1975

Maxwell's Predictions

MAXWELL'S PREDICTIONS

INTRODUCTION

With this module, you will reach a milestone in your study of electromagnetic phenomena. From past modules, you now have (at your fingertips, hopefully!) the same basic laws of electromagnetism that Maxwell collected together in the nineteenth century. However, as powerful as these laws were, Maxwell found that there was a basic flaw - a logical inconsistency - in the one known as Ampère's law. He was able to deduce (in advance of any direct experimental test) precisely the correction that was needed. With this correction, the addition of what is called the "displacement-current" term to Ampère's law, it follows that a changing electric field gives rise to a magnetic field, just as a changing magnetic field gives rise to an electric field according to Faraday's law.

After he had predicted this mutual relationship, Maxwell was able to go on and predict that the right combination of oscillating electric and magnetic fields could literally kick itself through empty space. This is the phenomenon that we now call electromagnetic waves - which include, along with TV and radio waves, the sunlight that we receive across 93 000 000 miles of space without any significant loss of intensity other than that which necessarily follows from its spreading out in all directions.

The development of the theory of electromagnetic waves from the basic laws of electricity and magnetism that you have studied in past modules is one of the most beautiful in physics, and at the same time one of the most mathematically difficult that you will meet in this course. Thus if the arguments at times seem long - bear with it! - the total module is fairly short.

PREREQUISITES

Before you begin this module, you should be able to:	Location of Prerequisite Content
*State and apply Ampère's law (needed for Objectives 1 through 3 of this module	Ampère's Law Module
*State and interpret Gauss' law (needed for Objectives 2 and 3 of this module	Flux and Gauss' Law Module
*State Faraday's law, and apply it to calculate the emf induced around a closed path (needed for Objectives 2 and 3 of this module)	Ampère's Law Module
*Describe a simple form of electrical oscillator (needed for Objective 3 of this module)	Inductance Module
*Use and interpret mathematical descriptions of one-dimensional waves (needed for Objective 3 of this module)	Traveling Waves Module
*Calculate partial derivatives of functions of two variables (needed for Objective 3 of this module)	Partial Derivatives Review

LEARNING OBJECTIVES

After you have mastered the content of this module, you will be able to:

1. Displacement current - Use Ampère's law (including the displacement current) to find the $\vec{B}$ field produced by a changing $\vec{E}$ field, or vice versa.
2. Maxwell's equations - State Maxwell's equations in vacuum (i.e., in the presence of charges and currents, but with no dielectrics or magnetic materials), and indicate the physical significance of each.
3. Electromagnetic waves - For a plane electromagnetic wave, use information about $\vec{E}$ or $\vec{B}$ at given times or places, the direction the wave moves, the frequency, and/or the wavelength to determine other information in this list; also, write down mathematical expressions for the components of $\vec{E}$ and $\vec{B}$, and show that your expressions satisfy the appropriate simplified differential form of Maxwell's equations.

TEXT: Frederick J. Bueche, Introduction to Physics for Scientists and Engineers (McGraw-Hill, New York, 1975), second edition

SUGGESTED STUDY PROCEDURE

Read the General Comments on the following pages of this study guide along with Sections 28.1 through 28.5 and 29.1. Optional: Read Sections 28.6 through 28.8.

When you compare Maxwell's equations (28.1) with the same equations in General Comment 2, you will find slight differences of notation: $\sum q \rightarrow q$, $d\vec{s} \rightarrow d\vec{\ell}$, and $\int \vec{J} \cdot d\vec{A} \rightarrow i$. Also note that in the absence of a dielectric, in the last term of Eq. (28.1d), ε becomes just ε_0. You will find that the derivation of the simplified differential form of Maxwell's equations, Eqs. (28.6) and (28.7), is quite similar to that given in General Comment 3, between Eqs. (9) and (15); take your pick!

Study the Problems with Solutions and work the Assigned Problems. Then take the Practice Test, and work some Additional Problems if necessary, before trying a Mastery Test.

BUECHE

Objective Number	Readings	Problems with Solutions: Study Guide	Problems with Solutions: Text	Assigned Problems: Study Guide	Additional Problems (Chap. 28)
1	General Comment 1; Secs. 28.1, 28.2	A	Illus.[a] 28.1	D, E	Quest.[a] 4 thru 9
2	General Comment 2; Eqs. (28.1) in Sec. 28.2	B		F	
3	General Comment 3; Secs. 28.3 thru 28.5, 29.1	C	Illus. 28.2	G, H, I	Probs. 1, 2, 4 thru 9, 12, 14

[a]Illus. = Illustration(s). Quest. = Question(s).

TEXT: David Halliday and Robert Resnick, Fundamentals of Physics (Wiley, New York, 1970; revised printing, 1974)

SUGGESTED STUDY PROCEDURE

Read the General Comments on the following pages of this study guide, along with Sections 34-4 through 34-6 and 35-1 through 35-3.

You will find that Maxwell's equations stated in Table 34-2 on p. 636 are exactly the same as in General Comment 2, except that we have used $d\vec{A}$ (instead of $d\vec{S}$) for the element of area. Your text gives a complete and accurate derivation of the simplified differential form of Maxwell's equations, Eqs. (35-4) and (35-8). However, the discussion is rather involved; you should find the corresponding derivation between Eqs. (9) and (15) of General Comment 3 easier to follow. When reading the derivation in your text, note that E means E_y and B means B_z.

Study the Problems with Solutions and work the Assigned Problems. Then take the Practice Test, and work some Additional Problems if necessary, before trying a Mastery Test.

HALLIDAY AND RESNICK

Objective Number	Readings	Problems with Solutions: Study Guide	Problems with Solutions: Text	Assigned Problems: Study Guide	Additional Problems
1	General Comment 1; Secs. 34-4, 34-5	A	Chap. 34, Ex.[a] 4, 5	D, E	Chap. 34, Probs. 19 thru 28; Quest.[a] 6 thru 11
2	General Comment 2; Sec. 34-6	B		F	
3	General Comment 3; Secs. 35-1 thru 35-3	C		G, H, I	Chap. 35, Probs. 1, 5 thru 9; Quest. 2

[a]Ex. = Example(s). Quest. = Question(s).

TEXT: Francis Weston Sears and Mark W. Zemansky, University Physics (Addison-Wesley, Reading, Mass., 1970), fourth edition

SUGGESTED STUDY PROCEDURE

Read the General Comments on the following pages of this study guide, along with Sections 36-8 and 36-9. Optional: Read Sections 32-8, 36-1, 36-2, 36-5, and 36-7.

Section 36-8 of your text is devoted to a derivation of the simplified Maxwell's equations, Eqs. (36-17) through (36-19), which is more detailed than the derivation given in General Comment 3 between Eqs. (9) and (18), but equivalent to it. You will probably find you do not need to read both these derivations; take your choice! In your text's derivation, note that H means H_z and E means E_y.

Your text uses the auxilliary quantities $\vec{H}$ and $\vec{D}$; in the absence of dielectric and magnetic materials (which will be the case in this module), these are simply proportional to the more familiar fields $\vec{B}$ and $\vec{E}$: $\vec{H} = \vec{B}/\mu_0$ and $D = \varepsilon_0\vec{E}$. In Section 36-9 and in the optional readings, you will also encounter the polarization $\vec{P} = \vec{D} - \varepsilon_0\vec{E}$ and the magnetization $\vec{M} = \vec{B}/\mu_0 - \vec{H}$; but these vanish when there are no dielectric or magnetic materials, and so need not concern you. Maxwell's equations given in Eqs. (36-20) through (36-23) reduce to those of General Comment 2, when the conditions $\vec{P} = \vec{M} = 0$ are used, along with the identifications of $\vec{H}$ and $\vec{D}$ above and the notation changes $Q_f \to q$, $I_C \to i$, and $d\vec{s} \to d\vec{\ell}$. Some other notation changes you will encounter are $\psi_D \to \varepsilon_0\Phi_E$, $\Phi \to \Phi_B$, and $I_D \to i_d$.

Study Problems A to C and work Problems D to I. Take the Practice Test before trying a Mastery Test.

SEARS AND ZEMANSKY

Objective Number	Readings	Problems with Solutions — Study Guide	Assigned Problems — Study Guide
1	General Comment 1	A	D, E
2	General Comment 2; Eqs. (36-20) thru (36-23) in Sec. 36-9	B	F
3	General Comment 3; Sec. 36-8	C	G, H, I

TEXT: Richard T. Weidner and Robert L. Sells, Elementary Classical Physics (Allyn and Bacon, Boston, 1973), second edition, Vol. 2

SUGGESTED STUDY PROCEDURE

Read the General Comments on the following pages of this study guide, along with Sections 35-1 through 35-3.

