Annales Mathematicae et Informaticae
37 *(2010) pp. 39–49*
http://ami.ektf.hu

Periodic fixed points of random operators

Ismat Beg[a], Mujahid Abbas[a], Akbar Azam[b]

[a]Center for Advanced Studies in Mathematics
Lahore University of Management Sciences

[b]Department of Mathematics
COMSATS Institute of Information Technology

Submitted 10 February 2010; Accepted 19 April 2010

Abstract

Sufficient conditions for existence of random fixed point of a nonexpansive rotative random operator are obtained and existence of random periodic points of a random operator is proved. We also derive random periodic point theorem for ϵ- expansive random operator.

Keywords: Random periodic point; random fixed point; ϵ- contractive random operator; ϵ- expansive random operator; rotative random operator; metric space; Banach space; measurable space.

MSC: 47H09, 47H10, 47H40, 54H25, 60H25

1. Introduction

Random nonlinear analysis has grown into an active research area closely associated with the study of random nonlinear operators and their properties needed in solving nonlinear random operator equations (see [7, 18, 21]). The study of random fixed point theory was initiated by the Prague school of probabilists in the 1950's ([15, 24]). Random fixed point theorems are of tremendous importance in probabilistic functional analysis as they provide a convenient way of modelling many real life problems and random methods have also revolutionized the financial markets. The survey article by Bharucha -Reid [8] in 1976 attracted the attention of several mathematician and gave wings to this theory. Itoh [17] extended Spacek's and Hans's theorems to random multivalued contraction mappings. In recent years, a lot of efforts have been made ([2, 3, 4, 5, 6, 16, 22, 23], and references therein) to show the existence of random fixed points of certain random single valued and multivalued operators and various applications in diverse area from pure mathematics to applied sciences have been explored. The aim of this paper is to establish the existence of random fixed point of nonexpansive rotative random operator in the setting of Banach spaces. A random analogue of Edelstein theorem to establish the existence of random periodic points for random single valued ϵ- contractive operator is proved. These results are then used to obtain the random periodic point of ϵ- expansive random operators. The results proved in this paper improve and generalize several well known results in the literature [9, 12, 17].

2. Preliminaries

We begin with some definitions and state the notations used throughout this paper. Let (Ω, Σ) be a measurable space (Σ- sigma algebra) and F be a nonempty subset of a separable metric space (X, d). A single valued mapping $T\colon \Omega \to X$ is *measurable* if $T^{-1}(U) \in \Sigma$ for each open subset U of X, where $T^{-1}(U) = \{\omega \in \Omega : T(\omega) \cap U \neq \emptyset\}$. A mapping $T\colon \Omega \times X \to X$ is a *random operator* if and only if for each fixed $x \in X$, the mapping $T(.,x)\colon \Omega \to X$ is measurable and it is *continuous* if for each $\omega \in \Omega$, the mapping $T(\omega, .)\colon X \to X$ is continuous. A measurable mapping $\xi\colon \Omega \to X$ is a *random fixed point* of a random operator $T\colon \Omega \times X \to X$ if and only if $\xi(\omega) = T(\omega, \xi(\omega)))$ for each $\omega \in \Omega$. We denote the set of random fixed points of a random operator T by $RF(T)$ and the set of all measurable mappings from Ω into X by $M(\Omega, X)$. For the random operator $f\colon \Omega \times X \to X$, the map $f_\omega^{-1}\colon X \to X$ is defined by $f_\omega^{-1}(y) = x$ if and only if $f(\omega, x) = y$.

We denote the *nth* iterate $T(\omega, T(\omega, T(\omega, \ldots, T(\omega, x) \cdots)))$ of random operator $T\colon \Omega \times X \to X$ by $T^n(\omega, x)$. The letter I denotes the random operator $I\colon \Omega \times X \to X$ defined by $I(\omega, x) = x$ and $T^0 = I$. The random operator $T\colon \Omega \times X \to X$ is called *random periodic operator* with period $p \in N$, if for each $x \in X$ and $\omega \in \Omega$ we obtain $T^p(\omega, x) = I(\omega, x)$. Let $B(x_0, r)$ denotes the spherical ball centred at x_0 with radius r, defined as the set $\{x \in X : d(x, x_0) \leqslant r\}$.

Definition 2.1. Let F be a nonempty subset of a separable metric space X. The random operator $T\colon \Omega \times F \to F$ is said to be:

(a) $k(\omega)$- *contraction random operator* if for any $x, y \in F$ and $\omega \in \Omega$, we have

$$d(T(\omega, x), T(\omega, y)) \leqslant k(\omega) d(x, y),$$

where $k\colon \Omega \to [0, 1)$ is a measurable map. If $k(\omega) = 1$ for any $\omega \in \Omega$, then T is called *nonexpansive random operator.*

(b) *contractive random operator* if for any $x, y \in F$ and $\omega \in \Omega$, we have

$$d(T(\omega, x), T(\omega, y)) < d(x, y).$$

(c) ϵ-*contractive random operator* if for $\epsilon > 0$ and $x, y \in F$ with $x \neq y$ and $d(x, y) < \epsilon$, we have,

$$d(T(\omega, x), T(\omega, y)) < d(x, y),$$

for every $\omega \in \Omega$. Obviously, every contractive random operator is ϵ- contractive random operator for any $\epsilon > 0$.

