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Properties of balancing, cobalancing and generalized balancing numbers*

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Dedicated to professor Béla Pelle on his 80th birthday

Abstract

A positive integer n is called a balancing number if

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$$

for some positive integer r.

Several authors investigated balancing numbers and their various generalizations.

The goal of this paper is to survey some interesting properties and results on balancing, cobalancing and all types of generalized balancing numbers.

Keywords: balancing and cobalancing number, recurrence relation, sequence balancing number, power numerical center, (a,b)-type balancing number

MSC: 11D25, 11D41

1. Introduction

The sequence $R = \{R_i\}_{i=0}^{\infty} = R(A, B, R_0, R_1)$ is called a second order linear recurrence if the recurrence relation

$$R_i = AR_{i-1} + BR_{i-2} \qquad (i > 1)$$

holds for its terms, where A, $B \neq 0$, R_0 and R_1 are fixed rational integers and $|R_0| + |R_1| > 0$. The polynomial $f(x) = x^2 - Ax - B$ is called the companion polynomial

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of the sequence $R = R(A, B, R_0, R_1)$. Let $D = A^2 + 4B$ be the discriminant of f. The roots of the companion polynomial will be denoted by α and β. As it is well-known, if $D > 0$ then sequence can be written in the form

$$R_i = \frac{a\alpha^i - b\beta^i}{\alpha - \beta}, \quad (i \geqslant 2),$$

where $a = R_1 - R_0\beta$ and $b = R_1 - R_0\alpha$.

In [3] A. Behera and G. K. Panda gave the notion of balancing number.

Definition 1.1 ([3]). A positive integer n is called a balancing number if

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r)$$

for some positive integer r. This number is called the balancer corresponding to the balancing number n. The mth term of the sequence of balancing numbers is denoted by B_m.

Remark 1.2. It can be derived from Definition 1.1 that the following statements are equivalent to each other (see also [3]):

- n is a balancing number,
- n^2 is a triangular number (i.e. $n^2 = 1 + 2 + \cdots + k$ for some $k \in \mathbb{N}$),
- $8n^2 + 1$ is a perfect square.

It is easy to see that 6, 35, and 204 are balancing numbers with balancers 2, 14 and 84, respectively.

2. Properties of balancing numbers

2.1. Generating balancing numbers

In [3] A. Behera and G. K. Panda proved other interesting properties about balancing numbers.

Let us consider the following functions:

$$F(x) = 2x\sqrt{8x^2+1} \tag{2.1}$$

$$G(x) = 3x + \sqrt{8x^2+1} \tag{2.2}$$

$$H(x) = 17x + 6\sqrt{8x^2+1} \tag{2.3}$$

They proved that these functions always generate balancing numbers.

Theorem 2.1 (Theorem 2.1 in [3]). *For any balancing number n, $F(n)$, $G(n)$, and $H(n)$ are also balancing numbers.*

Remark 2.2. Using the theorem above we get that if n is a balancing number, then $G(F(n)) = 6n\sqrt{8n^2+1}+16n^2+1$ is an odd balancing number, because $F(n)$ is always even and $G(n)$ is odd when n is even.

For generating balancing numbers they proved the following theorems:

Theorem 2.3 (Theorem 3.1 in [3]). *If n is any balancing number, then there is no balancing number k such that $n < k < 3n + \sqrt{8n^2+1}$.*

Its corollary is the following:

Corollary 2.4 (Corollary 3.2 in [3]). *If $n = B_m$ is a balancing number with $m > 1$, then we have $B_{m-1} = 3n - \sqrt{8n^2+1}$.*

They proved that a balancing number can also be generated by two balancing numbers.

Theorem 2.5 (Theorem 4.1 in [3]). *If n and k are balancing numbers, then*

$$f(n,k) = n\sqrt{8k^2+1} + k\sqrt{8n^2+1} \tag{2.4}$$

is also a balancing number.

2.2. A recurrence relation and other properties

In [3] Behera and Panda proved that the balancing numbers fulfill the following recurrence relation

$$B_{m+1} = 6B_m - B_{m-1} \quad (m \geqslant 1)$$

where $B_0 = 1$ and $B_1 = 6$. Using this recurrence relation they get interesting relations between balancing numbers.