Your text gives a very nice and direct demonstration in Section 35-3 that a traveling pulse of crossed $\vec{E}$ and $\vec{B}$ fields is a solution of Maxwell's equations. However, it does not derive the differential form of Maxwell's equations for plane waves; if you want to see more discussion of this topic than is found in General Comment 3, refer to one of the texts listed under Additional Learning Materials below. The first page of Section 35-1 in your text and General Comment 1 in this study guide give alternate versions of the argument for the displacement-current term in Ampère's law; take your choice! When reading your text's discussion, note that there is no reason for the hemispherical surface in Figure 35-2 to touch the edge of the plate; this is just an accident of the drawing.

Study the Problems with Solutions before working Problems D through I. Then take the Practice Test, and work Problem 35-1 if necessary, before taking a Mastery Test.

WEIDNER AND SELLS

Objective Number	Readings	Problems with Solutions: Study Guide	Problems with Solutions: Text	Assigned Problems: Study Guide	Additional Problems
1	General Comment 1; Sec. 35-1	A	Ex.[a] 35-1	D, E	
2	General Comment 2; Sec. 35-2	B		F	
3	General Comment 3; Sec. 35-3	C		G, H, I	35-1

[a]Ex. = Example.

GENERAL COMMENTS

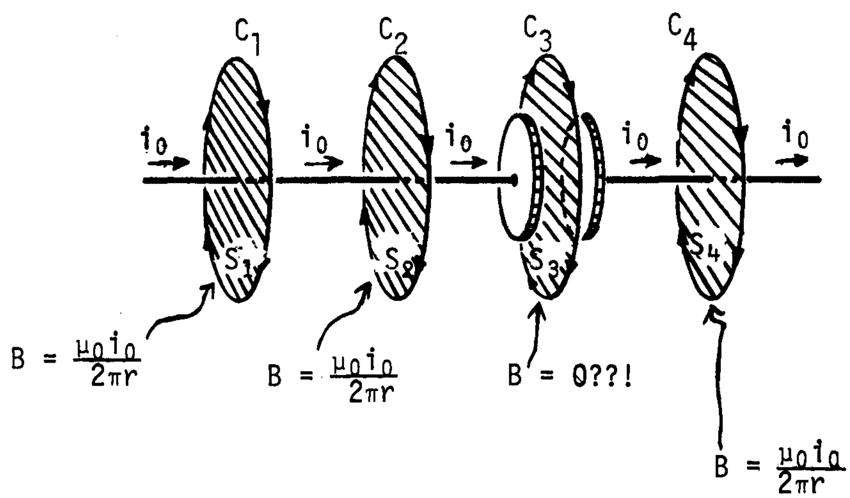
1. The Consistency Argument for the "Displacement Current"

Let us calculate the magnetic field around the long wire in Figure 1. It is a little different from the long wires you have seen in this course before: there is a capacitor in the middle of it. But this does not, of course, preclude a pulse of current for a short time, as the capacitor charges up - or a pulsating back-and-forth current for an indefinite period of time.

We first construct a circle of radius r, such as C_1. The current through the surface S_1 bounded by C_1 is just i_0; and thus Ampère's law yields

$$\mu_0 i_0 = \oint_{C_1} \vec{B} \cdot d\vec{\ell} = 2\pi r B(r) \quad \text{or} \quad B(r) = \frac{\mu_0 i_0}{2\pi r}. \tag{1}$$

Figure 1



We are not at all surprised, of course, when a repetition of this calculation using C_2 and S_2 yields the same result. But we are in for a rude shock when we try C_3 and S_3; there is no current through S_3; therefore the original form of Ampère's law yields

$$0 = \oint_{C_3} \vec{B} \cdot d\vec{\ell} = 2\pi r B(r) \quad \text{or} \quad B(r) = 0 \qquad \text{at } C_3! \tag{2}$$

When we get down to C_4, its surface again cuts through a current i_0, and once again we get the result (1).

Is this possible? Can B really suddenly drop to zero just when we get opposite the gap of the capacitor? It hardly seems so; we must have somehow missed some-thing when we used S_3. The most obvious thing that we did not use was the flux (or $\vec{E}$ field) in the space between the capacitor plates - and clearly this flux is related to the current i_0 in the wire. Since the electric field of a capacitor lies mostly between the plates, and the charge resides mostly on the inner surfaces of the plates, Gauss' law applied to a closed surface containing the upper plate yields

$$q = \varepsilon_0 \oint_S \vec{E} \cdot d\vec{A} = \varepsilon_0 \Phi_0. \tag{3}$$

Since i_0 is just the derivative of q,

$$i_0 = dq/dt = \varepsilon_0 \, d\Phi_E/dt. \tag{4}$$

Wonderful - this solves our problem!! If we define the "total" current by

$$i_{tot} \equiv i + i_d, \tag{5}$$

where $i_d \equiv \varepsilon_0 \, d\Phi_0/dt$, and use this instead of just i in Ampère's law:

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 i_{tot} \equiv \mu_0(i + i_d), \tag{6}$$

then it does not matter which of the circles C_1 through C_4 we use - we always find, consistently, that

$$B(r) = \mu_0 i_0/2\pi r! \tag{7}$$

If we happen to use a surface that intersects the wire, then we pick up $i = i_0$ and i_d vanishes ($E \approx 0$ outside the capacitor); if we use a surface that passes through the capacitor gap, we pick up $i_d = i_0$ and i vanishes. (There is no "true" current between the capacitor plates.) For historical reasons, i_d is called the "displacement current."

You may wonder how it is that we (or Maxwell, for that matter) can get away with making changes like this to an equation, such as Ampère's law, which was based on experimental observation. The reason is that the added term i_d was too small to be observed in the phenomena studied up to Maxwell's time. On the other hand, it is absolutely essential to the now-familiar phenomenon of electromagnetic waves!

2. Maxwell's Equations

Here we record, for reference, the integral form of Maxwell's equations in empty space - i.e., where there are no dielectric or magnetic materials.

Gauss' law for electricity: $\varepsilon_0 \oint_S \vec{E} \cdot d\vec{S} = q(\text{inside } S).$ (8a)

Gauss' law for magnetism: $\oint_S \vec{B} \cdot d\vec{S} = 0.$ (8b)

Faraday's law: $\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}.$ (8c)

Ampère's law: $\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 i + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}.$ (8d)

In the last two, Φ means the flux through any surface bounded by the curve C, and i is the current through such a surface.

Your first reaction may be that Gauss' law for magnetism is new to you; but really it is not! In regions where there are no electric charges, the right-hand side of Eq. (8a) is zero, and Gauss' law for electricity becomes just the statement that the electric flux through any closed surface is zero - or, equivalently, that in such regions electric field lines never end. You tacitly used the corresponding property of magnetic fields when you calculated, say, the field $\vec{B}$ inside a toroid by using Ampère's law: you assumed the flux was constant around the toroid - that field lines did not abruptly end. This is just the content of Eq. (8b). The difference between the right-hand sides of Eqs. (8a) and (8b) arises from the fact that (as far as we know) no magnetic "charges" exist anywhere.

3. Plane Electromagnetic Waves

Undoubtedly, electromagnetic waves are a difficult phenomenon to comprehend properly. This difficulty starts with the problem of visualizing just exactly what is going on: There are $\vec{E}$ and $\vec{B}$ fields oscillating throughout three-dimensional space, and these oscillations somehow travel through space - difficult enough to visualize in 3-space, let alone describe by diagrams on a flat sheet of paper!

Nonetheless, the diagrams in Figures 2, 3, and 4, contrived for this purpose by various people, may help to explain what is going on. The first diagram, Figure 2, shows a pulse of constant amplitude traveling along the x axis with velocity $\vec{c}$. It can, in fact, be shown that an electric field pulse traveling along as in that diagram must be accompanied by a magnetic field pulse, according to Maxwell's equations - and vice-versa! That is, a pulse of $\vec{E}$ and $\vec{B}$ fields together is a valid solution to Maxwell's equations. To the extent that we have confidence that those equations are correct, they predict the existence of such pulses as an observable phenomenon.

