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Signed (k, k)-domatic number of a graph

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Abstract

Let G be a finite and simple graph with vertex set $V(G)$, and let $f\colon V(G) \to \{-1, 1\}$ be a two-valued function. If $k \geqslant 1$ is an integer and $\sum_{x\in N[v]} f(x) \geqslant k$ for each $v \in V(G)$, where $N[v]$ is the closed neighborhood of v, then f is a signed k-dominating function on G. A set $\{f_1, f_2, \dots, f_d\}$ of signed k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(x) \leqslant k$ for each $x \in V(G)$, is called a signed (k, k)-dominating family (of functions) on G. The maximum number of functions in a signed (k, k)-dominating family on G is the signed (k, k)-domatic number on G, denoted by $d_S^k(G)$.

In this paper we initiate the study of the signed (k, k)-domatic number, and we present different bounds on $d_S^k(G)$. Some of our results are extensions of well-known properties of the signed domatic number $d_S(G) = d_S^1(G)$.

Keywords: Signed (k, k)-domatic number, signed k-dominating function, signed k-domination number

MSC: 05C69

1. Terminology and introduction

Various numerical invariants of graphs concerning domination were introduced by means of dominating functions and their variants. In this paper we define the *signed (k, k)-domatic number* in an analogous way as Volkmann and Zelinka [6] have introduced the signed domatic number.

We consider finite, undirected and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. The cardinality of the vertex set of a graph G is called the *order* of G and is denoted by $n(G)$. If $v \in V(G)$, then $N(v)$ is the *open neighborhood* of v, i.e., the set of all vertices adjacent to v. The *closed neighborhood* $N[v]$ of a vertex

v consists of the vertex set $N(v) \cup \{v\}$. The number $d(v) = |N(v)|$ is the *degree* of the vertex v. The *minimum* and *maximum degree* of a graph G are denoted by $\delta(G)$ and $\Delta(G)$. The *complement* of a graph G is denoted by $\overline{G}$. We write K_n for the *complete graph* of order n and C_n for a *cycle* of length n. A *fan* and a *wheel* is a graph obtained from a path and a cycle by adding a new vertex and edges joining it to all the vertices of the path and cycle, respectively. If $A \subseteq V(G)$ and f is a mapping from $V(G)$ into some set of numbers, then $f(A) = \sum_{x \in A} f(x)$.

If $k \geqslant 1$ is an integer, then the *signed k-dominating function* is defined in [7] as a two-valued function $f\colon V(G) \to \{-1, 1\}$ such that $\sum_{x \in N[v]} f(x) \geqslant k$ for each $v \in V(G)$. The sum $f(V(G))$ is called the weight $w(f)$ of f. The minimum of weights $w(f)$, taken over all signed k-dominating functions f on G, is called the *signed k-domination number* of G, denoted by $\gamma_{kS}(G)$. As the assumption $\delta(G) \geqslant k - 1$ is necessary, we always assume that when we discuss $\gamma_{kS}(G)$, all graphs involved satisfy $\delta(G) \geqslant k - 1$ and thus $n(G) \geqslant k$. The special case $k = 1$ was defined and investigated in [1]. Further information on $\gamma_{1S}(G) = \gamma_S(G)$ can be found in the monographs [2] and [3] by Haynes, Hedetniemi, and Slater.

Rall [4] has defined a variant of the domatic number of G, namely the fractional domatic number of G, using functions on $V(G)$. Analogous to the fractional domatic number we may define the signed (k, k)-domatic number.

A set $\{f_1, f_2, \ldots, f_d\}$ of signed k-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(x) \leqslant k$ for each $x \in V(G)$, is called a *signed (k, k)-dominating family* on G. The maximum number of functions in a signed (k, k)-dominating family on G is the *signed (k, k)-domatic number* of G, denoted by $d_S^k(G)$.

First we study basic properties of $d_S^k(G)$. Some of them are extensions of well-known results on the signed domatic number $d_S(G) = d_S^1(G)$ given in [6]. Using these results, we determine the signed (k, k)-domatic numbers of fans, wheels and grids.

