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Note on formal contexts of generalized one-sided concept lattices*

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Abstract

Generalized one-sided concept lattices represent one of the conceptual data mining methods, suitable for an analysis of object-attribute models with the different types of attributes. It allows to create FCA-based output in form of concept lattice with the same interpretation of concept hierarchy as in the case of classical FCA. The main aim of this paper is to investigate relationship between formal contexts and generalized one-sided concept lattices. We show that each one uniquely determines the other one and we also derive the number of generalized one-sided concept lattices defined within the given framework of formal context. The order structure of all mappings involved in some Galois connections between a power set and a direct product of complete lattices is also dealt with.

Keywords: Galois connection, generalized one-sided concept lattice, formal context.

MSC: 06A15

1. Introduction

Handling uncertainty, imprecise data or incomplete information has become an important research topic in the recent years. One of the frequent solutions, how to

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deal with "imperfect" information, usually leads to the development of the fuzzified versions of several well-known standard structures or approaches. In this paper, we focus on the area of the formal concept analysis, specifically, on the approach known as generalized one-sided concept lattices.

Formal Concept Analysis (FCA [9]) represents a method of data analysis for identifying conceptual structures among data sets. As an efficient tool, Formal Concept Analysis has been successfully applied to domains such as decision systems, information retrieval, data mining and knowledge discovery. Classical FCA is suitable for crisp case, where object-attribute model is based on binary relation (object has/has-not the attribute). In practice there are natural examples of object-attribute models for which relationship between objects and attributes are represented by many-valued (fuzzy) relations. Therefore, several attempts to fuzzify FCA have been proposed. As an example we mention work of Bělohlávek [2, 3, 4] or other approaches [12, 14, 15]. One-sided concept lattices play a special role in fuzzy FCA, where usually objects are considered as crisp subsets and attributes obtain fuzzy values. In this case the interpretation of object clusters is straightforward as in classical FCA, instead of fuzzy approaches with fuzzy subsets of objects, where interpretability often becomes problematic.

Recently, there was a generalization of all known one-sided approaches [1, 10, 11], so called generalized one-sided concept lattices, cf. [7, 8]. This approach is, in contrary with the previous one-sided approaches, convenient for the analysis of object-attribute models with different truth value structures. From this point of view it is applicable to a wide spectrum of real object-attribute models where methods of classical FCA are appropriate, cf. [5, 6, 16, 17]. In this note we deal with theoretical question, whether correspondence between formal contexts, which represent object-attribute models, and concept lattices on the other side is one-to-one or equivalently injective.

In order to make this paper as self-contained as possible, in the next section we give a brief overview of the notions like formal context, Galois connections, complete lattices, direct product, etc. We also describe the basic definitions and the results concerning generalized one-sided concept lattices.

Our main results are in Section 3. Firstly we prove that the correspondence between formal context and generalized one-sided concept lattices is injective, i.e., that each generalized one-sided concept lattice also uniquely determines formal context. Based on this result, we deduce the formula expressing number of generalized one-sided concept lattices defined within the fixed framework of a given formal context. Further, we are studying the order structure of mappings involved in some Galois connections between a power set and a direct product of complete lattices. In particular, we show that the lattice of all such mappings and the lattice of all incidence relations are isomorphic.

2. Formal contexts and generalized one-sided concept lattices

In this section we examine the notion of the object-attribute model and its mathematical counterpart formal context. Further, based on the notion of formal context we define generalized one-sided concept lattices as fuzzy generalization of classical concept lattices.

Firstly, we briefly describe the object-attribute models. Generally, by object we understand any item that can be individually selected and manipulated, e.g., person, car, document, etc. In general, an attribute is a property or characteristic of given object, e.g., height of a person, colour of a car or frequency of occurrence of a given word in some document. We will consider that each particular attribute under consideration has defined its range of possible values. Hence, if we measure the height in cm, then any person has assigned the height as integer value from interval $[0, 280]$. Similarly, color of a car can be from some given set of prescribed colors $\{\mathsf{red}, \mathsf{blue}, \mathsf{white}, \dots\}$ and frequency of occurrence of some word w can be given as the ratio $\frac{N_w}{N_{\text{all}}}$ from the interval $[0, 1]$ of rationals. In this case N_w denotes the number of the occurrences of the word w and N_{all} denotes the number of all words in the considered document.