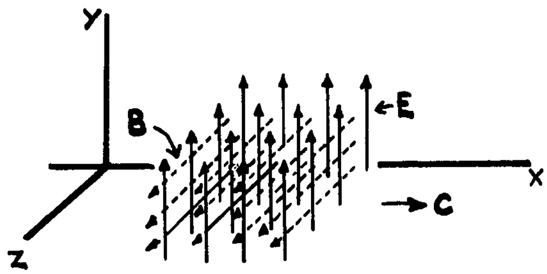


Figure 2

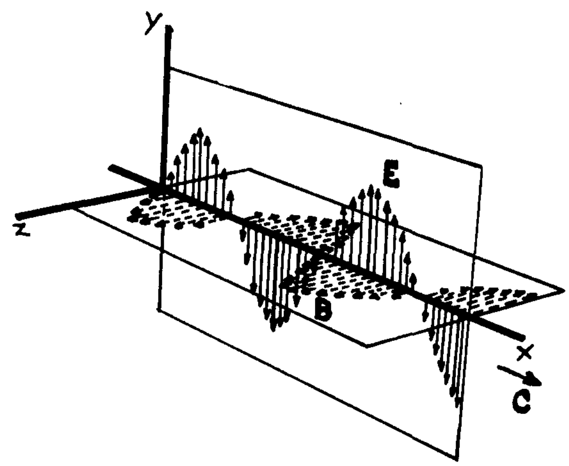
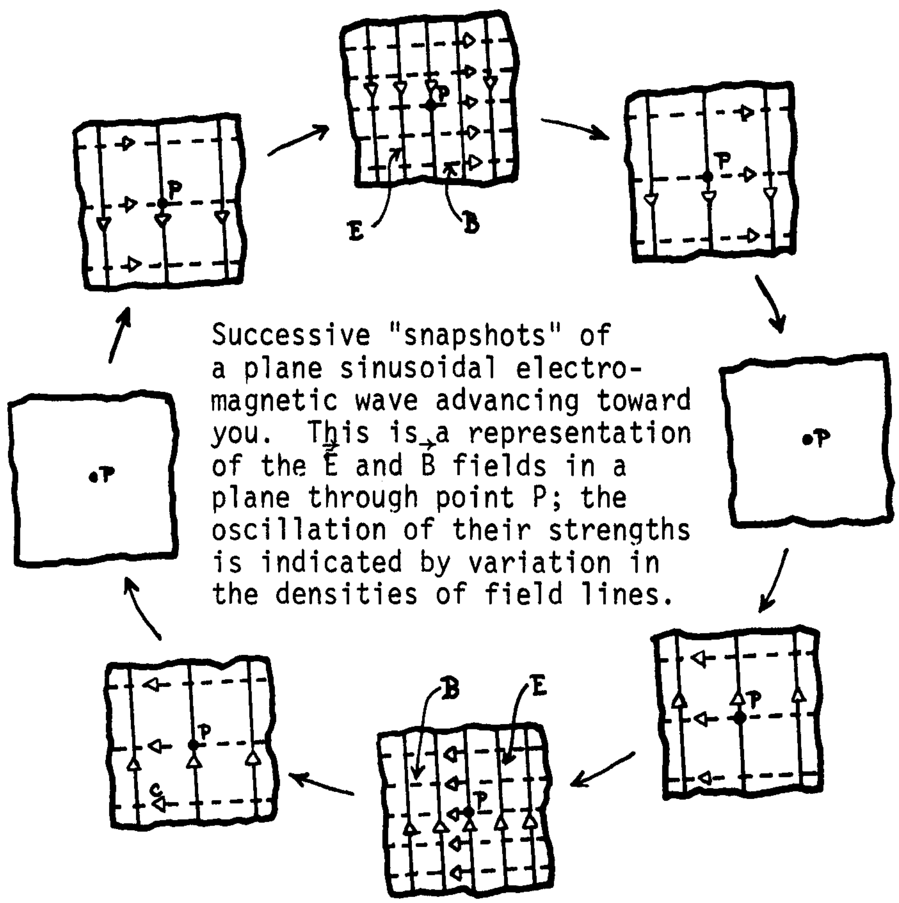


Figure 3



Successive "snapshots" of a plane sinusoidal electromagnetic wave advancing toward you. This is a representation of the $\vec{E}$ and $\vec{B}$ fields in a plane through point P; the oscillation of their strengths is indicated by variation in the densities of field lines.

Figure 4*

*This diagram was taken from <u>Fundamentals of Physics</u>, by David Halliday and Robert Resnick (Wiley, New York, 1970; revised, 1974), with permission of the publisher.

If we now imagine a series of such pulses traveling along one after another, we have a wave train. In fact, the most easily produced electromagnetic waves are a kind of wave train known as a sinusoidal wave. Since it is even harder to draw pictures of wave trains than of individual pulses, it is customary to draw only the field vectors for points on the axis along the direction of propagation; such a diagram for a sinusoidal plane wave is shown in Figure 3. (You will now see why this is called a sinusoidal wave.) The term "plane wave" refers to the fact that the $\vec{E}$ and $\vec{B}$ fields are the same throughout any one plane perpendicular to the axis of propagation (the x axis in Figure 3).

Another view of a plane sinusoidal electromagnetic wave is shown in Figure 4. Here you are looking at the $\vec{E}$ and $\vec{B}$ fields in a plane parallel to the yz plane. The y axis points up in this picture, and the z axis points to your left; the wave is advancing toward you, along the positive x axis. Of course, there is nothing special about the x axis as far as an electromagnetic wave is concerned; the waves shown in the pictures could just as well be traveling along the y or z axis, or in some arbitrary direction. However, there are several characteristics of the waves shown that are required by Maxwell's equations to be true of any plane electromagnetic wave:

(1) Electromagnetic waves in vacuum always travel with the speed of light $c = 3.00 \times 10^8$ m/s.
(2) The $\vec{E}$ and $\vec{B}$ fields are in phase, i.e., their maxima occur at the same place.
(3) Electromagnetic waves are transverse, and furthermore $\vec{E}$ is perpendicular to $\vec{B}$. Also, the directions of $\vec{E}$, $\vec{B}$, and $\vec{c}$ form a right-handed set. In terms of unit vectors, $\hat{E} \times \hat{B} = \hat{c}$.

When we come to a quantitative treatment of electromagnetic waves, we are immediately faced with the second part of the complexity of understanding these waves. You probably feel, justifiably, that the various Maxwell's equations that have served so well in solving problems up to this point are complicated enough to apply. However, using them directly on electromagnetic waves becomes much harder. In actual fact, when people deal with electromagnetic waves, they customarily use an apparently different, but mathematically equivalent, set of equations known as the "differential form" of Maxwell's equations - that is, a set of equations expressed in terms of derivatives. Sad to say, proving the equivalence is a very involved piece of work; and furthermore, these latter equations are too cumbersome to write down in their generality without using notation with which you are not likely to be familiar!

But the good news is that much of this complexity (though not all of it!) goes away if you look just for solutions of a very particular form, namely, plane waves traveling along, say, the x coordinate axis. Explicitly, let us look for solutions in which the fields are of the forms

$$B_x = B_y = E_x = E_z = 0 \quad \text{everywhere,}$$

$$B_z = B_z(x, t) \quad \text{and} \quad E_y = E_y(x, t) \qquad \text{(i.e., no dependence on y or z).} \qquad (9)$$

First, we note that the first two of Maxwell's equations, Gauss' laws for $\vec{E}$ and $\vec{B}$, are immediately satisfied by such fields: the flux lines are continuous, as you can see by drawing a sketch of them, and as they are required to be in the absence of charges.

