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On $\mathcal{I}_c^{(q)}$-convergence

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Abstract

In this paper we will study the properties of ideals $\mathcal{I}_c^{(q)}$ related to the notion of $\mathcal{I}$-convergence of sequences of real numbers. We show that $\mathcal{I}_c^{(q)}$ and $\mathcal{I}_c^{(q)*}$-convergence are equivalent. We prove some results about modified Olivier's theorem for these ideals. For bounded sequences we show a connection between $\mathcal{I}_c^{(q)}$-convergence and regular matrix method of summability.

1. Introduction

In papers [9],and [10] the notion of $\mathcal{I}$-convergence of sequences of real numbers is introduced and its basic properties are investigated. The $\mathcal{I}$-convergence generalizes the notion of the statistical convergence (see[5]) and it is based on the ideal $\mathcal{I}$ of subsets of the set $\mathbb{N}$ of positive integers.

Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$. $\mathcal{I}$ is called an admissible ideal of subsets of positive integers, if $\mathcal{I}$ is additive (i.e. $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$), hereditary (i.e. $A \in \mathcal{I},\ B \subset A \Rightarrow B \in \mathcal{I}$), containing all singletons and it doesn't contain $\mathbb{N}$. Here we present some examples of admissible ideals. More examples can be found in the papers [7, 9, 10, 12].

(a) The class of all finite subsets of $\mathbb{N}$ form an admissible ideal usually denote by $\mathcal{I}_f$.

(b) Let ϱ be a density function on $\mathbb{N}$, then the set $\mathcal{I}_\varrho = \{A \subseteq \mathbb{N} : \varrho(A) = 0\}$ is an admissible ideal. We will use the ideals $\mathcal{I}_d, \mathcal{I}_\delta, \mathcal{I}_u$ related to asymptotic,logarithmic,uniform density,respectively. For those densities for definitions see [9, 10, 12, 13].

(c) For any $q \in (0,1\rangle$ the set $\mathcal{I}_c^{(q)} = \{A \subseteq \mathbb{N} : \sum_{a\in A} a^{-q} < \infty\}$ is an admissible ideal. The ideal $\mathcal{I}_c^{(1)} = \{A \subseteq \mathbb{N} : \sum_{a\in A} a^{-1} < \infty\}$ is usually denoted by $\mathcal{I}_c$. It is easy to see, that for any $q_1 < q_2; q_1, q_2 \in (0,1)$

$$\mathcal{I}_f \subsetneq \mathcal{I}_c^{(q_1)} \subsetneq \mathcal{I}_c^{(q_2)} \subsetneq \mathcal{I}_c \subsetneq \mathcal{I}_d \tag{1.1}$$

In this paper will we study the ideals $\mathcal{I}_c^{(q)}$. In particular the equivalence between $\mathcal{I}_c^{(q)}, \mathcal{I}_c^{(q)*}$, Olivier's like theorems for this ideals and characterization of $\mathcal{I}_c^{(q)}$-convergent sequences by regular matrices.

2. The equivalence between $\mathcal{I}_c^{(q)}$ and $\mathcal{I}_c^{(q)*}$-convergence

Let us recall the notion of $\mathcal{I}$-convergence of sequences of real numbers, (cf.[9, 10]).

Definition 2.1. We say that a sequence $x = (x_n)_{n=1}^{\infty}$ $\mathcal{I}$-converges to a number L and we write $\mathcal{I} - \lim\ x_n = L$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n : |x_n - L| \geq \varepsilon\}$ belongs to the ideal $\mathcal{I}$.

$\mathcal{I}$-convergence satisfies usual axioms of convergence i.e. the uniqueness of limit, arithmetical properties etc. The class of all $\mathcal{I}$-convergent sequences is a linear space. We will also use the following elementary fact.

Lemma 2.2. *Let $\mathcal{I}_1, \mathcal{I}_2$ be admissible ideals such that $\mathcal{I}_1 \subset \mathcal{I}_2$. If $\mathcal{I}_1 - \lim x_n = L$ then $\mathcal{I}_2 - \lim x_n = L$.*

In the papers [9, 10] there was defined yet another type of convergence related to the ideal $\mathcal{I}$.

