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Simulation of vibrations of a rectangular membrane with random initial conditions

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Dedicated to Mátyás Arató on his eightieth birthday

Abstract

A new method is proposed in this paper to construct models for solutions of boundary-value problems for hyperbolic equations with random initial conditions. We assume that the initial conditions are strictly sub-Gaussian random fields (in particular, Gaussian random fields with zero mean). The models approximate solutions with a given accuracy and reliability in the uniform metric.

Keywords: Rectangular Membrane's Vibrations, Stochastic processes, Model of solution, Accuracy and Reliability.

MSC: 60G60; Secondary 60G15.

1. Introduction

We construct a model that approximates a solution of the boundary-value problem (2.1)–(2.3) for the hyperbolic equation with random initial conditions. The model is convenient to use when developing a software for computers. It approximates a solution with a given reliability and accuracy in the uniform metric.

We consider a strictly sub-Gaussian random field to model initial conditions in problem (2.1)–(2.3). Note that Gaussian fields are particular cases of strictly sub-Gaussian random fields.

It is known that a solution of the boundary-value problem can be represented, under certain conditions in the form of an infinite series, namely

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} V_{nm}(x,y)\left[a_{nm}\cos\sqrt{\lambda_{nm}}t + b_{nm}\sin\sqrt{\lambda_{nm}}t\right],$$

where $V_{nm}(x,y)$ are known functions and a_{nm} and b_{nm} are random variables whose joint distributions are known.

One can consider the following model for a solution of the boundary-value problem:

$$u(x,y,t,N)=\sum_{n=1}^{N}\sum_{m=1}^{N}V_{nm}(x,y)\left[a_{nm}\cos\sqrt{\lambda_{nm}}t+b_{nm}\sin\sqrt{\lambda_{nm}}t\right],$$

One can find the values of N for which $u(x,\ y,\ t,\ N)$ approximates the field $u(x;\ y,\ t)$ with a given reliability and accuracy.

The main disadvantage of this method is that the random variables a_{nm} and b_{nm} are independent only for very special initial conditions. Therefore it is practically impossible to apply this method for large N.

A new method is proposed in this paper to construct a model for a solution of the boundary problem (2.1)–(2.3). The idea of the method is, first, to model the initial conditions with a given accuracy and, second, to compute approximate values $\widetilde{a}_{nm}$ and $\widetilde{b}_{nm}$ of the coefficients a_{nm} and b_{nm}, respectively, by using the model for the initial conditions. The finite sum

$$\widetilde{u}(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}V_{nm}(x,y)\left[\widetilde{a}_{nm}\cos\sqrt{\lambda_{nm}}t+\widetilde{b}_{nm}\sin\sqrt{\lambda_{nm}}t\right],$$

is considered as a model for the solution. We find values of N and an accuracy of the approximation of a_{nm} and b_{nm} by $\widetilde{a}_{nm}$ and $\widetilde{b}_{nm}$ for which this model approximates the solution of the boundary-value problem with a given reliability and accuracy in the uniform metric.

Note the paper consists of five section. The main result, Theorem 2.2 is stated in Section 2. The proof of the theorem is given in Section 3, and some examples are considered in Section 4. The model of a solution of a hyperbolic type equation with random initial conditions was investigated in the paper [7].

Note that all the results of the paper hold for the case where the initial conditions are zero mean Gaussian random fields. Some methods to model Gaussian and sub-Gaussian random processes and random fields can be found in the articles [4], [5] and the book [3].

2. Main result

Consider the problem of vibrations of a rectangular membrane [8] $0<x<p$, $0<y<q$:

$$u_{xx}+u_{yy}=u_{tt}, \tag{2.1}$$

$$u|_{t=0}=\xi(x,y),\ \frac{\partial u}{\partial t}|_{t=0}=\eta(x,y), \tag{2.2}$$

$$u|_{s}=0, \tag{2.3}$$

where u is the deviation of the membrane from its equilibrium position, which coincides with the plane x, y, S is boundary of a rectangle $0 < x < p$, $0 < y < q$.

Let the initial conditions $\{\xi(x,y), x \in [0,p], y \in [0,q]\}$, $\{\eta(x,y), x \in [0,p], y \in [0,q]\}$ be an independent strictly sub-Gaussian stochastic processes (see [1]).

