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# Positive trigonometric sums and applications

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## Abstract

Some new positive trigonometric sums that sharpen Vietoris’s classical inequalities are presented. These sharp inequalities have remarkable applications in geometric function theory. In particular, we obtain information for the partial sums of certain analytic functions that correspond to starlike functions in the unit disk. We also survey some earlier results with additional remarks and comments.

*Keywords:* Positive trigonometric sums, starlike functions.

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## 1. Introduction

The problem of constructing positive trigonometric sums is very old and has been dealt with by many authors. The most familiar examples are the Fejér-Jackson-Gronwall inequality

$$\sum_{k=1}^n \frac{\sin k\theta}{k} > 0, \quad \text{for all } n \in \mathbb{N} \text{ and } 0 < \theta < \pi, \quad (1.1)$$

and Young’s inequality

$$1 + \sum_{k=1}^n \frac{\cos k\theta}{k} > 0, \quad \text{for all } n \in \mathbb{N} \text{ and } 0 < \theta < \pi. \quad (1.2)$$

As it is known, inequality (1.1) conjectured by Fejér in 1910, and its first proof published by D. Jackson [20], in 1911 and also was proved independently by T. H. Gronwall in [22] few months later. Inequality (1.2) was established by W. H. Young

[35], in 1913. Since then these inequalities have attracted the attention of several mathematicians who offered new and simpler proofs, various generalizations and extensions of different type. A complete account of related results, historical comments and an extensive bibliography can be found in monograph [10] and also in [26, Ch.4] and in the survey article [12]. It is worth mentioning that these inequalities are naturally incorporated in the context of more general results for classical orthogonal polynomials. (cf. [10, 11, 12]). Several applications have indicated which generalizations of (1.1) and (1.2) are essential and have led to a deeper understanding of these results. Conversely, a variety of problems reduces to positivity results for trigonometric or other orthogonal sums of this type. Indeed, these inequalities have remarkable applications in the theory of Fourier series, summability theory, approximation theory, positive quadrature methods, the theory of univalent functions and many others.

We refer the reader to the recently published research articles [1, 2, 3, 4, 5, 6, 7], [13], [16], [19], [21] and [23] for some new results on positive trigonometric sums including refinements and extensions of (1.1) and (1.2) and various applications. Of course, we cannot survey this whole subject here and we restrict ourselves on some recently found refinements of a far reaching extension of (1.1) and (1.2) due to Vietoris and some applications of them in geometric function theory. We also summarize some earlier closely related results. We note that positivity results for trigonometric sums and geometric function theory have been closely related subjects over the past century. Both areas have taken and given to each other and this paper intends to present few more results of this interplay.

## 2. Vietoris's inequalities

In 1958 L. Vietoris [34] gave a striking generalization of both (1.1) and (1.2). In particular, he showed that if  $a_k$ ,  $k = 0, 1, 2, \dots$  is a decreasing sequence of positive real numbers such that

$$2ka_{2k} \leq (2k-1)a_{2k-1}, \quad k \geq 1, \quad (2.1)$$

then for all positive integers  $n$ , we have

$$\sum_{k=0}^n a_k \cos k\theta > 0, \quad 0 < \theta < \pi, \quad (2.2)$$

and

$$\sum_{k=1}^n a_k \sin k\theta > 0, \quad 0 < \theta < \pi. \quad (2.3)$$

Vietoris observed that (2.2) and (2.3) follow by a partial summation from the special case  $a_k = c_k$ , where

$$c_0 = c_1 = 1 \quad \text{and} \quad c_{2k} = c_{2k+1} = \frac{1.3.5 \dots (2k-1)}{2.4.6 \dots 2k}, \quad k \geq 1. \quad (2.4)$$

Conversely, this  $c_k$  is an extreme sequence in (2.1). It is clear that the sequence  $a_0 = 1$ ,  $a_k = \frac{1}{k}$ ,  $k \geq 1$  satisfies (2.1), hence inequalities (1.1) and (1.2) are obtained by Vietoris's result.

