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Acoustic reciprocity: An extension to spherical harmonics domain

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Abstract: Acoustic reciprocity is a fundamental property of acoustic wavefields that is commonly used to simplify the measurement process of many practical applications. Traditionally, the reciprocity theorem is defined between a monopole point source and a point receiver. Intuitively, it must apply to more complex transducers than monopoles. In this paper, the authors formulate the acoustic reciprocity theory in the spherical harmonics domain for directional sources and directional receivers with higher order directivity patterns.

[DHC]
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1. Introduction

Reciprocity is a fundamental property that is obeyed by waves propagating between two points in space, given the propagation medium is at rest. If waves can propagate from a source to an observer, the opposite propagation path, from the observer to the source, is equally possible. Waves travel in a symmetric manner, ensuring reciprocity, such that the source and receiver locations may be interchanged without changing the waveform observed at the receiver end.

The reciprocity property is often used in theoretical and practical applications involving acoustic waves.[1] It is also used in other areas involving vibro-acoustics and seismic waves. Some examples for acoustic applications where the measurement process is simplified by interchanging the microphone and loudspeaker locations are head-related transfer function measurements,[2] engine/tyre noise modeling in automobiles/trains,[3] and noise field measurement in submarines.[1] Other applications of reciprocity often involve seismic applications[4] and transducer calibrations.[5] The intention behind using reciprocity in almost all of the above applications is to overcome two key limitations in the measurement process, namely, space limitations (in some locations, placing a small microphone is more practical than a loudspeaker) and high cost (utilizing an array of microphones is cheaper than an array of loudspeakers). A simple exchange of source and receiver often overcomes these limitations.

In acoustic applications, the act of interchanging source and receiver assumes reciprocity in acoustic waves between omnidirectional sources and omnidirectional receivers. However in practice, commercial loudspeakers and microphones are often directional due to their physical structure and orientation. Therefore, to successfully apply the reciprocity principle to real acoustic devices, it is necessary to incorporate the transducer directionality in a generalized manner. The generalization of reciprocity theorem to directional transducers is conceptually not too difficult. However, there exists no closed form relationship describing the reciprocity principle for directional transducers that can be directly used with their individual directivities to predict reciprocal properties. In this paper, we show that by formulating both the reciprocity theory and the transducer directionality in the spherical harmonics domain, the aforementioned task can be easily achieved.

Another application where reciprocity in the spherical harmonics domain can be widely used is as follows. Higher Order (HO) microphones[6] and loudspeakers[7,8] are recently becoming popular due to their ability to record and reproduce spatial soundfields with desired directional properties. While HO microphones are commercially available for some time, the design and implementation of HO speakers is comparably

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challenging, mainly due to the requirement of large aperture sizes to produce low frequency sound waves. In such instances, the radiation pattern measurements [e.g., Room Transfer Function (RTF)] due to the lower order modes of an HO speaker can be carried out by replacing the HO speaker with an HO microphone and by using the reciprocity principle. This task also requires the directional properties of the transducers and the reciprocity principle to be generalized in a common spatial domain.

In this paper, we formulate the acoustic reciprocity principle in the spherical harmonics domain such that the radiation pattern of the source/receiver, when expressed in terms of spherical harmonic functions, can be effectively integrated with the reciprocity principle to assist related practical applications. We choose the spherical harmonics domain because spherical harmonics are spatial basis functions that can be used to describe any directionality pattern in the three-dimensional space.

2. Rayleigh reciprocity theorem

The first known work on reciprocity was carried out by Hermann von Helmholtz. In a paper published in 1860,[9] he asserted that the acoustic behavior inside open ended-pipes exhibited reciprocity. He later formalized his expression for point sources (omnidirectional) and point pressures (as recorded by an omnidirectional microphone) in the presence of any arbitrary number and form of rigid scatterers in the fluid. However, the most comprehensive proposition of the general principle of reciprocity for vibrating systems was presented by Lord Rayleigh in 1873.[10] He presented a general theory of vibrational reciprocity as applied to passive linear distributed elastic systems having time-invariant physical parameters. It is shown that, if a point harmonic force is applied to one coordinate and the oscillatory displacement is observed at another, the complex ratio of displacement to force is invariant with respect to exchange of input and output coordinates.