(d) ϵ-*expansive random operator* if for $\epsilon > 0$ and $x, y \in F$ with $x \neq y$ and $d(x, y) < \epsilon$, we have

$$d(T(\omega, x), T(\omega, y)) > d(x, y), \tag{2.1}$$

for every $\omega \in \Omega$. If inequality (2.1) holds for every $x, y \in X$ with $x \neq y$ then T is called an *expansive random operator*.

Obviously, every expansive random operator is ϵ- expansive random operator for any $\epsilon > 0$.

Definition 2.2. Let $T\colon \Omega \times F \to F$ be a random operator, where F is a nonempty subset of a separable complete metric space X. A measurable mapping $\xi\colon \Omega \to F$ is called a *random periodic point* of T there exists $n \geqslant 1$ such that $T^n(\omega, \xi(\omega)) = \xi(\omega)$, for every $\omega \in \Omega$. That is, random periodic point is random fixed point of nth iterate of T for some $n \geqslant 1$. The least such positive integer n is called *period* of random periodic point ξ.

Note that random fixed point of T is also random periodic point of T of period 1 but there exists a random periodic point of T which fails to be the random fixed point of T as shown in the examples presented below. It is also shown that there exists a random operator having random periodic point of period 5 but does not posses the random periodic point of period 3.

Example 2.3. Let $\Omega = [0, 1]$ and Σ be the sigma algebra of Lebesgue's measurable subsets of Ω. Take $X = R$ with $d(x, y) = |x - y|$, for $x, y \in R$. Define random operator T from $\Omega \times X$ to X as,

$$T(\omega, x) = \begin{cases} \omega^2 - x, & \text{if } (\omega, x) \in \Omega \times [0, 1] \\ \omega^2 - x - 1, & \text{otherwise.} \end{cases}$$

Define the measurable mapping $\xi\colon \Omega \to X$ as $\xi(\omega) = \frac{1}{2}(3\omega^2 - 1)$, for every $\omega \in \Omega$. Now ξ is a random periodic point of T with period 2 but it fails to be a random fixed point of T.

Example 2.4. Let $\Omega = [0, 1]$ and Σ be the sigma algebra of Lebesgue's measurable subsets of Ω. Take $X = R$ with $d(x, y) = |x - y|$, for $x, y \in R$. Define random operator T from $\Omega \times X$ to X as, $T(\omega, 1) = 3$, $T(\omega, 2) = 5$, $T(\omega, 3) = 4$, $T(\omega, 4) = 2$, $T(\omega, 5) = 1$ and $T(\omega, x) = x - \omega$, when $x \notin \{1, 2, 3, 4, 5\}$.

Define measurable mapping $\xi\colon \Omega \to X$ as $\xi(\omega) = 1$, for every $\omega \in \Omega$. Note that ξ is a random periodic point of period 5. It is also noted that random operator T in this example does not posses random fixed point because for any ξ to be the random fixed point, we must have $T(\omega, \xi(\omega)) = \xi(\omega)$, for every $\omega \in \Omega$. But this random operator equation holds only for $\omega = 0$.

Remark 2.5. Let F be a closed subset of a complete separable metric space X and the sequence of measurable mappings $\{\xi_n\}$ from Ω to F be point wise convergent, that is, $\xi_n(\omega) \to q := \xi(\omega)$ for each $\omega \in \Omega$. Then ξ being the limit of the sequence of measurable mappings is measurable and closedness of F implies ξ is a mapping from Ω to F. Since F is a subset of a complete separable metric space X, also if T is a continuous random operator from $\Omega \times F$ to F then by the lemma 8.2.3 of [1], the map $\omega \to T^n(\omega, f(\omega))$ is measurable for any measurable mapping f from Ω to F.

Definition 2.6. Let F be a nonempty subset of a Banach space X. The random operator $T\colon \Omega \times F \to F$ is said to be $(k, n)-$ *rotative random operator* for $k < n$, if for each $\omega \in \Omega$,

$$\|\xi(\omega) - T^n(\omega, \xi(\omega))\| \leqslant k \|\xi(\omega) - T(\omega, \xi(\omega))\|,$$

where ξ is a mapping from Ω to F and $n \in N$. The operator T is said to be $n-$ *rotative random operator* if it $(k, n)-$ rotative random operator for some $k < n$ and T is called *rotative random operator* if it is an n- rotative random operator for some $n \in N$. Note that any random periodic operator with period p is $(0, p)$-rotative random operator.