Theorem 2.6 (Therem 5.1 in [3]). *For any $m > 1$ we have*

- $B_{m+1} \cdot B_{m-1} = (B_m + 1)(B_m - 1)$,
- $B_m = B_k \cdot B_{m-k} - B_{k-1} \cdot B_{m-k-1}$ *for any positive integer $k < m$,*
- $B_{2m} = B_m^2 - B_{m-1}^2$,
- $B_{2m+1} = B_m(B_{m+1} - B_{m-1})$.

In [26] G. K. Panda established other interesting arithmetic-type, de-Moivre's-type and trigonometric-type properties of balancing numbers.

Theorem 2.7 (Theorem 2.1 in [26]). *If m and k are natural numbers and $m > k$, then $(B_m + B_k)(B_m - B_k) = B_{m+k} \cdot B_{m-k}$.*

Remark 2.8. The Fibonacci numbers F_m satisfy a similar property (see [16] p. 59)

$$F_{m+k} \cdot F_{m-k} = F_m^2 - (-1)^{m+k} F_k^2.$$

We know that if m is natural number, then $1+3+\cdots+(2m-1)=m^2$. In [26] G. K. Panda proved three properties of balancing numbers similar to the identity above. For balancing numbers we get:

Theorem 2.9 (Theorem 2.2 in [26]).

- $B_1 + B_3 + \cdots + B_{2m-1} = B_m^2$,
- $B_2 + B_4 + \cdots + B_{2m} = B_m B_{m+1}$,
- $B_1 + B_2 + \cdots + B_{2m} = B_m(B_m + B_{m+1})$.

The identity $(\cos x + i \sin x)^n = \cos nx + i \sin nx$ for complex numbers is known as the de-Moivre's formula. The following theorem gives a de-Moivre's-type property of balancing numbers. Let $C_m = \sqrt{8B_m^2+1}$.

Theorem 2.10 (Theorem 2.3 in [26]). *If m and k are natural numbers, then*

$$(C_m + \sqrt{8}B_m)^k = C_{mk} + \sqrt{8}B_{mk}.$$

Remark 2.11. The Fibonacci (F_m) and Lucas (L_m) numbers satisfy a similar property

$$\left[\frac{L_m + \sqrt{5}F_m}{2}\right]^r = \frac{L_{mr} + \sqrt{5}F_{mr}}{2}.$$

Panda proved another interesting result about the greatest common divisor of balancing numbers.

Theorem 2.12 (Theorem 2.5 in [26]). *If m and k are natural numbers then*

$$\gcd(B_m, B_k) = B_{(m,k)}.$$

In [3] we can find nonrecursive forms to obtain balancing numbers. One of these results is the following:

Theorem 2.13 (Theorem 7.1 in [3]). *If B_m is the mth balancing number then*

$$B_m = \frac{\lambda_1^{m+1} - \lambda_2^{m+1}}{\lambda_1 - \lambda_2}, \quad m = 0, 1, 2, \dots,$$

where $\lambda_1 = 3 + \sqrt{8}$ and $\lambda_2 = 3 - \sqrt{8}$.

Remark 2.14. We get this formula easily using the companion polynomial of the recurrence relation of B_m.

2.3. Fibonacci and Lucas balancing numbers

In [21] K. Liptai obtained several results about special type of balancing numbers. Let us consider the definition below:

Definition 2.15 ([21] and [22]). We call a balancing number a *Fibonacci* or a *Lucas balancing number* if it is a Fibonacci or a Lucas number, too.

Using this definition and companion polynomial of B_m K. Liptai proved that the balancing numbers are solutions of a Pell's equation.

Theorem 2.16 (Theorem 1 in [21]). *The terms of the second order linear recurrence $R(6,-1,1,6)$ are the solutions of the equation*

$$x^2 - 8y^2 = 1$$

for some integer y.

There is also a connection between Fibonacci or Lucas numbers and Pell's equation. The following theorem is due to D. E. Ferguson:

Theorem 2.17 (Theorem in [7]). *The only solutions of the equation*

$$x^2 - 5y^2 = \pm 4$$

are $x = \pm L_m$, $y = \pm F_m$ $(n = 0, 1, 2 \ldots)$, where L_m and F_m are the mth terms of the Lucas and Fibonacci sequences, respectively.

