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João Guerra, Manuel Guerra, Zachary Polaski

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Market Timing with Option-Implied Distributions in an Exponentially Tempered Stable Lévy Market

Zachary Polaski[*], João Guerra[†], and Manuel Guerra[‡]

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Abstract:

This paper explores the empirical implementation of a dynamic asset allocation strategy using option-implied distributions when the underlying risky asset price is modeled by an exponential Lévy process. One month risk-neutral densities are extracted from option prices and are subsequently transformed to the risk-adjusted, or real-world densities. Optimal portfolios consisting of a risky and risk-free asset rebalanced on a monthly basis are then constructed and their performance analyzed. It is found that the portfolios formed using option-implied expectations under the Lévy market assumption, which are flexible enough to capture the higher moments of the implied distribution, are far more robust to left-tail market risks and offer statistically significant improvements to risk-adjusted performance when investor risk aversion is low, however this diminishes as risk aversion increases.

Keywords: Asset Allocation, Lévy Processes, Option-Implied Distributions, Portfolio Optimization

JEL Classification: C10, C51, G11, G13, G17

* At the time this research was completed, the author was a student at ISEG - Universidade de Lisboa, Department of Mathematics. e-mail: zack.polaski1@gmail.com.

†CEMAPRE, REM - Research in Economics and Mathematics, ISEG - Universidade de Lisboa. The author was partially supported by the Project CEMAPRE - UID/MULTI/00491/2013 financed by FCT/MEC through national funds. e-mail: jguerra@iseg.ulisboa.pt.

‡CEMAPRE, REM - Research in Economics and Mathematics, ISEG - Universidade de Lisboa. The author was partially supported by the Project CEMAPRE - UID/MULTI/00491/2013 financed by FCT/MEC through national funds. e-mail: mguerra@iseg.ulisboa.pt.

Contents

1 Introduction

1.1 Motivations and Literature Review

This paper undertakes a study of the fundamental optimal asset allocation problem in finance and extends the current literature by assessing the empirical performance of a dynamic strategy which relies on signals generated from extracting information from option prices under an exponential Lévy market assumption. The extensive literature regarding multi-period portfolio selection was catalyzed by the seminal papers of Mossin [42], Samuelson [51], and Merton [40] in the 1960s. In [37] much of the current framework for using Lévy processes for price modeling was first developed with the Variance Gamma process, although the idea of modeling prices with infinite activity and discontinuities had already been considered years earlier in [38], [47], [46], and [5]. Since the Variance Gamma process, a number of additional Lévy processes have been proposed to more realistically capture stylized facts of the markets, perhaps the most popular being the Hyperbolic process of [20], the Normal Inverse Gaussian process of [2], the CGMY process of [9], the Kou process of [32], and the Meixner process of [54]. This study involves the merging of the optimal portfolio selection literature and the literature on Lévy processes in finance within a systematic investment strategy. The problem of optimizing a portfolio with Lévy drivers was first solved using the dynamic programming approach in [3] and [22]. In [27], the problem was also solved using the martingale method. More recently, in [45], the optimal portfolio problem was studied in more depth numerically for exponential Lévy processes under CRRA utility.

We build mainly off of the results and methodology of [31], but also [19], [23], and [24], which showed that portfolio performance could be improved by using forward-looking rather than historical distributions, and [45], where as mentioned above a number of results were derived for the optimal control strategy of exponential Lévy processes. However, here the authors only considered the problem cross-sectionally, that is they compared optimal portfolios under the jump assumption to the pure diffusion assumption using a single parameterization and studied the implications. We take a different approach here by effectively analyzing the problem across time by means of a periodically recalibrated and rebalanced investment strategy. Further, in for instance [21] it was found that market information extracted under an exponential Lévy model showed that the risk preferences of equity investors were signaling anomalous behavior prior to the U.S. subprime crisis of late 2007. This result is extended in the current analysis to answer the question of whether or not these signals provide any value in an investable strategy. Given that the parameterizations of Lévy processes used in financial modeling have economic interpretations and are deemed to offer improvements over more simple classical models, forward-looking estimates of the parameters could reasonably be expected to improve realized portfolio performance within a framework which incorporates this information. The goal is thus to investigate whether a number of theoretical results in stochastic processes, option pricing, and optimal stochastic control together have any value in a quite simple and implementable investment strategy. In essence, the objective of the research was to demonstrate the value of option-implied information in constructing portfolios, using a flexible parametric approach (i.e. an option pricing model) to periodically model the distribution implied in option prices and selecting portfolios according to the solution to a stochastic optimal control problem.

1.2 Organization of Paper

The rest of the paper is organized as follows. Section 2 begins with a brief overview of Lévy processes, and then details the Exponentially Tempered Stable process with application to financial markets and an important result regarding transformations between the risk-neutral and real-world parameterization. Section 3 offers a quick review of the Carr-Madan transform option pricing methodology and how this is used to calibrate the model. Section 4 discusses the optimal strategy in the context of a stochastic optimal control problem. Section 5 presents the main results of the paper. Section 6 concludes. Finally, appendices offer proofs for a number of theorems regarding the Exponentially Tempered Stable Lévy process, including the characteristic function of the process (Appendix A), the characteristic function of the log-stock price under the model (Appendix B), equivalent martingale measure conditions for the process (Appendix C), and the relative entropy of the process (Appendix D).

2 Model Setup

2.1 Lévy Processes

We begin with a brief review of the essentials of Lévy processes. For a more detailed treatment, the reader can refer to [52], and [17] for applications to finance.

A Lévy process is a real valued and adapted stochastic process $X = \{X_t, t \geq 0\}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that has stationary independent increments, is stochastically continuous, has right-continuous sample paths with left limits (càdlàg), and satisfies $X_0 = 0$. More simply, a Lévy process can be thought of as the continuous-time analog of a random walk. The Lévy-Khintchine formula, a necessary and sufficient condition for an infinitely divisible distribution, gives the following representation for the characteristic function of every Lévy process X_t:

$$\mathbb{E}[e^{iuX_t}] = \phi_{X_t}(u) = e^{t\psi(u)}, \tag{2.1}$$

where $\psi(u)$ is the characteristic exponent, given by the expression

$$\psi(u) = i\mu u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbb{1}_{\{|x|\leq 1\}}(x))\nu(dx), \tag{2.2}$$

with $\mu \in \mathbb{R}, \sigma \in \mathbb{R}_0^+$, and ν a positive sigma-finite measure on $\mathbb{R}$. The measure ν gives the expected number of jumps of a certain size per unit of time and is referred to as the Lévy measure of X, satisfying

$$\nu\{0\} = 0, \quad \int_{\mathbb{R}} \min(1, x^2)\nu(dx) < \infty. \tag{2.3}$$

The Lévy measure need not be finite, and if $\nu(\mathbb{R}) = \infty$ then infinitely many small jumps occur, and the Lévy process is said to have infinite activity. If $\nu(\mathbb{R}) < \infty$ then almost all paths have a finite number of jumps on any bounded non-trivial interval, and the Lévy process is said to have finite activity.

2.2 Exponential Lévy Price Models

Let $S_t, 0 \leq t \leq T$, denote the stock price process on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. An exponential Lévy model represents the dynamics of S_t under the real-world probability measure $\mathbb{P}$ as

$$S_t = S_0 e^{Y_t}, \tag{2.4}$$

with

$$Y_t = X_t + bt, \tag{2.5}$$

where X_t is a Lévy process with triplet (κ, σ^2, ν) and b is an asset-specific drift term. That is, the Lévy process X is taken as the martingale component of the log-stock price process. Assuming no arbitrage, we can construct a price process under the risk-neutral measure $\mathbb{Q}$ as

$$\tilde{S}_t = S_0 e^{\tilde{Y}_t}, \tag{2.6}$$

with

$$\tilde{Y}_t = \tilde{X}_t + (r - d)t + \tilde{\omega}t, \tag{2.7}$$

where tilde denotes a variable is under the risk-neutral measure, r is the interest rate, d the (continuous) dividend yield, and $\tilde{\omega}$ a so-called (as in [53]) "mean-correcting measure", such that the discounted price $\bar{S}_t = e^{-(r-d)t}\tilde{S}_t$ is a martingale, or

$$\mathbb{E}^{\mathbb{Q}}[\tilde{S}_t] = S_0 e^{(r-d)t}, \tag{2.8}$$

which requires that $\tilde{\omega} = -\log[\phi_{\tilde{X}_t}(-i)]$. However, it is important to note that this equivalent martingale measure is not unique in most cases involving Lévy processes (i.e., the market is incomplete). We will address this consideration with more specifics later.

2.3 The Exponentially Tempered Stable (ETS) Process

The Lévy process used throughout this study is an Exponentially Tempered Stable (hereafter ETS) process. The ETS process (as studied previously in [6], [7], [14], and [30], for example) is in fact just the CGMY process introduced in [9] with a generalization of the activity or kurtosis parameter depending on the sign of the random variable (equivalently, the CGMY process is a special case of the ETS process). Later, in for instance [33], another special case known as Bilateral Gamma was studied.

Definition 2.1. *An infinitely divisible distribution is called an Exponentially Tempered Stable (ETS) distribution with parameters λ_+, λ_-, β_+, β_-, and α if its Lévy triplet (κ, σ^2, ν) is given by $\sigma^2 = 0, \kappa \in \mathbb{R}$, and*

$$\nu(dx) = \left(\lambda_+ \frac{\exp(-\beta_+ x)}{x^{\alpha+1}} \mathbb{1}_{x>0} + \lambda_- \frac{\exp(-\beta_-|x|)}{|x|^{\alpha+1}} \mathbb{1}_{x<0}\right) dx, \tag{2.9}$$

where $\lambda_+, \lambda_-, \beta_+, \beta_- > 0$ and $\alpha \in (-1, 2)\backslash\{0, 1\}$.