In our understanding object-attribute model consists of the set of objects, set of the attributes with prescribed ranges and values which characterizes objects by the given attributes, e.g., John is tall 183 cm.

In order to apply methods of FCA, we will need one restriction on the ranges of all attributes belonging to object-attribute models. This restriction is given by the usage of fuzzy logic in the theory of fuzzy concept lattices. The main idea of fuzzifications of classical FCA is the usage of graded truth. In classical logic, each proposition is either true or false, hence classical logic is bivalent. In fuzzy logic, to each proposition there is assigned a truth degree from some scale L of truth degrees. The structure L of the truth degrees is partially ordered and contains the smallest and the greatest element. If to the propositions ϕ and ψ are assigned truth degrees $\parallel \phi \parallel = a$ and $\parallel \psi \parallel = b$, then $a \leq b$ means that ϕ is considered less true than ψ. In the object-attribute models typical propositions are of the form "object has attribute in degree a".

In the theory of fuzzy concept lattices it is always assumed that the structure L of the truth degrees assigned to each attribute forms complete lattice.

Now we recall some basic facts concerning partially ordered sets and lattices. By the partially ordered set $(P, \leq)$ we understand non-empty set $P \neq \emptyset$ together with binary relation $\leq$ satisfying:

i) $x \leq x$ for all $x \in P$, i.e., the relation $\leq$ is reflexive,

ii) $x \leq y$ and $y \leq x$ then $x = y$, i.e., antisymmetry of $\leq$,

iii) $x \leq y$ and $y \leq z$ then $x \leq z$, i.e., transitivity of the relation $\leq$.

Let $(P, \leq)$ be a partially ordered set and $H \subseteq P$ be an arbitrary subset. An element $a \in P$ is said to be the *least upper bound* or *supremum* of H, if a is the upper bound of the subset H ($h \leq a$ for all $h \in H$) and a is the least of all elements majorizing H ($a \leq x$ for any upper bound x of H). We shall write $a = \sup H$ or $a = \bigvee H$. The concepts of the *greatest lower bound* or *infimum* is similarly defined and it will be denoted by $\inf H$ or $\bigwedge H$.

A partially ordered set $(L, \leq)$ is a *lattice* if $\sup\{a, b\} = a \vee b$ and $\inf\{a, b\} = a \wedge b$ exist for all $a, b \in L$. A lattice L is called *complete* if $\bigvee H$ and $\bigwedge H$ exist for any subset $H \subseteq L$. Obviously, each finite lattice is complete. Note that any complete lattice contains the greatest element $1_L = \sup L = \inf \emptyset$ and the smallest element $0_L = \inf L = \sup \emptyset$. In what follows we will denote the class of all complete lattices by CL.

Now we are able to define formal context which represents mathematical formalization of the notion object-attribute model.

Definition 2.1. A 4-tuple $\big(B, A, \mathcal{L}, R\big)$ is said to be a generalized one-sided formal context if the following conditions are fulfilled:

a) B is a non-empty set of objects and A is a non-empty set of attributes.

b) $\mathcal{L}\colon A \to \mathsf{CL}$,

c) $R\colon B \times A \to \bigcup_{a \in A} \mathcal{L}(a)$ is a mapping satisfying $R(b, a) \in \mathcal{L}(a)$ for all $b \in B$ and $a \in A$.

Second condition says that $\mathcal{L}$ is a mapping from the set of attributes to the class of all complete lattices. Hence, for any attribute a, $\mathcal{L}(a)$ denotes the complete lattice, which represents structure of truth values for attribute a, i.e., $\mathcal{L}(a)$ denotes the range of attribute a. As it is explicitly given, we require that all ranges form complete lattices. The symbol R denotes so-called (generalized) incidence relation, i.e., $R(b, a)$ represents a degree from the structure $\mathcal{L}(a)$ in which the element $b \in B$ has the given attribute a.