Remark 2.7. If $T\colon \Omega \times F \to F$ is $k(\omega)$ contraction random operator where F is a closed subset of Banach space X and $n > 1$. For any $\xi\colon \Omega \to F$, consider,

$$\begin{aligned}
\|\xi(\omega) - T^n(\omega, \xi(\omega))\| &\leqslant \sum_{k=1}^{n} \left\|T^{k-1}(\omega, \xi(\omega)) - T^k(\omega, \xi(\omega))\right\| \\
&\leqslant (1 + k(\omega) + (k(\omega))^2 + \cdots \\
&\qquad + (k(\omega))^{n-1}) \|\xi(\omega) - T(\omega, \xi(\omega))\| \\
&< n \|\xi(\omega) - T(\omega, \xi(\omega))\|,
\end{aligned}$$

for every $\omega \in \Omega$. Thus T is a rotative random operator.

3. Periodic and fixed points of rotative random operators

In this section, we first show an existence of a random fixed point of a nonexpansive rotative random operator which not only provides a random analogue of theorem 17.1 of [11] (see also [12]) but also improves theorem 2.1 of [17] in the sense that it does not require the boundedness of $T(\omega, F)$ for any $\omega \in \Omega$. Moreover we replace continuous condensing random operator by nonexpansive rotative random operator.

Periodic point problems were systematically studied since the beginning of fifties (see [9, 10, 13, 14, 19, 20]). We show some results on the existence of random periodic points of random single valued ϵ- contractive operator in the setting of a separable metric space.

Theorem 3.1. *Let F be a nonempty closed and convex subset of a separable Banach space X and $T\colon \Omega \times F \to F$ be a nonexpansive rotative random operator. Then T has a random fixed point.*

Proof. Let $\xi\colon \Omega \to F$ be any fixed measurable mapping. For $0 < \alpha < 1$ and any arbitrary measurable mapping $\eta\colon \Omega \to F$, define $T_\alpha\colon \Omega \times F \to F$ as,

$$T_\alpha(\omega, \eta(\omega)) = (1-\alpha)\xi(\omega) + \alpha T(\omega, \eta(\omega)).$$

Note that for each α, the random operator T_α has Lipschitz constant α. we may apply [8] to obtain the sequence of random operators $F_\alpha\colon \Omega \times F \to F$ such that $T_\alpha(\omega, F_\alpha(\omega, \xi(\omega))) = F_\alpha(\omega, \xi(\omega))$, for every $\omega \in \Omega$. Consequently, we have

$$F_\alpha(\omega, \xi(\omega)) = (1-\alpha)\xi(\omega) + \alpha T(\omega, F_\alpha(\omega, \xi(\omega))).$$

It can be verified that each F_α is nonexpansive random operator. By iterating F_α we obtain

$$F_\alpha^k(\omega, \xi(\omega)) = (1-\alpha)F_\alpha^{k-1}(\omega, \xi(\omega)) + \alpha T(\omega, F_\alpha^k(\omega, \xi(\omega))),\ k \in N. \tag{3.1}$$

Note that,

$$\begin{aligned}
&(1-\alpha)F_\alpha(\omega, \xi(\omega)) \\
&= (1-\alpha)\xi(\omega) + \alpha T(\omega, F_\alpha(\omega, \xi(\omega))) - \alpha F_\alpha(\omega, \xi(\omega)) \\
&= (1-\alpha)\xi(\omega) + \alpha(T(\omega, F_\alpha(\omega, \xi(\omega))) - F_\alpha(\omega, \xi(\omega))).
\end{aligned}$$

Thus for each $\omega \in \Omega$

$$\begin{aligned}
&(1-\alpha)(\xi(\omega) - F_\alpha(\omega, \xi(\omega))) \\
&= \alpha(F_\alpha(\omega, \xi(\omega)) - T(\omega, F_\alpha(\omega, \xi(\omega)))).
\end{aligned} \tag{3.2}$$

Now suppose T is a (a, n)-rotative random operator, that is

$$\|\xi(\omega) - T^n(\omega, \xi(\omega))\| \leqslant a \|\xi(\omega) - T(\omega, \xi(\omega))\|,$$