When solving problems similar (2.1)–(2.3) by Fourier's method, regardless of whether initial conditions are random or nonrandom, we look for a solution of the form

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} V_{nm}(x,y)\left[a_{nm}\cos\sqrt{\lambda_{nm}}t + b_{nm}\sin\sqrt{\lambda_{nm}}t\right], \tag{2.4}$$

where

$$a_{nm} = \int_0^p\int_0^q \xi(x,y)V_{nm}(x,y)dxdy,$$

$$b_{nm} = \frac{1}{\sqrt{\lambda_{nm}}}\int_0^p\int_0^q \eta(x,y)V_{nm}(x,y)dxdy,$$

λ_{nm} and $V_{nm}(x,y)$ are eigenvalues and eigenfunctions of the Sturm-Liouville problem [8]:

$$V_{xx} + V_{yy} + \lambda V = 0,$$

$$V|_s = 0.$$

where λ_{nm} and $V_{nm}(x,y)$ have the following forms

$$\lambda_{nm} = \pi^2\left(\frac{n^2}{p^2} + \frac{m^2}{q^2}\right),$$

$$V_{nm}(x,y) = \frac{2}{\sqrt{pq}}\sin\frac{n\pi}{p}x\sin\frac{m\pi}{q}y, \tag{2.5}$$

where $n, m = 1, 2, \ldots$

In the papers by [6] (see also [2]) the theorems are formulated according to the conditions of which series (2.4) is the solution of problem (2.1)–(2.3).

Let's construct a model for a solution of problem (2.1)–(2.3) approximating the solution with a given reliability and accuracy in the uniform metric.

Let $\{\widehat{\xi}(x,y), x \in [0,p], y \in [0,q]\}$ and $\{\widehat{\eta}(x,y), x \in [0,p], y \in [0,q]\}$ be models of processes $\{\xi(x,y), x \in [0,p], y \in [0,q]\}$ and $\{\eta(x,y), x \in [0,p], y \in [0,q]\}$, respectively. Note that the models $\widehat{\xi}(x,y)$ and $\widehat{\eta}(x,y)$ are independent stochastic processes.

Put

$$\widehat{a}_{nm} = \int_0^p\int_0^q \widehat{\xi}(x,y)V_{nm}(x,y)dxdy,$$

$$\widehat{b}_{nm} = \frac{1}{\sqrt{\lambda_{nm}}} \int_0^p \int_0^q \widehat{\eta}(x,y) V_{nm}(x,y) dx dy.$$

The sum

$$u^N(x,y,t) = \sum_{n=1}^{N} \sum_{m=1}^{N} V_{nm}(x,y) \left[\widehat{a}_{nm} \cos \sqrt{\lambda_{nm}} t + \widehat{b}_{nm} \sin \sqrt{\lambda_{nm}} t \right] \tag{2.6}$$

is called a model of the process $u(x,y,t)$.

Definition 2.1. Let a solution $u(x,y,t)$ of problem (2.1)–(2.3) be represented in the form of series (2.4). We say that a model $u^N(x,y,t)$ approximates $u(x,y,t)$ with a given reliability $1-\gamma$ and accuracy δ in the uniform metric in the domain $D = [0,p] \times [0,q] \times [0,T]$ if

$$P \left\{ \sup_{(x,y,t) \in D} |u^N(x,y,t) - u(x,y,t)| > \delta \right\} \leq \gamma.$$

Put

$$\Delta_N(x,y,t,N) = u(x,y,t) - u^N(x,y,t) = u_N(x,y,t) + V_N(x,y,t),$$

where

$$u_N(x,y,t) = \sum_{n=N+1}^{\infty} \sum_{m=N+1}^{\infty} V_{nm}(x,y) \left[a_{nm} \cos \sqrt{\lambda_{nm}} t + b_{nm} \sin \sqrt{\lambda_{nm}} t \right],$$

$$V_N(x,y,t) = \sum_{n=1}^{N} \sum_{m=1}^{N} V_{nm}(x,y) \Big[(\widehat{a}_{nm} - a_{nm}) \cos \sqrt{\lambda_{nm}} t + (\widehat{b}_{nm} - b_{nm}) \sin \sqrt{\lambda_{nm}} t \Big].$$

Below is the main result of the paper.

Theorem 2.2. *Let $\{\xi(x,y), x \in [0,p], y \in [0,q]\}$ and $\{\eta(x,y), x \in [0,p], y \in [0,q]\}$ be independent $SSub(\Omega)$ processes. Let the models $\{\widehat{\xi}(x,y), x \in [0,p], y \in [0,q]\}$ and $\{\widehat{\eta}(x,y), x \in [0,p], y \in [0,q]\}$ be such that*

$$\frac{1}{\sqrt{pq}} \int_0^p \int_0^q \sqrt{E \left(\widehat{\xi}(x,y) - \xi(x,y) \right)^2} dx dy \leq \Lambda,$$

$$\frac{1}{\sqrt{pq}} \int_0^p \int_0^q \sqrt{E \left(\widehat{\eta}(x,y) - \eta(x,y) \right)^2} dx dy \leq \Lambda.$$

