

Monodromy dependence and connection formulae for isomonodromic tau functions

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Abstract

We discuss an extension of the Jimbo-Miwa-Ueno differential 1-form to a form closed on the full space of extended monodromy data of systems of linear ordinary differential equations with rational coefficients. This extension is based on the results of M. Bertola generalizing a previous construction by B. Malgrange. We show how this 1-form can be used to solve a long-standing problem of evaluation of the connection formulae for the isomonodromic tau functions which would include an explicit computation of the relevant constant factors. We explain how this scheme works for Fuchsian systems and, in particular, calculate the connection constant for generic Painlevé VI tau function. The result proves the conjectural formula for this constant proposed in [ILT13]. We also apply the method to non-Fuchsian systems and evaluate constant factors in the asymptotics of Painlevé II tau function.

1 Introduction

Consider a system of linear ordinary differential equations with rational coefficients,

$$\frac{d\Phi}{dz} = A(z)\Phi, \quad (1.1)$$

where $A(z)$ is an $N \times N$, $N > 1$ matrix-valued rational function. We are concerned with its isomonodromy deformations. More precisely, the object of our study is the global asymptotic analysis of the associated Jimbo-Miwa-Ueno tau function. Let us remind, following [JMU], the general set-up associated with this notion.

Denote the poles of matrix function $A(z)$ on $\mathbb{P} \equiv \mathbb{P}^1(\mathbb{C})$ by a_1, \dots, a_n, ∞ and write the leading terms of the Laurent expansions of $A(z)$ at these points in the form

$$A(z) = \begin{cases} \frac{A_\nu}{(z - a_\nu)^{r_\nu+1}} + \dots & \text{as } z \rightarrow a_\nu, \\ z^{r_\infty-1} A_\infty + \dots & \text{as } z \rightarrow \infty, \end{cases}$$

where $r_1, \dots, r_n, r_\infty \in \mathbb{Z}_{\geq 0}$, and the dots stand for less singular terms. We are going to make the standard assumption that all A_ν with $\nu = 1, \dots, n, \infty$ are diagonalizable and have distinct eigenvalues, non-resonant whenever $r_\nu = 0$. Fix the diagonalizations

$$A_\nu = G_\nu \Theta_{\nu, -r_\nu} G_\nu^{-1}, \quad \Theta_{\nu, -r_\nu} = \text{diag}\{\theta_{\nu,1}, \dots, \theta_{\nu,N}\}.$$

At each singular point, the system (1.1) admits a unique *formal* solution

$$\Phi_{\text{form}}^{(\nu)}(z) = G^{(\nu)}(z) e^{\Theta_\nu(z)}, \quad \nu = 1, \dots, n, \infty, \quad (1.2)$$

where $G^{(\nu)}(z)$ is a formal series,

$$G^{(\nu)}(z) = G_\nu \hat{\Phi}^{(\nu)}(z), \quad \hat{\Phi}^{(\nu)}(z) = \begin{cases} \mathbf{1} + \sum_{k=1}^{\infty} g_{\nu,k} (z - a_\nu)^k, & \nu = 1, \dots, n, \\ \mathbf{1} + \sum_{k=1}^{\infty} g_{\infty,k} z^{-k}, & \nu = \infty, \end{cases}$$

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and $\Theta_\nu(z)$ are diagonal matrix-valued functions,

$$\Theta_\nu(z) = \sum_{k=-r_\nu}^{-1} \frac{\Theta_{\nu,k}}{k} (z - a_\nu)^k + \Theta_{\nu,0} \ln(z - a_\nu), \quad \Theta_\infty(z) = - \sum_{k=1}^{r_\infty} \frac{\Theta_{\infty,-k}}{k} z^k - \Theta_{\infty,0} \ln z.$$

For every $\nu \in \{1, \dots, n, \infty\}$, the matrix coefficients $g_{\nu,k}$ and $\Theta_{\nu,k}$ can be explicitly computed in terms of the coefficients of the matrix-valued rational function $G_\nu^{-1} A(z) G_\nu$, see [JMU].

The non-formal global properties of solutions of the equation (1.1) are described by its *monodromy data* M which include: i) formal monodromy exponents $\Theta_{\nu,0}$, ii) appropriate connection matrices between canonical solutions at different singular points, and iii) relevant Stokes matrices at irregular singular points. Let us denote the space of monodromy data of the system (1.1) by \mathcal{M} . It will be described in more detail in the main body of the paper.

Assume that irregular singular points (i.e. the points with $r_\nu > 0$) are ∞ and the first $m \leq n$ among the singular points a_1, \dots, a_n . Introduce the set \mathcal{T} of *isomonodromic times*

$$a_1, \dots, a_n, \quad (\Theta_{\nu,k})_{ll}, \quad k = -r_\nu, \dots, -1, \quad \nu = 1, \dots, m, \infty, \quad l = 1, \dots, N. \quad (1.3)$$

Let us also denote by \mathcal{A} the variety of all rational matrix-valued functions $A(z)$ with a fixed number of poles of fixed orders. The so-called Riemann-Hilbert correspondence states that, up to submanifolds where the inverse monodromy problem for (1.1) is not solvable, the space \mathcal{A} can be identified with the product $\widetilde{\mathcal{T}} \times \mathcal{M}$, where $\widetilde{\mathcal{T}}$ denotes the universal covering of \mathcal{T} . We shall loosely write,

$$\mathcal{A} \simeq \widetilde{\mathcal{T}} \times \mathcal{M}.$$

The *Jimbo-Miwa-Ueno 1-form* is defined as the following differential form on \mathcal{A} :

$$\omega_{\text{JMU}} = - \sum_{\nu=1, \dots, n, \infty} \text{res}_{z=a_\nu} \text{Tr}(\hat{\Phi}^{(\nu)}(z)^{-1} \partial_z \hat{\Phi}^{(\nu)}(z) d_{\mathcal{T}} \Theta_\nu(z)), \quad (1.4)$$

where we put $a_\infty \equiv \infty$. The notation $d_{\mathcal{T}} \Theta_\nu(z)$ stands for

$$d_{\mathcal{T}} \Theta_\nu(z) = \sum_{k=1}^L \frac{\partial \Theta_\nu(z)}{\partial t_k} dt_k, \quad L = n + N \left(\sum_{\nu=1}^m r_\nu + r_\infty \right),$$

where t_1, \dots, t_L are parameters from (1.3). The significance of this form is that, being restricted to any isomonodromic family in the space \mathcal{A} ,

$$A(z) \equiv A(z; \vec{t}; M), \quad \vec{t} = (t_1, \dots, t_L),$$

it becomes closed with respect to times \mathcal{T} , i.e.

$$d_{\mathcal{T}} \left(\omega_{\text{JMU}}|_{A(z; \vec{t}; M)} \right) = 0. \quad (1.5)$$

The isomonodromicity of the family $A(z; \vec{t}; M)$ means that all equations from it have the same set $M \in \mathcal{M}$ of monodromy data. The fact that \mathcal{T} can be taken as a complete set of independent isomonodromic deformation parameters is nontrivial, and has been proved in [JMU].

The closedness of the 1-form ω_{JMU} with respect to \mathcal{T} in turn implies that locally there is a function $\tau \equiv \tau(\vec{t}; M)$ on $\mathcal{T} \times \mathcal{M}$ such that

$$d_{\mathcal{T}} \ln \tau = \omega_{\text{JMU}}. \quad (1.6)$$

A remarkable property of this *tau function* $\tau(\vec{t}; M)$, which was established in [Mal] and [Miw], is that it admits analytic continuation as an entire function to the whole universal covering $\widetilde{\mathcal{T}}$ of the parameter space \mathcal{T} . Furthermore, zeros of $\tau(\vec{t}; M)$ correspond to the points in \mathcal{T} where the inverse monodromy problem for (1.1) is not solvable for a given set M of monodromy data (or, equivalently, where certain holomorphic vector bundle over \mathbb{P}^1 determined by M becomes nontrivial). Hence a central role of the concept of tau function in the monodromy theory of systems of linear differential equations.

The tau function has several other striking properties. Among them we shall single out the hamiltonian aspect. A key fact of the monodromy theory of linear systems is that the isomonodromic family $A(z; \vec{t}; M)$ can be

described in terms of solutions of an integrable (in the sense of Frobenius) *monodromy preserving deformation equation*,

$$d_{\mathcal{G}} A = \partial_z U + [U, A]. \quad (1.7)$$

Here U is a matrix-valued differential form, $U \equiv \sum_{k=1}^L U_k(z) dt_k$, whose coefficients $U_k(z)$ are rational matrix-valued functions of z uniquely determined by the coefficients of the system (1.1). The space \mathcal{A} can be equipped with a canonical symplectic structure so that the isomonodromy equation (1.7) induces L commuting hamiltonian flows on \mathcal{A} . It turns out that the logarithm of the tau function serves as the *generating function* of their Hamiltonians H_k :

$$\frac{\partial \ln \tau(\vec{t}; M)}{\partial t_k} = H_k|_{A(z; \vec{t}; M)}. \quad (1.8)$$

Isomonodromy equation (1.7) is of great interest on its own. Indeed, it includes as special cases practically all known integrable differential equations. The first nontrivial cases of (1.7), where the set of isomonodromic times effectively reduces to a single variable t , cover all six classical Painlevé equations. Solutions of the latter are dubbed as *nonlinear special functions*, and they indeed play this role in many areas of modern nonlinear science. Besides the canonical applications of Painlevé transcendents such as integrable systems [WMTB, JMMS, AS], two-dimensional quantum gravity [BK, DS, GrM] and random matrix theory [TW1, TW2], we would like to mention a few very recent examples concerned with black hole scattering [NC], Rabi model [CCR] and Fermi gas spectral determinants arising in supersymmetric Yang-Mills theory [BGT].

The principal analytic issue concerning the tau function, in particular from the point of view of applications, is its behavior near the critical hyperplanes, where either $a_\mu = a_\nu$ for some $\mu \neq \nu$, or $\theta_{\nu, \alpha} = \theta_{\nu, \beta}$ for some ν and some $\alpha \neq \beta$. This is the question we are addressing in this paper. We are going to study two nontrivial examples corresponding to the sixth and the second Painlevé equations. The critical hyperplanes reduce to three branching points $t = 0, 1, \infty$ in the case of Painlevé VI, and to one essential singularity $t = \infty$ in the case of Painlevé II. Our goal is to express the parameters of the asymptotic behavior of the corresponding tau functions at these critical points explicitly in terms of monodromy data of the associated linear systems (1.1).

A convenient tool of the global asymptotic analysis of Painlevé transcendents as well as solutions of an arbitrary monodromy preserving deformation equation (1.7) is provided by the Riemann-Hilbert method. It is based on the *Riemann-Hilbert representation* of solutions, i.e. on the representation of the coefficients of matrix $A(z)$ in terms of the inverse monodromy map,

$$\mathcal{R}\mathcal{H}^{-1}: \mathcal{M} \rightarrow \mathcal{A}. \quad (1.9)$$

Analytically, this map is realized as a matrix Riemann-Hilbert problem. It has been proven to be extremely efficient in the asymptotic analysis of Painlevé equations; the reader is referred to the monograph [FIKN] for detailed exposition and history of the subject. The Riemann-Hilbert technique, however, addresses directly the coefficients of the matrix $A(z)$, i.e. in the 2×2 case, it deals with conventional Painlevé functions and not the associated tau functions. In order to obtain a complete asymptotic information about the latter, one has to evaluate, according to (4.6), integrals of certain combinations of Painlevé transcendents and their derivatives. This would mean the evaluation of the tau function asymptotics including *constant factors*. More precisely, since the tau function is itself defined up to a multiplicative constant, we are actually talking about the evaluation, in terms of monodromy data, of the ratios of constant factors corresponding to different critical points (Painlevé VI) or to different critical directions (Painlevé II).

For a long time, the “constant problem” has been successfully handled only for rather special solutions of Painlevé equations whose tau functions admit additional representations in terms of certain Fredholm or Toeplitz determinants. The aim of the present paper is to develop a technique which would be applicable to general two-parameter families of Painlevé tau functions and which would not rely on determinant formulae. The key idea of our approach is to find an extension of the Jimbo-Miwa-Ueno differential form ω_{JMU} to a closed 1-form on the whole space $\mathcal{A} \simeq \widetilde{\mathcal{T}} \times \mathcal{M}$. This means a construction of a differential 1-form $\hat{\omega} \equiv \hat{\omega}(A) \equiv \hat{\omega}(\vec{t}; M)$ such that

$$d\hat{\omega} \equiv d_{\mathcal{G}} \hat{\omega} + d_{\mathcal{M}} \hat{\omega} = 0,$$

and such that the compatibility condition

$$\hat{\omega}(\partial_{t_k}) = \omega_{\text{JMU}}(\partial_{t_k})$$

is satisfied for all isomonodromic times $t_1, \dots, t_L \in \mathcal{T}$. Having such a 1-form expressed in terms of the fundamental matrix solution of (1.1), we will be able to define the tau function by the formula

$$\ln \tau = \int \hat{\omega}. \quad (1.10)$$

Equation (1.10) allows one to use the asymptotic behavior of $\Phi(z)$ to evaluate the asymptotics of the associated tau function up to a numerical (i.e. independent of monodromy data) constant. The latter can be calculated by applying the final formulae to trivial solutions of deformation equations.

The program outlined above has been first realized in [IP] for Painlevé III equation of the most degenerate type D_8 , where it allowed to give a proof of the connection formula for PIII (D_8) tau function earlier conjectured in [ILT14]. In the present paper, we use it to solve the “constant problem” for the sixth and second Painlevé equations. The key ingredient of our approach is yet another 1-form which we shall denote ω :

$$\omega = \sum_{v=1, \dots, n, \infty} \operatorname{res}_{z=a_v} \operatorname{Tr} \left(G^{(v)}(z)^{-1} A(z) dG^{(v)}(z) \right), \quad d = d_{\mathcal{T}} + d_{\mathcal{M}}. \quad (1.11)$$

Its expression is inspired by the works of Malgrange [Mal] and Bertola [Ber]. It extends the Jimbo-Miwa-Ueno form ω_{JMU} and can be defined for an arbitrary system (1.7). However, ω is *not* closed. Instead, its exterior differential turns out to be a 2-form on \mathcal{M} only and it furthermore turns out to be independent of isomonodromic times \mathcal{T} . This fact in conjunction with computable asymptotics of $\Phi(z)$ determines what should be added to the form ω to make it closed, i.e. to transform it into the form $\hat{\omega}$.

In the case of Painlevé III (D_8), a very important simplification occurs thanks to the fact that the form ω is completely localized. By this we mean that it can be expressed in terms of relevant Painlevé function and its time and monodromy derivatives, see e.g. [IP, equation (42)]. This observation relates the critical asymptotics of ω (and hence the explicit form of $\hat{\omega}$!) to the asymptotics of Painlevé transcendent, which in its turn can be computed to any desired order starting from the known leading terms and using the appropriate differential equation. We have not managed to achieve complete localization for Painlevé VI equation, and it is not clear whether it should occur in general. We are going to demonstrate that this difficulty can be overcome by a fully-fledged asymptotic analysis of the fundamental solution $\Phi(z)$.

Let us now describe the organization of the paper. The next two sections are devoted to the general Fuchsian case of the system (1.1). The main result of Section 2 is Proposition 2.3. In this proposition we perform the first step of the program: that is, we construct a 1-form ω which extends the Fuchsian Jimbo-Miwa-Ueno form to the space $\tilde{\mathcal{T}} \times \mathcal{M}$ and whose exterior differential, $\Omega := d\omega$, does not depend on the isomonodromic times (in the Fuchsian case they are just the positions of singular points a_1, \dots, a_n). The form $\hat{\omega}$ is then defined formally as $\hat{\omega} := \omega - \omega_0$, where ω_0 is a 1-form on \mathcal{M} such that its differential is again Ω . According to the scheme outlined above, the form ω_0 should be determined by analyzing the asymptotics of the forms ω and Ω . The relevant analysis is carried out in Section 3 for the case of 4-point Fuchsian systems. It is based on the Riemann-Hilbert method. In Subsection 3.1, the Riemann-Hilbert problem which represents the inverse monodromy map (1.9) for the 4-point system is formulated. In Subsections 3.2 and 3.3, its asymptotic solution is constructed in terms of solutions of the certain 3-point Fuchsian systems. The result is used in Subsection 3.4 to derive the asymptotics of the forms ω and Ω and hence determine the forms ω_0 (see Lemma 3.7) and $\hat{\omega}$ (Proposition 3.9). Solution of the constant problem for the tau function of 4-point Fuchsian systems thereby reduces to 3-point inverse monodromy problems. Explicit solution of the latter for general Fuchsian system of rank $N > 2$ is not known and therefore Proposition 3.9 is the best we could do for the generic 4-point tau function. For $N = 2$, however, the 3-point systems may be solved in terms of contour integrals, i.e. in terms of hypergeometric functions. This yields explicit solutions of the corresponding 3-point inverse monodromy problems in terms of gamma functions and an explicit solution of the constant problem for the 4-point isomonodromic tau function in terms of Barnes G -functions. Computational details are presented in Subsections 3.5–3.9. In the 4-point $N = 2$ Fuchsian case, matrix monodromy preserving deformation equation (1.7) is known to be equivalent to a single scalar nonlinear 2nd order ODE — the sixth Painlevé equation. Hence the results of these subsections provide us with the constant factor of relative normalization of the Painlevé VI tau function asymptotics at different critical points. The expression for this constant in terms of monodromy data is given in Theorem 3.29. It coincides with a conjectural formula previously suggested in [ILT13].

The fourth section of the paper is concerned with the non-Fuchsian case. We begin with a detailed description of monodromy data for non-Fuchsian systems, that is, general systems (1.1) which allow for the presence

of irregular singular points. We then introduce the form ω in this general case, and check that its exterior differential is a 2-form on \mathcal{M} independent of isomonodromic times \mathcal{T} (Theorem 4.2). We do not pursue the general case further. Instead, in Subsection 4.2, we move to a nontrivial example of a 2×2 non-Fuchsian system whose isomonodromic deformations are described by the second Painlevé equation. The evaluation of the asymptotics of the form ω , its further transformation to a closed form $\hat{\omega}$ and the evaluation of the connection constant for the corresponding Painlevé II tau function are done in Subsections 4.3 and 4.4. This time the constant in question is the ratio of constant factors corresponding to the asymptotic behaviors of the tau function $\tau(t)$ along the rays $t \rightarrow +\infty$ and $t \rightarrow -\infty$. The result is given in Theorem 4.7.

There is a very interesting additional observation related to the form ω in the Painlevé II case. Equation (4.42) indicates that 1-form ω , up to addition of an explicit total differential, is an extension to the space $\widetilde{\mathcal{T}} \times \mathcal{M}$ of the differential of classical action. The same fact has already been noticed in [IP] in the case of Painlevé III (D_8) equation. We conjecture that this relation of the extended tau function (1.10) to the classical action is a general fact of the monodromy theory of linear systems. This conjecture is closely related to another observation that can be made about ω . As follows from Remark 3.19 and the calculations in Subsection 4.2, the 2-form $\Omega = d\omega$ is nothing but (up to a numerical coefficient and restriction to symplectic leaves) the symplectic form on monodromy manifolds of Painlevé VI and II, respectively. The same fact has also been observed in the case of Painlevé III (D_8) equation in [IP] and we again conjecture it to be general. We intend to study this issue in more detail in a future work.

We would like to close this introduction with some historical remarks. Since its birth in 1980, the concept of tau function has been playing an increasingly important role in the theory of integrable systems and its numerous applications. Correlation functions of various exactly solvable quantum mechanical and statistical models are tau functions associated to special examples of the linear system (1.1). Partition functions of matrix models and 2D quantum gravity, the generating functions in the intersection theory of moduli spaces of algebraic curves are again special examples of tau functions. Yet more examples arise in the study of Hurwitz spaces and quantum cohomology. The evaluation of constant terms in the asymptotics of these correlation, distribution and generating functions has always been a great analytic challenge. The first rigorous solution of a constant problem for Painlevé equations (a special Painlevé III transcendent appearing in the Ising model) has been obtained in the work of Tracy [T91]. Other constant problems have been studied in the works [BT], [BB], [Kra], [Ehr], [DIKZ], [DIK], [DKV], [L09] and [BBDiF].

The tau functions that appear in the papers quoted above correspond to very special families of Painlevé functions. The first results concerning the general two-parameter families of solutions of Painlevé equations have been obtained only recently in [ILT13, ILT14]. These works are based on *conformal block representations* of isomonodromic tau functions — see [GIL12, GIL13, ILTe] and also [BS, Gav, GM, Nag] for subsequent developments. Although very powerful, conformal block approach is still to be put on a rigorous ground. In this paper, we show that with the help of Riemann-Hilbert technique the conjectural formula of [ILT13] for the constant factor in the asymptotics of the Painlevé VI tau function can be proven. In a sequel, we plan to understand within the Riemann-Hilbert formalism the other key results provided by conformal field theory; first of all the novel series representations for isomonodromic tau functions.

2 Fuchsian systems

Let us start by fixing the notations. We are going to consider monodromy preserving deformations of rank N Fuchsian systems with $n + 1$ regular singular points $a_1, \dots, a_n, a_{n+1} = \infty$ on \mathbb{P} :

$$\partial_z \Phi = \sum_{v=1}^n \frac{A_v}{z - a_v} \Phi, \quad A_1, \dots, A_n \in \mathfrak{sl}_N(\mathbb{C}). \quad (2.1)$$

Define $A_\infty := -\sum_{v=1}^n A_v$ and for $v = 1, \dots, n, \infty$ denote by $\theta_{v,k}$ the eigenvalues of A_v .

Assumption 2.1. *All eigenvalues satisfy a non-resonancy condition $\theta_{v,k} - \theta_{v,l} \notin \mathbb{Z}$ for $k \neq l$. It implies in particular that all A_v are diagonalizable.*

Fix the diagonalizations of A_v by introducing the matrices G_v, Θ_v such that

$$A_v = G_v \Theta_v G_v^{-1}, \quad \Theta_v = \text{diag}\{\theta_{v,1}, \dots, \theta_{v,N}\}. \quad (2.2)$$

The choice of G_ν is not unique, as there remains an ambiguity of right multiplication by a diagonal matrix. Local behavior of the fundamental matrix solution near the singular points may be written as

$$\begin{aligned}\Phi(z \rightarrow a_\nu) &= G_\nu \left[\mathbf{1} + \sum_{m=1}^{\infty} g_{\nu,m} (z - a_\nu)^m \right] (z - a_\nu)^{\Theta_\nu} C_\nu, \\ \Phi(z \rightarrow \infty) &= G_\infty \left[\mathbf{1} + \sum_{m=1}^{\infty} g_{\infty,m} z^{-m} \right] z^{-\Theta_\infty} C_\infty.\end{aligned}\tag{2.3}$$

Connection matrices C_ν are determined by the Fuchsian system, the initial conditions and the choice of diagonalisations. They also depend on the choice of branch cuts making the solution single-valued, and on the determination of fractional powers $(z - a_\nu)^{\Theta_\nu}$. The series (2.3) have non-zero radii of convergence, and their coefficients $g_{\nu,m}$ can be calculated recursively from (2.1). In particular, $g_{\nu,1}$ may be found from

$$g_{\nu,1} + [g_{\nu,1}, \Theta_\nu] = \sum_{\mu \neq \nu} \frac{G_\nu^{-1} A_\mu G_\nu}{a_\nu - a_\mu}.$$

The idea of isomonodromic deformation is to vary a_ν and A_ν (with $\nu = 1, \dots, n$) simultaneously keeping constant the local monodromy exponents Θ_ν and the connection matrices C_ν . The matrix G_∞ will also be fixed. The singularity at ∞ then plays a role of the normalization point of the fundamental matrix solution $\Phi(z)$. The product $\partial_{a_\nu} \Phi \cdot \Phi^{-1}$ is a meromorphic matrix function on \mathbb{P} with poles only possible at a_1, \dots, a_n, ∞ . Local analysis shows that

$$\partial_{a_\nu} \Phi = -\frac{A_\nu}{z - a_\nu} \Phi, \quad \nu = 1, \dots, n.\tag{2.4}$$

The compatibility of (2.1) and (2.4) yields the classical Schlesinger system of nonlinear matrix PDEs:

$$\begin{aligned}\partial_{a_\mu} A_\nu &= \frac{[A_\mu, A_\nu]}{a_\mu - a_\nu}, \quad \mu \neq \nu, \\ \partial_{a_\nu} A_\nu &= -\sum_{\mu \neq \nu}^n \frac{[A_\mu, A_\nu]}{a_\mu - a_\nu}, \quad \nu = 1, \dots, n.\end{aligned}\tag{2.5}$$

A slightly more refined problem is to describe the isomonodromic evolution of diagonalization matrices G_ν . It can be addressed using the same linear equations (2.1), (2.4). The result is

$$\begin{aligned}\partial_{a_\mu} G_\nu \cdot G_\nu^{-1} &= \frac{A_\mu}{a_\mu - a_\nu}, \quad \mu \neq \nu, \\ \partial_{a_\nu} G_\nu \cdot G_\nu^{-1} &= -\sum_{\mu \neq \nu}^n \frac{A_\mu}{a_\mu - a_\nu}, \quad \nu = 1, \dots, n.\end{aligned}\tag{2.6}$$

Definition 2.2. We denote by $\mathcal{M} = (\mathbb{C}^\times)^{(N-1)(n+1)} \times (GL_N(\mathbb{C}))^{n+1} \times GL_N(\mathbb{C})$ the space of monodromy data parameterizing local monodromy exponents Θ_ν , connection matrices C_ν and normalization matrix G_∞ . We also introduce the space of isomonodromic times $\mathcal{T} = \{(a_1, \dots, a_n) \in \mathbb{C}^n \mid a_\mu \neq a_\nu\}$ and denote by $\widetilde{\mathcal{T}}$ its universal cover.

