Holonomy and parallel transport for Abelian gerbes

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July 17, 2001

Abstract

In this paper we establish a one-to-one correspondence between $U(1)$-gerbes with connections, on the one hand, and their holonomies, for simply-connected manifolds, or their parallel transports, in the general case, on the other hand. This result is a higher-order analogue of the familiar equivalence between bundles with connections and their holonomies for connected manifolds. The holonomy of a gerbe with

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group $U(1)$ on a simply-connected manifold M is a group morphism from the thin second homotopy group to $U(1)$, satisfying a smoothness condition, where a homotopy between maps from $[0,1]^2$ to M is thin when its derivative is of rank ≤ 2. For the non-simply connected case, holonomy is replaced by a parallel transport functor between two special Lie groupoids, which we call Lie 2-groups. The reconstruction of the gerbe and connection from its holonomy is carried out in detail for the simply-connected case.

1 Introduction

In [3] Barrett studied the holonomy of connections in principal bundles and proved a reconstruction theorem which showed that in a very precise sense all information about the connections and the bundles is contained in their holonomy. In this paper we obtain analogous results for Abelian gerbes with connections and their holonomy.

We review the background of our work. Caetano and the second author [10] used a slightly different approach to obtain Barrett's results, which has some technical advantages. Let us sketch these results. It is well known that the holonomy of a connection in a principal G-bundle, P, defined over a connected smooth manifold M, assigns an element of G to each smooth (based) loop in M. The holonomy of the composite of two based loops is exactly the product of the two holonomies. Unless the connection is flat, two homotopic loops have different holonomies in general. However, when there is a homotopy between the loops whose differential has rank at most 1 everywhere, the holonomies around the two loops are the same. We call these homotopies *thin* homotopies. One glance at any introductory book on algebraic topology shows that the homotopies used in the proof that the fundamental group obeys the group axioms are all thin (after smoothing at a finite number of non-differentiable points). Therefore the holonomy map descends to a group homomorphism from the *thin* fundamental group of M, $\pi_1^1(M)$, to G. We should remark here that we are using the equivalence relation on loops from [10], while we are borrowing Barrett's terminology, which he used, strictly speaking, for a slightly different equivalence relation. In [10] the term *intimacy relation* was used for what we call a *thin* homotopy. In the present paper this terminological twist should not lead to any confusion, since the paper is intended to be self-contained. Barrett's main result, also obtained in the setup of [10], shows that it is possible to reconstruct the bundle and the connection, up to equivalence, from the holonomy. This result is a nice strengthening of the well-known Ambrose-Singer theorem [1].

Caetano and the second author [11] also defined the higher order thin homotopy groups of a manifold. The definition of the nth-order thin homotopy group, $\pi_n^n(M)$, is quite simple: one takes the definition of the ordinary $\pi_n(M)$, but, instead of dividing by all homotopies, one only divides by homotopies whose differentials have rank at most n. Just like the ordinary homotopy groups, all thin homotopy groups of order at least 2 are Abelian. Once we understand that a G-bundle with connection is equivalent to a smooth group homomorphism $\pi_1^1(M) \to G$, we can ask for a geometrical interpretation of a smooth group homomorphism from the second thin homotopy group, $\pi_2^2(M)$, to an Abelian group, for example the circle group, $U(1)$. As a matter of fact this was the main question left open in [11]. As we show in this paper the answer is that, for a 1-connected manifold M, smooth group homomorphisms $\pi_2^2(M) \to U(1)$ correspond bijectively to equivalence classes of $U(1)$-*gerbes with connections*.

Gerbes were first introduced by Giraud [16] in an attempt to understand non-Abelian cohomology. Ironically only Abelian gerbes have developed into a nice geometric theory so far. Several people [9, 12, 14, 19] have studied Abelian gerbes. In this paper when we say gerbe we always mean a $U(1)$-gerbe. Just as a line-bundle on M can be defined by a set of transition functions on double intersections of open sets in a covering of M, a gerbe can be defined by a set of "transition line-bundles" on double intersections. This point of view was worked out by Chatterjee [12] (see also [19]). The interest of gerbes resides in the possibility of doing differential geometry with them. One can define gerbe-connections and gerbe-curvatures, just as for bundles. Gerbe-connections and line-bundle connections have a lot in common: for example, the Kostant-Weil integrality theorem has a gerbe analogue [9, 12]. The main difference is a shift in dimension: equivalence classes of line-bundles on M are classified by $H^2(M, \mathbb{Z})$, whereas equivalence classes of gerbes on M are classified by $H^3(M, \mathbb{Z})$. The curvature of an ordinary connection is a 2-form, the curvature of a gerbe is given by a 3-form. The holonomy of an ordinary connection associates a group element to each loop, the holonomy of a gerbe-connection associates a group element to each 2-loop, i.e. a smooth map from S^2 to the manifold. Analogously one can define n-gerbes for any $n \in \mathbb{N}$, which are classified by the elements of $H^{n+2}(M, \mathbb{Z})$. Note that in this convention an ordinary gerbe is a 1-gerbe.

The main thing to notice is the existence of a "geometrical ladder" [12, 19]: a line-bundle is given by a set of transition functions, a gerbe is given by a set of transition line-bundles, a 2-gerbe is given by a set of transition gerbes, etc. Whereas two functions are either equal or not, two different line-bundles can also be isomorphic. In the definition of a gerbe it is this extra degree of freedom that is of most interest. On a triple intersection of open sets one requires the product of the transition bundles to be isomorphic to the trivial bundle by a given isomorphism and requires these isomorphisms to satisfy a cocycle condition on every four-fold intersection. The higher one gets up the ladder, the more intricate the notion of isomorphism becomes. One gets isomorphisms between isomorphisms, etc, up to the highest level where one requires a cocycle condition to hold. This feature is known as *categorification*, and we recommend the reader to read [4, 5] on the general concept of categorification. Finally we note that there is also a notion of n-gerbe connection, and the parallel transport of an n-gerbe connection is defined along $n+1$-dimensional paths. Some details of these notions remain to be worked out completely, but Gajer [14, 15] has worked out a considerable part already.

The truly categorical nature of gerbe-connections comes to light when we study the general setting of their parallel transport. To understand the need for this categorical language, one ought to think first about the right way of formulating parallel transport for a connection on a principal bundle over a manifold which is not necessarily connected. Since the concept of holonomy involves the choice of a base-point, one cannot expect to recover all information about the connection by looking only at its holonomy around based closed loops. One has to give up working with closed loops only and start working with arbitrary paths with arbitrary endpoints. This way one finds that the right language is that of Lie groupoids and groupoid morphisms, i.e. functors, instead of Lie groups and group homomorphisms. A thorough account of Lie groupoids and their history can be found in [23]. We note that a Lie groupoid with only one object is precisely a Lie group. We will show how to translate Barrett's [3] results to this more general framework. As pointed out by Brylinski [9], a gerbe with gerbe-connection on M gives rise to a line-bundle with connection on $\Omega(M)$, the free loop space of M. Now the point is that, if M is not 1-connected, then $\Omega(M)$ is not connected, so we cannot expect to recover the gerbe-connection just from its holonomy around based 2-loops. Our result about the parallel transport of gerbe-connections is that, if M is connected at least, their proper setting is that of Lie groupoids with a monoidal structure satisfying the group laws, which we therefore propose to call *Lie 2-groups*, and functors between them. As we show, the notion of thin homotopy is central in this whole story.

We should remark that, when finishing this paper, we found an article by Gajer [15] in which he already obtains the characterization of Abelian n-gerbes with n-gerbe connections in terms of holonomy maps. However, Gajer's approach is very different from ours. Let us briefly explain where they differ. The main motivation for our approach is that we are trying to understand the differential geometry behind the four-dimensional state-sum models defined by the first author of the present paper [22, 21]. For the understanding of these state-sums it would be helpful to find a categorical way of perceiving the relation between homotopy theory and differential geometry. Concretely the motivation was to understand what the homotopy 2-type of a manifold has to do with differential geometry. We believe that the setup in this paper provides such a link, because in the most general case the parallel transport of a gerbe with gerbe-connection on M yields a functor whose domain is the *thin homotopy 2-type* of M. Gajer defines everything in terms of groups and group homomorphisms and therefore his approach does not make this link with homotopy theory. For more information about our motivation see the beginning of Sect. 6.

We work out the simply-connected case "by hand", i.e. without relying on abstract cohomological arguments as Gajer does. Given the gerbe-holonomy we compute explicitly the gerbe and the gerbe-connection that correspond to it. We also remark that Gajer does not use some type of higher thin homotopy groups, which conceptually are easy to understand, but he uses the rather more complicated iterated construction, $G^{(n-1)}\left(G_{ab}(M)\right)$, where G is a certain variant of the thin fundamental group and G_{ab} its Abelianization. We hope that our approach makes the subject accessible to a broader group of mathematicians and physicists.

Finally, our setup provides a link with the theory of double Lie groupoids which is not necessarily restricted to the Abelian case [24, 25]. We are only aware of one type of concrete examples of non-Abelian gerbes, which define the obstruction to lifting a principal G-bundle to a $\hat{G}$-bundle, where $\hat{G}$ is a non-abelian extension of G. To our knowledge no way has been found to even start a theory of connections on non-Abelian gerbes, and in the aforementioned concrete examples one can easily see that a straightforward attempt to generalize the Abelian approach to connections fails. Maybe our link with double Lie groupoids can shed some new light on non-Abelian gerbes and the possibility of defining connections on them. We should remark that Thompson [27] has worked out in detail the definition of quaternionic gerbes and some of their properties. Unfortunately his only examples, related to conformal 4-manifolds, are actually just Čech 1-coboundaries with values in $SU(2)$ and therefore trivial as gerbes. For an introduction to the general theory of non-Abelian gerbes and its history one can read [8]. However, the only concrete example in the book is the one we mentioned above.

We now give a short table of contents:

1) **Introduction**.

2) **Line-bundles and gerbes with connections**. In this section we recall some basic definitions and facts about the objects mentioned in the title. We claim no originality, but we hope that this section helps the non-specialist to understand the paper. In the same spirit we have tried to make the paper as self-contained as possible.

3) **Ordinary holonomy**. We show in detail the equivalence between line-bundles with connections, given in terms of transition functions and local 1-forms, and their holonomies, as a preparation for the discussion of gerbe-holonomy and parallel transport.

4) **Thin higher homotopy groups**. We recall the definition of the thin higher homotopy groups.

5) **Gerbe-holonomy**. We show how the holonomy of a gerbe-connection on a gerbe on M gives rise to a smooth group homomorphism $\pi_2^2(M) \to U(1)$. We also give a local formula for the holonomy in terms of the connection 0- and 1-forms, which constitute the gerbe-connection.

6) **Parallel transport in gerbes**. In this section we explain the key idea of the whole paper. We first show how parallel transport in ordinary bundles can be formulated in terms of Lie groupoids and smooth functors. Then we show how a gerbe with connection on M gives rise to a Lie 2-group (a Lie 2-groupoid with only one object) on M and how the parallel transport yields a smooth functor between Lie 2-groups. Specialists can jump to this section right after the introduction.

7) **Barrett's lemma for 2-loops**. This is just a technical intermezzo which is necessary for the sequel. On a first reading one can jump over this section without loss of comprehension.

8) **The 1-connected case**. Here we reconstruct explicitly the Čech 2-cocycle of a gerbe and the 0- and 1-connections from a given holonomy map $\pi_2^2(M) \to U(1)$ for a 1-connected manifold M. In this case the reconstruction statement is much easier to formulate and understand than in the general case, because the language of groups and group homomorphisms is more familiar than the language of Lie 2-groups and functors between those to most mathematicians. We have therefore decided to prove this case in great detail and only indicate in Sect. 6 what changes have to be made to obtain the proof for the general case (which are only small when the change in language is well understood). On a first reading one might try to read this section before Sect. 6.

2 Line bundles and gerbes with connections

This section is just meant to recall some of the basic facts about line-bundles and gerbes. We claim no originality in this section. As a matter of fact we follow Hitchin [19] and Chatterjee [12] closely. However, we feel that a section like this is necessary, because gerbes are still rather unfamiliar mathematical objects to most mathematicians, and therefore the lack of an introductory section might scare off people who would like to read this paper. Of course we assume some familiarity with the differential geometry of principal bundles and connections.

Throughout this paper let M be a smooth connected finite-dimensional manifold and $\mathcal{U} = \{U_i \mid i \in J\}$ an open cover of M such that every non-empty p-fold intersection $U_{i_1 \dots i_p} = U_{i_1} \cap \dots \cap U_{i_p}$ is contractible. We also assume the existence of a partition of unity $(\rho_i \colon U_i \to \mathbb{R})$ subordinate to $\mathcal{U}$. In this paper a *complex line-bundle* L on M can come in essentially two different but equivalent forms: as a complex vector bundle of rank 1 or as a set of transition functions $g_{ij} \colon U_{ij} \to U(1)$. Note that we take the values of our transition functions to be in $U(1)$ rather than $\mathbb{C}^*$, thinking of them as transition functions of a principal bundle rather than a vector bundle. Recall that the g_{ij} have to satisfy $g_{ji} = g_{ij}^{-1}$ and the Čech cocycle condition $\delta g_{ijk} = g_{ij} g_{jk} g_{ki} = 1$ on U_{ijk}. Two sets of transition functions g_{ij} and g'_{ij} define equivalent line-bundles if and only if there exist functions $h_i \colon U_i \to U(1)$ such that $g'_{ij} g_{ij}^{-1} = h_i^{-1} h_j$. In this case we say that g_{ij} and g'_{ij} are cohomologous and $\delta h_{ij} = h_i^{-1} h_j$ is the coboundary by which they differ.

Thus equivalence classes of line-bundles correspond bijectively to cohomology classes in the first Čech cohomology group $\check{H}^1(M, \underline{U(1)}_M)$. Following the usual notation in the literature on gerbes, we write $\underline{G}_M$ for the sheaf of smooth G-valued functions on (open sets of) M, where G is any Lie group. Following the same convention, the sheaf of *constant* G-valued functions is denoted by G_M. The bijection above defines a group homomorphism, where the group operation for cochains is defined by pointwise multiplication and for line-bundles by their tensor product. Similarly we use two different but equivalent ways to write down a *connection* in L: as a covariant derivative $\nabla\colon \Gamma(M, L) \to \Omega^1(M, \mathbb{C})$ or as a set of of local 1-forms $A_i \in \Omega^1(U_i)$. For each i the 1-form A_i is of course the pull-back of the connection 1-form associated to ∇ via a local section $\sigma_i\colon U_i \to p^{-1}(U_i) \subset L$. We will not say more about covariant derivatives, but we recall that the A_i have to satisfy the rule

$$i(A_j - A_i) = d \log g_{ij}.$$

Two line-bundles, g_{ij} and g'_{ij}, with connections, A_i and A'_i respectively, are equivalent exactly when g_{ij} and g'_{ij} define equivalent bundles as above and $iA'_i = iA_i + d \log h_i$. Given a line-bundle, g_{ij}, with connection, A_i, we can define its *curvature 2-form* $F \in \Omega^2(M)$ by $F|_{U_i} = dA_i$. Using the partition of unity (ρ_i) we can easily define a connection for a given line-bundle, g_{ij}, by $A_i = -i \sum_\alpha \rho_\alpha d \log g_{\alpha i}$. An immediate consequence of the definitions is that the cohomology class of F is independent of the chosen connection. The invariant $[F]/2\pi$ is called the Chern class of the line-bundle. A well known fact about line-bundles is the Kostant-Weil integrality theorem (see [9] and references therein), which says that any closed 2-form F on M is the curvature 2-form of a connection in a line-bundle if and only if $[F]/2\pi$ is the image of an integer singular cohomology class in $H^2(M, \mathbb{Z})$. Basically this theorem is the statement that the cohomology groups $\check{H}^1(M, \underline{U(1)}_M)$ and $H^2(M, \mathbb{Z})$ are isomorphic, which follows from the exact sequence of sheaves

$$0 \longrightarrow \mathbb{Z}_M \xrightarrow{\times 2\pi i} \underline{\mathbb{C}}_M \xrightarrow{\exp} \underline{\mathbb{C}}^*_M \longrightarrow 1$$

and from the isomorphism

$$\check{H}^2(M, \mathbb{Z}_M) \cong H^2(M, \mathbb{Z}).$$

Finally we should remark that Deligne (see references in [9]) found a nice way to encode line-bundles with connections in a cohomological terminology.

Equivalence classes of line-bundles with connections correspond bijectively to cohomology classes in the first (smooth) Deligne hypercohomology group $H^1(M, \underline{\mathbb{C}}^* \stackrel{d\log}{\to} \underline{\mathcal{A}}^1_{M,\mathbb{C}})$. This means little more than that a line-bundle with connection can be defined by a pair of local data (g_{ij}, A_i) with respect to the covering $\mathcal{U}$ such that

$$D\,(g_{ij}, A_i) = (\delta g_{ijk}, i(\delta A_{ij} - d\log g_{ij})) = (1, 0)$$

and that two such pairs $(g_{ij}, A_i)\,, \left(g'_{ij}, A'_i\right)$ represent equivalent line-bundles with equivalent connections precisely if there is a set of functions h_i such that

$$\left(g'_{ij} g_{ij}^{-1}, i(A'_i - A_i)\right) = (\delta h_{ij}, d\log h_i) = D(h_i).$$

The difference between this remark and the precise definition of Deligne cohomology lies in the fact that the local data are not just a family of sets but a *sheaf*. This distinction is important for a rigorous treatment because one wants all definitions to be independent of the choice of open cover in the end. However, this is not the right place to define the whole machinery of sheaves and sheaf cohomology. For a rigorous introduction to Deligne hypercohomology see Brylinski's book [9].

Just as in the case of line-bundles we can define a gerbe by two different but equivalent kinds of data. The first alternative is to say that a gerbe is given by a set of *transition line-bundles* Λ_{ij} on U_{ij} together with a nowhere zero section (also called trivialization) $\theta_{ijk} \in \Gamma(U_{ijk}, \Lambda_{ijk})$ of the tensor product line-bundle $\Lambda_{ijk} = \Lambda_{ij}\Lambda_{jk}\Lambda_{ki}$. The transition line-bundles have to satisfy $\Lambda_{ji} \cong \Lambda_{ij}^{-1}$, where the latter is the inverse of Λ_{ij} with respect to the tensor product, and the cocycle condition which says that

$$\Lambda_{ijk} = \Lambda_{ij}\Lambda_{jk}\Lambda_{ki} \cong 1,$$

where the last line-bundle is the trivial line-bundle over U_{ijk}. The trivializations have to satisfy $\theta_{s(i)s(j)s(k)} = \theta_{ijk}^{\epsilon(s)}$, for any permutation $s \in S_3$, where $\epsilon(s)$ is the sign of s, and the cocycle condition

$$\delta\theta_{ijkl} = \theta_{jkl}\theta_{ikl}^{-1}\theta_{ijl}\theta_{ijk}^{-1} = 1.$$

To understand the last equation one has to note that on U_{ijkl} the tensor product of all the line-bundles involved is identical to the trivial line-bundle, of course up to the fixed isomorphisms $\Lambda_{ji} \cong \Lambda_{ij}^{-1}$ and $\Lambda_{ij}\Lambda_{ij}^{-1} \cong 1$, and the canonical isomorphism which reorders the factors in the tensor product. This holds because every line-bundle appears twice in the product with opposite signs. The last equation means that the product of the sections has to be equal, forgetting about the uninteresting isomorphisms above, to the canonical section in the trivial line-bundle over U_{ijkl} (i.e. the one which associates to each $x \in U_{ijkl}$ the point $(x, 1) \in U_{ijkl} \times U(1)$).