Then the stochastic process $u^N(x,y,t)$ defined by (2.6), is a model of the stochastic process $u(x,y,t)$ that approximates it with reliability $1-\gamma$ and accuracy δ in the uniform metric in the domain $D = [0,p] \times [0,q] \times [0,T]$ if γ and N a such that

$$\left(T^{1/2} + p^{1/2} + q^{1/2} \right) A_N^2 \epsilon_0^2(N) < \delta,$$

$$\frac{1}{2}\left(\frac{\delta^{1/3}\left(\delta^{2/3}-3\left(T^{1/2}+p^{1/2}+q^{1/2}\right)^{\frac{2}{3}}A_N^{1/3}\epsilon_0^{1/3}(N)\right)}{\epsilon_0(N)}\right)^2 \geq \ln\frac{1}{\gamma},$$

where

$$A_N = \frac{2\pi}{p^{3/2}q^{3/2}}\left\{\left(\left(\sum_{n=N+1}^{\infty}\sum_{m=N+1}^{\infty}\sqrt{Ea_{nm}^2}(nq+mp)\right)^2 + \left(\sum_{n=N+1}^{\infty}\sum_{m=N+1}^{\infty}\sqrt{Eb_{nm}^2}(nq+mp)\right)^2\right)^{1/2} + 2\Lambda\left(\left(\sum_{n=1}^{N}\sum_{m=1}^{N}(nq+mp)\right)^2 + \left(\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{pq}{\pi\sqrt{n^2q^2+m^2p^2}}(nq+mp)\right)^2\right)^{1/2}\right\},$$

$$\epsilon_0(N) = \frac{4}{\sqrt{pq}}\left\{\left(\left(\sum_{n=N+1}^{\infty}\sum_{m=N+1}^{\infty}\sqrt{Ea_{nm}^2}\right)^2 + \left(\sum_{n=N+1}^{\infty}\sum_{m=N+1}^{\infty}\sqrt{Eb_{nm}^2}\right)^2\right)^{1/2} + \Lambda\left(N^4 + \left(\frac{1}{\pi^2}\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{pq}{\sqrt{n^2p^2+m^2q^2}}\right)^2\right)^{1/2}\right\}.$$

Remark 2.3. If the conditions of Theorem 4.3 in the paper [6] are hold true the series in definitions A_N and $\varepsilon_0(N)$ will converge.

3. Proof of Theorem 2.2

Since $\Delta_N(x,y,t,N)$ is a strictly sub-Gaussian stochastic process, we apply the result of the paper [6], and conclude that

$$P\left\{\sup_{(x,y,t)\in D}|\Delta_N(x,y,t,N)| > \delta\right\} \leq 2\widetilde{A}(\delta,\theta), \tag{3.1}$$

for all $0 < \theta < 1$, where

$$\widetilde{A}(\delta,0) = \exp\left\{-\frac{\left(\delta(1-\theta)-\frac{2}{\theta}I(\theta\epsilon_0)\right)^2}{2\epsilon_0^2}\right\}, \tag{3.2}$$

ϵ_0 is an arbitrary number such that

$$\epsilon_0 \geq \sup_{(x,y,t)\in D}\left(E|\Delta_N(x,y,t,N)|^2\right)^{1/2},$$

$$I(\theta\epsilon_0) = \frac{1}{\sqrt{2}} \int_0^{\theta\epsilon_0} \left(\ln\left(\frac{p}{2\sigma^{-1}(x)} + 1 \right) + \ln\left(\frac{q}{2\sigma^{-1}(x)} + 1 \right) \right.$$
$$\left. + \ln\left(\frac{T}{2\sigma^{-1}(x)} + 1 \right) \right)^{1/2} dx, \tag{3.3}$$

and where $\sigma(h)$ is a continuous increasing function such that $\sigma(0) = 0$ and

$$\sup_{\substack{|x-x_1|\le h\\|y-y_1|\le h\\|t-t_1|\le h}} \left(E|\Delta_N(x,y,t,N) - \Delta_N(x_1,y_1,t_1,N)|^2\right)^{1/2} \le \sigma(h),$$

$$\sup_{\substack{|x-x_1|\le h\\|y-y_1|\le h\\|t-t_1|\le h}} \left(E|u_N(x,y,t) + V_N(x,y,t) - u_N(x_1,y_1,t_1) - V_N(x_1,y_1,t_1)|^2\right)^{1/2}$$
$$\le \sup_{\substack{|x-x_1|\le h\\|y-y_1|\le h\\|t-t_1|\le h}} \left[\left(E|u_N(x,y,t) - u_N(x_1,y_1,t_1)|^2\right)^{1/2} \right.$$
$$\left. + \left(E|V_N(x,y,t) - V_N(x_1,y_1,t_1)|^2\right)^{1/2} \right],$$

$$\sup_{(x,y,t)\in D} \left(E|\Delta_N(x,y,t,N)|^2\right)^{1/2} = \sup_{(x,y,t)\in D} \left(E|u_N(x,y,t) + V_N(x,y,t)|^2\right)^{1/2}$$
$$\le \sup_{(x,y,t)\in D} \left[\left(E|u_N(x,y,t)|^2\right)^{1/2} + \left(E|V_N(x,y,t)|^2\right)^{1/2} \right].$$