The importance of Vietoris's inequalities became widely known after the work of R. Askey and J. Steinig [9], (see also [8, p. 375]), who gave a simplified proof of them and showed that they have some nice applications in estimating the zeros of certain trigonometric polynomials. Askey and Steinig also observed that these inequalities are better viewed in the context of more general inequalities concerning positive sums of Jacobi polynomials and they play a role in problems dealing with quadrature methods. (See the comments and remarks in [10, p. 87]). Vietoris's theorem is nowadays one of the most cited results in the area having received attention from several authors who offered various extensions and generalizations. Several new applications of these inequalities have also been given. For instance, S. Ruscheweyh [31] used them to derive some coefficient conditions for starlike univalent functions. In a recent work, S. Ruscheweyh and L. Salinas [33] gave a beautiful interpretation of Vietoris theorem in geometric function theory and they pointed a new direction of applications for this type of results. For more background information, we refer the reader to the recent paper [23].

It is the aim of this article to present some recently found generalizations and extensions of Vietoris's inequalities which are useful in applications and to provide some additional comments and remarks on these results.

We first observe that Vietoris's sine inequality cannot be much improved if we require all the sums in (2.3) to be positive. Indeed, suppose that  $a_0 \geq a_1 \geq \dots \geq a_n > 0$ . The condition

$$\sum_{k=1}^n (-1)^{k-1} k a_k \geq 0, \text{ for all } n \geq 1 \quad (2.5)$$

is necessary for the positivity of all these sine sums in  $(0, \pi)$ . (Divide (2.3) by  $\sin \theta$  and take the limit as  $\theta \rightarrow \pi$  to obtain (2.5)). Obviously, for a non-negative sequence  $(a_n)$ , the condition (2.5) is equivalent to

$$\sum_{k=1}^n ((2k-1)a_{2k-1} - 2ka_{2k}) \geq 0 \text{ for all } n \geq 1,$$

which holds as an equality for the extreme sequence (2.4).

In an impressive paper, A. S. Belov [13] proved that the condition (2.5) is also sufficient for the validity of (2.3) and it also implies the positivity of the corresponding cosine sums (2.2). Clearly, Belov's result implies Vietoris's theorem. It should be noted that if either (2.1) or (2.5) is weakened then the sums in (2.3) are not everywhere positive in  $(0, \pi)$ . It is possible, however, to have everywhere positive sine sums in (2.3) under weaker conditions on the coefficients in the case where  $n$  is odd. This will be discussed in the next section.

We now turn to Vietoris's cosine inequality (2.2) which has received a substantial improvement and sharpening over the past twenty years.

The first result in this direction is due to G. Brown and E. Hewitt [15] who proved that all the cosine sums in (2.2) remain positive when (2.1) is replaced by the weaker condition  $(2k+1)a_{2k} \leq 2k a_{2k-1}$ ,  $k \geq 1$ . It is interesting to observe that their result follows from the particular case  $a_k = p_k$ , where  $p_{2k} = p_{2k+1} = \frac{2.4.6 \dots (2k)}{3.5.7 \dots (2k+1)}$ .

In [13], A. S. Belov established another sufficient condition for the positivity of cosine sums (2.2). Namely, let  $a_{2k} = a_{2k+1} = \gamma_k$ , where  $\gamma_k$  is a decreasing sequence of non-negative real numbers satisfying

$$\sum_{k=0}^m \gamma_k - \sum_{k=m}^n \gamma_k + \frac{2}{3}(n-3m)\gamma_m \geq 0 \text{ for all } n \geq 1, \quad (2.6)$$

where  $m = \left[ \frac{n+1}{3} \right]$ , the square brackets denoting the integer part of  $(n+1)/3$ , then inequality (2.2) holds. Take now as  $\gamma_k$  the sequence  $\frac{1.3.5 \dots (2k-1)}{2.4.6 \dots 2k}$  and see that this sequence satisfies both (2.5) and (2.6). Then take as  $\gamma_k$  the sequence  $\frac{2.4.6 \dots (2k)}{3.5.7 \dots (2k+1)}$  and observe that this satisfies only (2.6). So, Belov's result implies both the Vietoris and the Brown-Hewitt theorem. Condition (2.6), however, provides no best possible extension of Vietoris's cosine inequality. Consider, for example  $a_0 = a_1 = 1$  and  $a_{2k} = a_{2k+1} = \gamma_k = \frac{3.5 \dots (2k+1)}{4.6 \dots (2k+2)}$ ,  $k = 1, 2, \dots$ . G. Brown and Q. Yin showed in [17] that for this choice of coefficients the cosine sums (2.2) are positive. This is a further extension of both Vietoris and Brown-Hewitt cosine inequalities which is, still, not best possible. We observe that for this sequence  $\gamma_k$  inequality (2.6) fails to hold for some  $n$ , therefore this sharpening is not deduced from Belov's result. Other examples of this type will be given in the next section.