For any point in \mathcal{M} there exists a unique invertible matrix $\Phi(z)$ holomorphic on the universal cover of $\mathbb{P} \setminus \{a_1, \dots, a_{n+1}\}$ with singular behavior (2.3) at the branch points. It in turn uniquely determines the local solution $\{A_1, \dots, A_n\}$ of the corresponding Schlesinger system. Solving this system thus amounts to constructing an inverse of the Riemann-Hilbert map

$$\mathcal{RH} : \{G_1, \dots, G_n\} \mapsto \{C_1, \dots, C_n\}$$

for given Θ_ν , G_∞ and C_∞ . Note that the solution of the Schlesinger system remains invariant under the right action $C_\nu \mapsto C_\nu H$ with $H \in GL_N(\mathbb{C})$ and $\nu = 1, \dots, n, \infty$. Gauge transformations $G_\nu \mapsto H G_\nu$ (with fixed C_∞) preserve the connection matrices C_1, \dots, C_n .

Proposition 2.3. Let $\omega \in \Lambda^1(\widetilde{\mathcal{F}} \times \mathcal{M})$ be a 1-form locally defined by

$$\omega = \sum_{\nu < \mu}^n \text{Tr} A_\mu A_\nu d \ln(a_\mu - a_\nu) + \sum_{\nu=1, \dots, n, \infty} \text{Tr}(\Theta_\nu G_\nu^{-1} d_{\mathcal{M}} G_\nu), \quad (2.7)$$

where $d_{\mathcal{M}}$ denotes the differential with respect to monodromy data. Its exterior differential $\Omega := d\omega$ is a closed 2-form on \mathcal{M} independent of a_1, \dots, a_n .

Proof. Straightforward calculation using Schlesinger equations (2.5) shows that $\Omega(\partial_{a_\mu}, \partial_{a_\nu}) = 0$. Let M be a local coordinate on \mathcal{M} . It can be deduced from (2.6) that

$$\begin{aligned} \partial_{a_\mu}(G_\nu^{-1} \partial_M G_\nu) &= \frac{G_\nu^{-1}(\partial_M A_\mu) G_\nu}{a_\mu - a_\nu}, & \mu \neq \nu, \\ \partial_{a_\nu}(G_\nu^{-1} \partial_M G_\nu) &= - \sum_{\mu \neq \nu} \frac{G_\nu^{-1}(\partial_M A_\mu) G_\nu}{a_\mu - a_\nu}, & \nu = 1, \dots, n, \end{aligned} \quad (2.8)$$

which in turn implies that $\Omega(\partial_{a_\mu}, \partial_M) = 0$. Since Ω is a total differential, it follows that $d_{\mathcal{F}} \Omega(\partial_{M_1}, \partial_{M_2})$ vanishes for any pair M_1, M_2 of monodromy parameters.

The last assertion can also be checked directly. Indeed, we have

$$\Omega(\partial_{M_2}, \partial_{M_1}) = \sum_\nu \text{Tr}(\Theta_\nu [G_\nu^{-1} \partial_{M_1} G_\nu, G_\nu^{-1} \partial_{M_2} G_\nu] + \partial_{M_2} \Theta_\nu G_\nu^{-1} \partial_{M_1} G_\nu - \partial_{M_1} \Theta_\nu G_\nu^{-1} \partial_{M_2} G_\nu). \quad (2.9)$$

The relations (2.8) may be repackaged into a more compact expression

$$d_{\mathcal{F}}(G_\nu^{-1} \partial_M G_\nu) = \sum_{\mu \neq \nu} G_\nu^{-1}(\partial_M A_\mu) G_\nu d \ln(a_\mu - a_\nu), \quad (2.10)$$

which can be used to differentiate Ω . For example:

$$\begin{aligned} & \sum_\nu \text{Tr}(\Theta_\nu [d_{\mathcal{F}}(G_\nu^{-1} \partial_{M_1} G_\nu), G_\nu^{-1} \partial_{M_2} G_\nu]) = \\ &= \sum_\nu \sum_{\mu \neq \nu} \text{Tr}(\Theta_\nu [G_\nu^{-1}(\partial_{M_1} A_\mu) G_\nu, G_\nu^{-1} \partial_{M_2} G_\nu]) d \ln(a_\mu - a_\nu) = \\ &= \sum_\nu \sum_{\mu \neq \nu} \text{Tr}(A_\nu [\partial_{M_1} A_\mu, \partial_{M_2} G_\nu \cdot G_\nu^{-1}]) d \ln(a_\mu - a_\nu) = \\ &= \sum_\nu \sum_{\mu \neq \nu} \text{Tr}(\partial_{M_1} A_\mu [\partial_{M_2} G_\nu \cdot G_\nu^{-1}, A_\nu]) d \ln(a_\mu - a_\nu) = \\ &= \sum_\nu \sum_{\mu \neq \nu} \text{Tr}(\partial_{M_1} A_\mu \cdot \partial_{M_2} A_\nu - \partial_{M_1} A_\mu \cdot G_\nu \partial_{M_2} \Theta_\nu G_\nu^{-1}) d \ln(a_\mu - a_\nu). \end{aligned}$$

Similarly,

$$d_{\mathcal{F}} \sum_\nu \text{Tr}(\partial_{M_2} \Theta_\nu G_\nu^{-1} \partial_{M_1} G_\nu) = \sum_\nu \sum_{\mu \neq \nu} \text{Tr}(\partial_{M_2} \Theta_\nu G_\nu^{-1}(\partial_{M_1} A_\mu) G_\nu) d \ln(a_\mu - a_\nu)$$

Since the sum $\sum_\nu \sum_{\mu \neq \nu} \text{Tr}(\partial_{M_1} A_\mu \cdot \partial_{M_2} A_\nu) d \ln(a_\mu - a_\nu)$ of the last two expressions is symmetric with respect to the exchange $M_1 \leftrightarrow M_2$, we finally obtain the expected result $d_{\mathcal{F}} \Omega(\partial_{M_2}, \partial_{M_1}) = 0$. \square

The first term in (2.7) is the usual definition of the Jimbo-Miwa-Ueno tau function [JMU], while the second sum incorporates its dependence on monodromy.

Definition 2.4. Let $\omega_0 \in \Lambda^1(\mathcal{M})$ be a 1-form such that $d\omega_0 = \Omega$. The extended isomonodromic tau function $\tau_{\omega_0} : \widetilde{\mathcal{F}} \times \mathcal{M} \rightarrow \mathbb{C}$ is defined by

$$d \ln \tau_{\omega_0} = \omega - \omega_0 \equiv \hat{\omega}$$

In the next sections, this construction is explicitly carried out in the case $n = 3$, $N = 2$ corresponding to Painlevé VI equation.

Remark 2.5. Left multiplication of all G_v by a matrix $H \in GL_N(\mathbb{C})$ possibly depending on monodromy parameters leads to transformation $A_v \mapsto HA_vH^{-1}$. It obviously preserves the first term in (2.7). Since

$$\sum_v \text{Tr}(G_v \Theta_v G_v^{-1} H^{-1} d_{\mathcal{M}} H) = \text{Tr}\left(\sum_v A_v H^{-1} d_{\mathcal{M}} H\right) = 0$$

due to the relation $\sum_v A_v = 0$, the second term also remains invariant. Hence the form ω is preserved by the gauge transformations.

Remark 2.6. Right $GL_N(\mathbb{C})$ -action $C_v \mapsto C_v H$ does not affect the solution of the Schlesinger system. Therefore

$$d \ln \tau_{\omega_0}^H - d \ln \tau_{\omega_0} = \omega_0 - \omega_0^H \in d(\Lambda^0(\mathcal{M})).$$

The corresponding tau functions thus necessarily coincide up to a factor depending only on monodromy data (but not on the isomonodromic times!). In other words, the tau function depends in a nontrivial way only on the conjugacy class of monodromy.

3 Four-point tau function

3.1 Riemann-Hilbert problem

It is always possible to explicitly integrate the isomonodromic flows associated to global conformal transformations. This allows to fix 3 of the singular points at 0, 1, and ∞ . The simplest nontrivial case of isomonodromy equations therefore corresponds to $n = 3$ (4 regular singularities). The position of the 4th singular point is the only remaining time variable, to be denoted by t . The Fuchsian system (2.1) then acquires the form

$$\partial_z \Phi = A(z) \Phi, \quad A(z) = \frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}, \quad (3.1)$$

and the Schlesinger system consists of two matrix ODEs

$$\frac{dA_0}{dt} = \frac{[A_t, A_0]}{t}, \quad \frac{dA_1}{dt} = \frac{[A_t, A_1]}{t-1},$$

where $A_{0,t,1}$ satisfy the constraint $A_0 + A_t + A_1 = -A_\infty$. The 1-form ω from Proposition 2.3 becomes

$$\omega = \mathcal{P} dt + \sum_{v=0,t,1,\infty} \text{Tr}(\Theta_v G_v^{-1} d_{\mathcal{M}} G_v), \quad (3.2)$$

where the time part of ω is defined by

$$\mathcal{P} = \frac{\text{Tr} A_t A_0}{t} + \frac{\text{Tr} A_t A_1}{t-1} = \frac{1}{2} \text{res}_{z=t} \text{Tr} A^2(z).$$

One of our tasks is to compute the exterior differential $\Omega = d\omega$. It was already shown to be independent of t , therefore it suffices to determine the asymptotics of ω when two singular points of the Fuchsian system (3.1) collide. We will therefore attempt to analyze the behavior of the fundamental matrix solution $\Phi(z)$ as $t \rightarrow 0$, extract from it the asymptotics of $G_{0,t,1}(t)$ and use the latter to calculate Ω .

The most convenient framework for realization of this plan is provided by the Riemann-Hilbert method. Instead of working with a Fuchsian system, $\Phi(z)$ may be related to the unique solution $\Psi(z)$ of the following Riemann-Hilbert problem (RHP):

- ♣ Given an oriented contour $\Gamma_\Psi \subset \mathbb{C}$ and a prescribed jump matrix $J_\Psi : \Gamma_\Psi \rightarrow GL_N(\mathbb{C})$, find a holomorphic matrix $\Psi : \mathbb{C} \setminus \Gamma_\Psi \rightarrow GL_N(\mathbb{C})$ such that its boundary values on Γ_Ψ satisfy $\Psi_+(z) = \Psi_-(z) J_\Psi(z)$ and $\Psi(\infty) = G_\infty$.

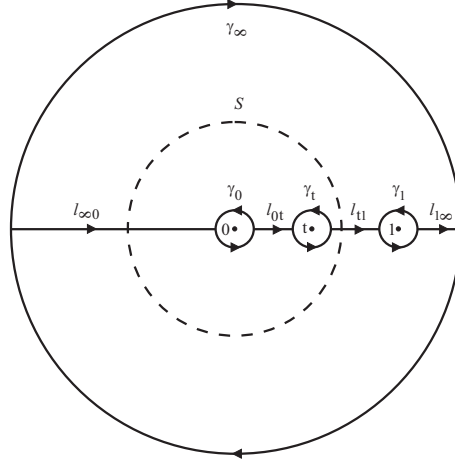


Figure 1: Contour Γ_Ψ of the Riemann-Hilbert problem for $\Psi(z)$

The contour Γ_Ψ consists of 4 circles and 4 segments⁴ represented by solid lines in Fig. 1. The jump matrix $J_\Psi(z)$ is defined as

$$\begin{aligned} J_\Psi(z)|_{\ell_{\infty 0}} &= \mathbf{1}, & J_\Psi(z)|_{\ell_{0t}} &= M_0^{-1}, & J_\Psi(z)|_{\ell_{t1}} &= (M_t M_0)^{-1}, \\ J_\Psi(z)|_{\ell_{1\infty}} &= (M_1 M_t M_0)^{-1} = M_\infty, & J_\Psi(z)|_{\gamma_\infty} &= C_\infty^{-1}(-z)^{\Theta_\infty}, \\ J_\Psi(z)|_{\gamma_0} &= C_0^{-1}(-z)^{-\Theta_0}, & J_\Psi(z)|_{\gamma_t} &= C_t^{-1}(t-z)^{-\Theta_t}, & J_\Psi(z)|_{\gamma_1} &= C_1^{-1}(1-z)^{-\Theta_1}, \end{aligned} \quad (3.3)$$

where M_ν are counterclockwise monodromies of $\Phi(z)$ around ν with basepoint chosen on the negative real axis. Fractional powers will always be understood in terms of their principal branches. Expressions for the connection matrices C_ν differ on the upper and lower halves of γ_ν . We have

$$\begin{aligned} M_0 &= C_{0,+}^{-1} e^{2\pi i \Theta_0} C_{0,-} = C_{t,+}^{-1} C_{t,-}, & M_t M_0 &= C_{t,+}^{-1} e^{2\pi i \Theta_t} C_{t,-} = C_{1,+}^{-1} C_{1,-}, \\ M_\infty^{-1} &= C_{1,+}^{-1} e^{2\pi i \Theta_1} C_{1,-} = C_{\infty,+}^{-1} e^{-2i\pi \Theta_\infty} C_{\infty,-}, & \mathbf{1} &= C_{\infty,+}^{-1} C_{\infty,-} = C_{0,+}^{-1} C_{0,-}. \end{aligned}$$

where the indices \pm correspond to $\Im z \gtrless 0$. Below the indices of this type are omitted whenever it may not lead to confusion. The solution $\Phi(z)$ of the Fuchsian system (3.1) is given by $\Psi(z)$ outside the circles γ_ν and by $\Psi(z) J_\Psi^{-1}(z)$ in their interior. We adopt the convention that the interior of the circle of largest radius (here γ_∞) is a disk around ∞ .

Assumption 3.1. *The monodromy matrix $M_{t0} := M_t M_0$ along S is assumed to be diagonalizable. Fix a matrix $C_S \in GL_N(\mathbb{C})$ such that*

$$M_{t0} = C_S^{-1} e^{2\pi i \mathfrak{S}} C_S, \quad \mathfrak{S} = \text{diag}\{\sigma_1, \dots, \sigma_N\} \in \mathfrak{sl}_N(\mathbb{C}). \quad (3.4)$$

The logarithms σ_k of the eigenvalues of M_{t0} are assumed to satisfy the conditions $|\Re(\sigma_j - \sigma_k)| < 1$ for $j, k = 1, \dots, N$. It is furthermore assumed that all σ_k are distinct.

Remark 3.2. The condition $|\Re(\sigma_j - \sigma_k)| < 1$ involves almost no loss in generality. Indeed, for any choice of logarithms satisfying $\text{Tr } \mathfrak{S} = 0$ let σ_{\max} and σ_{\min} denote the eigenvalues of \mathfrak{S} with maximal and minimal real part. If $\Re(\sigma_{\max} - \sigma_{\min}) > 1$, then replace $\sigma_{\max} \mapsto \sigma_{\max} - 1$, $\sigma_{\min} \mapsto \sigma_{\min} + 1$ and iterate the procedure. After a finite number of steps we will reach the situation where $|\Re(\sigma_j - \sigma_k)| \leq 1$ for all $j, k = 1, \dots, N$. The values with $\Re(\sigma_j - \sigma_k) = \pm 1$ are excluded to avoid some technicalities in what follows.

The RHP could also be formulated directly in terms of $\Phi(z)$. In this case the circles γ_ν are not needed, the contour Γ_Φ may be identified with the real line and the jump matrix is piecewise constant:

$$\begin{aligned} J_\Phi(z)|_{] -\infty, 0[} &= \mathbf{1}, & J_\Phi(z)|_{] 0, t[} &= M_0^{-1}, \\ J_\Phi(z)|_{] t, 1[} &= (M_{t0})^{-1}, & J_\Phi(z)|_{] 1, \infty[} &= M_\infty. \end{aligned} \quad (3.5)$$

⁴To avoid unnecessary complications it is assumed that $t \in]0, 1[$ and the segments belong to the real line.

However, extra conditions (2.3) should be imposed on the behavior of $\Phi(z)$ in the vicinity of $0, t, 1, \infty$ to ensure the uniqueness of solution.

The idea of the Riemann-Hilbert method is to factorize the original RHP into a sequence of simpler RHPs that can be solved exactly or asymptotically. We start by constructing the matrices that mimic the monodromy properties of $\Psi(z)$ inside and outside an auxiliary circle $S = \{z \in \mathbb{C} : |z| = \delta\}$ of fixed finite radius $\delta \in]t, 1[$ represented by dashed line in Fig. 1. The respective complex domains will be denoted by S_i and S_e .

3.2 Parametrics

Inside S_e , the matrix $\Psi(z)$ will be approximated by the solution $\Psi^e(z)$ of the Riemann-Hilbert problem with contour Γ_Ψ^e shown in Fig. 2a. The corresponding jump matrix $J_\Psi^e(z)$ is defined by

$$\begin{aligned} J_\Psi^e(z)|_{\gamma_1} &= C_1^{-1} (1-z)^{-\Theta_1}, & J_\Psi^e(z)|_{\gamma_\infty} &= C_\infty^{-1} (-z)^{\Theta_\infty}, & J_\Psi^e(z)|_S &= C_S^{-1} (-z)^{-\mathfrak{S}}, \\ J_\Psi^e(z)|_{\ell_{\sigma\sigma}} &= \mathbf{1}, & J_\Psi^e(z)|_{\ell_{\sigma 1}} &= (M_{t0})^{-1}, & J_\Psi^e(z)|_{\ell_{1\infty}} &= M_\infty. \end{aligned} \quad (3.6)$$

Together with the normalization $\Psi^e(0) = \mathbf{1}$, the jumps fix $\Psi^e(z)$ uniquely. Outside the circles $S, \gamma_1, \gamma_\infty$ this matrix can be expressed as $\Psi^e(z) = \Phi^e(z) C_S$ in terms of the solution $\Phi^e(z)$ of a Fuchsian system with 3 regular singular points:

$$\partial_z \Phi^e = A^e(z) \Phi^e, \quad A^e(z) = \frac{A_0^e}{z} + \frac{A_1^e}{z-1}, \quad (3.7)$$

normalized as $\Phi^e(z \rightarrow 0) \simeq (-z)^{\mathfrak{S}}$. In particular, $A_0^e = \mathfrak{S}$ and the spectra of $A_1^e, A_\infty^e := -A_0^e - A_1^e$ coincide with those of A_1 and A_∞ of the 4-point Fuchsian system (3.1). Local behavior of $\Phi^e(z)$ near the singular points $0, 1, \infty$ is given by

$$\Phi^e(z) = \begin{cases} G_0^e(z) (-z)^{\mathfrak{S}} & \text{as } z \rightarrow 0, \\ G_1^e(z) (1-z)^{\Theta_1} C_1^e & \text{as } z \rightarrow 1, \\ G_\infty^e(z) (-z)^{-\Theta_\infty} C_\infty^e & \text{as } z \rightarrow \infty, \end{cases} \quad (3.8a)$$

where

$$G_0^e(z) = \mathbf{1} + \sum_{m=1}^{\infty} g_{0,m}^e z^m, \quad G_1^e(z) = G_1^e \left(\mathbf{1} + \sum_{m=1}^{\infty} g_{1,m}^e (z-1)^m \right), \quad G_\infty^e(z) = G_\infty^e \left(\mathbf{1} + \sum_{m=1}^{\infty} g_{\infty,m}^e z^{-m} \right), \quad (3.8b)$$

connection matrices of $\Phi^e(z)$ are related to those of $\Phi(z)$ by

$$C_1^e = C_1 C_S^{-1}, \quad C_\infty^e = C_\infty C_S^{-1}, \quad (3.8c)$$

and G_1^e, G_∞^e are diagonalizing transformations for A_1^e, A_∞^e :

$$A_1^e = G_1^e \Theta_1 G_1^{e-1}, \quad A_\infty^e = G_\infty^e \Theta_\infty G_\infty^{e-1}. \quad (3.8d)$$

Let us next introduce in a similar way an approximation $\tilde{\Psi}^i(z)$ which reproduces the monodromy properties of $\Psi(z)$ inside S_i . The appropriate contour Γ_Ψ^i is represented in Fig. 2b and the jump matrix $J_\Psi^i(z)$ is

$$\begin{aligned} J_\Psi^i(z)|_{\gamma_0} &= C_0^{-1} (-z)^{-\Theta_0}, & J_\Psi^i(z)|_{\gamma_t} &= C_t^{-1} (t-z)^{-\Theta_t}, & J_\Psi^i(z)|_S &= C_S^{-1} (-z)^{-\mathfrak{S}}, \\ J_\Psi^i(z)|_{\ell_{\sigma 0}} &= \mathbf{1}, & J_\Psi^i(z)|_{\ell_{0t}} &= M_0^{-1}, & J_\Psi^i(z)|_{\ell_{t\sigma}} &= M_{t0}^{-1}. \end{aligned} \quad (3.9)$$

Outside the circles γ_0, γ_t, S , the interior parametrix can be written as $\tilde{\Psi}^i(z) = \Phi^i\left(\frac{z}{t}\right) C_S$, where $\Phi^i(z)$ is a solution of the Fuchsian system

$$\partial_z \Phi^i = A^i(z) \Phi^i, \quad A^i(z) = \frac{A_0^i}{z} + \frac{A_1^i}{z-1}, \quad (3.10)$$

with appropriate monodromy. The matrices A_0^i, A_1^i satisfy the constraint $A_\infty^i := -A_0^i - A_1^i = -\mathfrak{S}$ and their spectra coincide with those of A_0, A_t in (3.1). It is convenient to fix the normalization of $\tilde{\Psi}^i(z)$ by normalizing this

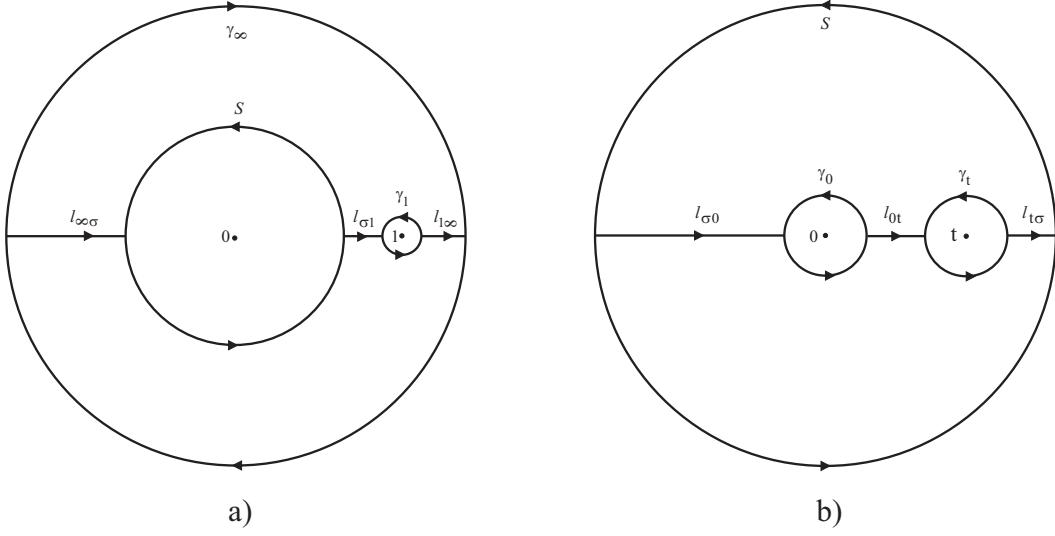


Figure 2: Contours a) Γ_{Ψ}^e and b) Γ_{Ψ}^i for exterior and interior parametrix

solution as $\Phi^i(z \rightarrow \infty) \simeq (-z)^{\mathfrak{S}}$, which amounts to setting $\tilde{\Psi}^i(\infty) = t^{-\mathfrak{S}}$. Let us also record for further reference the local expansions

$$\Phi^i(z) = \begin{cases} G_0^i(z) (-z)^{\Theta_0} C_0^i & \text{as } z \rightarrow 0, \\ G_1^i(z) (1-z)^{\Theta_t} C_1^i & \text{as } z \rightarrow 1, \\ G_{\infty}^i(z) (-z)^{\mathfrak{S}} & \text{as } z \rightarrow \infty, \end{cases} \quad (3.11a)$$

where

$$G_0^i(z) = G_0^i \left(\mathbf{1} + \sum_{m=1}^{\infty} g_{0,m}^i z^m \right), \quad G_1^i(z) = G_1^i \left(\mathbf{1} + \sum_{m=1}^{\infty} g_{1,m}^i (z-1)^m \right), \quad G_{\infty}^i(z) = \mathbf{1} + \sum_{m=1}^{\infty} g_{\infty,m}^i z^{-m}. \quad (3.11b)$$

Similarly to the above, connection matrices of $\Phi^i(z)$ are expressed in terms of connection matrices of the original Fuchsian system (3.1) as

$$C_0^i = C_0 C_S^{-1}, \quad C_1^i = C_t C_S^{-1}, \quad (3.11c)$$

whereas G_0^i, G_1^i diagonalize A_0^i, A_1^i :

$$A_0^i = G_0^i \Theta_0 G_0^{i-1}, \quad A_1^i = G_1^i \Theta_t G_1^{i-1}. \quad (3.11d)$$

In the next subsections it will be shown that the asymptotics of the 4-point isomonodromic tau function at the critical points can be derived given the inverse of the Riemann-Hilbert map for auxiliary 3-point solutions $\Phi^{i,e}(z)$. More precisely, it turns out that all one needs is an expression for the matrix coefficients $g_{0,1}^e, g_{\infty,1}^i$ in terms of monodromy data: see, for instance, Proposition 3.9.