Since the stochastic processes $\xi(x,y)$ and $\eta(x,y)$ are independent, that is, a_{nm} and b_{nm} are independent, we obtain

$$E|u_N(x,y,t) - u_N(x_1,y_1,t_1)|^2$$
$$= E\left| \sum_{n=N+1}^{\infty} \sum_{m=N+1}^{\infty} V_{nm}(x,y) \left[a_{nm} \cos\sqrt{\lambda_{nm}} t + b_{nm} \sin\sqrt{\lambda_{nm}} t \right] \right.$$
$$\left. - \sum_{n=N+1}^{\infty} \sum_{m=N+1}^{\infty} V_{nm}(x_1,y_1) \left[a_{nm} \cos\sqrt{\lambda_{nm}} t_1 + b_{nm} \sin\sqrt{\lambda_{nm}} t_1 \right] \right|^2$$
$$= E\left| \sum_{n=N+1}^{\infty} \sum_{m=N+1}^{\infty} a_{nm} \frac{2}{\sqrt{pq}} \left[\sin\frac{n\pi}{p} x \sin\frac{m\pi}{q} y \cos\sqrt{\lambda_{nm}} t \right. \right.$$
$$\left. - \sin\frac{n\pi}{p} x_1 \sin\frac{m\pi}{q} y_1 \cos\sqrt{\lambda_{nm}} t_1 \right]$$
$$+ \sum_{n=N+1}^{\infty} \sum_{m=N+1}^{\infty} b_{nm} \frac{2}{\sqrt{pq}} \left[\sin\frac{n\pi}{p} x \sin\frac{m\pi}{q} y \sin\sqrt{\lambda_{nm}} t \right.$$
$$\left. \left. - \sin\frac{n\pi}{p} x_1 \sin\frac{m\pi}{q} y_1 \sin\sqrt{\lambda_{nm}} t_1 \right] \right|^2$$

$$
\begin{aligned}
&\leq \frac{4}{pq} \sum_{n=N+1}^{\infty} \sum_{m=N+1}^{\infty} \sum_{k=N+1}^{\infty} \sum_{l=N+1}^{\infty} |Ea_{nm}a_{kl}| \cdot \left| \sin \frac{n\pi}{p} x \sin \frac{m\pi}{q} y \cos \sqrt{\lambda_{nm}} t \right. \\
&\left. - \sin \frac{n\pi}{p} x_1 \sin \frac{m\pi}{q} y_1 \cos \sqrt{\lambda_{nm}} t_1 \right| \cdot \left| \sin \frac{k\pi}{p} x \sin \frac{l\pi}{q} y \cos \sqrt{\lambda_{kl}} t \right. \\
&\left. - \sin \frac{k\pi}{p} x_1 \sin \frac{l\pi}{q} y_1 \cos \sqrt{\lambda_{kl}} t_1 \right| \\
&+ \frac{4}{pq} \sum_{n=N+1}^{\infty} \sum_{m=N+1}^{\infty} \sum_{k=N+1}^{\infty} \sum_{l=N+1}^{\infty} |Eb_{nm}b_{kl}| \cdot \left| \sin \frac{n\pi}{p} x \sin \frac{m\pi}{q} y \sin \sqrt{\lambda_{nm}} t \right. \\
&\left. - \sin \frac{n\pi}{p} x_1 \sin \frac{m\pi}{q} y_1 \sin \sqrt{\lambda_{nm}} t_1 \right| \cdot \left| \sin \frac{k\pi}{p} x \sin \frac{l\pi}{q} y \sin \sqrt{\lambda_{kl}} t \right. \\
&\left. - \sin \frac{k\pi}{p} x_1 \sin \frac{l\pi}{q} y_1 \sin \sqrt{\lambda_{kl}} t_1 \right| \\
&= \frac{4}{pq} \left(\sum_{n=N+1}^{\infty} \sum_{m=N+1}^{\infty} \sqrt{Ea_{nm}^2} \left| \sin \frac{n\pi}{p} x \sin \frac{m\pi}{q} y \cos \sqrt{\lambda_{nm}} t \right. \right. \\
&\left. \left. - \ \sin \frac{n\pi}{p} x_1 \sin \frac{m\pi}{q} y_1 \cos \sqrt{\lambda_{nm}} t_1 \right| \right)^2 \\
&+ \frac{4}{pq} \left(\sum_{n=N+1}^{\infty} \sum_{m=N+1}^{\infty} \sqrt{Eb_{nm}^2} \left| \sin \frac{n\pi}{p} x \sin \frac{m\pi}{q} y \sin \sqrt{\lambda_{nm}} t \right. \right. \\
&\left. \left. - \ \sin \frac{n\pi}{p} x_1 \sin \frac{m\pi}{q} y_1 \sin \sqrt{\lambda_{nm}} t_1 \right| \right)^2 .
\end{aligned}
$$