To find all Fibonacci or Lucas balancing numbers K. Liptai proved that there are finitely many common solutions of the Pell's equations above using a method of A. Baker and H. Davenport.

The main theorem in [21] and [22] are the following:

Theorem 2.18 (Theorem 4 in [21] and [22]). *There is no Fibonacci or Lucas balancing number.*

Remark 2.19. Using another method L. Szalay got the same result for the solutions of simultaneous Pell equations in [35]. In this method he converted simultaneous Pell's equations into a family of Thue equations which could be solved completely.

3. Properties of cobalancing numbers

3.1. Introduction

By slightly modifying the definition 1.1 we get:

Definition 3.1 ([27]). We call $n \in \mathbb{N}$ a cobalancing number if

$$1 + 2 + \cdots + n = (n+1) + (n+2) + \cdots + (n + r^c)$$

for some $r^c \in \mathbb{N}$. Here we call r^c the cobalancer corresponding to the cobalancing number n. Denote n by B_m^c if n is the mth term of the sequence of cobalancing numbers.

Remark 3.2. The first three cobalancing numbers are 2, 14 and 84 with cobalancers 1, 6, 35, respectively.

3.2. Properties of cobalancing numbers

Cobalancing numbers B_m^c have similar properties to balancing numbers B_m. In [27] G. K. Panda and P. K. Ray proved the following properties:

Theorem 3.3 (Theorem 2.2 in [27]). *If $n = B_m^c$ is a cobalancing number with $m > 1$ then $B_{m+1}^c = 3n + \sqrt{8n^2 + 8n + 1} + 1$ and $B_{m-1}^c = 3n - \sqrt{8n^2 + 8n + 1} + 1$.*

By Theorem 3.3 they get a recurrence relation for cobalancing numbers that is

$$B_{m+1}^c = 6B_m^c - B_{m-1}^c + 2, \quad (m = 2, 3, \ldots)$$

where they set $B_1^c = 0$. The following theorem is a consequence of the relation above.

Theorem 3.4 (Theorem 3.1 in [27]). *Every cobalancing number is even.*

We also denote by r_m the balancer belonging to B_m and r_m^c the cobalancer belongig to B_m^c. Then by using the definition 1.1 and 3.1 the following theorems are valid:

Theorem 3.5 (Theorem 6.1 in [27]). *Every balancer is a cobalancing number and every cobalancer is a balancing number.*

Using our notation we get:

Theorem 3.6 (Theorem 6.2 in [27]). *We have $r_m = B_m^c$ and $r_{m+1}^c = B_m$ for every $m = 1, 2, \ldots$.*

Panda and Ray got a corollary from the theorems above.

Corollary 3.7 (Corollary 6.4 in [27]). $r_{m+1} = r_m + 2B_m$.

3.3. Connection between (co)balancing and Pell numbers

In [28] we can find interesting results about the connection of Pell, balancing or cobalancing numbers. Let P_m be the mth Pell number ($m = 1, 2 \ldots$). It is well known that

$$P_1 = 1, P_2 = 2, P_{m+1} = 2P_m + P_{m-1}.$$

The authors call $C_m = \sqrt{8B_m^2 + 1}$ the mth Lucas-balancing number and $c_m = \sqrt{8\left(B^c\right)_m^2 + 8B_m^c + 1}$ the mth Lucas-cobalancing number. The first result of them is the following:

Theorem 3.8 (Theorem 2.2 in [28]). *The sequences of Lucas-balancing and Lucas-cobalancing numbers satisfy recurrence relations with identical balancing numbers. More precisely,* $C_1 = 3$, $C_2 = 17$, $C_{m+1} = 6C_m - C_{m-1}$ *and* $c_1 = 1$, $c_2 = 7$, $c_{m+1} = 6c_m - c_{m-1}$ *for* $m = 2, 3, \ldots$.

In [28] the authors get a formula how to calculate balancing or cobalancing numbers from Pell numbers.

Theorem 3.9 (Theorem 3.2 in [28]). *If* P *is a Pell number then* $\lceil P/2 \rceil$ *is either a balancing number or a cobalancing number. More precisely* $P_{2m}/2 = B_m$ *and* $\lceil P_{2m-1}/2 \rceil = B_m^c$ $(m = 1, 2, \ldots)$.

