extracting a subsequence, we have $v_n \to v_0$. a.e. Ω and, since j is continuous, one obtains $j(v_n) \to j(v_0)$, a.e. Ω. By the Generalized Lebesgue Dominated Convergence Theorem, we infer $j(v_n) \to j(v_0)$ in $L^1(\Omega)$. Since $\int_\Omega (uv_n - j(v_n))dx \leq \theta$, we can pass to the limit by the Lebesgue Dominated Convergence Theorem to obtain $\int_\Omega (uv_0 - j(v_0))dx \leq \theta$. It follows from (5.1.13) that $\int_\Omega j^*(u)dx - \epsilon \leq \theta \leq \int_\Omega j^*(u)dx$, and therefore, $\int_\Omega j^*(u)dx = \theta$. $\square$

Similar to Proposition 5.1.3 we can deduce the following result.

Proposition 5.1.4. *Let $j : \mathbb{R} \to [0, \infty)$ be a convex function with $j(0) = 0$. If $u \in L^1(\Gamma)$, then*

$$\int_\Gamma j^*(u)d\Gamma = \sup\left\{\int_\Gamma (uv - j(v))d\Gamma : v \in W^{1,\infty}(\Gamma)\right\}.$$

5.2 Proof of Theorem 1.3.22

We carry out the proof in three steps.

Step 1: Since j_0 and j_1 are continuous on $\mathbb{R}$, then if $\rho > 0$ is given, then there exists $\eta > 0$ such that $j_0(s), j_1(s) \leq \eta$, whenever $|s| \leq \rho$. Thus, if $v \in C^1(\overline{\Omega})$ with $\|v\|_{C(\overline{\Omega})} \leq \rho$, then $j_0(v(x)) \leq \eta$ for all $x \in \Omega$ and $j_1(v(x)) \leq \eta$ for all $x \in \Gamma$. Therefore, by Fenchel's inequality

$$\begin{aligned}\langle T, v\rangle \leq J^*(T) + J(v) &= J^*(T) + \int_\Omega j_0(v)dx + \int_\Gamma j_1(\gamma v)d\Gamma \\ &\leq J^*(T) + \eta(|\Omega| + |\Gamma|) < \infty, \end{aligned} \tag{5.2.1}$$

for all $v \in C^1(\overline{\Omega})$ with $\|v\|_{C(\overline{\Omega})} \leq \rho$. By Hahn-Banach theorem, we can extend T to be a bounded linear functional on $C(\overline{\Omega})$, and since $C^1(\overline{\Omega})$ is dense in $C(\overline{\Omega})$, the extension is unique, which we still denote it by T. That is, $T \in (C(\overline{\Omega}))'$, and so, T is a signed Radon measure on $\overline{\Omega}$. Then we have the following Radon-Nikodym decomposition of T:

$$T = T_a d\Omega + T_{\Omega,s}, \tag{5.2.2}$$

where $T_a \in L^1(\Omega)$ and $T_{\Omega,s}$ is singular with respect to $d\Omega$, the Lebesgue measure on $\overline{\Omega}$.

Now, let $d\Gamma$ denote the Lebesgue measure on $(\Gamma, \mathcal{L}_\Gamma)$ where $\mathcal{L}_\Gamma$ is the class of Lebesgue measurable subset of Γ. We extend $d\Gamma$ to the interior of Ω by defining the measure $d\tilde{\Gamma}$ on $(\overline{\Omega}, \mathcal{L}_{\overline{\Omega}})$ via

$$d\tilde{\Gamma}(A) = d\Gamma(A \cap \Gamma),$$

for $A \in \mathcal{L}_{\overline{\Omega}}$. Notice, $d\tilde{\Gamma}$ is a well-defined measure since one can show that $A \cap \Gamma \in \mathcal{L}_\Gamma$ for all $A \in \mathcal{L}_{\overline{\Omega}}$. Subsequently, we decompose $T_{\Omega,s}$ with respect to $d\tilde{\Gamma}$:

$$T_{\Omega,s} = T_{\Gamma,a} d\tilde{\Gamma} + T_s, \tag{5.2.3}$$

where $T_{\Gamma,a} \in L^1(d\tilde{\Gamma})$ and T_s is singular with respect to both $d\tilde{\Gamma}$ and $d\Omega$. It follows from (5.2.2)-(5.2.3) that,

$$T = T_a d\Omega + T_{\Gamma,a} d\tilde{\Gamma} + T_s. \tag{5.2.4}$$

Clearly, $T_{\Gamma,a} \in L^1(\Gamma)$. Thus, for all $v \in C(\overline{\Omega})$, we have

$$\begin{aligned}\langle T, v\rangle &= \int_\Omega T_a v dx + \int_{\overline{\Omega}} T_{\Gamma,a} v d\tilde{\Gamma} + \langle T_s, v\rangle \\ &= \int_\Omega T_a v dx + \int_\Gamma T_{\Gamma,a} \gamma v d\Gamma + \langle T_s, v\rangle. \end{aligned} \tag{5.2.5}$$