It is easy to check that

$$
\begin{aligned}
&\left| \sin \frac{n\pi}{p} x \sin \frac{m\pi}{q} y \cos \sqrt{\lambda_{nm}} t - \sin \frac{n\pi}{p} x_1 \sin \frac{m\pi}{q} y_1 \cos \sqrt{\lambda_{nm}} t_1 \right| \\
&\leq \left| \sin \frac{n\pi}{p} x - \sin \frac{n\pi}{p} x_1 \right| + \left| \sin \frac{m\pi}{q} y - \sin \frac{m\pi}{q} y_1 \right| + \left| \cos \sqrt{\lambda_{nm}} t - \cos \sqrt{\lambda_{nm}} t_1 \right| \\
&\leq 2 \left| \sin \frac{\frac{n\pi}{p}(x - x_1)}{2} \right| + 2 \left| \sin \frac{\frac{m\pi}{q}(y - y_1)}{2} \right| + 2 \left| \sin \frac{\sqrt{\lambda_{nm}}(t - t_1)}{2} \right| \\
&\leq \frac{n\pi}{p} h + \frac{m\pi}{q} h + \sqrt{\lambda_{nm}} h = \pi h \left(\frac{n}{p} + \frac{m}{q} + \sqrt{\frac{n^2}{p^2} + \frac{m^2}{q^2}} \right) \\
&\leq 2\pi h \left(\frac{n}{p} + \frac{m}{q} \right) = 2\pi h \left(\frac{nq + pm}{pq} \right).
\end{aligned}
$$

Similarly

$$
\left| \sin \frac{n\pi}{p} x \sin \frac{m\pi}{q} y \sin \sqrt{\lambda_{nm}} t - \sin \frac{n\pi}{p} x_1 \sin \frac{m\pi}{q} y_1 \sin \sqrt{\lambda_{nm}} t_1 \right|
$$

$$\leq 2\pi h\left(\frac{n}{p}+\frac{m}{q}\right)=\frac{2\pi h}{pq}(nq+mp).$$

Then

$$\begin{aligned}&\left(E|u_N(x,y,t)-u_N(x_1,y_1,t_1)|^2\right)^{1/2}\\&\leq\left(\frac{4}{pq}\left(\sum_{n=N+1}^{\infty}\sum_{m=N+1}^{\infty}\sqrt{Ea_{nm}^2}\left(\frac{2\pi h}{pq}(nq+mp)\right)\right)^2\right.\\&\quad\left.+\frac{4}{pq}\left(\sum_{n=N+1}^{\infty}\sum_{m=N+1}^{\infty}\sqrt{Eb_{nm}^2}\left(\frac{2\pi h}{pq}(nq+mp)\right)\right)^2\right)^{1/2}\\&=\frac{4\pi h}{p^{\frac{3}{2}}q^{\frac{3}{2}}}\left(\left(\sum_{n=N+1}^{\infty}\sum_{m=N+1}^{\infty}\sqrt{Ea_{nm}^2}(nq+mp)\right)^2\right.\\&\quad\left.+\left(\sum_{n=N+1}^{\infty}\sum_{m=N+1}^{\infty}\sqrt{Eb_{nm}^2}(nq+mp)\right)^2\right)^{1/2}.\end{aligned}\tag{3.4}$$

We also have

$$\begin{aligned}&\left(E|V_N(x,y,t)-V_N(x_1,y_1,t_1)|^2\right)^{1/2}\\&\leq\frac{4\pi h}{p^{\frac{3}{2}}q^{\frac{3}{2}}}\left(\left(\sum_{n=1}^{N}\sum_{m=1}^{N}\sqrt{E(\widehat{a}_{nm}-a_{nm})^2}(nq+mp)\right)^2\right.\\&\quad\left.+\left(\sum_{n=1}^{N}\sum_{m=1}^{N}\sqrt{E(\widehat{b}_{nm}-b_{nm})^2}(nq+mp)\right)^2\right)^{1/2}.\end{aligned}$$

One can easily obtain that

$$\begin{aligned}E(\widehat{a}_{nm}-a_{nm})^2&=E\left(\int_0^p\int_0^q\left(\widehat{\xi}(x,y)-\xi(x,y)\right)V_{nm}(x,y)dxdy\right)^2\\&=E\left(\frac{2}{\sqrt{pq}}\int_0^p\int_0^q\left(\widehat{\xi}(x,y)-\xi(x,y)\right)\sin\frac{n\pi}{p}x\sin\frac{m\pi}{q}y\right)^2\\&\leq\left(\frac{2}{\sqrt{pq}}\int_0^p\int_0^q\sqrt{E\left(\widehat{\xi}(x,y)-\xi(x,y)\right)^2}dxdy\right)^2\leq 4\Lambda^2.\end{aligned}\tag{3.5}$$