3.3 Global approximation

Let us now consider the matrix $\Psi^S(z)$ defined by

$$\Psi^S(z) = \begin{cases} \Psi(z) \Psi^e(z)^{-1}, & z \in S_e, \\ \Psi(z) \tilde{\Psi}^i(z)^{-1}, & z \in S_i. \end{cases}$$

It is holomorphic and invertible on $\mathbb{P} \setminus S$; in particular, it has no jumps on Γ_{Ψ} . The normalization and the non-constant jump of $\Psi^S(z)$ on S are given by

$$J_S(z) = \Psi_-^e(z) \tilde{\Psi}_+^i(z)^{-1}, \quad \Psi^S(\infty) = G_{\infty} \Psi^e(\infty)^{-1}.$$

If it were possible to analytically continue $J_S(z)$ from S to its interior S_i , the Riemann-Hilbert problem for $\Psi^S(z)$ would be trivially solved by

$$\Psi^S(z) = \begin{cases} G_\infty \Psi^e(\infty)^{-1}, & z \in S_e, \\ G_\infty \Psi^e(\infty)^{-1} J_S(z), & z \in S_i. \end{cases}$$

The crucial point for our analysis is that this effectively becomes true as $t \rightarrow 0$. In the sequel, we in fact go far beyond the leading order approximation to show how one can systematically construct a perturbative solution for $\Psi^S(z)$ for small t .

First observe that on S we have

$$\begin{aligned} \tilde{\Psi}_+^i(z) &= G_\infty^i \left(\frac{z}{t}\right) \left(-\frac{z}{t}\right)^\mathfrak{S} C_S = \left[\mathbf{1} + \sum_{m=1}^{\infty} g_{\infty,m}^i \left(\frac{t}{z}\right)^m \right] \left(-\frac{z}{t}\right)^\mathfrak{S} C_S, \\ \Psi_-^e(z) &= G_0^e(z) (-z)^\mathfrak{S} C_S = \left[\mathbf{1} + \sum_{m=1}^{\infty} g_{0,m}^e z^m \right] (-z)^\mathfrak{S} C_S, \end{aligned}$$

which in turn implies that

$$J_S(z) = G_0^e(z) t^\mathfrak{S} G_\infty^i \left(\frac{z}{t}\right)^{-1}. \quad (3.12)$$

Here $G_0^e(z)$ and $G_\infty^i(z)$ are matrices that appear in the characterizations (3.8a), (3.11a) of asymptotic behavior of $\Phi^e(z)$ as $z \rightarrow 0$ and, respectively, $\Phi^i(z)$ as $z \rightarrow \infty$. We now have to factorize the jump matrix (3.12) as $J_-^S(z)^{-1} J_+^S(z)$, where $J_+^S(z)$ and $J_-^S(z)$ admit analytic continuation inside and outside S . Such factorization immediately yields a non-normalized solution of the RHP for $\Psi^S(z)$ by

$$\Psi^S(z) = \begin{cases} J_-^S(z), & z \in S_e, \\ J_+^S(z), & z \in S_i. \end{cases}$$

The factors analytic inside and outside S appear in the exact jump (3.12) in reverse order as compared to the desired one. However, the desired factorization can be calculated asymptotically to arbitrary order in small parameter t . Asymptotic factorization including the terms up to linear order $O(t)$ is described by the following lemma.

Lemma 3.3. *Under conditions on the spectrum of \mathfrak{S} specified in Assumption 3.1, one has*

$$\begin{aligned} J_-^S(z)^{-1} &= \mathbf{1} - \frac{\mathcal{E}(t)}{z} + o(t), \\ J_+^S(z) &= \left[\mathbf{1} - \mathfrak{q}(z, t) + o(t) \right] G_0^e(z) t^\mathfrak{S}. \end{aligned} \quad (3.13)$$

where

$$\mathcal{E}(t) = \varepsilon(t) \left[\mathbf{1} - g_{0,1}^e \varepsilon(t) \right]^{-1}, \quad (3.14a)$$

$$\varepsilon(t) = t^{1+\mathfrak{S}} g_{\infty,1}^i t^{-\mathfrak{S}}, \quad (3.14b)$$

$$\mathfrak{q}(z, t) = \mathfrak{G}_1(z, t) + \mathcal{E}(t) \mathfrak{G}_2(z, t), \quad (3.14c)$$

$$\mathfrak{G}_1(z, t) = \frac{G_0^e(z) \varepsilon(t) G_0^e(z)^{-1} - \varepsilon(t)}{z}, \quad (3.14d)$$

$$\mathfrak{G}_2(z, t) = \frac{\mathfrak{G}_1(z, t) - \mathfrak{G}_1(0, t)}{z}. \quad (3.14e)$$

Proof. The statement of the lemma can be checked directly. We are not going to carry out the details of this lengthy but straightforward calculation. Let us instead outline a constructive approach to derivation of the above result, which also makes clear how to calculate next order contributions.

Consider (3.13) as an ansatz for $J_\pm^S(z)$ where $\mathcal{E}(t)$ and $\mathfrak{q}(z, t)$ are to be determined. These quantities are considered as small corrections including all terms up to $o(t)$. Comparing $J_-^S(z)^{-1} J_+^S(z)$ with the exact jump matrix (3.12), we see that it suffices to solve the equation

$$\frac{\mathcal{E}(t)}{z} + \mathfrak{q}(z, t) - \frac{\mathcal{E}(t) \mathfrak{q}(z, t)}{z} = \frac{G_0^e(z) \varepsilon(t) G_0^e(z)^{-1}}{z} + o(t), \quad (3.15)$$

with $\varepsilon(t)$ defined as in (3.14b). In what follows, the contributions of order $o(t)$ will sometimes be omitted.

Under conventions of Assumption 3.1, $\varepsilon(t) = O\left(t^{1-\max_{j,k}|\Re(\sigma_j - \sigma_k)|}\right) = o(1)$ is a small matrix parameter, with respect to which we are going to expand $\mathcal{E}(t)$ and $q(z, t)$. Let us assign to any product of matrices a degree equal to the number of $\varepsilon(t)$ -factors it contains, and decompose $\mathcal{E}(t)$ and $q(z, t)$ accordingly as

$$\mathcal{E}(t) = \sum_{k=1}^{\infty} \mathcal{E}_k(t), \quad q(z, t) = \sum_{k=1}^{\infty} q_k(z, t). \quad (3.16)$$

Observe that since $\varepsilon := \max_{j,k}|\Re(\sigma_j - \sigma_k)| < 1$ and the $o(t)$ -contributions are to be neglected, the above sums can in fact be truncated to contain a finite number of terms, e.g. bounded by $(1 - \varepsilon)^{-1}$. Substituting the decompositions (3.16) into (3.15) yields recurrence relations

$$\begin{aligned} \mathcal{E}_k(t) &= \sum_{m=1}^{k-1} \mathcal{E}_m(t) q_{k-m}(0, t), \\ q_k(z, t) &= \sum_{m=1}^{k-1} \mathcal{E}_m(t) \frac{q_{k-m}(z, t) - q_{k-m}(0, t)}{z}, \end{aligned} \quad (3.17)$$

valid for $k > 1$ and subject to the initial conditions

$$\mathcal{E}_1(t) = \varepsilon(t), \quad q_1(z, t) = \mathfrak{G}_1(z, t),$$

with $\mathfrak{G}_1(z, t)$ defined by (3.14d). This in principle allows to determine all terms in the sums (3.15) up to $o(t)$ -corrections.

A crucial simplification of the resulting expressions is related to the observation that $\varepsilon^2(t) = t^2 \cdot t^{\mathfrak{G}} \left(g_{\infty,1}^i\right)^2 t^{-\mathfrak{G}} = O(t^{2-2\varepsilon}) = o(t)$. It implies that all terms containing $\varepsilon^2(t)$ can be dropped out from the sums (3.17). At the same time one has to keep e.g. expressions of the form $\varepsilon(t) A \varepsilon(t)$ with non-diagonal A , since their order can only be estimated as $O(t^{2-2\varepsilon})$. With this in mind, let us write down explicitly a few more terms in the expansion of $\mathcal{E}(t)$ and $q(z, t)$:

$$\begin{aligned} \mathcal{E}_2(t) &= \mathcal{E}_1(t) q_1(0, t) = \varepsilon(t) \mathfrak{G}_1(0, t) = \varepsilon(t) \left[g_{0,1}^e, \varepsilon(t)\right] = \varepsilon(t) g_{0,1}^e \varepsilon(t) + o(t), \\ q_2(z, t) &= \mathcal{E}_1(t) \frac{q_1(z, t) - q_1(0, t)}{z} = \varepsilon(t) \mathfrak{G}_2(z, t), \\ \mathcal{E}_3(t) &= \mathcal{E}_1(t) q_2(0, t) + \mathcal{E}_2(t) q_1(0, t) = \mathcal{E}_2(t) q_1(0, t) + o(t) = \varepsilon(t) g_{0,1}^e \varepsilon(t) g_{0,1}^e \varepsilon(t) + o(t), \\ q_3(z, t) &= \mathcal{E}_1(t) \frac{q_2(z, t) - q_2(0, t)}{z} + \mathcal{E}_2(t) \frac{q_1(z, t) - q_1(0, t)}{z} = \varepsilon(t) g_{0,1}^e \varepsilon(t) \mathfrak{G}_2(z, t) + o(t), \\ &\dots \quad \dots \quad \dots \end{aligned}$$

The general pattern now becomes manifest. It remains to employ an inductive argument to show that for $k > 1$ the only term in the sums (3.17) which is not of order $o(t)$ corresponds to $m = k - 1$. Indeed, in all other terms $q_{k-m}(z, t)$ contain $\varepsilon(t)$ as the leftmost factor and simultaneously $\mathcal{E}_m(t)$ has $\varepsilon(t)$ on the right. The formulae (3.14a) and (3.14c) for $\mathcal{E}(t)$ and $q(z, t)$ follow immediately. \square

Remark 3.4. The above formal procedure can be justified in a usual way. Put

$$\begin{aligned} J_-^{0S}(z) &:= \mathbf{1} + \frac{\mathcal{E}(t)}{z}, \\ J_+^{0S}(z) &:= \left[\mathbf{1} - q(z, t)\right] G_0^e(z) t^{\mathfrak{G}}, \end{aligned} \quad (3.18)$$

and define

$$R(z) := \begin{cases} J_-^S(z) J_-^{0S}(z)^{-1}, & z \in S_e, \\ J_+^S(z) J_+^{0S}(z)^{-1}, & z \in S_i. \end{cases}$$

Our goal is to show that the matrix function $R(z)$ satisfies the following uniform estimate:

$$R(z) = \mathbf{1} + o(t), \quad \text{as } t \rightarrow 0, \quad z \in \mathbb{C}. \quad (3.19)$$

To this end we observe that $R(z)$ satisfies the Riemann-Hilbert problem posed on S with the jump matrix

$$J_R(z) = J_-^{0S}(z) J_S(z) J_+^{0S}(z)^{-1}, \quad (3.20)$$

and it is normalized at ∞ as $R(\infty) = \mathbf{1}$. By the nature of ansatz (3.13) we have that

$$J_R(z) = \mathbf{1} + o(t), \quad (3.21)$$

as $t \rightarrow 0$ and uniformly for all $z \in S$. We also note that the estimate $\varepsilon^2(t) = o(t)$ and defining relations (3.14a) imply that

$$J_-^{0S}(z)^{-1} = \left[\mathbf{1} + \frac{\mathcal{E}(t)}{z} \right]^{-1} = \mathbf{1} - \frac{\mathcal{E}(t)}{z} + o(t).$$

The estimate (3.21) transforms into (3.19) by standard arguments involving the singular integral operator associated with the R -Riemann-Hilbert problem (see e.g. [DZ1] or [FIKN, Chapter 8, Theorem 8.1]).

Corollary 3.5. *A uniform approximation for $\Psi(z)$ as $t \rightarrow 0$ is given by*

$$\Psi(z) = \begin{cases} G_\infty \Psi^e(\infty)^{-1} \left(\mathbf{1} + \frac{\mathcal{E}(t)}{z} + o(t) \right) \Psi^e(z), & z \in S_e, \\ G_\infty \Psi^e(\infty)^{-1} \left(\mathbf{1} - q(z, t) + o(t) \right) G_0^e(z) t^{\mathfrak{S}} \tilde{\Psi}^i(z), & z \in S_i. \end{cases}$$

Equivalently, the solution of the Fuchsian system (3.1) can be approximated by

$$\Phi(z) = \begin{cases} G_\infty G_\infty^e^{-1} \left(\mathbf{1} + \frac{\mathcal{E}(t)}{z} + o(t) \right) \Phi^e(z) C_S, & z \in S_e, \\ G_\infty G_\infty^e^{-1} \left(\mathbf{1} - q(z, t) + o(t) \right) \Phi^e(z) \left(-\frac{z}{t} \right)^{-\mathfrak{S}} \Phi^i\left(\frac{z}{t}\right) C_S, & z \in S_i. \end{cases} \quad (3.22)$$

Proof. As it has already been noticed, $\mathcal{E}^2(t) = o(t)$. This in turn gives

$$J_-^S(z) = \left[\mathbf{1} - \frac{\mathcal{E}(t)}{z} + o(t) \right]^{-1} = \mathbf{1} + \frac{\mathcal{E}(t)}{z} + o(t).$$

The constant prefactors $G_\infty \Psi^e(\infty)^{-1}$, $G_\infty G_\infty^e^{-1}$ ensure required normalization of the approximate solutions of the RHP and Fuchsian system at ∞ . \square

We are now going to use this approximation to derive the $t \rightarrow 0$ asymptotics of the 1-form ω defined by (3.2). As a byproduct, we will derive the famous Jimbo asymptotic formula for Painlevé VI [Jim]. Our main concern is however to describe the dependence of the Painlevé VI tau function on monodromy data.

3.4 Asymptotics of ω and Ω

The short-distance behavior of the form ω is described by the following two lemmata.

Lemma 3.6. *The asymptotics of the time part of ω as $t \rightarrow 0$ is given by*

$$\mathcal{P} = \frac{\text{Tr}(\mathfrak{S}^2 - \Theta_0^2 - \Theta_t^2)}{2t} + \partial_t \ln \det \left(\mathbf{1} - g_{0,1}^e t^{1+\mathfrak{S}} g_{\infty,1}^i t^{-\mathfrak{S}} \right) + o(1). \quad (3.23)$$

Proof. The estimates (3.22) imply that

$$\mathcal{P} = \frac{1}{2} \text{res}_{z=t} \text{Tr}(\partial_z \Phi \cdot \Phi^{-1})^2 = \frac{1}{2} \text{res}_{z=t} \text{Tr} \left(\frac{1}{t} A^i \left(\frac{z}{t} \right) \right)^2 + \text{res}_{z=t} \text{Tr} \left(\frac{1}{t} A^i \left(\frac{z}{t} \right) \mathcal{G}^{-1} \partial_z \mathcal{G} \right), \quad (3.24)$$

with

$$\mathcal{G} := \left(\mathbf{1} - q(z, t) + o(t) \right) G_0^e(z) t^{\mathfrak{S}}.$$

The first term in (3.24) is readily computed in terms of critical exponents:

$$\mathcal{P}_{-1} := \frac{1}{2} \operatorname{res}_{z=t} \operatorname{Tr} \left(\frac{1}{t} A^i \left(\frac{z}{t} \right) \right)^2 = \frac{\operatorname{Tr} A_0^i A_1^i}{t} = \frac{\operatorname{Tr} \left((A_0^i + A_1^i)^2 - (A_0^i)^2 - (A_1^i)^2 \right)}{2t} = \frac{\operatorname{Tr} (\mathfrak{S}^2 - \Theta_0^2 - \Theta_t^2)}{2t}.$$

The second term can be transformed as

$$\begin{aligned} \mathcal{P}_0 &:= \operatorname{res}_{z=t} \operatorname{Tr} \left(\frac{1}{t} A^i \left(\frac{z}{t} \right) \mathcal{G}^{-1} \partial_z \mathcal{G} \right) = \operatorname{Tr} \left(A_1^i \mathcal{G}^{-1} \partial_z \mathcal{G} \Big|_{z=t} \right) = \\ &= \operatorname{Tr} \left(A_1^i t^{-\mathfrak{S}} \left(g_{0,1}^e - \frac{1}{\mathbf{1} - \mathfrak{q}(0,t)} \partial_z \mathfrak{q}(z,t) \Big|_{z=0} \right) t^{\mathfrak{S}} \right) + o(1). \end{aligned}$$

Recall that according to Assumption 3.1 we have $|\Re(\sigma_j - \sigma_k)| < 1$. Taking into account the expression (3.14c) for $\mathfrak{q}(z,t)$ and $\varepsilon^2(t) = o(t)$, it can be deduced that

$$\begin{aligned} t^{-\mathfrak{S}} \frac{1}{\mathbf{1} - \mathfrak{q}(0,t)} \partial_z \mathfrak{q}(z,t) \Big|_{z=0} t^{\mathfrak{S}} &= t^{-\mathfrak{S}} \frac{1}{\mathbf{1} - \mathfrak{G}_1(0,t)} \left(\partial_z \mathfrak{G}_1(0,t) + \mathcal{E}(t) \partial_z \mathfrak{G}_2(0,t) \right) t^{\mathfrak{S}} + o(t) = \\ &= t^{-\mathfrak{S}} \frac{1}{\mathbf{1} - g_{0,1}^e \varepsilon(t)} \partial_z \mathfrak{G}_1(0,t) t^{\mathfrak{S}} + o(t) = -t^{-\mathfrak{S}} \frac{1}{\mathbf{1} - g_{0,1}^e \varepsilon(t)} g_{0,1}^e \varepsilon(t) g_{0,1}^e t^{\mathfrak{S}} + o(t), \end{aligned}$$

and, consequently,

$$\mathcal{P}_0 = \operatorname{Tr} \left(A_1^i t^{-\mathfrak{S}} \frac{1}{\mathbf{1} - g_{0,1}^e \varepsilon(t)} g_{0,1}^e t^{\mathfrak{S}} \right) + o(1).$$

The quantities $g_{0,1}^e$ and $g_{\infty,1}^i$ (the latter matrix appears in the definition of $\varepsilon(t)$) can be related to the coefficients of the linear systems for Φ^e and Φ^i , namely:

$$\begin{aligned} g_{0,1}^e + [g_{0,1}^e, \mathfrak{S}] + A_1^e &= 0, \\ g_{\infty,1}^i - [g_{\infty,1}^i, \mathfrak{S}] + A_1^i &= 0. \end{aligned} \tag{3.25}$$

This allows to rewrite \mathcal{P}_0 as

$$\mathcal{P}_0 = \partial_t \operatorname{Tr} \ln(\mathbf{1} - g_{0,1}^e \varepsilon(t)) + o(1) = \partial_t \ln \det \left(\mathbf{1} - g_{0,1}^e t^{1+\mathfrak{S}} g_{\infty,1}^i t^{-\mathfrak{S}} \right) + o(1),$$

which finishes the proof. \square

Lemma 3.7. *The asymptotics of the monodromy part of ω as $t \rightarrow 0$ is given by*

$$\sum_{v=0,t,1,\infty} \operatorname{Tr} (\Theta_v G_v^{-1} d_{\mathcal{M}} G_v) = \frac{\ln t}{2} d_{\mathcal{M}} \operatorname{Tr} (\mathfrak{S}^2 - \Theta_0^2 - \Theta_t^2) + \omega_0 + o(1), \tag{3.26}$$

where $\omega_0 \in \Lambda^1(\mathcal{M})$ is a 1-form on \mathcal{M} defined as

$$\omega_0 = \operatorname{Tr} \left(\Theta_0 G_0^{i-1} d_{\mathcal{M}} G_0^i + \Theta_t G_t^{i-1} d_{\mathcal{M}} G_t^i + \Theta_1 G_1^{e-1} d_{\mathcal{M}} G_1^e + \Theta_{\infty} G_{\infty}^{e-1} d_{\mathcal{M}} G_{\infty}^e \right). \tag{3.27}$$

Proof. From the approximation (3.22) it also follows that

$$\begin{aligned} G_0 &= G_{\infty} G_{\infty}^{e-1} \left(\mathbf{1} - \mathfrak{q}(0,t) + o(t) \right) t^{\mathfrak{S}} G_0^i t^{-\Theta_0}, \\ G_t &= G_{\infty} G_{\infty}^{e-1} \left(\mathbf{1} - \mathfrak{q}(0,t) + g_{0,1}^e t + o(t) \right) t^{\mathfrak{S}} G_t^i t^{-\Theta_t}, \\ G_1 &= G_{\infty} G_{\infty}^{e-1} \left(\mathbf{1} + \mathcal{E}(t) + o(t) \right) G_1^e. \end{aligned} \tag{3.28}$$

The statement follows by straightforward computation combining the above estimates with the relations $A_0^i + A_1^i = \mathfrak{S}$ and $\mathfrak{S} + A_1^e = -A_{\infty}^e$. \square

Compatibility of Lemmata 3.6 and 3.7 is manifest. The estimate (3.26) may seem too rough to give the $O(1)$ short-distance behavior of $\Omega = d\omega$ by directly computing the differential: naively for that one would need the asymptotics of the left side up to $o(t)$. However, we already know from Proposition 2.3 that $\Omega(\partial_t, \partial_M) = 0$ for any local coordinate M on \mathcal{M} and that Ω does not depend on t . Therefore this 2-form is completely determined by the $O(1)$ asymptotics of the monodromy part of ω .

Corollary 3.8. *The 2-form Ω coincides with $d\omega_0$, where $\omega_0 \in \Lambda^1(\mathcal{M})$ is defined by (3.27).*

The results of this subsection can now be summarized as follows.

Proposition 3.9. *Given ω defined by (3.2) and ω_0 by (3.27), the difference $\hat{\omega} := \omega - \omega_0$ is a closed 1-form on $\widetilde{\mathcal{F}} \times \mathcal{M}$. Its short-distance ($t \rightarrow 0$) asymptotics is given by*

$$\hat{\omega} = d \ln \left(t^{\frac{1}{2} \text{Tr}(\mathfrak{S}^2 - \Theta_0^2 - \Theta_t^2)} \det \left(\mathbf{1} - g_{0,1}^e t^{1+\mathfrak{S}} g_{\infty,1}^i t^{-\mathfrak{S}} \right) \right) + o(1). \quad (3.29)$$

Definition 3.10. *The four-point tau function $\tau_{\omega_0} : \widetilde{\mathcal{F}} \times \mathcal{M} \rightarrow \mathbb{C}$ is defined by*

$$d \ln \tau_{\omega_0} = \omega - \omega_0. \quad (3.30)$$

This relation defines τ_{ω_0} up to a multiplicative constant independent of monodromy data, including the local monodromy exponents $\Theta_{0,t,1,\infty}$.