Step 2: Let $v \in H^2(\Omega)$, then Fenchel's inequality yields:

$$\begin{cases} T_a(x)v(x) - j_0(v(x)) \le j_0^*(T_a(x)) \text{ a.e. } x \in \Omega, \\ T_{\Gamma,a}(x)\gamma v(x) - j_1(\gamma v(x)) \le j_1^*(T_{\Gamma,a}(x)) \text{ a.e. } x \in \Gamma. \end{cases} \tag{5.2.6}$$

Integrate the two inequalities in (5.2.6) over Ω and Γ, respectively, and add the results to obtain:

$$\langle T, v\rangle - \int_\Omega j_0(v)dx - \int_\Gamma j_1(\gamma v)d\Gamma \le \int_\Omega j_0^*(T_a)dx + \int_\Gamma j_1^*(T_{\Gamma,a})d\Gamma + \langle T_s, v\rangle, \tag{5.2.7}$$

where we have used (5.2.5).

Now, notice Lemma 5.1.1 implies

$$\begin{aligned} J^*(T) &= \sup\{\langle T, v\rangle - J(v) : v \in D(J)\} \\ &= \sup\left\{\langle T, v\rangle - \int_\Omega j_0(v)dx - \int_\Gamma j_1(\gamma v)d\Gamma : v \in H^2(\Omega)\right\}. \end{aligned} \tag{5.2.8}$$

Therefore, if we set

$$\begin{aligned} A &= \int_\Omega j_0^*(T_a)dx + \int_\Gamma j_1^*(T_{\Gamma,a})d\Gamma, \\ B &= \sup\{\langle T_s, v\rangle : v \in H^2(\Omega)\}, \end{aligned}$$

then (5.2.7) and (5.2.8) yield $J^*(T) \le A + B$.

Step 3: Since $T_a \in L^1(\Omega)$, then by Proposition 5.1.3 there exists a sequence $v_1^n \in C_0(\Omega) \cap W^{1,\infty}(\Omega)$ such that

$$\int_\Omega (T_a v_1^n - j_0(v_1^n))dx \uparrow \int_\Omega j_0^*(T_a)dx, \text{ as } n \to \infty. \tag{5.2.9}$$

Also, since $T_{\Gamma,a} \in L^1(\Gamma)$, then by Proposition 5.1.4 there exists a sequence $v_2^n \in W^{1,\infty}(\Gamma)$ such that

$$\int_\Gamma (T_{\Gamma,a} v_2^n - j_1(v_2^n))d\Gamma \uparrow \int_\Gamma j_1^*(T_{\Gamma,a})d\Gamma, \text{ as } n \to \infty. \tag{5.2.10}$$

Since each v_1^n has compact support, let $K_n := \operatorname{supp} v_1^n \subset \Omega$. Put $\alpha_n = \|v_2^n\|_{C(\Gamma)}$ and $\beta_n = \sup\{j_0(s) : |s| \le \alpha_n\}$. Since $T_a \in L^1(\Omega)$, then for each n, there exists a open set E_n with smooth boundary such that, $K_n \subset E_n \subset \overline{E_n} \subset \Omega$ and

$$\int_{\Omega\setminus E_n} (\alpha_n|T_a| + \beta_n)\, dx < \frac{1}{n}. \tag{5.2.11}$$

Now, for each n, we can construct a function $v_3^n \in C(\overline{\Omega}) \cap H^1(\Omega)$ as follows:

$$v_3^n = \begin{cases} v_1^n & \text{on } K_n, \\ 0 & \text{on } \overline{E_n} \backslash K_n, \\ \xi^n & \text{in } \Omega \backslash \overline{E_n}, \\ v_2^n & \text{on } \Gamma, \end{cases}$$

where $\xi^n \in C^2(\Omega\backslash\overline{E_n}) \cap C(\overline{\Omega}\backslash E_n) \cap H^1(\Omega\backslash\overline{E_n})$ is the unique solution of the Dirichlét problem:

$$\begin{cases} \Delta \xi^n = 0 & \text{in } \Omega\backslash\overline{E_n}, \\ \xi^n = 0 & \text{on } \partial E_n, \\ \xi^n = v_2^n \in W^{1,\infty}(\Gamma) & \text{on } \Gamma. \end{cases}$$