Rayleigh's vibrational reciprocity principle is applicable to acoustics because air at rest is a fluid that behaves like a linear-elastic medium in response to small applied disturbances. Therefore, the input variable of an elementary acoustic source (point source) is its volume velocity Q, which is the rate of displacement of air when producing sound, and the output variable of an omnidirectional pressure microphone is the force F applied by the fluid to the transducer's diaphragm. Thus, in acoustics, Rayleigh's reciprocity is applicable to the ratio F/Q. In simpler terms, if we consider an acoustic medium at rest, as represented in Fig. 1, at any point A, acoustic waves may be excited and the resulting sound pressure and potential velocity observed at a second point B is the same both in magnitude and phase, as it would have been at A, had B been the source of sound. This statement has been proven to hold for arbitrarily reflective media (as represented by the dark blue region in Fig. 1) even in the presence of absorption losses. The proof is provided in Volume II of Rayleigh's *The Theory of Sound* (1878),[11] and it is typically discussed in most books in acoustics.[12]

Rayleigh also extended the reciprocity relationship to acoustic dipoles and particle velocities in the resulting soundfields. Naturally, it must be extendable for arbitrary multi-pole transducers as well. For example, the reciprocity principle for a directional source can be conceptualized as follows. Consider a source region in which there are L monopoles, and a receiver region in which the pressure is known at R locations. Reciprocity applies for each source and receiver pair. One can then imagine exciting all L monopoles with a signal S which is weighted by w_ℓ for each monopole $(\ell = 1, \ldots, L)$, producing a general source radiation pattern. Similarly the receiver outputs can be assumed to be weighted by $\tilde{w}_r$ where $(r = 1, \ldots, R)$ producing a general receiver beam pattern. The reciprocal arrangement would be to place a monopole at each receiver

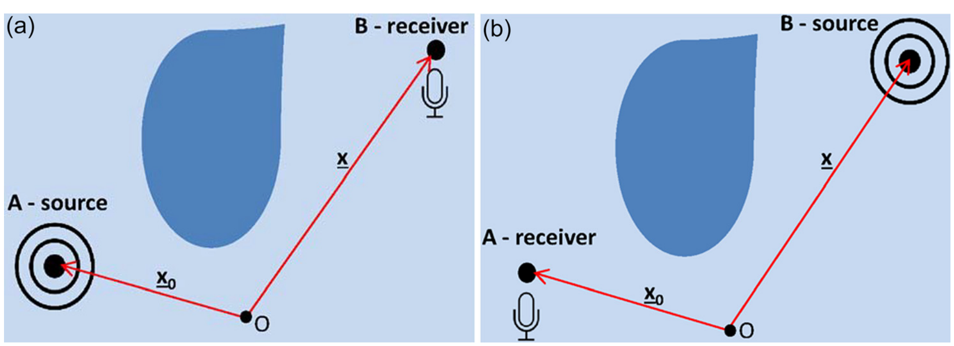

Fig. 1. (Color online) Rayleigh reciprocity theorem states that the pressure and velocity potential induced at point B by a time-harmonic source placed at point A (a) is the same in both magnitude and phase as the one induced at point A if the same source is placed at point B (b).

point R producing S weighted with $\tilde{w}_r$, and to replace the L transmit monopoles with microphones weighted with w_ℓ. Utilizing the reciprocity theorem for point transducers, it can be shown that the total microphone output in both cases will be identical.

2.1 Reciprocity in terms of the Green's function

The acoustic reciprocity principle for monopole transducers can also be described in terms of the Green's function, which is a solution to the inhomogeneous wave equation. Figure 1(a) shows an inhomogeneous soundfield with respect to origin $\boldsymbol{O}$ caused by a monopole point source. Reciprocity between the source at $\boldsymbol{x}_0$ and a point receiver at $\boldsymbol{x}$ can also be expressed in terms of the Green's function,[13] as

$$G(\boldsymbol{x}|\boldsymbol{x}_0, k) = G(\boldsymbol{x}_0|\boldsymbol{x}, k), \tag{1}$$

where $k = 2\pi f/c$ is the wave number with f denoting the frequency and c denoting the speed of sound. In a free field with no obstacles (where the dark blue region in Fig. 1 is non-existent), a solution to the Green's function that satisfies the inhomogeneous wave equation is

$$G(\boldsymbol{x}|\boldsymbol{x}_0, k) = Q(k)\frac{e^{-ik\|\boldsymbol{x}-\boldsymbol{x}_0\|}}{4\pi\|\boldsymbol{x}-\boldsymbol{x}_0\|}, \tag{2}$$

where $Q(k)$ represents the source signal.