Definition 2.3. Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. A sequence $x = (x_n)_{n=1}^{\infty}$ of real numbers is said to be $\mathcal{I}^*$-convergent to $L \in \mathbb{R}$ (shortly $\mathcal{I}^* - \lim x_n = L$) if there is a set $H \in \mathcal{I}$, such that for $M = \mathbb{N} \setminus H = \{m_1 < m_2 < \ldots\}$ we have, $\lim\limits_{k\to\infty} x_{m_k} = L$.

It is easy to prove, that for every admissible ideal $\mathcal{I}$ the following relation between $\mathcal{I}$ and $\mathcal{I}^*$-convergence holds:

$$\mathcal{I}^* - \lim x_n = L \Rightarrow \mathcal{I} - \lim x_n = L.$$

Kostyrko, Šalát and Wilczynski in [9] give an algebraic characterization of ideals $\mathcal{I}$, for which the $\mathcal{I}$ and $\mathcal{I}^*$-convergence are equal; it turns out that these ideals are with the property *(AP)*.

Definition 2.4. An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to satisfy the property *(AP)* if for every countable family of mutually disjoint sets $\{A_1, A_2, \ldots\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{B_1, B_2, \ldots\}$ such that $A_j \triangle B_j$ is a finite set for $j \in \mathbb{N}$ and $\bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.$(A \triangle B = (A \setminus B) \cup (B \setminus A))$.

For some ideals it is already known whether they have property *(AP)*(see [9, 10, 12, 13]). Now, will show the equivalence between $\mathcal{I}_c^{(q)}$ and $\mathcal{I}_c^{(q)*}$-convergence.

Theorem 2.5. *For any* $0 < q \leq 1$ *the ideal* $\mathcal{I}_c^{(q)}$ *has a property (AP).*

Proof. It suffices to prove that any sequences $(x_n)_{n=1}^{\infty}$ of real numbers such that $\mathcal{I}_c^{(q)} - \lim x_n = \xi$ there exist a set $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \subseteq \mathbb{N}$ such that $\mathbb{N} \setminus M \in \mathcal{I}_c^{(q)}$ and $\lim\limits_{k\to\infty} x_{m_k} = \xi$.

For any positive integer k let $\varepsilon_k = \frac{1}{2^k}$ and $A_k = \{n \in \mathbb{N} : |x_n - \xi| \geq \frac{1}{2^k}\}$. As $\mathcal{I}_c^{(q)} - \lim x_n = \xi$, we have $A_k \in \mathcal{I}_c^{(q)}$, i.e.

$$\sum_{a\in A_k} a^{-q} < \infty.$$

Therefore there exist an infinite sequence $n_1 < n_2 < \ldots < n_k \ldots$ of integers such that for every $k = 1, 2, \ldots$

$$\sum_{\substack{a>n_k \\ a\in A_k}} a^{-q} < \frac{1}{2^k}$$

Let $H = \bigcup_{k=1}^{\infty}[(n_k, n_{k+1}\rangle \cap A_k]$. Then

$$\sum_{a\in H} a^{-q} \leq \sum_{\substack{a>n_1 \\ a\in A_1}} a^{-q} + \sum_{\substack{a>n_2 \\ a\in A_2}} a^{-q} + \ldots + \sum_{\substack{a>n_k \\ a\in A_k}} a^{-q} + \ldots <$$

$$< \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^k} + \ldots < +\infty$$

Thus $H \in \mathcal{I}_c^{(q)}$. Put $M = \mathbb{N} \setminus H = \{m_1 < m_2 < \ldots < m_k < \ldots\}$. Now it suffices to prove that $\lim\limits_{k\to\infty} x_{m_k} = \xi$. Let $\varepsilon > 0$. Choose $k_0 \in \mathbb{N}$ such that $\frac{1}{2^{k_0}} < \varepsilon$. Let $m_k > n_{k_0}$. Then m_k belongs to some interval $(n_j, n_{j+1}\rangle$ where $j \geq k_0$ and doesn't belong to A_j $(j \geq k_0)$. Hence m_k belongs to $\mathbb{N} \setminus A_j$, and then $|x_{m_k} - \xi| < \varepsilon$ for every $m_k > n_{k_0}$, thus $\lim\limits_{k\to\infty} x_{m_k} = \xi$. □

3. Olivier's like theorem for the ideals $\mathcal{I}_c^{(q)}$

In 1827 L. Olivier proved the results about the speed of convergence to zero of the terms of a convergent series with positive and decreasing terms.(cf.[8, 11])

Theorem A. *If* $(a_n)_{n=1}^{\infty}$ *is a non-increasing sequences and* $\sum_{n=1}^{\infty} a_n < +\infty$, *then* $\lim\limits_{n\to\infty} n \cdot a_n = 0$.