In [17] the following direction for a further improvement of (2.2) was suggested. Suppose that  $a_0 \geq a_1 \geq \dots \geq a_n > 0$  such that

$$\frac{a_{2k}}{a_{2k-1}} \leq \frac{2k + \beta - 1}{2k + \beta}, \quad k \geq 1, \quad (2.7)$$

and determine the maximum value of  $\beta > 0$ , such that condition (2.7) implies (2.2) for all  $n$ . The authors observed in [17] that this value of  $\beta$  does not exceed 2.34. Clearly, it is sufficient to consider the extreme sequence  $a_k = e_k$  for which we have equality in (2.7). This can be written as  $e_0 = e_1 = 1$  and

$$e_{2k} = e_{2k+1} = \delta_k := \frac{\left(\frac{1+\beta}{2}\right)_k}{\left(\frac{2+\beta}{2}\right)_k}, \quad k = 0, 1, 2, \dots, \quad (2.8)$$

using the Pochhammer symbol,

$$(a)_0 = 1, \text{ and } (a)_k = a(a+1) \dots (a+k-1) = \frac{\Gamma(k+a)}{\Gamma(a)}, \text{ for } k = 1, 2, \dots$$

Observe that for  $\beta = 0, 1, 2$  in (2.8), we obtain the Vietoris, Brown-Hewitt and Brown-Yin results respectively.

The maximum value of  $\beta$  for which condition (2.7) implies the positivity of cosine sums (2.2) is  $\beta = 2.33088\dots$ , and it is determined by the case  $n = 6$ . This has been obtained in [23]. A different extension of Vietoris cosine inequality is used in the proof of this result. This extension and some of its consequences will be presented in the next section.

We complete this section by observing that for the sequence  $\delta_k$  in (2.8) we have  $\delta_k \sim \frac{1}{k^{\frac{1}{2}}}$ , as  $k \rightarrow \infty$  for all  $\beta > 0$ .

If we replace  $\delta_k$  in (2.8) by  $d_k := \frac{(1-\alpha)_k}{k!}$ ,  $0 < \alpha < 1$ , we see that  $d_k \sim \frac{1}{k^\alpha}$ , as  $k \rightarrow \infty$ , and that Vietoris's sequence (2.4) corresponds to the case  $\alpha = \frac{1}{2}$ .

### 3. Extensions of Vietoris cosine inequality

Vietoris's cosine inequality admits the following sharpening.

**Theorem 3.1.** *Let  $0 < \alpha < 1$  and*

$$\begin{aligned} c_0 &= c_1 = 1 \\ c_{2k} &= c_{2k+1} = \frac{(1-\alpha)_k}{k!}, \quad k = 1, 2, \dots \end{aligned}$$

*For all positive integers  $n$  and  $0 < \theta < \pi$ , we have*

$$\sum_{k=0}^n c_k \cos k\theta > 0,$$

*when  $\alpha \geq \alpha_0$ , where  $\alpha_0$  is the unique solution in  $(0, 1)$  of the equation*

$$\int_0^{\frac{3\pi}{2}} \frac{\cos t}{t^\alpha} dt = 0.$$

*Also for  $\alpha < \alpha_0$*

$$\lim_{n \rightarrow \infty} \min \left\{ \sum_{k=0}^n c_k \cos k\theta : \theta \in (0, \pi) \right\} = -\infty. \quad (3.1)$$

*Numerical methods give  $\alpha_0 = 0.3084437\dots$*

Let us denote

$$d_k = \frac{(1-\alpha)_k}{k!}, \quad k = 0, 1, \dots$$

Notice that for the sequence  $c_k$  of the theorem above we have

$$\sum_{k=0}^{2n+1} c_k \cos k\theta = 2 \cos \frac{\theta}{2} \sum_{k=0}^n d_k \cos \left(2k + \frac{1}{2}\right)\theta, \quad 0 < \theta < \pi,$$

therefore an immediate consequence of this theorem is

$$\sum_{k=0}^n d_k \cos \left(2k + \frac{1}{2}\right)\theta > 0 \quad \text{for all } n \text{ and } 0 < \theta < \pi, \quad (3.2)$$

if and only if  $\alpha \geq \alpha_0$ .