There is another result for calculating balancing number and its balancer, too.

Theorem 3.10 (Theorem 3.4 in [28]). *The sum of the first* $2m - 1$ *Pell numbers is equal to the sum of the* m*th balancing number and its balancer.*

4. Generalizations

4.1. Sequence balancing and cobalancing numbers

In [25] G. K. Panda defined sequence balancing and sequence cobalancing numbers.

Definition 4.1 ([25]). Let $\{s_m\}_{m=1}^{\infty}$ be a sequence of real numbers. We call a number s_m of this sequence a sequence balancing number if

$$s_1 + s_2 + \cdots + s_{m-1} = s_{m+1} + s_{m+2} + \cdots + s_{m+r}$$

for some natural number r. Similarly, we call s_m a sequence cobalancing number if

$$s_1 + s_2 + \cdots + s_m = s_{m+1} + s_{m+2} + \cdots + s_{m+r}$$

for some natural number r.

Remark 4.2. For example, if we take $s_m = 2m$ then the sequence balancing numbers of this sequence are 12, 70, 408,... which are twice the balancing numbers. It is also true for sequence cobalancing numbers and similarly in the case when $s_m = \frac{m}{2}$.

In [25] the author investigated the existence of sequence balancing or cobalancing numbers in the sequence of odd natural numbers. So, let $s_m = 2m - 1$. Using simple technics he got that the sequence of sequence balancing numbers in the sequence of odd natural numbers is given by $\{2B^c_{m+1} + r^c_{m+1} + 1\}_{m=1}^{\infty}$ (see Theorem 2.1.4 in [25]). So, let the mth sequence balancing number in the sequence of odd natural numbers be denoted by x_m. Then by this fact above G. K. Panda got the following recurrence relation for these solutions.

Theorem 4.3 (Theorem 2.1.5 in [25]). *The sequence $\{x_m\}_{m=1}^{\infty}$ satisfies the recurrence relation $x_{m+1} = 6x_m - x_{m-1}$ for $m \geqslant 2$.*

Remark 4.4. The author in [25] investigated also the existance of sequence balancing or cobalancing numbers in the cases when $a_m = m+1$ and $a_m = F_m$ (among Fibonacci numbers). In the first case the sequence balancing numbers among the numbers $a_m = m + 1$ can be given by a linear combination of balancing numbers.

In the second one he gets that the only sequence cobalancing number in the Fibonacci sequence is $F_2 = 1$.

4.2. Generalized balancing sequences

In [4] A. Bérczes, K. Liptai and I. Pink generalized the definition 4.1 due to G. K. Panda.

Definition 4.5 ([4]). We call a binary recurrence $R_i = R(A, B, R_0, R_1)$ a balancing sequence if

$$R_1 + R_2 + \cdots + R_{m-1} = R_{m+1} + R_{m+2} + \cdots + R_{m+k} \tag{4.1}$$

holds for some $k \geqslant 1$ and $m \geqslant 2$.

In that paper they proved that any sequence $R_i = R(A, B, 0, R_1)$ with conditions $D = A^2 + 4B > 0$, $(A, B) \neq (0, 1)$ is not a balancing sequence.

Theorem 4.6 (Theorem 1 in [4]). *There is no balancing sequence of the form $R_i = R(A, B, 0, R_1)$ with $D = A^2 + 4B > 0$ except for $(A, B) = (0, 1)$ in which case (4.1) has infinitely many solutions $(m, k) = (m, m - 1)$ and $(m, k) = (m, m)$ for $m \geqslant 2$.*

By this theorem they got the following corollary.

Corollary 4.7 (Corollary 1 in [4]). *Let $R_i = R(A, B, 0, 1)$ be a Lucas-sequence with $A^2 + 4B > 0$. Then R_i is not a balancing sequence.*

4.3. (k, l)-numerical centers

Definition 4.8 ([23]). Let y, k and l be fixed positive integers with $y \geqslant 4$. A positive integer x $(x \leqslant y - 2)$ is called a (k, l)-power numerical center for y, or a (k, l)-balancing number for y if

$$1^k + 2^k + \cdots + (x - 1)^k = (x + 1)^l + \cdots + (y - 1)^l.$$

Remark 4.9. In [8] R. Finkelstein studied ”The house problem” and introduced the notion of first-power numerical center which coincides with the notion of balancing number B_m. He proved that infinitely many integers y possess $(1,1)$-power centers and there is no integer $y > 1$ with a $(2,2)$-power numerical center. In his paper, he conjectured that if $k > 1$ then there is no integer $y > 1$ with (k,k)-power numerical center. Later in [33] his conjeture was confirmed for $k = 3$. Recently, Ingram in [17] proved Finkelstein's conjecture for $k = 5$.