Simple example $a_n = \frac{1}{n}$ if n is a square i.e. $n = k^2$, $(k = 1, 2, \ldots)$ and $a_n = \frac{1}{2^n}$ otherwise shows that monotonicity condition on the sequence $(a_n)_{n=1}^{\infty}$ can not be in general omitted.

In [14] T.Šalát and V.Toma characterized the class $\mathcal{S}(T)$ of ideals such that

$$\sum_{n=1}^{\infty} a_n < +\infty \Rightarrow \mathcal{I} - \lim_{n\to\infty} n \cdot a_n = 0 \tag{3.1}$$

for any convergent series with positive terms.

Theorem B. *The class $\mathcal{S}(T)$ consists of all admissible ideals $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ such that $\mathcal{I} \supseteq \mathcal{I}_c$.*

From inclusions (1.1) is obvious that ideals $\mathcal{I}_c^{(q)}$ do not belong to the class $\mathcal{S}(T)$.

In what follows we show that it is possible to modify the Olivier's condition $\sum_{n=1}^{\infty} a_n < +\infty$ in such a way that the ideal $\mathcal{I}_c^{(q)}$ will play the role of ideal $\mathcal{I}_c$ in Theorem B.

Lemma 3.1. *Let $0 < q \leq 1$. Then for every sequence $(a_n)_{n=1}^{\infty}$ such that $a_n > 0, n = 1, 2, \ldots$ and $\sum_{n=1}^{\infty} {a_n}^q < +\infty$ we have $\mathcal{I}_c^{(q)} - \lim n \cdot a_n = 0$.*

Proof. Let the conclusion of the Lemma 3.1 doesn't hold. Then there exists $\varepsilon_0 > 0$ such that the set $A(\varepsilon_0) = \{n : n \cdot a_n \geq \varepsilon_0\}$ doesn't belong to $\mathcal{I}_c^{(q)}$. Therefore

$$\sum_{k=1}^{\infty} m_k^{-q} = +\infty, \tag{3.2}$$

where $A(\varepsilon_0) = \{m_1 < m_2 < \ldots < m_k < \ldots\}$. By the definition of the set $A(\varepsilon_0)$ we have $m_k \cdot a_{m_k} \geq \varepsilon_0 > 0$, for each $k \in N$. From this $m_k^q \cdot a_{m_k}^q \geq \varepsilon_0^q > 0$ and so for each $k \in N$

$$a_{m_k}^q \geq \varepsilon_0^q \cdot m_k^{-q} \tag{3.3}$$

From (3.2) and (3.3) we get $\sum_{k=1}^{\infty} a_{m_k}^q = +\infty$, and hence $\sum_{n=1}^{\infty} a_n^q = +\infty$. But it contradicts the assumption of the theorem. □

Let's denote by $\mathcal{S}_q(T)$ the class of all admissible ideals $\mathcal{I}$ for which an analog Lemma 3.1 holds. From Lemma 2.2 we have:

Corollary 3.2. *If $\mathcal{I}$ is an admissible ideal such that $\mathcal{I} \supseteq \mathcal{I}_c^{(q)}$ then $\mathcal{I} \in \mathcal{S}_q(T)$.*

Main result of this section is the reverse of Corollary 3.2.