Two gerbes are said to be equivalent if for each $i \in J$ there exists a line-bundle L_i on U_i, such that for each $i, j \in J$ we have bundle isomorphisms

$$m_{ij} \colon L_j \xrightarrow{\cong} \Lambda'_{ij}\Lambda_{ij}^{-1} \otimes L_i$$

such that

$$m_{ij} \circ m_{jk} \circ m_{ki} = \theta'_{ijk}\theta_{ijk}^{-1} \otimes \mathrm{id}$$

on U_{ijk}. The data defining an equivalence are called an *object* by Chatterjee [12].

In order to relate the definition above to Čech cohomology we only have to remember that all our p-fold intersections are contractible so all line-bundles above are necessarily equivalent to the trivial line-bundle. This means that we can choose a nowhere zero section σ_{ij} in Λ_{ij} for each $i, j \in J$. With respect to these sections we now get $\theta_{ijk} = g_{ijk}\sigma_{ijk}$, where we take $\sigma_{ijk} = \sigma_{ij}\sigma_{jk}\sigma_{ki}$ in the tensor product line-bundle. By the definition above we see that g defines a Čech 2-cocycle, i.e., for all $i, j, k, l \in J$ we have

$$\delta g_{ijkl} = g_{jkl}g_{ikl}^{-1}g_{ijl}g_{ijk}^{-1} = 1$$

on U_{ijkl}. Given two equivalent gerbes $(\Lambda'_{ij}, \theta'_{ijk}), (\Lambda_{ij}, \theta'_{ijk})$ and an object (L_i, m_{ij}) for $\Lambda'_{ij}\Lambda_{ij}^{-1}$, we can also choose nowhere zero sections σ_i in L_i for each $i \in J$. This gives us two nowhere zero sections in $\Lambda'_{ij}\Lambda_{ij}^{-1} \otimes L_j$: $\sigma'_{ij}\sigma_{ij}^{-1}\sigma_j$ and $m_{ij} \circ \sigma_i$. For each $i, j \in J$ we can take the quotient of these sections which defines a function $h_{ij} \colon U_{ij} \to U(1)$ such that for each $i, j, k \in J$ we have

$$g'_{ijk}g_{ijk}^{-1} = \delta h_{ijk} = h_{jk}h_{ik}^{-1}h_{ij}$$

on U_{ijk}. Thus every equivalence class of gerbes induces a Čech cohomology class in $\check{H}^2(M, \underline{U(1)}_M)$. Hitchin [19] shows that the converse is true as well, which leads to the conclusion that equivalence classes of gerbes correspond bijectively to the elements in $\check{H}^2(M, \underline{U(1)}_M)$.

Gerbe-connections are defined by two sets of data: the 0-connections and the 1-connections (Chatterjee's terminology). Let $\mathcal{G}$ be a gerbe given by a set of line-bundles Λ_{ij} and trivializations θ_{ijk}. A *0-connection* in $\mathcal{G}$ consists of a covariant derivative ∇_{ij} in Λ_{ij} for each $i, j \in J$ such that for each $i, j, k \in J$

$$\nabla_{ijk}\theta_{ijk} = 0$$

on U_{ijk}. Here ∇_{ijk} is the covariant derivative in the tensor product of the corresponding line-bundles induced by the covariant derivatives in these line-bundles. Given two line-bundles , L and L', with connections, ∇ and ∇' respectively, Brylinski [9] denotes the induced connection in $L \otimes L'$ by $\nabla + \nabla'$, which is defined by

$$(\nabla + \nabla')\, \sigma \otimes \sigma' = \nabla\sigma \otimes \sigma' + \sigma \otimes \nabla'\sigma',$$

for any sections $\sigma \in L$ and $\sigma' \in L'$. We also require that $\nabla_{ij} + \nabla_{ji} = 0$ for all $i, j \in J$. The alternative definition of a 0-connection is obtained in a straightforward manner by taking the pull-back of the connection 1-form associated to ∇_{ij} via σ_{ij} for each $i, j \in J$. Let $\mathcal{G}$ correspond to the 2-cocycle g_{ijk}, and choose a logarithm of g_{ijk}. A 0-connection can then be defined by a set of 1-forms $A_{ij} \in \Omega^1(U_{ij})$ such that

$$i(A_{ij} + A_{jk} + A_{ki}) = -d\log g_{ijk}.$$

Of course we assume that $A_{ji} = -A_{ij}$. A *1-connection* in $\mathcal{G}$ is defined by a set of local 2-forms $F_i \in \Omega^2(U_i)$ such that

$$F_j - F_i = \sigma_{ij}^* K(\nabla_{ij}),$$

where $K(\nabla_{ij})$ denotes the curvature of ∇_{ij}. Alternatively we get

$$F_j - F_i = dA_{ij}.$$

In the sequel we sometimes denote dA_{ij} by F_{ij}. A 0-connection and a 1-connection on $\mathcal{G}$ together form what we call a *gerbe-connection*. Our typical notation for a gerbe-connection is $\mathcal{A}$. In a natural way a gerbe-connection, $\mathcal{A}$, leads to the notion of a gerbe-curvature 3-form, $G \in \Omega^3(M)$, which is defined by $G|_{U_i} = dF_i$. Again using the partition of unity we see that, for a given gerbe g_{ijk}, it is easy to define a 0-connection by $A_{ij} = -i\sum_\alpha \rho_\alpha d\log g_{\alpha ij}$ and a 1-connection by $F_i = \sum_\beta \rho_\beta dA_{\beta i}$. Just as for line-bundles one can show that the cohomology class of the gerbe-curvature, $[G]$, does not depend on the chosen 0- and 1-connection (the proof is a bit harder though), and that any closed 3-form G on M is a gerbe-curvature 3-form if and only if $[G]/2\pi$ is the image of a class in $H^3(M,\mathbb{Z})$. The cohomology class $[G]/2\pi$ is called the Dixmier-Douady class of the gerbe.

To complete the picture we have to define when two gerbes, $\mathcal{G}$ and $\mathcal{G}'$, with gerbe-connections, $\mathcal{A}$ and $\mathcal{A}'$ respectively, are equivalent. First of all, such an equivalence requires $\mathcal{G}$ and $\mathcal{G}'$ to be equivalent as gerbes. Let (L_i, m_{ij}) define an object for this equivalence and let h_{ij} be the Čech cochain corresponding to this object. $\mathcal{A}$ and $\mathcal{A}'$ are now equivalent if for every $i \in J$ there exists a connection ∇_i in L_i such that m_{ij} maps

$$\nabla_j \mapsto \nabla'_{ij} - \nabla_{ij} + \nabla_i \tag{1}$$

and such that

$$F'_i = F_i + \sigma_i^* K(\nabla_i). \tag{2}$$

Equivalently this means that for every $i \in J$ there exists a 1-form $A_i \in \Omega^1(U_i)$ such that

$$iA'_{ij} = i(A_{ij} + A_j - A_i) - d\log h_{ij} \tag{3}$$

on U_{ij} and

$$F'_i = F_i + dA_i \tag{4}$$

on U_i. Now there is a subtlety in Chatterjee's [12] terminology. Suppose a gerbe $\mathcal{G}$ can be trivialized by an object (L_i, m_{ij}) (or, equivalently, by h_{ij}). He calls local data ∇_i (or, equivalently, A_i) which satisfy (1) (resp. (3)) an *object* 0*-connection*. In general such an object 0-connection need not satisfy (2) (resp. (4)). That would only be possible if the object-connection were trivializable. As Chatterjee proves, on a 2-dimensional surface any gerbe with any gerbe-connection admits an object with an object 0-connection, but in general the latter does not satisfy (2). This is of course analogous to the fact that any line-bundle on the circle admits a trivialization, but an arbitrary connection in such a line-bundle need not be trivializable. In this article we call an object 0-connection simply an object connection. It is important to keep these remarks in mind for Sect. 5.

Finally we should remark that there is a bijective correspondence between equivalence classes of gerbes with gerbe-connections and cohomology classes in $H^2(M, \underline{\mathbb{C}}^* \overset{d\log}{\to} \underline{\mathcal{A}}^1_{M,\mathbb{C}} \overset{d}{\to} \underline{\mathcal{A}}^2_{M,\mathbb{C}})$, the next order Deligne hypercohomology group. Again this amounts to little more than saying that a gerbe with gerbe-connection is defined by a triple of local data (g_{ijk}, A_{ij}, F_i) satisfying the conditions we have explained above.

Example 2.1 Let G be a Lie group and $1 \to U(1) \xrightarrow{i} \hat{G} \xrightarrow{\pi} G \to 1$ a central extension. It is well known that any central extension is locally split. This means that $\hat{G} \xrightarrow{\pi} G$ is a locally trivial principal $U(1)$-bundle. Given a principal G-bundle over M, denoted by P, in the form of its transition functions $g_{ij} \colon U_{ij} \to G$, we can locally lift these transition functions to obtain $\hat{g}_{ij} \colon U_{ij} \to \hat{G}$ (by assuming that the image of g_{ij} is sufficiently small so that the central extension can be trivialized over it). In general the $\hat{g}_{ij}$ do not define a cocycle, but $\pi(\delta\hat{g}_{ijk}) = \delta g_{ijk} = 1$ of course, so we have $\delta\hat{g}_{ijk} \colon U_{ijk} \to \ker\pi \cong U(1)$. As a matter of fact $\delta\hat{g}_{ijk}$ defines a gerbe, because $\delta^2 = 1$ always. Thus the obstruction to lifting P to a principal $\hat{G}$-bundle defines a gerbe.

Example 2.2 Let $S^3 \subset \mathbb{R}^4$ be the three-dimensional sphere and take $N = S^3 - \{(0,0,0,1)\}$ and $S = S^3 - \{(0,0,0,-1)\}$. The intersection $N \cap S$ is homotopy equivalent to S^2. For example, $(x,y,w,z) \mapsto (x,y,w)/\sqrt{x^2+y^2+w^2}$ defines a homotopy equivalence. Therefore equivalence classes of line-bundles on $N \cap S$ are determined by cohomology classes in $H^2(S^2, \mathbb{Z}) = \mathbb{Z}$. Since there are no 3-fold intersections, any line-bundle on S^2 defines a gerbe on S^3. Let Λ_{NS} be such a line-bundle. Any connection A_{NS} in Λ_{NS} defines a 0-connection in the gerbe. A 1-connection is also easy to obtain, because the curvature 2-form of A_{NS}, which we denote by F_{NS}, can always be extended to S, since S is contractible, and therefore we can define F_S to be this extension of F_{NS} and define F_N to be zero. Chatterjee [12] and Hitchin [19] show how to obtain a gerbe for any codimension-3 submanifold and how to define a gerbe-connection for such a gerbe. There is a little subtlety that we should explain: in this example $U_N \cap U_S$ is not contractible. However our definition of a gerbe in terms of line-bundles does not use that condition at all. Only when one wants to pass to the corresponding Čech-cocycle and the local forms which define the gerbe-connection one has to assume that all intersections are contractible. If one feels happier with contractible intersections one can subdivide N and S, but of course this makes the definition of the gerbe a little bit more complicated. For a detailed treatment see [12].

3 Ordinary holonomy

It is well-known that a principal G-bundle with connection over M allows one to define the notion of holonomy around any smooth closed curve on M (Kobayashi and Nomizu [20]). In particular, given a fixed basepoint $*$ in M, and a point in the fibre over $*$, this data induces an assignment of an element of G to each loop in M, based at $*$. Such an assignment is called a holonomy map, or simply a holonomy. In Barrett [3], and in a slightly different fashion in Caetano-Picken [10], it was shown that suitably defined holonomy maps are in one-to-one correspondence with G-bundles plus connection plus the choice of a point in the fibre over $*$, up to isomorphism. This result should be seen as a geometric version of the well-known equivalence between flat G-bundles modulo gauge transformations over M and $\mathrm{Hom}(\pi_1(M), G)/G$, the group homomorphisms from $\pi_1(M)$ to G, modulo conjugation by elements of G.

The reconstruction of the bundle and connection from a holonomy map in Barrett [3] and Caetano-Picken [10] was carried out in the total space of the bundle, using Ehresmann connections. The main result of this section is to prove the same equivalence using instead the local data defining a bundle and connection from section 2. Since the aim is to prepare the ground for the gerbe discussion in sections 5 and 8, we will only show the result for the case $G = U(1)$.

We start by recalling briefly the definition of holonomy map from [10]. Let $\Omega^\infty(M)$ be the space of smooth loops $\ell : [0,1] \to M$ based at $*$ such that $\ell(t) = *$ for $0 \leq t < \epsilon$ and $1 - \epsilon < t \leq 1$ for some $0 < \epsilon < 1/2$. We say that the loop *sits*, or has a *sitting instant*, at $t = 0$ and $t = 1$. In [10] it is shown how to reparametrize any path to sit at its endpoints using a smoothly increasing function $\beta\colon [0,1] \to [0,1]$ with $\beta(t) = 0$, for $t \in [0, \frac{1}{3}]$, and $\beta(t) = 1$, for $t \in [\frac{2}{3}, 1]$. As usual we may define the product and inverse of loops, and the sitting property means that the product closes in $\Omega^\infty(M)$. We will denote the product of loops by $\star$. An equivalence relation between loops appropriate for parallel transport purposes is the following:

Definition 3.1 *Two loops ℓ and ℓ' belonging to $\Omega^\infty(M)$ are said to be* rank-1 homotopic*, or* thin homotopic*, written $\ell \overset{1}{\sim} \ell'$, if there exists a map H : $[0,1] \times [0,1] \to M$ such that:*

1. *H is smooth throughout $[0,1] \times [0,1]$*

2. $\mathrm{rank}(DH_{(s,t)}) \leq 1 \ \forall (s,t) \in [0,1]\times[0,1]$

3. there exists $0 < \epsilon < 1/2$ *such that*

$$\begin{aligned} 0 \leq s \leq \epsilon &\Rightarrow H(s,t) = \ell(t) \\ 1-\epsilon \leq s \leq 1 &\Rightarrow H(s,t) = \ell'(t) \\ 0 \leq t \leq \epsilon &\Rightarrow H(s,t) = * \\ 1-\epsilon \leq t \leq 1 &\Rightarrow H(s,t) = * \end{aligned}$$

Now the space of equivalence classes $\pi_1^1(M,*) = \Omega^\infty(M)/ \stackrel{1}{\sim}$ acquires the structure of a group in exactly the same way as $\pi_1(M)$ does, but using rank-1 homotopy instead of ordinary homotopy, since all homotopies used in the construction of $\pi_1(M,*)$ are in fact rank-1. Again the function β is used to give the usual homotopies sitting endpoints. In the rest of this paper all paths and homotopies are understood to have sitting endpoints, which can always be achieved by reparametrization with β, as shown in [10]. We remark that in [10] the terms *intimate* and the *group of loops*, $\mathcal{GL}^\infty(M)$, were used instead of *rank*-1 *homotopic* and $\pi_1^1(M,*)$ respectively. When the basepoint $*$ is understood we will frequently write $\pi_1^1(M)$ instead of $\pi_1^1(M,*)$.

In an analogous fashion we may introduce the smooth path groupoid $P_1^1(M)$, consisting of smooth paths $p : I \to M$ which are constant in a neighbourhood of $t = 0$ and $t = 1$, identified up to rank-1 homotopy, defined as in Def. 3.1 with the obvious modification $0 \leq t \leq \epsilon \Rightarrow H(s,t) = p(0)$, $1-\epsilon \leq t \leq 1 \Rightarrow H(s,t) = p(1)$. Multiplication in $P_1^1(M)$, when possible, will also be denoted by $\star$. The set of paths without dividing by the thin homotopy relation we denote by $P^\infty(M)$.

We define a *smooth family of loops* to be a map $\psi\colon U \subseteq \mathbb{R}^k \to \Omega^\infty(M)$ defined on an open set $U \subseteq \mathbb{R}^k$ such that the function $\psi(x,t) = \psi(x)(t)$ is smooth on $U \times I$.

Definition 3.2 *A* holonomy *is a group morphism* $\mathcal{H} : \pi_1^1(M) \to U(1)$*, which is smooth in the following sense: for every smooth family of loops* $\psi : U \subseteq \mathbf{R}^k \to \Omega^\infty(M)$*, the composite*

$$U \xrightarrow{\psi} \Omega^\infty(M) \xrightarrow{\mathrm{proj}} \pi_1^1(M) \xrightarrow{\mathcal{H}} U(1),$$

where proj *is the natural projection, is smooth throughout* U*.*

For later it will be useful to have the following result.

Lemma 3.3 *(Recentering a holonomy) Let $\mathcal{H} : \pi_1^1(M, *) \to U(1)$ be a holonomy, $m \in M$ and $p \in P^\infty(M)$ a path from $*$ to m. Then $\tilde{\mathcal{H}} : \pi_1^1(M, m) \to U(1)$ defined by $\tilde{\mathcal{H}}(\ell) = \mathcal{H}(p \star \ell \star p^{-1})$ is a holonomy.*

Proof: Since the smoothness property is clear, we only have to show that $\tilde{\mathcal{H}}$ is a group morphism, which follows from the fact that products $p^{-1} \star p$ may be cancelled up to rank-1 homotopy. □

Suppose now that a $U(1)$ bundle L with connection $\mathcal{A}$ is given in terms of local data, g_{ij} and A_i, as in section 2. Define their holonomy map $\mathcal{H}$ from $\Omega^\infty(M)$ to $U(1)$ as follows:

$$\mathcal{H}(\ell) = \exp i \int_I A^\ell$$

where A^ℓ is a 1-form on I defined on each open set of the pullback cover via ℓ of the interval, $\mathcal{V} = \{V_i, i \in J\}$, by

$$iA^\ell = i\ell^*(A_i) - d \log k_i$$

where $k_i : V_i \to U(1)$ is an arbitrary trivialization of $\ell^* L$, i.e. $\ell^*(g_{ij}) = k_j k_i^{-1}$ on V_{ij}. The 1-form A^ℓ is globally defined on I since

$$\begin{aligned} i\ell^*(A_j - A_i) &= \ell^*(g_{ij}^{-1} dg_{ij}) \\ &= (k_j k_i^{-1})^{-1} d(k_j k_i^{-1}) \\ &= d \log k_j - d \log k_i. \end{aligned}$$

Furthermore $\mathcal{H}(\ell)$ doesn't depend on the choice of trivialization k_i, since, if $A'^\ell = i\ell^*(A_i) - k_i'^{-1} dk_i'$, then $k_i/k_i' = k_j/k_j'$ on V_{ij}, so that f, defined locally by $f = k_i/k_i'$, is a function on I. Therefore

$$\exp i \int_I (A^\ell - A'^\ell) = \exp \int_I f^{-1} df = f(1) f(0)^{-1} = 1.$$

Lemma 3.4 *$\mathcal{H}$ descends to $\pi_1^1(M)$ and defines a holonomy, which is independent of the choice of data $L, \mathcal{A}$ up to equivalence.*

Proof: Suppose that $\ell \overset{1}{\sim} \ell'$, and that $H : I^2 \to M$ is a rank-1 homotopy between ℓ and ℓ', as in Def. 3.1. Let k_i be a trivialization of $H^*(L)$ over I^2, and A^H be the 1-form on I^2 defined locally on $H^{-1}(U_i)$ by $iA^H = iH^*(A_i) - d\log k_i$. Now,

$$\begin{aligned} \exp i \int_I (A^\ell - A^{\ell'}) &= \exp i \int_{I^2} dA^H \\ &= \exp i \int_{I^2} H^*(F) \\ &= 1, \end{aligned}$$

using Stokes' theorem in the first equality and the fact that H is of rank ≤ 1, whereas F is a 2-form, in the final equality. Thus $\mathcal{H}$ descends to $\pi_1^1(M)$.