Our next step is to cast the second two of Maxwell's equations in the absence of charges,

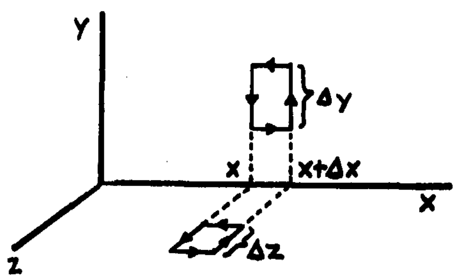
$$\oint \vec{E} \cdot d\vec{\ell} = -d\Phi_B/dt \qquad \text{(Faraday's)},$$

$$\oint \vec{B} \cdot d\vec{\ell} = \varepsilon_0\mu_0(d\Phi_E/dt) \qquad \text{(Ampère's)}, \tag{10}$$

into their differential form for the simplified case of fields satisfying Eq. (9). Applying, first, Faraday's law to the upper rectangle in Figure 5, remembering that $E_x = 0$ by Eq. (9) above, we get

$$E_y(x + \Delta x, t)\ \Delta y - E_y(x, t)\ \Delta y \simeq -[\partial B_z(x, t)/\partial t]\ \Delta x\ \Delta y. \tag{11}$$

Figure 5



You will notice that B_z has been evaluated at the left edge of the rectangle, whereas its average value is needed for exact equality; however, Eq. (11) will become exact when we take the limit $\Delta x \rightarrow 0$ below. Also, note that the time derivative is a partial derivative, since B_z depends on x as well as t. The left-hand side can be simplified by setting

$$E_y(x + \Delta x, t) \simeq E_y(x, t) + [\partial E_y(x, t)/\partial x]\ \Delta x, \tag{12}$$

which also is allowable because of the limit $\Delta x \rightarrow 0$ to be taken below. Thus, Faraday's law reduces to

$$E_y(x, t) + [\partial E_y(x, t)/\partial x]\ \Delta x\ \Delta y - E_y(x, t)\ \Delta y \simeq -[\partial B_z(x, t)/\partial t]\ \Delta x\ \Delta y. \tag{13}$$

Canceling a term, dividing by $\Delta x\ \Delta y$, and taking the limit $\Delta x \rightarrow 0$ to validate the approximations above yields the simplified differential form of Faraday's law, valid under the conditions (9):

$$\partial E_y(x, t)/\partial x = -\partial B_z(x, t)/\partial t. \qquad \text{[Simplified differential form of Faraday's law in empty space.]} \tag{14}$$

In exactly the same way, applying Ampère's law (with i = 0) to the lower rectangle in Figure 5 yields

$$\partial B_z(x, t)/\partial x = -\mu_0\varepsilon_0[\partial E_y(x, t)/\partial t]. \quad \text{[Simplified differential form of Ampère's law in empty space.]} \tag{15}$$

Another important equation can be obtained by differentiating the first of these equations with respect to x and the second with respect to t; this makes the right-hand side of Eq. (14) just the negative of the left-hand side of Eq. (15). They can then be combined to yield

$$\partial^2 E_y(x, t)/\partial x^2 = \mu_0\varepsilon_0[\partial^2 E_y(x, t)/\partial t^2]. \quad [\underline{\text{Wave equation}} \text{ for } E_y.] \tag{16}$$

Expressions of the form

$$E_y(x, t) = E_m \sin(kx \pm \omega t). \quad [\text{Wave traveling in the } \mp x \text{ direction.}] \tag{17}$$

satisfy the above wave equation, as you can readily check by direct substitution, provided $k^2 = \mu_0\varepsilon_0\omega^2$. This expression should look familiar to you, from the module Traveling Waves; and, hopefully, you will remember that the speed of such a wave is given by $v = \omega/k$. (If you do not remember the argument leading to this result, you should really look it up.) Since the speed of an electromagnetic wave is usually denoted by c, we have

$$c = \omega/k = \sqrt{1/\mu_0\varepsilon_0}. \tag{18}$$

This fundamental relationship between the speed of electromagnetic waves and the constants occuring in the equations of basic electromagnetism (since then verified experimentally to a high degree of accuracy) was one of the very impressive successes of Maxwell's theory.

Combining Eqs. (14) and (15) the other way around (this is left as a problem) yields an equation of the same form as Eq. (16) except that E_y is replaced by B_z. That is, B_z satisfies the same differential equation as E_y, and it can thus be expressed in a similar form:

$$B_z = B_m \sin(k'x \pm \omega' t + \phi). \tag{19}$$

(The phase constant ϕ is necessary because we do not yet know the phase relation between E_y and B_z.) Substituting Eqs. (17) and (19) into Eq. (14) yields

$$kE_m \cos(kx \pm \omega t) = \mp\omega' B_m \cos(k'x \pm \omega' t + \phi). \tag{20}$$

This will be satisfied for all values of x and t if and only if

$$k = k', \quad \omega' = \omega, \quad \phi = 0, \quad B_m = \mp(k/\omega)E_m = \mp E_m/c. \tag{21}$$

Thus we have verified the claim [No. (2) on p. 8] that the $\vec{E}$ and $\vec{B}$ fields are in phase ($\phi = 0$). The $\mp$ sign in the last of Eqs. (21) is just what we need for the right-hand property noted above [No. (3)]; and we have also found another characteristic property of electromagnetic waves:

(4) $|E_m| = c|B_m|$.

Since you will be using expressions of the form of Eq. (17) in working the problems of this module, we close these comments by recalling the relations among k, ω, the wavelength λ, the frequency f, and the wave speed c that you learned in the module Traveling Waves:

$$\lambda = 2\pi/k; \quad \omega = 2\pi f; \quad c = \omega/k = \lambda f. \tag{22}$$

If you cannot recall how these relations are obtained, refer to Traveling Waves to refresh your memory.

ADDITIONAL LEARNING MATERIALS

Auxilliary Reading

Stanley Williams, Kenneth Brownstein, and Robert Gray, Student Study Guide with Programmed Problems to Accompany Fundamentals of Physics and Physics, Parts I and II by David Halliday and Robert Resnick (Wiley, New York, 1970).
Objective 1: Section 33-4;
Objective 2: Section 33-5;
Objective 3: Sections 34-1 and 34-2.

Various Texts

Frederick J. Bueche, Introduction to Physics for Scientists and Engineers (McGraw-Hill, New York, 1975), second edition: Sections 28.1 through 28.5 and 29.1.

David Halliday and Robert Resnick, Fundamentals of Physics (Wiley, New York, 1970; revised printing, 1974): Sections 34-4 through 34-6 and 35-1 through 35-3.

Francis Weston Sears and Mark W. Zemansky, University Physics (Addison-Wesley, Reading, Mass., 1970), fourth edition: Sections 32-8 and 36-7 through 36-9.

Richard T. Weidner and Robert L. Sells, Elementary Classical Physics (Allyn and Bacon, Boston, 1973), second edition, Vol. 2: Sections 35-1 through 35-3.

Your attention is especially directed to Section 35-1 of Weidner and Sells for an alternate presentation of the arguments for the displacement-current term, and to Section 35-3 of the same text for a demonstration that a moving pulse of crossed

$\vec{E}$ and $\vec{B}$ fields is a solution to Maxwell's equations. Also, Section 28.4 of Bueche seems to give the most straightforward derivation of the simplified differential form of Maxwell's equations.

PROBLEM SET WITH SOLUTIONS

Some Facts You May Wish to Use While Working These Problems

$c = 3.00 \times 10^{8}$ m/s. $\mu_0 = 4\pi \times 10^{-7}$ Wb/A m. $\varepsilon_0 = 8.9 \times 10^{-12}$ $C^2/N\ m^2$.

For fields satisfying $B_x = B_y = E_x = E_z = 0$ everywhere, and $B_z = B_z(x, t)$ and $E_y = E_y(x, t)$ (no dependence on y or z), Maxwell's equations simplify to the conditions

$$\partial E_y/\partial x = -\partial B_z/\partial t \qquad \text{(Faraday's)}$$

and

$$\partial B_z/\partial x = -\varepsilon_0\mu_0(\partial E_y/\partial t) \qquad \text{(Ampère's)}.$$

A(1). A long cylindrical conducting rod with radius a is centered on the x axis as in Figure 6. A narrow saw cut is made in the rod at x = b. An increasing current $i_1 = At$ (with $A > 0$) flows in the rod toward the right; by some ingenious means, it is arranged that this current is uniformly distributed over the cross section of the rod. At t = 0, there is no charge on the cut faces near x = b.
(a) Find the magnitude of the total charge on these faces, as a function of time.
(b) Use Gauss' law to find E in the gap at x = b as a function of time.
(c) Sketch or describe the magnetic lines of force for r < a, where r is the distance from the x axis.
(d) Use Ampère's law to find B(r) in the gap for r < a.
(e) Compare with what you get for B(r) <u>in the rod</u> for r < a.