for every $\omega \in \Omega$. Now,

$$\begin{aligned}
&\left\|F_\alpha(\omega, \xi(\omega)) - F_\alpha^2(\omega, \xi(\omega))\right\| \\
&= \left\| \begin{matrix} (1-\alpha)\xi(\omega) + \alpha T(\omega, F_\alpha(\omega, \xi(\omega))) - (1-\alpha)F_\alpha(\omega, \xi(\omega)) \\ - \alpha T(\omega, F_\alpha^2(\omega, \xi(\omega))) \end{matrix} \right\| \\
&= \left\| \begin{matrix} (1-\alpha)(\xi(\omega) - F_\alpha(\omega, \xi(\omega))) + \alpha(T(\omega, F_\alpha(\omega, \xi(\omega))) \\ - \alpha T(\omega, F_\alpha^2(\omega, \xi(\omega))) \end{matrix} \right\| \\
&= \left\| \begin{matrix} \alpha(F_\alpha(\omega, \xi(\omega)) - T(\omega, F_\alpha(\omega, \xi(\omega)))) + \alpha(T(\omega, F_\alpha(\omega, \xi(\omega))) \\ - \alpha T(\omega, F_\alpha^2(\omega, \xi(\omega))) \end{matrix} \right\| \\
&= \alpha \left\|F_\alpha(\omega, \xi(\omega)) - T(\omega, F_\alpha^2(\omega, \xi(\omega)))\right\|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \alpha \left\| F_\alpha(\omega, \xi(\omega)) - T^n(\omega, F_\alpha(\omega, \xi(\omega))) \right\| \\
&\quad + \alpha \left\| T^n(\omega, F_\alpha(\omega, \xi(\omega))) - T(\omega, F_\alpha^2(\omega, \xi(\omega))) \right\| \\
&\leqslant \alpha a \left\| F_\alpha(\omega, \xi(\omega)) - T(\omega, F_\alpha(\omega, \xi(\omega))) \right\| \\
&\quad + \alpha \left\| T^{n-1}(\omega, F_\alpha(\omega, \xi(\omega))) - F_\alpha^2(\omega, \xi(\omega)) \right\| \\
&= (1-\alpha) a \left\| F_\alpha(\omega, \xi(\omega)) - \xi(\omega) \right\| \\
&\quad + \alpha \left\| T^{n-1}(\omega, F_\alpha(\omega, \xi(\omega))) - F_\alpha^2(\omega, \xi(\omega)) \right\|,
\end{aligned}
$$

for every $\omega \in \Omega$. Now we claim that the following inequality holds for every $\omega \in \Omega$ and $m \geqslant 2$.

$$
\begin{aligned}
&\alpha \left\| T^{m-1}(\omega, F_\alpha(\omega, \xi(\omega))) - F_\alpha^2(\omega, \xi(\omega)) \right\| \\
&\leqslant (m-1) - m\alpha + \alpha^m \left\| \xi(\omega) - F_\alpha(\omega, \xi(\omega)) \right\| \\
&\quad + \alpha^m \left\| F_\alpha(\omega, \xi(\omega)) - F_\alpha^2(\omega, \xi(\omega)) \right\|.
\end{aligned} \tag{3.3}
$$

For this consider,

$$
\begin{aligned}
&\alpha \left\| T(\omega, F_\alpha(\omega, \xi(\omega))) - F_\alpha^2(\omega, \xi(\omega)) \right\| \\
&= \alpha \left\| T(\omega, F_\alpha(\omega, \xi(\omega))) - (1-\alpha) F_\alpha(\omega, \xi(\omega)) - \alpha T(\omega, F_\alpha^2(\omega, \xi(\omega))) \right\| \\
&= \alpha \left\| \begin{array}{c} (1-\alpha)(T(\omega, F_\alpha(\omega, \xi(\omega))) - F_\alpha(\omega, \xi(\omega))) - \alpha(T(\omega, F_\alpha^2(\omega, \xi(\omega))) \\ - T(\omega, F_\alpha(\omega, \xi(\omega)))) \end{array} \right\| \\
&\leqslant (1-\alpha) \left\| \alpha(T(\omega, F_\alpha(\omega, \xi(\omega))) - F_\alpha(\omega, \xi(\omega))) \right\| \\
&\quad + \alpha^2 \left\| T(\omega, F_\alpha^2(\omega, \xi(\omega))) - T(\omega, F_\alpha(\omega, \xi(\omega))) \right\| \\
&= (1-\alpha)^2 \left\| \xi(\omega) - F_\alpha(\omega, \xi(\omega)) \right\| + \alpha^2 \left\| T(\omega, F_\alpha^2(\omega, \xi(\omega))) - T(\omega, F_\alpha(\omega, \xi(\omega))) \right\| \\
&\leqslant (1-\alpha)^2 \left\| \xi(\omega) - F_\alpha(\omega, \xi(\omega)) \right\| + \alpha^2 \left\| F_\alpha^2(\omega, \xi(\omega)) - F_\alpha(\omega, \xi(\omega)) \right\|.
\end{aligned}
$$

So (3.3) is valid for $m = 2$ and for any $\omega \in \Omega$.