We have thus expressed 4-point tau function asymptotics in terms of parameters of two auxiliary 3-point Fuchsian systems with appropriate monodromy. Given the solutions $\Phi^i(z)$, $\Phi^e(z)$ of the inverse monodromy problem for these systems, Proposition 3.9 provides an explicit asymptotic expression for $d \ln \tau_{\omega_0}$. Finding 3-point solutions remains an open problem in general. However, in a number of cases the 3-point inverse monodromy problem can be solved in terms of generalized hypergeometric functions. Below we discuss the simplest situation of this type which corresponds to generic monodromy in rank $N = 2$ and leads to Gauss hypergeometric system.

3.5 Painlevé VI monodromy data

The space of essential monodromy data of the 4-point Fuchsian system (3.1) is the space of conjugacy classes of triples $(M_0, M_t, M_1) \in SL_N(\mathbb{C})^3$. Here the matrices $M_{0,t,1}$ and $M_\infty := (M_1 M_t M_0)^{-1}$ represent monodromy of $\Phi(z)$ around the singular points. The spectra of M_ν coincide with those of $e^{2\pi i \Theta_\nu}$ ($\nu = 0, t, 1, \infty$) and are considered as fixed. Define

$$\mathcal{M}_4^\Theta = \left\{ (M_0, M_t, M_1) \in SL_N(\mathbb{C})^3 : M_\nu \sim e^{2\pi i \Theta_\nu} \text{ for } \nu = 0, t, 1, \infty \right\} / SL_N(\mathbb{C}).$$

To compute the dimension of \mathcal{M}_4^Θ and introduce on it a convenient set of local coordinates, it is useful to start with a simpler case of 3 points. Having diagonalized one of the two generators of monodromy group, the second is defined up to diagonal conjugation and has fixed spectrum, which gives $(N-1)^2$ parameters. Fixing the spectrum of monodromy around the third singular point subtracts $N-1$ parameters so that

$$\dim \mathcal{M}_3^\Theta = (N-1)(N-2).$$

Note in particular that in rank 2 the conjugacy class of monodromy is completely determined by the local monodromy exponents — the corresponding Fuchsian system is rigid. In higher rank $N > 2$, monodromy data have an even number of nontrivial internal moduli.

In the case of 4 or more poles a convenient parameterization is suggested by decompositions of the punctured sphere into pairs of pants, such as the one represented in Fig. 3. It is instructive to compare this picture with Figs. 1 and 2.

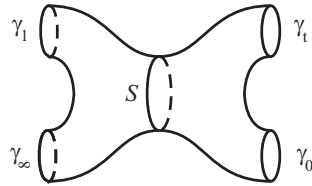


Figure 3: Pants decomposition of the 4-punctured Riemann sphere

The spectrum of the product $M_t M_0$ (monodromy along S) contains $N-1$ monodromy parameters, encoded in the matrix \mathfrak{S} . Let us fix in each of the conjugacy classes $[(M_0, M_t)]$, $[(M_1, M_\infty)]$ the representatives

$$\tilde{M}_v = C_S M_v C_S^{-1}, \quad v = 0, t, 1, \infty$$

with diagonal $\tilde{M}_t \tilde{M}_0 = (\tilde{M}_\infty \tilde{M}_1)^{-1} = e^{2\pi i \mathfrak{S}}$. This involves fixing $(N-1)(N-2)$ parameters for each pair of pants and choosing a representative in each of the two $(N-1)$ -dimensional orbits of diagonal conjugation. The possibility of simultaneous diagonal conjugation of $(\tilde{M}_0, \tilde{M}_t, \tilde{M}_1)$ reduces the latter $2(N-1)$ coordinates to $N-1$ parameters of relative twist, to be collectively denoted as \mathfrak{T} . The resulting dimension is

$$\dim \mathcal{M}_4^\Theta = \underbrace{(N-1)}_{\text{spectrum of } \mathfrak{S}} + 2 \times \underbrace{(N-1)(N-2)}_{\text{moduli of pants}} + 2 \times \underbrace{(N-1) - (N-1)}_{\text{twist parameters } \mathfrak{T}} = 2(N-1)^2.$$

Let us now make this parameterization completely explicit for rank $N = 2$, where \mathcal{M}_4^Θ is two-dimensional. The three-point moduli are absent and two local coordinates are provided by the eigenvalue of $\mathfrak{S} = \text{diag}\{\sigma, -\sigma\}$ and one twist parameter. Indeed, for $v = 0, t, 1, \infty$ denote the eigenvalues of Θ_v by $\pm\theta_v$. The knowledge of $\text{Tr } \tilde{M}_0 = 2 \cos 2\pi\theta_0$ and $\text{Tr } M_0 M_{t0}^{-1} = \text{Tr } \tilde{M}_0 e^{-2\pi i \mathfrak{S}} = 2 \cos 2\pi\theta_t$ determines the diagonal elements of \tilde{M}_0 . The unit determinant fixes the product of its off-diagonal elements.

In addition to Assumption 3.1 (implying that $|\Re\sigma| < \frac{1}{2}$ and $\sigma \neq 0$), it is convenient to impose further genericity constraints on monodromy:

Assumption 3.11. *Parameters $\theta_{0,t,1,\infty}$ and σ satisfy*

$$\theta_0 + \theta_t \pm \sigma, \theta_0 - \theta_t \pm \sigma, \theta_\infty + \theta_1 \pm \sigma, \theta_\infty - \theta_1 \pm \sigma \notin \mathbb{Z}. \quad (3.31)$$

We can then write

$$\tilde{M}_0 = \frac{1}{i \sin 2\pi\sigma} \begin{pmatrix} e^{2\pi i\sigma} \cos 2\pi\theta_0 - \cos 2\pi\theta_t & s_i [\cos 2\pi(\theta_t - \sigma) - \cos 2\pi\theta_0] \\ s_i^{-1} [\cos 2\pi\theta_0 - \cos 2\pi(\theta_t + \sigma)] & \cos 2\pi\theta_t - e^{-2\pi i\sigma} \cos 2\pi\theta_0 \end{pmatrix}. \quad (3.32a)$$

Next, from $\tilde{M}_t = e^{2\pi i \mathfrak{S}} M_0^{-1}$ it follows that

$$\tilde{M}_t = \frac{1}{i \sin 2\pi\sigma} \begin{pmatrix} e^{2\pi i\sigma} \cos 2\pi\theta_t - \cos 2\pi\theta_0 & s_i e^{2\pi i\sigma} [\cos 2\pi\theta_0 - \cos 2\pi(\theta_t - \sigma)] \\ s_i^{-1} e^{-2\pi i\sigma} [\cos 2\pi(\theta_t + \sigma) - \cos 2\pi\theta_0] & \cos 2\pi\theta_0 - e^{-2\pi i\sigma} \cos 2\pi\theta_t \end{pmatrix}. \quad (3.32b)$$

In these two expressions, s_i is a coordinate on the one-dimensional orbit of diagonal conjugation of $(\tilde{M}_0, \tilde{M}_t)$ with fixed product $\tilde{M}_t \tilde{M}_0 = e^{2\pi i \mathfrak{S}}$. In a similar fashion, we can write a parameterization of $\tilde{M}_1, \tilde{M}_\infty$: it suffices to replace in the above formulas $\theta_0 \mapsto \theta_1, \theta_t \mapsto \theta_\infty, \sigma \mapsto -\sigma$ so that

$$\tilde{M}_1 = \frac{1}{i \sin 2\pi\sigma} \begin{pmatrix} \cos 2\pi\theta_\infty - e^{-2\pi i\sigma} \cos 2\pi\theta_1 & s_e e^{2\pi i\sigma} [\cos 2\pi(\theta_1 + \sigma) - \cos 2\pi\theta_\infty] \\ s_e^{-1} e^{-2\pi i\sigma} [\cos 2\pi\theta_\infty - \cos 2\pi(\theta_1 - \sigma)] & e^{2\pi i\sigma} \cos 2\pi\theta_1 - \cos 2\pi\theta_\infty \end{pmatrix}, \quad (3.32c)$$

$$\tilde{M}_\infty = \frac{1}{i \sin 2\pi\sigma} \begin{pmatrix} \cos 2\pi\theta_1 - e^{-2\pi i\sigma} \cos 2\pi\theta_\infty & s_e [\cos 2\pi\theta_\infty - \cos 2\pi(\theta_1 + \sigma)] \\ s_e^{-1} [\cos 2\pi(\theta_1 - \sigma) - \cos 2\pi\theta_\infty] & e^{2\pi i\sigma} \cos 2\pi\theta_\infty - \cos 2\pi\theta_1 \end{pmatrix}. \quad (3.32d)$$

The possibility of simultaneous conjugation of $\tilde{M}_{0,t,1,\infty}$ by any diagonal matrix implies that the role of the twist parameter is played by the ratio

$$s_i / s_e \equiv e^{i\eta}. \quad (3.33)$$

The variables (σ, η) provide a pair of convenient local coordinates on \mathcal{M}_4^Θ which can be thought of as complexified Fenchel-Nielsen coordinates on the Teichmüller space for spheres with 4 punctures. They are closely related to monodromy parameterization in [Jim] and can be written in terms of trace functions. For instance, introduce

$$\begin{aligned} p_v &= 2 \cos 2\pi\theta_v = \text{Tr } M_v, & v &= 0, t, 1, \infty, \\ p_{\mu\nu} &= 2 \cos 2\pi\sigma_{\mu\nu} = \text{Tr } M_\mu M_\nu, & \mu\nu &= 0t, t1, 01. \end{aligned} \quad (3.34)$$

We can identify $\sigma_{0t} = \sigma$. Straightforward calculation also shows that

$$(4 - p_{0t}^2) p_{t1} = 2(p_0 p_\infty + p_t p_1) - p_{0t}(p_0 p_1 + p_t p_\infty) \quad (3.35a)$$

$$\begin{aligned}
& - \sum_{\epsilon=\pm 1} (p_\infty - 2 \cos 2\pi (\theta_1 - \epsilon\sigma))(p_0 - 2 \cos 2\pi (\theta_t - \epsilon\sigma)) e^{i\epsilon\eta}, \\
(4 - p_{0t}^2) p_{01} &= 2(p_0 p_1 + p_t p_\infty) - p_{0t} (p_0 p_\infty + p_t p_1) \\
& + \sum_{\epsilon=\pm 1} (p_\infty - 2 \cos 2\pi (\theta_1 - \epsilon\sigma))(p_0 - 2 \cos 2\pi (\theta_t - \epsilon\sigma)) e^{i\epsilon\eta - 2\pi i \epsilon\sigma}.
\end{aligned} \tag{3.35b}$$

The functions (3.34) satisfy a quartic relation

$$\begin{aligned}
p_0 p_t p_1 p_\infty + p_{0t} p_{t1} p_{01} - (p_0 p_t + p_1 p_\infty) p_{0t} - (p_t p_1 + p_0 p_\infty) p_{t1} - (p_0 p_1 + p_t p_\infty) p_{01} + \\
+ p_{0t}^2 + p_{t1}^2 + p_{01}^2 + p_0^2 + p_t^2 + p_1^2 + p_\infty^2 = 4.
\end{aligned} \tag{3.36}$$

and generate the algebra of invariant polynomial functions on \mathcal{M}_4 . There exists a canonical quadratic Poisson bracket $\{ \cdot, \cdot \}$ of geometric origin on \mathcal{M}_4 [Gol]. Trace functions p_0, p_t, p_1, p_∞ are its Casimirs and the nontrivial brackets are given by

$$\begin{aligned}
\{p_{0t}, p_{t1}\} &= 2p_{01} + p_{0t} p_{t1} - p_0 p_1 - p_t p_\infty, \\
\{p_{t1}, p_{01}\} &= 2p_{0t} + p_{t1} p_{01} - p_0 p_t - p_1 p_\infty, \\
\{p_{01}, p_{0t}\} &= 2p_{t1} + p_{0t} p_{01} - p_t p_1 - p_0 p_\infty.
\end{aligned} \tag{3.37}$$

Lemma 3.12. *The pair of local coordinates (σ, η) satisfies $\{\eta, \sigma\} = \frac{1}{2\pi}$, i.e. (σ, η) are canonical Darboux coordinates on the symplectic leaf \mathcal{M}_4^Θ with respect to the Poisson bracket (3.37).*

Proof. Direct calculation. For instance, from (3.35) it follows that

$$e^{i\eta} = \frac{2i \sin 2\pi\sigma (e^{2\pi i\sigma} p_{t1} + p_{01}) + f(\theta, \sigma)}{g(\theta, \sigma)}, \tag{3.38a}$$

where the quantities $f(\theta, \sigma), g(\theta, \sigma)$ are independent of η :

$$f(\theta, \sigma) = (p_0 p_\infty + p_t p_1) - (p_0 p_1 + p_t p_\infty) e^{2\pi i\sigma}, \tag{3.38b}$$

$$g(\theta, \sigma) = (p_\infty - 2 \cos 2\pi (\theta_1 - \sigma))(p_0 - 2 \cos 2\pi (\theta_t - \sigma)). \tag{3.38c}$$

But then (3.37) implies that

$$\{p_{0t}, e^{i\eta}\} = \frac{2i \sin 2\pi\sigma}{g(\theta, \sigma)} \{p_{0t}, e^{2\pi i\sigma} p_{t1} + p_{01}\} = 2i \sin 2\pi\sigma e^{i\eta}.$$

which is equivalent to the statement of the lemma. \square

3.6 Inverse monodromy problem in rank 2

We are now going to find explicit form of the parametrices $\Phi^e(z), \Phi^i(z)$ in the case $N = 2$. Recall that $\Phi^e(z)$ is the solution of the Fuchsian system (3.7) with the following properties:

- its monodromy is described by (3.8); in particular, the connection matrices at $1, \infty$ are given by $C_1^e = C_1 C_5^{-1}, C_\infty^e = C_\infty C_5^{-1}$.
- the normalization is fixed by $\Phi^e(z \rightarrow 0) \simeq (-z)^\mathfrak{S}$;
- in (3.7), $A_0^e = \mathfrak{S} = \text{diag}\{\sigma, -\sigma\}$; the eigenvalues of $A_1^e, A_\infty^e = -A_0^e - A_1^e$ are $\pm\theta_1$ and $\pm\theta_\infty$, respectively.

The known spectrum of $A_{1,\infty}^e \in \mathfrak{sl}_2(\mathbb{C})$ determines these matrices almost completely. Indeed, computing the trace

$$\text{Tr} A_0^e A_1^e = \frac{1}{2} \text{Tr} \left((A_\infty^e)^2 - (A_0^e)^2 - (A_1^e)^2 \right) = \theta_\infty^2 - \theta_1^2 - \sigma^2,$$

one can find the diagonal elements of A_1^e . The known determinant $\det A_1^e = -\theta_1^2$ then gives the product of the off-diagonal elements. As a result of this calculation, one finds

$$A_1^e = \frac{1}{2\sigma} \begin{pmatrix} \theta_\infty^2 - \theta_1^2 - \sigma^2 & r_e ((\theta_1 + \sigma)^2 - \theta_\infty^2) \\ r_e^{-1} (\theta_\infty^2 - (\theta_1 - \sigma)^2) & \sigma^2 + \theta_1^2 - \theta_\infty^2 \end{pmatrix}, \tag{3.39}$$

where $r_e \in \mathbb{C}^\times$ is the only remaining unknown parameter related to the freedom of simultaneous conjugation of $A_{0,1,\infty}^e$ by a non-degenerate diagonal matrix. Let us write the general form of diagonalizing transformations for $A_1^e = G_1^e \Theta_1 G_1^{e-1}$ and $A_\infty^e = G_\infty^e \Theta_\infty G_\infty^{e-1}$ as

$$G_1^e = \begin{pmatrix} r_e & (\theta_1 + \sigma)^2 - \theta_\infty^2 \\ 1 & r_e^{-1}((\theta_1 - \sigma)^2 - \theta_\infty^2) \end{pmatrix} \begin{pmatrix} c_{1a}^{e-1} & 0 \\ 0 & c_{1b}^e \end{pmatrix}, \quad (3.40a)$$

$$G_\infty^e = \begin{pmatrix} r_e(\theta_1 - \theta_\infty + \sigma) & \theta_1 + \theta_\infty + \sigma \\ \theta_1 - \theta_\infty - \sigma & r_e^{-1}(\theta_1 + \theta_\infty - \sigma) \end{pmatrix} \begin{pmatrix} c_{\infty a}^{e-1} & 0 \\ 0 & c_{\infty b}^e \end{pmatrix}. \quad (3.40b)$$

Our main remaining task is to relate r_e in (3.39)–(3.40) to monodromy parameter s_e in (3.32c), (3.32d). This can be done using the explicit solution for $\Phi^e(z)$.

Lemma 3.13. *Let $\Phi^e(z)$ denote the solution of (3.7) normalized as $\Phi^e(z \rightarrow 0) \simeq (-z)^\mathfrak{S}$, with $A_0^e = \text{diag}\{\sigma, -\sigma\}$ and A_1^e parameterized as in (3.39). Then*

$$\Phi_{11}^e(z) = (-z)^\sigma (1-z)^{\theta_1} {}_2F_1(\theta_1 + \theta_\infty + \sigma, \theta_1 - \theta_\infty + \sigma; 2\sigma; z), \quad (3.41a)$$

$$\Phi_{12}^e(z) = r_e \frac{\theta_\infty^2 - (\theta_1 + \sigma)^2}{2\sigma(2\sigma - 1)} (-z)^{1-\sigma} (1-z)^{\theta_1} {}_2F_1(1 + \theta_1 + \theta_\infty - \sigma, 1 + \theta_1 - \theta_\infty - \sigma; 2 - 2\sigma; z), \quad (3.41b)$$

$$\Phi_{21}^e(z) = r_e^{-1} \frac{\theta_\infty^2 - (\theta_1 - \sigma)^2}{2\sigma(2\sigma + 1)} (-z)^{1+\sigma} (1-z)^{\theta_1} {}_2F_1(1 + \theta_1 + \theta_\infty + \sigma, 1 + \theta_1 - \theta_\infty + \sigma; 2 + 2\sigma; z), \quad (3.41c)$$

$$\Phi_{22}^e(z) = (-z)^{-\sigma} (1-z)^{\theta_1} {}_2F_1(\theta_1 + \theta_\infty - \sigma, \theta_1 - \theta_\infty - \sigma; -2\sigma; z), \quad (3.41d)$$

where ${}_2F_1(\alpha, \beta; \gamma; z)$ denotes the Gauss hypergeometric function.

Corollary 3.14. *The relation between s_e and r_e is as follows:*

$$s_e = \frac{\Gamma(1-2\sigma)\Gamma(1+\theta_\infty+\theta_1+\sigma)\Gamma(1-\theta_\infty+\theta_1+\sigma)}{\Gamma(1+2\sigma)\Gamma(1+\theta_\infty+\theta_1-\sigma)\Gamma(1-\theta_\infty+\theta_1-\sigma)} r_e. \quad (3.42)$$

Proof. As $z \rightarrow -\infty$, it becomes convenient to replace (3.41) by an equivalent representation

$$\Phi^e(z) = G_\infty^e \hat{\Phi}^e(z) C_\infty^e, \quad (3.43)$$

where G_∞^e is defined by (3.40b) and

$$\hat{\Phi}_{11}^e(z) = (-z)^{-\theta_\infty} (1-z^{-1})^{\theta_1} {}_2F_1(\theta_1 + \theta_\infty + \sigma, \theta_1 + \theta_\infty - \sigma; 2\theta_\infty; z^{-1}), \quad (3.44a)$$

$$\hat{\Phi}_{12}^e(z) = \frac{c_{\infty a}^e}{r_e c_{\infty b}^e} \frac{(\theta_1 + \theta_\infty)^2 - \sigma^2}{2\theta_\infty(2\theta_\infty - 1)} (-z)^{\theta_\infty - 1} (1-z^{-1})^{\theta_1} {}_2F_1(1 + \theta_1 - \theta_\infty + \sigma, 1 + \theta_1 - \theta_\infty - \sigma; 2 - 2\theta_\infty; z^{-1}), \quad (3.44b)$$

$$\hat{\Phi}_{21}^e(z) = \frac{r_e c_{\infty b}^e}{c_{\infty a}^e} \frac{(\theta_1 - \theta_\infty)^2 - \sigma^2}{2\theta_\infty(2\theta_\infty + 1)} (-z)^{-\theta_\infty - 1} (1-z^{-1})^{\theta_1} {}_2F_1(1 + \theta_1 + \theta_\infty + \sigma, 1 + \theta_1 + \theta_\infty - \sigma; 2 + 2\theta_\infty; z^{-1}), \quad (3.44c)$$

$$\hat{\Phi}_{22}^e(z) = (-z)^{\theta_\infty} (1-z^{-1})^{\theta_1} {}_2F_1(\theta_1 - \theta_\infty + \sigma, \theta_1 - \theta_\infty - \sigma; -2\theta_\infty; z^{-1}). \quad (3.44d)$$

The connection matrix C_∞^e can be derived from the standard connection formulas for hypergeometric functions. Explicitly, its expression reads

$$C_\infty^e = \begin{pmatrix} c_{\infty a}^e & 0 \\ 0 & c_{\infty b}^e \end{pmatrix} \begin{pmatrix} \frac{r_e^{-1}\Gamma(-2\theta_\infty)\Gamma(2\sigma)}{\Gamma(-\theta_1 - \theta_\infty + \sigma)\Gamma(1 + \theta_1 - \theta_\infty + \sigma)} & \frac{\Gamma(-2\theta_\infty)\Gamma(-2\sigma)}{\Gamma(-\theta_1 - \theta_\infty - \sigma)\Gamma(1 + \theta_1 - \theta_\infty - \sigma)} \\ \frac{\Gamma(2\theta_\infty)\Gamma(2\sigma)}{\Gamma(-\theta_1 + \theta_\infty + \sigma)\Gamma(1 + \theta_1 + \theta_\infty + \sigma)} & \frac{r_e\Gamma(2\theta_\infty)\Gamma(-2\sigma)}{\Gamma(-\theta_1 + \theta_\infty - \sigma)\Gamma(1 + \theta_1 + \theta_\infty - \sigma)} \end{pmatrix}. \quad (3.45)$$

Now it suffices to compute the monodromy matrix $\tilde{M}_\infty = C_\infty^{e-1} e^{2\pi i \Theta_\infty} C_\infty^e$. Comparing the result with (3.32d), we deduce the identification (3.42). \square

Remark 3.15. One can similarly derive an expression for the connection matrix C_1^e associated to the choice (3.40a) of diagonalizing matrix G_1^e . In contrast to C_0^e and C_∞^e , this matrix is given by different expressions for $\Im z \geq 0$, which are related by $C_{1,+}^e = C_{1,-}^e e^{-2\pi i \Im}$. In fact we have

$$C_{1,\pm}^e = \begin{pmatrix} c_{1a}^e & 0 \\ 0 & c_{1b}^e \end{pmatrix} \left(\begin{array}{cc} \frac{r_e^{-1} \Gamma(2\sigma) \Gamma(-2\theta_1)}{\Gamma(-\theta_1 - \theta_\infty + \sigma) \Gamma(-\theta_1 + \theta_\infty + \sigma)} & \frac{\Gamma(-2\sigma) \Gamma(-2\theta_1)}{\Gamma(-\theta_1 - \theta_\infty - \sigma) \Gamma(-\theta_1 + \theta_\infty - \sigma)} \\ \frac{\Gamma(2\sigma) \Gamma(2\theta_1)}{\Gamma(1 + \theta_1 - \theta_\infty + \sigma) \Gamma(1 + \theta_1 + \theta_\infty + \sigma)} & \frac{r_e \Gamma(-2\sigma) \Gamma(2\theta_1)}{\Gamma(1 + \theta_1 - \theta_\infty - \sigma) \Gamma(1 + \theta_1 + \theta_\infty - \sigma)} \end{array} \right) e^{\mp i\pi \Im}. \quad (3.46)$$

An explicit expression for the interior parametrix, as well as the relation between monodromy and coefficient matrices of the auxiliary Fuchsian system (3.10), can be obtained in a completely analogous way. The known spectra $\text{sp } A_0^i = \{\pm\theta_0\}$, $\text{sp } A_1^i = \{\pm\theta_t\}$ and the diagonal form of $A_0^i + A_1^i = \Im = \text{diag}\{\sigma, -\sigma\}$ allow to parameterize A_0^i, A_1^i by setting

$$A_1^i = \frac{1}{2\sigma} \begin{pmatrix} \sigma^2 + \theta_t^2 - \theta_0^2 & r_i (\theta_0^2 - (\sigma - \theta_t)^2) \\ r_i^{-1} ((\sigma + \theta_t)^2 - \theta_0^2) & \theta_0^2 - \theta_t^2 - \sigma^2 \end{pmatrix}, \quad (3.47)$$

with $r_i \in \mathbb{C}^\times$. The diagonalizing transformations for $A_0^i = G_0^i \Theta_0 G_0^{i-1}$ and $A_1^i = G_1^i \Theta_1 G_1^{i-1}$ can be chosen as

$$G_0^i = \begin{pmatrix} r_i (\theta_t - \theta_0 - \sigma) & \theta_t + \theta_0 - \sigma \\ \theta_t - \theta_0 + \sigma & r_i^{-1} (\theta_t + \theta_0 + \sigma) \end{pmatrix} \begin{pmatrix} c_{0a}^{i-1} & 0 \\ 0 & c_{0b}^{i-1} \end{pmatrix}, \quad (3.48a)$$

$$G_1^i = \begin{pmatrix} r_i & (\theta_t - \sigma)^2 - \theta_0^2 \\ 1 & r_i^{-1} ((\theta_t + \sigma)^2 - \theta_0^2) \end{pmatrix} \begin{pmatrix} c_{1a}^{i-1} & 0 \\ 0 & c_{1b}^{i-1} \end{pmatrix}. \quad (3.48b)$$