As an example of simple formal context, consider four-element set of objects $B = \{a, b, c, d\}$ and eight-element set of attributes $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$. We will assume that the attributes in our model are binary or real, i.e., ranges of these attributes are represented either two-element chain $\mathbf{2} = \{0, 1\}$ with $0 < 1$ or real unit interval $[0, 1]$. Particularly we have $\mathcal{L}(a_1) = \mathcal{L}(a_3) = \mathcal{L}(a_5) = \mathcal{L}(a_6) = \mathbf{2}$ and $\mathcal{L}(a_2) = \mathcal{L}(a_4) = \mathcal{L}(a_7) = \mathcal{L}(a_8) = [0, 1]$. The generalized incidence relation R of each formal context is usually described as data table. In this case the value $R(b, a)$ can be found on the intersection of b-th row and a-th column of the table. The incidence relation of our example is depicted in Table 1.

Further we define generalized one-sided concept lattices derived from given generalized one-sided formal context. Since the theory of concept lattices is based on the notion of Galois connections, we recall this notion at first, cf. [13] or [9].

Definition 2.2. Let $(P, \leq)$ and $(Q, \leq)$ be partially ordered sets and let

$$\varphi\colon P \to Q \quad \text{and} \quad \psi\colon Q \to P$$

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
a	0	0.2	1	0.3	1	0	0.1	0.5
b	1	0.6	0	0.6	0	1	0.5	0.3
c	1	1.0	0	0.7	0	0	0.5	0.0
d	0	0.2	0	0.3	1	0	0.1	0.5

Table 1: Data table of object-attribute model

be maps between these ordered sets. Such a pair (φ, ψ) of mappings is called a Galois connection between the ordered sets if:

(a) $p_1 \leq p_2$ implies $\varphi(p_1) \geq \varphi(p_2)$,

(b) $q_1 \leq q_2$ implies $\psi(q_1) \geq \psi(q_2)$,

(c) $p \leq \psi(\varphi(p))$ and $q \leq \varphi(\psi(q))$.

Let us remark that the conditions (a), (b) and (c) are equivalent to the following one:

$$p \leq \psi(q) \quad \text{iff} \quad \varphi(p) \geq q. \tag{2.1}$$

These two maps are also called dually adjoint to each other. An important property of Galois connections is captured in the following expressions (see [9] for the proof).

$$\varphi = \varphi \circ \psi \circ \varphi \quad \text{and} \quad \psi = \psi \circ \varphi \circ \psi \tag{2.2}$$

Moreover the dual adjoint is determined uniquely, i.e., if (φ_1, ψ) forms Galois connection as well as (φ_2, ψ) then $\varphi_1 = \varphi_2$. The same is true if (φ, ψ_1) and (φ, ψ_2) form Galois connections, then $\psi_1 = \psi_2$.

Now we describe the partially ordered sets, where we define appropriate Galois connection. On the side of objects, we will consider the set $\mathbf{P}(B)$ as a domain of one part of Galois connection. Let us note that $\mathbf{P}(B)$ denotes the power set of all subsets of the set B partially ordered by the set theoretical inclusion. It is well known fact that $\mathbf{P}(B)$ forms complete lattice. In this case, clusters of objects are represented by classical subsets, hence this is the reason for the name "one-sided concept lattices".

If L_i for $i \in I$ is a family of lattices the *direct product* $\prod_{i \in I} L_i$ is defined as the set of all functions

$$f : I \to \bigcup_{i \in I} L_i \tag{2.3}$$

such that $f(i) \in L_i$ for all $i \in I$ with the "componentwise" order, i.e, $f \leq g$ if $f(i) \leq g(i)$ for all $i \in I$. If $L_i = L$ for all $i \in I$ we get a direct power L^I. In this case the direct power L^I represents the structure of L-fuzzy sets, hence direct product of lattices can be seen as a generalization of the notion of L-fuzzy sets. The direct product of lattices forms complete lattice if and only if all members of the family are complete lattices. The straightforward computations show that the lattice operations in the direct product $\prod_{i\in I} L_i$ of complete lattices are calculated componentwise, i.e., for any subset $\{f_j : j \in J\} \subseteq \prod_{i\in I} L_i$ we obtain

$$\big(\bigvee_{j\in J} f_j\big)(i) = \bigvee_{j\in J} f_j(i) \quad \text{and} \quad \big(\bigwedge_{j\in J} f_j\big)(i) = \bigwedge_{j\in J} f_j(i), \tag{2.4}$$

where these equalities hold for each index $i \in I$.

Generalized one-sided concept lattices were designed to handle with different types of attributes, hence the appropriate domain for second part of Galois connection consists of direct product of attribute lattices $\prod_{a\in A} \mathcal{L}(a)$.