2. Basic properties of the signed (k, k)-domatic number

Theorem 2.1. *The signed (k, k)-domatic number $d_S^k(G)$ is well-defined for each graph G with $\delta(G) \geqslant k - 1$.*

Proof. Since $\delta(G) \geqslant k - 1$, the function $f\colon V(G) \to \{-1, 1\}$ with $f(v) = 1$ for each $v \in V(G)$ is a signed k-dominating function on G. Thus the family $\{f\}$ is a signed (k, k)-dominating family on G. Therefore the set of signed k-dominating functions on G is non-empty and there exists the maximum of their cardinalities, which is the signed (k, k)-domatic number of G. □

Theorem 2.2. *If G is a graph of order n, then*

$$\gamma_{kS}(G) d_S^k(G) \leqslant kn.$$

Proof. If $\{f_1, f_2, \ldots, f_d\}$ is a signed (k,k)-dominating family on G such that $d = d_S^k(G)$, then the definitions imply

$$\begin{aligned} d\gamma_{kS}(G) &= \sum_{i=1}^{d} \gamma_{kS}(G) \leqslant \sum_{i=1}^{d} \sum_{x\in V(G)} f_i(x) \\ &= \sum_{x\in V(G)} \sum_{i=1}^{d} f_i(x) \leqslant \sum_{x\in V(G)} k = kn. \end{aligned}$$

□

Theorem 2.3. *If G is a graph with minimum degree $\delta(G) \geqslant k-1$, then*

$$d_S^k(G) \leqslant \delta(G) + 1.$$

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed (k,k)-dominating family on G such that $d = d_S^k(G)$. If $v \in V(G)$ is a vertex of minimum degree $\delta(G)$, then it follows that

$$\begin{aligned} dk &= \sum_{i=1}^{d} k \leqslant \sum_{i=1}^{d} \sum_{x\in N[v]} f_i(x) \\ &= \sum_{x\in N[v]} \sum_{i=1}^{d} f_i(x) \\ &\leqslant \sum_{x\in N[v]} k = k(\delta(G)+1), \end{aligned}$$

and this implies the desired upper bound on the signed (k,k)-domatic number. □

The special case $k = 1$ in Theorems 2.2 and 2.3 can be found in [6]. As an application of Theorem 2.3, we will prove the following Nordhaus-Gaddum type result.

Theorem 2.4. *If $k \geqslant 1$ is an integer and G a graph of order n such that $\delta(G) \geqslant k-1$ and $\delta(\overline{G}) \geqslant k-1$, then*

$$d_S^k(G) + d_S^k(\overline{G}) \leqslant n + 1.$$

If $d_S^k(G) + d_S^k(\overline{G}) = n + 1$, then G is regular.

Proof. Since $\delta(G) \geqslant k-1$ and $\delta(\overline{G}) \geqslant k-1$, it follows from Theorem 2.3 that

$$\begin{aligned} d_S^k(G) + d_S^k(\overline{G}) &\leqslant (\delta(G)+1) + (\delta(\overline{G})+1) \\ &= (\delta(G)+1) + (n - \Delta(G) - 1 + 1) \\ &\leqslant n+1, \end{aligned}$$

and this is the desired Nordhaus-Gaddum inequality. If G is not regular, then $\Delta(G) - \delta(G) \geqslant 1$, and the above inequality chain leads to the better bound $d_S^k(G) + d_S^k(\overline{G}) \leqslant n$. This completes the proof. □