The condition $\alpha > -1$ ensures a completely monotone Lévy density (see [9]), and the condition $\alpha < 2$ ensures that the Lévy density will integrate x^2 around 0. It can also be shown that the process is of infinite activity for $\alpha > 0$ and of infinite variation for $\alpha > 1$. Although the probability density function of the ETS process is not explicitly known, the characteristic function admits a rather simple form, and is given in the following theorem (for Proof see Appendix A). For ease of notation, the vector $\overrightarrow{\mathbf{v}} = \{\lambda_+, \lambda_-, \beta_-, \beta_+, \alpha\}$ will be used to denote the ETS parameters.

Theorem 2.2 (ETS Characteristic Function) *The characteristic function of the ETS random variable is given by:*

$$\phi_{ETS}(u, t; \overrightarrow{\mathbf{v}}) = \exp(t\lambda_+\Gamma(-\alpha)[(\beta_+ - iu)^\alpha - \beta_+^\alpha] + t\lambda_-\Gamma(-\alpha)[(\beta_- + iu)^\alpha - \beta_-^\alpha]). \tag{2.10}$$

As the name suggests, the ETS process and a stable process with stability index $\alpha \in (0, 2)$ have similar Lévy measures, but importantly the measure of the ETS process includes the additional exponential "tempering" factors. Because of this exponential tempering within the Lévy measure, the distribution has finite moments of all orders and the large jumps do not require truncation. The ETS process is quite versatile and the parameters of the process play an important role in capturing various desirable aspects in a stochastic process with applications to financial problems. The parameters λ_+ and λ_- can be seen as a measure of the overall level of activity of the process. The parameters β_+ and β_-, respectively, control the rate of exponential decay on the right and left of the Lévy density, allowing for the construction of a skewed distribution. For example, if $\beta_- < \beta_+$ the exponential tempering factor for negative values of the random variable is inducing slower decay than the positive factor and thus we have a left-skewed distribution, which is consistent with the risk-neutral distribution typically implied from option prices. In the special case where $\beta_- = \beta_+$, the Lévy measure is symmetric, although non-normal distributions can still be generated through the parameters λ_+ and λ_-, which provide control over the kurtosis of the random variable. Finally, the parameter α describes the behavior of the Lévy density near zero. Note that the case when $\alpha = 0$ is the Bilateral Gamma process of [33], which can also be seen as a generalized version of the Variance Gamma process of [36].

2.3.1 Risk Neutral Price Process

The rather general formulations for the stock price process introduced in section 2.2 can now be specified to an ETS model. We will consider a risk-neutral jump-diffusion ETS model for the stock price process, which takes an extended ETS random variable,

$$X_{ETS_e}(t; \tilde{\vec{\mathbf{v}}}, \tilde{\sigma}) = X_{ETS}(t; \tilde{\vec{\mathbf{v}}}) + \tilde{\sigma} W_t, \tag{2.11}$$

with W_t a standard Wiener process independent of X_{ETS}, as the martingale component of the log-stock price, giving the risk-neutral price process as

$$\tilde{S}_t = S_0 \cdot \exp[((r-d) + \tilde{\omega} - \tilde{\sigma}^2/2)t + X_{ETS_e}(t; \tilde{\vec{\mathbf{v}}}, \tilde{\sigma})], \tag{2.12}$$

with $\tilde{\omega}$ the mean-correction term introduced earlier, given here by

$$\begin{aligned}\tilde{\omega} = -\log(\phi_{ETS}(-i; \tilde{\vec{\mathbf{v}}})) = \\ -\tilde{\lambda_+} \cdot \Gamma(-\tilde{\alpha})[(\tilde{\beta_+} - 1)^{\tilde{\alpha}} - \tilde{\beta_+}^{\tilde{\alpha}}] - \tilde{\lambda}_- \cdot \Gamma(-\tilde{\alpha})[(\tilde{\beta_-} + 1)^{\tilde{\alpha}} - \tilde{\beta_-}^{\tilde{\alpha}}].\end{aligned} \tag{2.13}$$

We also now give the characteristic function of the risk-neutral log-stock price under the model in the following theorem (for Proof see Appendix B).

Theorem 2.3 (Characteristic Function of ETS Log-Stock Price) *The characteristic function of the risk-neutral log-stock price under the extended ETS model is given by:*

$$\tilde{\phi}_{\log(\tilde{S}_t)}(u,t) = \mathbb{E}^{\mathbb{Q}}[e^{iu\log(\tilde{S}_t)}] =$$

$$\exp(iu\{\log(S_0) + ((r-d) + \tilde{\omega} - \tilde{\sigma}^2/2)t\}) \cdot \phi_{ETS}(u; \tilde{\overrightarrow{\mathbf{v}}}) \cdot exp(-\tilde{\sigma}^2 u^2/2). \quad (2.14)$$

As in section 2.2, the discounted risk-neutral price process of equation (2.12) is a martingale, and we further clarify the equivalence of this martingale measure with the objective measure in the following theorem (for proof see Appendix C).

Theorem 2.4 (EMM Condition for the ETS Model) *Assume $S_t,\ 0 \leq t \leq T$ is the real-world ETS stock price process with parameters $(\overrightarrow{\mathbf{v}}, b)$ under the measure $\mathbb{P}$, and with parameters $(\tilde{\overrightarrow{\mathbf{v}}}, (r-d))$ under the measure $\mathbb{Q}$. Then $\mathbb{Q}$ is an equivalent martingale measure (EMM) of $\mathbb{P}$ if and only if $\alpha = \tilde{\alpha}, \lambda_+ = \tilde{\lambda_+}, \lambda_- = \tilde{\lambda_-}$, and*

$$r - d - \log(\phi_{ETS}(-i; \lambda_+, \lambda_-, \tilde{\beta_-}, \tilde{\beta_+}, \alpha)) = b - \log(\phi_{ETS}(-i; \overrightarrow{\mathbf{v}})). \quad (2.15)$$

Choosing some θ such that $-\beta_- < \theta < \beta_+$, we can express this as

$$r - d - \log(\phi_{ETS}(-i; \lambda_+, \lambda_-, \beta_- + \theta, \beta_+ - \theta, \alpha)) = b - \log(\phi_{ETS}(-i; \overrightarrow{\mathbf{v}})). \quad (2.16)$$

2.3.2 Minimal Entropy Martingale Measure of the ETS Process

As mentioned in section 2.2, there are generally an infinite number of equivalent martingale measures when dealing with Lévy processes, and we must therefore have a selection rule or criterion to reduce the class of possible measures $\mathbb{Q}$ to an appropriate subset and then obtain a unique equivalent measure. In other words, the equivalent martingale measure of theorem 2.4 is not unique, and we thus select the unique EMM minimizing the relative entropy with respect to $\mathbb{P}$ and satisfying equation (2.16). Recall that relative entropy, or Kullback-Leibler divergence, is a measure of the distance between two equivalent probability measures, defined as (with $H(\mathbb{P}|\mathbb{Q})$ the relative entropy between $\mathbb{P}$ and $\mathbb{Q}$, see [28]):

$$H(\mathbb{P}|\mathbb{Q}) = \mathbb{E}^{\mathbb{P}}\left[\ln \frac{d\mathbb{P}}{d\mathbb{Q}}\right], \quad (2.17)$$

where $\frac{d\mathbb{P}}{d\mathbb{Q}}$ is the Radon-Nikodym derivative of $\mathbb{P}$ with respect to $\mathbb{Q}$. The relative entropy of the ETS process can be expressed explicitly and is thus given in the following theorem (for Proof see Appendix D).

Theorem 2.5 (Relative Entropy of the ETS Process) *Let $(X_t, \mathbb{P}),\ 0 \leq t \leq T$ and $(\tilde{X}_t, \mathbb{Q}),\ 0 \leq t \leq T$ be ETS processes with parameters $\overrightarrow{\mathbf{v}}$ and $\tilde{\overrightarrow{\mathbf{v}}}$, respectively. Suppose $\mathbb{P}$ and $\mathbb{Q}$ are equivalent measures, that is $\alpha = \tilde{\alpha}, \lambda_+ = \tilde{\lambda}_+$, and $\lambda_- = \tilde{\lambda}_-$. The relative entropy between these processes is given by:*

$$H(\mathbb{P}|\mathbb{Q}) = t\lambda_+\Gamma(-\alpha)((\alpha-1)\tilde{\beta_+}^{\alpha} - \alpha\beta_+\tilde{\beta_+}^{\alpha-1} + \beta_+^{\alpha})$$

$$+t\lambda_-\Gamma(-\alpha)((\alpha-1)\tilde{\beta_-}^{\alpha} - \alpha\beta_-\tilde{\beta_-}^{\alpha-1} + \beta_-^{\alpha}). \tag{2.18}$$

3 Calibrating the Model

3.1 A Suitable Pricing Model for Lévy Processes

The ETS model can be calibrated to option prices, that is under the risk-neutral measure, by inverting the pricing problem and finding those model parameters which allow closest replication of market observed prices. Accordingly, we first need a suitable pricing formula. Consider the classical case of pricing a European call option. Risk-neutral valuation yields

$$C_T(k) = e^{-(r-d)T}\mathbb{E}^{\mathbb{Q}}[\max(S_T - K, 0)] = e^{-(r-d)T}\int_k^{\infty}(e^s - e^k)q_T(s)ds, \tag{3.1}$$

where C_T is the time 0 price of the claim maturing at time T, $s = \log(S)$, $k = \log(K)$, and q_T is the risk-neutral density of s_T. However, we do not have an expression for q_T for most cases involving Lévy processes (and importantly including the ETS process). Thus, we need an alternative pricing formula. Carr and Madan [10] lay the framework for using the Fast Fourier transform to solve the option pricing problem, a method which takes advantage of the fact that a bijection, or a unique one-to-one relationship exists between probability densities and characteristic functions (recall that we *do* know the characteristic function of log returns when working with the ETS process). We very briefly summarize the methodology below. Consider the Fourier transform of a modified or dampened call (this remedies a square-integrability concern) $c_T(k) = e^{\alpha k} C_T(k)$,

$$\Psi_T(\vartheta) = \int_{-\infty}^{\infty} e^{i\vartheta k} c_T(k) dk. \tag{3.2}$$