Definition 2.3. Let $\big(B, A, \mathcal{L}, R\big)$ be a generalized one-sided formal context. We define a pair of mappings $\uparrow\colon \mathbf{P}(B) \to \prod_{a\in A} \mathcal{L}(a)$ and $\downarrow\colon \prod_{a\in A} \mathcal{L}(a) \to \mathbf{P}(B)$ as follows:

$$\uparrow\big(X\big)(a) = \bigwedge_{b\in X} R(b, a),\ \text{for all } X \subseteq B, \tag{2.5}$$

$$\downarrow(g) = \{b \in B : \forall a \in A,\ g(a) \le R(b, a)\},\ \text{for all } g \in \prod_{a\in A} \mathcal{L}(a). \tag{2.6}$$

The main result concerning such defined pair of mappings is stated in the following proposition.

Proposition 2.4. *The pair* $(\uparrow, \downarrow)$ *forms a Galois connection between* $\mathbf{P}(B)$ *and* $\prod_{a\in A} \mathcal{L}(a)$.

Proof. We prove that $\uparrow(X) \ge g$ if and only if $X \subseteq \downarrow(g)$ for all $X \subseteq B$ and all $g \in \prod_{a\in A} \mathcal{L}(a)$.

Since $\uparrow\big(X\big) \ge g$ if and only if $\uparrow\big(X\big)(a) \ge g(a)$ for all $a \in A$, according to the Definition (2.5) of the map $\uparrow$ and expression (2.4) we obtain

$$\forall a \in A, \uparrow\big(X\big)(a) = \bigwedge_{b\in X} R(b, a) \ge g(a) \quad \text{iff} \quad \forall a \in A, \forall b \in X,\ R(b, a) \ge g(a).$$

Due to the definition (2.6) of the map $\downarrow$, this is equivalent to

$$X \subseteq \{b \in B : \forall a \in A,\ g(a) \le R(b, a)\} = \downarrow(g). \qquad \square$$

The result of this proposition allows to define generalized one-sided concept lattices. Let $\big(B, A, \mathcal{L}, R\big)$ be a generalized one-sided formal context. Denote by $\mathcal{C}\big(B, A, \mathcal{L}, R\big)$ the set of all pairs (X, g), $X \subseteq B$, $g \in \prod_{a\in A} \mathcal{L}(a)$ which form fixed points of the Galois connection $(\uparrow, \downarrow)$, i.e., satisfying

$$\uparrow\big(X\big) = g \quad \text{and} \quad \downarrow(g) = X.$$

In this case the ordered pair (X, g) is said to be a concept, the set X is usually referred as extent and g as intent of the concept (X, g).

Further we define partial order on the set $\mathcal{C}\big(B, A, \mathcal{L}, R\big)$ as follows:

$$(X_1, g_1) \le (X_2, g_2) \quad \text{iff} \quad X_1 \subseteq X_2 \quad \text{iff} \quad g_1 \ge g_2. \tag{2.7}$$

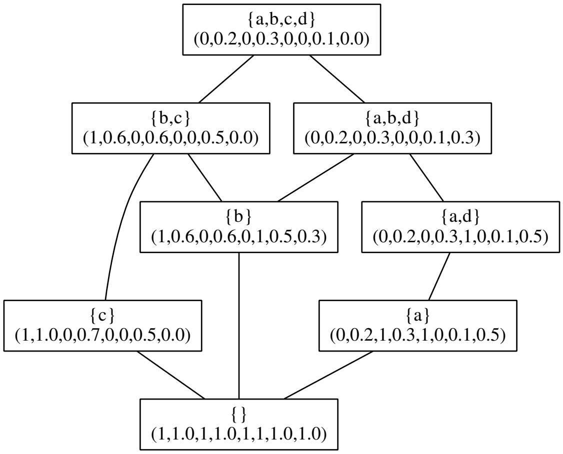

Figure 1: Generalized one-sided concept lattice

Proposition 2.5. *The set* $\mathcal{C}\big(B, A, \mathcal{L}, R\big)$ *with the partial order defined by (2.7) forms a complete lattice, where*

$$\bigwedge_{i\in I}(X_i, g_i) = \Big(\bigcap_{i\in I} X_i, \uparrow\downarrow\big(\bigvee_{i\in I} g_i\big)\Big) \quad \textit{and} \quad \bigvee_{i\in I}(X_i, g_i) = \Big(\downarrow\uparrow\big(\bigcup_{i\in I} X_i\big), \bigwedge_{i\in I} g_i\Big)$$

for each family $(X_i, g_i)_{i\in I}$ *of elements from* $\mathcal{C}\big(B, A, \mathcal{L}, R\big)$.