Similarly

$$E(\widehat{b}_{nm}-b_{nm})^2=4\Lambda^2\frac{p^2q^2}{\pi^2(n^2q^2+m^2p^2)}.\tag{3.6}$$

Then

$$\left[|V_N(x,y,t)-V_N(x_1,y_1,t_1)|^2\right]^{1/2} \le \frac{4\pi h}{p^{3/2}q^{3/2}}\left(\left(\sum_{n=1}^{N}\sum_{m=1}^{N}2\Lambda(nq+mp)\right)^2 + \left(\sum_{n=1}^{N}\sum_{m=1}^{N}2\Lambda\frac{pq}{\pi\sqrt{n^2q^2+m^2p^2}}(nq+mp)\right)^2\right)^{1/2}. \quad (3.7)$$

Thus we obtain from (3.4), (3.5), (3.6) and (3.7) that $\sigma(h)=hA_N$, where

$$A_N = \frac{2\pi}{p^{3/2}q^{3/2}}\left\{\left(\left(\sum_{n=N+1}^{\infty}\sum_{m=N+1}^{\infty}\sqrt{Ea_{nm}^2}(nq+mp)\right)^2 + \left(\sum_{n=N+1}^{\infty}\sum_{m=N+1}^{\infty}\sqrt{Eb_{nm}^2}(nq+mp)\right)^2\right)^{1/2} + 2\Lambda\left(\left(\sum_{n=1}^{N}\sum_{m=1}^{N}(nq+mp)\right)^2 + \left(\sum_{n=1}^{N}\sum_{m=1}^{N}\frac{pq}{\pi\sqrt{n^2q^2+m^2p^2}}(nq+mp)\right)^2\right)^{1/2}\right\}.$$

It is easy to see that

$$\begin{aligned}
&E|u_N(x,y,t)|^2 \\
&\le E\left|\sum_{n=N+1}^{\infty}\sum_{m=N+1}^{\infty}V_{nm}(x,y)\left[a_{nm}\cos\sqrt{\lambda_{nm}}t+b_{nm}\sin\sqrt{\lambda_{nm}}t\right]\right|^2 \\
&= \left|\sum_{n=N+1}^{\infty}\sum_{m=N+1}^{\infty}\sum_{k=N+1}^{\infty}\sum_{l=N+1}^{\infty}V_{nm}(x,y)V_{kl}(x,y)\left[Ea_{nm}a_{kl}\cos\sqrt{\lambda_{nm}}\cos\sqrt{\lambda_{kl}}t\right.\right. \\
&\qquad \left.\left. +\ Eb_{nm}b_{kl}\sin\sqrt{\lambda_{nm}}\sin\sqrt{\lambda_{kl}}t\right]\right| \\
&\le \frac{4}{pq}\left(\left(\sum_{n=N+1}^{\infty}\sum_{m=N+1}^{\infty}\sqrt{Ea_{nm}^2}\right)^2 + \left(\sum_{n=N+1}^{\infty}\sum_{m=N+1}^{\infty}\sqrt{Eb_{nm}^2}\right)\right) \qquad (3.8)
\end{aligned}$$

and

$$\begin{aligned}
&E|V_N(x,y,t)|^2 \\
&\le \frac{4}{pq}\left(\left(\sum_{n=1}^{N}\sum_{m=1}^{N}\sqrt{E\left(\widehat{a}_{nm}-a_{nm}\right)^2}\right)^2 + \left(\sum_{n=1}^{N}\sum_{m=1}^{N}\sqrt{E\left(\widehat{b}_{nm}-b_{nm}\right)^2}\right)^2\right). \qquad (3.9)
\end{aligned}$$

Thus we obtain from (3.8) and (3.9) that

$$\epsilon_0(N) = \frac{4}{\sqrt{pq}} \left\{ \left(\left(\sum_{n=N+1}^{\infty} \sum_{m=N+1}^{\infty} \sqrt{a_{nm}^2} \right)^2 + \left(\sum_{n=N+1}^{\infty} \sum_{m=N+1}^{\infty} \sqrt{b_{nm}^2} \right)^2 \right)^{1/2} + \Lambda \left(N^4 + \left(\frac{1}{\pi^2} \sum_{n=1}^{N} \sum_{m=1}^{N} \frac{pq}{\sqrt{n^2p^2 + m^2q^2}} \right)^2 \right)^{1/2} \right\}.$$