Lemma 3.16. *Let $\Phi^i(z)$ be the solution of the Fuchsian system (3.10), normalized as $\Phi^i(z \rightarrow -\infty) \simeq (-z)^\Im$, with A_0^i, A_1^i defined by (3.47) and $A_1^i = \Im - A_0^i$. Then*

$$\Phi_{11}^i(z) = (-z)^\sigma (1 - z^{-1})^{\theta_t} {}_2F_1(\theta_t + \theta_0 - \sigma, \theta_t - \theta_0 - \sigma; -2\sigma; z^{-1}), \quad (3.49a)$$

$$\Phi_{12}^i(z) = r_i \frac{\theta_0^2 - (\theta_t - \sigma)^2}{2\sigma(2\sigma + 1)} (-z)^{-\sigma-1} (1 - z^{-1})^{\theta_t} {}_2F_1(1 + \theta_t + \theta_0 + \sigma, 1 + \theta_t - \theta_0 + \sigma; 2 + 2\sigma; z^{-1}), \quad (3.49b)$$

$$\Phi_{21}^i(z) = r_i^{-1} \frac{\theta_0^2 - (\theta_t + \sigma)^2}{2\sigma(2\sigma - 1)} (-z)^{\sigma-1} (1 - z^{-1})^{\theta_t} {}_2F_1(1 + \theta_t + \theta_0 - \sigma, 1 + \theta_t - \theta_0 - \sigma; 2 - 2\sigma; z^{-1}), \quad (3.49c)$$

$$\Phi_{22}^i(z) = (-z)^{-\sigma} (1 - z^{-1})^{\theta_t} {}_2F_1(\theta_t + \theta_0 + \sigma, \theta_t - \theta_0 + \sigma; 2\sigma; z^{-1}). \quad (3.49d)$$

An equivalent form of the solution (3.49) suitable for study of its local behavior as $z \rightarrow 0$ is given by

$$\Phi^i(z) = G_0^i \hat{\Phi}^i(z) C_0^i, \quad (3.50)$$

where G_0^i is defined by (3.48a) and $\hat{\Phi}^i(z)$ by

$$\hat{\Phi}_{11}^i(z) = (-z)^{\theta_0} (1 - z)^{\theta_t} {}_2F_1(\theta_t + \theta_0 + \sigma, \theta_t + \theta_0 - \sigma, 2\theta_0, z), \quad (3.51a)$$

$$\hat{\Phi}_{12}^i(z) = r_i^{-1} \frac{c_{0a}^i (\theta_t + \theta_0)^2 - \sigma^2}{c_{0b}^i 2\theta_0(2\theta_0 - 1)} (-z)^{1-\theta_0} (1 - z)^{\theta_t} {}_2F_1(1 + \theta_t - \theta_0 + \sigma, 1 + \theta_t - \theta_0 - \sigma, 2 - 2\theta_0, z), \quad (3.51b)$$

$$\hat{\Phi}_{21}^i(z) = r_i \frac{c_{0b}^i (\theta_t - \theta_0)^2 - \sigma^2}{c_{0a}^i 2\theta_0(2\theta_0 + 1)} (-z)^{\theta_0+1} (1 - z)^{\theta_t} {}_2F_1(1 + \theta_t + \theta_0 + \sigma, 1 + \theta_t + \theta_0 - \sigma, 2 + 2\theta_0, z), \quad (3.51c)$$

$$\hat{\Phi}_{22}^i(z) = (-z)^{-\theta_0} (1 - z)^{\theta_t} {}_2F_1(\theta_t - \theta_0 + \sigma, \theta_t - \theta_0 - \sigma, -2\theta_0, z). \quad (3.51d)$$

The connection matrix $C_0^i = C_0 C_S^{-1}$ is given by

$$C_0^i = \begin{pmatrix} c_{0a}^i & 0 \\ 0 & c_{0b}^i \end{pmatrix} \left(\begin{array}{cc} \frac{r_i^{-1} \Gamma(-2\theta_0) \Gamma(-2\sigma)}{\Gamma(-\theta_0 - \theta_t - \sigma) \Gamma(1 - \theta_0 + \theta_t - \sigma)} & \frac{\Gamma(-2\theta_0) \Gamma(2\sigma)}{\Gamma(-\theta_0 - \theta_t + \sigma) \Gamma(1 - \theta_0 + \theta_t + \sigma)} \\ \frac{\Gamma(2\theta_0) \Gamma(-2\sigma)}{\Gamma(\theta_0 - \theta_t - \sigma) \Gamma(1 + \theta_0 + \theta_t - \sigma)} & \frac{r_i \Gamma(2\theta_0) \Gamma(2\sigma)}{\Gamma(\theta_0 - \theta_t + \sigma) \Gamma(1 + \theta_0 + \theta_t + \sigma)} \end{array} \right). \quad (3.52)$$

Connection matrices $C_{1,\pm}^i$ are related by $C_{1,+}^i = C_{1,-}^i e^{-2\pi i \mathfrak{S}}$ and can be written as

$$C_{1,\pm}^i = e^{\pm i\pi\Theta_t} \begin{pmatrix} c_{1a}^i & 0 \\ 0 & c_{1b}^i \end{pmatrix} \begin{pmatrix} \frac{r_i^{-1} \Gamma(-2\theta_t) \Gamma(-2\sigma)}{\Gamma(-\theta_0 - \theta_t - \sigma) \Gamma(\theta_0 - \theta_t - \sigma)} & \frac{\Gamma(-2\theta_t) \Gamma(2\sigma)}{\Gamma(-\theta_0 - \theta_t + \sigma) \Gamma(\theta_0 - \theta_t + \sigma)} \\ \frac{\Gamma(2\theta_t) \Gamma(-2\sigma)}{\Gamma(1 - \theta_0 + \theta_t - \sigma) \Gamma(1 + \theta_0 + \theta_t - \sigma)} & \frac{r_i \Gamma(2\theta_t) \Gamma(2\sigma)}{\Gamma(1 - \theta_0 + \theta_t + \sigma) \Gamma(1 + \theta_0 + \theta_t + \sigma)} \end{pmatrix} e^{\mp i\pi \mathfrak{S}}. \quad (3.53)$$

Comparing the monodromy matrix $\tilde{M}_0 = C_0^{i-1} e^{2\pi i \Theta_0} C_0^i$ with the parameterization (3.32a), we obtain the following result:

Corollary 3.17. *The quantities s_i and r_i are related by*

$$s_i = \frac{\Gamma(1+2\sigma) \Gamma(1+\theta_0+\theta_t-\sigma) \Gamma(1-\theta_0+\theta_t-\sigma)}{\Gamma(1-2\sigma) \Gamma(1+\theta_0+\theta_t+\sigma) \Gamma(1-\theta_0+\theta_t+\sigma)} r_i. \quad (3.54)$$

Corollary 3.18. *Results obtained above enable us to compute explicitly the 1-form ω_0 defined by (3.27) for $N=2$. By straightforward calculation, it follows from the parameterizations (3.40), (3.48) that*

$$\omega_0 = \theta_\infty d_{\mathcal{M}} \ln \frac{r_e c_{\infty b}^e}{c_{\infty a}^e} + \theta_1 d_{\mathcal{M}} \ln \frac{r_e c_{1b}^e}{c_{1a}^e} + \theta_t d_{\mathcal{M}} \ln \frac{r_i c_{1b}^i}{c_{1a}^i} + \theta_0 d_{\mathcal{M}} \ln \frac{r_i c_{0b}^i}{c_{0a}^i} + \sigma d_{\mathcal{M}} \ln \frac{r_i}{r_e} - 2d_{\mathcal{M}}(\theta_1 + \theta_t). \quad (3.55)$$

Remark 3.19. If we consider parameters $\theta_{0,t,1,\infty}$ as fixed, then the differential $\Omega = d\omega_0$ is determined by single term $\sigma d_{\mathcal{M}} \ln \frac{r_i}{r_e}$ in (3.55). In this case it follows from (3.42), (3.54) and (3.33) that $\Omega = i d\sigma \wedge d\eta$, i.e. the 2-form Ω coincides with symplectic form on \mathcal{M}_4^Θ induced by Goldman bracket (3.37). However, for the computation of the connection constant for Painlevé VI tau function in Subsection 3.9 we need to keep track of the dependence of ω_0 on the spectra of local monodromies.

3.7 Jimbo asymptotic formula

We are now in a position to express the short-distance asymptotics of the Painlevé VI tau function (3.30) in terms of monodromy data of the associated Fuchsian system. Indeed, the asymptotics of the 4-point tau function in arbitrary rank is described by Proposition 3.9 and involves matrices $g_{0,1}^e, g_{\infty,1}^i$ appearing in the local expansions of 3-point parametrices $\Phi^e(z), \Phi^i(z)$ around 0 and ∞ . Relations (3.25) express these matrices in terms of matrix coefficients A_1^e, A_1^i of the corresponding 3-point Fuchsian systems. In the case $N=2$, the latter quantities are parameterized by two variables $r_e, r_i \in \mathbb{C}^\times$ as indicated in (3.39) and (3.47). The parameters r_e, r_i are related to monodromy of the initial 4-point Fuchsian system by (3.42) and (3.54), see also (3.33) and (3.35). Altogether, this leads to the following claim.

Proposition 3.20. *For $N=2$, let (σ, η) be two local coordinates on the space \mathcal{M}_4^Θ of monodromy data defined in Subsection 3.5. Let $\tau_{\omega_0}(t)$ denote the Painlevé VI tau function normalized as in (3.30). Its $t \rightarrow 0$ asymptotics is given by*

$$\tau_{\omega_0}(t \rightarrow 0) = \mathcal{C}_{\omega_0} \cdot t^{\sigma^2 - \theta_0^2 - \theta_t^2} \left[1 - \sum_{\epsilon=\pm 1} \frac{((\theta_t - \epsilon\sigma)^2 - \theta_0^2)((\theta_1 - \epsilon\sigma)^2 - \theta_\infty^2)}{4\sigma^2(1+2\epsilon\sigma)^2} \kappa^\epsilon t^{1+2\epsilon\sigma} + \frac{(\sigma^2 - \theta_\infty^2 + \theta_1^2)(\sigma^2 - \theta_0^2 + \theta_t^2)}{2\sigma^2} t + o(t) \right], \quad (3.56)$$

where the coefficient κ is defined by

$$\kappa = \frac{\Gamma^2(1-2\sigma) \Gamma(1+\theta_0+\theta_t+\sigma) \Gamma(1-\theta_0+\theta_t+\sigma) \Gamma(1+\theta_\infty+\theta_1+\sigma) \Gamma(1-\theta_\infty+\theta_1+\sigma)}{\Gamma^2(1+2\sigma) \Gamma(1+\theta_0+\theta_t-\sigma) \Gamma(1-\theta_0+\theta_t-\sigma) \Gamma(1+\theta_\infty+\theta_1-\sigma) \Gamma(1-\theta_\infty+\theta_1-\sigma)} e^{i\eta}, \quad (3.57)$$

and the constant prefactor \mathcal{C}_{ω_0} does not depend on monodromy data.

Proof. For $N=2$, one can replace (3.29) by the estimate

$$\tau_{\omega_0}(t) = \mathcal{C}_{\omega_0} \cdot t^{\sigma^2 - \theta_0^2 - \theta_t^2} \left[1 - \text{Tr} \left(g_{0,1}^e t^{1+\mathfrak{S}} g_{\infty,1}^i t^{-\mathfrak{S}} \right) + o(t) \right]. \quad (3.58)$$

(In higher rank, one needs to take into account more terms in the determinant expansion). From (3.25), (3.39) and (3.47) it follows that

$$g_{0,1}^e = \begin{pmatrix} \frac{\sigma^2 + \theta_1^2 - \theta_\infty^2}{2\sigma} & r_e \frac{(\theta_1 + \sigma)^2 - \theta_\infty^2}{2\sigma(2\sigma - 1)} \\ r_e^{-1} \frac{(\theta_1 - \sigma)^2 - \theta_\infty^2}{2\sigma(2\sigma + 1)} & \frac{\theta_\infty^2 - \theta_1^2 - \sigma^2}{2\sigma} \end{pmatrix},$$

$$g_{\infty,1}^i = \begin{pmatrix} \frac{\theta_0^2 - \theta_t^2 - \sigma^2}{2\sigma} & r_i \frac{(\theta_t - \sigma)^2 - \theta_0^2}{2\sigma(2\sigma + 1)} \\ r_i^{-1} \frac{(\theta_t + \sigma)^2 - \theta_0^2}{2\sigma(2\sigma - 1)} & \frac{\theta_t^2 - \theta_0^2 + \sigma^2}{2\sigma} \end{pmatrix}.$$

Upon substitution into (3.58), the diagonal parts of $g_{0,1}^e$, $g_{\infty,1}^i$ give the linear contribution in (3.56). The off-diagonal elements determine the coefficients of $t^{1 \pm 2\sigma}$, with identification $\kappa = r_i/r_e$. Finally, the latter coefficient is related to invariant monodromy data by (3.42), (3.54) and (3.33). \square

Remark 3.21. Proposition 3.20 is a slightly upgraded version of Jimbo asymptotic formula for Painlevé VI (Theorem 1.1 in [Jim]). The improvement concerns the error estimate: e.g. the terms such as $t^{2 \pm 4\sigma}$, $t^{3 \pm 6\sigma}$, ..., a priori present in the short-distance asymptotics of $\ln \tau_{\omega_0}$, disappear from the expansion of the tau function itself. This fact was already noticed and played an important role in [GIL12]. Similar nontrivial cancellations have been experimentally observed to happen in higher rank [Gav]. This was one of our motivations for establishing the results of Subsection 3.4, in particular, Proposition 3.9.

3.8 Crossing to $t \rightarrow 1$

To be able to deal with the connection problem for Painlevé VI tau function, we now have to investigate the asymptotics of the form (3.2) as $t \rightarrow 1$. It can be obtained by a suitable exchange of parameters, which is not trivial even for the “time part” of the tau function: in the latter case the initial suggestion of [Jim, Theorem 1.2] should be modified as explained in [L09, Remark 7.1]. The tau function extended to the space of monodromy data needs even more care as here at intermediate steps one has to manipulate with quantities that are not preserved by conjugation.

The basic idea is to replace the RHP for $\Psi(z)$ by a RHP for a new matrix function $\bar{\Psi}(z)$. The corresponding contour is represented in Fig. 4 and $\bar{\Psi}(z)$ is defined by

$$\bar{\Psi}(z) = \begin{cases} \Psi(1-z)M_\infty, & z \text{ outside } \tilde{\gamma}_\nu, \Im z > 0, \\ \Psi(1-z) & \text{otherwise.} \end{cases}$$

This function is designed to have the structure of jump matrix $J_{\bar{\Psi}} = \bar{\Psi}_-^{-1} \bar{\Psi}_+$ analogous to (3.3):

$$\begin{aligned} J_{\bar{\Psi}}(z) \Big|_{\bar{\ell}_{\infty 0}} &= \mathbf{1}, & J_{\bar{\Psi}}(z) \Big|_{\bar{\ell}_{0,1-t}} &= M_1^{-1}, & J_{\bar{\Psi}}(z) \Big|_{\bar{\ell}_{1-t,1}} &= (M_1 M_t)^{-1}, & J_{\bar{\Psi}}(z) \Big|_{\bar{\ell}_{1\infty}} &= M_\infty, \\ J_{\bar{\Psi}}(z) \Big|_{\bar{\gamma}_0} &= \bar{C}_0^{-1} (-z)^{-\Theta_1}, & J_{\bar{\Psi}}(z) \Big|_{\bar{\gamma}_{1-t}} &= \bar{C}_{1-t}^{-1} (1-t-z)^{-\Theta_t}, \\ J_{\bar{\Psi}}(z) \Big|_{\bar{\gamma}_1} &= \bar{C}_1^{-1} (1-z)^{-\Theta_0}, & J_{\bar{\Psi}}(z) \Big|_{\bar{\gamma}_\infty} &= \bar{C}_\infty^{-1} (-z)^{\Theta_\infty}. \end{aligned} \quad (3.59)$$

Note that — in spite of certain similarity — the first line of (3.59) is *not* obtained from (3.3) by exchange $M_0 \leftrightarrow M_1$ because of reversed order of factors in the corresponding jumps. New connection matrices are expressed in terms of C_ν 's in the following way:

$$\begin{aligned} \bar{C}_{0,+} &= e^{i\pi\Theta_1} C_{1,-} M_\infty, & \bar{C}_{0,-} &= e^{-i\pi\Theta_1} C_{1,+}, & \bar{C}_{1-t,+} &= e^{i\pi\Theta_t} C_{t,-} M_\infty, & \bar{C}_{1-t,-} &= e^{-i\pi\Theta_t} C_{t,+}, \\ \bar{C}_{1,+} &= e^{i\pi\Theta_0} C_{0,-} M_\infty, & \bar{C}_{1,-} &= e^{-i\pi\Theta_0} C_{0,+}, & \bar{C}_{\infty,+} &= e^{-i\pi\Theta_\infty} C_{\infty,-} M_\infty, & \bar{C}_{\infty,-} &= e^{i\pi\Theta_\infty} C_{\infty,+}. \end{aligned}$$

The function $\bar{\Psi}(1-z)$ in the exterior of circles γ_ν coincides with the analytic continuation of the solution $\Phi(z)$ of the initial Fuchsian system from the upper half-plane $\Im z > 0$ to the cut Riemann sphere $\mathbb{P}^1 \setminus [-\infty, 1]$.

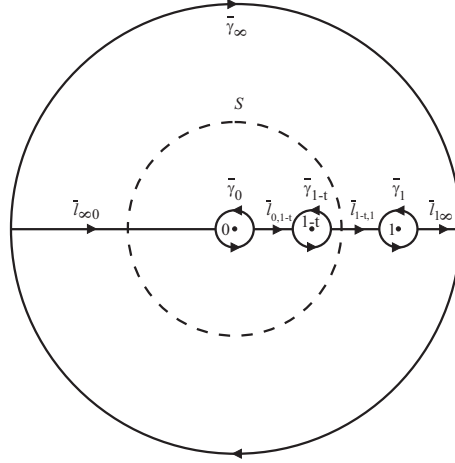


Figure 4: Contour $\Gamma_{\bar{\Psi}}$ of the Riemann-Hilbert problem for $\bar{\Psi}(z)$

Approximate solution for $\bar{\Psi}(z)$ is constructed in the same way as for $\Psi(z)$. The main building blocks are solutions $\bar{\Phi}^i(z)$ and $\bar{\Phi}^e(z)$ of two hypergeometric systems which model the jumps inside and outside the auxiliary circle S . To simplify the exposition we consider only sufficiently generic situation described by an analog of Assumptions 3.1 and 3.11:

Assumption 3.22. *The matrix $M_{1t} := M_1 M_t$ is assumed to be diagonalizable. It will be parameterized as*

$$M_{1t} = \bar{C}_S^{-1} e^{2\pi i \bar{\mathfrak{S}}} \bar{C}_S, \quad \bar{\mathfrak{S}} = \text{diag}\{\bar{\sigma}, -\bar{\sigma}\}.$$

It is furthermore assumed that $|\Re \bar{\sigma}| < \frac{1}{2}$, $\bar{\sigma} \neq 0$, and that

$$\theta_1 + \theta_t \pm \bar{\sigma}, \theta_1 - \theta_t \pm \bar{\sigma}, \theta_\infty + \theta_0 \pm \bar{\sigma}, \theta_\infty - \theta_0 \pm \bar{\sigma} \notin \mathbb{Z}.$$

Referring to the notations of Subsection 3.5, $\bar{\sigma}$ can be identified with σ_{t1} . The latter quantity is related to previously used coordinates σ, η by formula (3.35a).

Exterior and interior parametrices are solutions of the Fuchsian systems

$$\partial_z \bar{\Phi}^e = \left(\frac{\bar{A}_0^e}{z} + \frac{\bar{A}_1^e}{z-1} \right) \bar{\Phi}^e, \quad \partial_z \bar{\Phi}^i = \left(\frac{\bar{A}_0^i}{z} + \frac{\bar{A}_1^i}{z-1} \right) \bar{\Phi}^i,$$

normalized as $\bar{\Phi}^e(z \rightarrow 0) \simeq (-z)^{\bar{\mathfrak{S}}}$, $\bar{\Phi}^i(z \rightarrow \infty) \simeq (-z)^{\bar{\mathfrak{S}}}$. Furthermore, one has $\bar{A}_0^e = \bar{A}_0^i + A_1^i = \bar{\mathfrak{S}}$. The spectra of $\bar{A}_0^i, \bar{A}_1^i, \bar{A}_1^e$ and $\bar{A}_0^e + \bar{A}_1^e$ are given by $\pm\theta_1, \pm\theta_t, \pm\theta_0$ and $\pm\theta_\infty$ respectively. All results of previous subsections have their $t \rightarrow 1$ counterparts, most of which are obtained by the replacements (the notation should be sufficiently self-explanatory)

$$\begin{aligned} \Phi^e(z) &\mapsto \bar{\Phi}^e(z), & \Phi^i(z) &\mapsto \bar{\Phi}^i(z), \\ G_\nu^e(z) &\mapsto \bar{G}_\nu^e(z), & G_\nu^i(z) &\mapsto \bar{G}_\nu^i(z), & \nu &= 0, 1, \infty, \\ G_{1,\infty}^e &\mapsto \bar{G}_{1,\infty}^e, & C_{1,\infty}^e &\mapsto \bar{C}_{1,\infty}^e, & G_{0,1}^i &\mapsto \bar{G}_{0,1}^i, & C_{0,1}^i &\mapsto \bar{C}_{0,1}^i, \\ & & (\theta_0, \theta_t, \theta_1, \theta_\infty, \sigma, r_e, r_i) &\mapsto (\theta_1, \theta_t, \theta_0, \theta_\infty, \bar{\sigma}, \bar{r}_e, \bar{r}_i), \\ & & (c_{0,q}^e, c_{1,q}^e, c_{0,q}^i, c_{0,q}^i) &\mapsto (\bar{c}_{0,q}^e, \bar{c}_{1,q}^e, \bar{c}_{0,q}^i, \bar{c}_{0,q}^i), & q &= a, b, \end{aligned} \tag{3.60}$$

In particular, one can prove the following statement:

Lemma 3.23. *The form Ω can be alternatively written as $d\bar{\omega}_0$, where $\bar{\omega}_0 \in \Lambda^1(\mathcal{M})$ is given by*

$$\bar{\omega}_0 = \text{Tr} \left(\Theta_1 \bar{G}_0^{i-1} d_{\mathcal{M}} \bar{G}_0^i + \Theta_t \bar{G}_1^{i-1} d_{\mathcal{M}} \bar{G}_1^i + \Theta_0 \bar{G}_1^{e-1} d_{\mathcal{M}} \bar{G}_1^e + \Theta_\infty \bar{G}_\infty^{e-1} d_{\mathcal{M}} \bar{G}_\infty^e \right) = \tag{3.61a}$$

$$= \theta_\infty d_{\mathcal{M}} \ln \frac{\bar{r}_e \bar{c}_{\infty b}^e}{\bar{c}_{\infty a}^e} + \theta_1 d_{\mathcal{M}} \ln \frac{\bar{r}_i \bar{c}_{0b}^i}{\bar{c}_{0a}^i} + \theta_t d_{\mathcal{M}} \ln \frac{\bar{r}_i \bar{c}_{1b}^i}{\bar{c}_{1a}^i} + \theta_0 d_{\mathcal{M}} \ln \frac{\bar{r}_e \bar{c}_{1b}^e}{\bar{c}_{1a}^e} + \bar{\sigma} d_{\mathcal{M}} \ln \frac{\bar{r}_i}{\bar{r}_e} - 2d_{\mathcal{M}} (\theta_0 + \theta_t). \quad (3.61b)$$

The first equation remains valid in arbitrary rank and the second is its specialization to $N = 2$.