Proof of this proposition is based on the fact that any Galois connection between complete lattices induces dually isomorphic closure systems (see [13]). Consequently, this dual isomorphism maps infima on the one side onto suprema in a closure system on the other side and vice versa.

Remark that the algorithm for generation of generalized one-sided concept lattices can be found in [7] or [8].

The Hasse diagram of the generalized one-sided concept lattice determined by Table 1 is shown on Figure 1. Let us remark that we denote the elements of direct product as ordered tuples, as it is common in lattice theory.

3. On relationship between incidence relations and generalized one-sided concept lattices

In this section we present our results concerning incidence relations and corresponding one-sided concept lattices. We also describe the order structure of the set of all mappings involving in some Galois connection between power set and the direct product of complete lattices. Firstly, we show that the correspondence

$$\text{generalized one-sided context} \;\mapsto\; \text{generalized one-sided concept lattice}$$

is injective or equivalently one-to-one. We already know how to define generalized one-sided lattice from given formal context. However, there is an interesting theoretical question, whether different formal contexts yield different one-sided concept lattices. The positive answer means that not only formal context fully characterizes generalized one-sided context, but the converse is also true, i.e., given generalized one-sided concept lattice fully determines formal context. Hence, generalized one-sided concept lattice contains all information about object-attribute model.

We recall the definition of injective mapping. A mapping $f : A \to B$ is said to be injective (one-to-one) if

$$x \neq y \quad \text{implies} \quad f(x) \neq f(y)$$

Evidently, this condition is equivalent to the condition $f(x) = f(y)$ implies $x = y$.

In what follows, we will consider that the set of objects B is fixed, as well as the set of all attributes A (together with truth value structures $\mathcal{L}(a)$). Consider that we have two generalized one-sided formal contexts $(B, A, \mathcal{L}, R_1)$ and $(B, A, \mathcal{L}, R_2)$. The corresponding concept lattices are denoted by $C_1 = \mathcal{C}\big(B, A, \mathcal{L}, R_1\big)$ and $C_2 = \mathcal{C}\big(B, A, \mathcal{L}, R_2\big)$.

Theorem 3.1. *The correspondence $(B, A, \mathcal{L}, R) \mapsto \mathcal{C}\big(B, A, \mathcal{L}, R\big)$, which assign to each generalized one-sided formal context the corresponding generalized one-sided concept lattice is injective.*

Proof. We prove this theorem in two steps. Firstly we show that the correspondence $(B, A, \mathcal{L}, R) \mapsto (\uparrow, \downarrow)$, which maps formal context onto the Galois connection given by (2.5) and (2.6) respectively, is injective. Next we show that the correspondence $(\uparrow, \downarrow) \mapsto \mathcal{C}\big(B, A, \mathcal{L}, R\big)$, which maps Galois connection to the concept lattice is injective too. Since the composition of two injective mappings is injective, this will satisfy to prove our result.

Suppose that incidence R_1 and R_2 differ, i.e., there exist $b \in B$, $a \in A$ such that $R_1(b, a) \neq R_2(b, a)$. Note, that we will recognize the corresponding Galois connection by subscript. According to the definition (2.5) of mapping $\uparrow$ we obtain:

$$\uparrow_1(\{b\}) = \bigwedge_{b' \in \{b\}} R_1(b', a) = R_1(b, a) \neq R_2(b, a) = \bigwedge_{b' \in \{b\}} R_2(b', a) = \uparrow_2(\{b\}).$$

This equation shows that we have found one-element subset $\{b\}$ with $\uparrow_1(\{b\}) \neq \uparrow_2(\{b\})$ and consequently $(\uparrow_1, \downarrow_1) \neq (\uparrow_2, \downarrow_2)$. Hence, the first correspondence between formal contexts and Galois connections is injective.