Theorem 2.5. *If v is a vertex of a graph G such that $d(v)$ is odd and k is odd or $d(v)$ is even and k is even, then*

$$d_S^k(G) \leqslant \frac{k}{k+1}(d(v)+1).$$

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed (k,k)-dominating family on G such that $d = d_S^k(G)$. Assume first that $d(v)$ and k are odd. The definition yields to $\sum_{x \in N[v]} f_i(x) \geqslant k$ for each $i \in \{1, 2, \ldots, d\}$. On the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore it is an even number, and as k is odd, we obtain $\sum_{x \in N[v]} f_i(x) \geqslant k+1$ for each $i \in \{1, 2, \ldots, d\}$. It follows that

$$\begin{aligned} k(d(v)+1) = \sum_{x \in N[v]} k &\geqslant \sum_{x \in N[v]} \sum_{i=1}^{d} f_i(x) \\ &= \sum_{i=1}^{d} \sum_{x \in N[v]} f_i(x) \\ &\geqslant \sum_{i=1}^{d} (k+1) = d(k+1), \end{aligned}$$

and this leads to the desired bound. Assume next that $d(v)$ and k are even. Note that $\sum_{x \in N[v]} f_i(x) \geqslant k$ for each $i \in \{1, 2, \ldots, d\}$. On the left-hand side of this inequality a sum of an odd number of odd summands occurs. Therefore it is an odd number, and as k is even, we obtain $\sum_{x \in N[v]} f_i(x) \geqslant k+1$ for each $i \in \{1, 2, \ldots, d\}$. Now the desired bound follows as above, and the proof is complete. □

The next result is an immediate consequence of Theorem 2.5.

Corollary 2.6. *If G is a graph such that $\delta(G)$ and k are odd or $\delta(G)$ and k are even, then*

$$d_S^k(G) \leqslant \frac{k}{k+1}(\delta(G)+1).$$

As an Application of Corollary 2.6, we will improve the Nordhaus-Gaddum bound in Theorem 2.4 for many cases.

Theorem 2.7. *Let $k \geqslant 1$ be an integer, and let G be a graph of order n such that $\delta(G) \geqslant k-1$ and $\delta(\overline{G}) \geqslant k-1$. If $\Delta(G) - \delta(G) \geqslant 1$ or k is even or k and $\delta(G)$ are odd or k is odd and $\delta(G)$ and n are even, then*

$$d_S^k(G) + d_S^k(\overline{G}) \leqslant n.$$

Proof. If $\Delta(G) - \delta(G) \geqslant 1$, then Theorem 2.4 implies the desired bound. Thus assume now that G is $\delta(G)$-regular.

Case 1: Assume that k is even. If $\delta(G)$ is even, then it follows from Theorem 2.3 and Corollary 2.6 that

$$\begin{aligned} d_S^k(G) + d_S^k(\overline{G}) &\leqslant \frac{k}{k+1}(\delta(G)+1) + (\delta(\overline{G})+1) \\ &= \frac{k}{k+1}(\delta(G)+1) + (n-\delta(G)-1+1) \\ &< n+1, \end{aligned}$$

and we obtain the desired bound. If $\delta(G)$ is odd, then n is even and thus $\delta(\overline{G}) = n-\delta(G)-1$ is even. Combining Theorem 2.3 and Corollary 2.6, we find that

$$\begin{aligned} d_S^k(G) + d_S^k(\overline{G}) &\leqslant (\delta(G)+1) + \frac{k}{k+1}(\delta(\overline{G})+1) \\ &= (n-\delta(\overline{G})) + \frac{k}{k+1}(\delta(\overline{G})+1) \\ &< n+1, \end{aligned}$$

and this completes the proof of Case 1.

Case 2: Assume that k is odd. If $\delta(G)$ is odd, then it follows from Theorem 2.3 and Corollary 2.6 that

$$d_S^k(G) + d_S^k(\overline{G}) \leqslant \frac{k}{k+1}(\delta(G)+1) + (n-\delta(G)) < n+1.$$

If $\delta(G)$ is even and n is even, then $\delta(\overline{G}) = n-\delta(G)-1$ is odd, and we obtain the desired bound as above. □