Notice the regularity of ξ^n follows from Theorem 6.1 (p.55) and Corollary 7.1 (p.361) in [19]. By the maximal principle, we know $|\xi^n(x)| \le \alpha_n = \|v_2^n\|_{C(\Gamma)}$ for all $x \in \Omega\backslash\overline{E_n}$. Therefore,

$$\begin{aligned} &\left| \int_\Omega (T_a v_3^n - j_0(v_3^n))dx - \int_\Omega (T_a v_1^n - j_0(v_1^n))dx \right| \\ &\le \int_{\Omega\backslash E_n} (\alpha_n |T_a| + \beta_n)\, dx < \frac{1}{n}. \end{aligned} \tag{5.2.12}$$

By combining (5.2.9)-(5.2.12) together with the fact $\gamma v_3^n = v_2^n$, we have

$$\begin{aligned} &\int_\Omega (T_a v_3^n - j_0(v_3^n))dx + \int_\Gamma (T_{\Gamma,a} \gamma v_3^n - j_1(\gamma v_3^n))d\Gamma \\ &\uparrow \int_\Omega j_0^*(T_a)dx + \int_\Gamma j_1^*(T_{\Gamma,a})d\Gamma = A \ \text{ as } \ n \to \infty. \end{aligned} \tag{5.2.13}$$

Let us also remark here that while each v_3^n belongs to $H^1(\Omega)$, the result in (5.2.13) does not require the H^1 norm to be bounded in n, so the blow up of ξ_n in $H^1(\Omega)$ as $n \to \infty$ is irrelevant.

Recall $B = \sup\{\langle T_s, v\rangle : v \in H^2(\Omega)\}$, so there exists a sequence $v_4^n \in H^2(\Omega)$ such that

$$\langle T_s, v_4^n \rangle \uparrow B \quad \text{as} \quad n \to \infty. \tag{5.2.14}$$

Since the measure T_s is singular with respect to both $d\Omega$ and $d\tilde{\Gamma}$, there exists a measurable set $S \subset \overline{\Omega}$ such that $\overline{\Omega}\backslash S$ is null for T_s and S is null for both $d\Omega$ and $d\tilde{\Gamma}$.

So for any $\delta > 0$, there exists U relatively open in $\overline{\Omega}$ such that $S \subset U$ with

$$\int_U dx < \delta \ \text{ and } \ \int_U d\tilde{\Gamma} = \int_{\Gamma \cap U} d\Gamma < \delta. \tag{5.2.15}$$

We may extend U to U_{ext} such that U_{ext} is open and bounded in $\mathbb{R}^3$, $U \subset U_{ext}$ and $U \cap \overline{\Omega} = U_{ext} \cap \overline{\Omega}$.

Given the preceding general observation we can claim that there exist open sets $\{V_k\}$ and $\{U_k\}$ such that V_k and U_k both having smooth boundaries and satisfy $V_k \subset \overline{V_k} \subset U_k \subset U_{ext}$ with

$$\int_{U_{ext} \setminus V_k} dx < \frac{1}{k}, \quad \int_{\Gamma \cap (U_{ext} \setminus V_k)} d\Gamma < \frac{1}{k} \ \text{ and } \ \int_{U \cap (U_{ext} \setminus V_k)} dT_s < \frac{1}{k}. \tag{5.2.16}$$

Now fix n; one may extend $v_3^n \in C(\overline{\Omega}) \cap H^1(\Omega)$ and $v_4^n \in H^2(\Omega)$ to functions on $\mathbb{R}^3$, i.e., there exist $\tilde{v}_3^n \in C_0(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$ and $\tilde{v}_4^n \in C_0(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)$ such that $\tilde{v}_3^n|_{\overline{\Omega}} = v_3^n$ and $\tilde{v}_4^n|_{\overline{\Omega}} = v_4^n$.