Further, assume that $C_1 = C_2$, i.e., that the generalized one-sided concept lattices equal. This means that the sets of fixed points coincide, i.e., for all $X \subseteq B$ and $g \in \prod_{a \in A} \mathcal{L}(a)$ it holds

$$\uparrow_1(X) = g \text{ and } \downarrow_1(g) = X \quad \text{iff} \quad \uparrow_2(X) = g \text{ and } \downarrow_2(g) = X. \tag{3.1}$$

Let $X \subseteq B$ be an arbitrary subset. From the property (2.2) of Galois connections we have $\uparrow_1(X) = \uparrow_1(\downarrow_1(\uparrow_1(X)))$, thus ordered pair $(\downarrow_1(\uparrow_1(X)), \uparrow_1(X))$ forms a fixed point of Galois connection $(\uparrow_1, \downarrow_1)$. Then, due to condition (3.1) we obtain that $\downarrow_2(\uparrow_1(X)) = \downarrow_1(\uparrow_1(X))$. Consequently, we have $X \subseteq \downarrow_1(\uparrow_1(X)) = \downarrow_2(\uparrow_1(X))$ which yields the first half of the condition (c) of the Definition 2.2.

Similarly, using (2.2) we obtain for each element $g \in \prod_{a\in A} \mathcal{L}(A)$ the pair $(\downarrow_2(g), \uparrow_2(\downarrow_2(g))$ forms fixed point of $(\uparrow_2, \downarrow_2)$. Again, due to condition (3.1) we obtain $\uparrow_2(\downarrow_2(g)) = \uparrow_1(\downarrow_2(g))$, which yields $g \leq \uparrow_2(\downarrow_2(g)) = \uparrow_1(\downarrow_2(g))$. Since the mappings $\uparrow_1$ and $\downarrow_2$ are order reversing, we have proved that the pair $(\uparrow_1, \downarrow_2)$ forms Galois connection. Now using the fact that dual adjoint is unique, we obtain $\uparrow_1 = \uparrow_2$ and $\downarrow_1 = \downarrow_2$, which completes the proof. □

It was proved in [7] that for any Galois connection (Φ, Ψ) between $\mathbf{P}(B)$ and $\prod_{a\in A} \mathcal{L}(a)$ there exists a generalized formal context $(B, A, \mathcal{L}, R)$ that $\uparrow = \Phi$ and $\downarrow = \Psi$. Hence the correspondence between formal contexts and generalized one-sided concept lattices is surjective, too. Since we have shown that it is injective, in fact this correspondence is bijective. Using this fact we can prove the following theorem about number of all concept lattices.

Theorem 3.2. *Let $B \neq \emptyset$ be set of objects, $A = \{a_1, a_2, \ldots, a_m\}$ be set of attributes. Denote by $n = |B|$ number of objects and for all $i = 1, \ldots, m$ denote by $n_i = |\mathcal{L}(a_i)|$ the cardinality of the complete lattice $\mathcal{L}(a_i)$. Then there is $(\prod_{i=1}^{m} n_i)^n$ generalized one-sided concept lattices.*

Proof. There is a bijection between set of all generalized incidence relations and one-sided concept lattices, thus it is sufficient to count all generalized incidence relations. For each object b and each attribute a the value $R(b, a)$ can obtain $n_i = |\mathcal{L}(a_i)|$ values. Since we have n objects, there is ${n_i}^n$ possibilities for columns in data table (which represents incidence relation). Together we have

$$ {n_1}^n \cdot {n_2}^n \cdot \ldots \cdot {n_m}^n \cdot \ldots = (\prod_{i=1}^{m} n_i)^n $$

possibilities to define incidence relation. □

This result generalizes the similar assertion for classical concept lattices. Suppose there is given a formal context (B, A, I). If we have n objects and m attributes, then there is $2^{n\cdot m}$ concept lattices. Any classical concept lattice can be characterized as generalized one-sided concept lattice by setting $\mathcal{L}(a) = \mathbf{2}$ ($\mathbf{2} = \{0, 1\}$ denotes two-element chain) and $R(b, a) = 1$ if and only if $(b, a) \in I$ (see [14] for details). Hence applying the result of Theorem 3.2 we obtain $\prod_{i=1}^{m} 2^n = (2^n)^m = 2^{m\cdot n}$.