Assuming the validity of (3.3) for $m = j$ and for any $\omega \in \Omega$, consider

$$
\begin{aligned}
&\alpha \left\| T^j(\omega, F_\alpha(\omega, \xi(\omega))) - F_\alpha^2(\omega, \xi(\omega)) \right\| \\
&= \alpha \left\| T^j(\omega, F_\alpha(\omega, \xi(\omega))) - (1-\alpha) F_\alpha(\omega, \xi(\omega)) - \alpha T(\omega, F_\alpha^2(\omega, \xi(\omega))) \right\| \\
&= \alpha \left\| \begin{array}{c} (1-\alpha)(T^j(\omega, F_\alpha(\omega, \xi(\omega))) - F_\alpha(\omega, \xi(\omega))) + \alpha(T^j(\omega, F_\alpha(\omega, \xi(\omega))) \\ - T(\omega, F_\alpha^2(\omega, \xi(\omega)))) \end{array} \right\| \\
&\leqslant \alpha(1-\alpha) \left\| T^j(\omega, F_\alpha(\omega, \xi(\omega))) - F_\alpha(\omega, \xi(\omega)) \right\| \\
&\quad + \alpha^2 \left\| T^j(\omega, F_\alpha(\omega, \xi(\omega))) - T(\omega, F_\alpha^2(\omega, \xi(\omega))) \right\| \\
&\leqslant j\alpha(1-\alpha) \left\| F_\alpha(\omega, \xi(\omega)) - T(\omega, F_\alpha(\omega, \xi(\omega))) \right\| \\
&\quad + \alpha^2 \left\| T^{j-1}(\omega, F_\alpha(\omega, \xi(\omega))) - F_\alpha^2(\omega, \xi(\omega)) \right\| \\
&\leqslant j\alpha(1-\alpha) \left\| F_\alpha(\omega, \xi(\omega)) - T(\omega, F_\alpha(\omega, \xi(\omega))) \right\| \\
&\quad + \alpha[(j-1) - j\alpha + \alpha^j] \left\| \xi(\omega) - F_\alpha(\omega, \xi(\omega)) \right\| \\
&\quad + \alpha^{j+1} \left\| F_\alpha(\omega, \xi(\omega)) - F_\alpha^2(\omega, \xi(\omega)) \right\| \\
&\leqslant j(1-\alpha)^2 + \alpha^2[(j-1) - j\alpha + \alpha^j] \left\| \xi(\omega) - F_\alpha(\omega, \xi(\omega)) \right\|
\end{aligned}
$$

$$
\begin{aligned}
&+ \alpha^{j+1} \left\| F_\alpha(\omega, \xi(\omega)) - F_\alpha^2(\omega, \xi(\omega)) \right\| \\
\leqslant\ & [j - (j+1)\alpha + \alpha^{j+1}] \left\| \xi(\omega) - F_\alpha(\omega, \xi(\omega)) \right\| \\
&+ \alpha^{j+1} \left\| F_\alpha(\omega, \xi(\omega)) - F_\alpha^2(\omega, \xi(\omega)) \right\|.
\end{aligned}
$$

So by induction inequality (3.3) is valid for every $\omega \in \Omega$ and $m \geqslant 2$.

Now consider, for $\omega \in \Omega$

$$
\begin{aligned}
&\left\| F_\alpha(\omega, \xi(\omega)) - F_\alpha^2(\omega, \xi(\omega)) \right\| \\
\leqslant\ & (1-\alpha)a \left\| F_\alpha(\omega, \xi(\omega)) - \xi(\omega) \right\| \\
&+ \alpha \left\| T^{n-1}(\omega, F_\alpha(\omega, \xi(\omega))) - F_\alpha^2(\omega, \xi(\omega)) \right\| \\
\leqslant\ & (1-\alpha)a \left\| F_\alpha(\omega, \xi(\omega)) - \xi(\omega) \right\| \\
&+ [(n-1) - n\alpha + \alpha^n] \left\| \xi(\omega) - F_\alpha(\omega, \xi(\omega)) \right\| \\
&+ \alpha^n \left\| F_\alpha(\omega, \xi(\omega)) - F_\alpha^2(\omega, \xi(\omega)) \right\|.
\end{aligned}
$$

It further implies that

$$
\begin{aligned}
&(1-\alpha^n) \left\| F_\alpha(\omega, \xi(\omega)) - F_\alpha^2(\omega, \xi(\omega)) \right\| \\
\leqslant\ & [(1-\alpha)a + (n-1) - n\alpha + \alpha^n] \left\| \xi(\omega) - F_\alpha(\omega, \xi(\omega)) \right\|,
\end{aligned}
$$

for every $\omega \in \Omega$. Now we arrive at

$$
\begin{aligned}
&\left\| F_\alpha(\omega, \xi(\omega)) - F_\alpha^2(\omega, \xi(\omega)) \right\| \\
\leqslant\ & (1-\alpha^n)^{-1}[(1-\alpha)a + (n-1) - n\alpha + \alpha^n] \left\| \xi(\omega) - F_\alpha(\omega, \xi(\omega)) \right\| \\
\leqslant\ & (a+n)(1-\alpha)(1-\alpha^n)^{-1} - 1 \left\| \xi(\omega) - F_\alpha(\omega, \xi(\omega)) \right\| \\
=\ & [(a+n)(\sum_{i=0}^{n-1} \alpha^i)^{-1} - 1] \left\| \xi(\omega) - F_\alpha(\omega, \xi(\omega)) \right\| \\
=\ & g(\alpha) \left\| \xi(\omega) - F_\alpha(\omega, \xi(\omega)) \right\|,
\end{aligned}
$$