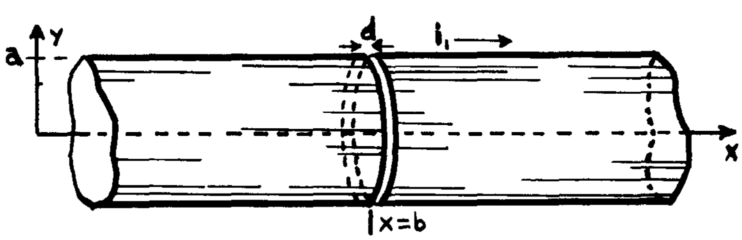


Figure 6

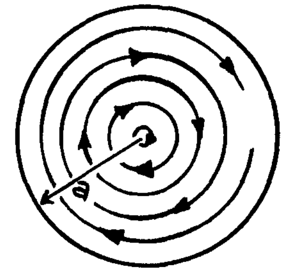


Figure 7

Solution

(a) Since $i_1 = At = dq/dt$, and $q(0) = 0$, we must have $q = \int i_1\, dt = (1/2)At^2$.
(b) Applying Gauss' law to a closed surface enclosing the left-hand face of the

cut yields $q = \varepsilon_0\Phi_E = \varepsilon_0\pi a^2E$. Therefore $E = \dfrac{q}{\varepsilon_0\pi a^2} = \dfrac{(1/2)At^2}{\varepsilon_0\pi a^2}$.

(c) See Figure 7: the current is assumed to be <u>into</u> the paper and increasing. This diagram is valid both inside the rod and in the gap.

(d) Apply Ampère's law, $\oint \vec{B} \cdot d\vec{\ell} = \mu_0(i + i_d)$, to a circular path of radius r in the diagram. In the gap, $i = 0$ and $i_d = \varepsilon_0(d\Phi_E/dt) = \varepsilon_0\pi r^2(dE/dt) = Atr^2/a^2$. We thus get $2\pi rB = \mu_0Atr^2/a^2$; and

$$B = \frac{(1/2)\mu_0Atr}{\pi a^2}.$$

(e) Inside the rod, $i_d = 0$, and through a circular path of radius r the current i will be (area of path/cross-sectional area of rod) $\times\ i_1 = (r^2/a^2)i_1$. So applying Ampère's law to such a path yields $2\pi rB = \mu_0(r^2/a^2)i_1$; and the field is again

$$B = \frac{(1/2)\mu_0Atr}{\pi a^2}.$$

B(2). Identify the Maxwell equation that is equivalent to or includes:
(a) Electric lines of force end only on electric charges.
(b) The displacement current.
(c) Under static conditions, there cannot be any charge inside a conductor.
(d) A changing electric field must be accompanied by a magnetic field.
(e) The net magnetic flux through a closed surface is always zero.
(f) A changing magnetic field must be accompanied by an electric field.
(g) Magnetic flux lines have no ends.
(h) The net electric flux through a closed surface is proportional to the total charge inside.
(i) An electric charge is always accompanied by an electric field.
(j) There are no true magnetic poles.
(k) An electric current is always accompanied by a magnetic field.
(l) Coulomb's law, if the equation for the electric force is assumed.
(m) The electro<u>static</u> field is conservative.

<u>Solution</u>

(1) $\varepsilon_0 \oint_S \vec{E} \cdot d\vec{S} = q$ (inside S) (Gauss' Law).

(2) $\oint_S \vec{B} \cdot d\vec{S} = 0$ (Gauss' Law for Magnetism).

(3) $\oint_C \vec{E} \cdot d\vec{\ell} = -d\Phi_B/dt$ (Faraday's Law).

(4) $\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 i + \mu_0\varepsilon_0(d\Phi_E/dt)$ (Ampère's Law).

In terms of the equation numbers above: (a) ≡ (1); (b) ≡ (4); (c) ≡ (1); (d) ≡ (4); (e) ≡ (2); (f) ≡ (3); (g) ≡ (2); (h) ≡ (1); (i) ≡ (1); (j) ≡ (2); (k) ≡ (4); (l) ≡ (1); (m) ≡ (3). [In regard to (m), dig back in your memory to recall that a conservative force can be defined by the requirement that $\oint \vec{F} \cdot d\vec{\ell} = 0$.]

C(3). The plane electromagnetic wave from a distant radio station produces a vertical magnetic field with amplitude B_0. The radio station is directly north of you, and transmits on a frequency f_s.

(a) How should you orient your coordinate system to make use of the simplified differential form of Maxwell's equations (SDME) derived in this module?

(b) With respect to this coordinate system, write expression(s) for the components of the magnetic field as a function of x, y, z, and t.

(c) Use the SDME to obtain a wave equation for the nonzero component of $\vec{B}$. What information does this wave equation give you regarding the parameters in your expression(s) in part (b)?

(d) Which components of $\vec{E}$ must be zero?

(e) Apply the SDME to your expression(s) in part (b) to obtain expression(s) for the derivatives of the nonzero component of $\vec{E}$.

(f) Write a suitable expression for this component, and show that it satisfies your expression(s) in part (e).

Solution

(a) The x axis should point either away from or toward the station; let us make it point south, so that the wave travels in the +x direction. The z axis should point up or down, so that $\vec{B}$ lies along it; let us make it point up. Then the y axis must point east, for a right-handed coordinate system.

(b) $B_x = B_y = 0$; $B_z = B_0 \sin(kx - \omega t)$. [Of course, we could also use $\cos(kx - \omega t)$, or $\sin(kx - \omega t + \phi)$, etc.]

(c) Differentiating the simplified form of Faraday's law with respect to t yields $\partial^2 E_y/\partial x\, \partial t = -\partial^2 B_z/\partial t^2$; and differentiating the simplified Ampère's law with respect to x yields $\partial^2 B_z/\partial x^2 = -\varepsilon_0\mu_0\, \partial^2 E_y/\partial t\, \partial x$. The term $\partial^2 E_y/\partial x\, \partial t$ occurs in both these equations; we can thus combine them to obtain $\partial^2 B_z/\partial x^2 = \varepsilon_0\mu_0\, \partial^2 B_z/\partial t^2$. If the expression for B_z in part (b) is substituted into this equation, we get $-k^2 B_0 \sin(kx - \omega t) = -\omega^2 \varepsilon_0\mu_0 B_0 \sin(kx - \omega t)$; this requires that $k^2 = \omega^2\varepsilon_0\mu_0$, or $k/\omega = \sqrt{\varepsilon_0\mu_0} (= 1/c)$.

(d) $E_x = E_z = 0$, in order to make $\hat{E} \times \hat{B} = \hat{c}$.
(e) Direct differentiation and the SDME yield $\partial E_y/\partial x = -\partial B_z/dt = +\omega B_0 \cos(kx - \omega t)$ and $\partial E_y/\partial t = -(1/\mu_0\varepsilon_0)(\partial B_y/\partial x) = -(k/\mu_0\varepsilon_0)B_0 \cos(kx - \omega t)$.
(f) $E_y = cB_0 \sin(kx - \omega t)$. This satisfies the first equation above because $ck = c(\omega/c) = \omega$ and $c\omega = c(ck) = c^2k = k/\varepsilon_0\mu_0$.

Problems

D(1). A parallel-plate capacitor with circular plates 20.0 cm in diameter is being charged as in Figure 8. The displacement current density throughout the region is uniform, into the paper in the diagram, and has a value of 20.0 A/m².
(a) Calculate the magnetic field strength B at a distance R = 5.0 cm from the axis of symmetry of the region.
(b) Calculate dE/dt in this region.

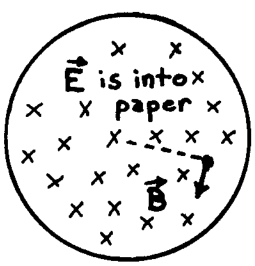


Figure 8

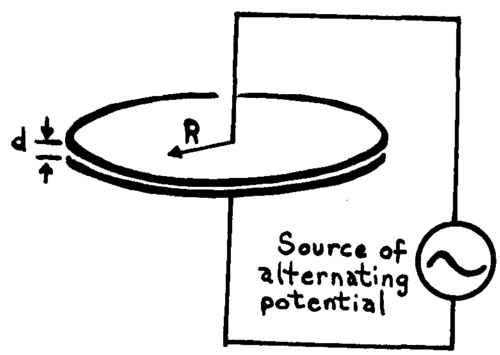


Figure 9

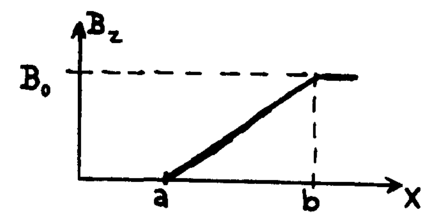


Figure 10

E(1). The capacitor in Figure 9 consisting of two circular plates with area $A = 0.100\ m^2$ is connected to a source of potential $V = V_{max} \sin \omega t$, where $V_{max} = 200$ V and $\omega = 100$ rad/s. The maximum value of the displacement current is $i_d = 8.9 \times 10^{-6}$ A. Neglect "fringing" of the electric field at the edges of the plates.
(a) What is the maximum value of the current i?
(b) What is the maximum value of $d\Phi_E/dt$, where Φ_E is the electric flux through the region between the plates?
(c) What is the separation d between the plates?
(d) Find the maximum value of the magnitude of $\vec{B}$ between the plates at a distance R = 0.100 m from the center.