Also $\mathcal{H}$ is a group homomorphism from $\pi_1^1(M)$ to $U(1)$ since

$$\int_I A^{\ell\star\ell'} = \int_I A^\ell + \int_I A^{\ell'}.$$

Suppose $\psi : U \subset \mathbf{R}^k \to \Omega(M)$ is a smooth family of loops in the sense of Def. 3.2. Without loss of generality we suppose that U is contractible, and let k_i be a trivialization of the pull-back bundle $\psi^*(L)$. Once again we define a 1-form on $U \times I$ by $iA^\psi = i\psi^*(A_i) - d\log k_i$ on each open set of the pullback cover under ψ. Now

$$\mathcal{H}(\psi(s_1, \ldots, s_k)) = (\exp i \int_I A^\psi)(s_1, \ldots, s_k)$$

is a smooth function of $s_1, \ldots, s_k$, since all functions are smooth and integration is a smooth operation. Thus $\mathcal{H}$ defines a holonomy.

Finally $\mathcal{H}$ does not depend on the choice of $L, \mathcal{A}$ up to equivalence, which follows immediately from Stokes' theorem and the local formula for holonomy which we give below.

$\square$

For later purposes it is convenient to have a local expression for the holonomy $\mathcal{H}$, defined directly in terms of g_{ij} and A_i. Let $\ell : I \to M$ be a smooth loop, based at $*$. Fix an element of the open cover, U_0, such that $* \in U_0$. Consider again the pull-back cover $\mathcal{V}$ of the interval I. Since I is a compact metric space it has a Lebesgue number $\lambda > 0$ such that $\forall t \in I$ we have $]t-\lambda, t+\lambda[\subseteq V_i$ for some i. Thus, given a decomposition of the unit interval

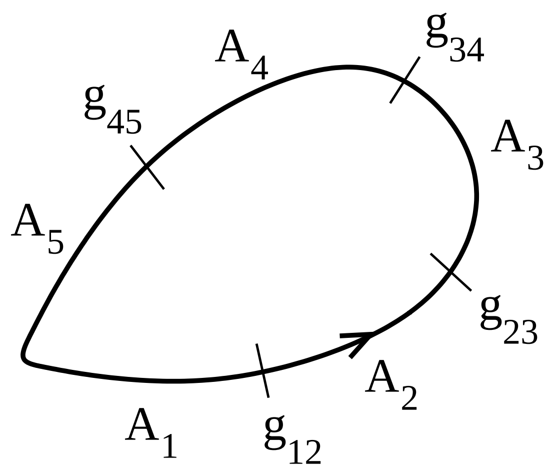

Figure 1: local formula for holonomy

$0 = x_0 < x_1 < \cdots < x_N = 1$ such that $x_\alpha - x_{\alpha-1} < \lambda$, $\forall \alpha = 1, \ldots, N$, each subinterval $I_\alpha = [x_{\alpha-1}, x_\alpha]$ is contained in V_α for some α, and furthermore, by choosing a smaller λ if necessary, or adjusting the decomposition, we can ensure $V_1 = V_N = V_0$. Since

$$\exp \int_{I_\alpha} (i\ell^*(A_\alpha) - d \log k_\alpha) = k_\alpha(x_{\alpha-1}) \left(\exp \int_{I_\alpha} i\ell^*(A_\alpha) \right) k_\alpha^{-1}(x_\alpha)$$

and $k_\alpha^{-1} k_{\alpha+1} = \ell^* g_{\alpha,\alpha+1}$ on $V_{\alpha,\alpha+1}$, we arrive at the following local formula

$$\mathcal{H}(\ell) = \prod_{\alpha=1}^{N} \exp i \int_{I_\alpha} \ell^*(A_\alpha) \cdot g_{\alpha,\alpha+1}(\ell(x_\alpha)), \tag{5}$$

where we set $U_{N+1} = U_1$, and thus $g_{N,N+1}(\ell(x_N)) = 1$. Fig. 1 sketches this local formula. We may also extend this formula by an identical procedure to smooth paths $p : I \to M$. However, for a path that is not closed we can not in general choose the initial and final open sets to be equal and the formula depends on the specific choice (but not on the intermediate open sets in the covering of the path). Thus we define

$$\mathcal{H}^{1,N}(p) = \left(\prod_{\alpha=1}^{N-1} \exp i \int_{I_\alpha} p^*(A_\alpha) \cdot g_{\alpha,\alpha+1}(p(x_\alpha)) \right) \cdot \exp i \int_{I_N} p^*(A_N),$$

and we have the multiplicative formula:

$$\mathcal{H}^{ij}(p \star q) = \mathcal{H}^{ik}(p) \cdot g_{kl}(p(1)) \cdot \mathcal{H}^{lj}(q). \tag{6}$$

Now we turn to the reconstruction of a bundle with connection from a given holonomy $\mathcal{H}$. Assume that the cover $\mathcal{U}$ is such that for each i we have a diffeomorphism $\phi_i : U_i \to B(0,1)$, where $B(0,1)$ is the open unit ball in $\mathbf{R}^n$. Since M is path-connected we may choose a smooth path $p_i \in P^\infty(M)$ from $*$ to $x_i = \phi^{-1}(0)$, the centre of U_i. For U_0 we set p_0 to be the constant path at $*$. Given $x \in U_i$ there is a natural path $\gamma_{i,x} \in P^\infty(M)$ from x_i to x, namely the pullback under ϕ_i of the radial path from the origin to $\phi_i(x)$ in $\mathbf{R}^n$, reparametrized to be constant in a neighbourhood of $t = 0, 1$. For future use it is practical to define

$$p_{i,x} = p_i \star \gamma_{i,x}.$$

Now we define the transition functions, g_{ij}, of the bundle L corresponding to $\mathcal{H}$ by setting (see Fig. 2)

$$\ell_{ij}(x) = p_{i,x} \star p_{j,x}^{-1}$$

and defining

$$g_{ij}(x) = \mathcal{H}(\ell_{ij}(x)) \tag{7}$$

Lemma 3.5 *The transition functions g_{ij} satisfy the cocycle condition.*

Proof: For x in a triple overlap U_{ijk}, we have $g_{ij}g_{jk}g_{ki}(x) = 1$, since the product of the corresponding three loops is rank-1 homotopic to the trivial constant loop, using the fact that we may cancel products of the form $p \star p^{-1}$ up to rank-1 homotopy. Note that we have $g_{ji} = g_{ij}^{-1}$ by definition as well. □

To define the 1-forms A_i corresponding to $\mathcal{H}$, let $x \in U_i$, $v \in T_xU_i$, and let $q :]-\epsilon, \epsilon[\to U_i$ be a smooth path, such that $q(0) = x$, $\dot{q}(0) = v$. Let q_k denote a path following q from x to $q(k)$, reparametrized at $t = 0, 1$ so as to belong to $P^\infty(M)$. Concretely we define $q_k(t) = q(\beta(t)k)$, where β is the previously mentioned smoothly increasing function which is equal to 0 on $[0, \frac{1}{3}]$ and equal to 1 on $[\frac{2}{3}, 1]$. Note that $d/dk\ (q_k(1))\,|_{k=0} = v$. Define the loop (see Fig. 3)

$$\ell_{i,q}(k) = p_{i,x} \star q_k \star p_{i,q(k)}^{-1}$$

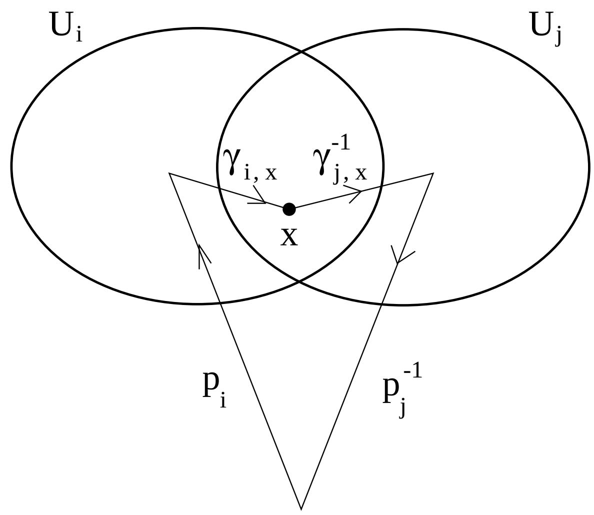

Figure 2: transition function

and set

$$f_{i,q}(k) = \mathcal{H}(\ell_{i,q}(k)).$$

Now we define:

$$A_i(v) = -i\frac{d}{dk}\log f_{i,q}(k)\mid_{k=0}.$$

Lemma 3.6 *The* 1*-forms* A_i *are well-defined and they satisfy*

$$i\,(A_j - A_i) = d\log g_{ij}$$

on double overlaps U_{ij}.

Proof: From Fig. 4 we have the equality

$$g_{ij}(q(k)) = f_{i,q}(k)^{-1}g_{ij}(x)f_{j,q}(k). \tag{8}$$

Taking the derivative of the logarithm at $k = 0$ we derive

$$g_{ij}^{-1}\,d\,g_{ij}(v) = i(A_j(v) - A_i(v)).$$

Now we introduce a new open set U_j in the atlas and corresponding ϕ_j : $U_j \to B(0,1)$, such that U_j is centred around $x_j = x$ and contained in U_i. Such a pair U_j, ϕ_j may easily be constructed from ϕ_i. Take the path from $*$

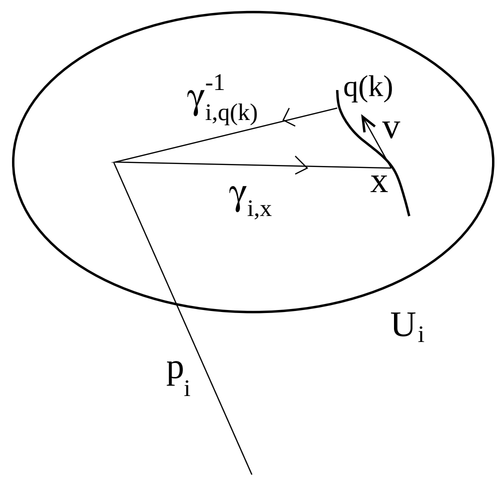

Figure 3: connection

to the center of U_j to be $p_j = p_{i,x}$. Now let r be a second path in U_i satisfying $r(0) = x$, $\dot{r}(0) = v$. Set

$$\begin{aligned} A_i^{(q)}(v) &= -i\frac{d}{dk}\log f_{i,q}(k)\,|_{k=0} \\ A_i^{(r)}(v) &= -i\frac{d}{dk}\log f_{i,r}(k)\,|_{k=0}\,. \end{aligned}$$

Now

$$A_j^{(q)}(v) - A_i^{(q)}(v) = A_j^{(r)}(v) - A_i^{(r)}(v),$$

since $g_{ij}^{-1}\,d\,g_{ij}(v)$ is the evaluation of a 1-form on a vector and does not depend on which path is used. Let $\tilde{\mathcal{H}} : \pi_1^1(M, x) \to U(1)$ be the recentered holonomy at x using the path p_j, i.e. $\tilde{\mathcal{H}}(\ell) = \mathcal{H}(p_j \star \ell \star p_j^{-1})$ for any loop based at x. Set $\tilde{f}_{j,q}(k) = q_k \star \gamma_{j,q(k)}^{-1}$. Then

$$A_j^{(q)}(v) = -i\frac{d}{dk}\log\tilde{\mathcal{H}}(\tilde{f}_{j,q}(k))\,|_{k=0} = 0$$

where the last equality follows from Barrett's lemma to be proved in Sect. 7, since $\tilde{f}_{j,q}(k)$ becomes the trivial loop at x when k goes to zero. Of course, by the same argument, $A_j^{(r)}(v) = 0$, and thus $A_i^{(q)}(v) = A_i^{(r)}(v)$. $\square$

The reconstruction described above involved choices of paths.

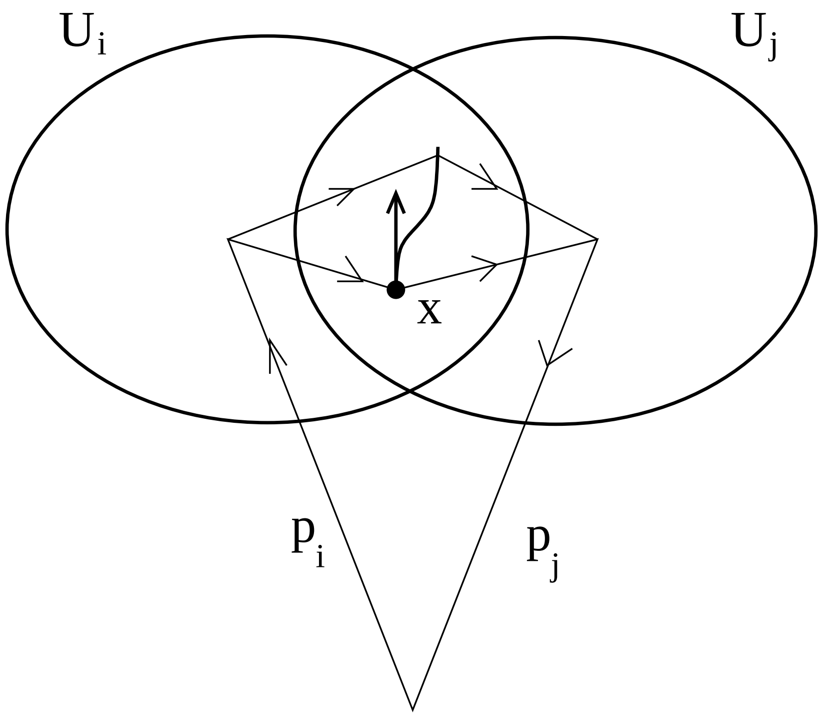

Figure 4: Proof of Eq. 8

Lemma 3.7 *The reconstructed bundle and connection are independent of the paths p_i up to equivalence.*

Proof: Let g'_{ij} and A'_i be the reconstructed data using paths p'_i instead of p_i. Define $h_i \in U(1)$ by $h_i = \mathcal{H}(p_i \star (p'_i)^{-1})$, for each i. Then

$$g'_{ij}(x) = h_i^{-1} g_{ij}(x) h_j$$

and

$$A'_i(v) = -i \frac{d}{dk} \log\, h_i^{-1} \star f_i(k) \star h_i \mid_{k=0} = A_i(v).$$

Thus the data g'_{ij} and A'_i are equivalent to the original data g_{ij} and A_i. □

The following result concerning the reconstructed connection will be used below.

Lemma 3.8 *Suppose $[a, b] \subset I$ is contained in V_i. Set*

$$\ell_{a,b} = p_{i,\ell(a)} \star \ell \mid_{[a,b]} \star p^{-1}_{i,\ell(b)}.$$

Then

$$i \int_a^b \ell^*(A_i) = \log \mathcal{H}(\ell_{a,b}).$$

Proof: Set $\chi(k) = \mathcal{H}(\ell_{a,k})$. Now

$$\begin{aligned} \frac{d}{dk} \log \chi(k) &= \lim_{\epsilon \to 0} \log \mathcal{H}(\ell_{k,k+\epsilon})/\epsilon \\ &= iA_i(\dot{\ell}(k)). \end{aligned}$$

Thus, integrating from a to b, the result follows. □

It remains to show that the above assignments from bundle and connection data $L, \mathcal{A}$ to holonomies $\mathcal{H}$ and vice-versa are mutual inverses up to equivalence. Let $\mathcal{H}$ be the holonomy obtained from $(L, \mathcal{A}) = (g_{ij}, A_i)$. The data reconstructed from this holonomy are given by:

$$\tilde{g}_{ij}(x) = \mathcal{H}(\ell_{ij}(x)) \tag{9}$$

$$i\tilde{A}_i(v) = \frac{d}{dk} \log \mathcal{H}(\ell_{i,q}(k)) \mid_{k=0} \tag{10}$$

Now using the local formula for $\mathcal{H}$ in (5) and the multiplicative property in (6), and setting $h_i(x) = \mathcal{H}^{0i}(p_{i,x})$, we have

$$\begin{aligned} \tilde{g}_{ij}(x) &= \mathcal{H}^{0i}(p_{i,x}) g_{ij}(x) \mathcal{H}^{j0}(p_{j,x}^{-1}) \\ &= h_i(x) g_{ij}(x) h_j^{-1}(x) \end{aligned}$$

and

$$\begin{aligned} i\tilde{A}_i(v) &= \frac{d}{dk} \log \mathcal{H}^{0i}(p_{i,x}) \mathcal{H}^{ii}(q_k) \mathcal{H}^{i0}(p_{i,q(k)}^{-1}) \mid_{k=0} \\ &= \frac{d}{dk} i \int_I q_k^*(A_i) \mid_{k=0} + \frac{d}{dk} \log \mathcal{H}^{i0}(p_{i,q(k)}^{-1}) \mid_{k=0} \\ &= iA_i(v) + d \log h_i(v). \end{aligned}$$

The final equality follows from

$$\begin{aligned} \frac{d}{dk} \int_0^1 q_k^*(A_i) \mid_{k=0} &= \frac{d}{dk} \int_0^k q^*(A_i) \mid_{k=0} \\ &= A_i(v). \end{aligned}$$

Thus $(\tilde{g}_{ij},\, \tilde{A}_i)$ is equivalent to the original data $(g_{ij},\, A_i)$.

Conversely, let $(L, \mathcal{A}) = (g_{ij}, A_i)$ be the line bundle and connection obtained from the holonomy $\mathcal{H}$. Let $\tilde{\mathcal{H}}$ be the holonomy obtained from these data. To show $\tilde{\mathcal{H}} = \mathcal{H}$ we start with the local formula for $\tilde{\mathcal{H}}$:

$$\tilde{\mathcal{H}}(\ell) = \prod_{\alpha=1}^{N} \exp i \int_{I_\alpha} \ell^*(A_\alpha) \cdot g_{\alpha,\alpha+1}(\ell(x_\alpha)). \tag{11}$$

Now using the definition of the transition functions (9) and Lem. 3.8 all paths cancel except for the subpaths of ℓ between $x_{\alpha-1}$ and x_α (see Fig. 5), and therefore $\tilde{\mathcal{H}}(\ell) = \mathcal{H}(\ell), \forall \ell$.