Theorem 2.8. *If G is a graph such that k is odd and $d_S^k(G)$ is even or k is even and $d_S^k(G)$ is odd, then*

$$d_S^k(G) \leqslant \frac{k-1}{k}(\delta(G)+1).$$

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a signed (k,k)-dominating family on G such that $d = d_S^k(G)$. Assume first that k is odd and d is even. If $x \in V(G)$ is an arbitrary vertex, then $\sum_{i=1}^d f_i(x) \leqslant k$. On the left-hand side of this inequality a sum of an even number of odd summands occurs. Therefore it is an even number, and as k is odd, we obtain $\sum_{i=1}^d f_i(x) \leqslant k-1$ for each $x \in V(G)$. If v is a vertex of minimum degree, then it follows that

$$\begin{aligned} dk = \sum_{i=1}^{d} k &\leqslant \sum_{i=1}^{d} \sum_{x\in N[v]} f_i(x) \\ &= \sum_{x\in N[v]} \sum_{i=1}^{d} f_i(x) \\ &\leqslant \sum_{x\in N[v]} (k-1) = (\delta(G)+1)(k-1), \end{aligned}$$

and this yields to the desired bound. Assume second that k is even and d is odd. If $x \in V(G)$ is an arbitrary vertex, then $\sum_{i=1}^{d} f_i(x) \leqslant k$. On the left-hand side of this inequality a sum of an odd number of odd summands occurs. Therefore it is an odd number, and as k is even, we obtain $\sum_{i=1}^{d} f_i(x) \leqslant k-1$ for each $x \in V(G)$. Now the desired bound follows as above, and the proof is complete. □

According to Theorem 2.1, $d_S^k(G)$ is a positive integer. If we suppose in the case $k = 1$ that $d_S(G) = d_S^1(G)$ is an even integer, then Theorem 2.8 leads to the contradiction $d_S(G) \leqslant 0$. Consequently, we obtain the next known result.

Corollary 2.9 (Volkmann, Zelinka [6] 2005). *The signed domatic number $d_S(G)$ is an odd integer.*

Corollary 2.10. *If T is a nontrivial tree, then $d_S(T) = 1$ and $d_S^2(T) \leqslant 2$. In addition, if the diameter of T is at most three, then $d_S^2(T) = 1$.*

Proof. Theorem 2.3 implies that $d_S(T) \leqslant 2$ and $d_S^2(T) \leqslant 2$. Applying Corollary 2.9, we obtain $d_S(T) = 1$. Now let f be a signed 2-dominating function on T. Then we observe that $f(x) = 1$ if x is a leaf or x is neighbor of a leaf. However, if the diameter of T is at most three, then each vertex of T is a leaf or a neighbor of a leaf and thus $f(x) = 1$ for every vertex $x \in V(T)$. This shows that $d_S^2(T) = 1$ in that case, and the proof is complete. □

The following example demonstrates that the bound $d_S^2(T) \leqslant 2$ in Corollary 2.10 is sharp.

Let T' be a tree of order 10 with the leaves $u_1, u_2, v_1, v_2, w_1, w_2$ and the vertices u_3, v_3, w_3 and z such that u_3 is adjacent to u_1 and u_2, v_3 is adjacent to v_1 and v_2, w_3 is adjacent to w_1 and w_2 and z is adjacent to u_3, v_3 and w_3. Then the functions $f_i \colon V(T') \to \{-1, 1\}$ such that $f_1(x) = 1$ for each $x \in V(T')$ and $f_2(z) = -1$ and $f_2(x) = 1$ for each vertex $x \in V(T') \setminus \{z\}$ are signed 2-dominating functions on T' such that $f_1(x) + f_2(x) \leqslant 2$ for each vertex $x \in V(T')$. Using Corollary 2.10, we conclude that $d_S^2(T') = 2$.

Theorem 2.11. *Let $k \geqslant 2$ be an integer, and let G be a graph with minimum degree $\delta(G) \geqslant k-1$. Then $d_S^k(G) = 1$ if and only if for every vertex $v \in V(G)$ the closed neighborhood $N[v]$ contains a vertex of degree at most k.*

Proof. Assume that $N[v]$ contains a vertex of degree at most k for every vertex $v \in V(G)$, and let f be a signed k-dominating function on G. If $d(v) \leqslant k$, then it follows that $f(v) = 1$. If $d(x) \leqslant k$ for a neighbor x of v, then we observe $f(v) = 1$ too. Hence $f(v) = 1$ for each $v \in V(G)$ and thus $d_S^k(G) = 1$.