After inverting this transform and expressing $\Psi_T(\vartheta)$ fully in terms of $\phi_T(\vartheta)$, the characteristic function of log returns, we have for the call price:

$$C_T(k) = \frac{e^{-\alpha k} e^{-(r-d)T}}{\pi} \mathrm{Re}[\int_0^{\infty} \frac{e^{-i\vartheta k} \phi_T(\vartheta - (\alpha+1)i)}{\alpha^2 + \alpha - \vartheta^2 + i(2\alpha+1)\vartheta} d\vartheta]. \tag{3.3}$$

We evaluate this expression using the Fast Fourier Transform (FFT) algorithm, which recall is a method for computing sums of the form

$$\sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}jk} x_j, \quad k = 0, ..., N-1. \tag{3.4}$$

Using a trapezoidal approximation and incorporating the Simpson weighting rule yields

$$C_T(k) \approx \frac{e^{-\alpha k}}{\pi} \sum_{j=0}^{N-1} e^{-i\eta j \lambda u} \underbrace{\Psi_T(\eta j) e^{i\eta j(b-\theta)} \frac{\eta}{3}(3 + (-1)^{j+1} - \delta_j)}_{=x_j}, \tag{3.5}$$

with δ_j the Kronecker delta function that is unity for $j = 0$ and zero otherwise, $\eta > 0$ the integration step, k a vector of strikes ranging from $\theta - b$ to $\theta + b$ with $\theta = \ln(S_0)$, $b = \frac{\pi}{\eta}$ and N steps of size $\lambda = \frac{2b}{N-1}$, or $k = \theta - b + \lambda u$ with $u = 0, ..., N-1$. According to equation (3.4), we need $\eta\lambda = \frac{2\pi}{N}$. As long as N is chosen as a power of 2, and all other parameters are set as defined above, this condition will be satisfied.

3.2 Inverting the Option Pricing Problem

Taking market prices as a reference, the transform pricing method presented above can be used to calibrate the parameters of a given Lévy model. That is, the option pricing problem is inverted, with market prices assumed as given, and the ETS process parameters calibrated so as to most closely replicate these given data points. To this end, an objective or cost function is constructed to achieve the best approximation to the data. As proposed in for instance [1], we will use an average pricing error (APE), given by

$$APE = \min_{\tilde{\vec{v}},\tilde{\sigma}} \frac{1}{N} \sum_{i=1}^{N} \left| \frac{\text{market price}_i - \text{calculated price}_i}{\text{market price}_i} \right|. \tag{3.6}$$

A significant issue remains due to the fact that equation (3.6) can have many local minimizers, and thus the calibration may be non-unique and numerical schemes can be highly sensitive to the initial value choices. There are a few ways we can deal with this. The first and perhaps most intuitive, though heuristic, is to simply run the optimization multiple times with different initial parameter vectors, and choose the eventual calibration which most minimizes the cost function of equation (3.6). Alternatively, one could utilize a regularization framework, such as that developed in [16], [17], and [18], which for instance adds an entropy-based regularization parameter to the cost function. For simplicity and to avoid the necessary technical discussion of a regularized approach, the multiple initial values approach was used throughout this study.

4 Calculating the Optimal Portfolio

We consider the decision problem to be that of a risk-averse investor following a one-dimensional portfolio strategy (i.e. a single risky asset and without consumption) subject to some utility function of wealth $U(\cdot)$. Thus, the investor's goal can be thought of as maximizing terminal or end-of-period wealth,

$$\mathbb{E}^{\mathbb{P}}_{t,x}[U(X^{\pi}_{T})], \tag{4.1}$$

where the notation $\mathbb{E}_{t,x}[\cdot] = \mathbb{E}[\cdot|X_t = x]$ and $\mathbb{P}$ denotes the problem is set under the objective or real-world probability measure. The variable of which it is necessary to know the current value, or current state, is conveniently referred to as the state variable, here denoted x. The variables chosen at any given point in time are known as the control variables, and here our control variable is π, the weight of the portfolio to be invested in the risky asset. Assuming the state process or variable x is driven by a stochastic process, equation (4.1) is known as a stochastic optimal control problem. We assume that the dynamics of wealth X are given by

$$dX^{\pi}_t = \mu(t, X^{\pi}_t, \pi)dt + \sigma(t, X^{\pi}_t, \pi)dW_t + \int_{\mathbb{R}} g(t, X^{\pi}_{t^-}, z, \pi)\tilde{N}_t(dt, dz), \quad X_0 = x > 0, \tag{4.2}$$

with $\mu(t, X^{\pi}_t, \pi)$ and $\sigma(t, X^{\pi}_t, \pi)$ functions possibly depending on t, X^{π}_t, and π, $g(t, X^{\pi}_t, z, \pi)$ a function possibly depending on t, X^{π}_t, π, and also the jump size z, and $\tilde{N}$ is a centered or "compensated" Poisson random measure (see [17] for a more comprehensive treatment of these random measures). The suitable Hamilton-Jacobi-Bellman (HJB) equation for this problem (see [4], [43]) is:

$$\begin{cases} 1) \quad \frac{\partial V}{\partial t}(t,x) + \sup\limits_{\pi \in \Pi} \Big\{ \{\mu(t,x,\pi)\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2(t,x,\pi)\frac{\partial^2}{\partial x^2}\} V(t,x) \\ \quad + \int_{\mathbb{R}} \{V(t, x + g(t,x,z,\pi)) - V(t,x) - g(t,x,z,\pi)\frac{\partial V}{\partial x}(t,x)\}\nu_t(dz) \Big\} = 0 \\ 2) \quad V(T,x) = U(x) \quad \forall x \in \mathbb{R}. \end{cases} \tag{4.3}$$

4.1 CRRA Utility

We consider the case of iso-elastic or power utility, where

$$U(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma} & \text{for } \gamma > 0, \gamma \neq 1 \\ \log(x) & \text{for } \gamma = 1, \end{cases} \tag{4.4}$$

in a market with two possible investments,
1) A risk-free asset with dynamics

$$dB_t = r_t B_t dt, \quad B_0 = b > 0. \tag{4.5}$$

2) A risky asset with dynamics

$$dS_t = S_{t^-}[\mu_t dt + \sigma_t dW_t + \int_{-1}^{\infty} z_t \tilde{N}(dt, dz)], \quad S_0 = s > 0, \tag{4.6}$$

with $r_t > 0$, $\mu_t > 0$, and $\sigma_t > 0 \in \mathbb{R}$. Note that the bounds on the integral term imply that the random jump variable has support on $[-1, \infty]$, which guarantees the positivity of the stock price. We additionally assume that

$$\int_{-1}^{\infty} |z_t| d\nu(z) < \infty, \quad \text{and} \quad \mu_t > r_t \; \forall t. \tag{4.7}$$

Consider now a wealth or portfolio process P given by

$$dP_t = P_t \left[\pi \frac{dS_t}{S_t} + (1 - \pi) \frac{dB_t}{B_t} \right], \quad P_0 = p. \tag{4.8}$$

Substituting the asset dynamics from (4.5) and (4.6), we have

$$dP_t = P_t \left[\pi \frac{S_{t^-}[\mu_t dt + \sigma_t dW_t + \int_{-1}^{\infty} z_t \tilde{N}(dt, dz)]}{S_t} + (1 - \pi) \frac{r_t B_t dt}{B_t} \right], \tag{4.9}$$

which can be simplified to the following form, with underbraces clarifying analogous terms to equation (4.2):

$$dP_t = \underbrace{P_t[r_t + (\mu_t - r_t)\pi]}_{=\mu(t,p,\pi)} dt + \int_{-1}^{\infty} \underbrace{P_t \pi z_t}_{=g(t,p,z,\pi)} \tilde{N}(dt, dz))dt + \underbrace{P_t \sigma_t \pi}_{=\sigma(t,p,\pi)} dW_t, \tag{4.10}$$

and also with terminal condition

$$V(T, p) = U(p). \tag{4.11}$$

Thus, according to equation (4.3), our HJB equation is, for $\gamma \neq 1$ (the equation for when $\gamma = 1$ will be handled shortly):

$$\begin{cases} 1) \quad \frac{\partial V}{\partial t}(t,p) + \sup\limits_{\pi \in \Pi} \Big\{ \{p_t[r_t + (\mu_t - r_t)\pi] \frac{\partial}{\partial x} + \frac{1}{2}(p_t \sigma_t \pi)^2 \frac{\partial^2}{\partial x^2}\} V(t,p) \\ \quad + \int_{-1}^{\infty} \{V(t, p_t + p_t \pi z_t) - V(t,p) - p_t \pi z_t \frac{\partial V}{\partial x}(t,p)\} \nu_t(dz) \Big\} = 0 \\ 2) \quad V(T,p) = U(p) \quad \forall p \in \mathbb{R}. \end{cases} \tag{4.12}$$

We now search for a solution of the form

$$V(t,p) = U(e^{\delta_t} p), \tag{4.13}$$

where δ_t is a C^1 deterministic function of time such that $\delta_T = 0$, giving

$$V(t,p) = \frac{(e^{\delta_t} p)^{1-\gamma}}{1-\gamma} \text{ for } \gamma > 0, \gamma \neq 1. \tag{4.14}$$

After some manipulation and simplification, we find the following form for our HJB equation when $\gamma > 0, \gamma \neq 1$:

$$\begin{cases} 1) \quad \dot{\delta}_t + \sup\limits_{\pi \in \Pi} \Big\{ r_t + (\mu_t - r_t)\pi - \frac{1}{2}\gamma\sigma_t^2\pi^2 \\ \quad + \int_{-1}^{\infty} \{ \frac{1}{1-\gamma} \left[(1+z_t\pi)^{(1-\gamma)} - 1 \right] - z_t\pi \} \nu_t(dz) \Big\} = 0 \\ 2) \quad \delta_T = 0. \end{cases} \tag{4.15}$$

We can now also address the limiting case when $\gamma = 1$, invoking the Cauchy/L'Hôpital theorems to obtain:

$$\begin{cases} 1) \quad \dot{\delta}_t + \sup\limits_{\pi \in \Pi} \Big\{ r_t + (\mu_t - r_t)\pi - \frac{1}{2}\gamma\sigma_t^2\pi^2 \\ \quad + \int_{-1}^{\infty} \{ [\log(1+z_t\pi) - z_t\pi] \, \nu_t(dz) \Big\} = 0 \\ 2) \quad \delta_T = 0. \end{cases} \tag{4.16}$$

For the proof of existence of a solution, including a suitable verification theorem, refer to [45].