for every $\omega \in \Omega$, where $g(\alpha) = [(a+n)(\sum_{i=0}^{n-1} \alpha^i)^{-1} - 1]$. Since g is continuous and decreasing for $\alpha \in (0,1]$ with $g(1) = \frac{a}{n} < 1$, there exists $b \in (0,1]$ such that $g(1) < 1$ for $\alpha \in (b,1]$. For such α, the sequence of measurable mappings defined by $\eta_n(\omega) = F_\alpha^n(\omega, \xi(\omega)) \to \eta(\omega)$, for each $\omega \in \Omega$, $\eta\colon \Omega \to F$, being the limit of the sequence of measurable functions, is also measurable (see remark 2.6). From (3.1) it follows that η is a random fixed point of T. $\square$

Example 3.2. Let $\Omega = [0,1]$ and Σ be the sigma algebra of Lebesgue's measurable subsets of Ω. Take $X = R$ with $d(x,y) = |x-y|$, for $x, y \in R$. Define random operator T from $\Omega \times X$ to X as, $T(\omega, x) = \omega - x$.

Define a fixed measurable mapping $\xi\colon \Omega \to X$ as $\xi(\omega) = \frac{\omega}{3}$, for every $\omega \in \Omega$. Note that T is nonexpansive random operator. Since random operator equation $T^2(\omega, \xi(\omega)) = \xi(\omega)$ holds for every $\omega \in \Omega$, therefore it is $(2,1)-$rotative random operator. Thus the conditions of Theorem 3.1 are satisfied. Moreover a measurable mapping $\eta\colon \Omega \to X$ defined as $\eta(\omega) = \frac{\omega}{2}$, for every $\omega \in \Omega$, serve as a unique random fixed point of T.

Theorem 3.3. *Let X be a separable metric space and $T\colon \Omega \times X \to X$ be a ϵ-contractive random operator. Let $\xi_0\colon \Omega \to X$ be any measurable mapping such that a sequence $\{T^n(\omega, \xi_0(\omega))\}$ has a point wise convergent subsequence of measurable mappings. Then T has a random periodic point.*

Proof. Let $\{T^{n_i}(\omega, \xi_0(\omega))\}$ be a subsequence of $\{T^n(\omega, \xi_0(\omega))\}$ such that $T^{n_i}(\omega, \xi_0(\omega)) \to \xi(\omega)$ for each $\omega \in \Omega$ as $n_i \to \infty$ where $\{n_i\}$ is a strictly increasing sequence of positive integers. The mapping $\xi\colon \Omega \to X$ being point wise limit of sequence of measurable mappings is measurable. Define sequence of measurable mappings $\xi_i\colon \Omega \to X$ as $\xi_i(\omega) = T^{n_i}(\omega, \xi_0(\omega))$. Given $\epsilon > 0$, there exists an integer n_0 such that

$$d(\xi_i(\omega), \xi(\omega)) < \frac{\epsilon}{4}, \text{ for } i \geqslant n_0 \text{ and } \omega \in \Omega.$$

Put $k = n_{i+1} - n_i$. Consider,

$$\begin{aligned} d(\xi_{i+1}(\omega), T^k(\omega, \xi(\omega))) &= d(T^k(\omega, \xi_i(\omega)), T^k(\omega, \xi(\omega))) \\ &< d(\xi_i(\omega), \xi(\omega)) < \frac{\epsilon}{4}, \text{ for each } \omega \in \Omega. \end{aligned}$$

Now,

$$\begin{aligned} &d(\xi(\omega), T^k(\omega, \xi(\omega))) \\ &\leqslant d(\xi_{i+1}(\omega), T^k(\omega, \xi(\omega))) + d(\xi_{i+1}(\omega), \xi(\omega)) \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}, \text{ for every } \omega \in \Omega. \end{aligned}$$

Now we claim that ξ is a random periodic point of T. To prove this, assume that $\eta\colon \Omega \to X$ be any measurable mapping such that $\eta(\omega) = T^k(\omega, \xi(\omega))$ but

$$\eta(\omega) \neq \xi(\omega), \text{ for some } \omega \in \Omega. \tag{3.4}$$

Which implies that $0 < d(\eta(\omega), \xi(\omega)) < \epsilon$. As T is a ϵ- contractive random operator therefore for $\omega \in \Omega$ for which (3.4) holds, we have

$$d(T(\omega, \xi(\omega)), T(\omega, \eta(\omega))) < d(\xi(\omega), \eta(\omega)).$$

Define $h\colon \Omega \times X^2 \to R$ as, $h(\omega, x(\omega), y(\omega)) = \frac{d(T(\omega, x(\omega)), T(\omega, y(\omega)))}{d(x(\omega), y(\omega))}$, where $x(\omega) \neq y(\omega) \in X$ for each $\omega \in \Omega$. Now $h(\omega, ., .)$ is continuous at $(\xi(\omega), \eta(\omega))$ for every $\omega \in \Omega$ for which (3.4) is valid.