We also observe that

$$\sin \frac{\theta}{2} \sum_{k=0}^{2n+1} c_k \cos k\theta = \cos \frac{\theta}{2} \sum_{k=1}^{2n+1} c_k \sin k(\pi - \theta)$$

which is to say that inequalities

$$\sum_{k=0}^{2n+1} c_k \cos k\theta > 0, \quad 0 < \theta < \pi \quad (3.3)$$

and

$$\sum_{k=1}^{2n+1} c_k \sin k\theta > 0, \quad 0 < \theta < \pi \quad (3.4)$$

are equivalent, hence inequality (3.4) holds for all positive integers  $n$  and  $0 < \theta < \pi$ , if and only if  $\alpha \geq \alpha_0$ .

The situation is different when we are looking for a corresponding result for even sine sums, that is,

$$\sum_{k=1}^{2n} c_k \sin k\theta > 0, \quad 0 < \theta < \pi. \quad (3.5)$$

Taking into consideration the necessary and sufficient condition (2.5) (or its equivalent version given in Section 2) we infer that (3.5) holds precisely when  $\alpha \geq 1/2$ , hence Viëtoris result is, in this case, best possible.

We note that for sine sums having coefficients  $c_k$  there is no analogue of (3.1) when  $\alpha < \alpha_0$ . Indeed, these sums are uniformly bounded below on  $(0, \pi)$  because the conjugate Dirichlet kernel  $\tilde{D}_n(\theta) = \sum_{k=1}^n \sin k\theta$  satisfies  $\tilde{D}_n(\theta) \geq -\frac{1}{2}$  for all  $n$  and  $\theta \in (0, \pi)$ .

In order to prove Theorem 3.1 we have to consider only the critical case  $\alpha = \alpha_0$ , then the full result follows by a partial summation. Another interesting observation here is that for the sequence  $d_k$ , with  $\alpha = \alpha_0$ , condition (2.6) fails to hold.

A proof of Theorem 3.1 is given in [23]. A different proof was independently obtained in [18].

In the section that follows, we shall see that the sharpening of Vietoris inequality given in Theorem 3.1 is not artificial and that sharp results of this type are necessary in the resolution of some specific problems in geometric function theory. Inequality (3.2) was an important ingredient in the proof of a conjecture regarding subordination of certain starlike functions, originally presented in [24], then settled in [25]. This, as well as a generalized version of (3.2) will be given next.

## 4. Applications to geometric function theory

We first, recall some necessary definitions, notations, and background results.

Let  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  be the unit disk in the complex plane  $\mathbb{C}$  and  $A(\mathbb{D})$  be the space of analytic functions in  $\mathbb{D}$ . It is well known that  $A(\mathbb{D})$  is a locally convex linear topological space with respect to the topology given by uniform convergence on compact subsets of  $\mathbb{D}$ . For  $\lambda < 1$  let  $\mathcal{S}_\lambda$  be the family of functions  $f$  starlike of order  $\lambda$ , i.e.

$$\mathcal{S}_\lambda = \left\{ f \in A(\mathbb{D}) : f(0) = f'(0) - 1 = 0 \text{ and } \operatorname{Re} \left( z \frac{f'(z)}{f(z)} \right) > \lambda, \ z \in \mathbb{D} \right\}.$$

