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Orthogonal Polynomials with Orthogonal Derivatives

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ORTHOGONAL POLYNOMIALS WITH ORTHOGONAL DERIVATIVES*

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1. **Introduction.** Let $\{\phi_n(x) \equiv x^n + \cdots\}$ be a set of orthogonal polynomials satisfying the relations

$$(1)\quad \begin{gathered} \int_a^b p(x)\phi_m(x)\phi_n(x)dx = \int_a^b q(x)\phi_m'(x)\phi_n'(x)dx = 0, \\ m \neq n;\, m, n = 0, 1, \cdots, \\ \alpha_i \equiv \int_a^b p(x)x^i dx, \qquad \beta_i \equiv \int_a^b q(x)x^i dx, \qquad i = 0, 1, \cdots, \\ p(x) \geqq 0, \qquad q(x) \geqq 0, \qquad \alpha_0 > 0, \qquad \beta_0 > 0. \end{gathered}$$

Lebesgue integrals are used and the interval (a, b)† may be finite or infinite.

We are concerned with the following assertion:

THEOREM. *If $\{\phi_n(x)\}$ and $\{\phi_n'(x)\}$ are orthogonal systems of polynomials, then $\{\phi_n(x)\}$ may be reduced to the classical polynomials of Jacobi, Laguerre, or Hermite by means of a linear transformation on x.*

This result was first proved by W. Hahn‡ who obtained a differential equation of the second order for $\phi_n(x)$. When (a, b) is finite, Krall§ derived the Jacobi polynomials by using the moments β_i to determine the weight function $q(x)$. The present paper extends his method to the case (a, b) infinite, thus obtaining the Laguerre and Hermite polynomials.

2. **Weight function for $\{\phi_n'(x)\}$.** Krall's proof shows that constants r, s, t (not all zero) may be determined so that

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† There is no loss of generality in assuming the intervals of orthogonality for $\{\phi_n(x)\}$ and for $\{\phi_n'(x)\}$ to be the same, since the definitions of $p(x)$, $q(x)$ may always be extended to a common interval (a, b). More generally, $p(x)dx$ may be replaced by $d\psi_1(x) \equiv Ap(x) + dT(x)$, where A is a constant, and $\int_a^b x^i dT(x) = 0$, $(i = 0, 1, \cdots)$; $q(x)dx$ may be replaced by $d\psi_2(x)$, where $\psi_2(x)$ is monotone nondecreasing.

‡ W. Hahn, *Über die Jacobischen Polynome und zwei verwandte Polynomklassen*, Mathematische Zeitschrift, vol. 39 (1935), pp. 634–638.

§ H. Krall, *On derivatives of orthogonal polynomials*, this Bulletin, vol. 42 (1936), pp. 423–428.

$$(2) \qquad \int_a^b q_1(x)x^i dx = \int_a^b q(x)x^i dx, \qquad q_1(x) \equiv (rx^2 + sx + t)p(x), \qquad i = 0, 1, \cdots .$$

We suppose that (a, b) is the smallest interval in the sense that no number h, $(a<h<b)$, exists such that $\int_a^h p(x)dx=0$ or $\int_h^b p(x)dx=0$. There is no restriction in assuming likewise that either $(a, b)=(0, \infty)$ or $(a, b)=(-\infty, \infty)$. (Perform, if necessary, a linear transformation on x.)

Following Krall we let

$$(3) \qquad S(x) = K\int_a^x (z - L)p(z)dz, \qquad a \leqq x \leqq b,$$

where K, L are constants determined by the conditions $S(b)=0$, $\int_a^b S(x)dx=\int_a^b q(x)dx$. The boundary conditions on $S(x)$ require that the integrand $(z-L)p(z)$ change sign so that $a<L<b$. Then $\int_a^x(z-L)p(z)dz$ decreases in (a, L) and increases in (L, b), therefore this integral is always less than or equal to zero. Hence,