The form $\bar{\omega}_0$ can be used to define a second tau function $\tau_{\bar{\omega}_0}(t)$ via $d \ln \tau_{\bar{\omega}_0} = \omega - \bar{\omega}_0$. This tau function is of course proportional to $\tau_{\omega_0}(t)$ but its asymptotics is normalized at $t = 1$ instead of $t = 0$. Let us formulate an analog of Proposition 3.20 for $\tau_{\bar{\omega}_0}(t)$:

Proposition 3.24. *Normalized Painlevé VI tau function $\tau_{\bar{\omega}_0}(t)$ has the following asymptotics as $t \rightarrow 1$:*

$$\tau_{\bar{\omega}_0}(t \rightarrow 1) = \mathcal{E}_{\bar{\omega}_0} \cdot (1-t)^{\bar{\sigma}^2 - \theta_1^2 - \theta_t^2} \left[1 - \sum_{\epsilon=\pm 1} \frac{((\theta_t - \epsilon \bar{\sigma})^2 - \theta_1^2)((\theta_0 - \epsilon \bar{\sigma})^2 - \theta_\infty^2)}{4\bar{\sigma}^2(1+2\epsilon\bar{\sigma})^2} \bar{\kappa}^\epsilon (1-t)^{1+2\epsilon\bar{\sigma}} + \frac{(\bar{\sigma}^2 - \theta_\infty^2 + \theta_0^2)(\bar{\sigma}^2 - \theta_1^2 + \theta_t^2)}{2\bar{\sigma}^2} (1-t) + o(1-t) \right]. \quad (3.62)$$

Here the coefficient $\bar{\kappa}$ is defined by

$$\bar{\kappa} = \frac{\Gamma^2(1-2\bar{\sigma}) \Gamma(1+\theta_1+\theta_t+\bar{\sigma}) \Gamma(1-\theta_1+\theta_t+\bar{\sigma}) \Gamma(1+\theta_\infty+\theta_0+\bar{\sigma}) \Gamma(1-\theta_\infty+\theta_0+\bar{\sigma})}{\Gamma^2(1+2\bar{\sigma}) \Gamma(1+\theta_1+\theta_t-\bar{\sigma}) \Gamma(1-\theta_1+\theta_t-\bar{\sigma}) \Gamma(1+\theta_\infty+\theta_0-\bar{\sigma}) \Gamma(1-\theta_\infty+\theta_0-\bar{\sigma})} e^{i\bar{\eta}}, \quad (3.63)$$

$$e^{i\bar{\eta}} = \frac{-2i \sin 2\pi \bar{\sigma} (p_{01} + p_{0t} e^{-2\pi i \bar{\sigma}}) + (p_0 p_t + p_1 p_\infty) - (p_0 p_1 + p_t p_\infty) e^{-2\pi i \bar{\sigma}}}{(p_\infty - 2 \cos 2\pi(\theta_0 - \bar{\sigma})) (p_1 - 2 \cos 2\pi(\theta_t - \bar{\sigma}))}, \quad (3.64)$$

and the constant prefactor $\mathcal{E}_{\bar{\omega}_0}$ does not depend on monodromy data.

Proof. The only subtlety that has to be taken into account as compared to derivation of Proposition 3.20 concerns the quantity $e^{i\bar{\eta}}$ and is as follows. The coefficient $\bar{\kappa}$ coincides with the ratio \bar{r}_i/\bar{r}_e , and $e^{i\bar{\eta}}$ with \bar{s}_i/\bar{s}_e . The last ratio is expressed in terms of traces of products of monodromy matrices $\bar{M}_{0,t,1,\infty}$ associated with $\bar{\Psi}(z)$ by overlining parameters in (3.38). These new monodromy matrices can be expressed in terms of $M_{0,t,1,\infty}$ using the first line of (3.59). One has

$$\bar{M}_0 = M_1, \quad \bar{M}_t = M_1 M_t M_1^{-1}, \quad \bar{M}_1 = M_1 M_t M_0 M_t^{-1} M_1^{-1}, \quad \bar{M}_\infty = M_\infty.$$

This implies simple transformation formulas for most of the trace functions:

$$(\bar{p}_0, \bar{p}_t, \bar{p}_1, \bar{p}_\infty, \bar{p}_{0t}, \bar{p}_{t1}) = (p_1, p_t, p_0, p_\infty, p_{t1}, p_{0t}). \quad (3.65a)$$

The only exception is

$$\bar{p}_{01} = \text{Tr} \bar{M}_0 \bar{M}_1 = \text{Tr} M_1 M_t M_0 M_t^{-1} = p_0 p_1 + p_t p_\infty - p_{01} - p_{0t} p_{t1}. \quad (3.65b)$$

The last equality follows from the fact that any $M \in SL_2(\mathbb{C})$ satisfies skein relation $M + M^{-1} = \text{Tr} M \cdot \mathbf{1}$. Relation (3.64) is then obtained from (3.38) by identification (3.65). \square

3.9 Connection problem for Painlevé VI tau function

The tau functions $\tau_{\omega_0}(t)$ and $\tau_{\bar{\omega}_0}(t)$ can differ only by a constant (i.e. independent of t) factor of relative normalization

$$\Upsilon(M) := \frac{\tau_{\omega_0}(t)}{\tau_{\bar{\omega}_0}(t)}, \quad (3.66)$$

which is our main quantity of interest in this section. It can also be written for generic non-normalized Painlevé VI tau function $\tau(t)$ as

$$\Upsilon(M) = \lim_{t \rightarrow 0} \frac{(1-t)^{\theta_t^2 + \theta_1^2 - \bar{\sigma}^2} \tau(1-t)}{t^{\theta_0^2 + \theta_t^2 - \bar{\sigma}^2} \tau(t)}. \quad (3.67)$$

This constant is completely determined by Painlevé VI equation and appropriate initial conditions — that is, it depends only on the conjugacy class of monodromy. The latter property is by no means manifest in the representation

$$d_{\mathcal{M}} \ln Y = \bar{\omega}_0 - \omega_0, \quad (3.68)$$

and becomes yet more implicit if we rewrite the right side using previous results:

$$\begin{aligned} d_{\mathcal{M}} \ln Y &= \bar{\sigma} d_{\mathcal{M}} \ln \frac{\bar{r}_i}{\bar{r}_e} - \sigma d_{\mathcal{M}} \ln \frac{r_i}{r_e} + 2d_{\mathcal{M}} (\theta_1 - \theta_0) \\ &+ \theta_{\infty} d_{\mathcal{M}} \ln \frac{\bar{r}_e \bar{c}_{\infty b}^e c_{\infty a}^e}{r_e \bar{c}_{\infty a}^e c_{\infty b}^e} + \theta_1 d_{\mathcal{M}} \ln \frac{\bar{r}_i \bar{c}_{0b}^i c_{1a}^i}{r_e \bar{c}_{0a}^i c_{1b}^i} + \theta_t d_{\mathcal{M}} \ln \frac{\bar{r}_i \bar{c}_{1b}^i c_{1a}^i}{r_i \bar{c}_{1a}^i c_{1b}^i} + \theta_0 d_{\mathcal{M}} \ln \frac{\bar{r}_e \bar{c}_{1b}^e c_{0a}^e}{r_i \bar{c}_{1a}^e c_{0b}^e}. \end{aligned} \quad (3.69)$$

It is even less obvious that the last expression is a closed 1-form! Our task is now to rewrite (3.69) in terms of more convenient local coordinates on \mathcal{M} , such as $\theta_{0,t,1,\infty}$, σ and η .

The right side of (3.69) is expressed in terms of parameters

$$\theta_{0,t,1,\infty}, \sigma, \bar{\sigma}, r_{e,i}, \bar{r}_{e,i}, c_{\infty q}^e, c_{1q}^e, \bar{c}_{\infty q}^e, \bar{c}_{1q}^e, c_{0q}^i, c_{1q}^i, \bar{c}_{0q}^i, \bar{c}_{1q}^i,$$

which appear in the monodromy and connection matrices. Of course, not all of them are independent. Connection matrices of the original 4-point problem can be related to 3-point ones in two different ways:

$$\begin{aligned} C_{\infty}^e &= C_{\infty} C_S^{-1}, & C_{1,+}^e &= C_{1,+} C_S^{-1}, & C_{1,+}^i &= C_{t,+} C_S^{-1}, & C_0^i &= C_0 C_S^{-1}, \\ \bar{C}_0 &= e^{-i\pi\theta_1} C_{1,+}, & \bar{C}_{1,-} &= e^{-i\pi\theta_t} C_{t,+}, & \bar{C}_{1,-} &= e^{-i\pi\theta_0} C_0, & \bar{C}_{\infty} &= e^{i\pi\theta_{\infty}} C_{\infty}, \\ \bar{C}_{\infty}^e &= \bar{C}_{\infty} \bar{C}_S^{-1}, & \bar{C}_{1,-}^e &= \bar{C}_{1,-} \bar{C}_S^{-1}, & \bar{C}_{1,-}^i &= \bar{C}_{1,-} \bar{C}_S^{-1}, & \bar{C}_0^i &= \bar{C}_0 \bar{C}_S^{-1}. \end{aligned} \quad (3.70)$$

Here the first line reproduces (3.8c), (3.11c), the third lists analogous relations arising in the study of $t \rightarrow 1$ tau function asymptotics and the middle one relates the two sets of connection matrices. It follows that

$$\bar{C}_{\infty}^e = e^{i\pi\theta_{\infty}} C_{\infty}^e C_{S\bar{S}}, \quad \bar{C}_{1,-}^e = e^{-i\pi\theta_0} C_0^i C_{S\bar{S}}, \quad \bar{C}_{1,-}^i = e^{-i\pi\theta_t} C_{1,+}^i C_{S\bar{S}}, \quad \bar{C}_0^i = e^{-i\pi\theta_1} C_{1,+}^e C_{S\bar{S}}. \quad (3.71)$$

where $C_{S\bar{S}} = C_S \bar{C}_S^{-1}$. Since by definition $\bar{C}_S M_1 M_t \bar{C}_S^{-1} = e^{2\pi i \bar{\sigma}}$, the transformation $C_{S\bar{S}}$ diagonalizes the product $\bar{M}_1 \bar{M}_t = C_{S\bar{S}} e^{2\pi i \bar{\sigma}} C_{S\bar{S}}^{-1}$, where \bar{M}_t, \bar{M}_1 are defined in (3.32b), (3.32c). This in turn fixes $C_{S\bar{S}}$ up to right multiplication by diagonal matrix. In fact one can write

$$C_{S\bar{S}} = S_e^{-1} C_{S\bar{S}} D_{S\bar{S}},$$

where $D_{S\bar{S}} = \text{diag}\{d_a, d_b\}$, $S_e = \text{diag}\{1, s_e\}$ and

$$\begin{aligned} C_{S\bar{S}} &\equiv \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} \alpha & \alpha \\ e^{2\pi i \bar{\sigma}} - \beta & e^{-2\pi i \bar{\sigma}} - \beta \end{pmatrix}, \\ 4\alpha \sin^2 2\pi\sigma &= (p_{\infty} - 2 \cos 2\pi(\theta_1 + \sigma)) (p_0 e^{2\pi i \sigma} - p_t) - (p_0 - 2 \cos 2\pi(\theta_t - \sigma)) (p_{\infty} e^{2\pi i \sigma} - p_1) e^{i\eta}, \\ 4\beta \sin^2 2\pi\sigma &= (p_t e^{2\pi i \sigma} - p_0) (p_1 e^{-2\pi i \sigma} - p_{\infty}) - (p_0 - 2 \cos 2\pi(\theta_t + \sigma)) (p_{\infty} - 2 \cos 2\pi(\theta_1 + \sigma)) e^{-i\eta}. \end{aligned} \quad (3.72)$$

The fact that $C_{S\bar{S}}$ depends only on the conjugacy class of monodromy will be important below. On the other hand, explicit form of $d_{a,b}$ does not play any role in subsequent computation.

Recall that the ratios r_i/r_e and \bar{r}_i/\bar{r}_e from the first line of (3.69) coincide with κ and $\bar{\kappa}$ defined by (3.57) and (3.63). They are thus preserved by conjugation. We are now going to check that the ratios from the second line share this important property. In order to derive invariant expressions suitable for subsequent computation of Y , it is crucial to rewrite 3-point connection matrices in a different form. As we will see in a moment, their nontrivial dependence on monodromy parameters is trigonometric. The key is the following elementary observation.

Let us denote by \mathfrak{M} the set of 2×2 matrices with non-zero elements. Let $\rho : \mathfrak{M} \rightarrow \mathbb{C}^{\times}$ be defined by

$$\rho(M) := \frac{M_{11} M_{22}}{M_{12} M_{21}}.$$

Two matrices $M, M' \in \mathfrak{M}$ will be called ρ -equivalent if $\rho(M) = \rho(M')$.

Lemma 3.25. *Two matrices $M, M' \in \mathfrak{M}$ are ρ -equivalent iff there exists a pair of diagonal matrices $D_{L,R}$ such that $M' = D_L M D_R$.*

This basic fact can be used to compute four ratios from the second line of (3.69). Below we explain the details of the procedure for one of these, namely, $\frac{\bar{r}_e \bar{c}_{\infty b}^e c_{\infty a}^e}{r_e \bar{c}_{\infty a}^e c_{\infty b}^e}$.

The ρ -invariant of the connection matrix C_∞^e given by (3.45) is equal to

$$\rho(C_\infty^e) = \frac{\sin \pi (\theta_\infty + \theta_1 - \sigma) \sin \pi (\theta_\infty - \theta_1 - \sigma)}{\sin \pi (\theta_\infty + \theta_1 + \sigma) \sin \pi (\theta_\infty - \theta_1 + \sigma)}.$$

It is therefore not surprising that we can parameterize C_∞^e as

$$C_\infty^e = D_{\infty,L}^e C_\infty^e D_{\infty,R}^e R_e,$$

with

$$\begin{aligned} C_\infty^e &= \begin{pmatrix} \sin \pi (\theta_\infty + \theta_1 - \sigma) & \sin \pi (\theta_\infty - \theta_1 + \sigma) \\ \sin \pi (\theta_\infty + \theta_1 + \sigma) & \sin \pi (\theta_\infty - \theta_1 - \sigma) \end{pmatrix}, \quad R_e = \text{diag}\{1, r_e\}, \\ D_{\infty,L}^e &= \text{diag} \left\{ \frac{c_{\infty a}^e \Gamma(-2\theta_\infty) \Gamma(\theta_\infty - \theta_1 + \sigma)}{r_e \Gamma(1 - \theta_\infty + \theta_1 + \sigma)}, \frac{c_{\infty b}^e \Gamma(2\theta_\infty) \Gamma(-\theta_\infty - \theta_1 - \sigma)}{\Gamma(1 + \theta_\infty + \theta_1 - \sigma)} \right\}, \\ D_{\infty,R}^e &= \text{diag} \left\{ -\frac{\Gamma(2\sigma) \Gamma(1 + \theta_\infty + \theta_1 - \sigma)}{\pi \Gamma(\theta_\infty - \theta_1 + \sigma)}, \frac{\Gamma(-2\sigma) \Gamma(1 - \theta_\infty + \theta_1 + \sigma)}{\pi \Gamma(-\theta_\infty - \theta_1 - \sigma)} \right\}. \end{aligned} \quad (3.73)$$

Analogous expressions for \bar{C}_∞^e is obtained by the exchange $\theta_0 \leftrightarrow \theta_1$, $\sigma \rightarrow \bar{\sigma}$, $r_e \rightarrow \bar{r}_e$, $c_{\infty q}^e \rightarrow \bar{c}_{\infty q}^e$. We can now rewrite the first of relations (3.71) in the form

$$\left(C_\infty^e D_{\infty,R}^e R_e S_e^{-1} C_{S\bar{S}} \right)^{-1} \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \bar{C}_\infty^e = \begin{matrix} \text{diagonal} \\ \text{matrix} \end{matrix},$$

where $\text{diag}\{x, y\} = D_{\infty,L}^e e^{-i\pi\theta_\infty} \bar{D}_{\infty,L}^e$. We are interested in the ratio

$$\frac{y}{x} = e^{2\pi i \theta_\infty} \frac{\Gamma(-\theta_1 + \theta_\infty + \sigma) \Gamma(1 + \theta_1 + \theta_\infty - \sigma) \Gamma(-\theta_0 - \theta_\infty - \bar{\sigma}) \Gamma(1 + \theta_0 - \theta_\infty + \bar{\sigma}) \bar{r}_e \bar{c}_{\infty b}^e c_{\infty a}^e}{\Gamma(-\theta_1 - \theta_\infty - \sigma) \Gamma(1 + \theta_1 - \theta_\infty + \sigma) \Gamma(-\theta_0 + \theta_\infty + \bar{\sigma}) \Gamma(1 + \theta_0 + \theta_\infty - \bar{\sigma}) r_e \bar{c}_{\infty a}^e c_{\infty b}^e}, \quad (3.74)$$

which is readily computed as

$$\frac{y}{x} = -\frac{\left(C_\infty^e D_{\infty,R}^e R_e S_e^{-1} C_{S\bar{S}} \right)^{-1}_{11} \left(\bar{C}_\infty^e \right)_{12}}{\left(C_\infty^e D_{\infty,R}^e R_e S_e^{-1} C_{S\bar{S}} \right)^{-1}_{12} \left(\bar{C}_\infty^e \right)_{22}} = \frac{\left(C_\infty^e D_{\infty,R}^e R_e S_e^{-1} C_{S\bar{S}} \right)_{22} \left(\bar{C}_\infty^e \right)_{12}}{\left(C_\infty^e D_{\infty,R}^e R_e S_e^{-1} C_{S\bar{S}} \right)_{12} \left(\bar{C}_\infty^e \right)_{22}}. \quad (3.75)$$

All quantities in the last expression are manifestly invariant: $C_\infty^e, \bar{C}_\infty^e, D_{\infty,R}^e$ are given by (3.73), $C_{S\bar{S}}$ by (3.72), and $R_e S_e^{-1}$ by (3.42). Moreover, up to irrelevant scalar multiple the matrix $C_\infty^e D_{\infty,R}^e R_e S_e^{-1}$ is given by trigonometric expression

$$C_\infty^e D_{\infty,R}^e R_e S_e^{-1} \sim \begin{pmatrix} \cos 2\pi \theta_\infty - \cos 2\pi (\theta_1 - \sigma) & \cos 2\pi \theta_1 - \cos 2\pi (\theta_\infty + \sigma) \\ \cos 2\pi (\theta_\infty + \sigma) - \cos 2\pi \theta_1 & \cos 2\pi (\theta_1 + \sigma) - \cos 2\pi \theta_\infty \end{pmatrix}. \quad (3.76)$$

After some algebra involving gamma function reflection formula $\Gamma(z) \Gamma(1-z) = \frac{\pi}{\sin \pi z}$, relations (3.74)–(3.76) yield the following result:

Lemma 3.26. *One has*

$$\begin{aligned} \frac{\bar{r}_e \bar{c}_{\infty b}^e c_{\infty a}^e}{r_e \bar{c}_{\infty a}^e c_{\infty b}^e} &= \frac{\Gamma(1 + \theta_1 - \theta_\infty - \sigma) \Gamma(1 + \theta_1 - \theta_\infty + \sigma) \Gamma(1 + \theta_0 + \theta_\infty - \bar{\sigma}) \Gamma(-\theta_0 + \theta_\infty - \bar{\sigma})}{\Gamma(1 + \theta_1 + \theta_\infty + \sigma) \Gamma(1 + \theta_1 + \theta_\infty - \sigma) \Gamma(1 + \theta_0 - \theta_\infty - \bar{\sigma}) \Gamma(-\theta_0 - \theta_\infty - \bar{\sigma})} \times \\ &\times e^{-2\pi i \theta_\infty} \frac{c_{12} \sin \pi (\theta_1 - \theta_\infty - \sigma) - c_{22} \sin \pi (\theta_1 - \theta_\infty + \sigma)}{c_{12} \sin \pi (\theta_1 + \theta_\infty - \sigma) - c_{22} \sin \pi (\theta_1 + \theta_\infty + \sigma)}, \end{aligned} \quad (3.77a)$$

where matrix elements c_{12}, c_{22} are given by (3.72).

Explicit invariant expressions for the other three ratios from the second line of (3.69) can be derived in a completely analogous fashion. The result is

$$\frac{\bar{r}_i \bar{c}_{0b}^i c_{1a}^e}{r_e \bar{c}_{0a}^i c_{1b}^e} = \frac{\Gamma(-\theta_1 + \theta_\infty + \sigma) \Gamma(-\theta_1 - \theta_\infty - \sigma) \Gamma(1 + \theta_1 + \theta_t + \bar{\sigma}) \Gamma(\theta_1 - \theta_t + \bar{\sigma})}{\Gamma(1 + \theta_1 + \theta_\infty - \sigma) \Gamma(1 + \theta_1 - \theta_\infty + \sigma) \Gamma(1 - \theta_1 + \theta_t + \bar{\sigma}) \Gamma(-\theta_1 - \theta_t + \bar{\sigma})} \times \quad (3.77b)$$

$$\times \frac{e^{2\pi i \theta_1} (\cos 2\pi \theta_1 - \cos 2\pi (\theta_\infty + \sigma)) (c_{12} e^{-\pi i \sigma} - c_{22} e^{\pi i \sigma})}{c_{12} e^{-\pi i \sigma} (\cos 2\pi (\theta_1 - \sigma) - \cos 2\pi \theta_\infty) - c_{22} e^{\pi i \sigma} (\cos 2\pi (\theta_1 + \sigma) - \cos 2\pi \theta_\infty)},$$

$$\frac{\bar{r}_i \bar{c}_{1b}^i c_{1a}^i}{r_i \bar{c}_{1a}^i c_{1b}^i} = \frac{\Gamma(\theta_0 - \theta_t - \sigma) \Gamma(-\theta_0 - \theta_t + \sigma) \Gamma(1 + \theta_1 + \theta_t + \bar{\sigma}) \Gamma(1 - \theta_1 + \theta_t + \bar{\sigma})}{\Gamma(1 + \theta_0 + \theta_t + \sigma) \Gamma(1 - \theta_0 + \theta_t - \sigma) \Gamma(\theta_1 - \theta_t + \bar{\sigma}) \Gamma(-\theta_1 - \theta_t + \bar{\sigma})} \times \quad (3.77c)$$

$$\times \frac{e^{-2\pi i \theta_t} (\cos 2\pi \theta_t - \cos 2\pi (\theta_0 - \sigma)) (c_{12} e^{-\pi i \sigma} - c_{22} e^{\pi i \sigma + i\eta})}{c_{12} e^{-\pi i \sigma} (\cos 2\pi (\theta_t + \sigma) - \cos 2\pi \theta_0) - c_{22} e^{\pi i \sigma + i\eta} (\cos 2\pi (\theta_t - \sigma) - \cos 2\pi \theta_0)},$$

$$\frac{\bar{r}_e \bar{c}_{1b}^e c_{0a}^i}{r_i \bar{c}_{1a}^e c_{0b}^i} = \frac{\Gamma(1 - \theta_0 + \theta_t + \sigma) \Gamma(1 - \theta_0 + \theta_t - \sigma) \Gamma(1 + \theta_0 + \theta_\infty - \bar{\sigma}) \Gamma(1 + \theta_0 - \theta_\infty - \bar{\sigma})}{\Gamma(1 + \theta_0 + \theta_t + \sigma) \Gamma(1 + \theta_0 + \theta_t - \sigma) \Gamma(-\theta_0 + \theta_\infty - \bar{\sigma}) \Gamma(-\theta_0 - \theta_\infty - \bar{\sigma})} \times \quad (3.77d)$$

$$\times \left(-e^{2\pi i \theta_0} \right) \frac{c_{12} \sin \pi (\theta_0 - \theta_t - \sigma) - c_{22} e^{i\eta} \sin \pi (\theta_0 - \theta_t + \sigma)}{c_{12} \sin \pi (\theta_0 + \theta_t + \sigma) - c_{22} e^{i\eta} \sin \pi (\theta_0 + \theta_t - \sigma)}.$$

We now have at our disposal all ingredients of the 1-form (3.69) expressed in terms of local coordinates on the space \mathcal{M} of monodromy data.

It is still very far from obvious that the right side of (3.69) is indeed a *closed* 1-form. To check that this is the case and to find the antiderivative Υ_M , we will proceed in several steps. First let us eliminate the gamma function factors appearing in (3.77) and in the ratios $\kappa = r_i/r_e$, $\bar{\kappa} = \bar{r}_i/\bar{r}_e$ given by (3.57), (3.63) by introducing the notation

$$\Upsilon(M) = \hat{\Upsilon}(M) \prod_{\epsilon, \epsilon' = \pm} \frac{G(1 + \epsilon \bar{\sigma} + \epsilon' \theta_t - \epsilon \epsilon' \theta_1) G(1 + \epsilon \bar{\sigma} + \epsilon' \theta_0 - \epsilon \epsilon' \theta_\infty)}{G(1 + \epsilon \sigma + \epsilon' \theta_t + \epsilon \epsilon' \theta_0) G(1 + \epsilon \sigma + \epsilon' \theta_1 + \epsilon \epsilon' \theta_\infty)} \prod_{\epsilon = \pm} \frac{G(1 + 2\epsilon \sigma)}{G(1 + 2\epsilon \bar{\sigma})}, \quad (3.78)$$

where $G(z)$ denotes Barnes G -function.