Theorem 3.3. *For any $q \in (0, 1\rangle$ the class $\mathcal{S}_q(T)$ consists of all admissible ideals such that $\mathcal{I} \supseteq \mathcal{I}_c^{(q)}$.*

Proof. It this sufficient to prove that for any infinite set $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \in \mathcal{I}_c^{(q)}$ we have $M \in \mathcal{I}$, too. Since $M \in \mathcal{I}_c^{(q)}$ we have

$$\sum_{k=1}^{\infty} m_k^{-q} < +\infty.$$

Now we define the sequence $(a_n)_{n=1}^{\infty}$ as follows

$$\begin{aligned} a_{m_k} &= \frac{1}{m_k} \quad (k = 1, 2, \ldots), \\ a_n &= \frac{1}{10^n} \quad \text{for} \quad n \in \mathbb{N} \setminus M. \end{aligned}$$

Obviously $a_n > 0$ and $\sum_{n=1}^{\infty} a_n{}^q < +\infty$ by the definition of numbers a_n. Since $\mathcal{I} \in \mathcal{S}_q(T)$ we have

$$\mathcal{I} - \lim n \cdot a_n = 0.$$

This implies that for each $\varepsilon > 0$ we have

$$A(\varepsilon) = \{n : n \cdot a_n \geq \varepsilon\} \in \mathcal{I},$$

in particular $M = A(1) \in \mathcal{I}$. □

4. $\mathcal{I}_c^{(q)}$-convergence and regular matrix transformations

$\mathcal{I}_c^{(q)}$-convergence is an example of a linear functional defined on a subspace of the space of all bounded sequences of real numbers. Another important family of such functionals are so called matrix summability methods inspired by [1, 6]. We will study connections between $\mathcal{I}_c^{(q)}$-convergence and one class of matrix summability methods. Let us start by introducing a notion of regular matrix transformation (see [4]).

Let $\mathbf{A} = (a_{nk})$ $(n, k = 1, 2, \ldots)$ be an infinite matrix of real numbers. The sequence $(t_n)_{n=1}^{\infty}$ of real numbers is said to be **A**-limitable to the number s if $\lim_{n\to\infty} s_n = s$, where

$$s_n = \sum_{k=1}^{\infty} a_{nk} t_k \quad (n = 1, 2, \ldots).$$

If $(t_n)_{n=1}^{\infty}$ is **A**-limitable to the number s, we write $\mathbf{A} - \lim_{n\to\infty} t_n = s$.

We denote by $F(\mathbf{A})$ the set of all **A**-limitable sequences. The set $F(\mathbf{A})$ is called the convergence field. The method defined by the matrix **A** is said *to be regular* provided that $F(\mathbf{A})$ contains all convergent sequences and $\lim_{n\to\infty} t_n = t$ implies $\mathbf{A} - \lim_{n\to\infty} t_n = t$. Then **A** is called a *regular matrix.*

It is well-known that the matrix **A** is regular if and only if satisfies the following three conditions (see [4]):

(A) $\exists K > 0, \forall n = 1, 2, \ldots \sum_{k=1}^{\infty} |a_{nk}| \leq K$;

(B) $\forall k = 1, 2, \ldots \lim_{n\to\infty} a_{nk} = 0$

(C) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} = 1$

Let‘s ask the question: Is there any connection between $\mathcal{I}$-convergence of sequence of real numbers and **A**-limit of this sequence? It is well know that a sequence $(x_k)_{k=1}^{\infty}$ of real numbers $\mathcal{I}_d$-converges to real number ξ if and only if the sequence is strongly summable to ξ in Caesaro sense. The complete characterization of statistical convergence ($\mathcal{I}_d$-convergence) is described by Fridy-Miller in the paper [6]. They defined a class of lower triangular nonnegative matrices $\mathcal{T}$ with properties:

$$\sum_{k=1}^{n} a_{nk} = 1 \quad \forall n \in \mathbb{N}$$

$$\text{if } \ C \subseteq \mathbb{N} \ \text{ such that } \ d(C) = 0, \quad \text{then } \lim_{n\to\infty} \sum_{k \in C} a_{nk} = 0.$$

They proved the following assertion:

Theorem C. *The bounded sequence $x = (x_n)_{n=1}^{\infty}$ is statistically convergent to L if and only if $x = (x_n)_{n=1}^{\infty}$ is $\boldsymbol{A}$-summable to L for every $\boldsymbol{A}$ in $\mathcal{T}$.*

Similar result for $\mathcal{I}_u$-convergence was shown by V. Baláž and T. Šalát in [1]. Here we prove analogous result for $\mathcal{I}_c^{(q)}$-convergence. Following this aim let’s define the class $\mathcal{T}_q$ lower triangular nonnegative matrices in this way:

Definition 4.1. Matrix $\mathbf{A} = (a_{nk})$ belongs to the class $\mathcal{T}_q$ if and only if it satisfies the following conditions:

(I) $\lim\limits_{n\to\infty} \sum_{k=1}^{n} a_{nk} = 1$

(q) If $C \subset \mathbb{N}$ and $C \in \mathcal{I}_c^{(q)}$, then $\lim\limits_{n\to\infty} \sum_{k \in C} a_{nk} = 0, \quad 0 < q \le 1.$

It is easy to see that every matrix of class $\mathcal{T}_q$ is regular. As the following example shows the converse does not hold.

Example 4.2. Let $C = \{1^2, 2^2, 3^2, 4^2, \ldots, n^2, \ldots\}$ and $q = 1$. Obviously $C \in \mathcal{I}_c^{(1)} = \mathcal{I}_c$. Now define the matrix **A** by:

$$a_{11} = 1, a_{1k} = 0, \quad k > 1$$

$$a_{nk} = \frac{1}{2k \cdot \ln n}, \quad k \ne l^2, k \le n$$

$$a_{nk} = \frac{1}{l \ln n}, \quad k = l^2, k \le n$$

$$a_{nk} = 0, \quad k > n$$

It is easy to show that **A** is lower triangular nonnegative regular matrix but does not satisfy the condition (q) from Definition 4.1.

$$\sum_{\substack{k<n^2 \\ k \in C}} a_{n^2 k} = \frac{1}{\ln n^2}(1 + \frac{1}{2} + \ldots + \frac{1}{n}) \ge \frac{\ln n}{2 \ln n} = \frac{1}{2} \nrightarrow 0$$

for $n \to \infty$. Therefore $\mathbf{A} \notin \mathcal{T}_1$.

Lemma 4.3. *If the bounded sequence $x = (x_n)_{n=1}^{\infty}$ is not $\mathcal{I}$-convergent then there exist real numbers $\lambda < \mu$ such that neither the set $\{n \in \mathbb{N} : x_n < \lambda\}$ nor the set $\{n \in \mathbb{N} : x_n > \mu\}$ belongs to ideal $\mathcal{I}$.*

As the proof is the same as the proof on Lemma in [6] we will omit it.

Next theorem shows connection between $\mathcal{I}_c^{(q)}$-convergence of bounded sequence of real numbers and **A**-summability of this sequence for matrices from the class $\mathcal{T}_q$.

Theorem 4.4. *Let $q \in (0,1\rangle$. Then the bounded sequence $x = (x_n)_{n=1}^{\infty}$ of real numbers $\mathcal{I}_c^{(q)}$-converges to $L \in \mathbb{R}$ if and only if it is $\boldsymbol{A}$-summable to $L \in \mathbb{R}$ for each matrix $\boldsymbol{A} \in \mathcal{T}_q$.*

Proof. Let $\mathcal{I}_c^{(q)} - \lim x_n = L$ and $\mathbf{A} \in \mathcal{T}_q$. As $\mathbf{A}$ is regular there exists a $K \in \mathbb{R}$ such that $\forall n = 1, 2, \ldots \sum_{k=1}^{\infty} |a_{nk}| \leq K$.

It is sufficient to show that $\lim\limits_{n\to\infty} b_n = 0$, where $b_n = \sum_{k=1}^{\infty} a_{nk}.(x_k - L)$. For $\varepsilon > 0$ put $B(\varepsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$. By the assumption we have $B(\varepsilon) \in \mathcal{I}_c^{(q)}$. By condition (q) from Definition 4.1 we have

$$\lim_{n\to\infty} \sum_{k\in B(\varepsilon)} |a_{nk}| = 0 \tag{4.1}$$

As the sequence $x = (x_n)_{n=1}^{\infty}$ is bounded, there exists $M > 0$ such that

$$\forall k = 1, 2, \ldots : |x_k - L| \leq M \tag{4.2}$$

Let $\varepsilon > 0$. Then

$$\begin{aligned} |b_n| &\leq \sum_{k\in B(\frac{\varepsilon}{2K})} |a_{nk}||x_k - L| \; + \sum_{k\notin B(\frac{\varepsilon}{2K})} |a_{nk}||x_k - L| \leq \\ &\leq M \sum_{k\in B(\frac{\varepsilon}{2K})} |a_{nk}| \; + \; \frac{\varepsilon}{2K} \sum_{k\notin B(\frac{\varepsilon}{2K})} |a_{nk}| \leq \\ &\leq M \sum_{k\in B(\frac{\varepsilon}{2K})} |a_{nk}| \; + \; \frac{\varepsilon}{2} \end{aligned} \tag{4.3}$$