The family  $\mathcal{S}_\lambda$  was introduced by M. S. Robertson in [29], and since then it has been the subject of systematic study by several researchers. We note that  $\mathcal{S}_\lambda$  is a compact subset of  $A(\mathbb{D})$  and that  $f_\lambda(z) := \frac{z}{(1 - e^{-it}z)^{2-2\lambda}}$  belong to  $\mathcal{S}_\lambda$ , for all  $t \in \mathbb{R}$ , and they represent the extreme points of the closed convex hull of  $\mathcal{S}_\lambda$ . We have

$$\overline{\operatorname{conv}}(\mathcal{S}_\lambda) = \left\{ \int_0^{2\pi} \frac{z}{(1 - e^{-it}z)^{2-2\lambda}} d\mu(t) : \mu \in \mathcal{P}(\mathbb{T}) \right\}, \quad (4.1)$$

where  $\mathcal{P}(\mathbb{T})$  denotes the set of all probability measures on the unit circle  $\mathbb{T}$ . Also

$$\operatorname{ex}(\overline{\operatorname{conv}}(\mathcal{S}_\lambda)) = \left\{ \frac{z}{(1 - \chi z)^{2-2\lambda}} : \chi \in \mathbb{T} \right\} \subset \mathcal{S}_\lambda$$

(cf. [14] and [30]). Suppose that

$$f(z) = z \sum_{k=0}^{\infty} a_k z^k \in \mathcal{S}_\lambda.$$

Then we have

$$\operatorname{Re} \left\{ f(z)/z \right\} = \operatorname{Re} \sum_{k=0}^{\infty} a_k z^k > \frac{1}{2}, \quad \text{when } \frac{1}{2} \leq \lambda < 1,$$

but this conclusion is not necessarily true for all the partial sums of such a function. For an analytic function  $f(z) = \sum_{k=0}^{\infty} a_k z^k$  and  $n \in \mathbb{N}$  we set  $s_n(f, z) = \sum_{k=0}^n a_k z^k$ , for the  $n$ -th partial sum of  $f$ .

It has been shown in [32] that  $\operatorname{Re} s_n(f/z, z) > 0$  holds in  $\mathbb{D}$  for all  $n \in \mathbb{N}$  and for all  $f \in \mathcal{S}_{3/4}$  and it has been pointed out that the number  $\frac{3}{4}$  can probably be replaced by a smaller one. The smallest value of  $\lambda$  such that  $\operatorname{Re} s_n(f/z, z) > 0$  holds in  $\mathbb{D}$  for all  $n \in \mathbb{N}$  and for all  $f \in \mathcal{S}_\lambda$ , has been determined in [24]. Indeed, we have the following.

**Theorem 4.1.** *For all  $n \in \mathbb{N}$  and  $z \in \mathbb{D}$ , we have*

$$\operatorname{Re} s_n(f/z, z) > 0, \quad \text{for all } f \in \mathcal{S}_\lambda,$$

*if and only if  $\lambda_0 \leq \lambda < 1$ , where  $\lambda_0$  is the unique solution in  $(\frac{1}{2}, 1)$  of the equation*

$$\int_0^{3\pi/2} t^{1-2\lambda} \cos t \, dt = 0.$$

*In fact,*

$$\lambda_0 = \frac{1 + \alpha_0}{2} = 0.654222\dots,$$

*where  $\alpha_0$  is as in Theorem 3.1.*

We give the idea of the proof of this theorem, in order to see that the sharp versions of positive trigonometric sums are indispensable.

Let

$$f(z) = z \sum_{k=0}^{\infty} a_k z^k \in \mathcal{S}_\lambda.$$

It follows from (4.1) that

$$a_k = \hat{\mu}(k) \frac{(2-2\lambda)_k}{k!}, \quad k = 0, 1, \dots$$

where  $\hat{\mu}(k)$  are the Fourier coefficients of the measure  $\mu$ . Since

$$s_n(f/z, z) = \sum_{k=0}^n a_k z^k,$$

we deduce from the above that

$$\operatorname{Re} s_n(f/z, e^{i\theta}) = \int_0^{2\pi} \sum_{k=0}^n \frac{(2-2\lambda)_k}{k!} \cos k(\theta - t) \, d\mu(t).$$