In [23] the authors proved a general result about (k,l)-balancing numbers, but they could not deal with Finkelstein's conjecture in its full generality. Their main results are the following theorems.

Theorem 4.10 (Theorem 1 in [23]). *For any fixed positive integer $k > 1$, there are only finitely many positive pairs of integers (y,l) such that y possesses a (k,l)-power numerical center.*

For the proof of this theorem they used a result from [31]. Thus Theorem 4.10 is ineffective in case $l \leqslant k$ in the sense that no upper bound was made for possible numerical centers except for the cases when $l = 1$ or $l = 3$.

Theorem 4.11 (Theorem 2 in [23]). *Let k be a fixed positive integer with $k \geqslant 1$ and $l \in \{1,3\}$. If $(k,l) \neq (1,1)$, then there are only finitely many (k,l)-balancing numbers, and these balancing numbers are bounded by an effectively computable constant depending only on k.*

Remark 4.12. In [23] the authors gave an example for numerical centers in the case when $(k,l) = (2,1)$. After solving an elliptic equation by MAGMA [24] they got three $(2,1)$-power numerical centers x, namely 5, 13 and 36.

4.4. (a,b)-type balancing numbers

Another generalization is the following by T. Kovács, K. Liptai and P. Olajos:

Definition 4.13 ([20]). Let a, b be nonnegative coprime integers. We call a positive integer $an + b$ an (a,b)-type balancing number if

$$(a+b) + (2a+b) + \cdots + (a(n-1)+b) = (a(n+1)+b) + \cdots + (a(n+r)+b)$$

for some $r \in \mathbb{N}$. Here r is called the balancer corresponding to the balancing number. We denote the positive integer $an + b$ by $B_m^{(a,b)}$ if this number is the mth among the (a,b)-type balancing numbers.

Remark 4.14. We have to mention that if we use notation $a_n = an + b$ then we get sequence balancing numbers and if $a = 1$ and $b = 0$ for (a,b)-type balancing numbers than we get balancing numbers B_m.

Using the definition the authors in [20] get the following proposition:

Lemma 4.15 (Proposition 1 in [20]). *If $B_m^{(a,b)}$ is an (a,b)-type balancing number then the following equation*

$$z^2 - 8\left(B_m^{(a,b)}\right)^2 = a^2 - 4ab - 4b^2 \tag{4.2}$$

is valid for some $z \in \mathbb{Z}$.

4.4.1. Polynomial values among balancing numbers

Let us consider the following equation for (a,b)-type balancing numbers

$$B_m^{(a,b)} = f(x) \tag{4.3}$$

where $f(x)$ is a monic polynomial with integer coefficients. By Proposition 4.15 and the result from Brindza [5] Kovács, Liptai and Olajos proved the following theorem:

Theorem 4.16 (Theorem 1 in [20]). *Let $f(x)$ be a monic polynomial with integer coefficients, of degree $\geqslant 2$. If a is odd, then for the solutions of (4.3) we have $\max(m, |x|) < c_0(f,a,b)$, where $c_0(f,a,b)$ is an effectively computable constant depending only on a, b and f.*

Let us consider a special case of Theorem 4.16 with $f(x) = x^l$. Using one of the results from Bennett [1] the authors in [20] get the following theorem:

Theorem 4.17 (Theorem 2 in [20]). *If $a^2 - 4ab - 4b^2 = 1$, then there is no perfect power (a,b)-balancing number.*

Remark 4.18. There are infinitely many integer solutions of the equation $a^2 - 4ab - 4b^2 = 1$.