Conversely, assume that $d_S^k(G) = 1$. If G contains a vertex w such $d(x) \geqslant k+1$ for each $x \in N[w]$, then the functions $f_i \colon V(G) \to \{-1, 1\}$ such that $f_1(x) = 1$ for each $x \in V(G)$ and $f_2(w) = -1$ and $f_2(x) = 1$ for each vertex $x \in V(G) \setminus \{w\}$ are signed k-dominating functions on G such that $f_1(x) + f_2(x) \leqslant 2 \leqslant k$ for each vertex $x \in V(G)$. Thus $\{f_1, f_2\}$ is a signed $(2,2)$-dominating family on G, a contradiction to $d_S^k(G) = 1$. □

Next we present a lower bound on the signed (k,k)-domatic number.

Theorem 2.12. *Let $k \geqslant 1$ be an integer, and let G be a graph with minimum degree $\delta(G) \geqslant k-1$. If G contains a vertex $v \in V(G)$ such that all vertices of $N[N[v]]$ have degree at least $k+1$, then $d_S^k(G) \geqslant k$.*

Proof. Let $\{u_1, u_2, \ldots, u_k\} \subset N(v)$. The hypothesis that all vertices of $N[N[v]]$ have degree at least $k+1$ implies that the functions $f_i \colon V(G) \to \{-1, 1\}$ such that $f_i(u_i) = -1$ and $f_i(x) = 1$ for each vertex $x \in V(G) \setminus \{u_i\}$ are signed k-dominating functions on G for $i \in \{1, 2, \ldots, k\}$. Since $f_1(x) + f_2(x) + \cdots + f_k(x) \leqslant k$ for each vertex $x \in V(G)$, we observe that $\{f_1, f_2, \ldots, f_k\}$ is a signed (k,k)-dominating family on G, and Theorem 2.12 is proved. $\square$

Corollary 2.13. *If G is a graph of minimum degree $\delta(G) \geqslant k+1$, then $d_S^k(G) \geqslant k$.*

Theorem 2.14. *Let $k \geqslant 1$ be an integer, and let G be a $(k+1)$-regular graph of order n. If $n \not\equiv 0 \pmod{k+2}$, then $d_S^k(G) = k$.*

Proof. Let f be an arbitrary signed k-dominating function on G. If we define the sets $P = \{v \in V(G) \mid f(v) = 1\}$ and $M = \{v \in V(G) \mid f(v) = -1\}$, then we firstly show that

$$|P| \geqslant \left\lceil \frac{n(k+1)}{k+2} \right\rceil. \tag{2.1}$$

Because of $\sum_{x \in N[y]} f(x) \geqslant k$ for each vertex $y \in V(G)$, the $(k+1)$-regularity of G implies that each vertex $u \in P$ is adjacent to at most one vertex in M and each vertex $v \in M$ is adjacent to exactly $k+1$ vertices in P. Therefore we obtain

$$|P| \geqslant |M|(k+1) = (n - |P|)(k+1),$$

and this leads to (2.1) immediately.

Now let $\{f_1, f_2, \ldots, f_d\}$ be a signed (k,k)-dominating family on G with $d = d_S^k(G)$. Since $\sum_{i=1}^{d} f_i(u) \leqslant k$ for every vertex $u \in V(G)$, each of these sums contains at least $\lceil (d-k)/2 \rceil$ summands of value -1. Using this and inequality (2.1), we see that the sum

$$\sum_{x \in V(G)} \sum_{i=1}^{d} f_i(x) = \sum_{i=1}^{d} \sum_{x \in V(G)} f_i(x) \tag{2.2}$$

contains at least $n\lceil (d-k)/2 \rceil$ summands of value -1 and at least $d\lceil n(k+1)/(k+2) \rceil$ summands of value 1. As the sum (2.2) consists of exactly dn summands, it follows that

$$n \left\lceil \frac{d-k}{2} \right\rceil + d \left\lceil \frac{n(k+1)}{k+2} \right\rceil \leqslant dn. \tag{2.3}$$