Take $0 < \alpha < 1$, then there exists $\delta > 0$ such that $x(\omega) \in B(\xi(\omega), \delta)$ and $y(\omega) \in B(\eta(\omega), \delta)$ gives

$$d(T(\omega, x(\omega)), T(\omega, y(\omega))) < \alpha d(x(\omega), y(\omega)).$$

As, $\lim\limits_{r \to \infty} T^k(\omega, \xi_r(\omega)) = T^k(\omega, \xi(\omega)) = \eta(\omega)$, for every $\omega \in \Omega$. So there exists $n_1 \geqslant n_0$ such that

$$d(\xi_r(\omega), \xi(\omega)) < \delta$$

and

$$d(T^k(\omega, \xi_r(\omega)), \eta(\omega)) < \delta,$$

for $r \geqslant n_1$ and $\omega \in \Omega$. Hence we have

$$d(T(\omega, \xi_r(\omega)), T(\omega, T^k(\omega, \xi_r(\omega)))) < \alpha d(\xi_r(\omega), T^k(\omega, \xi_r(\omega))). \tag{3.5}$$

Consider,

$$\begin{aligned} &d(\xi_r(\omega), T^k(\omega, \xi_r(\omega))) \\ &\leqslant d(\xi_r(\omega), \xi(\omega)) + d(\xi(\omega), T^k(\omega, \xi(\omega))) + d(T^k(\omega, \xi(\omega)), T^k(\omega, \xi_r(\omega))) \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{2} + \frac{\epsilon}{4} = \epsilon, \end{aligned} \tag{3.6}$$

for $r \geqslant n_1 \geqslant n_0$ and $\omega \in \Omega$ for which (3.4) holds. Now using (3.5) and (3.6), we have

$$\begin{aligned} &d(T(\omega, \xi_r(\omega)), T(\omega, T^k(\omega, \xi_r(\omega)))) \\ &< \alpha d(\xi_r(\omega), T^k(\omega, \xi_r(\omega))) < d(\xi_r(\omega), T^k(\omega, \xi_r(\omega))) < \epsilon, \end{aligned}$$

for $r \geqslant n_1$. Since T is a ϵ- contractive random operator so for $r \geqslant n_1$ and $q > 0$, we have

$$\begin{aligned} &d(T^q(\omega, \xi_r(\omega)), T^q(\omega, T^k(\omega, \xi_r(\omega)))) \\ &< d(\xi_r(\omega), T^k(\omega, \xi_r(\omega))) < \frac{\epsilon}{\alpha}. \end{aligned}$$

Put $q = n_{r+1} - n_r$, we have $d(\xi_{r+1}(\omega), T^k(\omega, \xi_{r+1}(\omega))) < \frac{\epsilon}{\alpha}$. Hence,

$$d(\xi_s(\omega), T^k(\omega, \xi_s(\omega))) < \epsilon\alpha^{s-r}.$$

Now,

$$\begin{aligned} d(\xi(\omega), \eta(\omega)) &\leqslant d(\xi(\omega), \xi_s(\omega)) + d(\xi_s(\omega), T^k(\omega, \xi_s(\omega))) \\ &+ d(T^k(\omega, \xi_s(\omega)), \eta(\omega)) \to 0, \text{ as } s \to \infty. \end{aligned}$$

for those $\omega \in \Omega$ for which (3.4) holds. This contradiction concludes the result. □

Corollary 3.4. *If in theorem 3.2, the random periodic point ξ (say) of T satisfies*

$$d(\xi(\omega), T(\omega, \xi(\omega))) < \epsilon, \text{ for every } \omega \in \Omega. \tag{3.7}$$

Then ξ is a random fixed point of T.

Proof. Let k be the positive integer such that $T^k(\omega, \xi(\omega)) = \xi(\omega)$, for every $\omega \in \Omega$. If ξ is not a random fixed point of T, then $\xi(\omega) \neq T(\omega, \xi(\omega)$ for some $\omega \in \Omega$. Since T is ϵ- contractive random operator, using (3.7) we have

$$\begin{aligned} d(\xi(\omega), T(\omega, \xi(\omega))) &= d(T^k(\omega, \xi(\omega)), T^{k+1}(\omega, \xi(\omega))) \\ &< d(\xi(\omega), T(\omega, \xi(\omega))). \end{aligned}$$

This contradiction concludes the proof. □

Remark 3.5. If X is a separable compact metric space and $T\colon \Omega \times X \to X$ is an ϵ- contractive random operator. Then applying theorem 3.3, we conclude that T has a random periodic point.