Again, an extension of this relationship to multi-pole transducers can be conceptually visualized, at least for directional sources. If a directional source is represented by a continuous distribution of point sources, and if one finds the Green's function for each transmit position x_s and then finds a general response by integrating the Green's function over the volume of space, weighted by a function $w(x_s)$, then the discrete array model discussed earlier is a discrete approximation to the continuous theory.

While the generalization of reciprocity principle to multi-pole transducers is conceptually explainable, a closed form definition that is directly applicable to non-ideal transducers is non-existent. In this paper, we develop such a formula in the spherical harmonics domain, which we believe is a very useful addition to the literature.

3. Summary of the framework extending the reciprocity principle

In the process of extending the reciprocity principle, we first divide the inhomogeneous soundfield (see Fig. 1) into two homogeneous soundfields (see Fig. 2), (i) exterior to the source and (ii) incident at the receiver, with their respective origins defined at the local origin of each transducer. Then the outgoing and incoming soundfields are decomposed in terms of the spherical harmonic decomposition of homogeneous soundfields. Then the inherent coupling between these two soundfields is defined in the spherical harmonics domain, which acts as the main element on which the reciprocity theory is developed. The authors have recently developed a spherical harmonics domain parameterization for the room response (or RTF) between directional/HO transducers using a soundfield configuration as described above. Therefore, we base the formulation of the reciprocity principle on the aforementioned RTF parameterization. However, since the parameterization proposed in Ref. 14 is applicable to any linear system, it can be generalized to represent the transfer function between directional transducers in any arbitrary environment (free field or reflective). For this reason, we would like to remind the reader that the following theory is general for any acoustic transfer function, but by way of example is applicable to rooms.

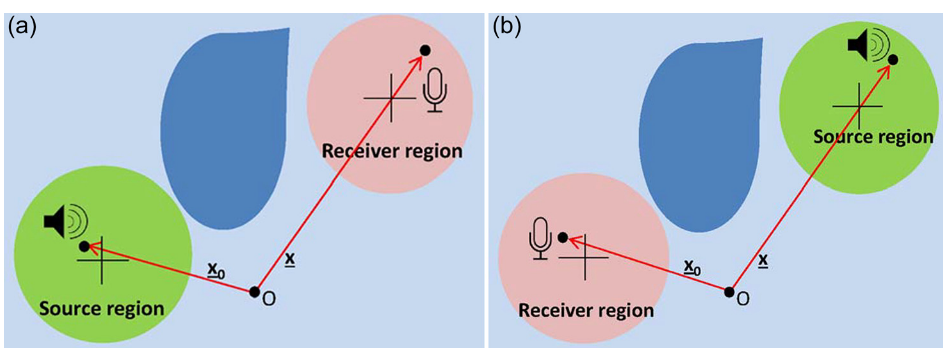

Fig. 2. (Color online) Framework used in the paper by defining two homogeneous soundfields inside Fig. 1, namely, (i) exterior to the source region and (ii) interior to the receiver region.

4. Spherical harmonics based parameterization of the RTF

As mentioned earlier, this formulation is based on RTF because the authors have previously developed a spherical harmonics based parameterization for the RTF,[14] which can be directly utilized to derive the reciprocity principle of interest. By definition, RTF is the frequency domain representation of the point pressure (as observed by an omnidirectional microphone) at the receiver location due to an impulse generated by a point source at a secondary location. RTF is inherently unique to the source and receiver locations, as well as the room characteristics.

In Ref. 14, the authors presented a parameterization of the RTF between any two arbitrary points from a pre-defined source region and a receiver region (see Fig. 3) in the spherical harmonics domain. This parameterization allows a finite set of coefficients to approximate the RTF between two arbitrary points within the said regions. As shown later, this parameterization can also be interpreted as the room response between a directional source and a directional receiver. Therefore, by formulating the reciprocity principle for this spherical harmonics based RTF, we generalize the reciprocity principle for directional (or HO) sources and receivers.