For each k, we construct a function $\tilde{w}_k^n \in C_0(\mathbb{R}^3) \cap H^1(\mathbb{R}^3)$:

$$\tilde{w}_k^n = \begin{cases} \tilde{v}_3^n & \text{in } \mathbb{R}^3 \setminus U_k, \\ \zeta_k^n & \text{in } U_k \setminus \overline{V_k}, \\ \tilde{v}_4^n & \text{in } \overline{V_k}, \end{cases} \tag{5.2.17}$$

where $\zeta_k^n \in C^2(U_k \setminus \overline{V_k}) \cap C(\overline{U_k} \setminus V_k) \cap H^1(U_k \setminus \overline{V_k})$ is the unique solution of the Dirichlét problem:

$$\begin{cases} \Delta \zeta_k^n = 0 & \text{in } U_k \setminus \overline{V_k}, \\ \zeta_k^n = \tilde{v}_3^n & \text{on } \partial U_k, \\ \zeta_k^n = \tilde{v}_4^n & \text{on } \partial V_k. \end{cases}$$

Again, notice the regularity of ζ_k^n follows from Theorem 6.1 (p.55) and Corollary 7.1 (p.361) in [19].

Define $w_k^n = \tilde{w}_k^n|_{\overline{\Omega}}$, then $w_k^n \in C(\overline{\Omega}) \cap H^1(\Omega)$. By Fenchel's inequality and (5.2.5) we obtain

$$\begin{aligned} &J^*(T) \geq \langle T, w_k^n \rangle - J(w_k^n) \\ &= \int_\Omega T_a w_k^n dx + \int_\Gamma T_{\Gamma,a} \gamma w_k^n d\Gamma + \langle T_s, w_k^n \rangle - \int_\Omega j_0(w_k^n) dx - \int_\Gamma j_1(\gamma w_k^n) d\Gamma. \end{aligned} \tag{5.2.18}$$

By (5.2.17) and the maximum principle, one has $\|w_k^n\|_{C(\overline{\Omega})} \leq \max\{\|\tilde{v}_3^n\|_{C(\mathbb{R}^3)}, \|\tilde{v}_4^n\|_{C(\mathbb{R}^3)}\}$, for all k; and by (5.2.16) $w_k^n \to v_4^n$ $|T_s|-$a.e. on $\overline{\Omega}$ as $k \to \infty$, we infer $\lim_{k\to\infty} \langle T_s, w_k^n \rangle =$

$\langle T_s, v_4^n \rangle$. Also, by (5.2.16) we know $w_k^n \to v_4^n$ a.e. in U and $\gamma w_k^n \to \gamma v_4^n$ a.e. on $\Gamma \cap U$ as $k \to \infty$, thus the Lebesgue dominated convergence theorem implies

$$\lim_{k\to\infty} \int_\Omega T_a w_k^n dx = \lim_{k\to\infty} \int_U T_a w_k^n dx + \int_{\Omega\setminus U} T_a v_3^n dx = \int_U T_a v_4^n dx + \int_{\Omega\setminus U} T_a v_3^n dx,$$
$$\lim_{k\to\infty} \int_\Gamma T_{\Gamma,a} \gamma w_k^n d\Gamma = \int_{\Gamma\cap U} T_{\Gamma,a}\gamma v_4^n d\Gamma + \int_{\Gamma\setminus U} T_{\Gamma,a}\gamma v_3^n d\Gamma,$$

and

$$\lim_{k\to\infty} \int_\Omega j_0(w_k^n)dx = \int_U j_0(v_4^n)dx + \int_{\Omega\setminus U} j_0(v_3^n)dx,$$
$$\lim_{k\to\infty} \int_\Gamma j_1(\gamma w_k^n)d\Gamma = \int_{\Gamma\cap U} j_1(\gamma v_4^n)d\Gamma + \int_{\Gamma\setminus U} j_1(\gamma v_3^n)d\Gamma.$$

Therefore, taking the limit as $k \to \infty$ in (5.2.18) yields

$$\begin{aligned} J^*(T) \geq & \int_\Omega (T_a v_3^n - j_0(v_3^n))dx + \int_\Gamma (T_{\Gamma,a}\gamma v_3^n - j_1(\gamma v_3^n))d\Gamma + \langle T_s, v_4^n\rangle \\ & + \int_U (T_a v_4^n - T_a v_3^n - j_0(v_4^n) + j_0(v_3^n))dx \\ & + \int_{\Gamma\cap U} (T_{\Gamma,a}\gamma v_4^n - T_{\Gamma,a}\gamma v_3^n - j_1(\gamma v_4^n) + j_1(\gamma v_3^n))d\Gamma. \end{aligned}$$

By (5.2.15), if we let $\delta \to 0$, then the last two integrals on the right-hand side of the above inequality both converge to zero, hence one has

$$J^*(T) \geq \int_\Omega (T_a v_3^n - j_0(v_3^n))dx + \int_\Gamma (T_{\Gamma,a}\gamma v_3^n - j_1(\gamma v_3^n))d\Gamma + \langle T_s, v_4^n\rangle.$$

Finally, we let $n \to \infty$ and use (5.2.13)–(5.2.14) to obtain $J^*(T) \geq A + B$.