It follows from the hypothesis $n \not\equiv 0 \pmod{k+2}$ that

$$\left\lceil \frac{n(k+1)}{k+2} \right\rceil > \frac{n(k+1)}{k+2},$$

and thus (2.3) leads to

$$\frac{n(d-k)}{2} + \frac{dn(k+1)}{k+2} < dn.$$

A simple calculation shows that this inequality implies $d < k+2$ and so $d \leqslant k+1$. If we suppose that $d = k+1$, then we observe that d and k of different parity. Applying Theorem 2.8, we obtain the contradiction

$$k+1 = d \leqslant \frac{k-1}{k}(k+2) < k+1.$$

Therefore $d \leqslant k$, and Corollary 2.13 yields to the desired result $d = k$. □

On the one hand Theorem 2.14 demonstrates that the bound in Corollary 2.13 is sharp, on the other hand the following example shows that Theorem 2.14 is not valid in general when $n \equiv 0 \pmod{k+2}$.

Let $v_1, v_2, \ldots, v_{k+2}$ be the vertex set of the complete graph K_{k+2}. We define the functions $f_i\colon V(G) \to \{-1, 1\}$ such that $f_i(v_i) = -1$ and $f_i(x) = 1$ for each vertex $x \in V(G) \setminus \{v_i\}$ and each $i \in \{1, 2, \ldots, k+2\}$. Then we observe that f_i is a signed k-dominating function on K_{k+2} for each $i \in \{1, 2, \ldots, k+2\}$ and $\sum_{i=1}^{k+2} f_i(x) = k$ for each vertex $x \in V(K_{k+2})$. Therefore $\{f_1, f_2, \ldots, f_{k+2}\}$ is a signed (k,k)-dominating family on G and thus $d_S^k(K_{k+2}) \geqslant k+2$. Using Theorem 2.3, we obtain $d_S^k(K_{k+2}) = k+2$.

3. Signed (k,k)-domatic number of fans, wheels and grids

Volkmann and Zelinka [6] have proved that $d_S(G) = 1$ when G is a fan or a wheel of order $n \geqslant 4$. If a graph G has a vertex of degree 3, then Volkmann [5] showed that $d_S(G) = 1$. Therefore $d_S(G) = 1$ for each grid. Using the results of Section 2, we now determine the signed (k,k)-domatic numbers of fans, wheels and grids for $k \geqslant 2$.

Theorem 3.1. *Let G be a fan of order $n \geqslant 3$. Then $d_S^3(G) = 1$, $d_S^2(G) = 1$ when $3 \leqslant n \leqslant 5$ and $d_S^2(G) = 2$ when $n \geqslant 6$.*

Proof. Since $N[v]$ contains a vertex of degree at most 3 for every vertex $v \in V(G)$, it follows from Theorem 2.11 that $d_S^3(G) = 1$.

Let now $x_1, x_2, \ldots, x_n$ be the vertex set of the fan G such that $x_1 x_2 \ldots x_n x_1$ is a cycle of length n and x_n is adjacent to x_i for each $i = 2, 3, \ldots, n-2$.

If $n \leqslant 5$, then $N[v]$ contains a vertex of degree at most 2 for every vertex $v \in V(G)$, and Theorem 2.11 implies $d_S^2(G) = 1$.

If $n \geqslant 6$, then the functions $f_i\colon V(G) \to \{-1, 1\}$ such that $f_1(x) = 1$ for each $x \in V(G)$ and $f_2(x_3) = -1$ and $f_2(x) = 1$ for each vertex $x \in V(G) \setminus \{x_3\}$ are signed 2-dominating functions on G such that $f_1(x)+f_2(x)\leqslant 2$ for each vertex $x\in V(G)$. Thus $d_S^2(G)\geqslant 2$. In view of Corollary 2.6, we see that

$$d_S^2(G)\leqslant \frac{2}{3}(\delta(G)+1)=2,$$

and therefore $d_S^2(G)=2$. □

Theorem 3.2. *Let G be a wheel of order $n\geqslant 5$. Then $d_S^4(G)=d_S^3(G)=1$, $d_S^2(G)=4$ when $n-1\equiv 0 \pmod 3$ and $d_S^2(G)=2$ when $n-1\not\equiv 0 \pmod 3$.*

Proof. Since $N[v]$ contains a vertex of degree at most 3 for every vertex $v\in V(G)$, it follows from Theorem 2.11 that $d_S^4(G)=d_S^3(G)=1$.