In conclusion we have the following

Theorem 3.9 *The above assignments define a bijection between bundles and connections, modulo equivalence, on the one hand, and holonomies on the other.*

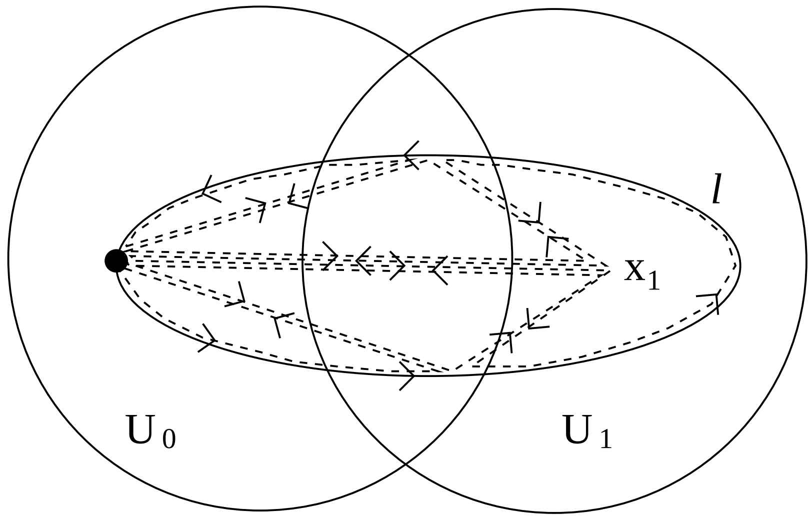

Figure 5: $\mathcal{H} = \tilde{\mathcal{H}}$

4 Thin higher homotopy groups

Caetano and the second author [11] also defined higher (relative) smooth homotopy groups with homotopies of restricted rank, generalizing the rank-1 homotopy group $\pi_1^1(M)$ of the previous section. In the present article we will only need to consider the special case of homotopy groups relative to the base point, which simplifies the definition.

Let M be a smooth manifold. Let I^n denote the unit n-cube, with coordinates $t_i \in [0, 1]$, $i = 1, \ldots, n$.

Definition 4.1 *An n-loop is a smooth map $\gamma : I^n \to M$ such that, for some $0 < \epsilon < 1/2$,*

$$\forall i = 1, \ldots, n, \quad t_i \in [0, \epsilon[\cup]1 - \epsilon, 1] \ \Rightarrow \ \gamma(t_1, \ldots, t_n) = *.$$

We denote the set of all n-loops by $\Omega_n^\infty(M)$.

Remark 4.2 The above condition generalizes the "sitting" condition $\gamma(t) = *, \forall t \in [0, \epsilon[\cup]1 - \epsilon, 1]$ for loops in Sect. 3. In [11] the weaker requirement

$$(t_1 \in [0, \epsilon[\cup]1 - \epsilon, 1]) \vee (t_i \in \{0, 1\} \text{ for some } i = 2, \ldots, n) \Rightarrow \gamma(t_1, \ldots, t_n) = *$$

was used. Definition 4.1 has the advantage that the n-loops can be multiplied smoothly (see below) along any of the t_i directions, and not just along t_1.

Definition 4.3 *The product, $\gamma_1 \star \gamma_2$, of two n-loops γ_1 and γ_2 is given by:*

$$\gamma_1 \star \gamma_2(t_1, \ldots, t_n) = \begin{cases} \gamma_1(2t_1, t_2, \ldots, t_n), & t_1 \in [0, 1/2], \\ \gamma_2(2t_1 - 1, t_2, \ldots, t_n), & t_1 \in]1/2, 1]. \end{cases}$$

The inverse, γ^{-1}, of an n-loop γ is given by:

$$\gamma^{-1}(t_1, \ldots, t_n) = \gamma(1 - t_1, t_2, \ldots, t_n)$$

Remark 4.4 $\gamma_1 \star \gamma_2$ is smooth because of the sitting condition in Def. 4.1, which implies that $\gamma_1 \star \gamma_2$ is constant in a neighbourhood of $t_1 = 1/2$.

We now define the thin homotopy relation, which was called intimacy relation by Caetano and Picken [11].

Definition 4.5 *Two n-loops γ_1 and γ_2 are said to be* rank-n homotopic *or* thin homotopic, *denoted $\gamma_1 \overset{n}{\sim} \gamma_2$, if there exists $\epsilon > 0$ and a homotopy $H : [0,1] \times I^n \to M$, such that:*

1. $t_i \in [0, \epsilon[\cup]1 - \epsilon, 1] \;\Rightarrow\; H(s, t_1, \ldots, t_n) = *, \quad i = 1, \ldots, n$
2. $s \in [0, \epsilon[\;\Rightarrow\; H(s, t_1, \ldots, t_n) = \gamma_1(t_1, \ldots, t_n)$
3. $s \in]1 - \epsilon, 1] \;\Rightarrow\; H(s, t_1, \ldots, t_n) = \gamma_2(t_1, \ldots, t_n)$
4. *H is smooth throughout its domain*
5. $\mathrm{rank} DH_{(s, t_1, \ldots, t_n)} \leq n$ *throughout its domain.*

It is straightforward to show that $\overset{n}{\sim}$ is an equivalence relation. Let us denote the set of equivalence classes of n-loops in M by $\pi_n^n(M, *)$, or just $\pi_n^n(M)$ when $*$ is understood.

Theorem 4.6 *$\pi_n^n(M, *)$ is an abelian group for $n \geq 2$.*

Proof: The product and inverse operations defined on n-loops descend to $\pi_n^n(M, *)$. The identity is the constant n-loop, which sends I^n to $*$. The group properties are shown in the same way as for $\pi_n(M)$, with the modifications introduced by Caetano and Picken in [10, 11] to accommodate smooth n-loops and homotopies. The group $\pi_n^n(M, *)$ is abelian by the standard geometric argument, since all homotopies involved are thin. $\square$

Remark 4.7 For dim$M \leq n$, $\pi_n^n(M) = \pi_n(M)$ and there is nothing new. For dim$M > n$ however, $\pi_n^n(M)$ is infinite-dimensional.

In the remainder of this paper we shall mainly be concerned with $\pi_2^2(M)$, the group of 2-loops, or surfaces, modulo rank-2 homotopy.

5 Gerbe-holonomy

Let $\mathcal{G}$ be a gerbe on M given by a set of transition line-bundles Λ_{ij} and trivializations θ_{ijk} , and let $\mathcal{A}$ be a gerbe-connection on $\mathcal{G}$ given by a set of connections, ∇_{ij}, on Λ_{ij} and local 2-forms F_i (see Sect. 2). We first define the gerbe-holonomy of $(\mathcal{G}, \mathcal{A})$, following Chatterjee [12]. Let $s\colon I^2 \to M$ be a 2-loop, then the pull-back of $\mathcal{G}$, $s^*(\mathcal{G})$, defines a gerbe on I^2. Since I^2 is two dimensional, the gerbe $s^*(\mathcal{G})$ is trivial and we can choose an arbitrary trivialization, i.e., an object $\mathcal{O}$, with object connection. Let $\mathcal{O}$ be given by the line-bundles L_i, trivialized by the sections σ_i over $V_i = s^{-1}(U_i)$, and let the object connection be given by ∇_i. We can now define a global 2-form on I^2 by the formula

$$\epsilon|_{V_i} = s^*(F_i) - \sigma_i^*(K(\nabla_i)).$$

Chatterjee [12] calls ϵ an error 2-form of the object connection.

Definition 5.1 *The* holonomy *of* $(\mathcal{G}, \mathcal{A})$ *around* s*, which we denote by* $\mathcal{H}(s)$*, is defined by*

$$exp\ i \int_{I^2} \epsilon.$$

Chatterjee proves that $\mathcal{H}(s)$ is independent of all the choices we made.

Lemma 5.2 *(*[12] Thm. 7.1.2) *The value of* $\mathcal{H}(s)$ *is independent of the choice of connection on the object, and of the choice of object.*

We now show that $\mathcal{H}$ is constant within each thin homotopy class.

Lemma 5.3 *Let* $s, s'\colon I^2 \to M$ *be two 2-loops. If* s *and* s' *are thin homotopic, then* $\mathcal{H}(s) = \mathcal{H}(s')$*.*

Proof: Let $H\colon I^3 \to M$ be a thin homotopy between s and s'. Note that only the two faces corresponding to s and s' of ∂H are mapped non-trivially to M, all other faces are mapped to the base-point. By the observation above and Stokes' theorem we get

$$\mathcal{H}(s)\mathcal{H}(s')^{-1} = \exp\, i \int_{\partial I^3} \epsilon = \exp\, i \int_{I^3} H^*(G).$$

Here ϵ is an error 2-form for an object of $H^*(\mathcal{G})$ defined on I^3 and G is the gerbe-curvature 3-form. Note that we can apply Stokes' theorem, to obtain the second equality, because $d\epsilon = H^*(G)$ (see [12]). The last expression is equal to 1 because the rank of the differential of H is at most 2 and G is a 3-form. □

Since it is also clear from the definition that the holonomy of the product of two 2-loops equals the product of the holonomies around each one of them, we arrive at the following Lemma.

Lemma 5.4 *The gerbe-holonomy defines a group homomorphism*

$$\mathcal{H}\colon \pi_2^2(M) \to U(1),$$

which only depends on $\mathcal{G}, \mathcal{A}$ up to equivalence.

Proof: The first part of the claim is a corollary to the previous lemma. The second part follows directly from the definition of equivalence between gerbes with gerbe-connections by applying Stokes' theorem repeatedly. □

As we showed in Sect. 3 the holonomy of a line-bundle with connection is smooth in a precise sense. The same is true for gerbe-holonomies. We define a *smooth family of 2-loops* to be a map $\psi\colon U \subseteq \mathbb{R}^n \to \Omega_2^\infty(M)$ defined on an open subset $U \subseteq \mathbb{R}^n$ such that $\psi(x; t^1, t^2) = \psi(x)(t^1, t^2)$ is smooth on $U \times I^2$.

Definition 5.5 *A* 2-holonomy *is a group homomorphism $\mathcal{H}\colon \pi_2^2(M) \to U(1)$ such that for every smooth family of 2-loops $\psi\colon U \subseteq \mathbb{R}^n \to \Omega_2^\infty(M)$ the composite*

$$U \subseteq \mathbb{R}^n \xrightarrow{\psi} \Omega_2^\infty(M) \xrightarrow{\text{proj}} \pi_2^2(M) \xrightarrow{\mathcal{H}} U(1)$$

is smooth throughout $U \subseteq \mathbb{R}^n$.

Lemma 5.6 *The gerbe-holonomy, $\mathcal{H}$, defines a 2-holonomy.*

Proof: This is an immediate consequence of the fact that $\mathcal{H}$ is defined by integration of a smooth 2-form. $\square$

Lemma 5.7 *(Recentering a 2-holonomy)*

a) Let $\ell \in \Omega^\infty(M,*)$ be a loop which is homotopic to the constant loop in $*$ via a homotopy $G\colon \ell \to *$. We define $P_2^2(M,\ell)$ as the group of all homotopies starting and ending at ℓ with fixed endpoints $*$ modulo thin homotopy. Given a group homomorphism $\tilde{\mathcal{H}}\colon P_2^2(M,\ell) \to U(1)$ we can recenter *$\tilde{\mathcal{H}}$ to obtain a 2-holonomy $\mathcal{H}\colon \pi_2^2(M) \to U(1)$ by defining $\mathcal{H}(s) = \tilde{\mathcal{H}}(G \star s \star G^{-1})$. We also say that we have* recentered *s by G.*

b) Choose $m \in M$ and let $p \in P^\infty(M)$ be a path from $*$ to m. Given a 2-holonomy $\mathcal{H}\colon \pi_2^2(M,*) \to U(1)$ we can recenter *$\mathcal{H}$ to obtain a 2-holonomy $\tilde{\mathcal{H}}\colon \pi_2^2(M,m) \to U(1)$ in the following way: choose $s \in \Omega_2^\infty(M,m)$ and let s' be the 2-path $Id_p \star s \star Id_{p^{-1}}$. Now recenter s' by the thin homotopy between the constant loop at $*$ and $p \star Id_m \star p^{-1}$, denoted by G_p. Note that for this recentering we have to use composition of 2-paths via the second coordinate. Thus we have obtained a 2-loop $\tilde{s} \in \Omega_2^\infty(M,*)$ and therefore we can define $\tilde{\mathcal{H}}(s) = \mathcal{H}(\tilde{s})$. We also say that we have* recentered *s by p.*

Proof: Both in a) and b) one only has to check that recentering is well defined modulo thin homotopy, which is straightforward. $\square$

Before going on to the next section it is worthwhile to have a look at a more concrete formula for $\mathcal{H}$. Analogously to what we did for line-bundles with connections in Sect. 3, we can define $\mathcal{H}$ completely in terms of the Čech cocycle g_{ijk}, the 0-connection A_{ij} and the 1-connection F_i. Let $s\colon I^2 \to M$ be a 2-loop in M as before. Let $\mathcal{V} = \{V_i \mid i \in J\}$ be the covering of I^2 obtained by taking the inverse image via s of all open sets in the cover $\mathcal{U}$. We define a *grid* on I^2 to be a rectangular subdivision of I^2. A rectangle in the grid is denoted by R_i, an edge by E_{ij} and a vertex by V_{ijkl}. The edges are all oriented from left to right and from bottom to top, the rectangles have the counter-clockwise orientation. Now take the grid sufficiently fine so that each rectangle F_i lies in at least one open set V_i. Note that this is possible because each covering of a compact metric space has a Lebesgue number. Note also that in general a small rectangle can be contained in more than one open set. A particular choice of open set for each rectangle we call a *labelling* of the grid. We always assume that the labels of the rectangles at the boundary are equal to 0, and we fix a choice of $V_0 = s^{-1}(U_0)$, such that U_0 contains the base point of M. Let $\epsilon \in \Omega^2(I^2)$ be the error 2-form of an object for $s^*(\mathcal{G}, \mathcal{A})$, given by a family of line-bundles L_i, with 0-connection given by a family of 1-forms $A_i \in \Omega^1(V_i)$. We write $\epsilon|_{V_i} = s^*F_i - dA_i$. We recall the identities

$$i(A_j - A_i) = is^*A_{ij} + d\log\lambda_{ij},$$

where $\lambda_{ij}\colon V_{ij} \to U(1)$ satisfies

$$\lambda_{ij}\lambda_{jk}\lambda_{ki} = s^*g_{ijk}.$$

Now pick a labelling of the grid. According to Def. 5.1 we have

$$\begin{aligned} \mathcal{H}(s) &= \exp\ i\textstyle\int_{I^2}\epsilon \\ &= \prod_\alpha \exp\ i\textstyle\int_{R_\alpha}(s^*F_\alpha - dA_\alpha) \\ &= \prod_\alpha \exp\ i\textstyle\int_{R_\alpha} s^*F_\alpha \cdot \prod_{\alpha,\beta} \exp\ i\textstyle\int_{E_{\alpha\beta}} s^*A_{\alpha\beta} \\ &\quad \times \prod_{\alpha,\beta,\gamma,\delta} g_{\alpha\beta\gamma}(s(V_{\alpha\beta\gamma\delta}))g^{-1}_{\alpha\delta\gamma}(s(V_{\alpha\beta\gamma\delta})). \end{aligned} \tag{12}$$

The last two products are to be taken over the labels of contiguous faces in the grid only and in such a way that each face, edge and vertex appears only once. The convention for the order of the labels is indicated in Fig. 6. Formula (12) follows from applying Stokes' theorem repeatedly. Note that Chatterjee's results for the global definition of the gerbe-holonomy [12] and the equalities above show that the value of $\mathcal{H}(s)$ is independent of the choice of grid and its labelling.

We have written down the explicit formula for gerbe-holonomy using a grid. Of course it is possible to obtain an analogous formula using other subdivisions of the unit square, for example, triangulations of I^2. The idea is exactly the same, but one has to take into account the different valencies of the vertices.

In Sect. 8 we show that every 2-holonomy is the gerbe-holonomy of some gerbe with gerbe-connection.

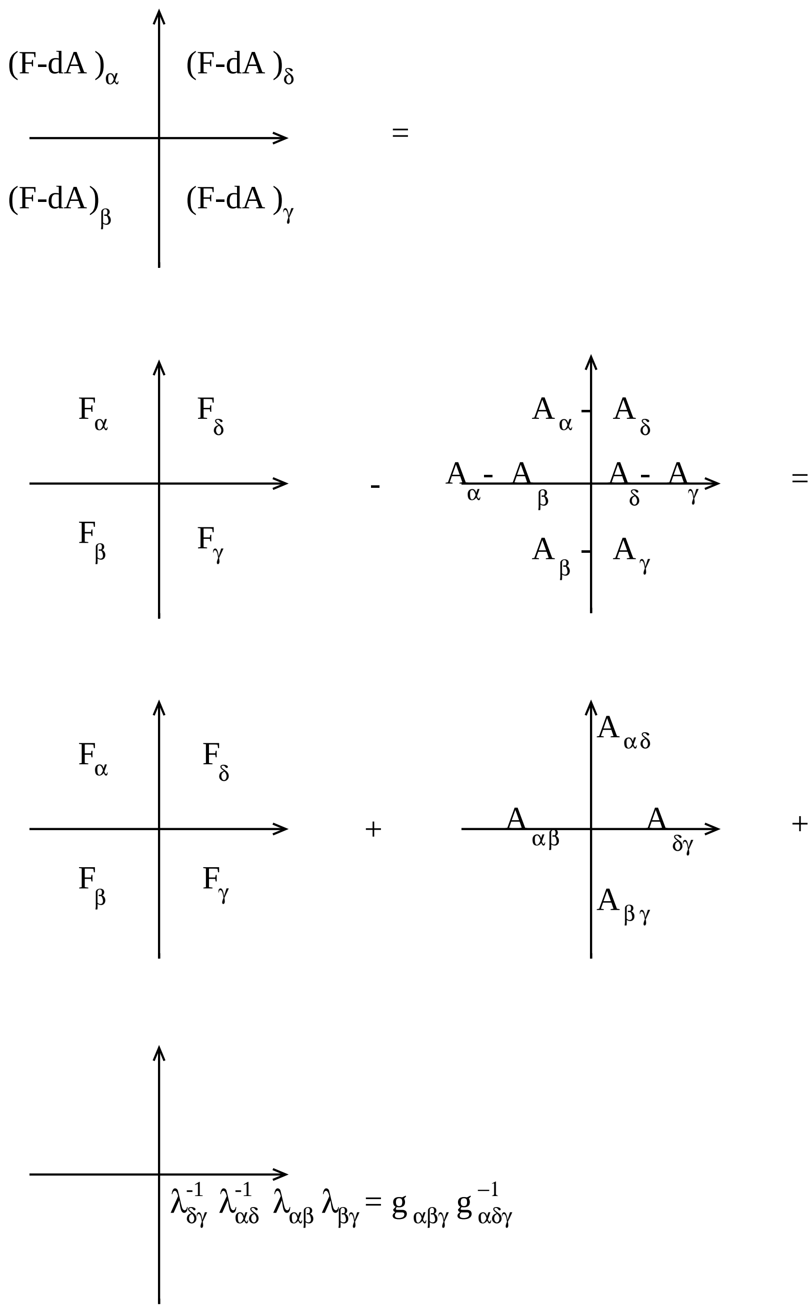

Figure 6: concrete formula for gerbe-holonomy

6 Parallel transport in gerbes

Let us now explain parallel transport for Abelian gerbe-connections in general along arbitrary homotopies between arbitrary loops. For this discussion we assume some basic knowledge about groupoids and 2-groupoids, which can be obtained by reading [2, 4, 23] for example. We should warn the reader that this section is different in flavour from the rest of the paper. Whereas we have followed a down-to-earth approach in the rest of the paper, here some readers might feel that we are trying to make up for that by being unnecessarily sophisticated. We have two arguments in our defense. First of all we do not know any simpler way of formulating the parallel transport of gerbes satisfactorily. Secondly, we believe that category theory is already at the heart of gerbes, since the original definition of gerbes is in terms of sheaves of categories [9]. Being at the heart of gerbes, we ought to understand the category theory that is involved a little better. Our formulation of the parallel transport of a gerbe on M shows the relation with what can be called the *thin homotopy* 2*-type* of M. Following ideas expressed by Grothendieck, homotopy theorists and category theorists have been endeavouring to define *weak* n*-categories*, of which a special sub-class, the *weak* n*-groupoids*, should model homotopy n-types of topological spaces. For an overview of n-category theory see [2, 4]. In [4] Baez and Dolan sketch the possible relevance of n-categories for the formulation of *Topological Quantum Field Theories* (TQFT's). Following these ideas the first author of this article investigated the possibility of defining four-dimensional TQFT's using monoidal 2-categories [22, 21]. Our formulation of parallel transport of gerbes in this section is also a first attempt to see if there is any link between n-categories and TQFT's on the one hand and differential geometry on the other hand. If only for this reason, we already feel that the effort of penetrating the relatively unfamiliar language of monoidal categories and 2-categories is not wasted. Due to this higher level of sophistication we cite results from the literature without giving direct proofs here, so that the full emphasis is placed on the change of language and not on mathematical detail. For mathematical detail we refer to Sect. 8, where we work out the simply-connected case in the more familiar language of groups and group homomorphisms. At the end of this section we indicate how an explicit proof in the general case can be deduced from the results in Sect. 8.