Substituting these values of $\sigma(h)$ and $\epsilon_0(N)$ in equality (3.3), we get for $z = \theta\epsilon_(N)$ that

$$\begin{aligned} I(z) &= \frac{1}{\sqrt{2}} \int_0^z \left(\ln\left(\frac{pA_N}{2x}+1\right) + \ln\left(\frac{qA_N}{2x}+1\right) + \ln\left(\frac{TA_N}{2x}+1\right) \right)^{1/2} dx \\ &\le \frac{1}{\sqrt{2}} \int_0^z \left(\ln\left(\frac{pA_N}{2x}+1\right) \right)^{1/2} dx + \frac{1}{\sqrt{2}} \int_0^z \left(\ln\left(\frac{qA_N}{2x}+1\right) \right)^{1/2} dx \\ &+ \frac{1}{\sqrt{2}} \int_0^z \left(\ln\left(\frac{TA_N}{2x}+1\right) \right)^{1/2} dx \\ &\le \left[\int_0^z \left(\frac{pA_N}{2x}\right)^{1/2} dx + \int_0^z \left(\frac{qA_N}{2x}\right)^{1/2} dx + \int_0^z \left(\frac{TA_N}{2x}\right)^{1/2} dx \right] \\ &= \left(T^{1/2}+p^{1/2}+q^{1/2}\right) A_N^{1/2} z_0^{1/2}(N), \end{aligned}$$

Then equality (3.2) can be rewritten as

$$\widetilde{A}(\delta,\theta) \le exp\left\{ -\frac{1}{2} \left(\frac{\delta(1-\theta) - \frac{2}{\theta^{1/2}} \left(T^{1/2}+p^{1/2}+q^{1/2}\right) A_N^{1/2} \epsilon_0^{\frac{1}{2}}(N)}{\epsilon_0(N)} \right)^2 \right\}.$$

If

$$\left(T^{1/2}+p^{1/2}+q^{1/2}\right) A_N^2 \epsilon_0^2(N) < \delta,$$

then $\widetilde{A}(\delta,\theta)$ attains its minimum at

$$\theta = \frac{\left(T^{1/2}+p^{1/2}+q^{1/2}\right)^{2/3} A_N^{1/3} \epsilon_0^{1/3}(N)}{\delta^{2/3}}.$$

Namely

$$\min_\theta \widetilde{A}(\delta,\theta)$$

$$= \exp\left\{-\frac{1}{2}\left(\frac{\delta^{1/3}\left(\delta^{2/3} - 3\left(T^{1/2}+p^{1/2}+q^{1/2}\right)^{\frac{2}{3}} A_N^{1/3}\epsilon_0^{1/2}(N)\right)}{\epsilon_0(N)}\right)^2\right\} \geq \ln\frac{1}{\gamma},$$

Therefore, given an accuracy δ, one can construct a model with reliability $1-\gamma$ if

$$\left(T^{1/2}+p^{1/2}+q^{1/2}\right) A_N^2 \epsilon_0^2(N) < \delta,$$

$$\frac{1}{2}\left(\frac{\delta^{1/3}\left(\delta^{2/3} - 3\left(T^{1/2}+p^{1/2}+q^{1/2}\right) A_N^{1/3}\epsilon_0^N\right)^2}{\epsilon_0(N)}\right)^2 \geq \ln\frac{1}{\gamma}.$$

4. Example

Let $\eta(x,y)=0$, $p=q=\pi$, $T=\pi$, then the solution of problem (2.1)–(2.3) may be represented as:

$$u(x,y,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\sum_{m=1}^{N} a_{nm}\sin nx \sin my \cos\left(\sqrt{n^2+m^2}t\right).$$

Let's construct a model of the solution in the form:

$$\hat{u}^N(x,y,t) = \frac{2}{\pi}\sum_{n=1}^{N}\sum_{m=1}^{N} \hat{a}_{nm}\sin nx \sin my \cos\left(\sqrt{n^2+m^2}t\right).$$

Let the assumptions of Theorem 2.2 hold and let $\xi(x,y)$ be a Gaussian stochastic process such that

$$\xi(x,y) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \xi_{ij}\sin(i\,(x))\sin(j\,(y)),$$

where ξ_{ij} are independent Gaussian random variable with $E\xi_{ij}=0$, $E\xi_{ij}^2 = d^{ij}$. Here d^{ij} is a number such that $0 < d^{ij} < 1$. Let

$$\hat{\xi}(x,y) = \hat{\xi}_M(x,y) = \sum_{i=1}^{M}\sum_{i=1}^{M} \xi_{ij}\sin(i\,(x))\sin(j\,(y)).$$

Then

$$E\left(\xi(x,y) - \hat{\xi}_M(x,y)\right)^2 = \sum_{i=M+1}^{\infty}\sum_{j=M+1}^{\infty} d^{ij}\sin^2(i\,(x))\sin^2(j\,(y))$$

$$\leq \sum_{i=M+1}^{\infty}\sum_{j=M+1}^{\infty} d^{ij} = \sum_{i=M+1}^{\infty}\frac{d^{i(M+1)}}{1-d^i} \leq \frac{1}{1-d}\sum_{i=M+1}^{\infty} d^{i(M+1)}$$

$$= \frac{1}{1-d} \cdot \frac{d^{(M+1)^2}}{1-d^{M+1}} \leq \frac{d^{(M+1)^2}}{(1-d)^2}.$$