Recall that in Step 2 we have shown that $J^*(T) \leq A + B$, so $J^*(T) = A + B$. Since $J^*(T) < \infty$ and $A > -\infty$, we know that $B < \infty$, and, being a supremum of a linear functional, must be zero. That is, $B = 0$ and $T_s = 0$. It follows that $J^*(T) = A$ and by (5.2.5) we obtain (1.3.30). This completes the proof of Theorem 1.3.12.

5.3 Proof of Theorem 1.3.23

First, we assume $T \in \partial J(u)$. Then, Fenchel's equality and the fact that $u \in D(\partial J) \subset D(J)$ yield that $J^*(T) = \langle T, u\rangle - J(u) < +\infty$. Then, by Theorem 1.3.12, T is a signed Radon measure on $\overline{\Omega}$ and there exist $T_a \in L^1(\Omega)$ and $T_{\Gamma,a} \in L^1(\Gamma)$ such that (1.3.30) holds.

Since $u \in D(J)$, by Lemma 5.1.1 there exists a sequence $v_n \in H^2(\Omega)$ such that $v_n \to u$ in $H^1(\Omega)$ and a.e. in Ω, $\gamma v_n \to \gamma u$ a.e. on Γ, $j_0(v_n) \to j_0(u)$ in $L^1(\Omega)$ and a.e. in Ω, $j_1(\gamma v_n) \to j_1(\gamma u)$ in $L^1(\Gamma)$ and a.e. on Γ.

Fenchel's inequality gives

$$\begin{aligned} &j_0^*(T_a) + j_0(v_n) - T_a v_n \geq 0 \ \text{ a.e. in } \Omega, \\ &j_1^*(T_{\Gamma,a}) + j_1(\gamma v_n) - T_{\Gamma,a}\gamma v_n \geq 0 \ \text{ a.e. on } \Gamma. \end{aligned}$$

Since $T \in (H^1(\Omega))'$, by (1.3.30) we have

$$\langle T, u\rangle = \lim_{n\to\infty} \langle T, v_n\rangle = \lim_{n\to\infty} \left(\int_\Omega T_a v_n dx + \int_\Gamma T_{\Gamma,a}\gamma v_n d\Gamma \right).$$

Therefore, Fatou's lemma yields

$$\begin{aligned} &\int_\Omega (j_0^*(T_a) + j_0(u) - T_a u)dx + \int_\Gamma (j_1^*(T_{\Gamma,a}) + j_1(\gamma u) - T_{\Gamma,a}\gamma u)d\Gamma \\ &\leq \liminf_{n\to\infty} \left(\int_\Omega (j_0^*(T_a) + j_0(v_n) - T_a v_n)dx + \int_\Gamma (j_1^*(T_{\Gamma,a}) + j_1(\gamma v_n) - T_{\Gamma,a}\gamma v_n)d\Gamma \right) \\ &= \int_\Omega (j_0^*(T_a) + j_0(u))dx + \int_\Gamma (j_1^*(T_{\Gamma,a}) + j_1(\gamma u))d\Gamma - \langle T, u\rangle \\ &= J^*(T) + J(u) - \langle T, u\rangle = 0 \end{aligned} \tag{5.3.1}$$

where we have used Theorem 1.3.12 and Fenchel's equality, since $T \in \partial J(u)$.

On the other hand, Fenchel's inequality implies

$$\begin{aligned} &j_0^*(T_a) + j_0(u) - T_a u \geq 0 \ \text{ a.e. in } \Omega, \\ &j_1^*(T_{\Gamma,a}) + j_1(\gamma u) - T_{\Gamma,a}\gamma u \geq 0 \ \text{ a.e. on } \Gamma. \end{aligned}$$

In order for (5.3.1) to hold, we must have

$$j_0^*(T_a) + j_0(u) = T_a u \ \text{ a.e. in } \Omega \ \text{ and } \ j_1^*(T_{\Gamma,a}) + j_1(\gamma u) = T_{\Gamma,a}\gamma u \ \text{ a.e. on } \Gamma.$$

So, $T_a u \in L^1(\Omega)$ and $T_{\Gamma,a}\gamma u \in L^1(\Gamma)$. Also (5.3.1) becomes equality, and thus (1.3.33) holds. Moreover, since $D(j_0)$ and $D(j_1) = \mathbb{R}$, the converse of Fenchel's equality theorem holds and we infer (1.3.31).