Theorem 3.6. *Let X be a separable compact metric space and $T\colon \Omega \times X \to X$ be an ϵ- contractive random operator. Then T has finitely many random periodic points.*

Proof. Let ξ, $\zeta\colon \Omega \to X$ be two random periodic points of T with $\xi(\omega) \neq \zeta(\omega)$ and $d(\xi(\omega), \zeta(\omega)) < \epsilon$ for some $\omega \in \Omega$. Let $m, n \geqslant 1$ be two integers such that $T^m(\omega, \xi(\omega)) = \xi(\omega)$ and $T^n(\omega, \zeta(\omega)) = \zeta(\omega)$ for every $\omega \in \Omega$. Obviously $T^{mn}(\omega, \xi(\omega)) = \xi(\omega)$ and $T^{mn}(\omega, \zeta(\omega)) = \zeta(\omega)$ for each $\omega \in \Omega$. Now consider,

$$\begin{aligned} d(\xi(\omega), \zeta(\omega)) &= d(T^{mn}(\omega, \xi(\omega)), T^{mn}(\omega, \zeta(\omega))) \\ &< d(\xi(\omega), \zeta(\omega)), \end{aligned}$$

which is contradiction. Therefore any two random periodic point of T must be at least ϵ- apart. Compactness of X prevents us defining infinitely many random periodic points from $\Omega \times X$ to X. □

Acknowledgement. The authors are thankful to referee for precise remarks to improve the presentation of the paper.

References

[1] Aubin, J. P., and Frankowska, H., *Set-Valued Analysis*, Birkhauser, Boston, 2009.

[2] Beg, I., Random fixed points of random operators satisfying semicontractivity conditions, *Math. Japan.*, 46 (1) (1997), 151–155.

[3] Beg, I., Random Edelstein theorem, *Bull.Greek Math. Soc.*, 45 (2001), 31–41.

[4] Beg, I., Minimal displacement of random variables under Lipschitz random maps, *Topological Methods in Nonlinear Anal.*, 19 (2002), 391–397.

[5] Beg, I., and Abbas, M., Iterative procedures for solutions of random operator equations in Banach spaces, *J. Math. Anal. Appl.*, 315(1)(2006), 181–201.

[6] Beg, I., and Thakur, B.S., Solution of random operator equations using general composite implicit iteration process, *Int. J. Modern Math.*, 4(1)(2009), 19–34.

[7] Bharucha-Reid, A.T., *Random Integral Equations*, Academic Press, New York and London, 1972.

[8] Bharucha- Reid, A.T., Fixed point theorems in probabilistic analysis, *Bull. Amer. Math. Soc.*, 82 (1976), 641–645.

[9] Edelstein, M., On fixed and periodic points under contractive mapping, *J. London Math. Soc.*, 37 (1962), 74–79.

[10] Fournier, G., and Górniewicz, L., The Lefschetz fixed point theorem for some non compact multivalued maps, *Fund. Math.*, 94 (1977), 245–254.

[11] Goebel, K., and Kirk, W.A., *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge 1990.

[12] Goebel, K., and Koter, M., A remark on nonexpansive mappings, *Canadian Math. Bull.*, 24 (1981), 113–115.

[13] Górniewicz, L., *Topological Fixed Point Theory of Multivalued Mappings*, Kluwer, Dordrecht, 1999.

[14] Halpern, B., Periodic points on tori, *Pacific J. Math.*, 83 (1979), 117–133.

[15] Hanš, O., Reduzierende zulliällige transformaten, *Czechoslovak Math. J.*, 7 (1957), 154–158.

[16] Hanš, O., Random operator equations, in: *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, Vol. II, Part I, University of California Press, Berkeley, 1961, 185–202.

[17] Itoh, S., Random fixed point theorems with an application to random differential equations in Banach spaces, *J. Math. Anal. Appl.*, 67 (1979), 261–273.

[18] Kalmoun, E.M., Some deterministic and random vector equilibrium problems, *J. Math. Anal. Appl.*, 267 (2002), 62–75.

[19] Kampen, J., On fixed points of maps and iterated maps and applications, *Nonlinear Anal.*, 42(3) (2000), 509–532.

[20] Matsuoka, T., The number of periodic points of smooth maps, *Ergod. Th. and Dynam. Sys.*, 8 (1989), 153–163.

[21] O'Regan, D., A continuation type result for random operators, *Proc. Amer. Math. Soc.*, 126, 7(1998), 1963–1971.

[22] Papageorgiou, N.S., Random fixed point theorems for measurable multifunctions in Banach spaces, *Proc. Amer. Math. Soc.*, 97 (1986), 507–514.

[23] Papageorgiou, N.S., On measurable multifunctions with stochastic domain, *J. Austral. Math. Soc.* (Ser. A), 45 (1988), 204–216.

[24] Spacek, A., Zufällige Gleichungen, *Czechoslovak Math. J.*, 5 (1955), 462–466.

Ismat Beg, Mujahid Abbas
Center for Advanced Studies in Mathematics,
Lahore University of Management Sciences,
Lahore-54792, Pakistan
Phone: 0092-42-35608229
Fax: 0092-42-35722591
e-mail: `ibeg@lums.edu.pk`
`mujahid@lums.edu.pk`

Akbar Azam
Department of Mathematics,
COMSATS Institute of Information Technology,
Islamabad, Pakistan