Lemma 3.27. *We have*

$$\begin{aligned} d_{\mathcal{M}} \ln \hat{\Upsilon} &= \bar{\sigma} d_{\mathcal{M}} \ln \left(\frac{\sin \pi (\theta_0 - \theta_\infty - \bar{\sigma}) \sin \pi (\theta_1 - \theta_t + \bar{\sigma})}{\sin \pi (\theta_0 + \theta_\infty + \bar{\sigma}) \sin \pi (\theta_1 + \theta_t + \bar{\sigma})} e^{i\eta} \right) - \sigma d_{\mathcal{M}} \ln \left(\frac{\sin \pi (\theta_0 + \theta_t - \sigma) \sin \pi (\theta_1 + \theta_\infty - \sigma)}{\sin \pi (\theta_0 - \theta_t - \sigma) \sin \pi (\theta_1 - \theta_\infty + \sigma)} e^{i\eta} \right) \\ &+ \theta_\infty d_{\mathcal{M}} \ln \left(\frac{\sin \pi (\theta_1 + \theta_\infty - \sigma) \sin \pi (\theta_0 - \theta_\infty - \bar{\sigma})}{\sin \pi (\theta_1 - \theta_\infty + \sigma) \sin \pi (\theta_0 - \theta_\infty + \bar{\sigma})} \frac{c_{12} \sin \pi (\theta_1 - \theta_\infty - \sigma) - c_{22} \sin \pi (\theta_1 - \theta_\infty + \sigma)}{c_{12} \sin \pi (\theta_1 + \theta_\infty - \sigma) - c_{22} \sin \pi (\theta_1 + \theta_\infty + \sigma)} e^{-2\pi i \theta_\infty} \right) \\ &+ \theta_1 d_{\mathcal{M}} \ln \left(\frac{\sin \pi (\theta_1 + \theta_t - \bar{\sigma})}{\sin \pi (\theta_1 + \theta_t + \bar{\sigma})} \frac{e^{2\pi i \theta_1} (\cos 2\pi \theta_1 - \cos 2\pi (\theta_\infty - \sigma)) (c_{12} e^{-\pi i \sigma} - c_{22} e^{\pi i \sigma})}{c_{12} e^{-\pi i \sigma} (\cos 2\pi (\theta_1 - \sigma) - \cos 2\pi \theta_\infty) - c_{22} e^{\pi i \sigma} (\cos 2\pi (\theta_1 + \sigma) - \cos 2\pi \theta_\infty)} \right) \\ &+ \theta_t d_{\mathcal{M}} \ln \left(\frac{\sin \pi (\theta_1 + \theta_t - \bar{\sigma})}{\sin \pi (\theta_1 + \theta_t + \bar{\sigma})} \frac{(\cos 2\pi \theta_t - \cos 2\pi (\theta_0 - \sigma)) (c_{12} e^{-\pi i \sigma} - c_{22} e^{\pi i \sigma + i\eta})}{c_{12} e^{-\pi i \sigma} (\cos 2\pi (\theta_t + \sigma) - \cos 2\pi \theta_0) - c_{22} e^{\pi i \sigma + i\eta} (\cos 2\pi (\theta_t - \sigma) - \cos 2\pi \theta_0)} e^{-2\pi i \theta_t} \right) \\ &+ \theta_0 d_{\mathcal{M}} \ln \left(\frac{\sin \pi (\theta_0 + \theta_t - \sigma) \sin \pi (\theta_0 - \theta_\infty + \bar{\sigma})}{\sin \pi (\theta_0 - \theta_t - \sigma) \sin \pi (\theta_0 - \theta_\infty - \bar{\sigma})} \frac{c_{12} \sin \pi (\theta_0 - \theta_t - \sigma) - c_{22} e^{i\eta} \sin \pi (\theta_0 - \theta_t + \sigma)}{c_{12} \sin \pi (\theta_0 + \theta_t + \sigma) - c_{22} e^{i\eta} \sin \pi (\theta_0 + \theta_t - \sigma)} e^{2\pi i \theta_0} \right). \quad (3.79) \end{aligned}$$

Proof. The crucial point here is that the logarithmic derivative of the Barnes G -function can be expressed in terms of the digamma function. It is convenient to write the corresponding formula as

$$d \ln G(1+z) = \frac{\ln 2\pi - 1}{2} dz - d \left(\frac{z^2}{2} \right) + z d \ln \Gamma(1+z). \quad (3.80)$$

The contributions of the first two terms on the right sum up to zero in the logarithmic derivative of the Barnes function factor from (3.78). The contributions of the third term in combination with gamma factors in (3.77) simplify to trigonometric expressions multiplied by rational combinations of $\theta_{0,t,1,\infty}$, σ , $\bar{\sigma}$. The latter rational part rather nontrivially simplifies and cancel out the term $2d_{\mathcal{M}}(\theta_1 - \theta_0)$ in (3.69). \square

Let us briefly recall the notations used in (3.79):

- Trace functions p_{t1} and p_{01} are expressed in terms of σ and η by (3.35a) and (3.35b); we can take for $\bar{\sigma}$ any of the two solutions of $p_{1t} = 2 \cos 2\pi \bar{\sigma}$ satisfying $|\Re \bar{\sigma}| < \frac{1}{2}$.
- The quantity $e^{i\bar{\eta}}$ is expressed in terms of $\sigma, \eta, \bar{\sigma}$ and p_{01} by (3.64).
- The coefficients c_{12}, c_{22} are expressed in terms of σ, η and $\bar{\sigma}$ by (3.72).

The dependence of $d_{\mathcal{M}} \ln \hat{Y}$ on monodromy parameters is therefore trigonometric. In particular, one can verify that the right side of (3.79) is indeed closed by direct (albeit very lengthy) calculation.

Somewhat cumbersome form of the derivatives and complicated relations between parameters may leave an impression that integrating $d_{\mathcal{M}} \ln \hat{Y}$ explicitly is a hopeless task. However, a conjectural answer for $Y(M)$ has been already produced in [ILT13]. Hence all we have to do is to compute its logarithmic derivatives and check that they coincide with those given by (3.79).

We start by preparing a convenient notation. Let us introduce an antisymmetrized combination of Barnes functions (it is related to classical dilogarithm but has much nicer analytic properties)

$$\hat{G}(z) = \frac{G(1+z)}{G(1-z)}. \quad (3.81)$$

Two main properties of $\hat{G}(z)$ that will be important for us are its differentiation formula and recursion relation. They can be written as

$$d \ln \hat{G}(z) = \ln 2\pi dz - z d \ln \sin \pi z, \quad (3.82)$$

$$\frac{\hat{G}(z+1)}{\hat{G}(z)} = -\frac{\pi}{\sin \pi z}. \quad (3.83)$$

Define the parameters $\nu_{1\dots 4}, \lambda_{1\dots 4}, \nu_{\Sigma}$ by

$$\begin{aligned} \nu_1 &= \sigma + \theta_0 + \theta_t, & \lambda_1 &= \theta_0 + \theta_t + \theta_1 + \theta_{\infty}, \\ \nu_2 &= \sigma + \theta_1 + \theta_{\infty}, & \lambda_2 &= \sigma + \bar{\sigma} + \theta_0 + \theta_1, \\ \nu_3 &= \bar{\sigma} + \theta_0 + \theta_{\infty}, & \lambda_3 &= \sigma + \bar{\sigma} + \theta_t + \theta_{\infty}, \\ \nu_4 &= \bar{\sigma} + \theta_t + \theta_1, & \lambda_4 &= 0, \\ 2\nu_{\Sigma} &= \sum_{k=1}^4 \nu_k = \sum_{k=1}^4 \lambda_k, \end{aligned} \quad (3.84)$$

Let us also introduce parameters ξ, ζ by

$$\xi \equiv e^{2\pi i \zeta} = \frac{2 \cos 2\pi (\sigma - \bar{\sigma}) - 2 \cos 2\pi (\theta_0 + \theta_1) - 2 \cos 2\pi (\theta_{\infty} + \theta_t) + p_{01}}{\sum_{k=1}^4 (e^{2\pi i (\nu_{\Sigma} - \nu_k)} - e^{2\pi i (\nu_{\Sigma} - \lambda_k)})}. \quad (3.85)$$

This equation defines ζ only up to integer shifts. As we shall see in a moment, this ambiguity turns out to be harmless. Note that ξ gives one of the two nontrivial roots of the equation

$$\prod_{k=1}^4 (1 - \xi e^{2\pi i \nu_k}) = \prod_{k=1}^4 (1 - \xi e^{2\pi i \lambda_k}). \quad (3.86)$$

Lemma 3.28. *We have*

$$\hat{Y}(M) = \prod_{k=1}^4 \frac{\hat{G}(\zeta + \nu_k)}{\hat{G}(\zeta + \lambda_k)}. \quad (3.87)$$

Proof. Let us first check that the right side of (3.87) is well-defined. Indeed, in view of the recurrence relation (3.83), the shift $\zeta \mapsto \zeta + 1$ is equivalent to multiplying the corresponding expression by

$$\prod_{k=1}^4 \frac{\sin \pi (\zeta + \lambda_k)}{\sin \pi (\zeta + \nu_k)} = 1, \quad (3.88)$$

where the equality is nothing but a rewrite of the equation (3.86).

Computing the differential with the help of (3.82), we get

$$\begin{aligned} d_{\mathcal{M}} \ln \prod_{k=1}^4 \frac{\hat{G}(\zeta + \nu_k)}{\hat{G}(\zeta + \lambda_k)} &= \sum_{k=1}^4 \left((\zeta + \lambda_k) d_{\mathcal{M}} \ln \sin \pi(\zeta + \lambda_k) - (\zeta + \nu_k) d_{\mathcal{M}} \ln \sin \pi(\zeta + \nu_k) \right) = \\ &= \sum_{k=1}^4 \left(\lambda_k d_{\mathcal{M}} \ln \sin \pi(\zeta + \lambda_k) - \nu_k d_{\mathcal{M}} \ln \sin \pi(\zeta + \nu_k) \right), \end{aligned}$$

where the last equality easily follows from (3.88). Taking into account the definition (3.84) of parameters ν_k, λ_k , the last expression can be rewritten as

$$\begin{aligned} d_{\mathcal{M}} \ln \prod_{k=1}^4 \frac{\hat{G}(\zeta + \nu_k)}{\hat{G}(\zeta + \lambda_k)} &= \bar{\sigma} d_{\mathcal{M}} \ln \frac{\sin \pi(\zeta + \lambda_2) \sin \pi(\zeta + \lambda_3)}{\sin \pi(\zeta + \nu_3) \sin \pi(\zeta + \nu_4)} - \sigma d_{\mathcal{M}} \ln \frac{\sin \pi(\zeta + \nu_1) \sin \pi(\zeta + \nu_2)}{\sin \pi(\zeta + \lambda_2) \sin \pi(\zeta + \lambda_3)} + \\ &+ \theta_{\infty} d_{\mathcal{M}} \ln \frac{\sin \pi(\zeta + \lambda_1) \sin \pi(\zeta + \lambda_3)}{\sin \pi(\zeta + \nu_2) \sin \pi(\zeta + \nu_3)} + \theta_1 d_{\mathcal{M}} \ln \frac{\sin \pi(\zeta + \lambda_1) \sin \pi(\zeta + \lambda_2)}{\sin \pi(\zeta + \nu_2) \sin \pi(\zeta + \nu_4)} + \\ &+ \theta_t d_{\mathcal{M}} \ln \frac{\sin \pi(\zeta + \lambda_1) \sin \pi(\zeta + \lambda_3)}{\sin \pi(\zeta + \nu_1) \sin \pi(\zeta + \nu_4)} + \theta_0 d_{\mathcal{M}} \ln \frac{\sin \pi(\zeta + \lambda_1) \sin \pi(\zeta + \lambda_2)}{\sin \pi(\zeta + \nu_1) \sin \pi(\zeta + \nu_3)}. \end{aligned}$$

To show that this coincides with 1-form in the right side of equation (3.79), it suffices to check six trigonometric identities, namely:

$$\begin{aligned} \frac{\sin \pi(\zeta + \lambda_2) \sin \pi(\zeta + \lambda_3)}{\sin \pi(\zeta + \nu_3) \sin \pi(\zeta + \nu_4)} &= \frac{\sin \pi(\theta_{\infty} - \theta_0 + \bar{\sigma}) \sin \pi(\theta_1 - \theta_t + \bar{\sigma})}{\sin \pi(\theta_{\infty} + \theta_0 + \bar{\sigma}) \sin \pi(\theta_1 + \theta_t + \bar{\sigma})} e^{i\bar{\eta}}, \\ \frac{\sin \pi(\zeta + \nu_1) \sin \pi(\zeta + \nu_2)}{\sin \pi(\zeta + \lambda_2) \sin \pi(\zeta + \lambda_3)} &= \frac{\sin \pi(\theta_0 + \theta_t - \sigma) \sin \pi(\theta_{\infty} + \theta_1 - \sigma)}{\sin \pi(\theta_0 - \theta_t - \sigma) \sin \pi(\theta_{\infty} - \theta_1 - \sigma)} e^{i\eta}, \\ \frac{\sin \pi(\zeta + \lambda_1) \sin \pi(\zeta + \lambda_3)}{\sin \pi(\zeta + \nu_2) \sin \pi(\zeta + \nu_3)} &= \frac{\sin \pi(\theta_{\infty} + \theta_1 - \sigma) \sin \pi(\theta_0 - \theta_{\infty} - \bar{\sigma})}{\sin \pi(\theta_{\infty} - \theta_1 - \sigma) \sin \pi(\theta_0 - \theta_{\infty} + \bar{\sigma})} \times \\ &\times \frac{c_{12} \sin \pi(\theta_1 - \theta_{\infty} - \sigma) - c_{22} \sin \pi(\theta_1 - \theta_{\infty} + \sigma)}{c_{12} \sin \pi(\theta_1 + \theta_{\infty} - \sigma) - c_{22} \sin \pi(\theta_1 + \theta_{\infty} + \sigma)} e^{-2\pi i \theta_{\infty}}, \\ \frac{\sin \pi(\zeta + \lambda_1) \sin \pi(\zeta + \lambda_2)}{\sin \pi(\zeta + \nu_2) \sin \pi(\zeta + \nu_4)} &= \frac{\sin \pi(\theta_1 + \theta_t - \bar{\sigma})}{\sin \pi(\theta_1 + \theta_t + \bar{\sigma})} \times \\ &\times \frac{(\cos 2\pi(\theta_{\infty} - \sigma) - \cos 2\pi\theta_1) (c_{12} e^{-\pi i \sigma} - c_{22} e^{\pi i \sigma}) e^{2\pi i \theta_1}}{c_{12} e^{-\pi i \sigma} (\cos 2\pi(\theta_1 - \sigma) - \cos 2\pi\theta_{\infty}) - c_{22} e^{\pi i \sigma} (\cos 2\pi(\theta_1 + \sigma) - \cos 2\pi\theta_{\infty})}, \\ \frac{\sin \pi(\zeta + \lambda_1) \sin \pi(\zeta + \lambda_3)}{\sin \pi(\zeta + \nu_1) \sin \pi(\zeta + \nu_4)} &= \frac{\sin \pi(\theta_1 + \theta_t - \bar{\sigma})}{\sin \pi(\theta_1 + \theta_t + \bar{\sigma})} \times \\ &\times \frac{(\cos 2\pi(\theta_0 - \sigma) - \cos 2\pi\theta_t) (c_{12} e^{-\pi i \sigma} - c_{22} e^{\pi i \sigma + i\eta}) e^{-2\pi i \theta_t}}{c_{12} e^{-\pi i \sigma} (\cos 2\pi(\theta_t + \sigma) - \cos 2\pi\theta_0) - c_{22} e^{\pi i \sigma + i\eta} (\cos 2\pi(\theta_t - \sigma) - \cos 2\pi\theta_0)}, \\ \frac{\sin \pi(\zeta + \lambda_1) \sin \pi(\zeta + \lambda_2)}{\sin \pi(\zeta + \nu_1) \sin \pi(\zeta + \nu_3)} &= \frac{\sin \pi(\theta_0 + \theta_t - \sigma) \sin \pi(\theta_0 - \theta_{\infty} + \bar{\sigma})}{\sin \pi(\theta_0 - \theta_t - \sigma) \sin \pi(\theta_0 - \theta_{\infty} - \bar{\sigma})} \times \\ &\times \frac{c_{12} \sin \pi(\theta_t - \theta_0 + \sigma) - c_{22} e^{i\eta} \sin \pi(\theta_t - \theta_0 - \sigma)}{c_{12} \sin \pi(\theta_t + \theta_0 + \sigma) - c_{22} e^{i\eta} \sin \pi(\theta_t + \theta_0 - \sigma)} e^{2\pi i \theta_0}. \end{aligned}$$

This can be done by direct algebraic manipulation using (3.35), (3.38), (3.64), (3.72) and (3.85).

The right side of (3.87) thus coincides with $\hat{Y}(M)$ up to a constant independent on monodromy data. To show that this constant is equal to 1, it suffices to compute it for any Painlevé VI solution with monodromy satisfying previous assumptions, see e.g. [KK, LT] for explicit elliptic and algebraic examples. A number of such checks has already been reported in [ILT13, ILST]. \square

We can finally summarize our developments in the following

Theorem 3.29. *Under conditions on monodromy data formulated in Assumptions 2.1, 3.1, 3.11 and 3.22, connection constant $Y(M)$ for the Painlevé VI tau function is given by*

$$Y(M) = \prod_{\epsilon, \epsilon' = \pm} \frac{G(1 + \epsilon\bar{\sigma} + \epsilon'\theta_t - \epsilon\epsilon'\theta_1) G(1 + \epsilon\bar{\sigma} + \epsilon'\theta_0 - \epsilon\epsilon'\theta_{\infty})}{G(1 + \epsilon\sigma + \epsilon'\theta_t + \epsilon\epsilon'\theta_0) G(1 + \epsilon\sigma + \epsilon'\theta_1 + \epsilon\epsilon'\theta_{\infty})} \prod_{\epsilon = \pm} \frac{G(1 + 2\epsilon\sigma)}{G(1 + 2\epsilon\bar{\sigma})} \prod_{k=1}^4 \frac{\hat{G}(\zeta + \nu_k)}{\hat{G}(\zeta + \lambda_k)}. \quad (3.89)$$

4 Systems with irregular singularities

4.1 Extended Jimbo-Miwa-Ueno differential

In this subsection we sketch the modifications that should be brought to the Fuchsian setup in the general case of systems with $n + 1$ irregular singularities at $a_1, \dots, a_n, a_\infty = \infty$ on \mathbb{P} . In this case, the system (1.1) can be rewritten as

$$\frac{d\Phi}{dz} = A(z)\Phi, \quad A(z) = \sum_{v=1}^n \sum_{k=1}^{r_v+1} \frac{A_{v,-k+1}}{(z-a_v)^k} + \sum_{k=0}^{r_\infty-1} z^k A_{\infty,-k-1}. \quad (4.1)$$

It may be assumed without any loss in generality that $\text{Tr} A(z) = 0$, i.e. that $A_{v,-k+1}, A_{\infty,-k-1} \in \mathfrak{sl}_N(\mathbb{C})$. As it has already been indicated in the Introduction, we shall also assume that all highest order matrix coefficients $A_v \equiv A_{v,-r_v}$ are diagonalizable

$$A_{v,-r_v} = G_v \Theta_{v,-r_v} G_v^{-1}; \quad \Theta_{v,-r_v} = \text{diag}\{\theta_{v,1}, \dots, \theta_{v,N}\},$$

and that their eigenvalues are distinct and non-resonant:

$$\begin{cases} \theta_{v,\alpha} \neq \theta_{v,\beta} & \text{if } r_v \geq 1, \quad \alpha \neq \beta, \\ \theta_{v,\alpha} \neq \theta_{v,\beta} \pmod{\mathbb{Z}} & \text{if } r_v = 0, \quad \alpha \neq \beta. \end{cases}$$

If the *Poincaré index* r_v of the pole a_v is greater or equal to 1, then the pole is called an *irregular singular point* of the system (4.1). In the neighborhood of such a point the asymptotic behavior of solution $\Phi(z)$ exhibits the *Stokes Phenomenon* which is described as follows.

Let a_v be an irregular singular point of index r_v . For $j = 1, \dots, 2r_v + 1$, let

$$\Omega_{j,v} = \left\{ z : 0 < |z - a_v| < \epsilon, \quad \theta_j^{(1)} < \arg(z - a_v) < \theta_j^{(2)}, \quad \theta_j^{(2)} - \theta_j^{(1)} = \frac{\pi}{r_v} + \delta \right\}, \quad (4.2)$$

be the *Stokes sectors* around a_v (see, e.g., [FIKN, Chapter 1] or [Was] for more details). According to the general theory of linear systems, in each sector $\Omega_{j,v}$ there exists a unique *canonical solution* $\Phi_j^{(v)}(z)$ of (4.1) which satisfies the asymptotic condition

$$\Phi_j^{(v)}(z) \simeq \Phi_{\text{form}}^{(v)}(z) \quad \text{as } z \rightarrow a_v, \quad z \in \Omega_{j,v}, \quad j = 1, \dots, 2r_v + 1, \quad (4.3)$$

where $\Phi_{\text{form}}^{(v)}(z)$ is the formal solution at the point a_v which has already been mentioned in the Introduction. For reader's convenience, we reproduce here the relevant formulae:

$$\Phi_{\text{form}}^{(v)}(z) = G^{(v)}(z) e^{\Theta_v(z)}, \quad G^{(v)}(z) = G_v \hat{\Phi}^{(v)}(z), \quad (4.4)$$

where

$$\hat{\Phi}^{(v)}(z) = \begin{cases} \mathbf{1} + \sum_{k=1}^{\infty} g_{v,k} (z - a_v)^k, & v = 1, \dots, n, \\ \mathbf{1} + \sum_{k=1}^{\infty} g_{\infty,k} z^{-k}, & v = \infty, \end{cases}$$

and $\Theta_v(z)$ are diagonal matrix-valued functions,

$$\Theta_v(z) = \begin{cases} \sum_{k=-r_v}^{-1} \frac{\Theta_{v,k}}{k} (z - a_v)^k + \Theta_{v,0} \ln(z - a_v), & v = 1, \dots, n \\ -\sum_{k=1}^{r_\infty} \frac{\Theta_{\infty,-k}}{k} z^k - \Theta_{\infty,0} \ln z, & v = \infty. \end{cases}$$

Among the identities that determine $\Theta_v(z)$, $\hat{\Phi}^{(v)}(z)$ and $G^{(v)}(z)$ in terms of $A(z)$ and G_v there is a particularly important family of relations that will be repeatedly used in what follows. Namely, the structure of the formal solution (4.4) implies that

$$A(z) - G^{(v)}(z) \Theta_v'(z) G^{(v)}(z)^{-1} = \begin{cases} O(1), & v = 1, \dots, n, \\ O(z^{-2}), & v = \infty. \end{cases} \quad (4.5)$$

Here and below the prime denotes the derivative with respect to z . The matrix $A(z)$ can thus be reconstructed by taking the sum of principal parts of Laurent series $G^{(v)}(z)\Theta'_v(z)G^{(v)}(z)^{-1}$ at $z = a_v$ (plus a constant part for the point at ∞).