By part (q) of Definition 4.1 there exists an integer n_0 such that for all $n > n_0$

$$\sum_{k\in B(\frac{\varepsilon}{2K})} |a_{nk}| < \frac{\varepsilon}{2M}$$

Together by (4.3) we obtain $\lim\limits_{n\to\infty} b_n = 0$.

Conversely, suppose that $\mathcal{I}_c^{(q)} - \lim x_n = L$ doesn't hold. We show that there exists a matrix $\mathbf{A} \in \mathcal{T}_q$ such that $\mathbf{A} - \lim\limits_{n\to\infty} x_n = L$ does not hold, too. If $\mathcal{I}_c^{(q)} - \lim x_n = L^{'} \neq L$ then from the firs part of proof it follows that $\mathbf{A} - \lim\limits_{n\to\infty} x_n = L^{'}$

$\neq L$ for any $\mathbf{A} \in \mathcal{T}_q$. Thus, we may assume that $(x_n)_{n=1}^{\infty}$ is not $\mathcal{I}_c^{(q)}$-convergent, and by the above Lemma 4.3 there exist λ and μ ($\lambda < \mu$), such that neither the set $U = \{k \in \mathbb{N} : x_k < \lambda\}$ nor $V = \{k \in \mathbb{N} : x_k > \mu\}$ belongs to the ideal $\mathcal{I}_c^{(q)}$. It is clear that $U \cap V = \emptyset$. If $U \notin \mathcal{I}_c^{(q)}$ then $\sum_{i\in U} i^{-q} = +\infty$ and if $V \notin \mathcal{I}_c^{(q)}$ then $\sum_{i\in V} i^{-q} = +\infty$. Let $U_n = U \cap \{1, 2, \ldots, n\}$ and $V_n = V \cap \{1, 2, \ldots, n\}$.

Now we define the matrix $\mathbf{A} = (a_{nk})$ by the following way: Let $s_{(1)n} = \sum_{i\in U_n} i^{-q}$ for $n \in U$, $s_{(2)n} = \sum_{i\in V_n} i^{-q}$ for $n \in V$ and $s_{(3)n} = \sum_{i=1}^{n} i^{-q}$ for $n \notin U \cap V$. As $U, V \notin \mathcal{I}_c^{(q)}$ we have $\lim_{n\to\infty} s_{(j)n} = +\infty, j = 1, 2, 3.$

$$a_{nk} = \begin{cases} a_{nk} = \frac{k^{-q}}{s_{(1)n}} & n \in U \text{ and } \ k \in U_n, \\ a_{nk} = 0 & n \in U \text{ and } \ k \notin U_n, \\ a_{nk} = \frac{k^{-q}}{s_{(2)n}} & n \in V \text{ and } \ k \in V_n, \\ a_{nk} = 0 & n \in V \text{ and } \ k \notin V_n, \\ a_{nk} = \frac{k^{-q}}{s_{(3)n}} & n \notin U \cap V, \\ a_{nk} = 0 & k > n, \end{cases}$$

Let's check that $\mathbf{A} \in \mathcal{T}_q$. Obviously $\mathbf{A}$ is a lower triangular nonnegative matrix. Condition (I) is clear from the definition of matrix $\mathbf{A}$. Condition (q): Let $B \in \mathcal{I}_c^{(q)}$ and $b = \sum_{k\in B} k^{-q} < +\infty$. Then

$$\sum_{k\in B} a_{nk} \leq \frac{1}{s_{(3)n}} \sum_{k\in B\cap\{1,\ldots,n\}} k^{-q}\chi_B(k) \leq \frac{b}{s_{(3)n}} \to 0$$

for $n \to \infty$. Thus $\mathbf{A} \in \mathcal{T}_q$.