Now let $x_1,x_2,\ldots,x_n$ be the vertex set of the wheel G such that $x_1x_2\ldots x_{n-1}x_1$ is a cycle of length $n-1$ and x_n is adjacent to x_i for each $i=1,2,\ldots,n-1$. It follows from Theorem 2.3 that $d_S^2(G)\leqslant 4$. Since $\delta(G)=3$, Corollary 2.13 implies that $d_S^2(G)\geqslant 2$. Let $\{f_1,f_2,\ldots,f_d\}$ be a signed $(2,2)$-dominating family on G with $d=d_S^2(G)$. If $d=3$, then Theorem 2.8 leads to the contradiction

$$3=d\leqslant \frac{1}{2}(\delta(G)+1)=2.$$

Consequently, $d=2$ or $d=4$. Assume that $d=4$. Since $f_1(x)+f_2(x)+f_3(x)+f_4(x)\leqslant 2$ for each vertex $x\in V(G)$, there exists at least one number $j\in\{1,2,3,4\}$ such that $f_j(x)=-1$ for each $x\in V(G)$. Assume, without loss of generality, that $f_1(x_n)=-1$. Because of $\sum_{x\in N[v]} f_1(x)\geqslant 2$ for each vertex v, we deduce that $f_1(x_1)=f_1(x_2)=\ldots=f_1(x_{n-1})=1$. If we assume, without loss of generality, that $f_2(x_1)=-1$, then it follows that $f_2(x_2)=f_2(x_3)=1$. If we assume next, without loss of generality, that $f_3(x_2)=-1$, then we observe that $f_3(x_3)=f_3(x_4)=1$ and therefore $f_4(x_3)=-1$ and thus $f_4(x_4)=f_4(x_5)=1$. This leads to $f_2(x_4)=-1$ and so $f_2(x_5)=f_2(x_6)=1$. Inductively, we see that $f_2(x_i)=-1$ if and only if $i\equiv 1 \pmod 3$, $f_3(x_i)=-1$ if and only if $i\equiv 2 \pmod 3$ and $f_4(x_i)=-1$ if and only if $i\equiv 0 \pmod 3$. This can be realized if and only if $n-1\equiv 0 \pmod 3$, and this completes the proof. □

The *cartesian product* $G=G_1\times G_2$ of two vertex disjoint graphs G_1 and G_2 has $V(G)=V(G_1)\times V(G_2)$ and two vertices (u_1,u_2) and (v_1,v_2) of G are adjacent if and only if either $u_1=v_1$ and $u_2v_2\in E(G_2)$ or $u_2=v_2$ and $u_1v_1\in E(G_1)$. The cartesian product of two paths $P_r=x_1x_2\ldots x_r$ and $P_t=y_1y_2\ldots y_t$ is called a *grid*.

Theorem 3.3. *Let $G=P_r\times P_t$ be a grid of order $n=rt\geqslant 2$ such that $r\leqslant t$. Then*

(1) If $r=1$, then $d_S^2(G)=1$.

(2) If $r=2$, then $d_S^3(G)=1$, $d_S^2(G)=1$ when $t\leqslant 4$ and $d_S^2(G)=2$ when $t\geqslant 5$.

(3) If $r \geqslant 3$, then $d_S^2(G) = 2$.

(4) If $3 \leqslant r \leqslant 4$, then $d_S^3(G) = 1$.

(5) If $r = 5$ and $t = 5$, then $d_S^3(G) = 2$.

(6) If $r = 5$ and $t \geqslant 6$ or $r \geqslant 6$, then $d_S^3(G) = 3$.