It is enlightening to go back to the case of principal bundles with connections first. Let $P \xrightarrow{p} M$ be a principal G-bundle on a not-necessarily-connected manifold M and let ω be a connection in P. Suppose in that case that we are trying to describe the parallel transport of ω. It is well known in homotopy theory that for general manifolds it is best to work with the *path groupoid* rather than the fundamental group, because the latter requires the choice of a basepoint in a certain connected component. Therefore it is natural to employ the thin path groupoid, $P_1^1(M)$ (see Sect. 3) to describe parallel transport. To formulate parallel transport along paths in terms of Lie groupoids and functors we also have to associate a groupoid to the bundle P. This is a well known construction which goes back to Ehresmann's work, and details can be found in Mackenzie's book [23], for example.

Let us sketch the construction of this groupoid, which we denote by $G(P, M)$.

Definition 6.1 *The objects of $G(P, M)$ are simply the points in M. The set of all morphisms is given by the manifold $P \times P/G$, where G acts by $(x, y)s = (xs, ys)$. We denote the equivalence classes by $[x, y]$, where we consider this to be a morphism from $p(x)$ to $p(y)$, the opposite of Mackenzie's [23] convention. The composite $[x, y][w, z]$ is defined by $[xs, z]$, where $w = ys$ in the fibre over $p(y) = p(w)$. The identity morphism for $m \in M$ is of course $[x, x]$, where $x \in p^{-1}(m)$ is arbitrary. Finally the inverse of $[x, y]$ is given by $[y, x]$.*

It is easy to check that this defines a groupoid, and it can be shown [23] that it is a locally trivial Lie groupoid. This last statement means that the operations just defined are smooth and that for every point in M there is an open neighborhood, say U, such that the restriction of this groupoid to U is isomorphic to $U \times G \times U$, with the trivial groupoid structure. Conversely one can construct a principal bundle from a locally trivial Lie groupoid by taking as the total space all morphisms with a fixed, but arbitrary, source, and as the projection the target map. These constructions are each other's inverses up to isomorphism.

The connection ω now gives rise to a functor $\mathcal{P}\colon P_1^1(M) \to G(P, M)$ which is the identity on objects, i.e. points of M, and is smooth in the sense of Sect. 3.

Definition 6.2 *The P(arallel) T(ransport) functor, denoted $\mathcal{P}$, of ω is defined on objects by $\mathcal{P}(m) = m$, for all $m \in M$. For a given path q in M, we define $\mathcal{P}(q) = [x, y]$, where $x \in p^{-1}(q(0))$ is arbitrary and $y \in p^{-1}(q(1))$ is obtained from x by parallel transport along the path q. Since parallel transport only depends on the thin homotopy class of q, the functor $\mathcal{P}$ is well-defined.*

Conversely, any such functor yields a *path-connection* in $G(P, M)$ in the sense of Def. 7.1 in [23], and is therefore equivalent to a connection in P. The path-connection for a given PT-functor, $\mathcal{P}$, is easy to describe: for any path q in M, the path-connection yields the path in $G(P, M)$ that is given by $k \mapsto \mathcal{P}(q_k)$, where q_k is as in Sect. 3. The definition of $\mathcal{P}$ and the observations above cannot be found in the literature on Lie groupoids, but they are an immediate consequence of Barrett's construction [3], so we will not spell them out here. It is easy to see that two connections in P are gauge-equivalent precisely if the corresponding PT-functors are naturally isomorphic. Note that the reconstruction result is less powerful than Barrett's original theorem, because the bundle is already part of the PT-functor in the form of its target. However, the upshot is that we can deal with the non-connected case directly by not fixing the choice of a base-point.

Back to gerbes again. Brylinski [9] (see Hitchin [19] too), explains how a gerbe on M with a gerbe-connection gives rise to a line-bundle on $\Omega(M)$ with an ordinary connection. This line-bundle has the special property that it is "multiplicative" with respect to the composition of loops (Prop. 6.2.5 in [9]). The connection in this line-bundle has the special property that it yields a parallel transport over "cylinders" which does not depend on the way the cylinder is made up out of a path of loops but only on the surface itself. This leads to the idea that a gerbe, $\mathcal{G}$, with gerbe-connection, $\mathcal{A}$, on M yields a groupoid on the *thin loop groupoid*, $L_1^1(M)$, which is the subgroupoid of $P_1^1(M)$ with only loops. If M is path-connected (something which we will assume for the rest of this section), one can fix a basepoint, $* \in M$, and work over $\pi_1^1(M, *)$. Before explaining the parallel transport functor for gerbes, let us first have a closer look at the construction of this line-bundle with connection over $\pi_1^1(M, *)$.

We follow Ch. 6 in Brylinski's book [9] mostly, but we consider $\Omega(M, *)$ to be smooth using smooth families of loops rather than trying to define an infinite-dimensional manifold structure on it. Let $\mathcal{G}$ be a gerbe on M and $\mathcal{A}$ a gerbe-connection. We first construct the line-bundle on $\Omega(M, *)$, as Brylinski does, and then show that it projects to a line-bundle on $\pi_1^1(M, *)$. Both bundles we denote by $L^{\mathcal{G}}$.

Definition 6.3 *The total space of $L^{\mathcal{G}}$ is given by the set of equivalence classes of quadruples (γ, F, ∇, z), where $\gamma \in \Omega(M, *)$, F is an object for $\gamma^*\mathcal{G}$ on the circle S^1 with ∇ an object connection in F, and finally $z \in \mathbb{C}^*$. Note that we use the word "object" as defined in Sect. 2, so it really is given* *by a set of data, as is ∇. The equivalence relation is generated by*

1. *$(\gamma, F, \nabla, z) \sim (\gamma, F', \nabla', z)$ if (F, ∇) and (F', ∇') are isomorphic pairs of objects with object-connections.*

2. *$(\gamma, F, \nabla + \alpha, z) \sim \left(\gamma, F, \nabla, z \exp\left(-\int_0^1 \alpha\right)\right)$ for any complex valued 1-form α on S^1.*

The action of $\mathbb{C}^*$ on $L^{\mathcal{G}}$ is given by $(\gamma, F, \nabla, z)w = (\gamma, F, \nabla, zw)$. Brylinski (Prop.6.2.1. in [9]) proves that there is a unique smooth structure on $L^{\mathcal{G}}$ such that

1. *The projection $(\gamma, F, \nabla, z) \mapsto \gamma$ defines a smooth principal $\mathbb{C}^*$-bundle.*

2. *For any $\gamma \in \Omega(M, *)$, any open contractible neighborhood of the origin $U \subset \mathbb{R}^n$, and any smooth family of loops $\Gamma \colon U \to \Omega(M, *)$ such that $\Gamma(0) = \gamma$, let (F, ∇) be an object with object-connection on $\Gamma(U) \subset M$. The map*
$$\sigma(\Gamma(x)) = (\Gamma(x), \Gamma(x)^* F, \Gamma(x)^* \nabla, 1)$$
defines a smooth local section of $L^{\mathcal{G}}$.

Recall that two objects with, in this case necessarily flat, object-connections of $\gamma^*\mathcal{G}$ on S^1 always differ by a line-bundle with a flat connection on S^1. The two objects with connections are isomorphic if and only if the flat connection in the line-bundle has trivial holonomy (in which case the line-bundle is necessarily trivializable as well). The fibre over a loop is now acted upon by line-bundles on S^1 with flat connections, whose isomorphism classes are the elements of $\mathrm{Hom}\,(\pi_1(S^1) = \mathbb{Z}, U(1)) \cong U(1)$. This is Hitchin's [19] description of $L^{\mathcal{G}}$. As remarked by Brylinski (Prop. 6.2.5. [9]) the fibres of $L^{\mathcal{G}}$ have a multiplicative property. Given the points $(\gamma, \gamma^*F, \gamma^*\nabla, 1)$ in $p^{-1}(\gamma)$ and $(\mu, \mu^*F, \mu^*\nabla, 1)$ in $p^{-1}(\mu)$, the product becomes $(\gamma \star \mu, (\gamma \star \mu)^* F, (\gamma \star \mu)^* \nabla, 1)$ in $p^{-1}(\gamma \star \mu)$. Here we have taken an object F and an object-connection ∇ which are defined on the image of $\gamma \star \mu$.

The connection on $L^{\mathcal{G}}$ can now be defined. Let v be any tangent vector to $\Omega(M, *)$ at γ. Let σ be the section defined above and let ϵ be the error 2-form (see Sect. 5) of $(\gamma^*F, \gamma^*\nabla)$ with the same notation as above.

Definition 6.4 *The covariant derivative in $L^{\mathcal{G}}$ is defined by*
$$\frac{D_v \sigma}{\sigma} = i \int_0^1 \epsilon_{\gamma(t)} \left(\dot{\gamma}(t), v(\gamma(t))\right) dt.$$

The curvature of this connection is equal to

$$K_\gamma(D)(u,v) = \int_0^1 \Omega_{\gamma(t)}\left(\dot\gamma(t), u(\gamma(t)), v(\gamma(t))\right) dt,$$

where Ω is the gerbe-curvature 3-form on M. It is also easy to describe the parallel transport of the connection D along a "cylinder". Let $H\colon I^2 \to M$ be a homotopy between two loops γ and μ. Choose an object F and an object connection ∇ for $H^*(\mathcal{G})$ on I^2 (or $S^1 \times I$). Let ϵ be the error 2-form for this object-connection.

Definition 6.5 *The parallel transport along H is given by*

$$\mathcal{P}(H)\left(\gamma, \gamma^* F, \gamma^*\nabla, 1\right) = \left(\mu, \mu^* F, \mu^*\nabla, 1\right) \exp\left(\int_{I^2} \epsilon\right).$$

Two observations show that $\left(L^{\mathcal{G}}, D\right)$ projects to a bundle on $\pi_1^1(M,*)$. Firstly the formula for parallel transport clearly shows that it is compatible with the multiplication. Secondly, as ϵ is a 2-form, the parallel transport along a thin homotopy is trivial, so there is a unique way to identify the fibres of $L^{\mathcal{G}}$ which lie over loops that are thin homotopic. By abuse of notation we denote this line-bundle over $\pi_1^1(M,*)$ with connection also by $(L^{\mathcal{G}}, D)$. Furthermore, Brylinski ([9], Thm. 6.2.4.(3)) shows that whenever two homotopies $G, H\colon I^2 \to M$ between a given pair of paths are homotopic themselves by a homotopy $J\colon I^3 \to M$, then the parallel transport around GH^{-1} is given by

$$\int_{I^3} J^*\Omega.$$

Thus we see that the parallel transport along G equals the parallel transport along H if they are thin homotopic, because in that case J can be chosen to have rank ≤ 2 everywhere.

One can now define the groupoid $L^{\mathcal{G}} \times L^{\mathcal{G}}/U(1)$ over $\pi_1^1(M,*)$ as we showed above, and this groupoid is equipped with a *monoidal structure* in the following sense. The equivalence classes of the loops can be multiplied, or tensored, $[\gamma] \star [\mu] = [\gamma \star \mu]$. Given $\alpha, \beta, \gamma, \mu \in \Omega(M,*)$, and given two morphisms in $L^{\mathcal{G}} \times L^{\mathcal{G}}/U(1)$, say $[a,b]\colon [\alpha] \to [\beta]$ and $[c,d]\colon [\gamma] \to [\mu]$, then one can tensor the morphisms to get $[a,b] \star [c,d] = [ac, bd]\colon [\alpha \star \gamma] \to [\beta \star \mu]$, where ac is defined by the product of the fibres as described above. This multiplication of the morphisms is compatible with the composition of morphisms in the following sense: for any quadruple of morphisms we have the equality

$$([a,b]\star[c,d])([e,f]\star[g,h]) = ([a,b][e,f])\star([c,d][g,h]),$$

whenever both sides of the equation are defined. This equation, which is an example of the *interchange law* for monoidal categories, is easily checked and only holds because $U(1)$ is Abelian. With respect to this monoidal structure the objects and the morphisms have inverses and there is a unit object, $[c_*]$, where c_* is the constant loop at $*$, and a unit morphism, $[a,a]$ for any a such that $p(a) = [c_*]$. Altogether we propose to call this structure a *Lie 2-group*, which is justified by the fact that all operations involved are smooth in the appropriate sense and that it can be seen as a Lie 2-groupoid with only one object. (This is analogous to the statement that a Lie groupoid with one object is nothing but a Lie group. For this kind of general remark about n-categories see [2, 4, 5] for example.) Without the smoothness condition this kind of groupoid goes under a variety of names in the homotopy literature. Yetter [28] calls them *categorical groups*, for example.

Definition 6.6 *Let $G(\mathcal{G},\mathcal{A},M)$ be the Lie 2-group given by $L^{\mathcal{G}}\times L^{\mathcal{G}}/U(1)$ over $\pi_1^1(M)$, as defined above.*

Let us now define the *thin Lie 2-group of cylinders*, denoted by $C_2^2(M,*)$. We define $C_2^2(M,*)$ as the quotient of a non-strict monoidal groupoid $C(M,*)$ by a *normal monoidal subgroupoid* $N(M,*)$.

Definition 6.7 *The objects of $C(M,*)$ are the elements of $\Omega(M,*)$. The morphisms are thin homotopy classes of homotopies between loops, through based loops. It is clear that this forms a groupoid under the obvious composition of homotopies. There is also a monoidal structure on $C(M,*)$ defined by the composition of loops and the corresponding composition of homotopies. For clarity, we refer to the monoidal composition of homotopies as horizontal composition, and write $\star$, and the other we call vertical and indicate by simple concatenation. We denote the vertical inverse of a homotopy H by H^{-1} and the horizontal inverse by $H^{\leftarrow}$.*

Recall that for homotopies of the trivial loop to itself both compositions are the same up to thin homotopy, which is why $\pi_2^2(M)$ is abelian. Concretely this follows from the interchange law which states that $(G_1\star H_1)(G_2\star H_2)$

is thin homotopic to $(G_1G_2) \star (H_1H_2)$, whenever both composites can be defined. This way the groupoid $C(M,*)$ is a *weak* monoidal groupoid, with weak inverses for the objects because loops only form a group up to thin homotopy. Instead of using the abstract *strictification theorem* [26], which is not very practical for the concrete application to gerbe-holonomy, we "strictify" $C(M,*)$ by hand by dividing out by the monoidal subgroupoid of only the thin homotopies, $N(M,*)$. Dividing out by a monoidal subgroupoid is only well-defined if the following conditions are satisfied, in which case we call it *normal*:

1. For any $\gamma \in \Omega(M,*)$: $\quad 1_\gamma \in N(M,*)$.
2. For any $\gamma, \mu \in \Omega(M,*)$, any homotopy $G\colon \gamma \to \mu$ and any **thin** homotopy $H\colon \mu \to \mu$, the **thin** homotopy class of $GHG^{-1}\colon \gamma \to \gamma$ belongs to $N(M,*)$.
3. For any $\gamma, \mu \in \Omega(M,*)$, any homotopy $G\colon \gamma \to \gamma$ and any **thin** homotopy $H\colon \mu \to \mu$, the **thin** homotopy class of $G \star H \star G^{\leftarrow}\colon \gamma \star \mu \star \gamma^{-1} \to \gamma \star \mu \star \gamma^{-1}$ belongs to $N(M,*)$.

It is easy to check that the conditions above are the right ones for our construction of the quotient monoidal groupoid, which we explain below, to be well-defined. The only reference for the definition of a normal monoidal subgroupoid that we know of is [6] (who give the more general definition of a normal monoidal subcategory), but it might be that it can be found in earlier papers on monoidal categories and 2-categories. We suspect that this definition goes back to the time when monoidal categories were defined for the first time [7], but we have been unable to find a precise written reference in the older literature.

Lemma 6.8 *The monoidal subgroupoid $N(M,*)$ is normal in $C(M,*)$.*

Proof The first condition is obviously satisfied.

We prove that the second condition holds. Denote the group of homotopies $\gamma \to \gamma$ in $C(M,*)$ by $C(M,*)(\gamma)$. This group is isomorphic to $C(M,*)(\gamma \star \mu)$, for any $\mu \in \Omega(M,*)$ (this is well known, see for example [18]). The isomorphism, which clearly preserves thinness, is given by

$$\phi\colon G \mapsto G \star 1_\mu.$$

Under ϕ the homotopy $GHG^{-1}\colon \gamma \to \gamma$ is mapped to $(G \star 1_\mu)(H \star 1_\mu)(G^{-1} \star 1_\mu)\colon \gamma\star\mu \to \gamma\star\mu$. Clearly this is thin homotopic to $(G\star 1_\mu)(1_\mu \star H)(G^{-1}\star 1_\mu)$. By the interchange law the latter is thin homotopic to $1_\gamma \star H\colon \gamma\star\mu \to \gamma\star\mu$ which is clearly a thin homotopy whenever H is thin.