4.2 Admissible Controls

Before finally deriving the associated optimal control strategy, it is useful to discuss the details of the set of admissible controls, which we will denote Π. Recall from equation (4.6) that the random jump variable is assumed to have support on $[-1, \infty)$ to guarantee positivity of, or equivalently to preserve the limited liability structure of, the stock price. Furthermore, as discussed in [34], there are additional concerns when considering jump processes in a wealth context. It would seem reasonable to assume that the wealth process itself inherits the behavior of the underlying drivers, that is importantly if all constituent securities are driven by pure diffusions the wealth process will also be a pure diffusion process, however portfolio assets with discontinuous paths would also introduce discontinuities into the overall wealth process. In the case of pure diffusion processes without jumps, the variance of returns over some time period dt is proportional to dt, implying that as one takes $dt \to 0$, the uncertainty associated with changes in wealth also goes to zero. Thus, investors retain complete control over the portfolio via continuous rebalancing, and leveraged positions can be adjusted quickly enough before wealth turns negative. However, this is not the case when discontinuities are present, as the investor has no time to rebalance when a jump occurs, and therefore negative wealth could arise due to leveraged long and short positions. This situation can be likened to illiquidity risk, since investors in illiquid assets may face large changes in the value of their portfolio before/without being able to rebalance the position. As such, the following theorem applies to the case of jump processes (see [34] or [35] for proof):

Theorem 4.1 (Bounds on Portfolio Weights) *For any $t, 0 < t \leq T$, and denoting X_{inf} and X_{sup} as the lower and upper bounds of the support of the random price jump X_t, the optimal portfolio weight $\hat{\pi}_t$ must satisfy*

$$1 + \hat{\pi}_t X_{inf} > 0 \quad and \quad 1 + \hat{\pi}_t X_{sup} > 0. \tag{4.17}$$

In particular, if $X_{inf} < 0$ and $X_{sup} > 0$,

$$-\frac{1}{X_{sup}} \leq \hat{\pi}_t \leq -\frac{1}{X_{inf}}. \tag{4.18}$$

Thus, the set of admissible controls is $\Pi = [0, 1]$ and the investor will never take leveraged long or short positions in the risky asset.

4.3 Optimal Strategy

Although solutions on the boundary are possible and must be evaluated case by case, we can analyze the first order condition and derive a more explicit solution by assuming an internal solution. We are thus interested in, for $\gamma \neq 1$,

$$\sup_{\pi\in\Pi}\left\{r_t + (\mu_t - r_t)\pi - \frac{1}{2}\gamma\sigma_t^2\pi^2 \quad + \int_{-1}^{\infty}\{\frac{1}{1-\gamma}\left[(1+z_t\pi)^{(1-\gamma)} - 1\right] - z_t\pi\}\nu_t(dz)\right\}, \tag{4.19}$$

and for $\gamma = 1$,

$$\sup_{\pi\in\Pi}\left\{r_t + (\mu_t - r_t)\pi - \frac{1}{2}\gamma\sigma_t^2\pi^2 \quad + \int_{-1}^{\infty}\{[\log(1+z_t\pi) - z_t\pi]\,\nu_t(dz)\right\}. \tag{4.20}$$

Working through the first order condition $\frac{\partial}{\partial\pi} = 0$ we obtain

$$(\mu_t - r_t) - \gamma\sigma_t^2\hat{\pi} + \int_{-1}^{\infty} z_t((1+z_t\hat{\pi})^{-\gamma} - 1)\nu_t(dz) = 0, \tag{4.21}$$

for all $\gamma > 0$. It is obvious from the equation that no explicit expression exists for the control $\hat{\pi}$, however numerical methods provide a simple solution.

5 Data, Methodology, and Results

We come now to the main contribution of our research. The first four sections of the paper addressed the theory and computations necessary to carry out the following analysis. Our approach to the portfolio construction process involves calibrating the extended (jump-diffusion) ETS process of section 2 to option data by means of the transform option pricing method made clear in section 3. After the appropriate transformation from the risk-neutral to the objective measure, the calibrated Lévy measure can then be used as an input to the stochastic control framework of section 4 to derive the optimal portfolio. Matching the investment horizon with the time to maturity of the options used in the calibration, we simulate a portfolio that iterates this process over a 14 year period. We proceed by detailing the dataset used, followed by a detailed explanation of the methodology, leading finally to the main results of the paper.

5.1 Dataset

The dataset for the empirical analysis covers the period of January 2004 - December 2017. A set of daily option chains on the Standard and Poor's 500 index (S&P 500) was obtained from the Chicago Board Options Exchange (CBOE), known as the SPX Options Traditional product.

The CBOE data has the benefit of including a quote at 3:45 PM U.S. Eastern time (15 minutes before the market close). The contract has a European exercise style and is AM-settled on the 3rd Friday of every month. Only call options were considered in the calibration of the ETS model. The average of the market bid and ask was used for the market price, and following [21] and [49], the following filters were imposed upon the data:

1) Moneyness is constrained to $-0.10 \leq (K/S_t - 1) \leq 0.10$,
2) Annualized implied volatility must be between 5% and 95%,
3) Remove options with premia below the lower bound (intrinsic value), in particular $\max(0, S_t - q_t S_t - Ke^{-r_t(T-t)})$ for call options,
4) Verify that call prices are a decreasing function of strike.

However, the option data was used in another important way, and in this case these filters proved far too restrictive, so it may be a good point to detail this consideration. In order to add context to the performance of the strategy, we of course need a suitable benchmark. In the spirit of using option-implied information to guide allocation decisions, the main benchmark will be a mean-variance version of the stochastic optimal control framework of section 4, where variance is implied from option prices using a "model-free" approach (see [8], [11], and [26]). This represents an interesting benchmark because, while similarly relying on option-implied information, it also contrasts the ETS model in a number of ways. First, it is of course a mean-variance type optimization, and therefore it allows us to differentiate the performance of a portfolio that is incorporating the option market's expectations of the higher moments of the distribution of forward returns from a portfolio that does not. Secondly, as stated, it is a non-parametric or model-free approach and therefore does not rely on any option pricing model. Again, the filters listed above are far too restrictive for this methodology, and produce implied variance numbers that are nonsensically low. Accordingly, model-free implied variance is calculated in similar fashion to the widely followed VIX index. The optimal control in this framework is analogous to the the well-known result of [40],

$$\hat{\pi} = \frac{\mu_t - r_t}{\gamma \sigma^2_{MFIV}}, \tag{5.1}$$

with σ^2_{MFIV} the implied variance estimate from the model-free approach. Two benchmark portfolios will actually be formed based on this mean-variance optimal control. Recall from theorem 4.1 that under the assumption that the risky asset (and by association the wealth process) can experience discontinuities, the investor never takes levered long or short positions in the risky asset. However, as also explained, in the case of pure diffusion processes, the investor retains a level of control over the portfolio that admits levered long and short positions. Accordingly, a non-levered version of the mean-variance portfolio, denoted "MFIV", which restricts the weights produced from equation (5.1) above to the same set of admissible controls for the ETS portfolio ($\hat{\pi} \in [0, 1]$) serves as one benchmark. Then, there is a second levered benchmark, denoted "MFIVHL", where "HL" stands for half-leverage, signifying that leverage is tempered by one popular rule-of-thumb in practice for managing the risk of levered positions. That is, at times the mathematics of the mean-variance approach may prescribe optimal portfolios with outsized percentages of wealth allocated to the risky asset, say 300% for illustrative purposes. A practitioner is unlikely to take this at face value, and will respond by limiting the leverage to more realistic levels, or $100\% + 0.5 \cdot (300\% - 100\%) = 200\%$ if doing so according to this stylized example of the "half-leverage" rule. For the interested reader, it turns out this rule-of-thumb isn't as arbitrary as one may assume, and in [55] the authors show that optimal bet sizes according to the Kelly criterion (equivalent to the Merton result for log-utility) must be adjusted down considerably in light of practical concerns. Note of course that this "half-leverage" rule is only one such approach. In for instance [31], leverage was constrained by restricting the optimal control to the set $[-1, 2]$.

Additionally, the underlying spot index price along with the associated index dividend yield as well as the risk-free interest rate, chosen to be one-month LIBOR, were obtained from Bloomberg. For the investable strategy to be detailed in the next section, daily adjusted close prices for the SPDR S&P 500 ETF (ticker SPY) were also obtained from Bloomberg. Finally, to apply theorem 2.4, we note that we need a number for the asset specific drift, b, to satisfy the equivalent martingale measure condition of equation (2.16). In order to keep with the theme of using forward looking implied information but also ensuring to satisfy the second condition of equation (4.7), this is taken to be a long-term implied market risk-premium for the US market plus LIBOR, with the first term being available from http://www.market-risk-premia.com/us.html. We will denote this as $\bar{b}$, and its exact use within our framework will be clarified shortly.