$$\int_a^b K\left(\int_a^x (z - L)p(z)dz\right)dx = \int_a^b q(x)dx > 0$$

requires $K<0$ and therefore $S(x)\geqq 0$. Suppose

$$\epsilon_x \equiv \int_x^\infty K(z - L)p(z)dz$$

and

$$\epsilon_x' \equiv \int_x^\infty Kz^i(z - L)p(z)dz,$$

i a positive integer. Then, $S(x)=-\epsilon_x$, $-\epsilon_x' \geqq -\epsilon_x x^i$ if $x>|L|$, and ϵ_x, $\epsilon_x' \to 0$ as $x\to\infty$. Therefore, $S(x)\leqq -\epsilon_x'/x^i$ if $x>|L|$, and $x^iS(x)\to 0$ as $x\to\infty$, $(i=0, 1, \cdots)$. Similarly, if $a=-\infty$, we prove that $x^iS(x)\to 0$ as $x\to-\infty$, $(i=0, 1, \cdots)$. In every case, $\int_a^b x^iS(x)dx$ exists, $(i=0, 1, \cdots)$. We conclude that $S(x)$ has the following properties:

$$(4) \qquad \begin{aligned} &K < 0, \ a < L < b, \ S(x) > 0, \ a < x < b, \ S(a) = S(b) = 0, \\ &S'(x) = K(x - L)p(x) \text{ exists almost everywhere,} \\ &S'(x) \geqq 0, \ a \leqq x \leqq L, \text{ almost everywhere,} \\ &S'(x) \leqq 0, \ L \leqq x \leqq b, \text{ almost everywhere,} \\ &x^iS(x) \to 0 \text{ as } x \to a \text{ or } b, \qquad i = 0, 1, \cdots . \end{aligned}$$

Again, with Krall, we obtain

$$\int_a^b S(x)x^i dx = \int_a^b q_1(x)x^i dx = \int_a^b q(x)x^i dx, \qquad i = 0, 1, \cdots ,$$

$$(5) \quad \int_a^b S(x)\phi_m'(x)\phi_n'(x)dx = \int_a^b q_1(x)\phi_m'(x)\phi_n'(x)dx = 0,$$

$$m \neq n;\ m, n = 0, 1, \cdots ,$$

$$q_1(x) \equiv S(x) + T(x), \quad \int_a^b T(x)x^i dx = 0, \qquad i = 0, 1, \cdots .$$

In the finite interval this requires that $T(x) \equiv 0$ almost everywhere, but in the infinite interval this result does not follow.* However, it is known that if $\int_{-\infty}^{\infty} T(x)x^i dx = 0$, $(i = 0, 1, \cdots)$, and if $\int_{-x}^{x} |T(z)| dz$ exists for every x, and $T(x) \geqq 0$ for $|x|$ sufficiently large, then $T(x) \equiv 0$ almost everywhere. We shall now prove this statement.

Suppose $T(x) \geqq 0$ for $|x| \geqq A$. In view of (5), $\int_{-x}^{x} |T(z)| dz$ exists for all x. Choose $A' > A$ and i even. Then

$$\int_{-\infty}^{\infty} T(x)x^i dx = \int_{-\infty}^{-A} T(x)x^i dx + \int_{-A}^{A} T(x)x^i dx + \int_{A}^{A'} T(x)x^i dx$$

$$+ \int_{A'}^{A'+1} T(x)x^i dx + \int_{A'+1}^{\infty} T(x)x^i dx = \sum_{n=1}^{5} I_n = 0,$$

where $I_1 \geqq 0$, $I_3 \geqq 0$, $I_4 \geqq 0$, $I_5 \geqq 0$, $I_2 \leqq 0$, $I_1 + I_3 + I_5 \geqq 0$, and $I_2 + I_4 \leqq 0$. Given $\epsilon > 0$, suppose $T(x) \geqq \epsilon$ on some set G of positive measure in (A, ∞). Choose A', $(A' > A)$, such that the interval $(A', A'+1)$ contains a subset of G of measure $\sigma > 0$. Then $I_4 > 0$ and

$$\frac{|I_2|}{I_4} \leqq \frac{\int_{-A}^{A} x^i \cdot |T(x)| dx}{\sigma \cdot \epsilon (A')^i} \leqq \frac{A^i \int_{-A}^{A} |T(x)| dx}{\sigma \cdot \epsilon (A')^i} < 1$$

if i is sufficiently large, since $A/A' < 1$. Then $|I_2| < I_4$, $I_2 + I_4 > 0$, which is a contradiction. Thus $T(x) \equiv 0$ almost everywhere in (A, ∞), and likewise in $(-\infty, -A)$. We conclude that $\int_{-A}^{A} T(x)x^i dx = 0$, $(i = 0, 1, \cdots)$, therefore $T(x) \equiv 0$ almost everywhere.