The third condition can be proved by a similar argument. $\square$

We can now define the quotient groupoid $C_2^2(M,*) = C(M,*)/N(M,*)$.

Definition 6.9 *The objects of $C_2^2(M,*)$ are the elements of $\pi_1^1(M,*)$, which we temporarily denote by $[\gamma]$.*

For any $\alpha, \beta, \gamma, \mu \in \Omega(M,*)$ and for any $G \in C(M,*)(\alpha,\beta)$ and $H \in C(M,*)(\gamma,\mu)$, we say that G and H are equivalent if there exist thin homotopies $A \in N(M,*)(\alpha,\gamma)$ and $B \in N(M,*)(\beta,\mu)$ such that $AHB^{-1} \overset{2}{\sim} G$.

The morphisms between $[\gamma]$ and $[\mu]$ are the equivalence classes of

$$\bigcup_{\alpha,\beta} \{C(M,*)(\alpha,\beta) \mid [\alpha]=[\gamma],\ [\beta]=[\mu]\}$$

modulo this equivalence relation.

The composition and the monoidal structure descend to the quotient precisely because $N(M,*)$ is normal. The smooth structure is defined by smooth families of loops and smooth families of homotopies.

In this way we have obtained a second example of a Lie 2-group.

Before going on we should explain a small technical gap that we have not been able to bridge yet. Although Lem. 6.8 is strong enough to conclude that $C_2^2(M,*)$ is well-defined, it is too weak to prove that $C(M,*)$ and $C_2^2(M,*)$ are equivalent as Lie 2-groups. To establish that equivalence one would have to prove that, for any $\gamma \in \Omega(M,*)$, the group $N(M,*)(\gamma)$ is trivial. We conjecture that this true, but have not been able to prove it for any manifold other than $\mathbb{R}^n$, where it is almost immediate. As it stands we cannot prove that $C_2^2(M,*)$ is a true strictification of $C(M,*)$, but it is clear that $C_2^2(M,*)$ suits our purpose in this paper very well indeed. In [17] the reader can find a more restricted notion of thin homotopy for which the conjecture can be shown. However, this notion of thin homotopy does not seem to be suited to the smooth context of parallel transport. Besides being right for the formulation of parallel transport, our notion of thinness has the extra benefit that it can be generalized immediately to homotopies of any dimension, which is necessary if one wants to formulate and prove the analogues of our results for n-gerbes in general.

From everything above it now follows that the following theorem holds:

Theorem 6.10 *The pair $(\mathcal{G}, \mathcal{A})$ gives rise to a smooth PT-functor of Lie 2-groups*

$$\mathcal{P}\colon C_2^2(M,*) \to G(\mathcal{G}, \mathcal{A}, M),$$

over the identity on $\pi_1^1(M,*)$.

Conversely, given any line-bundle L on $\pi_1^1(M,*)$ with the multiplicative property as described above, we obtain a Lie 2-group, $G(L,M)$. Any smooth functor $\mathcal{P}\colon C_2^2(M,*) \to G(L,M)$ over the identity on $\pi_1^1(M,*)$ gives rise to a gerbe-connection, unique up to gauge-equivalence, in the gerbe associated to $G(L,M)$, such that $\mathcal{P}$ is the PT-functor for that connection.

One can wonder about the meaning of the second part of this theorem. Where would one get such line-bundles on the loop space? Our construction of the line-bundle on $\pi_1^1(M,*)$ only depends on the gerbe and the 0-connection. Thus in this formulation the 1-connection is really the new information contained in the PT-functor. It is interesting to compare this to the case of line-bundles described in the beginning of this section, where the line-bundle is already given in the form of a Lie groupoid but the connection is determined by the PT-functor.

An explicit proof of the second part of Thm. 6.10 in which one recovers the Čech 2-cocycle of the gerbe and the local 1- and 2-forms of the gerbe-connection from the parallel transport is easily deduced from our results in Sect. 8, where we work out the simply-connected case explicitly and in great detail. There we close up all 2-paths to obtain 2-loops based at $*$ using certain auxiliary homotopies called P_{ij}, which exist when M is 1-connected. However, one can leave them out in general to obtain a direct proof of the second part of Thm. 6.10. Note that the proof for the general case is really the same as the one we give in Sect. 8, because $G(\mathcal{G}, \mathcal{A}, M)(\gamma) \cong U(1)$ for any loop γ. For completeness, we should remark that we have not proved that $C_2^2(M,*)(c_*) \cong \pi_2^2(M,*)$, because of the little technical gap that we explained above. Hypothetically $C_2^2(M,*)(c_*)$ might only be a "thin quotient" of $\pi_2^2(M,*)$. However, this technicality is of no importance for our approach and we can work with $\pi_2^2(M,*)$ in Sect. 8 without any difficulty.

7 Barrett's lemma for 2-loops

This section is a short intermezzo with two technical lemmas. The first one we needed in Sect. 3 for the reconstruction of the connection in a line-bundle from its holonomy, and the second lemma we will need for our reconstruction of the 1-connection in the gerbe obtained from a 2-holonomy in Sect 8. In [3] Barrett proved the following lemma, which henceforth we call *Barrett's lemma* (strictly speaking this is Caetano and Picken's [10] version, but the proofs are the same). We state and prove the theorem for the case $G = U(1)$, which we need here, but it is true for any Lie group G.

Lemma 7.1 *The trivial loop extremizes every holonomy $\mathcal{H}\colon \pi_1^1(M) \to U(1)$, i.e., given any smooth family of loops $\psi\colon [0,1] \to \Omega^\infty(M)$ such that $\psi(0)$ is the trivial loop, we have*

$$\frac{d\,\mathcal{H} \circ \psi}{ds}(0) = 0,$$

for any holonomy $\mathcal{H}$. Here we denote by ψ also the composite of ψ with the natural projection $\Omega^\infty(M) \to \pi_1^1(M)$.

Proof: In a neighborhood of $s = 0$ all loops in the family are contained in one coordinate chart, so it suffices to consider the case $M = \mathbb{R}^n$ with basepoint $0 \in \mathbb{R}^n$. Using the canonical coordinates in $\mathbb{R}^n$ we can write

$$\psi(s)(t) = (\psi_1(s,t), \psi_2(s,t), \ldots, \psi_n(s,t)).$$

This leads to the smooth function $\phi\colon [0,1]^n \times [0,1] \to \mathbb{R}^n$ defined by

$$\phi(s^1, s^2, \ldots, s^n, t) = (\psi_1(s^1,t), \psi_2(s^2,t), \ldots, \psi_n(s^n,t)).$$

Note that ϕ defines a smooth family of loops. We can now write $\psi = \phi \circ \Delta$, where $\Delta\colon [0,1] \to [0,1]^n$ is the diagonal map $\Delta(s) = (s, s, \ldots, s)$. A short calculation gives

$$\frac{d\,\mathcal{H} \circ \psi}{ds}(0) = \frac{d\,\mathcal{H} \circ \phi \circ \Delta}{ds}(0) = \sum_{i=1}^{n} \frac{\partial\,\mathcal{H} \circ \phi}{\partial s^i}(0, 0, \ldots, 0) = 0.$$

The last equality is a consequence of the fact that all partial derivatives are equal to zero. Let us show this for the first partial derivative. The value of $\partial\,\mathcal{H} \circ \phi / \partial s^1 (0, 0, \ldots, 0)$ only depends on the behaviour of $\mathcal{H} \circ \phi$ on the first axis. On this axis we have $\phi(s^1, 0, \ldots, 0)(t) = (\psi_1(s^1, t), 0 \ldots, 0)$. It is not hard to see that this is thin homotopic to the trivial loop: for example, a thin homotopy is given by $H(s,t) = (\beta(s)\psi_1(s^1,t), 0, \ldots, 0)$, where β is the function defined in Sect. 3. Therefore, $\mathcal{H} \circ \phi$ is constant on the first axis and its first partial derivative is zero. $\square$

There is a subtlety to be noted here: Lemma 7.1 is only true when $\psi(0)$ is equal to the trivial loop and not when it is thin homotopic to it. This is rather important to note, because otherwise one might get the wrong impression that the connection constructed in [3, 10] and the gerbe-connection in Sect. 8 of this paper vanish identically always.

In this paper we need the following analogue of Lem. 7.1 for 2-loops and 2-holonomies (see Def. 5.5).

Lemma 7.2 *Let $\psi\colon [0,1]^2 \to \Omega_2^\infty(M)$ be any smooth 2-parameter family of 2-loops such that $\psi(0,0)$ is the trivial 2-loop. Then we have*

$$\frac{\partial^2 \mathcal{H} \circ \psi}{\partial r \partial s}(0,0) = 0,$$

for any 2-holonomy $\mathcal{H}\colon \pi_2^2(M) \to U(1)$.

Proof: The proof follows the line of reasoning in Barrett's lemma. Again it suffices to consider the case $M = \mathbb{R}^n$ with basepoint $0 \in \mathbb{R}^n$. With respect to the standard coordinates in $\mathbb{R}^n$ we can write

$$\psi(r,s)(t^1,t^2) = (\psi_1(r,s,t^1,t^2), \psi_2(r,s,t^1,t^2), \ldots, \psi_n(r,s,t^1,t^2))$$

and

$$\phi(r^1,s^1;r^2,s^2;\ldots,r^n,s^n;t^1,t^2) =$$
$$(\psi_1(r^1,s^1;t^1,t^2), \psi_2(r^2,s^2;t^1,t^2), \ldots, \psi_n(r^n,s^n;t^1,t^2)).$$

The latter defines a smooth function from $[0,1]^{2n} \times [0,1]^2$ to $\mathbb{R}^n$. Note that ϕ defines a smooth family of 2-loops. Using the diagonal function $\Delta\colon [0,1]^2 \to [0,1]^{2n}$, defined by $\Delta(r,s) = (r,s;r,s;\ldots;r,s)$, we can write $\psi = \phi \circ \Delta$. Again a short calculation gives

$$\frac{\partial^2 \mathcal{H} \circ \psi}{\partial r \partial s}(0,0) = \frac{\partial^2 \mathcal{H} \circ \phi \circ \Delta}{\partial r \partial s}(0,0) = \sum_{i,j=1}^{n} \frac{\partial^2 \mathcal{H} \circ \phi}{\partial r^i \partial s^j}(0,0;\ldots;0,0) = 0.$$

The last equation is a consequence of the fact that all second order partial derivatives are equal to zero. Let us show this for the case $i = 1, j = 2$.

In this case the value of $\partial^2 \mathcal{H} \circ \phi / \partial r^1 \partial s^2(0,0;\dots;0,0)$ only depends on the behaviour of $\mathcal{H} \circ \phi$ on the plane spanned by the axes corresponding to r^1 and s^2. In this plane we have

$$\phi(r^1, 0; 0, s^2; 0, 0; \dots; 0, 0; t^1, t^2) = (\psi_1(r^1, 0; t^1, t^2), \psi_2(0, s^2; t^1, t^2), 0, \dots, 0).$$

Now $H(s, t^1, t^2) = (\beta(s)\psi_1(r^1, 0; t^1, t^2), \beta(s)\psi_2(0, s^2; t^1, t^2), 0, \dots, 0)$ defines a thin homotopy between the latter and the trivial 2-loop, so $\mathcal{H} \circ \phi$ is constant on our plane, whence $\partial^2 \mathcal{H} \circ \phi / \partial r^1 \partial s^2(0,0) = 0$. □

Again let us note that $\psi(0,0)$ in the previous lemma has to be equal to the trivial 2-loop and not just thin homotopic to it.

8 The 1-connected case

Let M be 1-connected.

Theorem 8.1 *Given an arbitrary 2-holonomy $\mathcal{H} \colon \pi_2^2(M) \to U(1)$, there exists a gerbe $\mathcal{G}$ with gerbe-connection $\mathcal{A}$ on M such that the holonomy of $(\mathcal{G}, \mathcal{A})$ is equal to $\mathcal{H}$. This construction establishes a bijective correspondence between equivalence classes of gerbes with gerbe-connections and 2-holonomies.*

Proof: We prove this theorem in several parts. First we show how to construct $\mathcal{G}$ and the 0-connection, i.e., the transition line-bundles with connections on double intersections and the covariantly constant sections on triple intersections. After that we show how to construct the 1-connection, $\mathcal{A}^1$, in $(\mathcal{G}, \mathcal{A}^0)$. In the final part we prove the last claim in Thm. 8.1.

Part 1: Let $\mathcal{H} \colon \pi_2^2(M) \to U(1)$ be an arbitrary 2-holonomy. We assume that the covering $\{U_i, i \in J\}$ of M is such that for every $i \in J$ there is a diffeomorphism $\phi_i \colon U_i \to B(0,1) \subset \mathbb{R}^n$, where $B(0,1)$ is the unit ball in $\mathbb{R}^n$, and that for every pair $i, j \in J$ there is a diffeomorphism $\phi_{ij} \colon U_{ij} \to B(0,1) \subset \mathbb{R}^n$ as well. We denote by x_i and x_{ij} the centers of U_i and U_{ij} respectively, i.e., $x_i = \phi_i^{-1}(0)$ and $x_{ij} = \phi_{ij}^{-1}(0)$. We also assume that every point $x \in M$ is the center of some open set in the covering, which we denote by U_x when needed. Finally we assume that all n-fold intersections are contractible. Since M is 1-connected, we can choose a path from $*$, the base point in M, to x_i for every $i \in J$, and a path from $*$ to x_{ij} for every pair $i, j \in J$ such that $U_{ij} \neq \emptyset$. For every $i \in J$ we can define a canonical path, from x_i to any other point $x \in U_i$ by $\phi_i^{-1}(r_x)$, where r_x is the straight line, the segment of a ray, in

$B(0,1)$ from $\phi_i(x_i) = 0$ to $\phi_i(x)$. In particular there is a canonical path from x_i to x_{ij}, for every $i, j \in J$. Similarly we define the canonical path from x_{ij} to any point $y \in U_{ij}$. Having chosen all these paths we now have to choose some homotopies. Note that the path, denoting the chosen paths from above by arrows,

$$* \to x_i \to x_{ij} \to *$$

is homotopic to $*$, the trivial loop, because M is simply-connected, so we can choose a homotopy, P_{ij}, between them (starting at $*$). By convention P_{ji} is the analogous homotopy for the loop

$$* \to x_j \to x_{ij} \to *.$$

We are now ready to start our construction.

Choose any pair $i, j \in J$. Let ℓ be a loop in U_{ij} based at x_{ij}. Consider the loop $\phi_i \circ \ell$ in $B(0,1)$. We can now define the cone on $\phi_i \circ \ell$ in $B(0,1)$ with top vertex $0 \in B(0,1)$. This is just the homotopy between $0 \in B(0,1)$ and $\phi_i \circ \ell$ obtained by taking all rays from 0 to any point on $\phi_i \circ \ell$ together. Now take the image of this cone in U_i. We obtain an analogous cone in U_j. Next we glue one cone onto the other, which corresponds to composing one homotopy with the inverse of the other, to obtain a double cone $C_{ij}(\ell)$. Finally we recenter $C_{ij}(\ell)$ by using P_{ij} and P_{ji} to obtain a 2-loop based at $*$, which we denote by $s_{ij}(\ell)$. See Fig. 7 for a graphical explanation of our construction. Two things are immediately clear from this construction: if ℓ and ℓ' are thin homotopic, then $s_{ij}(\ell)$ and $s_{ij}(\ell')$ are thin homotopic as well. This holds true, because $\mathrm{rk}D(CH)$, the rank of the differential of the cone on a homotopy H, is at most $\mathrm{rk}DH + 1$. Another obvious observation is that $s_{ij}(\ell \star \ell') \overset{2}{\sim} s_{ij}(\ell) \star s_{ij}(\ell')$ for any two loops ℓ and ℓ'. Thus we can define a holonomy $\mathcal{H}_{ij} \colon \pi_1^1(U_{ij}) \to U(1)$ by

$$\mathcal{H}_{ij}(\ell) = \mathcal{H}(s_{ij}(\ell)).$$

By the results explained in Sect. 3 we obtain a line-bundle Λ_{ij} with connection ∇_{ij} and curvature $K(\nabla_{ij})$ on U_{ij}. By construction $\Lambda_{ji} \cong (\Lambda_{ij})^{-1}$. Note also that our assumption that intersections are contractible implies that Λ_{ij} is equivalent to the trivial line-bundle. Choose a nowhere zero section σ_{ij} in Λ_{ij}. On a triple intersection U_{ijk} the tensor product $\Lambda_{ijk} = \Lambda_{ij} \otimes \Lambda_{jk} \otimes \Lambda_{ki}$ is trivial as well and $\sigma_{ijk} = \sigma_{ij} \otimes \sigma_{jk} \otimes \sigma_{ki}$ defines a nowhere zero section. Let ∇_{ijk} denote the connection on Λ_{ijk} induced by $\nabla_{ij}, \nabla_{jk}, \nabla_{ki}$ (note that

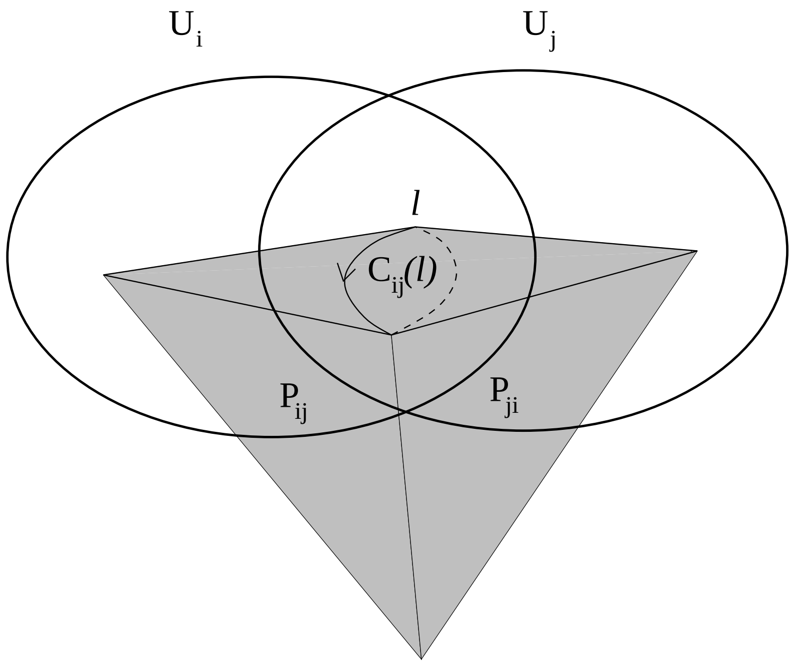

Figure 7: $s_{ij}(\ell)$

$\nabla_{ji} = -\nabla_{ij}$). By the results in Sect. 3 we know that the holonomy of $\Lambda_{ij}, \nabla_{ij}$ is exactly equal to $\mathcal{H}_{ij}$, so we conclude that the holonomy of $(\Lambda_{ijk}, \nabla_{ijk})$ around any loop in U_{ijk} is trivial, because in the construction above we go around each cone twice in opposite directions. This means that ∇_{ijk} is flat.