F(2). Name and state the four Maxwell equations in vacuum.

G(3). Under what conditions do the following expressions satisfy Maxwell's equations? (A, a, and b are constants.)

(a) $E_y = Ab(x - at)$, $B_z = A(x - at)$.
(b) $E_y = Ae^{(x - at)}$, $B_z = Abe^{(x - at)}$.

H(3). (a) Write an equation for the electric field component of a sinusoidal electromagnetic plane wave traveling in the negative x direction, having an amplitude of 1.40 V/m and a wavelength of 600 m.
(b) What is the frequency of this wave?
(c) How far apart are two points where the $\vec{E}$ fields are 60° out of phase?
(d) Find the amplitude of the magnetic field component of this wave.

I(3). The $\vec{B}$ field in Figure 10 at a given instant of time is independent of y and z, but points in the positive z direction and has a magnitude that increases linearly from zero to B_0 between x = a and x = b. What do Maxwell's equations tell you about $\vec{E}$, for a < x < b?

Solutions

D(1). (a) $B = (1/2)\mu_0 R \times$ (displacement-current density) $= 6.3 \times 10^{-7}$ T.
(b) $dE/dt = 2.20 \times 10^{-12}$ V/m s.

E(1). (a) 8.9×10^{-6} A. (b) 1.00×10^{6} V m/s. (c) 2.00 mm. (d) 5.6×10^{-12} T. [Did you get too large a value for (d)? If so, check that you used the correct displacement current.]

F(2). (a)(1) Gauss' law for electricity:

$$\varepsilon_0 \oint_S \vec{E} \cdot d\vec{S} = q \text{ (inside S)}.$$

(2) Gauss' law for magnetism:

$$\oint_S \vec{B} \cdot d\vec{S} = 0.$$

(3) Faraday's law of induction:

$$\oint_C \vec{E} \cdot d\vec{\ell} = -d\Phi_B/dt.$$

(4) Ampère's law (corrected):

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 i + \mu_0 \varepsilon_0 (d\Phi_E/dt).$$

In (3) and (4), Φ means the flux through any surface bounded by the curve C, and i is the current through such a surface.

G(3). (a) The simplified Maxwell's equations yield

$$Ab = +aA \text{ and } A = +\varepsilon_0 \mu_0 aAb.$$

These will be satisfied if $b = a = \sqrt{1/\varepsilon_0 \mu_0} = c$.

(b) In the same way, Maxwell's equations will be satisfied if

$$b = 1/a = \sqrt{\varepsilon_0\mu_0} = 1/c.$$

H(3). (a) Take the y axis along $\vec{E}$; then $E_y = E_m \sin(kx + \omega t + \phi)$ in general, though we can usually assume $\phi = 0$. $E_m = 1.40$ V/m, $k = 2\pi/\lambda = 1.05 \times 10^{-2}$/m, and $\omega = ck = 3.14 \times 10^6$/s.

(b) $f = \omega/2\pi = 5.0 \times 10^5$ Hz.

(c) $\lambda/6 = 100$ m.

(d) $B_m = E_m/c = 4.7 \times 10^{-9}$ T.

I(3). The graph Figure 10 tells us the value of $\partial B_z/\partial x$; so we refer to the simplified form of Ampère's law. This tells us that $\partial E_y/\partial t = -c^2(\partial B_z/\partial x) = -c^2B_0/(b - a)$ for $a < x < b$; that is, $\vec{E}$ is increasing with time in the <u>negative</u> y direction.

PRACTICE TEST

Some Facts You May Wish to Use While Working These Problems

$c = 3.00 \times 10^8$ m/s. $\mu_0 = 4\pi \times 10^{-7}$ Wb/A m. $\varepsilon_0 = 8.9 \times 10^{-12}$ C^2/N m^2.

For fields satisfying $B_x = B_y = E_x = E_z = 0$ everywhere, and $B_z = B_z(x, t)$ and $E_y = E_y(x, t)$ (no dependence on y or z), Maxwell's equations simplify to the conditions

$$\partial E_y/\partial x = -\partial B_z/\partial t \quad \text{(Faraday's)}$$

and

$$\partial B_z/\partial x = -\varepsilon_0\mu_0(\partial E_y/\partial t) \quad \text{(Ampère's).}$$

1. A parallel-plate capacitor has square plates 1.00 m on a side, as in Figure 11. There is a charging current $i = 2.00$ A flowing into the capacitor.
   (a) What is the displacement current through the region between the plates?
   (b) What is dE/dt in this region?
   (c) What is the displacement current through the square (dashed) path between the plates?
   (d) What is $\vec{B} \cdot d\vec{\ell}$ around this square path?

2. (a) State Maxwell's equations in vacuum.
   (b) In your answer to (a), identify:
      (i) Faraday's law of induction.
      (ii) The displacement-current term.

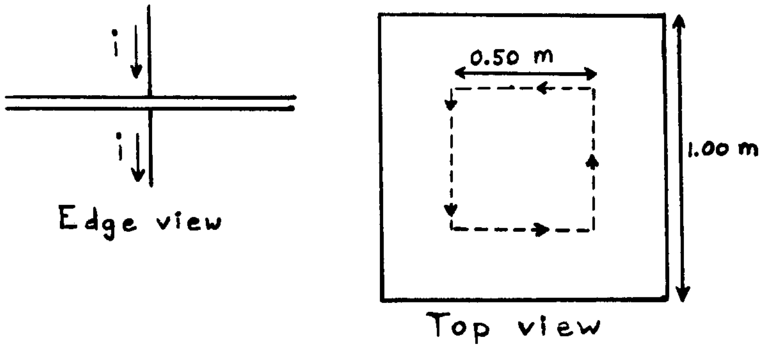


Figure 11

3. A plane electromagnetic wave has the electric field

$$E_y = (3.00\ \mu\text{V/m}) \cos[(15\pi x)\ \text{m} + (4.5\pi \times 10^9 t)\ \text{s}] \quad \text{and} \quad E_x = E_z = 0.$$

The x axis points up and the y axis points north.
(a) In which direction is this wave traveling?
(b) What is its frequency? Wavelength?
(c) What is the amplitude of its accompanying magnetic field?
(d) At $x = 0$, which way do $\vec{E}$ and $\vec{B}$ point when $t = 0$?
(e) Write expressions for all the components of $\vec{B}$. Show that the nonzero components of $\vec{E}$ and $\vec{B}$ satisfy Maxwell's equations.

Practice Test Answers

1. (a) 2.00 A. (b) 2.20×10^{11} V/m s. (c) 0.50 A. (d) 6.3×10^{-7} Wb/m.

2. $\varepsilon_0 \oint_S \vec{E} \cdot d\vec{A} = q.$

$\oint_S \vec{B} \cdot d\vec{A} = 0.$

$\oint_C \vec{E} \cdot d\vec{\ell} = -d\Phi_B/dt.$ (i)

$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0[i + \varepsilon_0(d\Phi_E/dt)].$ (ii)

3. (a) Down. (b) $f = 2.25 \times 10^9$ Hz. $\lambda = 13.3$ cm. (c) 1.00×10^{-14} T.
(d) North and east, respectively. (e) $B_x = B_y = 0$, $B_z = -E_y/c$.
The given equation is of the form $E_y = E_m \cos(kx + \omega t)$,
so $B_z = -(E_m/c)\cos(kx + \omega t)$. Substituting these into the two simplified Maxwell's equations gives $-kE_m \sin(kx + \omega t) = -(\omega E_m/c)\sin(kx + \omega t)$ and $+(kE_m/c)\sin(kx + \omega t) = +\varepsilon_0\mu_0\omega E_m \sin(kx + \omega t)$. Since $\varepsilon_0\mu_0 = 1/c^2$, these are both satisfied provided $\omega = ck$; the values given in the problem satisfy this condition.