While the work in Ref. 14 allowed the source and receiver regions to be arbitrarily positioned (overlapping, non-overlapping, and concentric), in this work we only consider the case when the source and receiver regions are non-overlapping. This is because we intend to later interpret the results in terms of a directional source and a directional microphone, which have to be physically non-overlapping. Hence, in this paper we can use a slightly simplified RTF parameterization as given below.

As shown in Fig. 3, let the source region be defined as a spherical region ζ of radius R_s within which a point source may be arbitrarily located. Similarly, let the receiver region be defined as a spherical region η of radius R_r within which an omnidirectional microphone may be arbitrarily located. The aim is to derive an RTF parameterization that is robust to source and receiver variations inside ζ and η. As mentioned earlier, spherical harmonics are a set of orthogonal spatial basis functions that can be utilized to decompose any arbitrary function defined on the sphere. Therefore, when observed on a sphere, the outgoing soundfield from ζ and the resulting room response arriving at η can be expressed in terms of spherical harmonic decompositions. We derive the desired parameterization by coupling the outgoing spherical harmonic modes from ζ and the incoming spherical harmonic modes at η. In the remainder of this section we describe the above process in three steps: (i) describe the outgoing field from the source in terms of spherical harmonics modes, (ii) describe the incident field at the receiver in terms of spherical harmonics modes, and (iii) formulate the coupling between the outgoing modes and incoming modes to parameterize the RTF.

4.1 Spherical harmonics representation of outgoing soundfield from the source

The expression for a homogeneous outgoing soundfield as observed by any arbitrary point $\boldsymbol{z}^{(s)} = (z^{(s)}, \theta_z^{(s)}, \phi_z^{(s)})$ outside of ζ with respect to its local origin $\boldsymbol{O_s}$ is[15]

$$S_{\text{out}}(\boldsymbol{z}^{(s)}, k) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \beta_{nm}(k) h_n(kz^{(s)}) Y_{nm}(\theta_z^{(s)}, \phi_z^{(s)}), \tag{3}$$

where $h_n(\cdot)$ denotes the spherical Hankel function of order n, $Y_{nm}(\cdot)$ denotes the spherical harmonic function of order n and mode m, and $\beta_{nm}(k)$ denotes the respective

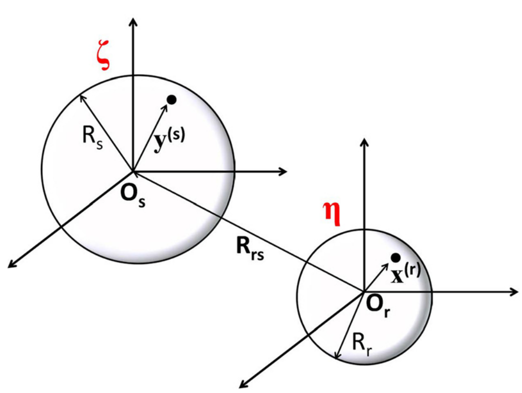

Fig. 3. (Color online) Spatial configurations of the source region ζ and receiver region η.

harmonic coefficients. When the outgoing field is produced by a point source within ζ, Eq. (3) can be expressed as

$$S_{\text{out}}(\boldsymbol{z}^{(s)},k)=\sum_{n=0}^{N}\sum_{m=-n}^{n}\underbrace{ikj_n(ky^{(s)})\,Y^*_{nm}(\theta_y^{(s)},\phi_y^{(s)})}_{\beta_{nm}(k)}\,h_n(kz^{(s)})\,Y_{nm}(\theta_z^{(s)},\phi_z^{(s)}), \tag{4}$$

where $(y^{(s)},\theta_y^{(s)},\phi_y^{(s)}):=\boldsymbol{y}^{(s)}$ is the point source location with respect to $\boldsymbol{O_s}$, and $j_n(\cdot)$ denotes the spherical Bessel function of order n. Due to the inherent properties of Bessel functions, the infinite summation given in Eq. (4) can be truncated at $N=\lceil ky^{(s)}\rceil$[6,16,17] with a minimal approximation error. Thus, the order of the outgoing soundfield at ζ depends on the location of the point source. If the point source is at $\boldsymbol{O_s}$ with $y^{(s)}=0$, the outgoing soundfield with respect to $\boldsymbol{O_s}$ is of order zero and omnidirectional. When the point source moves away from $\boldsymbol{O_s}$, the outgoing soundfield with respect to $\boldsymbol{O_s}$ becomes higher-order (or directional) due to the activation of additional soundfield modes. Note that a HO loudspeaker located at $\boldsymbol{O_s}$ can be interpreted in a similar manner to Eq. (3), where the order N and soundfield coefficients $\beta_{nm}(k)$ will have different values according to the loudspeaker properties.