By the minimum principle for harmonic functions it suffices to prove that

$$\sum_{k=0}^n \frac{(2-2\lambda)_k}{k!} \cos k\theta > 0, \quad \forall n \in \mathbb{N}, \quad \forall \theta \in \mathbb{R}, \quad (4.2)$$

if and only if  $\lambda_0 \leq \lambda < 1$ . This inequality is different from the one given in Theorem 3.1, in the sense that none implies the other. The proof of both requires



several sharp estimates and a delicate calculus work. To get the flavor of this and the common features of (4.2) with Theorem 3.1, we set  $\lambda = \frac{1+\alpha}{2}$  and consider the following limiting case

$$\lim_{n \rightarrow \infty} \left(\frac{\theta}{n}\right)^{1-\alpha} \sum_{k=0}^n \frac{(1-\alpha)_k}{k!} \cos k \frac{\theta}{n} = \frac{1}{\Gamma(1-\alpha)} \int_0^\theta \frac{\cos t}{t^\alpha} dt. \quad (4.3)$$

It follows from this that for  $\theta = 3\pi/2$  and  $\lambda < \lambda_0 = \frac{1+\alpha_0}{2}$ , the right hand side of (4.3) will be negative, therefore inequality (4.2) cannot hold for  $\lambda < \lambda_0$ , appropriate  $\theta$  and  $n$  sufficiently large. See also the discussion in [36, V, 2.29]. There is a simple way of proving (4.3), which reflects the idea of the proof of (4.2). Let

$$\Delta_k := \frac{1}{\Gamma(1-\alpha) k^\alpha} - \frac{(1-\alpha)_k}{k!}, \quad k = 1, 2, \dots, \quad 0 < \alpha < 1.$$

Since

$$\frac{\Gamma(x+1-\alpha)}{\Gamma(x+1)} x^\alpha = 1 - \frac{\alpha(1-\alpha)}{2} \frac{1}{x} + O\left(\frac{1}{x^2}\right), \quad \text{as } x \rightarrow \infty,$$

(See [8]), we have

$$\Delta_k = O\left(\frac{1}{k^{\alpha+1}}\right), \quad \text{as } k \rightarrow \infty.$$

On the other hand,

$$\sum_{k=1}^n \frac{1}{k^{\alpha+1}} = \zeta(\alpha+1) + O\left(\frac{1}{n^\alpha}\right), \quad \text{as } n \rightarrow \infty.$$

So that, putting everything together, we arrive at

$$\lim_{n \rightarrow \infty} \left(\frac{\theta}{n}\right)^{1-\alpha} \sum_{k=1}^n \Delta_k \cos k \frac{\theta}{n} = 0.$$

Using this and the results of [27, Part 2, Ch.1, Problems 20–21], the desired asymptotic formula (4.3) follows. The argument given above, reveals that in order to find estimates of the sums on the left hand side of (4.2) it is sufficient to look for appropriate estimates of the sums  $\sum_{k=1}^n \frac{\cos k\theta}{k^\alpha}$ , provided that sharp inequalities for the sequence  $\Delta_k$  are available. Details of all of this are in [24].

An immediate consequence of (4.2) is the following. Let

$$s_n^\lambda(z) := \sum_{k=0}^n \frac{(2-2\lambda)_k}{k!} z^k.$$

Then

$$\operatorname{Re} s_n^\lambda(z) > 0, \quad \forall n \in \mathbb{N}, \quad \forall z \in \mathbb{D}, \quad (4.4)$$

if and only if  $\lambda_0 \leq \lambda < 1$ . This is, of course, the particular case of Theorem 4.1 when applied to the extremal function  $f_\lambda(z) := \frac{z}{(1-z)^{2-2\lambda}}$  of  $\mathcal{S}_\lambda$ .

Next we shall give some other ways of extending (4.4). It turns out that inequalities of this type take a very nice and natural form when the notion of complex subordination is employed. We recall the definition of subordination of analytic functions. Let  $f(z), g(z) \in A(\mathbb{D})$ . We say that  $f(z)$  is *subordinate to*  $g(z)$ , if there exists a function  $\phi(z) \in A(\mathbb{D})$  satisfying  $\phi(0) = 0$  and  $|\phi(z)| < 1$  such that  $f(z) = g(\phi(z))$ ,  $\forall z \in \mathbb{D}$ . Subordination is denoted by  $f(z) \prec g(z)$ . If  $f(z) \prec g(z)$  then  $f(0) = g(0)$  and  $f(\mathbb{D}) \subset g(\mathbb{D})$ . Conversely, if  $g(z)$  is univalent and  $f(0) = g(0)$  and  $f(\mathbb{D}) \subset g(\mathbb{D})$  then  $f(z) \prec g(z)$ . See [28, Ch.2] for proofs and several properties of analytic functions associated with subordination.