5.2 Methodology of Investment Strategy

Beginning in January 2004, every 30 calendar days prior to the next month's SPX option expiry, using the transform pricing and calibration methods of section 3, the 3:45 PM price of the filtered range of option strikes is calibrated according to the risk-neutral jump-diffusion ETS price process of section 2. These option-implied parameters effectively give a one-month forward risk-neutral density forecast for the underlying asset. The calibrated parameters are then transformed to real-world parameters. Some elaboration on the numerical approach is useful here. First, the risk-neutral parameters calibrated from option prices are importantly *assumed to be under the model preserving minimal entropy martingale measure*. Thus, we adjust equation (2.16) to

$$r - d - \log(\phi_{ETS}(-i; \tilde{\vec{\mathbf{v}}}_{EMM})) = \bar{b} - \log(\phi_{ETS}(-i; \lambda_+, \lambda_-, \tilde{\beta_-} - \theta, \tilde{\beta_+} + \theta, \alpha)), \quad (5.2)$$

with $-\tilde{\beta_+} < \theta < \tilde{\beta_-}$, $\tilde{\vec{\mathbf{v}}}_{EMM} = (\lambda_+, \lambda_-, \tilde{\beta_-}, \tilde{\beta_+}, \alpha)$, and $\bar{b}$ is the implied expected market return introduced above. An "entropy cost function," or ECF, is thus constructed as

$$ECF = \min_{\theta}\Bigg\{[(r-d-\log(\phi_{ETS}(-i;\tilde{\vec{\mathbf{v}}}_{EMM}))-(\bar{b}-\log(\phi_{ETS}(-i;\lambda_+,\lambda_-,\tilde{\beta_-}-\theta,\tilde{\beta_+}+\theta,\alpha)))]^2$$
$$+ t\lambda_+\Gamma(-\alpha)((\alpha-1)\tilde{\beta_+}^{\alpha} - \alpha(\tilde{\beta_+}+\theta)\tilde{\beta_+}^{\alpha-1} + (\tilde{\beta_+}+\theta)^{\alpha})$$
$$+ t\lambda_-\Gamma(-\alpha)((\alpha-1)\tilde{\beta_-}^{\alpha} - \alpha(\tilde{\beta_-}-\theta)\tilde{\beta_-}^{\alpha-1} + (\tilde{\beta_-}-\theta)^{\alpha})\Bigg\}, \quad (5.3)$$

and evaluated numerically using a search algorithm. Once the real-world parameters have been calculated, they are utilized in the stochastic optimal control framework of section 4 to derive the optimal portfolio for an individual investor with some assumed level of risk aversion. Specifically, equation (4.21) is solved numerically for $\hat{\pi}$. Here it is also necessary to clarify an aspect of the implementation. First, the Lévy triplet presented for the ETS process in section 2 assumed ordinary exponential form, i.e. $S_t = S_0 e^{L_t}$ for the stock price. However, we assumed a geometric type stochastic differential equation for the risky asset in equation (4.6). Accordingly, we transform the triplet using the ordinary-stochastic exponential relation by proposition (8.22) of [17] to obtain (with subscript L denoting the variable under the ordinary exponential):

$$\sigma = \sigma_L,$$

$$\nu(dx) = \tfrac{\lambda_+(1+x)^{-\beta_+}}{(\log(1+x))^{\alpha+1}}\mathbb{1}_{\{x>0\}}dx + \tfrac{\lambda_-(1+x)^{\beta_-}}{(-\log(1+x))^{\alpha+1}}\mathbb{1}_{\{-1<x<0\}}dx,$$

$$\kappa = \kappa_L + \frac{\sigma_L^2}{2}$$
$$+ \Gamma(-\alpha)\Bigg\{-\frac{\lambda_+}{\beta_+}[\beta_+^{\alpha+1} - \alpha\beta_+^{\alpha} - \beta_+(\beta_+-1)^{\alpha}] - \frac{\lambda_-}{\beta_-}[\beta_-^{\alpha+1} + \alpha\beta_-^{\alpha} - \beta_-(\beta_-+1)^{\alpha}]\Bigg\}, \quad (5.4)$$

with $\kappa_L = \bar{b}$ as a result of the assumption that led to equation (5.2).

On the day of each calibration (recall this importantly happens at 3:45 PM U.S. Eastern time), the investor rebalances his portfolio at the market close (4:00 PM U.S. Eastern time) by buying the SPY ETF in proportions given by the portfolio optimization procedure, and is assumed to invest the remainder until the next rebalancing date at the prevailing one-month LIBOR rate (i.e. keep it safe in a money market account). This procedure is repeated once a month for the entire analysis period. The same methodology applies to the mean-variance (MFIV & MFIVHL) strategies introduced earlier. Additionally, the canonical buy-and-hold and 60/40 portfolios are included as benchmarks for some additional context, although these are truly not appropriate benchmarks since these strategies are not optimal in any sense. Finally, we address the importance to simulate at least some of the transaction costs involved with implementing live trades. A cursory analysis showed that the turnover of the ETS strategy (here defined as the proportion of the total portfolio required to be traded in the risky asset to bring it in line with the new optimal control) was materially higher than the other benchmarks, including even the mean-variance strategy, and thus it is imperative to investigate whether the additional costs mitigate any performance benefits. Nevertheless, building a full and realistic implementation shortfall model (assuming such is possible) is outside the scope and purpose of this study, and so a more parsimonious path is taken. In [15], a fairly recent study by the Chicago Mercantile Exchange, it was found that the all in costs of trading the SPY ETF were about 3.25 basis points, in addition to slightly less than 0.80 basis point holding fee per month. Similar results were also found in [41], a study by Morningstar from a few years earlier (therefore the results seem to be fairly stable). Accordingly, transaction costs are modeled as 4 basis points of the trade size, and the trade size is compensated such that the optimal control is achieved *after* transaction fees. That is, assume for illustrative purposes that at some time $t - \epsilon, \epsilon > 0$ just prior to a rebalance at time t the portfolio is worth $P_{t-\epsilon} = \$1$ with 50% allocated the risky asset. We can describe the makeup of the portfolio value by $P_{t-\epsilon} = \$1 = 0.5 \cdot \$1 + (1 - 0.5) \cdot \$1$. Now assume that the optimal control (weight allocated to the risky asset) is 60% at time t. Without transaction costs, we would simply take \$0.10 from the bank account and add this to the risky asset. With transaction costs, for a trade size $\$x$, we now have the simultaneous equations $P_t = (0.5 \cdot \$1 + \$x \cdot 0.9996) + ((1 - 0.5) \cdot \$1 - \$x);\ (0.5 \cdot \$1 + \$x \cdot 0.9996) = 0.6 \cdot P_t$.

5.3 Performance Results

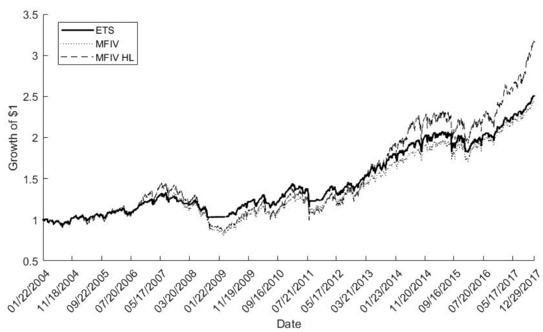
We now move to the performance of the backtested portfolios. The presentation for the main part of the text is restricted to risk aversion parameters of $\gamma = 1$ (i.e. log utility) and $\gamma = 5$, as these are illustrative of the main takeaway of the results as will be discussed. Figure 1 below shows the equity curves for investor risk aversion coefficients of 1 (log utility - relatively low risk aversion) and 5 (moderately high risk aversion), and the performance illustrated in these charts is quantified using a number of metrics in the preceding tables. Note that "Ann." stands for annual*ized*, and thus the first column is really showing excess CAGRs, for example. All calculations were carried out in MATLAB.

Table 1: Performance Metrics for $\gamma = 1$

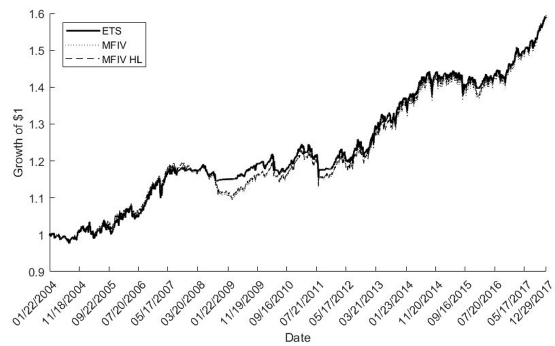
Strategy	Ann. Excess Return	Ann. Standard Deviation	Ann. Sharpe Ratio	Ann. Sortino Ratio	Max. DD	Avg. Turnover
	$\mu - r$	σ	$\frac{\mu-r}{\sigma}$	$\frac{\mu-r}{\sigma_-}$	$\max\{\frac{Trough}{Peak} - 1\}$	$\frac{1}{N}\sum_{i=1}^{N}\lvert w_i - w_{i-1}\rvert$
ETS	5.6927%	11.704%	0.48639	0.74348	-22.53%	12.58%
MFIV	5.4818%	13.395%	0.40926	0.64058	-39.76%	6.17%
MFIV HL	7.4679%	18.468%	0.40477	0.63313	-42.17%	31.04%
B&H	5.1407%	18.414%	0.27918	0.43551	-56.47%	-
60/40	3.2812%	10.9%	0.30102	0.46734	-37.53%	0.72%

Table 2: Performance Metrics for $\gamma = 5$

Strategy	Ann. Excess Return	Ann. Standard Deviation	Ann. Sharpe Ratio	Ann. Sortino Ratio	Max. DD	Avg. Turnover
	$\mu - r$	σ	$\frac{\mu-r}{\sigma}$	$\frac{\mu-r}{\sigma_-}$	$\max\{\frac{Trough}{Peak} - 1\}$	$\frac{1}{N}\sum_{i=1}^{N}\lvert w_i - w_{i-1}\rvert$
ETS	2.3021%	4.6498%	0.4951	0.75724	-7.30%	14.70%
MFIV	2.2826%	4.768%	0.47874	0.74597	-8.57%	11.40%
MFIV HL	2.3243%	4.7365%	0.49072	0.76664	-8.22%	10.55%
B&H	5.1407%	18.414%	0.27918	0.43551	-56.47%	-
60/40	3.2812%	10.9%	0.30102	0.46734	-37.53%	0.72%