4.2 Spherical harmonics representation of incident soundfield at the receiver

The spherical harmonic based expression for a homogeneous incident soundfield as observed by any arbitrary point $\boldsymbol{x}^{(r)}=(x^{(r)},\theta_x^{(r)},\phi_x^{(r)})$ inside η with respect to the local origin $\boldsymbol{O}_r$ is[15]

$$S(\boldsymbol{x}^{(r)},k)=\sum_{v=0}^{V}\sum_{u=-v}^{v}\delta_{vu}(k)j_v(kx^{(r)})\,Y_{vu}(\theta_x^{(r)},\phi_x^{(r)}), \tag{5}$$

where V is the respective truncation limit determined by $V=\lceil kx^{(r)}\rceil$ due to the presence of Bessel functions. When an omnidirectional microphone is at $\boldsymbol{O}_r$ it only observes the zeroth order soundfield with respect to $\boldsymbol{O}_r$, but when it moves away from $\boldsymbol{O}_r$ it observes the additional HO components active in the soundfield with respect to $\boldsymbol{O}_r$. A Vth order microphone located at $\boldsymbol{O}_r$ would be capable of successfully extracting the soundfield components $\delta_{vu}^{(r)}(k)$ for $v=0{:}V$ and $u=-v{:}v$.[6]

4.3 RTF parameterization by mode coupling

In order to derive the desired RTF parameterization, we couple the individual modes of the room response incident at η with the individual outgoing modes from ζ such that the total response at η due to any arbitrary excitation at ζ can be modally decomposed. That is, we define a parameter which quantifies the relationship between each outgoing mode from ζ and each incident mode at η. To illustrate this step, let us consider a unit amplitude outgoing wave of order n' and mode m',

$$\beta_{nm}(k)=\begin{cases}1, & n=n'\text{ and } m=m'\\ 0, & \text{otherwise,}\end{cases} \tag{6}$$

producing

$$S_{\text{out}}(\boldsymbol{z}^{(s)},k)=h_{n'}(kz^{(s)})\,Y_{n'm'}(\theta_z^{(s)},\phi_z^{(s)}). \tag{7}$$

For this particular outgoing source field, there will be a resulting room response present at η. We represent it with respect to the local origin $\boldsymbol{O}_r$ as

$$S_{n'm'}(\boldsymbol{x}^{(r)},k)=\sum_{v=0}^{V}\sum_{u=-v}^{v}\alpha_{vu}^{n'm'}(k)j_v(kx^{(r)})\,Y_{vu}(\theta_x^{(r)},\phi_x^{(r)}), \tag{8}$$

where $\alpha_{vu}^{n'm'}(k)$ denotes the vth order, uth mode soundfield coefficient of the room response incident at η caused by a unit amplitude n'th order and m'th mode outgoing soundfield from ζ. Note that the parameters $\alpha_{vu}^{n'm'}(k)$ express the coupling between each individual outgoing mode from ζ and each incident soundfield mode at η, and they are unique to the room enclosure of interest and the positioning of η and ζ. If these coefficients can be measured or simulated, then the RTF between any two points from ζ and η can be derived using Eqs. (7) and (8) as

$$H(\boldsymbol{x}^{(r)},\boldsymbol{y}^{(s)},k)=\sum_{n=0}^{N}\sum_{m=-n}^{n}\sum_{v=0}^{V}\sum_{u=-v}^{v}\underbrace{ikj_n(ky^{(s)})\,Y^*_{nm}(\theta_y^{(s)},\phi_y^{(s)})}_{\beta_{nm}(k)}\,\alpha_{vu}^{nm}(k)j_v(kx^{(r)})\,Y_{vu}(\theta_x^{(r)},\phi_x^{(r)}). \tag{9}$$