Proof. (1) Assume that $r = 1$. Then G is a path and it follows from Theorem 2.11 that $d_S^2(G) = 1$.

(2) Assume that $r = 2$. Then $2 \leqslant d(v) \leqslant 3$ for every $v \in V(G)$, and hence Theorem 2.11 implies that $d_S^3(G) = 1$. If $t \leqslant 4$, then $N[v]$ contains a vertex of degree at most 2 for every vertex $v \in V(G)$, and so $d_S^2(G) = 1$, by Theorem 2.11. If $t \geqslant 5$, then all vertices of $N[(x_1, y_3)]$ are of degree 3, and thus it follows from Theorem 2.11 that $d_S^2(G) \geqslant 2$. Since $\delta(G) = 2$, we deduce from Corollary 2.6 that $d_S^2(G) \leqslant 2$ and so $d_S^2(G) = 2$.

(3) Assume that $r \geqslant 3$. Then all vertices of $N[(x_2, y_2)]$ are of degree at least 3, and thus it follows from Theorem 2.11 that $d_S^2(G) \geqslant 2$. Since $\delta(G) = 2$, we deduce from Corollary 2.6 that $d_S^2(G) \leqslant 2$ and so $d_S^2(G) = 2$.

(4) Assume that $3 \leqslant r \leqslant 4$. This condition shows that $N[v]$ contains a vertex of degree at most 3 for every vertex $v \in V(G)$, and so Theorem 2.11 implies that $d_S^3(G) = 1$.

(5) Assume that $r = t = 5$. Then all vertices of $N[(x_3, y_3)]$ are of degree 4, and thus it follows from Theorem 2.11 that $d_S^3(G) \geqslant 2$. Since $N[v]$ contains a vertex of degree at most 3 for every vertex $v \in V(G) \setminus \{(x_3, y_3)\}$, we deduce that $f(v) = 1$ for every signed 3-dominating function f on G and every vertex $v \neq (x_3, y_3)$. This implies that $d_S^3(G) \leqslant 2$ and thus $d_S^3(G) = 2$.

(6) Assume that $r = 5$ and $t \geqslant 6$ or $r \geqslant 6$. In view of Theorem 2.3, we have $d_S^3(G) \leqslant 3$. Define now the functions $f_i \colon V(G) \to \{-1, 1\}$ such that $f_1(v) = 1$ for each vertex $v \in V(G)$, $f_2((x_3, y_3)) = -1$ and $f_2(v) = 1$ for each $v \in V(G) \setminus \{(x_3, y_3)\}$ and $f_3((x_3, y_4)) = -1$ and $f_3(v) = 1$ for each $v \in V(G) \setminus \{(x_3, y_4)\}$. Then $\{f_1, f_2, f_3\}$ is a family of signed 3-dominating functions on G such that $f_1(v) + f_2(v) + f_3(v) \leqslant 3$ for each vertex $v \in V(G)$. Therefore $d_S^3(G) = 3$, and the proof is complete. $\square$

References

[1] Dunbar, J. E., Hedetniemi, S. T., Henning, M. A., Slater, P. J., Signed domination in graphs. *Graph Theory, Combinatorics, and Applications*, John Wiley and Sons, Inc. 1 (1995), 311–322.

[2] Haynes, T. W., Hedetniemi, S. T., Slater, P. J., *Fundamentals of Domination in Graphs*, Marcel Dekker, Inc., New York (1998).

[3] Haynes, T. W., Hedetniemi, S. T., Slater, P. J., editors, *Domination in Graphs, Advanced Topics*, Marcel Dekker, Inc., New York (1998).

[4] Rall, D. F., A fractional version of domatic number, *Congr. Numer.* **74** (1990), 100–106.

[5] Volkmann, L., Some remarks on the signed domatic number of graphs with small minimum degree, *Appl. Math. Letters* **22** (2009), 1166-1169.

[6] Volkmann, L., Zelinka, B., Signed domatic number of a graph, *Discrete Appl. Math.* **150** (2005), 261–267.

[7] Wang, C. P., The signed k-domination numbers in graphs, *Ars Combin.*, to appear.

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