It is easily inferred that (4.4) is equivalent to

$$s_n^\lambda(z) \prec \frac{1+z}{1-z}, \quad \forall n \in \mathbb{N}, \quad \forall z \in \mathbb{D}.$$

Consider now the function

$$v(z) := \left( \frac{1+z}{1-z} \right)^{\frac{1}{2}} = \sum_{k=0}^{\infty} c_k z^k, \quad z \in \mathbb{D},$$

where

$$c_0 = c_1 = 1, \\ c_{2k} = c_{2k+1} = \frac{\left(\frac{1}{2}\right)_k}{k!} = \frac{1.3 \dots (2k-1)}{2.4 \dots 2k}, \quad k = 1, 2, \dots$$

This is a univalent function in  $\mathbb{D}$  and maps  $\mathbb{D}$  onto the sector

$$\left\{ \zeta \in \mathbb{C} : |\arg \zeta| < \frac{\pi}{4} \right\}.$$

Observe that these coefficients  $c_k$  are exactly the same as in relation (2.4) of Vietoris theorem. We note that the function  $v(z)$  plays, indeed, a key role in the proof of Vietoris result as it is given in [9], and its properties inspired the geometric interpretation of this theorem as presented in [33].

Now a strengthening of (4.4) reads as follows.

**Theorem 4.2.** *For all  $n \in \mathbb{N}$  and  $z \in \mathbb{D}$  we have*

$$(1-z)^{\frac{1}{2}} s_n^\lambda(z) \prec \left( \frac{1+z}{1-z} \right)^{\frac{1}{2}} \tag{4.5}$$

*if and only if  $\lambda_0 \leq \lambda < 1$ .*

This theorem was stated in [24] as a conjecture which was proved in [25]. It has several other consequences for the class of starlike functions  $\mathcal{S}_\lambda$ . Complete details can be found in [25]. Let us summarize here some of the important facts behind the proof of this result and its relevance to positive trigonometric sums discussed in the previous section.

It is clear that

$$\frac{1}{(1-z)^{1/2}} \prec \left(\frac{1+z}{1-z}\right)^{\frac{1}{2}}, \quad z \in \mathbb{D},$$

therefore (4.5) implies (4.4). Accordingly, (4.5) cannot hold for  $\lambda < \lambda_0$ . But it is not obvious that (4.5) holds, precisely when  $\lambda \geq \lambda_0$ . It is here that the extension of Vietoris's theorem given in Section 3 is applied. We observe that (4.5) is equivalent to

$$\operatorname{Re} \left\{ (1-z) [s_n^\lambda(z)]^2 \right\} > 0. \tag{4.6}$$

By the minimum principle for harmonic functions, it suffices to establish (4.6) for  $z = e^{2i\theta}$ ,  $0 < \theta < \pi$ . Let

$$P_n(\theta) := (1 - e^{2i\theta}) \left\{ \sum_{k=0}^n \frac{(2-2\lambda)_k}{k!} e^{2ik\theta} \right\}^2.$$

Then we find that

$$\begin{aligned} & \operatorname{Re} P_n(\theta) \\ &= \left( \sum_{k=0}^{2n+1} c_k \cos k\theta \right) \left( \sum_{k=0}^{2n+1} c_k \cos k(\pi - \theta) \right) + \left( \sum_{k=1}^{2n+1} c_k \sin k\theta \right) \left( \sum_{k=1}^{2n+1} c_k \sin k(\pi - \theta) \right), \end{aligned}$$

where  $c_0 = c_1 = 1$ ,  $c_{2k} = c_{2k+1} = \frac{(2-2\lambda)_k}{k!}$ ,  $k = 1, 2, \dots$ . Setting  $\lambda = \frac{1+\alpha}{2}$  and using (3.3) and (3.4) we conclude that

$$\operatorname{Re} P_n(\theta) > 0, \quad \text{for } 0 < \theta < \pi,$$

precisely when  $\lambda_0 \leq \lambda < 1$ , which is the desired result.