4.4 Response between a higher-order speaker and a HO microphone

If the outgoing modes $\beta_{nm}(k)$ are chosen to represent an Nth order directional speaker at $\boldsymbol{O}_s$, the above equation still delivers the room response present at η with

$$P(\boldsymbol{x}^{(r)},k) = \sum_{n=0}^{N}\sum_{m=-n}^{n}\sum_{v=0}^{V}\sum_{u=-v}^{v} \beta_{nm}(k)\alpha_{vu}^{nm}(k)j_v(kx^{(r)})\,Y_{vu}(\theta_x^{(r)},\phi_x^{(r)}). \tag{10}$$

A Vth order microphone at $\boldsymbol{O}_r$ would be capable of successfully extracting the spherical harmonic coefficients $\delta_{vu}(k) = \sum_{n=0}^{N}\sum_{m=-n}^{n}\beta_{nm}(k)\alpha_{vu}^{nm}(k)$. We name the new parameters $\alpha_{vu}^{n'm'}(k)$ as the room's spherical mode coupling as they comprehensively express the coupling between two spatial regions η and ζ in the room. They can also be interpreted as the spherical mode coupling between a HO source at $\boldsymbol{O}_s$ and a HO receiver at $\boldsymbol{O}_r$. It is important to remind that the parameterization given in Eq. (10) is not restricted to RTFs, but also applicable to any general soundfield observed at any arbitrary receiver due to an arbitrary source for any arbitrary medium (free-field, with obstacles, etc.), and any arbitrary enclosure including free-field. The mode coupling parameters $\alpha_{vu}^{n'm'}(k)$ define the coupling between the outgoing soundfield from the source and the incident soundfield at the receiver, which also characterize the properties of the enclosure and the medium. Thus the reciprocity principle formulated below is applicable to any general acoustic field.

5. Formulation of the reciprocity principle in spherical harmonics

In this section, we formulate the reciprocity principle for the spherical mode coupling parameters $\alpha_{vu}^{nm}(k)$ and introduce the following theorem.

Theorem 1. *Given $\alpha_{vu}^{nm}(k)$ is the spherical harmonic domain mode coupling Eq. (9) between an Nth order source region ζ (or Nth order source) and a Vth order receiver region η (or Vth order microphone), and $\tilde{\alpha}_{nm}^{vu}(k)$is the mode coupling between the inter-changed regions, that is where η produces a Vth order soundfield and ζ records an Nth order soundfield, then the reciprocity principle for the spherical harmonics domain mode coupling is*

$$\alpha_{vu}^{nm}(k) = \tilde{\alpha}_{n-m}^{v-u}(k). \tag{11}$$

Proof. In Eq. (9), RTF was parametrized between a point source at $\boldsymbol{y}^{(s)}$ and a point receiver at $\boldsymbol{x}^{(r)}$, where $\boldsymbol{y}^{(s)}$ and $\boldsymbol{x}^{(r)}$ are arbitrarily placed within a pre-defined source region ζ and a pre-defined receiver region η, respectively. Assume the source and receiver locations are interchanged with $\boldsymbol{x}^{(r)}$ within η representing the source position, and $\boldsymbol{y}^{(s)}$ within ζ representing the receiver position. The RTF can then be expressed by

$$H(\boldsymbol{y}^{(s)},\boldsymbol{x}^{(r)},k) = \sum_{n'=0}^{V}\sum_{m'=-n'}^{n'}\sum_{v'=0}^{N}\sum_{u'=-v'}^{v'} \underbrace{ikj_{n'}(kx^{(r)})\,Y_{n'm'}^{*}(\theta_x^{(r)},\phi_x^{(r)})}_{\beta_{n'm'}(k)}\tilde{\alpha}_{v'u'}^{n'm'}(k)j_{v'}(ky^{(s)})\,Y_{v'u'}(\theta_y^{(s)},\phi_y^{(s)}), \tag{12}$$

where $\tilde{\alpha}_{v'u'}^{n'm'}(k)$ are the room response coefficients for the interchanged source and receiver regions. From the reciprocity principle for point sources and point receivers, the RTF given by Eqs. (9) and (12) have the same value. Therefore