Since $S'(x) = K(x - L)p(x)$ almost everywhere, (2) and (5) lead to the differential equation

$$(6) \qquad (rx^2 + sx + t)S'(x) - K(x - L)S(x) = K(x - L)T(x).$$

The solution of (6) is

* Stieltjes' example is $\int_0^\infty x^n e^{-x^{1/4}} \sin (x^{1/4}) dx = 0$, $(n = 0, 1, \cdots)$.

$$
(7)\qquad
\begin{aligned}
S(x) &= S_1(x) + Cf(x), \qquad \log f(x) = \int^x \frac{K(z-L)dz}{rz^2+sz+t},\\
S_1(x) &= f(x)\int_c^x \frac{K(z-L)T(z)dz}{(rz^2+sz+t)f(z)}, \qquad c,\ C \text{ constants.}
\end{aligned}
$$

3. **Discussion of** rx^2+sx+t. (i) Suppose first that rx^2+sx+t has imaginary zeros. Then

$$f(x) = (rx^2+sx+t)^\alpha e^{\beta \arctan(\gamma x+\delta)}, \qquad \alpha,\ \beta,\ \gamma,\ \delta \text{ constants},$$

where $r[1+(\gamma x+\delta)^2]\equiv\gamma^2(rx^2+sx+t)$, $2\alpha r=K$, and $\beta r=-\alpha(2rL+s)$. Since $\beta_0=\int_a^b q_1(x)dx>0$, we conclude that $r>0$, $\alpha<0$, and $rx^2+sx+t>0$ in (a, b).

Let i be an integer such that $\alpha+i\geqq 0$, $i\geqq 1$, and let $f_1(x)\equiv(rx^2+sx+t)^i[(\alpha+i)(2rx+s)+\beta r/\gamma]f(x)$.

Integrating by parts we have

$$
\begin{aligned}
\int_a^b \frac{S(x)}{f(x)} f_1'(x)dx &= \left[f_1(x)\left\{\int_c^x \frac{K(z-L)T(z)dz}{(rz^2+sz+t)f(z)} + C\right\}\right]_a^b\\
&- K\int_a^b (x-L)(rx^2+sx+t)^{i-1}\left[(\alpha+i)(2rx+s)+\frac{\beta r}{\gamma}\right]T(x)dx,
\end{aligned}
$$

$$
\int_a^b S(x)(rx^2+sx+t)^{i-1}\left\{\left[(\alpha+i)(2rx+s)+\frac{\beta r}{\gamma}\right]^2 + 2r(\alpha+i)(rx^2+sx+t)\right\}dx = 0.
$$

Since the integrand does not change sign, we conclude that $S(x)\equiv 0$ almost everywhere, which is impossible in view of (4).

(ii) Suppose that $rx^2+sx+t=r(x-g)^2$. As in (i), $r>0$. Here

$$f(x) = (x-g)^\alpha e^{\beta/(x-g)}, \qquad \alpha,\ \beta \text{ constants}.$$

Let i be an integer such that $\alpha+i\geqq 0$, $i\geqq 2$, and

$$f_1(x) \equiv (x-g)^i[(\alpha+i)(x-g)-\beta]f(x).$$

As in (i),

$$\int_a^b \frac{S(x)}{f(x)} f_1'(x)dx = 0,$$

which is impossible.