Similarly, if one will consider $\mathcal{L}(a_i) = L$ for all $i = 1, \ldots, m$, than generalized one-sided concept lattices, represent one-sided concept lattices. Hence, applying Theorem 3.2 we obtain that there is $\prod_{i=1}^{m} |L|^n = |L|^{m\cdot n}$ different one-sided concept lattices.

Next we show that formal contexts also characterize order properties of the Galois connections between power sets and complete lattices. Firstly we prove the following lemma, concerning the closure property of Galois connections. Let L and

M be complete lattices. Denote by $\mathsf{Gal}(L,M)$ the set of all $\varphi\colon L \to M$ such that there exists $\psi\colon M \to L$ dually adjoint to φ.

Lemma 3.3. *Let L, M be complete lattices. The set $\mathsf{Gal}(L,M)$ forms a closure system in complete lattice M^L.*

Proof. We show that the set $\mathsf{Gal}(L,M)$ is closed under arbitrary infima. Let $\{\varphi_i : i \in I\} \subseteq \mathsf{Gal}(L,M)$ be an arbitrary system. Denote by $\varphi = \bigwedge_{i\in I}\varphi_i$. In this case $\varphi(x) = \bigwedge_{i\in I}\varphi(x)$ for all $x \in L$. In order to prove that $\varphi \in \mathsf{Gal}(L,M)$ we show that there is a dual adjoint $\psi\colon M \to L$. Define $\psi = \bigwedge_{i\in I}\psi_i$ where ψ_i is dually adjoint to φ_i for all $i \in I$.

Let $x_1, x_2 \in L$ be elements such that $x_1 \le x_2$. Since $\varphi_i(x_1) \ge \varphi_i(x_2)$ for all $i \in I$, we obtain

$$\varphi(x_1) = \bigwedge_{i\in I}\varphi_i(x_1) \ge \bigwedge_{i\in I}\varphi_i(x_2) = \varphi(x_2).$$

Similarly, for all $y_1, y_2 \in M$ condition $y_1 \le y_2$ implies $\psi(y_2) \ge \psi(y_1)$.

Finally, we show that $x \le \psi(\varphi(x))$ for all $x \in L$. Let $j \in I$ be an arbitrary index. Then for all $x \in L$ we have

$$x \le \psi_j\big(\varphi_j(x)\big) \le \psi_j\big(\bigwedge_{i\in I}\varphi(x)\big),$$

since ψ_j is order reversing and $\varphi_j(x) \ge \bigwedge_{i\in I}\varphi_i(x)$. This yields

$$x \le \bigwedge_{j\in I}\psi_j\big(\bigwedge_{i\in I}\varphi_i(x)\big) = \bigwedge_{j\in I}\psi_j\big(\varphi(x)\big) = \psi\big(\varphi(x)\big).$$

In similar way, one can prove $y \le \varphi(\psi(y))$ for all $y \in M$. $\square$

Since $\mathsf{Gal}(L,M)$ forms a closure system in complete lattice M^L, it forms complete lattice too. In this case meets in $\mathsf{Gal}(L,M)$ coincide with the meets in M^L, but this is not valid for joins in general. In particular, if $(\varphi_i : i \in I\} \subseteq \mathsf{Gal}(L,M)$ then

$$\sup\{\varphi_i : i \in I\} = \bigwedge\{\varphi \in \mathsf{Gal}(L,M) : \varphi \ge \bigvee_{i\in I}\varphi_i\}$$

where the symbols $\bigwedge$ and $\bigvee$ denote operations of meet and join in M^L.

Let us note that $\mathsf{Gal}(L,M)$ and $\mathsf{Gal}(M,L)$ forms isomorphic posets. This follows from the fact that the correspondence $\varphi \mapsto \psi$ where ψ denotes the dual adjoint of φ is bijective. Moreover it is order preserving in both directions. Suppose $\varphi_1(x) \le \varphi_2(x)$ for all $x \in L$. Let $y \in M$ be an arbitrary element. Then $y \le \varphi_1(\psi_1(y)) \le \varphi_2(\psi_1(y))$ and according to the condition (2.1) it follows $\psi_1(y) \le \psi_2(y)$. The opposite implication can be proved analogously, hence $\varphi_1 \le \varphi_2$ if and only if $\psi_1 \le \psi_2$.