MAXWELL'S PREDICTIONS Date ____________________

Mastery Test Form A

pass recycle

1 2 3

Name ______________________________ Tutor ____________________

Some Facts You May Wish to Use While Working These Problems

$c = 3.00 \times 10^8$ m/s. $\mu_0 = 4\pi \times 10^{-7}$ Wb/A m. $\varepsilon_0 = 8.9 \times 10^{-12}$ $C^2/N\ m^2$.

For fields satisfying $B_x = B_y = E_x = E_z = 0$ everywhere, and $B_z = B_z(x, t)$, $E_y = E_y(x, t)$ (no dependence on y or z), Maxwell's equations simplify to the conditions

$\partial E_y/\partial x = -\partial B_z/\partial t$ (Faraday's), $\partial B_z/\partial x = -\varepsilon_0\mu_0(\partial E_y/\partial t)$ (Ampère's).

1. The parallel-plate capacitor in Figure 1 is made from two rectangular metal plates of the dimensions shown, spaced 5.0 mm apart. Along the dotted rectangular path between the plates,

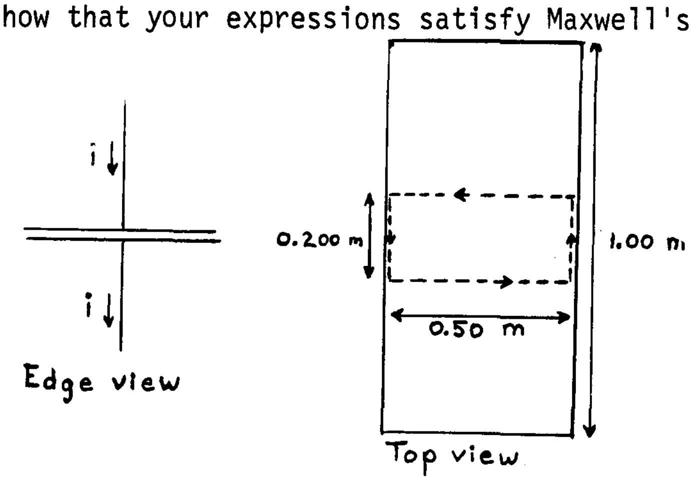
   $\oint \vec{B} \cdot d\vec{\ell} = 2.00 \times 10^{-8}$ Wb/m.

   (a) What is the displacement current through this path?
   (b) What is the (total) current i?
   (c) If the potential difference between the plates is V, what is dV/dt?

2. (a) State Maxwell's equations in vacuum.
   (b) In your answer to (a), identify:
   (i) Ampère's law.
   (ii) The equation that tells you whether electric field lines terminate, and where.

3. A plane electromagnetic waves is traveling to the right along the x axis, as shown in Figure 2. At x = a, $E_z(a, t) = 0$, and $E_y(a, t) = E_0 \cos(\omega t)$ with E_0 positive and $\omega = 3.00 \times 10^6$ rad/s.
   (a) At x = a, are any of B_x, B_y, and B_z identically zero (i.e., at all times)?
   (b) Write expressions for the nonzero components of $\vec{E}(x, t)$ and $\vec{B}(x, t)$. Evaluate the constants occurring in these expresssions as completely as possible.
   (c) Show that your expressions satisfy Maxwell's equations.

Figure 1



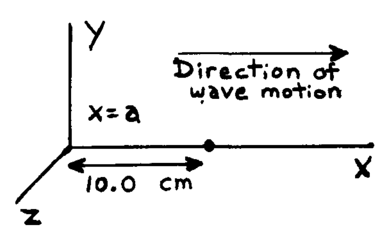


Figure 2

MAXWELL'S PREDICTIONS

Date ______________________

Mastery Test   Form B

pass   recycle
1   2   3

Name ______________________________________   Tutor ______________________

Some Facts You May Wish to Use While Working These Problems

$c = 3.00 \times 10^8$ m/s. $\mu_0 = 4\pi \times 10^{-7}$ Wb/A m. $\varepsilon_0 = 8.9 \times 10^{-12}$ $C^2/N\ m^2$.

For fields satisfying $B_x = B_y = E_x = E_z = 0$ everywhere, and $B_z = B_z(x, t)$, $E_y = E_y(x, t)$ (no dependence on y or z), Maxwell's equations simplify to the conditions

$\partial E_y/\partial x = -\partial B_z/\partial t$ (Faraday's),

$\partial B_z/\partial x = -\varepsilon_0\mu_0(\partial E_y/\partial t)$ (Ampère's).

1. The electric field between the circular plates of a plane, parallel-plate capacitor of radius 10.0 cm is given by $E_z = E_m \sin\omega t$, where $E_m = 2.00 \times 10^3$ V/m and $\omega = 6.0 \times 10^3$ rad/s.
   (a) What is the maximum displacement current through the region between the plates?
   (b) What is the maximum magnetic field at a radius of 5.0 cm from the axis of the circular plates?

2. (a) State Maxwell's equations in vacuum.
   (b) In your answer to (a), identify:
      (i) Gauss' law for the electric field.
      (ii) The condition that magnetic field lines do not terminate.

3. A distant radio station, transmitting at 1.50×10^6 Hz, produces a vertical electric field with $E_y = +2.00$ μV/m (its maximum value) at the origin of the coordinate system when t = 0. This wave is progressing in the negative direction along the x axis.
   (a) Obtain expressions for all the components of $\vec{E}$ and $\vec{B}$ as functions of x and t.
   (b) Express B_z as a function of t at the point x = y = z = 50 m.
   (c) Show that your expressions (a) satisfy Maxwell's equations.

MAXWELL'S PREDICTIONS   Date ____________________

Mastery Test   Form C   pass   recycle
1   2   3

Name ________________________________   Tutor ____________________

Some Facts You May Wish to Use While Working These Problems

$c = 3.00 \times 10^8$ m/s. $\mu_0 = 4\pi \times 10^{-7}$ Wb/A m. $\varepsilon_0 = 8.9 \times 10^{-12}$ $C^2/N\ m^2$.

For fields satisfying $B_x = B_y = E_x = E_z = 0$ everywhere, and $B_z = B_z(x, t)$, $E_y = E_y(x, t)$ (no dependence on y or z), Maxwell's equations simplify to the conditions

$$\partial E_y/\partial x = -\partial B_z/\partial t \quad \text{(Faraday's)},$$

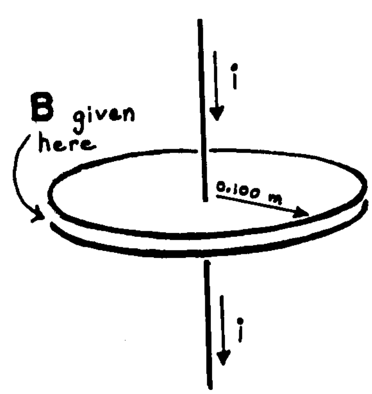
$$\partial B_z/\partial x = -\varepsilon_0\mu_0(\partial E_y/\partial t) \quad \text{(Ampère's)}.$$

1. The capacitor shown in Figure 1 is made from two circular plates with a radius $r = 0.100$ m, separated by a distance $d = 2.00 \times 10^{-3}$ m. Neglect "fringing" of the electric field at the edges of the plates. At a given instant there is a magnetic field of strength $B = 5.0 \times 10^{-10}$ T at a point midway between the edges of the two plates.
   (a) Find the displacement current i_d through the region between the plates.
   (b) Find the current i flowing into the capacitor.
   (c) Find dE/dt in the region between the plates.

2. (a) State Maxwell's equations in vacuum.
   (b) In your answer to (a), identify:
      (i) Gauss' law for the magnetic field.
      (ii) The conservative nature of the electrostatic field.

3. The star Betelgeuse is directly overhead (i.e., on your positive x axis). Assume it has emitted a sinusoidal electromagnetic wave with wavelength 6.0×10^{-5} m that is now striking the Earth.
   (a) Give as complete a mathematical description of this wave as you can; i.e., give expressions for the components of the electric and magnetic fields, and evaluate as many of the constants as possible with the information given.
   (b) Show that your expressions satisfy Maxwell's equations.

Figure 1



MASTERY TEST GRADING KEY - Form A

1. What To Look For: (b) Check that the correct displacement current is used, i.e., $i_{d(total)}$ rather than just i_d.

   Solution: (a) According to Ampère's law, $\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0(i + i_d)$. Between the plates, $i = 0$. Thus, the current passing through the rectangle is

   $i_d = (1/\mu_0) \oint_C \vec{B} \cdot d\vec{\ell} = (2.00 \times 10^{-8})/(4\pi \times 10^{-7}) = 15.9$ mA.