The authors are interested in combinatorial numbers (see also Kovács [19]), that is binomial coefficients, power sums, alternating power sums and products of consecutive integers. For all $k, x \in \mathbb{N}$ let

$$S_k(x) = 1^k + 2^k + \cdots + (x-1)^k,$$
$$T_k(x) = -1^k + 2^k - \cdots + (-1)^{x-1}(x-1)^k,$$
$$\Pi_k(x) = x(x+1)\ldots(x+k-1).$$

We mention that the degree of $S_k(x)$, $T_k(x)$ and $\Pi_k(x)$ are $k+1$, k and k, respectively and $\binom{x}{k}$, $S_k(x)$, $T_k(x)$ are polynomials with non-integer coefficients. Moreover, in the case when $f(x) = \Pi_k(x)$ Theorem 4.16 is valid but the parameter a is odd.

Let us consider the following equation

$$B_m^{(a,b)} = p(x), \tag{4.4}$$

where $p(x)$ is a polynomial with rational integer coefficients. In this case Kovács, Liptai and Olajos gave effective results for the solutions of equation (4.4).

Theorem 4.19 (Theorem 3 in [20]). *Let $k \geqslant 2$ and $p(x)$ be one of the polynomials $\binom{x}{k}$, $\Pi_k(x)$, $S_{k-1}(x)$, $T_k(x)$. Then the solutions of equation* (4.3) *satisfy* $\max(m, |x|) < c_1(a, b, k)$, *where $c_1(a, b, k)$ is an effectively computable constant depending only on a, b and k.*

4.4.2. Numerical results

In [20] T. Kovács, K. Liptai and the author completely solve the above type equations for some small values of k that lead to genus 1 or genus 2 equations. In this case the equation can be written as

$$y^2 = 8f(x)^2 + 1, \tag{4.5}$$

where $f(x)$ is one of the following polynomials. Beside binomial coefficients $\binom{x}{k}$, we consider power sums and products of consecutive integers, as well. We mention that in their results, for the sake of completeness, they provide all integral (even the negative) solutions to equation (4.5).

Genus 1 and 2 equations They completely solve equation (4.5) for all parameter values k in case when they can reduce the equation to an equation of genus 1. We have to mention that a similar argument has been used to solve several combinatorial Diophantine equations of different types, for example in [9], [10], [12], [13], [18], [19], [29], [30], [34], [37], [38]. Further they also solved a particular case ($f(x) = S_5(x)$) when equation (4.3) can be reduced to the resolution of a genus 2 equation. To solve this equation, they used the so-called Chabauty method. We have to note that the Chabauty method has already been successfully used to solve certain combinatorial Diophantine equations, see e.g. the corresponding results in the papers [6], [11], [14], [15], [32], [36] and the references given there.

Theorem 4.20 (Theorem 4 in [20]). *Suppose that $a^2 - 4ab - 4b^2 = 1$. Let $f(x) \in \{\binom{x}{2}, \binom{x}{3}, \binom{x}{4}, \Pi_2(x), \Pi_3(x), \Pi_4(x), S_1(x), S_2(x), S_3(x), S_5(x)\}$. Then the solutions (m, x) of equation* (4.3) *are those contained in Table 1. For the corresponding parameter values we have $(a, b) = (1, 0)$ in all cases.*

Remark 4.21. In [20] the authors considered some other related equations that led to genus 2 equations. However, because of certain technical problems, they could not solve them by the Chabauty method. They determined the "small" solutions(i.e. $|x| \leqslant 10000$) of equation (4.5) in cases

$$f(x) \in \left\{ \binom{x}{6}, \binom{x}{8}, \Pi_6(x), \Pi_8(x), S_7(x) \right\}.$$

Their conjecture is that that there is no solution for these equations.

$f(x)$	Solutions (m,x) of (4.3)
$\binom{x}{2}$	$(1,-3),(1,4)$
$\binom{x}{3}$	$(2,-5),(2,7)$
$\binom{x}{4}$	$(2,-4),(2,7)$
$\Pi_2(x)$	$(1,-3),(1,2)$
$\Pi_3(x)$	$(1,-3),(1,1)$
$\Pi_4(x)$	$\emptyset$
$S_1(x)$	$(1,-4),(1,3)$
$S_2(x)$	$(3,-8),(3,9),(5,-27),(5,28)$
$S_3(x)$	$\emptyset$
$S_5(x)$	$\emptyset$

Table 1

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