(iii) Suppose that $rx^2+sx+t=r(x-g)(x-h)$, (g, h real; $g<h$). (If (a, b) is finite, this is the only possible case, since $T(x)\equiv 0$ almost everywhere.) Here $f(x)=(x-g)^\alpha(x-h)^\beta$,

$$S(x) = (x-g)^\alpha (x-h)^\beta \left\{ \int_c^x \frac{K(z-L)T(z)dz}{r(z-g)^{\alpha+1}(z-h)^{\beta+1}} + C \right\},$$
$$\alpha, \beta, c, C \text{ constants}, r(\alpha+\beta) = K.$$

If i, j are integers such that $\alpha+i>0$, $\beta+j>0$, $i\geqq 1$, $j\geqq 1$, then, integrating by parts, we find that

$$\int_a^b \frac{S(x)}{(x-g)^\alpha(x-h)^\beta} \frac{d}{dx}\left[(x-g)^{\alpha+i}(x-h)^{\beta+j}\right]dx = 0,$$

$$\int_a^b S(x)(x-g)^{i-1}(x-h)^{j-1}(x-\bar{x})dx = 0, \quad \bar{x} = \frac{(\alpha+i)h+(\beta+j)g}{\alpha+\beta+i+j}.$$

This is impossible, as we shall show when $a=0$. (The proof is similar if $a=-\infty$.) If $\bar{S}(x)\equiv S(x)(x-g)^{i-1}(x-h)^{j-1}(x-\bar{x})$, and if constants A, A' are chosen so that $A'>A>3|h|+|L|+1$, then

$$\int_0^\infty \bar{S}(x)dx = \int_0^A \bar{S}(x)dx + \int_A^{A'} \bar{S}(x)dx + \int_{A'}^{A'+1} \bar{S}(x)dx$$
$$+ \int_{A'+1}^\infty \bar{S}(x)dx \equiv \sum_{n=1}^4 i_n = 0.$$

If i is so large that $|\bar{x}-h|<|h|+1$, we have

$$i_2>0, \quad i_3>0, \quad i_4>0, \quad i_1<0, \quad i_2+i_4>0, \quad i_1+i_3<0.$$

On the other hand,

$$\frac{|i_1|}{i_3} \leqq \frac{(A-g)^{i-1}(A-h)^{j-1}(A-\bar{x})}{(A'-g)^{i-1}(A'-h)^{j-1}(A'-\bar{x})S(A'+1)} \int_0^A S(x)dx < 1$$

if i is sufficiently large. Then $|i_1|<i_3$, contradicting $i_1+i_3<0$.

From these cases we conclude that $r=0$.

(iv) Suppose that $r=0$, $s\neq 0$. Let

$$\frac{K(x-L)}{sx+t} = \beta + \frac{\alpha s}{sx+t}, \qquad \alpha, \beta \text{ constants}, \beta s = K.$$

The condition $S(b)=0$ gives

$$\int_a^b [\beta(sx+t)p(x)+\alpha s p(x)]dx = \beta\int_a^b q_1(x)dx + \alpha s\int_a^b p(x)dx$$
$$= \beta\beta_0 + \alpha s \alpha_0 = 0.$$

We must have $\alpha>0$, because α_0, $\beta_0>0$, $\beta s<0$. In this case, $f(x) = (sx+t)^{\alpha}e^{\beta x}$, and

$$S(x) = (sx + t)^{\alpha}e^{\beta x}\left\{\int_c^x \frac{K(z - L)T(z)dz}{(sz + t)^{\alpha+1}e^{\beta z}} + C\right\}, \qquad c, C \text{ constants.}$$

If $a< -t/s \equiv -t'$, the existence of $S(x)$ near $-t'$ requires the existence of the integral

$$\int_c^{-t'} K(z - L)(sz + t)^{-\alpha-1}e^{-\beta z}T(z)dz,$$

so that $S(-t')=0$, which is impossible. Thus $sx+t$ does not change sign in (a, b), and $s>0$ because $\beta_0>0$. Then $a=0$, $\beta<0$, $t'\equiv t/s\geqq 0$.
Let

$$S_2(x) \equiv \begin{cases} 0 \quad \text{in} \quad (0, t'), \\ S(x - t') \quad \text{in} \quad (t', \infty), \end{cases}$$