It is readily shown that (4.5) implies

$$\operatorname{Re} \left\{ (1-z)^{\frac{1}{2}} s_n^\lambda(z) \right\} > 0, \tag{4.7}$$

for all  $n \in \mathbb{N}$  and  $\lambda_0 \leq \lambda < 1$ . It is then natural to ask for the maximum range of  $\lambda$  for which (4.7) is valid. For  $z = e^{2i\theta}$ ,  $0 < \theta < \pi$ , this is equivalent to

$$\sum_{k=0}^n \frac{(2-2\lambda)_k}{k!} \cos \left[ \left(2k + \frac{1}{2}\right)\theta - \frac{\pi}{4} \right] > 0. \tag{4.8}$$

On setting  $\lambda = \frac{1+\alpha}{2}$ , an argument similar to the proof of (4.3) yields the asymptotic formula

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left(\frac{\theta}{n}\right)^{1-\alpha} \sum_{k=0}^n \frac{(1-\alpha)_k}{k!} \cos \left[ \left(2k + \frac{1}{2}\right) \frac{\theta}{2n} - \frac{\pi}{4} \right] \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^\theta \frac{\cos\left(t - \frac{\pi}{4}\right)}{t^\alpha} dt. \end{aligned} \tag{4.9}$$

The integral in (4.9) is positive for all  $\theta > 0$  if and only if  $\alpha \geq \alpha'$ , where  $\alpha'$  is the unique solution in  $(0, 1)$  of the equation

$$\int_0^{\frac{7}{4}\pi} \frac{\cos(t - \frac{\pi}{4})}{t^\alpha} dt = 0,$$

whose numerical value is  $\alpha' = 0.0923103\dots$ . Then it can be shown that inequality (4.8) holds for all  $n$  and  $\theta \in (0, \pi)$  if and only if  $1 > \lambda \geq \lambda' = \frac{1+\alpha'}{2} = 0.546155\dots$ . See [25]. Note that  $\lambda' < \lambda_0 = 0.654222\dots$ .

The above results motivated us to consider the following more general problem. Let  $p \in [0, 1]$ . Determine the maximum range of  $\lambda$ , for which

$$\operatorname{Re} [(1-z)^p s_n^\lambda(z)] > 0, \quad (4.10)$$

for all  $n \in \mathbb{N}$  and  $z \in \mathbb{D}$ . The cases  $p = 0$  and  $p = 1/2$  have been completely solved, while the case  $p = 1$  follows by a partial summation from Fejér's classical inequality

$$\sum_{k=0}^n \sin(k + \frac{1}{2})\theta = \frac{1 - \cos(n+1)\theta}{2 \sin \frac{\theta}{2}} \geq 0, \quad 0 < \theta < 2\pi.$$

The conclusion is that for  $p = 1$ , inequality (4.10) holds for all  $1/2 \leq \lambda < 1$ . This also follows from [32, Theorem 1.1].

The general case of (4.10) reduces to a trigonometric inequality, a limiting case of which requires the positivity of the integral

$$\int_0^\theta \frac{\cos(t - \frac{p\pi}{2})}{t^\alpha} dt,$$

for all  $\theta > 0$ . This holds true if and only if  $\alpha \geq \alpha(p)$ , where  $\alpha(p)$  is the unique solution in  $(0, 1)$  of the equation

$$\int_0^{(3+p)\frac{\pi}{2}} \frac{\cos(t - \frac{p\pi}{2})}{t^\alpha} dt = 0.$$

In view of the above, we have led to the conjecture that (4.10) holds if and only if  $1 > \lambda \geq \lambda(p) = \frac{1 + \alpha(p)}{2}$ . This conjecture appears to be supported by numerical experimentation. For particular values of  $p \in [0, 1]$  this can be proved by the methods we followed in the cases  $p = 0, \frac{1}{2}$ . It would be interesting, however, to settle this conjecture by a method that comprises as a whole the values of  $p$  in  $[0, 1]$ .

Another interesting problem is the study of the function  $\alpha(p)$ . Numerical evidence suggests that this is a strictly decreasing function of  $p$  for  $p \in [0, 1]$ .

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