   (b) Assuming the displacement current is uniformly distributed over the area of the plates, $i_{d(total)} = 5.0 i_{d(rectangle)} = 80$ mA.

   (c) The potential difference V between the plates is V = ED, where D is their separation. Thus

   $$i_{d(total)} = \varepsilon_0 \frac{d\Phi_E}{dt} = \varepsilon_0 A \frac{dE}{dt} = \frac{\varepsilon_0 A}{D}\frac{dV}{dt},$$ where A is the area of the plates. Thus

   $$\frac{dV}{dt} = \frac{i_{d(total)}D}{\varepsilon_0 A} = \frac{(8.0 \times 10^{-2})(5.0 \times 10^{-3})}{(8.9 \times 10^{-12})0.50} = 9.0 \times 10^7 \text{ V/s}.$$

2. Solution:

   $$\varepsilon_0 \oint_S \vec{E} \cdot d\vec{A} = q \quad \text{(ii)};$$

   $$\oint_S \vec{B} \cdot d\vec{A} = 0; \qquad \oint_C \vec{E} \cdot d\vec{\ell} = -d\Phi_B/dt; \qquad \oint_C \vec{B} \cdot d\vec{\ell} = \mu_0\left(i + \varepsilon_0 \frac{d\Phi_E}{dt}\right) \quad \text{(i)}.$$

3. What To Look For: (b) Note that a phase constant (-ka) must be included in the argument of the cosine to obtain $E_y = E_0 \cos(\omega t)$ at $x = a$. The argument of the cosine could also be the negative of the one shown; either one works, since $\cos(-\alpha) = \cos(\alpha)$.

   Solution: (a) $B_x = B_y = o$ at all times (everywhere!).
   (b) $E_y = E_0 \cos[k(x - a) - \omega t]$. $B_z = +E_y/c$. (Note that $E_z = 0$.)
   $a = 0.100$ m, ω is given in the problem, and $k = \omega/c = 1.00 \times 10^{-2}$/m. Thus $ka = 1.00 \times 10^{-3}$ rad.

   (c) These expressions satisfy the conditions for the simplified form of Maxwell's equations given at the top of the test page. Substituting them into these equations gives us

   $$-kE_0 \sin[k(x - a) - \omega t] \stackrel{?}{=} -\omega(E_0/c) \sin[k(x - a) - \omega t]; \quad \text{and}$$

   $$-k(E_0/c) \sin[k(x - a) - \omega t] \stackrel{?}{=} -\varepsilon_0\mu_0\omega E_0 \sin[k(x - a) - \omega t].$$

   Since $\varepsilon_0\mu_0 = 1/c^2$, these equations are both satisfied because we set $k = \omega/c$, above.

MASTERY TEST GRADING KEY - Form B

1. What To Look For: (b) Check that the correct value is used for the displacement current, i.e., $i'_{d(max)}$ rather than $i_{d(max)}$.

   Solution: (a) $i_d = \varepsilon_0\, d\Phi_E/dt = \varepsilon_0 \pi R^2\, dE_z/dt$, where R is the radius of the plates. $i_d = -\varepsilon_0 \pi R^2 \omega E_m \cos \omega t$. The maximum value of this is $i_{d(max)} = \varepsilon_0 \pi R^2 \omega E_m = (8.9 \times 10^{-12})\pi(0.100)^2(6.0 \times 10^3)(2.00 \times 10^3) = 3.4$ pA.

   (b) By Ampère's law: $2\pi r B_{max} = \int_C \vec{B} \cdot d\vec{\ell} = \mu_0 i'_{d(max)} = (1/4)\mu_0 i_{d(max)}$. Thus $B_{max} = \dfrac{\mu_0 i_{d(max)}}{8\pi r} = \dfrac{(4\pi \times 10^{-7})(3.4 \times 10^{-6})}{8\,(5.0 \times 10^{-2})} = 3.4 \times 10^{-12}$ T.

2. Solution: (i) $\varepsilon_0 \oint_S \vec{E} \cdot d\vec{A} = q$. (ii) $\oint_S \vec{B} \cdot d\vec{A} = 0$.

   $\oint_C \vec{E} \cdot d\vec{\ell} = -d\Phi_B/dt$. $\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0(i + \varepsilon_0\, d\Phi/dt)$.

3. What To Look For: (a) Note that the argument of the cosine must be $kx + \omega t$, or $-kx - \omega t$, in order for the wave to travel in the negative x direction. There must also be a minus sign in the expression for B_z, to make $\hat{E} \times \hat{B} = \hat{c}$.

   Solution: (a) $E_x = E_z = B_x = B_y = 0$. $E_y = E_0 \cos(kx + \omega t)$.

   $B_z = -(E_0/c) \cos(kx + \omega t)$. $E_0 = 2.00\ \mu$V/m, $E_0/c = 6.7 \times 10^{-15}$ T, and $\omega = 2\pi f = 9.4 \times 10^6$ Hz. Thus $k = \omega/c = 3.14 \times 10^{-2}$/m.

   (b) $B_z(a, a, a, t) = -(E_0/c) \cos(ka + \omega t)$, where a = 50 m.

   $B_z(a, a, a, t) = -(E_0/c) \cos(1.57 \text{ rad} + \omega t)$ or $+(E_0/c) \sin \omega t$.

   (c) These expressions satisfy the conditions for the simplified form of Maxwell's equations given at the top of the test page. Substituting them yields

   $-kE_0 \sin(kx + \omega t) \stackrel{?}{=} -(E_0/c) \sin(kx + \omega t)$ and

   $+k(E_0/c) \sin(kx + \omega t) \stackrel{?}{=} +\varepsilon_0\mu_0 E_0 \sin(kx + \omega t)$.

   Since $\varepsilon_0\mu_0 = 1/c^2$ and we set $k = \omega/c$ above, these are both satisfied.

MASTERY TEST GRADING KEY - Form C

1. (a) By Ampère's law, $2\pi rB = \oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 i_d$ (i = 0 between the plates) where r is the radius of the circular plates, and of the path C. So

$$i_d = \frac{2\pi rB}{\mu_0} = \frac{2\pi(0.100)(5.0 \times 10^{-10})}{4\pi \times 10^{-7}} = 0.250 \text{ mA}.$$

(b) $i = i_d = 0.250$ mA.

(c) $i_d = \varepsilon_0 d\Phi_E/dt = \varepsilon_0 \pi r^2 dE/dt$; so

$$dE/dt = \frac{i_d}{\varepsilon_0 \pi r^2} = \frac{2.5 \times 10^{-4}}{(8.9 \times 10^{-12})\pi(0.100)^2} = 8.9 \times 10^8 \text{ V/m s}.$$

2. Solution: $\varepsilon_0 \oint_S \vec{E} \cdot d\vec{A} = q$. (i) $\oint_S \vec{B} \cdot d\vec{A} = 0$.

(ii) $\oint_C \vec{E} \cdot d\vec{\ell} = -d\Phi_B/dt$. $\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0(i + \varepsilon_0\, d\Phi_E/dt)$.

3. What To Look For: (a) Check that the argument of the cosine is $kx + \omega t$, or $-kx - \omega t$, so that the wave travels in the negative x direction. Also, there must be a minus sign in the expression for B_z, in order that $\hat{E} \times \hat{B} = \hat{c}$.

Solution: (a) Choose the y axis to lie along the direction of $\vec{E}$, and set your clock so that E_y is a maximum at t = 0 (this avoids a phase constant ϕ).

Then $E_x = E_z = B_x = B_y = 0$,

$E_y = E_0 \cos(kx + \omega t)$, $B_z = -(E_0/c)\cos(kx + \omega t)$ [where $k = 2\pi/\lambda$ $= 1.04 \times 10^5$/m; $\omega = kc = 3.14 \times 10^{13}$/s ($E_0$ is not determined).]

(b) These fields satisfy the conditions for the simplified form of Maxwell's equations; so we substitute the expressions for E_y and B_z into the equations at the top of the page.

$-kE_0 \sin(kx + \omega t) \stackrel{?}{=} -\omega(E_0/c)\sin(kx + \omega t)$ and

$+k(E_0/c)\sin(kx + \omega t) \stackrel{?}{=} +\varepsilon_0\mu_0\omega E_0 \sin(kx + \omega t)$.

Both these are satisfied, since $\varepsilon_0\mu_0 = 1/c^2$, and we set $\omega = kc$ above.