Note that given Λ, we chose M such that

$$\frac{1}{\pi} \int_0^\pi \int_0^\pi \sqrt{E\left(\xi(x,y) - \hat{\xi}_M(x,y)\right)^2} dxdy \leq \pi \sqrt{\frac{d^{(M+1)^2}}{(1-d)^2}} < \Lambda,$$

$$\frac{d^{(M+1)^2}}{(1-d)^2} < \frac{\Lambda^2}{\pi^2}$$

therefore,

$$M \geq \sqrt{\frac{\ln\left(\frac{\lambda^2}{\pi^2}(1-d)^2\right)}{\ln d}}.$$

In this case $b_{nm} = 0$,

$$a_{nm} = \int_0^\pi \int_0^\pi \xi(x,y) V_{nm}(x,y) dxdy = \frac{2}{\pi} \int_0^\pi \int_0^\pi \xi(x,y) \sin nx \sin my dxdy = 2\pi \xi_{nm},$$

that $Ea_{nm}^2 = 4\pi^2 d^{nm}$.

$$\hat{u}^N(x,y,t) = \frac{2}{\pi} \sum_{n=1}^N \sum_{m=1}^N \hat{a}_{nm} \sin nx \sin my \cos\left(\sqrt{n^2+m^2}t\right).$$

Thus

$$A_N = \frac{2}{\pi}\left\{2\pi \sum_{n=N+1}^\infty \sum_{m=N+1}^\infty \sqrt{d^{nm}}\,(n+m) + \Lambda \sum_{n=1}^N \sum_{m=1}^N (n+m)\right\}$$
$$\leq \frac{2}{\pi}\left\{\frac{4\pi d^{\frac{(N+1)^2}{2}}\left(1+N+Nd^{\frac{N+1}{2}}\right)}{(1-d)(d^{\frac{N+1}{2}})^2} + \Lambda(1+N)N^2\right\},$$

$$\epsilon_0 = \frac{4}{\pi^2}\left\{2\pi \sum_{n=N+1}^\infty \sum_{m=N+1}^\infty \sqrt{d^{nm}} + \Lambda N^2\right\} \leq \frac{4}{\pi}\left\{\frac{2\pi d^{\frac{(N+1)2}{2}}}{(1-d)(1-d^{\frac{N+1}{2}})} + \Lambda N^2\right\}.$$

So, we have received the model, where N and Λ satisfy the following inequality

$$A_N^2 \epsilon_0^2(N) < \frac{\delta}{3\sqrt{\pi}},$$

$$\left(\frac{\delta^{1/3}\left(\delta^{2/3} - (243\pi A_N \epsilon_0(N))^{1/3}\right)}{\epsilon_0(N)}\right)^2 \geq 2\ln\left(\frac{1}{\gamma}\right).$$

When some $\Lambda = 0.005$ and $N = 36$ the model $\hat{u}^N(x,y,t)$ approaches the random process $u(x,y,t)$ to reliability 0.99 and accuracy 0.01 in the uniform metric.

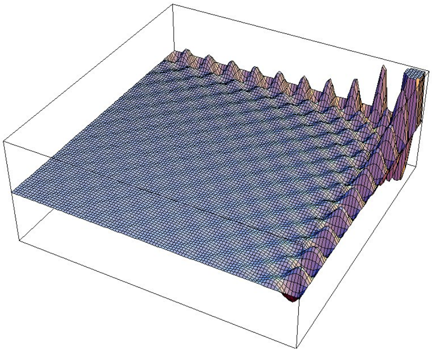

Figure 1: The model of membrane's vibration at the moment of time $t = 0$

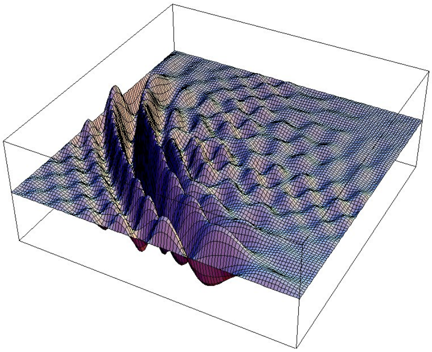

Figure 2: The model of membrane's vibration at the moment of time $t = 1$

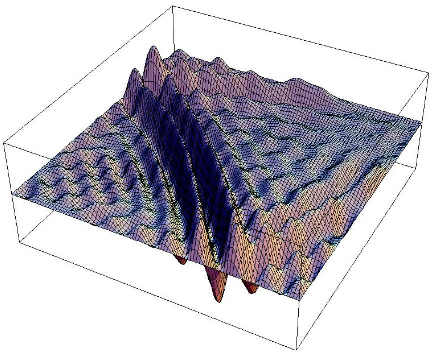

Figure 3: The model of membrane's vibration at the moment of time $t = 2$

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