where

$$S(x - t') \equiv \frac{1}{s}\, x^{\alpha}e^{\beta x}\left\{\int_{c'}^x \frac{K(z - L - t')T(z - t')dz}{z^{\alpha+1}e^{\beta z}} + C'\right\},$$

$$c', C' \text{ constants.}$$

The weight functions $S(x-t')$ in (t', ∞) and $S_2(x)$ in $(0, \infty)$ give rise to the same system of orthogonal polynomials $\{\phi_n'(x-t')\}$ since the moments are the same. Let

$$T_1(x) \equiv S_2(x) + C_1x^{\alpha}e^{\beta x},$$

where the constant C_1 is determined so that $\int_0^\infty T_1(x)dx=0$. Integrating by parts, we obtain

$$\int_0^\infty \frac{T_1(x)}{x^{\alpha}e^{\beta x}}\frac{d}{dx}\left[x^{\alpha+i}e^{\beta x}\right]dx = 0, \qquad i \geqq 1,$$

$$\int_0^\infty T_1(x)x^{i-1}[\alpha + i + \beta x]dx = \int_0^\infty T_1(x)x^{i-1}dx = 0,$$

$$i = 1, 2, \cdots.$$

Hence, if we neglect the function $T_1(x)$ whose moments vanish, the weight function is of the form $Cx^{\alpha}e^{\beta x}$ (C an arbitrary constant). Replacing x by $-x/\beta$ and putting $C=(-\beta)^{\alpha}$, we obtain the weight function $x^{\alpha}e^{-x}$ which is the weight function for the derivatives of the Laguerre polynomials with the property that

$$\int_0^\infty x^{\alpha-1}e^{-x}\phi_m(x)\phi_n(x)dx = \int_0^\infty x^\alpha e^{-x}\phi_m'(x)\phi_n'(x)dx = 0,$$

$$\alpha > 0;\, m \neq n;\, m,\, n = 0, 1, \cdots .$$

(v) Suppose that $r=s=0$, $t\neq 0$. Since $\beta_0>0$, we must have $t>0$. Here $f(x)=e^{K'(x^2-2Lx)}$, and

$$S(x) = e^{K'(x^2-2Lx)}\left\{\int_c^x 2K'(z-L)e^{-K'(z^2-2Lz)}T(z)dz + C\right\},$$

$$K',\, c,\, C \text{ constants, } K' = K/2t < 0.$$

If i is an odd positive integer, the function

$$e^{-K'(x^2-2Lx)}\int_\infty^x (z-L)^i e^{K'(z^2-2Lz)}dz$$

is a polynomial in x. Hence, integration by parts gives $\int_a^b S(x)(x-L)^i dx =0$, ($i$ odd), which requires $a=-\infty$. Since by (5)

$$\int_{-\infty}^\infty p(x)(x-L)^i dx = \int_{-\infty}^\infty p(x+L)x^i dx = \int_{-\infty}^\infty p(-x+L)x^i dx = 0,$$

$$i \text{ odd},$$

$$\int_{-\infty}^\infty p(x+L)x^i dx = \int_{-\infty}^\infty p(-x+L)x^i dx, \qquad i \text{ even},$$

it follows that $p_1(x)\equiv p(x+L)+p(-x+L)$ is a weight function for $\{\phi_n(x+L)\}$. Assuming that $p(x)$ has been replaced by $p_1(x)$, we find that $p_1(-x)\equiv p_1(x)$, $T(-x)\equiv T(x)$, $S(-x)\equiv S(x)$, and

$$S(x) = e^{K'x^2}\left\{\int_0^x 2K'ze^{-K'z^2}T(z)dz + C\right\}, \qquad K',\, C \text{ constants}.$$