$$\begin{aligned}&\sum_{n=0}^{N}\sum_{m=-n}^{n}\sum_{v=0}^{V}\sum_{u=-v}^{v} ikj_n(ky^{(s)})\,Y_{nm}^{*}(\theta_y^{(s)},\phi_y^{(s)})\alpha_{vu}^{nm}(k)j_v(kx^{(r)})\,Y_{vu}(\theta_x^{(r)},\phi_x^{(r)})\\ &= \sum_{n'=0}^{V}\sum_{m'=-n'}^{n'}\sum_{v'=0}^{N}\sum_{u'=-v'}^{v'} ikj_{n'}(kx^{(r)})\,Y_{n'm'}^{*}(\theta_x^{(r)},\phi_x^{(r)})\tilde{\alpha}_{v'u'}^{n'm'}(k)j_{v'}(ky^{(s)})\,Y_{v'u'}(\theta_y^{(s)},\phi_y^{(s)})\end{aligned} \tag{13}$$

In order to simplify Eq. (13), we multiply both sides with the term $Y_{pq}(\theta_y^{(s)},\ \phi_y^{(s)})$ $Y_{p'q'}^{*}(\theta_x^{(r)},\phi_x^{(r)})$ and double-integrate them over $\int_{\hat{\boldsymbol{y}}}\int_{\hat{\boldsymbol{x}}} d\hat{\boldsymbol{x}}d\hat{\boldsymbol{y}}$, where $\int_{\hat{\boldsymbol{x}}} d\hat{\boldsymbol{x}} = \int_{\hat{\boldsymbol{y}}} d\hat{\boldsymbol{y}}$ $= \int_0^{2\pi}\int_0^{\pi}\sin\theta d\theta d\phi$. Utilizing the orthogonality property of spherical harmonics given by

$$\int_{\hat{\boldsymbol{y}}} Y_{nm}(\theta_y^{(s)},\phi_y^{(s)})\,Y_{pq}^{*}(\theta_y^{(s)},\phi_y^{(s)})d\hat{\boldsymbol{y}} = \delta_{nmpq}, \tag{14}$$

and their evenness property

$$Y_{pq}^{*}(\theta_y^{(s)},\phi_y^{(s)}) = Y_{p-q}(\theta_y^{(s)},\phi_y^{(s)}), \tag{15}$$

Eq. (13) can be simplified into

$$\alpha_{vu}^{nm}(k) = \tilde{\alpha}_{n-m}^{v-u}(k). \tag{16}$$

This result proves that reciprocity holds in the spherical harmonics domain. Therefore, if directional sources/directional receivers are modeled in terms of spherical harmonic decompositions, then the reciprocity principle can be applied according to Eq. (16). □

Although the above proof is self-sufficient, we have verified the theory by simulations, but not presented due to lack of available place. Note that the spherical harmonic terms $Y_{nm}(\theta,\phi)$ discussed above are complex quantities, however the radiation pattern of a source/receiver is real. Real-valued spherical harmonics can be obtained by the combinations

$$\mathrm{Re}\{Y_{nm}(\theta,\phi)\} = (Y_{nm}(\theta,\phi) + Y_{n-m}(\theta,\phi))/2 \quad \text{and}$$
$$\mathrm{Im}\{Y_{nm}(\theta,\phi)\} = (Y_{nm}(\theta,\phi) - Y_{n-m}(\theta,\phi))/2i.$$

Therefore, for real-valued spherical harmonics, the reciprocity principle will be of the form $\gamma_{vu}^{nm}(k) = \tilde{\gamma}_{nm}^{vu}(k)$. This means that when the source and receiver are said to be interchanged, it is only the scalar magnitude which is interchanged; their real radiation patterns stay fixed at the same place. Thus, when applying reciprocity to directional sources and receivers, their radiation patterns can be regarded as attached to the medium, not as to the source and receiver.

As mentioned in Sec. 1, reciprocity theory in the spherical harmonics domain is largely useful in acoustic applications that involve non-ideal point transducers/ directional transducers, and in room measurements involving HO loudspeakers. Another important application made apparent from the RTF based proof in Sec. 5 is its ability act as a tool in validating the accuracy of RTF models such as the spherical harmonics based parametrization introduced in Ref. 14 and the image source model.[18]

6. Conclusion

In this paper, we have extended the acoustic reciprocity theory to the spherical harmonics domain such that it is applicable to directional sources and directional receivers of any radiation characteristic that can be described by spherical harmonics. The extended acoustic reciprocity principle can be directly used to simplify practical applications that require soundfield measurements with inter-changed source and receiver configurations.

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