Further assume that $B, A \ne \emptyset$ and $\mathcal{L}\colon A \to \mathsf{CL}$ are fixed. In order to describe the structure of the lattice $\mathsf{Gal}(\mathbf{P}(B), \prod_{a\in A}\mathcal{L}(a))$ we denote by $\mathsf{R}(B, A, \mathcal{L})$ the set of all relations R such that $(B, A, \mathcal{L}, R)$ forms generalized one-sided formal context. Obviously the set $\mathsf{R}(B, A, \mathcal{L})$ forms complete lattice. In this case, if $\{R_i : i \in I\}$ is a system of relations, then relation R where $R(b,a) = \bigwedge_{i\in I} R_i(b,a)$ ($R(b,a) = \bigvee_{i\in I} R_i(b,a)$) corresponds to the infimum (supremum).

Theorem 3.4. *The lattice* $\mathsf{Gal}(\mathbf{P}(B), \prod_{a\in A} \mathcal{L}(a))$ *is isomorphic to the lattice of all incidence relations* $\mathsf{R}(B, A, \mathcal{L})$.

Proof. Define $F\colon \mathsf{R}(B, A, \mathcal{L}) \to \mathsf{Gal}(\mathbf{P}(B), \prod_{a\in A} \mathcal{L}(a))$ for all $R \in \mathsf{R}(B, A, \mathcal{L})$ by $F(R) = \uparrow_R$, where $\uparrow_R$ is defined by (2.5). As we already know, the mapping F is bijective. We show, that it also preserves the lattice operations, i.e., $F\big(R_1 \wedge R_2\big) = F(R_1) \wedge F(R_2)$ and $F(R_1 \vee R_2) = \sup\{F(R_1), F(R_2)\}$.

Let $X \subseteq B$ be any subset and $a \in A$ be an arbitrary element. Then we obtain

$$\uparrow_{R_1\wedge R_2}(X)(a) = \bigwedge_{b\in X} \Big(R_1(b,a) \wedge R_2(b,a)\Big) =$$

$$= \bigwedge_{b\in X} R_1(b,a) \ \wedge \ \bigwedge_{b\in X} R_2(b,a) = \uparrow_{R_1}(X)(a) \ \wedge \ \uparrow_{R_2}(X)(a).$$

Hence the mapping F preserves meets.

In order to prove that F preserves joins, we use the fact that the mapping F is surjective, i.e., for any Galois connection (φ, ψ) between $\mathbf{P}(B)$ and $\prod_{a\in A} \mathcal{L}(a)$ there is some relation R with $\varphi = \uparrow_R$ and $\psi = \downarrow_R$.

Let $\varphi \in \mathsf{Gal}(\mathbf{P}(B), \prod_{a\in A} \mathcal{L}(a))$ be a mapping satisfying $\varphi \geq \uparrow_{R_1}, \uparrow_{R_2}$. Then $\varphi = \uparrow_R$ for some $R \in \mathsf{R}(B, A, \mathcal{L})$ and for all $b \in B$ and $a \in A$ we obtain

$$\varphi(\{b\})(a) = \uparrow_R(\{b\})(a) = \bigwedge_{b'\in\{b\}} R(b,a) = R(b,a).$$

Since $\varphi(\{b\}) \geq \uparrow_{R_1}(\{b\}), \uparrow_{R_2}(\{b\})$ for all $b \in B$ we have $R(b,a) \geq R_1(b,a) \vee R_2(b,a)$ for all $b \in B$ and $a \in A$. This yields

$$\varphi(X)(a) = \uparrow_R(X)(a) = \bigwedge_{b\in X} R(b,a) \geq \bigwedge_{b\in X} \Big(R_1(b,a) \vee R_2(b,a)\Big) = \uparrow_{R_1\vee R_2}(X)(a)$$

for all $X \subseteq B$ and for all $a \in A$. Obviously $\uparrow_{R_1\vee R_2}$ is the upper bound of $\uparrow_{R_1}$ and $\uparrow_{R_2}$ and we have shown that it is in fact the least upper bound of $\uparrow_{R_1}$ and $\uparrow_{R_2}$. Hence in the lattice $\mathsf{Gal}(\mathbf{P}(B), \prod_{a\in A} \mathcal{L}(a))$ the assertion $F(R_1 \vee R_2) = \sup\{F(R_1), F(R_2)\}$ is valid. □

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