Let

$$T_1(x) = e^{K'x^2}\left\{\int_0^x 2K'ze^{-K'z^2}T(z)dz + C_1\right\},$$

where C_1 is a constant to be determined. Then $T_1(-x)\equiv T_1(x)$, and $\int_{-\infty}^\infty T_1(x)x^i dx=0$, ($i$ odd). If i is even, then integration by parts shows that

$$u(x) \equiv e^{-K'x^2}\int_\infty^x z^i e^{K'z^2}dz = xP_{i-2}(x) + C_2 e^{-K'x^2}\int_\infty^x e^{K'z^2}dz,$$

where $P_{i-2}(x)$ is a polynomial of degree $i-2$ in x, ($i\geqq 2$, C_2 constant). It follows that

$$\int_0^\infty T_1(x)x^i dx = \left[\int_\infty^x z^i e^{K'z^2}dz\left\{\int_0^x 2K'ze^{-K'z^2}T(z)dz + C_1\right\}\right]_0^\infty$$

$$- 2K'\int_0^\infty xT(x)u(x)dx$$

$$= C_2\left\{C_1\int_\infty^x e^{K'z^2}dz\right.$$

$$\left.- 2K'\int_0^\infty xT(x)e^{-K'x^2}\left(\int_\infty^x e^{K'z^2}dz\right)dx\right\}$$

$$= 0, \qquad i \text{ even},$$

if C_1 is properly chosen. Thus $\int_{-\infty}^{\infty} T_1(x)x^i dx = 0$, $(i=0, 1, \cdots)$. Except for a function whose moments vanish, the weight function reduces to $C_3e^{K'x^2}$, (C_3 an arbitrary constant). Replacing x by $x/(-K')^{1/2}$ and putting $C_3=1$, we obtain e^{-x^2}, which is the weight function for Hermite polynomials.

4. **Conclusion.** Having completed a proof of the theorem, we give the following corollary:

COROLLARY. *If $\{\phi_n(x)\}$ is an orthogonal system of polynomials which is also an Appell system, so that $\phi_n'(x) \equiv n\phi_{n-1}(x)$ (that is, $p(x) \equiv q(x)$), then $\{\phi_n(x)\}$ is reducible to the system of Hermite polynomials by means of a linear transformation on x.*

Meixner* first proved this result, but other proofs have been given by W. Hahn,† the author,‡ and Shohat.§ Sheffer's|| recurrence relation for Appell polynomials and the recurrence relation for orthogonal polynomials¶ enable us to give a more direct proof.** Comparing Sheffer's relation

* J. Meixner, *Orthogonale Polynomsysteme mit einer besonderen Gestalt der erzeugenden Funktion*, Journal of the London Mathematical Society, vol. 9 (1934), pp. 6–13.

† Loc cit.

‡ M. Webster, *On the zeros of Jacobi polynomials with applications*, Duke Mathematical Journal, vol. 3 (1937), pp. 426–442.

§ J. Shohat, *The relation of the classical orthogonal polynomials to the polynomials of Appell*, American Journal of Mathematics, vol. 58 (1936), pp. 453–464.

|| I. Sheffer, *A differential equation for Appell polynomials*, this Bulletin, vol. 41 (1935), pp. 914–923.

¶ J. Shohat, *Théorie Générale des Polynômes Orthogonaux de Tchebichef*, Mémorial des Sciences Mathématiques, vol. 66, Paris, 1934.

** This Bulletin, abstract 42-3-127.

$$\phi_n(x) = (x + b_0)\phi_{n-1}(x) + (n-1)b_1\phi_{n-2} + (n-1)(n-2)b_2\phi_{n-3}(x) + \cdots + (n-1)(n-2)\cdots 1 b_{n-1}\phi_0(x)$$

with

$$\phi_n(x) = (x - c_n)\phi_{n-1}(x) - \lambda_n\phi_{n-2}(x),$$

we have

$$c_n = -b_0, \quad b_2 = b_3 = \cdots = b_{n-1} = 0, \quad \lambda_n = -b_1(n-1) > 0,$$

for $n>1$. Let $x=(-2b_1)^{1/2}y-b_0$; then $\phi_n(x)\equiv(-2b_1)^{n/2}\psi_n(y)$, where $\psi_n(y)\equiv y\psi_{n-1}(y)-[(n-1)/2]\psi_{n-2}(y)$, which proves that $\{\psi_n(y)\}$ is the set of Hermite polynomials.

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