Conversely, assume $T \in (H^1(\Omega))'$ such that there exist $T_a \in L^1(\Omega)$, $T_{\Gamma,a} \in L^1(\Gamma)$ satisfying (1.3.30) and (1.3.31). First, we claim that

$$\langle T, v\rangle = \int_\Omega T_a v dx + \int_\Gamma T_{\Gamma,a}\gamma v d\Gamma \text{ for all } v \in H^1(\Omega) \cap L^\infty(\Omega). \tag{5.3.2}$$

In fact, if $v \in H^1(\Omega) \cap L^\infty(\Omega)$, then there exists $v_n \in C(\overline{\Omega})$ such that $v_n \to v$ in $H^1(\Omega)$ and a.e. in Ω with $|v_n| \le M$ in Ω for some $M > 0$. By (1.3.30) and Lebesgue dominated convergence theorem, we obtain (5.3.2).

Since $u \in H^1(\Omega)$, if we set

$$u_n = \begin{cases} n & \text{if } u \ge n \\ u & \text{if } |u| < n \\ -n & \text{if } u \le -n \end{cases},$$

then $u_n \in H^1(\Omega) \cap L^\infty(\Omega)$. So by (5.3.2), one has

$$\langle T, u_n\rangle = \int_\Omega T_a u_n dx + \int_\Gamma T_{\Gamma,a}\gamma u_n d\Gamma. \tag{5.3.3}$$

Since j_0 and j_1 are convex functions, then it follows from (1.3.31) that, for all $v \in H^1(\Omega)$,

$$\begin{aligned} &T_a(x)(u(x) - v(x)) \ge j_0(u(x)) - j_0(v(x)) \text{ a.e. in } \Omega, \\ &T_{\Gamma,a}(x)(\gamma u(x) - \gamma v(x)) \ge j_1(\gamma u(x)) - j_1(\gamma v(x)) \text{ a.e. on } \Gamma. \end{aligned} \tag{5.3.4}$$

If $v = 0$, then $T_a(x)u(x) \ge j_0(u(x)) \ge 0$ a.e. in Ω and $T_{\Gamma,a}(x)\gamma u(x) \ge j_1(\gamma u(x)) \ge 0$ a.e. on Γ. Since $u_n(x)$ and $u(x)$ have the same sign a.e. in Ω, we obtain $T_a(x)u_n(x) \ge 0$ a.e. in Ω. Similarly, one has $T_{\Gamma,a}(x)\gamma u_n(x) \ge 0$ a.e. on Γ.

Since $u_n \to u$ in $H^1(\Omega)$ and a.e. in Ω with $\gamma u_n \to \gamma u$ a.e. on Γ, then by (5.3.3) and Fatou's lemma one has

$$\begin{aligned} \langle T, u\rangle &= \lim_{n\to\infty}\langle T, u_n\rangle = \lim_{n\to\infty}\left(\int_\Omega T_a u_n dx + \int_\Gamma T_{\Gamma,a}\gamma u_n d\Gamma\right) \\ &\ge \liminf_{n\to\infty}\int_\Omega T_a u_n dx + \liminf_{n\to\infty}\int_\Gamma T_{\Gamma,a}\gamma u_n d\Gamma \ge \int_\Omega T_a u dx + \int_\Gamma T_{\Gamma,a}\gamma u d\Gamma, \end{aligned}$$

and along with (1.3.30) and (5.3.4) we obtain for all $v \in H^2(\Omega)$,

$$\begin{aligned} \langle T, u - v\rangle &\ge \int_\Omega T_a(u - v)dx + \int_\Gamma T_{\Gamma,a}(\gamma u - \gamma v)d\Gamma \\ &\ge \int_\Omega (j_0(u) - j_0(v))dx + \int_\Gamma (j_1(\gamma u) - j_1(\gamma v))d\Gamma. \end{aligned}$$

By Lemma 5.1.1 we conclude that, for all $v \in D(J)$

$$\langle T, u - v \rangle \geq \int_{\Omega} (j_0(u) - j_0(v))dx + \int_{\Gamma} (j_1(\gamma u) - j_1(\gamma v))d\Gamma = J(u) - J(v).$$

Thus, $T \in \partial J(u)$, completing the proof of Theorem 1.3.13.

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