(a) Equity Curves for $\gamma = 1$



(b) Equity Curves for $\gamma = 5$

Figure 1: Equity Performance of ETS, MFIV, and MFIVHL Portfolios

In order to add statistical context to the risk-adjusted performance, so called "quality control" charts are next constructed which evaluate the significance of the Sharpe ratios of the ETS portfolios. The charts are based on the methods of [44] for assessing the statistical significance of Sharpe ratio differences. The Sharpe ratio of the ETS portfolio is compared against both the MFIV and MFIVHL portfolios in figures 2 and 3. Here, the null hypothesis is $(SR_a - SR_b) = SR_{diff} = 0$ against an alternative hypothesis of $SR_{diff} \neq 0$, and the confidence bands are

$$\pm 1.96 \times \sqrt{Var_{diff}/t}, \tag{5.5}$$

with

$$\begin{aligned} Var_{diff} = 2 + \frac{SR_a^2}{4}\left[\frac{\mu_{4a}}{\sigma_a^4} - 1\right] - SR_a\frac{\mu_{3a}}{\sigma_a^3} + \frac{SR_b^2}{4}\left[\frac{\mu_{4a}}{\sigma_b^4} - 1\right] - SR_b\frac{\mu_{3b}}{\sigma_b^3} \\ - 2\left[\rho_{a,b} + \frac{SR_aSR_b}{4}\left[\frac{\mu_{2a,2b}}{\sigma_a^2\sigma_b^2} - 1\right] - \frac{1}{2}SR_a\frac{\mu_{1b,2a}}{\sigma_b\sigma_a^2} - \frac{1}{2}SR_b\frac{\mu_{1a,2b}}{\sigma_a\sigma_b^2}\right]. \end{aligned} \tag{5.6}$$

Note that the reason the ETS portfolio is not compared statistically to the buy-and-hold or 60/40 portfolios is due to the lack of correlation, which, as noted in [44], highly affects the power of the tests (this should be intuitive; note the correlation between the ETS and MFIV and MFIVHL portfolios is over 90%). The inclusion of these portfolios above was largely conventional.

(a) ETS-MFIV SR Difference

(b) ETS-MFIVHL SR Difference

Figure 2: Sharpe Ratio Quality Control Charts for $\gamma = 1$

(a) ETS-MFIV SR Difference

(b) ETS-MFIVHL SR Difference

Figure 3: Sharpe Ratio Quality Control Charts for $\gamma = 5$

We see above that the ETS model results in statistically significant risk-adjusted performance improvements compared to the MFIV and MFIVHL portfolios when investor risk aversion is low with $\gamma = 1$, however we fail to reject the null hypothesis as risk aversion becomes moderately high with $\gamma = 5$ (the significance actually vanishes somewhere between $\gamma = 2$ and $\gamma = 3$). This phenomenon of convergence as risk aversion increases is one of the main findings of the study, and is driven by the limiting behavior of equation (4.21) as γ becomes very large. This drives $\hat{\pi}$ to zero, which has the effect of driving the entire integral term to zero. As a result, the difference in weights, $\triangle_{\hat{\pi}}$, reduces to

$$\triangle_{\hat{\pi}} = \gamma\hat{\pi}(\sigma^2_{MFIV} - \sigma^2_t), \tag{5.7}$$

which also goes to zero as $\hat{\pi}$ goes to zero. Visual proof of this convergence is offered in figure 4 below, which plots the average weight differential over all days of the ETS portfolio versus the two benchmarks on the vertical axis against an increasingly large risk aversion coefficient on the horizontal axis. As shown, the differences in the strategies rapidly decays as γ approaches 5 and continues on to effectively zero as risk aversion is progressively increased.

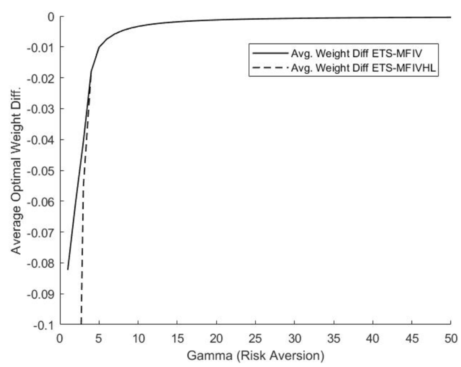
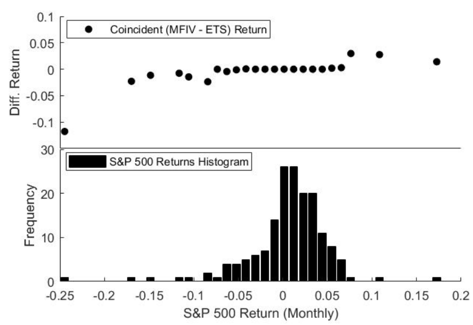
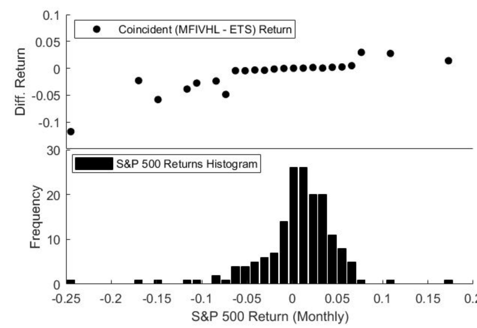


Figure 4: Average Optimal Control Differentials by γ

Continuing on, visual inspection of the Sharpe ratio difference charts shows that most of the outperformance of the ETS model came during 2007-2009, as it did a better job of getting out of the market before the lows of the U.S. financial crisis, then relative performance of the models slowly began to converge. This is a rather interesting result, and the implications are developed further in figures 5 and 6 below. These figures plot a histogram of the market's returns and overlay the coincidentally realized differential return between the ETS portfolio and the mean-variance benchmarks. If one focuses first on the left side of these charts the implication is clear and represents another major finding of the study - incorporating the options market's expectation for the entire forward distribution of returns, which of course includes implicitly the market's expectation of a left-tail event, *results in portfolios that are well positioned and outperform when left-tail events do occur.* However, one also sees evidence of some degree of upper tail dependence in these charts, meaning the ETS model *underperformed* when the market experienced a right-tail event. To better clarify and quantify the tail dependence, three different copulas were fit to the differential return series, and the results, by negative log-likelihood (NLL), are reported in table 3. Recall that Clayton copulas have lower-tail dependence but no upper-tail dependence, Gumbel copulas have upper-tail dependence but no lower-tail dependence, and t copulas have symmetric dependence in both tails (see [39] for a treatment of these and copulas in general). The idea is that if the Clayton copula offers the best fit to the data, lower-tail dependence is predominant (i.e. ETS outperformance), if the Gumbel copula offers the best fit, upper-tail dependence is predominant (i.e. ETS underperformance), and if the t copula offers the best fit there is dependence in both tails. As shown, although the Clayton copula fits the differential return series better than the Gumbel copula in all cases (inferring that left-tail dependence is more prevailing than right-tail dependence), except in the case of the MFIVHL-ETS series at $\gamma = 1$, the t copula offers the best fit. Although one wonders how these results would change if the frequency of the rebalancing period of the strategy were increased, we actually would not expect much improvement (in this context, reduction) in right-tail dependence. Note from the horizontal axis titles that these charts were constructed using monthly data specifically to abrogate the effect of v-shaped recoveries or the colloquial "dead cat bounce" on days in late 2008, only for the market to continue its broader trend.

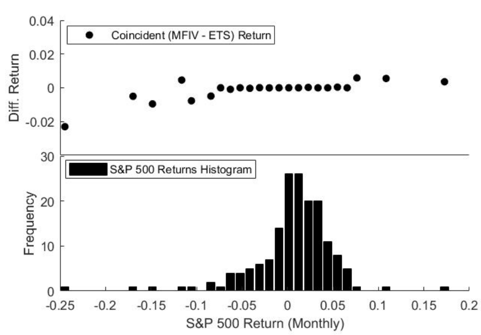


(a) MFIV-ETS Market Dependence

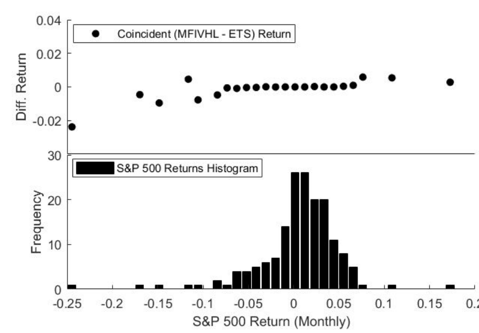


(b) MFIVHL-ETS Market Dependence

Figure 5: Market Dependence of Differential Returns for $\gamma = 1$



(a) MFIV-ETS Market Dependence



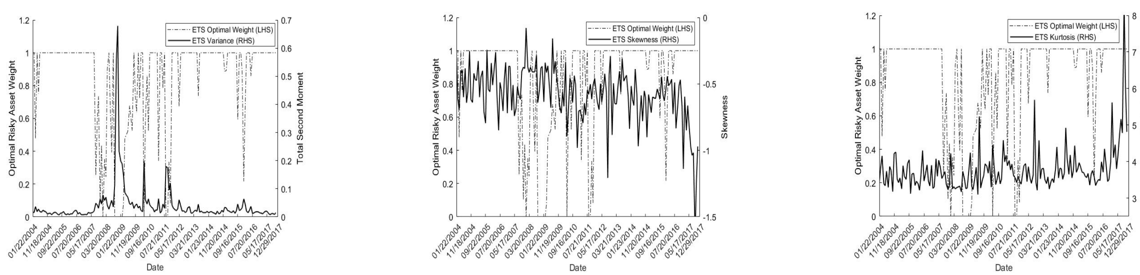
(b) MFIVHL-ETS Market Dependence

Figure 6: Market Dependence of Differential Returns for $\gamma = 5$

Table 3: Differential Return & S&P500 Return Copula Calibration NLLs

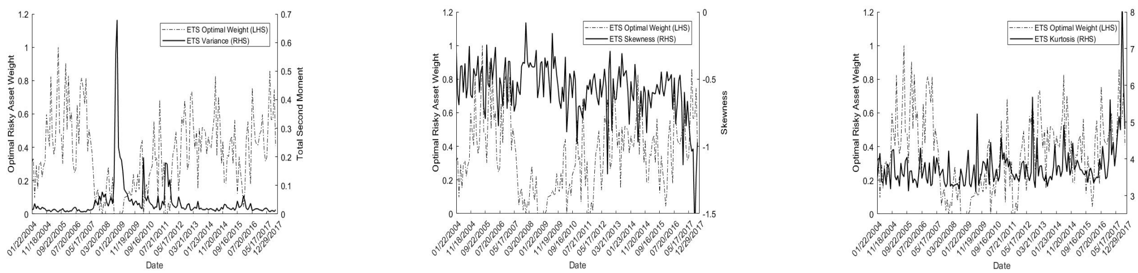
	MFIV-ETS	**MFIVHL-ETS**	**MFIV-ETS**	**MFIVHL-ETS**
	$\gamma = 1$	$\gamma = 1$	$\gamma = 5$	$\gamma = 5$
Clayton Copula	-29.531	-135.280	-34.760	-34.293
Gumbel Copula	-26.260	-99.832	-29.828	-33.834
t-Copula	-32.546	-126.630	-44.328	-52.015