For $n \in U$

$$\sum_{k=1}^{\infty} a_{nk}x_k = \frac{1}{s_{(1)n}} \sum_{k=1}^{n} k^{-q}\chi_U(k)x_k < \frac{\lambda}{s_{(1)n}} \sum_{k=1}^{n} k^{-q}\chi_U(k) = \lambda$$

on other hand for $n \in V$

$$\sum_{k=1}^{\infty} a_{nk}x_k = \frac{1}{s_{(2)n}} \sum_{k=1}^{n} k^{-q}\chi_V(k)x_k > \frac{\mu}{s_{(2)n}} \sum_{k=1}^{n} k^{-q}\chi_V(k) = \mu.$$

Therefore $\mathbf{A} - \lim_{n\to\infty} x_n$ does not exist. □

Corollary 4.5. *If* $0 < q_1 < q_2 \leq 1$, *then* $\mathcal{T}_{q_2} \subsetneqq \mathcal{T}_{q_1}$.

Proof. Let $B \in \mathcal{I}_c^{(q_2)} \setminus \mathcal{I}_c^{(q_1)}$ and let $(x_n) = \chi_B(n)$, $n = 1, 2, \ldots$ Clearly $\mathcal{I}_c^{(q_2)} - \lim x_n = 0$ and $\mathcal{I}_c^{(q_1)} - \lim x_n$ does not exist. Let $\mathbf{A}$ be the matrix constructed from the sequence $(x_n)_{n=1}^{\infty}$ as in the proof of Theorem 4.4. In particular $\mathbf{A} \in \mathcal{T}_{q_1}$ and $\mathbf{A} - \lim_{n\to\infty} x_n$ does not exist. Therefore $\mathbf{A} \notin \mathcal{T}_{q_2}$. □

Further we show some type well-known matrix which fulfills condition (I). Let $(p_j)_{j=1}^{\infty}$ be the sequence of positive real numbers. Put $P_n = p_1 + p_2 + \ldots + p_n$.

Now we define matrix $\mathbf{A} = (a_{nk})$ in this way:

$$a_{nk} = \frac{p_k}{P_n} \quad k \leq n$$

$$a_{nk} = 0 \quad k > n.$$

This type of matrix is called Riesz matrix.

Especially we put $p_n = n^{\alpha}$, where $0 < \alpha < 1$. Then

$$a_{nk} = \frac{k^{\alpha}}{1^{\alpha} + 2^{\alpha} + \ldots + n^{\alpha}} \quad k \leq n$$

$$a_{nk} = 0 \quad k > n.$$

This special class of matrix we denote by $(\mathbf{R}, n^{\alpha})$. It is clear that this matrix fulfills conditions (I) and (q). For this class of matrix is true following implication:

$$\mathcal{I}_c^{(q)} - \lim x_k = L \Rightarrow (\mathbf{R}, n^{\alpha}) - \lim x_k = L$$

where $(x_k)_{k=1}^{\infty}$ is a bounded sequence, $0 < q \leq 1$, $0 < \alpha < 1$. Converse does not hold. It is sufficient to choose the characteristic function of the set of all primes $\mathbb{P}$. Then $(\mathbf{R}, n^{\alpha}) - \lim x_k = 0$, but $\mathcal{I}_c^{(q)} - \lim x_k$ does not exist, because $\sum_{n \in \mathbb{P}} n^{-q} = +\infty$, where $\mathbb{P}$ is a se of all primes. Hence the class $(\mathbf{R}, n^{\alpha})$ of matrices belongs to $\mathcal{T} \setminus \mathcal{T}_q$.

Problem 4.6. If we take any admissible ideal $\mathcal{I}$ and define the class $\mathcal{T}_{\mathcal{I}}$ of matrices by replacing the condition (I) in Definition 4.1 by condition:if $C \subset \mathbb{N}$ and $C \in \mathcal{I}$, $\mathcal{I}$ admissible ideal on $\mathbb{N}$ then $\lim_{n \to \infty} \sum_{k \in C} |a_{nk}| = 0$ then it is easy to see that the if part of Theorem 4.4 holds for $\mathcal{I}$ too. The question is what about only if part.

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