Let us now define the desired horizontal section θ_{ijk} in $\Gamma(U_{ijk}, \Lambda_{ijk})$. For a given point $y \in U_{ijk}$ we define in Fig. 8 a 2-loop $s_{ijk}(y)$. For example, the part of $s_{ijk}(y)$ relating to the patches U_i and U_j is defined by the composite of the homotopies P_{ij}, $\Gamma_{ij}(y)$, and the inverses of $\Gamma_{ji}(y)$ and P_{ji}, where $\Gamma_{ij}(y)$ is one of the homotopies drawn in Fig. 8. We now define the function $g_{ijk}\colon U_{ijk} \to U(1)$ by

$$g_{ijk}(y) = \mathcal{H}(s_{ijk}(y)). \tag{13}$$

By construction we have $g_{p(i)p(j)p(k)} = g_{ijk}^{\epsilon(p)}$ for any permutation $p \in S_3$, where $\epsilon(p)$ is the sign of p. It is also easy to see that the collection of functions $g = \{g_{ijk} \mid i, j, k \in J\}$ defines a Čech cocycle, i.e., $\delta g \equiv 1$. We define $\theta_{ijk} = g_{ijk}\sigma_{ijk}$. The cocycle condition satisfied by g implies that $\delta\theta \equiv 1$, because $\delta\sigma$ is isomorphic to the canonical section in the trivial line bundle by definition.

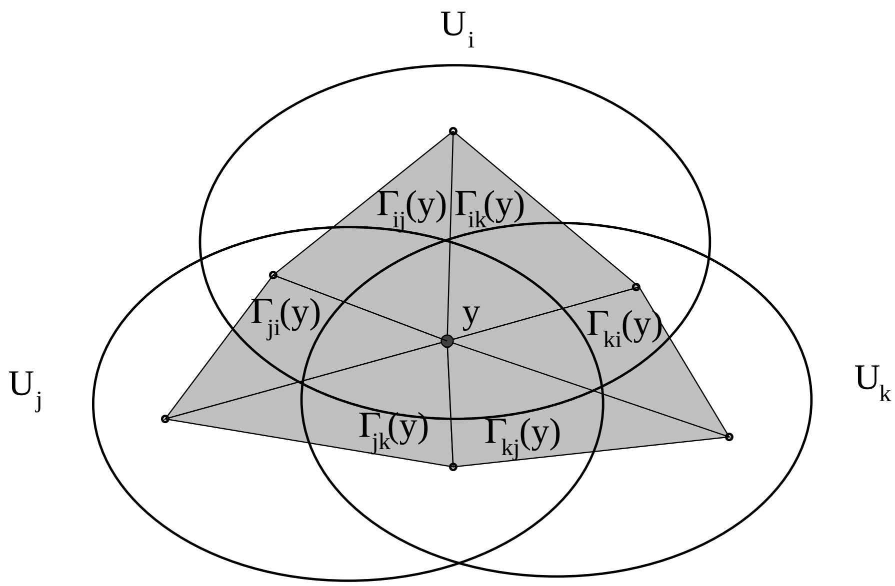

Figure 8: $s_{ijk}(y)$

We also claim that for each triple $i, j, k \in J$ the section $g_{ijk}\sigma_{ijk} \in \Gamma(U_{ijk}, \Lambda_{ijk})$ is covariantly constant with respect to ∇_{ijk}. In order to see why this holds true we first have to know what 1-form $A_{ij} \in \Omega^1(U_{ij})$ corresponds to ∇_{ij} (remember that we have chosen a section $\sigma_{ij} \in \Gamma(U_{ij}, \Lambda_{ij})$ which we can use to pull back the connection 1-form on the bundle to a 1-form on U_{ij}). The results in Sect. 3 and our construction of $(\Lambda_{ij}, \nabla_{ij})$ show that we can define A_{ij} in the following way: let v be any vector in $T_y(U_{ij})$, where $y \in U_{ij}$ is an arbitrary point. Represent v by a curve $q\colon I \to U_{ij}$, such that $q(0) = y$ and $\dot{q}(0) = v$. Let q_k be defined as in Sect. 3. We can now form the loop $\ell(k)$

$$x_{ij} \to y \xrightarrow{q_k} q_k(1) = q(k) \to x_{ij}.$$

Just as in the beginning of this section we can take the cones on $\ell(k)$ in U_i and U_j respectively and glue them together to form the 2-loop $s_{ij}(k)$ (see Fig. 9). The results in Sect. 3 now show that we have

$$A_{ij}(v) = -i\frac{d}{dk}\log \mathcal{H}(s_{ij}(k)|_{k=0}.$$

In order to show that $\nabla_{ijk}(g_{ijk}\sigma_{ijk}) = 0$ we now have to prove the equation

$$i(A_{ij} - A_{ik} + A_{jk}) = -d\log g_{ijk} = -g_{ijk}^{-1}dg_{ijk}. \tag{14}$$

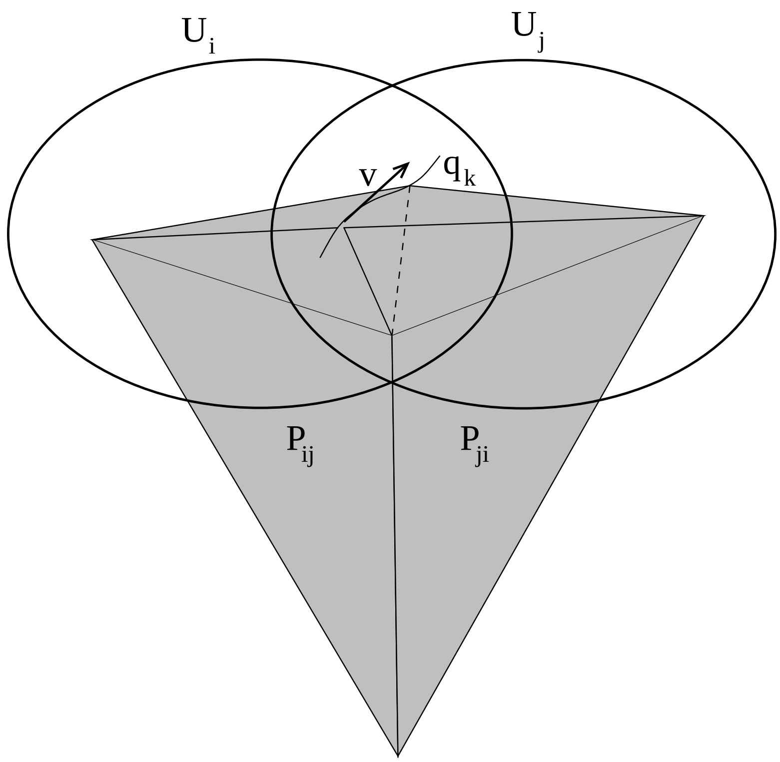

Figure 9: $s_{ij}(k)$

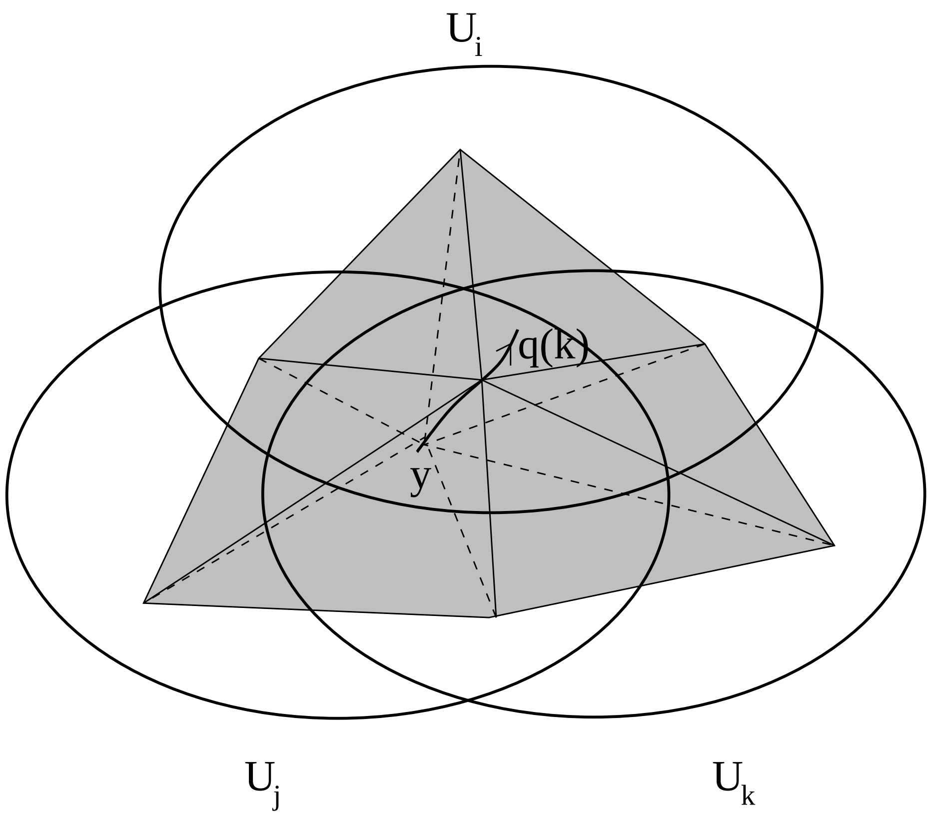

Figure 10: Proof of Eq. 14

Choose a point $y \in U_{ijk}$, a vector $v \in T_y(U_{ijk})$, and a curve $q\colon I \to U_{ijk}$ representing v. Fig. 10 shows that the 2-loops defined by $s_{ij}(k) \star s_{ik}(k)^{-1} \star s_{jk}(k)$ and $s_{ijk}(y) \star s_{ijk}(q(k))^{-1}$ are thin homotopic, so the holonomy $\mathcal{H}$ maps them to the same number. This proves the desired equation after taking derivatives of the logarithms.

Part 2: In this part we are going to define the 2-forms $F_i \in U_i$ which constitute the 1-connection on $\mathcal{G}$. Let $y \in U_i$ be an arbitrary point, and $v, w \in T_y(U_i)$ two (linearly independent) vectors. In a small neighborhood $W \subset U_i$ of y, we can choose two commuting flows $q, r\colon I \times W \to U_i$, representing v and w respectively, i.e., $q(0, y) = r(0, y) = y$, $\dot{q}(0, y) = v$, $\dot{r}(0, y) = w$, where $\dot{q}$ means the time derivative (first coordinate) of q. We say that q and r commute when $q(t_1, r(t_2, x)) = r(t_2, q(t_1, x))$ for all $t_1, t_2 \in I$ and all $x \in W$ (this corresponds exactly to saying that the two vector fields induced

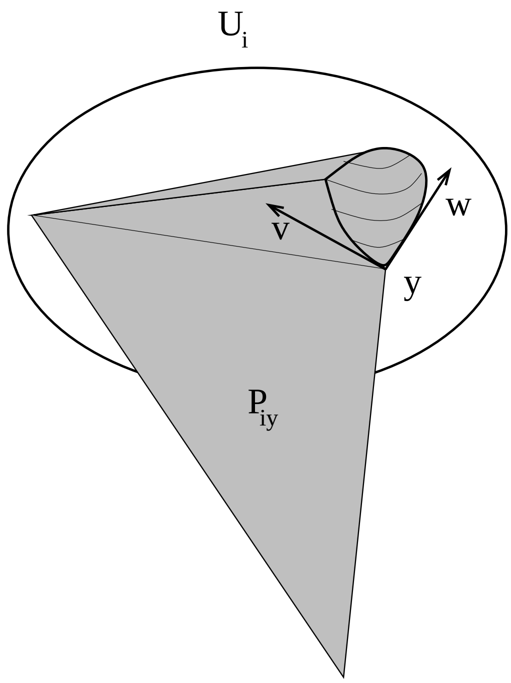

Figure 11: 1-connection

by q and r commute). Locally we can always choose such flows, because it is possible in $\mathbb{R}^n$. In particular, for any $k, l \in I$, we can define the 2-path, starting in y, by

$$r\left(\beta\left(t_2\right)l, q\left(\beta\left(t_1\right)k, y\right)\right).$$

See Sect. 3 for the definition of β. Now take the cone on the boundary of this 2-path with vertex in x_i and glue this cone on top of the 2-path in order to get a 2-loop, centered at y. As before we have to recenter this 2-loop. One of our assumptions was that every point x is the center of some open U_x, so we recenter using P_{iy} to obtain a 2-loop based at $*$ (see Fig. 11), which we denote by $s_i(k,l)$. Now define $F_i \in \Omega^2(U_i)$ by

$$F_i(v,w) = -i\frac{\partial^2}{\partial k \partial l} \log \mathcal{H}(s_i(k,l))|_{(k,l)=(0,0)}. \tag{15}$$

We have to show that $F_i(v,w)$ is well defined, i.e. independent of the choice of flows, and that the set $\{F_i \mid i \in J\}$ defines a 1-connection. Both facts are consequences of the same observations, which we explain now. Let y, v, w, q, r be as above. First of all we claim that $F_y(v,w) = 0$ for any q, r

representing v, w. Here F_y is defined using the open set U_y whose center is y. This follows from our next order version of Barrett's lemma, which is Lem. 7.2. Note that, in the notation from above, if we recenter our 2-loops so that they become based at y, the smooth 2-parameter family of 2-loops $s_y(k,l)$, depending on the parameters $k, l \in I$, satisfies the condition of Lem. 7.2 because $s_y(0,0) = y$, the constant 2-loop at y. Clearly this is true for any flows representing v and w. Next, let us have a look at Fig. 12. In this figure we show that the 2-loops $s_y(k,l) \star s_i(k,l)^{-1}$ and $s_{iy}(k,l)$ are thin homotopic. Here $s_{iy}(k,l)$ is defined analogously to $s_{iy}(k)$ (see Fig. 9) using the loop $\ell(k,l)$ around the boundary of the 2-path which we used in the definition of F_i. Therefore we have

$$\mathcal{H}(s_y(k,l))\mathcal{H}(s_i(k,l))^{-1} = \mathcal{H}(s_{iy}(k,l)). \tag{16}$$

Now, the right-hand side of this equation is exactly the holonomy of ∇_{iy} around the loop $\ell(k,l)$. Applying Stokes' theorem to the pull-back, $F_{iy} = dA_{iy} \in \Omega^2(U_{iy})$, of the curvature 2-form $K(\nabla_{iy})$ via the section σ_{iy}, and taking the second order partial derivative on the right-hand side gives

$$-i\frac{\partial^2}{\partial k \partial l} \log \mathcal{H}(s_{iy}(k,l))|_{(k,l)=(0,0)} = F_{iy}(v,w).$$

Taking also the corresponding second order partial derivative on the left-hand side of eq. 16 gives us

$$F_y(v,w) - F_i(v,w) = F_{iy}(v,w). \tag{17}$$

This equation shows two things at once. In the first place we conclude that F_i is well defined, because we have

$$F_i(v,w) = F_i(v,w) - F_y(v,w),$$

since the last term is zero, and $F_{iy}(v,w)$ does not depend on the choice of flows, because F_{iy} is an honest 2-form. Secondly eq. 17 implies that the F_i define a 1-connection in $\mathcal{G}$, because for any $j \in J$ we now get

$$\begin{aligned} F_j(v,w) - F_i(v,w) &= F_j(v,w) - F_y(v,w) + F_y(v,w) - F_i(v,w) \\ &= F_{yj}(v,w) + F_{iy}(v,w) \\ &= F_{ij}(v,w). \end{aligned}$$

The last equality follows from the fact that the F_{ij} are curvature 2-forms of the connections A_{ij} which define a 0-connection on $\mathcal{G}$, i.e., $\delta A = -id \log g$.

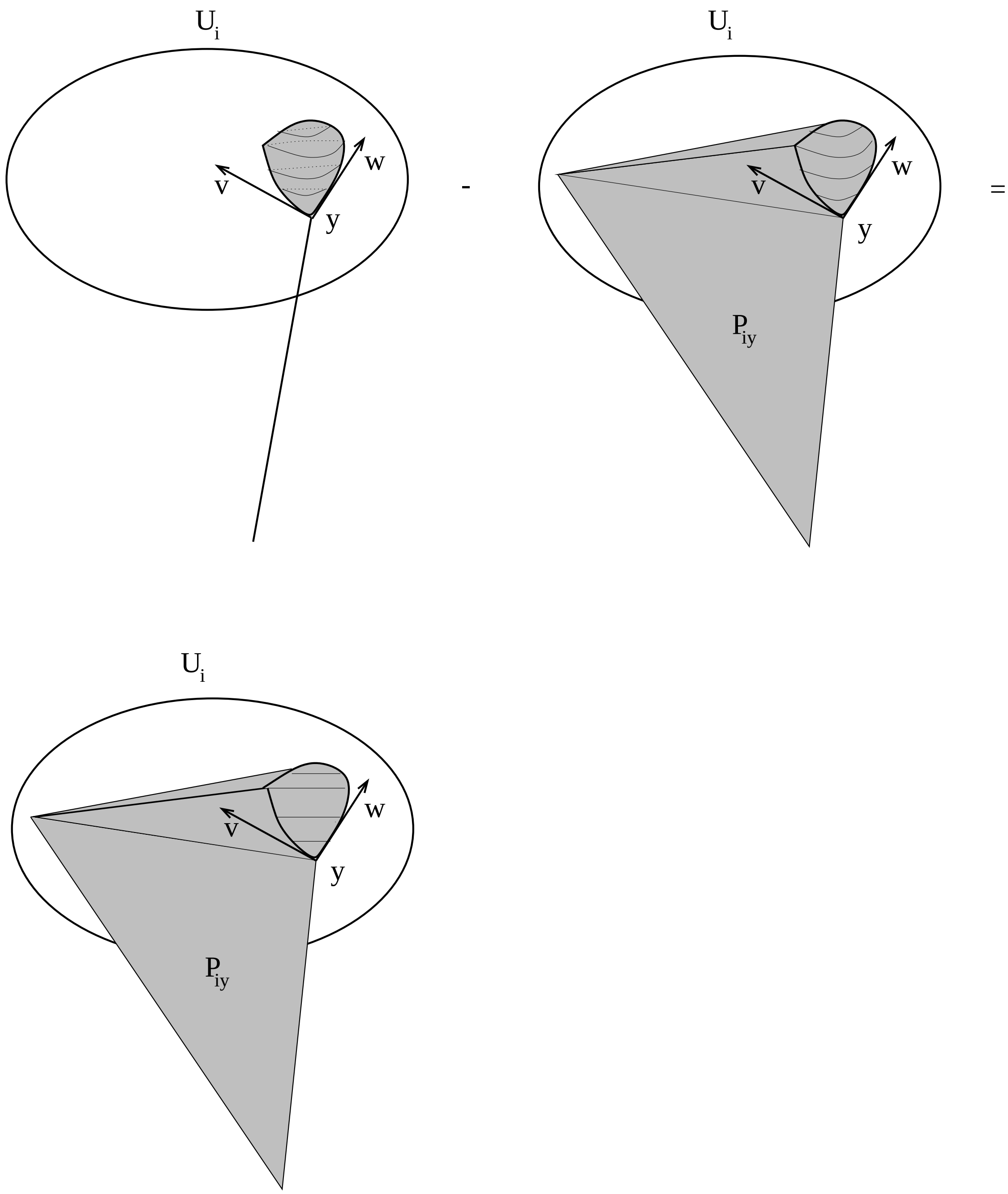

Figure 12: 1-connection is well defined

Part 3: In this final part of the proof of Thm. 8.1 we show that the construction above defines a bijection between equivalence classes of gerbes with gerbe-connections on the one hand and 2-holonomies on the other. Let $\mathcal{G}$ be a gerbe on M and $\mathcal{A}$ a gerbe-connection in $\mathcal{G}$, and let $\mathcal{H}\colon \pi_2^2(M) \to U(1)$ be the gerbe-holonomy of $\mathcal{G}, \mathcal{A}$. Using the construction above we obtain a new gerbe $\mathcal{G}'$ with a gerbe-connection $\mathcal{A}'$ from $\mathcal{H}$. Let us show that $\mathcal{G}, \mathcal{A}$ and $\mathcal{G}', \mathcal{A}'$ are equivalent. Our proof is local, so we assume that $\mathcal{G}$ (resp. $\mathcal{G}'$) is given by a cocycle g_{ijk} (resp. g'_{ijk}) and that $\mathcal{A}$ (resp. $\mathcal{A}'$) is given by A_{ij}, F_i (resp. A'_{ij}, F'_i). Let $y \in U_{ijk}$ be an arbitrary point. Recall (eq. 13) how we defined $g'_{ijk}(y) \in U(1)$:

$$g'_{ijk}(y) = \mathcal{H}(s_{ijk}(y)).$$

At the end of Sect. 5 we obtained a concrete formula for $\mathcal{H}(s)$, for any $s \in \pi_2^2(M)$. We define the function $h_{ij}\colon U_{ij} \to U(1)$ by

$$h_{ij}(y) = \exp\left(i \int_{I^2_{ij}} \epsilon_s\right) \cdot \lambda_{ij}^{-1}(y), \tag{18}$$

where I^2_{ij} is the part of I^2 which is mapped onto the part of $s_{ijk}(y)$ which goes from $*$ to U_i and U_j (see Fig. 13) and λ_{ij} is defined in Sect. 5. Likewise we define h_{ik} and h_{jk}. From formula (18) it is immediately clear that we have $h_{ji} = h_{ij}^{-1}$ and

$$g'_{ijk}(y) = g_{ijk}(y) h_{ij}(y) h_{jk}(y) h_{ki}(y).$$

This shows that $\mathcal{G}$ and $\mathcal{G}'$ are equivalent as gerbes. In order to establish the full equivalence between $\mathcal{A}$ and $\mathcal{A}'$ we also define the 1-forms $B_i \in \Omega^1(U_i)$ by

$$B_i(v) = -\frac{d}{dk}\left(\int_{C_i(k)} F_i\right)|_{k=0},$$

where $q(t)$ is a curve in U_i representing $v \in T_y(U_i)$ and $C_i(k)$ is the 2-path in U_i defined by the cone on q_k with vertex x_i (see Fig.14). A small and simple calculation in $\mathbb{R}^n$, which we omit, shows that the definition of $B_i(v)$ does not depend on the choice of $q(t)$. From our construction of A'_{ij} from $\mathcal{H}$, as explained in part 1 of this proof (see Fig. 9), we get the following equality:

$$\mathcal{H}(s) = \exp\left(i\left(\int_{q_k} A_{ij} + \int_{C_i(k)} F_i - \int_{C_j(k)} F_j\right)\right) h_{ij}(q(0)) h_{ij}^{-1}(q(k)).$$

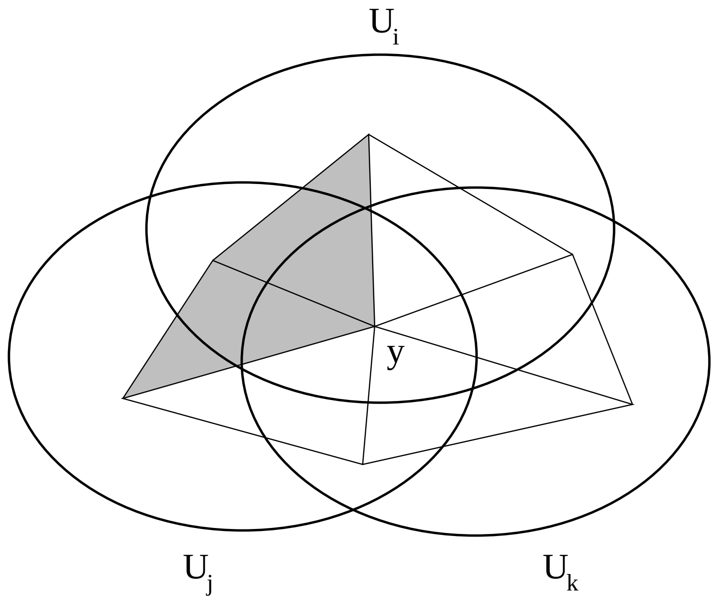

Figure 13: h_{ij}

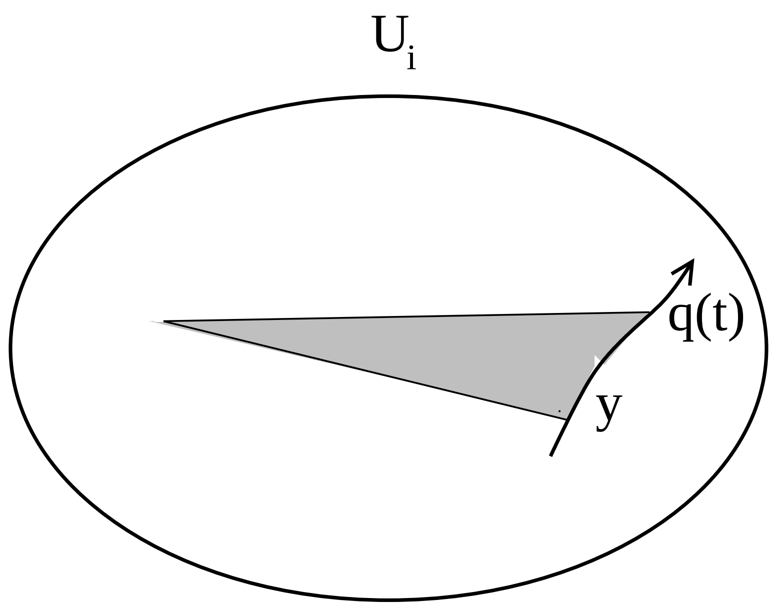

Figure 14: B_i

Here we take $s = s_{ij}(k)$, which we defined in Fig. 9. Taking the derivatives at $k = 0$ on both sides gives

$$i(A')_{ij}(v) = iA_{ij}(v) + iB_j(v) - iB_i(v) - (d\log h_{ij}(v))\,(v).$$

Finally we have to prove that $F'_i = F_i + dB_i$ for all i. Let $v, w \in T_y(U_i)$ be two arbitary vectors. Then

$$\begin{aligned}
F'_i(v,w) &= F'_i(v,w) - F'_y(v,w) \\
&= (F')_{yi}(v,w) \\
&= d(A')_{yi}(v,w) \\
&= d\left(A_{yi} + B_i - B_y + id\log h_{yi}\right)(v,w) \\
&= \left(F_{yi}(v,w) + dB_i - dB_y\right)(v,w) \\
&= \left(F_i + dB_i\right)(v,w) - \left(F_y + dB_y\right)(v,w) \\
&= \left(F_i + dB_i\right)(v,w).
\end{aligned}$$

The last equality follows from Barrett's lemma for 2-loops (Lem. 7.2), because $\left(F_y + dB_y\right)(v,w)$, according to our formula for $\mathcal{H}$ at the end of Sect. 5, is equal to

$$-i\frac{\partial^2}{\partial k\partial l}\log\mathcal{H}(s_y(k,l))|_{(k,l)=(0,0)} = 0,$$

following the notation in (15). The smooth 2-parameter family of 2-loops $s_y(k,l)$ starts at the trivial 2-loop at y after recentering, so we can indeed apply Lem. 7.2.

Conversely, let $\mathcal{H}$ be a 2-holonomy, reconstruct $\mathcal{G}, \mathcal{A}$ as above, and let $\mathcal{H}_{\mathcal{G},\mathcal{A}}$ be the gerbe-holonomy of $(\mathcal{G}, \mathcal{A})$. We show that $\mathcal{H}_{\mathcal{G},\mathcal{A}} = \mathcal{H}$. The analogous proof for line-bundles, which we gave in Sect. 3, relied on the fact that the holonomy around a loop, ℓ, can be written as the holonomies around many loops each of which only shares a part with ℓ, such that in the end all parts of the loops that do not belong to ℓ cancel out. The same idea underlies our proof for gerbe-holonomies. Let $s\colon I^2 \to M$ be a 2-loop. In Fig. 15 we have drawn a part of the image of s which is covered by four open sets, U_i, U_j, U_k, U_l, and which we denote by s_{ijkl}. The formula for $\mathcal{H}_{\mathcal{G},\mathcal{A}}(s)$ at the end of Sect. 5 shows that the part of $\mathcal{H}_{\mathcal{G},\mathcal{A}}(s)$ which corresponds to U_i, U_j, U_k, U_l is given by integration of the 2-forms F_i, F_j, F_k, F_l over that part of the image of s which intersects $U_i \cup U_j \cup U_k \cup U_l$, by integration of the 1-forms $A_{ij}, A_{jk}, A_{kl}, A_{li}$ over the edges $E_{ij}, E_{jk}, E_{kl}, E_{li}$ in the image

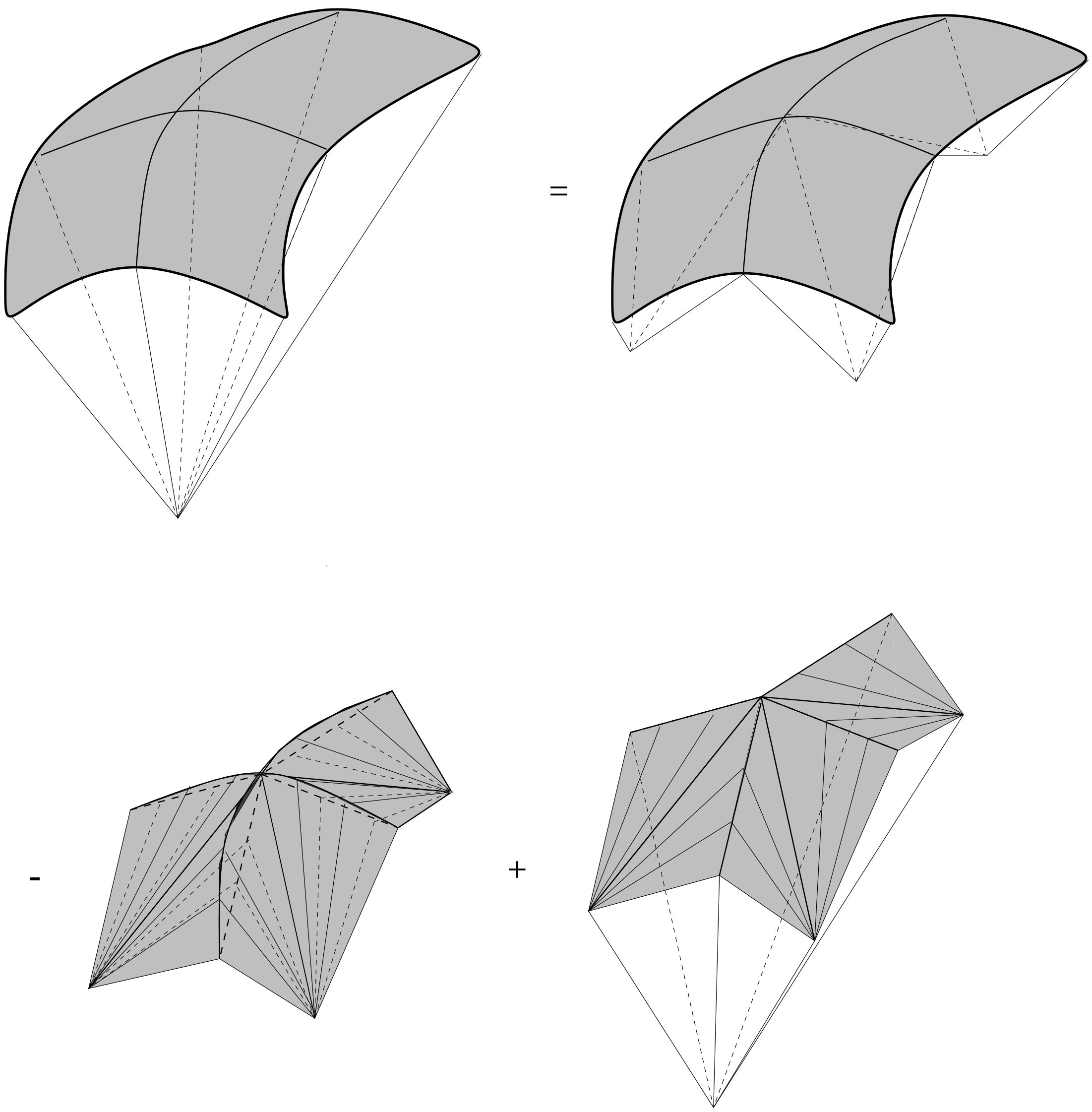

Figure 15: $\mathcal{H}_{\mathcal{G},\mathcal{A}}(s)$

of s, and finally by evaluating $g_{ijk}g_{ilk}^{-1}$ at y, a point in the image of s and in the intersection $U_i \cap U_j \cap U_k \cap U_l$. The key observation, just as in the case for line-bundles, is that F_i, A_{ij}, g_{ijk} are all defined in terms of $\mathcal{H}$. By close inspection of the definition in part 1 and 2 of this proof we find that the images of the 2-loops which define F_i, F_j, F_k, F_l contain all of s_{ijkl} but they contain more. This extra bit of 2-loop gets cancelled by the 2-loops defining $A_{ij}, A_{jk}, A_{kl}, A_{li}$ and $g_{ijk}g_{ilk}^{-1}$. In Fig. 15 the 2-loop in the first picture corresponds to s_{ijkl}. In the second picture we see the 2-loops corresponding to F_i, F_j, F_k, F_l. After composing with the inverse of the 2-loop in the third picture, which corresponds to $A_{ij}, A_{jk}, A_{kl}, A_{li}$, we are left with the 2-loop in the fourth picture which corresponds to $g_{ijk}g_{ilk}^{-1}(y)$. We see that all parts of the 2-loops above which are not part of s_{ijkl} cancel out and so we are left with s_{ijkl} only. This shows that we have $\mathcal{H}_{\mathcal{G},\mathcal{H}}(s) = \mathcal{H}(s)$. □

Remark 8.2 In Sect. 6 we recalled that a gerbe with gerbe-connection on M induces a line-bundle with connection on $\Omega(M, x)$, which actually quotients to a line-bundle on $\pi_1^1(M, x)$. We remark that in the simply-connected case there exists a different construction of that line-bundle which follows from the results in this section.

In [3, 10] the authors reconstructed a principal bundle with connection from a holonomy by a global method, i.e., they reconstructed the total space of the bundle first and showed that there is a natural lifting of paths built in which defines a connection. In the case of gerbes one could do the same for a given 2-holonomy, as we sketch in the following. Let $\mathcal{H}\colon \pi_2^2(M) \to U(1)$ be a 2-holonomy (M continues to be 1-connected). One can define the total space

$$P_2^\infty(M, *) \times U(1)/\sim .$$

Here $P_2^\infty(M, *)$ is the set of all 2-paths $s\colon I^2 \to M$ such that $s(r, 0) = s(r, 1) = * \; \forall r \in I$ and $s(0, t) = *, \; \forall t \in I$. The equivalence relation is defined by

$$(s_1, l_1) \sim (s_2, l_2) \Leftrightarrow \forall t \in I \;\, s_1(1, t) = s_2(1, t) \;\wedge\; l_2 = \mathcal{H}(s_2 \star s_1^{-1})l_1.$$

It is easy to check that this relation indeed defines an equivalence relation. The set of equivalence classes is a line-bundle on $\Omega^\infty(M, *)$, where the projection $\pi\colon P_2^\infty(M, *) \times U(1)/\sim \;\to \Omega^\infty(M, *)$ is defined by

$$\pi([(s, l)])(t) = s(1, t).$$

One can of course quotient this line-bundle to obtain one over $\pi_1^1(M,x)$, following the observations in Sect. 6. It looks likely that the whole construction carried out in [3, 10] works in this setting as well. For example, the connection would now come in the form of a lifting function of paths of loops. However, everything becomes infinite-dimensional in such an approach. To avoid that we have opted to do everything locally in M, which is very concrete although less elegant maybe.

Remark 8.3 In this remark we want to point out a relation between thin homotopy groups and hypercohomology groups that is a consequence of our results. In ordinary homotopy theory it is well known that for an $(n-1)$-connected manifold M, the Hurewicz map $\pi_n(M)_{ab} \to H_n(M)$ defines an isomorphism of groups. The results in [3, 10] show that for a connected manifold M there exists an isomorphism of groups between the group of holonomies $\{\pi_1^1(M) \to U(1)\}$ and the hypercohomology group $H^1(M, \underline{\mathbb{C}}^*_M \to \underline{A}^1_{M,\mathbb{C}})$. This isomorphism exists because both groups classify line-bundles with connections up to equivalence. In our case we see that for a 1-connected manifold M, there exists an isomorphism of groups between the group of 2-holonomies $\{\pi_2^2(M) \to U(1)\}$ and the hypercohomology group $H^2(M, \underline{\mathbb{C}}^*_M \to \underline{A}^1_{M,\mathbb{C}} \to \underline{A}^2_{M,\mathbb{C}})$ (both groups classify gerbes with gerbe-connections up to equivalence). It is likely that, in general, for an $(n-1)$-connected manifold M, the groups $\{\pi_n^n(M) \to U(1)\}$ and $H^n(M, \underline{\mathbb{C}}^*_M \to \underline{A}^1_{M,\mathbb{C}} \to \underline{A}^2_{M,\mathbb{C}} \to \cdots \to \underline{A}^n_{M,\mathbb{C}})$ are isomorphic.

Acknowledgements

The first author thanks John Barrett for helpful discussions and Ana Cannas da Silva for indicating Hitchin's paper.

This work was supported by *Programa Operacional "Ciência, Tecnologia, Inovação"* (POCTI) of the *Fundação para a Ciência e a Tecnologia* (FCT), cofinanced by the European Community fund FEDER. The first author is presently on leave from the Universidade do Algarve and working with a postdoctoral fellowship from FCT at the University of Nottingham (UK).

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