Finally, Figures 7 and 8 show how the optimal control (that is, the optimal weight at each rebalancing date) evolves along with the higher moments. The results here are also quite thought-provoking. First, notice that the higher moments each have their own broader trends, but with significant noise about these trends. This re-motivates the consideration of a regularized framework that results in more stable parameterizations. Secondly, the counter-trends in the variance versus the skewness and kurtosis raise interesting speculations about hidden risks and complacency in the market. That is, notice that since the crisis, (calibrated) skewness had been trending lower and (calibrated) kurtosis had been trending higher, both negative developments to the utility maximizing investor. Yet, the optimal weight in the risky asset had been trending higher as well. How? The progressive trend lower in the total variance had simply overwhelmed these effects. One interpretation of this is that risk was simply being pushed into the higher moments as returns had been strong and variance had become progressively more subdued. The market environment since the end point of our analysis perhaps vindicates such a conjecture (disregarding of course, for the sake of brevity, our benefit of hindsight in making it).



(a) Total Variance vs. Control (b) Skewness vs. Control (c) Kurtosis vs. Control

Figure 7: ETS Higher Moments vs. Optimal Control for $\gamma = 1$



(a) Total Variance vs. Control (b) Skewness vs. Control (c) Kurtosis vs. Control

Figure 8: ETS Higher Moments vs. Optimal Control for $\gamma = 5$

6 Conclusion

This paper examined the performance of a dynamic investment strategy which allocates between a risky and a risk-free asset by utilizing option market information and solving the decision problem under an exponential Lévy market assumption. Using a log-stock price process with an Exponentially Tempered Stable jump-diffusion process as the martingale component and a CRRA utility function as representative of the risky asset and investor preferences, respectively, the risk-neutral Lévy density was calibrated to option prices, transformed to the real-world or objective probability measure, and then the optimal portfolio was derived using the solution to a stochastic optimal control problem.

It was found that portfolios formed under the optimal strategy, when tested and compared to mean-variance alternatives, offered statistically significant improvements in risk-adjusted performance when investor risk-aversion was low, however these gains are lost as risk aversion increases. The use of option market data in calibrating the Lévy density and the flexibility of our method in fitting implied distribution does appear to have shown the ability to pick up on anomalous behavior some time prior to the financial crisis lows of 2008-2009, and the strategy offered the most value-added returns when the market experienced a left-tail event.

Broadly, the results suggest a performance improvement from using both more realistic models of asset prices and option implied expectations and motivates further investigation. The effect of the CRRA utility assumption on the optimal control as risk aversion increases seems particularly worthy of attention, as the notion that an investor cares progressively less about the higher moments of the distribution as they become more risk averse is counterintuitive. Incorporating a regularization framework into the calibration to improve stability (particularly as regards the implied higher moments) while retaining precision also appears a worthwhile extension. Finally, the underperformance of the strategy during right-tail market events motivates reflection on whether option implied distributions perhaps cause the investor to be systematically under-invested due to the enduring volatility skew. We conclude for now.

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Appendices

A Proof of Theorem 2.2

From the Lévy Khintchine formula, and considering $\mu = 0, \sigma^2 = 0$, we have that

$$\psi_{ETS}(u; \vec{\mathbf{v}}) = \int_{-\infty}^{\infty} (e^{iux} - 1)\nu(dx). \tag{A.1}$$

Given the structure of the ETS Lévy measure given in equation (2.9), this can be written as the sum

$$\psi_{ETS}(u; \vec{\mathbf{v}}) = \lambda_+ \int_0^{\infty} (e^{iux} - 1)\frac{e^{-\beta_+ x}}{x^{\alpha+1}} dx + \lambda_- \int_0^{\infty} (e^{-iux} - 1)\frac{e^{-\beta_- x}}{x^{\alpha+1}} dx. \tag{A.2}$$

Focusing on each term of equation (A.2) separately, we have for the first term,

$$\lambda_+ \int_0^{\infty} \frac{e^{iux-\beta_+ x}}{x^{\alpha+1}} - \frac{e^{-\beta_+ x}}{x^{\alpha+1}} dx$$

$$= \lambda_+ \int_0^{\infty} x^{-\alpha-1} e^{-(\beta_+ - iu)x} dx - \lambda_+ \int_0^{\infty} x^{-\alpha-1} e^{-\beta_+ x} dx. \tag{A.3}$$

It is now important to recall the gamma function, $\Gamma(\cdot)$, given by

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx, \tag{A.4}$$

or, equivalently

$$\Gamma(z) = r^z \int_0^{\infty} x^{z-1} e^{-rx} dx. \tag{A.5}$$

Thus, equation (A.3) yields

$$\lambda_+ \int_0^{\infty} x^{-\alpha-1} e^{-(\beta_+ - iu)x} dx - \lambda_+ \int_0^{\infty} x^{-\alpha-1} e^{-\beta_+ x} dx = \lambda_+ \Gamma(-\alpha)[(\beta_+ - iu)^{\alpha} - \beta_+^{\alpha}]. \tag{A.6}$$

Using the same argument, we evaluate now the second term of equation (A.2) by

$$\lambda_- \int_0^\infty (e^{-iux}-1)\frac{e^{-\beta_- x}}{x^{\alpha+1}}dx = \lambda_- \int_0^\infty x^{-\alpha-1}\left\{e^{-(\beta_-+iu)x}-e^{-\beta_- x}\right\}dx$$
$$= \lambda_- \int_0^\infty x^{-\alpha-1}e^{-(\beta_-+iu)x}dx - \lambda_- \int_0^\infty x^{-\alpha-1}e^{-\beta_- x}dx = \lambda_-\Gamma(-\alpha)[(\beta_-+iu)^\alpha-\beta_-^\alpha]. \tag{A.7}$$

So, adding equations (A.6) and (A.7), we are led to the characteristic exponent of the ETS process,

$$\psi_{ETS}(u;\overrightarrow{\mathbf{v}}) = \lambda_+\Gamma(-\alpha)[(\beta_+-iu)^\alpha-\beta_+^\alpha]+\lambda_-\Gamma(-\alpha)[(\beta_-+iu)^\alpha-\beta_-^\alpha]. \tag{A.8}$$

Multiplying this result by t and exponentiating leads to the final form of the characteristic function ϕ_{ETS} as shown in equation (2.10).

B Proof of Theorem 2.3

The characteristic function of the logarithm of the stock price is given by

$$\phi_{\ln(S_t)}(u,t) = \mathbb{E}^{\mathbb{P}}[e^{iu\ln(S_t)}]. \tag{B.1}$$

For the ETS stock price process, we thus have

$$\phi_{\ln(S_t)}(u,t) = \mathbb{E}^{\mathbb{P}}[e^{iu\ln(S_t)}]$$
$$= \mathbb{E}^{\mathbb{P}}\left[\exp\left(iu\{\ln(S_0)\cdot\exp(\mu+\omega-\eta^2/2)t+X_{ETS_e}(t;\overrightarrow{\mathbf{v}},\eta)\}\right)\right], \tag{B.2}$$

which decomposes to

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}}[e^{iu\ln(S_t)}] \\
&= \exp\{iu\ln(S_0)\}\cdot\mathbb{E}^{\mathbb{P}}\left[\exp\{iu(\mu+\omega-\eta^2/2)t+\eta W_t\}\right]\cdot\mathbb{E}^{\mathbb{P}}\left[\exp\{iuX_{ETS}(t;\vec{\mathbf{v}})\}\right]. \quad \text{(B.3)}
\end{aligned}
$$

By the definition of a characteristic function the second expectation is

$$
\mathbb{E}^{\mathbb{P}}\left[\exp\{iuX_{ETS_e}(t;\vec{\mathbf{v}})\}\right] = \phi_{ETS}(u;\vec{\mathbf{v}}). \quad \text{(B.4)}
$$

Further, since the first expectation term is the characteristic function of a Gaussian random variable, we also have

$$
\mathbb{E}^{\mathbb{P}}\left[\exp\{iu(\mu+\omega-\eta^2/2)t+\eta W_t\}\right] = \exp\{iu(\mu+\omega-\eta^2/2)t-\eta^2u^2/2\}, \quad \text{(B.5)}
$$

and we can now bring equation (B.2) to the form shown in equation (2.14), with tildes where necessary to denote a parameter under the risk-neutral measure.

C Proof of Theorem 2.4

We begin by proving that the discounted risk-neutral price process is indeed a martingale. Consider the risk-neutral price process $\tilde{S}_t,\ 0 \leq s \leq t \leq T$. We have

$$
\mathbb{E}^{\mathbb{Q}}[e^{-(r-d)t}\tilde{S}_t|\mathcal{F}_s] = e^{-(r-d)t}\tilde{S}_s e^{(r-d)(t-s)-(t-s)\log(\phi_{ETS}(-i))}\mathbb{E}^{\mathbb{Q}}[e^{X_t}-e^{X_s}]. \quad \text{(C.1)}
$$

By the stationarity of Lévy increments, we have

$$
\mathbb{E}^{\mathbb{Q}}[e^{X_t}-e^{X_s}] = \mathbb{E}^{\mathbb{Q}}[e^{X_{t-s}}] = e^{(t-s)\log(\phi_{ETS}(-i))}. \quad \text{(C.2)}
$$

Thus, equation (C.1) becomes

$$\mathbb{E}^{\mathbb{Q}}[e^{-(r-d)t}\tilde{S}_t|\mathcal{F}_s] =$$
$$e^{-(r-d)t}e^{(r-d)(t-s)}\tilde{S}_s e^{-(t-s)\log(\phi_{ETS}(-i))}e^{(t-s)\log(\phi_{ETS}(-i))} = e^{-(r-d)s}\tilde{S}_s. \quad \text{(C.3)}$$

From proposition 9.8 and example 9.1 in [17], the ETS process under the measure $\mathbb{P}$ and the risk-neutral measure $\mathbb{Q}$ will be equivalent for all t if

$$\lambda_+ \int_0^\infty \left(\sqrt{\frac{\tilde{\lambda}_+}{\lambda_+}} e^{-\frac{1}{2}(\tilde{\beta_+}-\beta_+)x} x^{(\alpha-\tilde{\alpha})/2} - 1\right)^2 \frac{e^{-\beta_+ x}}{x^{1+\alpha}} dx < \infty, \quad \text{(C.4)}$$

and

$$\tilde{\kappa} - \kappa = \int_{-1}^{1} x\tilde{\nu}(dx) - \int_{-1}^{1} x\nu(dx). \quad \text{(C.5)}$$

The integrand of equation (C.4) is only always integrable if $\tilde{\lambda}_+ = \lambda_+$ and $\tilde{\alpha} = \alpha$ (the condition $\tilde{\lambda}_- = \lambda_-$ follows by symmetry), and condition (C.5) leads to equation (2.15).

D Proof of Theorem 2.5

From proposition 2 in [17] (see [28]), it can be shown that

$$H(\mathbb{P}|\mathbb{Q}) = t\int_{-\infty}^{\infty} (\psi(x)e^{\psi(x)} - e^{\psi(x)} + 1)\nu(dx), \quad \text{(D.1)}$$

where $\psi(x) = (\beta_+ - \tilde{\beta_+})x\chi_{\{x>0\}} - (\beta_- - \tilde{\beta_-})x\chi_{\{x<0\}}$. A Taylor/MacLaurin expansion of the term attached to the Lévy measure in the integrand thus yields

$$
\begin{aligned}
&\psi(x)e^{\psi(x)} - e^{\psi(x)} + 1 \\
&\qquad = \left(\sum_{n=0}^{\infty} \frac{1}{n!}((\beta_+ - \tilde{\beta_+})x)^{n+1} - \sum_{n=0}^{\infty} \frac{1}{n!}((\beta_+ - \tilde{\beta_+})x)^{n} + 1\right)\mathbb{1}_{\{x>0\}} \\
&\qquad + \left(\sum_{n=0}^{\infty} \frac{1}{n!}(-(\beta_- - \tilde{\beta_-})x)^{n+1} - \sum_{n=0}^{\infty} \frac{1}{n!}(-(\beta_- - \tilde{\beta_-})x)^{n} + 1\right)\mathbb{1}_{\{x<0\}}. \quad \text{(D.2)}
\end{aligned}
$$

Now, since the first term of the second summation of each term cancels with the addition of 1, and since the first term of the first summation of each term cancels with the second term of the second summation of each term, and further since we can change the bounds of the negative part of the Lévy measure integration, the relative entropy can be written as

$$
\begin{aligned}
&H(\mathbb{P}|\mathbb{Q}) \\
&\quad = t\int_0^{\infty} \sum_{n=1}^{\infty} \frac{1}{n!}((\beta_+ - \tilde{\beta_+})x)^{n+1}\nu(dx) - t\int_0^{\infty} \sum_{n=2}^{\infty} \frac{1}{n!}((\beta_+ - \tilde{\beta_+})x)^{n}\nu(dx) \\
&\quad + t\int_0^{\infty} \sum_{n=1}^{\infty} \frac{1}{n!}((\beta_- - \tilde{\beta_-})x)^{n+1}\nu(dx) - t\int_0^{\infty} \sum_{n=2}^{\infty} \frac{1}{n!}((\beta_- - \tilde{\beta_-})x)^{n}\nu(dx). \quad \text{(D.3)}
\end{aligned}
$$

Let $H_1(\mathbb{P}|\mathbb{Q})$ be the first integral above, that is

$$
\begin{aligned}
&H_1(\mathbb{P}|\mathbb{Q}) = \int_0^{\infty} \sum_{n=1}^{\infty} \frac{1}{n!}((\beta_+ - \tilde{\beta_+})x)^{n+1}\nu(dx) \\
&= \int_0^{\infty} \sum_{n=1}^{\infty} \frac{1}{n!}((\beta_+ - \tilde{\beta_+})x)^{n+1}\left(\lambda_+ \frac{e^{-\beta_+ x}}{x^{\alpha+1}}dx\right) = \int_0^{\infty} \sum_{n=2}^{\infty} \frac{1}{(n-1)!}((\beta_+ - \tilde{\beta_+})x)^{n}\lambda_+ \frac{e^{-\beta_+ x}}{x^{\alpha+1}}dx.
\end{aligned}
$$

(D.4)

Recalling the Gamma function of equations (A.4-A.5), we have

$$
\begin{aligned}
H_1(\mathbb{P}|\mathbb{Q}) = \int_0^\infty \sum_{n=2}^\infty \frac{1}{(n-1)!}((\beta_+ - \tilde{\beta_+})x)^n \lambda_+ x^{-\alpha-1} e^{-\beta_+ x} dx \\
= \sum_{n=2}^\infty \frac{\lambda_+}{(n-1)!} \beta_+^\alpha \left(1 - \frac{\tilde{\beta_+}}{\beta_+}\right)^n \Gamma(n-\alpha) \\
= \lambda_+ \beta_+^\alpha \Gamma(-\alpha) \left(\frac{\tilde{\beta_+}}{\beta_+} - 1\right) \sum_{n=2}^\infty \frac{1}{(n-1)!} \left(\frac{\tilde{\beta_+}}{\beta_+} - 1\right)^{n-1} \alpha(\alpha-1)\cdots(\alpha-n+1). \quad \text{(D.5)}
\end{aligned}
$$

Now, since $\frac{\partial}{\partial x}\frac{x^n}{n!} = \frac{nx^{n-1}}{n!} = \frac{x^{n-1}}{\Gamma(n)} = \frac{x^{n-1}}{(n-1)!}$, we have

$$
H_1(\mathbb{P}|\mathbb{Q}) = \lambda_+ \beta_+^\alpha \Gamma(-\alpha) \left(\frac{\tilde{\beta_+}}{\beta_+} - 1\right) \left[\frac{\partial}{\partial x} \sum_{n=2}^\infty \frac{1}{n!} x^n \alpha(\alpha-1)\cdots(\alpha-n+1)\right]_{x=\frac{\tilde{\beta_+}}{\beta_+}-1}. \quad \text{(D.6)}
$$

We now need to recall the series expansion

$$
(1+x)^\delta = \sum_{n=0}^\infty \binom{\delta}{n} x^n, \quad \text{(D.7)}
$$

where

$$
\binom{\delta}{n} = \frac{\delta(\delta-1)\cdots(\delta-n+1)}{n!} = \frac{\Gamma(\delta+1)}{n!\,\Gamma(\delta-n+1)}. \quad \text{(D.8)}
$$

Thus,

$$
\begin{aligned}
H_1(\mathbb{P}|\mathbb{Q}) = \lambda_+ \beta_+^\alpha \Gamma(-\alpha) \left(\frac{\tilde{\beta_+}}{\beta_+} - 1\right) \left[\frac{\partial}{\partial x} \sum_{n=2}^\infty \binom{\alpha}{n} x^n\right]_{x=\frac{\tilde{\beta_+}}{\beta_+}-1} \\
= \lambda_+ \beta_+^\alpha \Gamma(-\alpha) \left(\frac{\tilde{\beta_+}}{\beta_+} - 1\right) \left[\frac{\partial}{\partial x}((1+x)^\alpha - 1 - \alpha x)\right]_{x=\frac{\tilde{\beta_+}}{\beta_+}-1} \\
= \lambda_+ \beta_+^\alpha \Gamma(-\alpha) \left(\frac{\tilde{\beta_+}}{\beta_+} - 1\right) \left[\alpha(1+x)^{\alpha-1} - \alpha\right]_{x=\frac{\tilde{\beta_+}}{\beta_+}-1}. \quad \text{(D.9)}
\end{aligned}
$$

Finally, a few more simplifications are made, yielding

$$H_1(\mathbb{Q}|\mathbb{P}) = \lambda_+\beta_+^\alpha\Gamma(-\alpha)\left[\alpha(\frac{\tilde{\beta_+}}{\beta_+})^\alpha - \alpha(\frac{\tilde{\beta_+}}{\beta_+})^{\alpha-1} - \alpha(\frac{\tilde{\beta_+}}{\beta_+}) + \alpha\right]$$
$$= \lambda_+\Gamma(-\alpha)(\alpha\tilde{\beta_+}^{\alpha} - \alpha\beta_+\tilde{\beta_+}^{\alpha-1} - \alpha\tilde{\beta_+}\beta_+^{\alpha-1} + \alpha\beta_+^\alpha). \quad \text{(D.10)}$$

Using analogous arguments the other three integrals of equation (D.3) are evaluated and can be simplified together as in equation (2.18).