Stokes and connection matrices relate the canonical solutions $\Phi_j^{(v)}(z)$ in different Stokes sectors and at different singular points:

$$\Phi_{j+1}^{(v)} = \Phi_j^{(v)} S_j^{(v)}, \quad j = 1, \dots, 2r_v, \quad \Phi_1^{(v)} = \Phi_1^{(\infty)} C_v, \quad v = 1, \dots, n.$$

Let us assume as before that the irregular singular points are ∞ and the first $m \leq n$ points among the singular points a_1, \dots, a_n . Denote by \mathcal{S}_v the collection of Stokes matrices at an irregular point a_v , i.e.

$$\mathcal{S}_v = \{S_1^{(v)}, \dots, S_{2r_v}^{(v)}\}. \quad (4.6)$$

The space \mathcal{M} of monodromy data of the system (4.1) consists of formal monodromy exponents $\Theta_{v,0}$, connection matrices C_v and Stokes matrices $S_j^{(v)}$. The latter constitute the main difference as compared to the Fuchsian case. More explicitly,

$$\mathcal{M} = \left\{ M \equiv \left(\Theta_{v,0}, v = 1, \dots, n, \infty; \quad C_v, v = 1, \dots, n; \quad \mathcal{S}_v, v = 1, \dots, m, \infty \right) \right\}. \quad (4.7)$$

Following the notations used in the Introduction, we denote by \mathcal{T} the set of times

$$a_1, \dots, a_n, \quad (\Theta_{v,k})_{ll}, \quad k = -r_v, \dots, -1, \quad v = 1, \dots, m, \infty, \quad l = 1, \dots, N. \quad (4.8)$$

Observe that in contrast to the Fuchsian case this set contains much more parameters than just the positions of singular points. Let us retain the notation

$$\vec{t} = (t_1, \dots, t_L), \quad L = n + N \left(\sum_{v=1}^m r_v + r_\infty \right),$$

for the points $\vec{t} \in \mathcal{T}$ and consider monodromy preserving deformations of the system (4.1) with respect to these times. We denote by $A(z) \equiv A(z; \vec{t}; M)$ the isomonodromic family of the systems (4.1) having the same set $M \in \mathcal{M}$ of monodromy data. The isomonodromy implies that the corresponding solution $\Phi(z) \equiv \Phi(z, \vec{t})$ satisfies an overdetermined system

$$\begin{cases} \partial_z \Phi = A(z, \vec{t}) \Phi(z, \vec{t}), \\ d_{\mathcal{T}} \Phi = U(z, \vec{t}) \Phi(z, \vec{t}) \end{cases} \quad (4.9)$$

The coefficients of the matrix-valued differential form $U \equiv \sum_{k=1}^L U_k(z, \vec{t}) dt_k$ are rational in z . Their explicit form may be algorithmically deduced from the expression for $A(z)$. The compatibility of the system (4.9) implies the monodromy preserving deformation equation (1.7):

$$d_{\mathcal{T}} A = \partial_z U + [U, A]. \quad (4.10)$$

The construction of an irregular analog of the 1-form ω defined in the Fuchsian case by the equation (2.7) is carried out in several steps. Let us first recall once again the standard definition of the Jimbo-Miwa-Ueno differential [JMU, equation (5.1)],

$$\omega_{\text{JMU}} = - \sum_{v=1, \dots, n, \infty} \text{res}_{z=a_v} \text{Tr} \left(\hat{\Phi}^{(v)}(z)^{-1} \partial_z \hat{\Phi}^{(v)}(z) d_{\mathcal{T}} \Theta_v(z) \right). \quad (4.11)$$

This 1-form is closed on solutions of the isomonodromy equation (4.10):

$$d_{\mathcal{T}} \omega_{\text{JMU}} = 0.$$

Our goal is to find an extension of it which would be closed on the whole space $\mathcal{A} \simeq \widetilde{\mathcal{T}} \times \mathcal{M}$ and which would coincide with (4.11) when restricted to \mathcal{T} . To this end, it is convenient to rewrite ω_{JMU} in a slightly different way, cf [JMU, Remark 5.2].

Lemma 4.1. *The 1-form ω_{JMU} can be alternatively written as*

$$\omega_{\text{JMU}} = \sum_{v=1, \dots, n, \infty} \text{res}_{z=a_v} \text{Tr} \left(A(z) d_{\mathcal{T}} G^{(v)}(z) G^{(v)}(z)^{-1} \right). \quad (4.12)$$

Proof. Formal series $G^{(v)}(z)$ appears in the asymptotic behavior of an actual solution $\Phi(z)$ in some Stokes sector as indicated in (4.4). The isomonodromy property then implies that $d_{\mathcal{T}} G^{(v)}(z) G^{(v)}(z)^{-1}$ can be replaced by the combination

$$d_{\mathcal{T}} G^{(v)} G^{(v)-1} = d_{\mathcal{T}} \Phi \Phi^{-1} - \Phi d_{\mathcal{T}} \Theta_v \Phi^{-1} = U - G^{(v)} d_{\mathcal{T}} \Theta_v G^{(v)-1}. \quad (4.13)$$

in the whole punctured neighborhood of a_v . The last equality follows from the second equation of the Lax representation (4.9) and the diagonal form of $\Theta_v(z)$. Since a_1, \dots, a_n, ∞ are the only possible poles of the rational matrix function $A(z)U(z)$, we necessarily have $\sum_{v=1, \dots, n, \infty} \text{res}_{z=a_v} \text{Tr}(A(z)U(z)) = 0$. Using this sum rule, the equation (4.13) and the relation

$$A = \Phi' \Phi^{-1} = G^{(v)'} G^{(v)-1} + G^{(v)} \Theta_v' G^{(v)-1}, \quad (4.14)$$

the right side of (4.12) can be rewritten as

$$- \sum_{v=1, \dots, n, \infty} \text{res}_{z=a_v} \text{Tr} \left(A G^{(v)} d_{\mathcal{T}} \Theta_v G^{(v)-1} \right) = \omega_{\text{JMU}} - \sum_{v=1, \dots, n, \infty} \text{res}_{z=a_v} \text{Tr} \left(\Theta_v' d_{\mathcal{T}} \Theta_v \right).$$

All residues in the second sum obviously vanish (Θ_v' and $d_{\mathcal{T}} \Theta_v$ are Laurent polynomials with only principal part), which proves the statement of the lemma. \square

The expression (4.12) is much better adapted for extension to $\widetilde{\mathcal{T}} \times \mathcal{M}$ than the original formula (4.11). Indeed, we have the following result:

Theorem 4.2. *Let $\omega \in \Lambda^1(\widetilde{\mathcal{T}} \times \mathcal{M})$ be a 1-form defined by*

$$\omega = \sum_{v=1, \dots, n, \infty} \text{res}_{z=a_v} \text{Tr} \left(A(z) dG^{(v)}(z) G^{(v)}(z)^{-1} \right), \quad (4.15)$$

where $d = d_{\mathcal{T}} + d_{\mathcal{M}}$. Its exterior differential $\Omega := d\omega$ is a closed 2-form on \mathcal{M} independent of isomonodromic times $\vec{t} \in \mathcal{T}$ listed in (4.8).

Proof. Notice that the residues in (4.15) are completely determined by the singular parts of the corresponding Laurent expansions of $A(z)$. Using their identification (4.5), we may replace $A(z)$ by $G^{(v)} \Theta_v' G^{(v)-1}$ in the computation of each residue. This yields another representation of the form ω :

$$\omega = \sum_{v=1, \dots, n, \infty} \text{res}_{z=a_v} \text{Tr} \left(\Theta_v'(z) G^{(v)}(z)^{-1} dG^{(v)}(z) \right). \quad (4.16)$$

Let us now pick an arbitrary isomonodromic time t_k and a parameter s which can be either a local coordinate on the space \mathcal{M} of monodromy data or another time variable. We are going to show that $\Omega(\partial_{t_k}, \partial_s) = 0$. First, from (4.16) it follows that

$$\begin{aligned} \Omega(\partial_{t_k}, \partial_s) &= \partial_{t_k} \omega(\partial_s) - \partial_s \omega(\partial_{t_k}) = \\ &= \sum_{v=1, \dots, n, \infty} \text{res}_{z=a_v} \text{Tr} \left(\partial_{t_k} \Theta_v' G^{(v)-1} \partial_s G^{(v)} - \partial_s \Theta_v' G^{(v)-1} \partial_{t_k} G^{(v)} + \Theta_v' \left[G^{(v)-1} \partial_s G^{(v)}, G^{(v)-1} \partial_{t_k} G^{(v)} \right] \right). \end{aligned}$$

Thanks to cyclic properties of the trace, we can make in the last line the following replacements:

$$\begin{aligned} \partial_s \Theta_v' G^{(v)-1} \partial_{t_k} G^{(v)} &\mapsto \partial_{t_k} G^{(v)} \partial_s \Theta_v' G^{(v)-1}, \\ \Theta_v' \left[G^{(v)-1} \partial_s G^{(v)}, G^{(v)-1} \partial_{t_k} G^{(v)} \right] &\mapsto \partial_{t_k} G^{(v)} \left[\Theta_v', G^{(v)-1} \partial_s G^{(v)} \right] G^{(v)-1}, \end{aligned}$$

so that the expression under trace becomes

$$\partial_{t_k} \Theta'_v G^{(v)-1} \partial_s G^{(v)} - \partial_{t_k} G^{(v)} G^{(v)-1} \partial_s \left(G^{(v)} \Theta'_v G^{(v)-1} \right).$$

Using once again the coincidence of singular parts of $A(z)$ and $G^{(v)} \Theta'_v G^{(v)-1}$, let us rewrite the expression for the curvature coefficient $\Omega(\partial_{t_k}, \partial_s)$ as

$$\Omega(\partial_{t_k}, \partial_s) = \sum_{v=1, \dots, n, \infty} \operatorname{res}_{z=a_v} \operatorname{Tr} \left(\partial_{t_k} \Theta'_v G^{(v)-1} \partial_s G^{(v)} - \partial_{t_k} G^{(v)} G^{(v)-1} \partial_s A \right). \quad (4.17)$$

The next step is to use isomonodromy and the equation $\partial_{t_k} \Phi = U_{t_k} \Phi$ in a way similar to the proof of Lemma 4.1. Since (cf equation (4.13))

$$\partial_{t_k} G^{(v)} G^{(v)-1} = U_{t_k} - G^{(v)} \partial_{t_k} \Theta_v G^{(v)-1},$$

and the sum of residues of rational function $U_{t_k}(z) \partial_s A(z)$ at $z = a_1, \dots, a_n, \infty$ is clearly equal to zero, the expression (4.17) can be reduced to

$$\Omega(\partial_{t_k}, \partial_s) = \sum_{v=1, \dots, n, \infty} \operatorname{res}_{z=a_v} \operatorname{Tr} \left(\partial_{t_k} \Theta'_v G^{(v)-1} \partial_s G^{(v)} + G^{(v)} \partial_{t_k} \Theta_v G^{(v)-1} \partial_s A \right). \quad (4.18)$$

Substitute the matrix A in the second term of this identity by its expression (4.14). This transforms the expression under trace in (4.18) into a sum of six terms:

$$\begin{aligned} & \partial_{t_k} \Theta'_v G^{(v)-1} \partial_s G^{(v)} + G^{(v)} \partial_{t_k} \Theta_v G^{(v)-1} \partial_s G^{(v)'} G^{(v)-1} - G^{(v)} \partial_{t_k} \Theta_v G^{(v)-1} G^{(v)'} G^{(v)-1} \partial_s G^{(v)} G^{(v)-1} + \\ & + G^{(v)} \partial_{t_k} \Theta_v G^{(v)-1} \partial_s G^{(v)} \Theta'_v G^{(v)-1} + G^{(v)} \partial_{t_k} \Theta_v \partial_s \Theta'_v G^{(v)-1} - G^{(v)} \partial_{t_k} \Theta_v \Theta'_v G^{(v)-1} \partial_s G^{(v)} G^{(v)-1}. \end{aligned} \quad (4.19)$$

The fourth and sixth term cancel each other thanks to cyclic property of the trace and the diagonal form of Θ_v implying that $[\partial_{t_k} \Theta_v, \Theta'_v] = 0$.

The last step consists essentially in integration by parts. Namely, we are going to use that

$$\operatorname{res}_{z=a_v} f' g = - \operatorname{res}_{z=a_v} f g' \quad (4.20)$$

for arbitrary pair of functions $f(z)$, $g(z)$ analytic in a punctured neighborhood of $z = a_v$. Applying this formula to, say, the first term in (4.19) yields two terms which cancel out with the second and third. The only contribution to the curvature thus comes from the fifth term:

$$\Omega(\partial_{t_k}, \partial_s) = \sum_{v=1, \dots, n, \infty} \operatorname{res}_{z=a_v} \operatorname{Tr} \left(\partial_{t_k} \Theta_v \partial_s \Theta'_v \right).$$

It however vanishes by the same argument as in the proof of Lemma 4.1. We have thus shown that Ω is a 2-form on \mathcal{M} only. Since in addition Ω is a total differential, it cannot depend on isomonodromic times, as otherwise $d\Omega$ would be non-zero. \square

Remark 4.3. The Fuchsian 1-form ω defined by (2.7) is a specialization of the general formula (4.15). This becomes completely manifest in the representation (4.16), since in the Fuchsian case $\Theta'_v(z)$ reduces to a simple pole contribution $\Theta_{v,0}'(z - a_v)$.

Remark 4.4. A related extension of the Jimbo-Miwa-Ueno form has been proposed by M. Bertola in the work [Ber] generalizing previous results of B. Malgrange [Mal]. It has been defined for solution $\Psi(z)$ of a general Riemann-Hilbert problem with contour Γ and jump matrix $J(z)$ as a contour integral

$$\omega_{\text{MB}}(\partial) = \frac{1}{4\pi i} \int_{\Gamma} \operatorname{Tr} \left(\Psi_-^{-1} \Psi'_- dJ J^{-1} + \Psi_+^{-1} \Psi'_+ J^{-1} dJ \right) dz. \quad (4.21)$$

That in the isomonodromic setting this Malgrange-Bertola form could localize (i.e. that the integral can be evaluated in terms of Ψ and its derivatives with respect to times and monodromy parameters) and become our form ω was first realized in the paper [IP] by two of the authors in the context of Painlevé III (D_8). Shortly

after M. Bertola pointed out how the localization should be carried out in general, see [IP, Remark 3]. The result coincides with ω introduced above up to addition of a monodromy- and contour-dependent term. An intriguing feature of the contour integral set-up is that it may allow for the computation of the differential $d\omega_{\text{MB}}$ in terms of monodromy data [Ber1] bypassing asymptotic analysis. It has thus a strong potential of simplifying the study of connection problems for isomonodromic tau functions, which we hope to explore in a future work. Also, we plan to relate the constructions of this section to the general Hamiltonian formalism of isomonodromic deformations developed by I. Krichever in [Kri]. It can be expected that the technique of [Kri] could provide another “asymptotics free” derivation of the monodromy representation of the form Ω , alternative to the contour integral method of [Ber1].

4.2 Painlevé II case

The first nontrivial case of a non-Fuchsian system is a 2×2 linear system with one irregular singular point of Poincaré rank 3. We shall place the singular point at infinity, so that the matrix $A(z)$ in (4.1) becomes a 2nd order polynomial in z ,

$$\partial_z \Phi = A(z) \Phi, \quad A(z) = A_{-3} z^2 + A_{-2} z + A_{-1}. \quad (4.22)$$

With the help of trivial affine and gauge transformations the system can be reduced to the following normal form:

$$\partial_z \Phi = \begin{pmatrix} -4iz^2 - it - 2iuw & 4izu - 2x \\ -4izw - 2y & 4iz^2 + it + 2iuw \end{pmatrix} \Phi. \quad (4.23)$$

Here t, u, w, x, y are complex parameters playing the role of coordinates on the space \mathcal{A} of systems (4.22).

The formal solution of this system at $z = \infty$ is given by

$$\Phi_{\text{form}}^{(\infty)}(z) \equiv \Phi_{\text{form}}(z) = \left(\mathbf{1} + \sum_{k=1}^{\infty} g_k z^{-k} \right) e^{-\left(\frac{4i}{3} z^3 + itz + \kappa \ln z\right) \sigma_3}, \quad (4.24)$$

where $\sigma_3 = \text{diag}\{1, -1\}$ and $\kappa = wx - uy$. The non-formal behavior of solutions of (4.23) is described by seven canonical solutions uniquely specified by the following asymptotic conditions, cf (4.3),

$$\Phi_j(z) \simeq \Phi_{\text{form}}(z) \quad \text{as } z \rightarrow \infty, \quad z \in \Omega_j, \quad j = 1, \dots, 7, \quad (4.25)$$

where the Stokes sectors are given by

$$\Omega_j = \left\{ z : \frac{\pi(j-2)}{3} < \arg z < \frac{\pi j}{3} \right\}.$$

The canonical solutions satisfy the formal monodromy condition

$$\Phi_7(z) = \Phi_1(z) e^{-2\pi i \kappa \sigma_3}. \quad (4.26)$$

There are six Stokes matrices S_1, \dots, S_6 defined by the equations

$$S_j = \Phi_j^{-1}(z) \Phi_{j+1}(z), \quad j = 1, \dots, 6.$$

These matrices have the familiar triangular structure (see [FIKN, chapter 2, section 1.6])

$$S_{2l+1} = \begin{pmatrix} 1 & 0 \\ s_{2l+1} & 1 \end{pmatrix}, \quad S_{2l} = \begin{pmatrix} 1 & s_{2l} \\ 0 & 1 \end{pmatrix},$$

and satisfy cyclic relation

$$S_1 S_2 S_3 S_4 S_5 S_6 = e^{-2\pi i \kappa \sigma_3}, \quad (4.27)$$

which follows from (4.26). A single matrix equation (4.27) implies three scalar equations

$$\begin{aligned} 1 + s_1 s_2 &= (1 + s_4 s_5) e^{2\pi i \kappa}, & 1 + s_2 s_3 &= (1 + s_5 s_6) e^{-2\pi i \kappa}, \\ s_1 + s_3 + s_1 s_2 s_3 &= -s_5 e^{2\pi i \kappa}. \end{aligned} \quad (4.28)$$

The space \mathcal{M} of the monodromy data of system (4.23) is parametrized by seven parameters κ, s_1, \dots, s_6 subject to three constraints (4.28). Hence

$$\dim \mathcal{M} = 4 = \dim \mathcal{A} - 1. \quad (4.29)$$

The exponential factor in the right side of the formula (4.24) shows that the parameter t is the only remaining time variable (cf with 4-point Fuchsian case considered in the previous sections). The fact that isomonodromic families in the case under consideration are one-parameter also follows from (4.29). The second linear differential equation in the Lax pair (4.9) is given by

$$\partial_t \Phi = U(z) \Phi \equiv \begin{pmatrix} -iz & iu \\ -iw & iz \end{pmatrix} \Phi. \quad (4.30)$$

and the corresponding isomonodromy equation (4.10) yields the following system of nonlinear ODEs satisfied by the scalar functions $u = u(t)$, $w = w(t)$, $x = x(t)$ and $y = y(t)$,

$$\begin{aligned} x = u_t, \quad y = w_t, \quad wu_t - uw_t = \text{const} \equiv \kappa, \\ u_t = tu + 2u^2 w, \quad w_t = tw + 2w^2 u. \end{aligned} \quad (4.31)$$

In what follows we will only consider the reduction

$$u \equiv w, \quad \kappa = 0, \quad (4.32)$$

under which the system (4.31) reduces to a special case of the second Painlevé equation ⁵,

$$u_{tt} = 2u^3 + tu, \quad (4.33)$$

and the linear equations (4.23) and (4.30) form the Flaschka-Newell Lax pair for this equation [FN1].

At the level of Stokes parameters, the reduction (4.32) is equivalent to imposing the following constraints (see again [FIKN]),

$$s_4 = -s_1, \quad s_5 = -s_2, \quad s_6 = -s_3.$$

The space of monodromy data then becomes two-dimensional, and is given by an affine cubic in \mathbb{C}^3 :

$$\mathcal{M}_{\text{PII}} = \{s \equiv (s_1, s_2, s_3) \in \mathbb{C}^3 : s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0\}. \quad (4.34)$$

Solutions $u(t) \equiv u(t; s)$ of the second Painlevé equation (4.33) are parametrized by the points $s \in \mathcal{M}_{\text{PII}}$ via the inverse monodromy map, $\mathcal{M}_{\text{PII}} \rightarrow \mathcal{A}$. The latter is realized as follows.

For $j = 1, \dots, 6$, let Γ_j denote the rays

$$\Gamma_j = \left\{ z \in \mathbb{C} : \arg z = \frac{\pi(2j-1)}{6} \right\}.$$

oriented towards infinity, and let $\Omega_j^{(0)}$ be the sectors between the rays Γ_{j-1} and Γ_j . Note that $\overline{\Omega_j^{(0)}} \subset \Omega_j$. Define a piecewise analytic function $\Psi(z)$ by the relations

$$\Psi(z) = \Phi_j(z) \quad \text{for } z \in \Omega_j^{(0)}.$$

The function $\Psi(z)$ satisfies the following Riemann-Hilbert problem posed on the contour $\Gamma = \bigcup_{j=1}^6 \Gamma_j$:

- $\Psi(z)$ is analytic for $z \in \mathbb{C} \setminus \Gamma$;
- $\Psi_+(z) = \Psi_-(z) J(z)$ for $z \in \Gamma$, where $J(z \in \Gamma_j) = S_j$;
- $\Psi(z) = \left(\mathbf{1} + O\left(\frac{1}{z}\right) \right) e^{-\left(\frac{4i}{3} z^3 + itz + \kappa \ln z\right) \sigma_3}$ as $z \rightarrow \infty$.

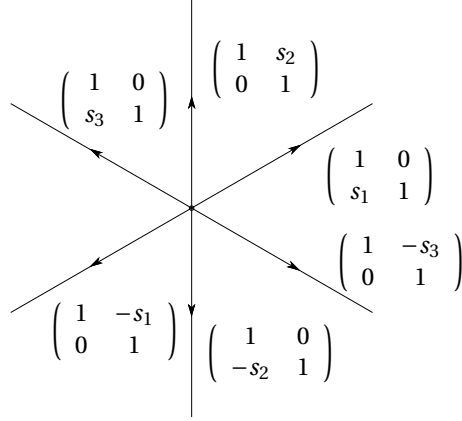


Figure 5: Contour and jump matrices of the PII-RH problem

The contour of the Riemann-Hilbert problem is depicted in Figure 5 and different components of the piecewise constant jump matrix $J(z)$ are given by the equations

$$L(s) = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}, \quad R(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0. \quad (4.35)$$

We shall refer to this Riemann-Hilbert problem as the PII-RH problem.

The PII-RH problem is meromorphically solvable. This means (see [BIK] and [FIKN], appendix A) that for every given $s \in \mathcal{M}_{\text{PII}}$ there exists a discrete set $\mathcal{K}_s = \{t_k\}_{k=1}^{\infty}$ of points in the complex t -plane such that for all $t \notin \mathcal{K}_s$ the solution $\Psi(z) \equiv \Psi(z; t)$ exists and is meromorphic in t with \mathcal{K}_s being the set of its poles. The corresponding solution $u(t; s)$ of the second Painlevé equation is given by the formula,

$$u = 2(g_1)_{12}, \quad (4.36)$$

where g_1 is the first matrix coefficient in the asymptotic expansion

$$\Psi(z) \simeq \left[\mathbf{1} + \sum_{k=1}^{\infty} g_k z^{-k} \right] e^{-\left(\frac{4i}{3}z^3 + itz + \kappa \ln z\right)\sigma_3}, \quad z \rightarrow \infty. \quad (4.37)$$

Let us now proceed to the calculation of the forms ω_{MJU} and ω corresponding to the linear system (4.23) under reduction (4.32). Since there is only one singular point, we can globally define $G(z) = \Psi(z) e^{\left(\frac{4iz^3}{3} + izt\right)\sigma_3}$ and the sum in (4.15) contains only one term,

$$\omega = \text{res}_{z=\infty} \text{Tr} \left(A(z) dG(z) G(z)^{-1} \right). \quad (4.38)$$

We shall also restrict our consideration to the part of \mathcal{M}_{PII} where Stokes parameters s_1 and s_2 can be taken as coordinates. This means that

$$dG = \frac{\partial G}{\partial t} dt + \frac{\partial G}{\partial s_1} ds_1 + \frac{\partial G}{\partial s_2} ds_2.$$

Observe that

$$\begin{aligned} \text{res}_{z=\infty} \text{Tr} \left(A_{-3} z^2 dG G^{-1} \right) &= -\text{Tr} \left(A_{-3} (dg_3 - dg_2 \cdot g_1 - dg_1 \cdot g_2 + dg_1 \cdot g_1^2) \right), \\ \text{res}_{z=\infty} \text{Tr} \left(A_{-2} z dG G^{-1} \right) &= -\text{Tr} \left(A_{-2} (dg_2 - dg_1 \cdot g_1) \right), \\ \text{res}_{z=\infty} \text{Tr} \left(A_{-1} dG G^{-1} \right) &= -\text{Tr} \left(A_{-1} dg_1 \right). \end{aligned}$$

Substitution of series (4.37) into the equation (4.23) yields the following formulae for the coefficients g_1, g_2, g_3 :

$$g_1 = \frac{u}{2} \sigma_1 + \frac{i\sigma_3}{2} (u^2 t + u^4 - u_t^2) \equiv \alpha_1 \sigma_1 + \alpha_3 \sigma_3,$$

⁵In the general case, the system (4.31) reduces to the Painlevé XXXIV equation for the product uw and to a pair of Painlevé II equations for functions u_t/u and w_t/w , with parameters $\frac{1}{2} - \kappa$ and $\frac{1}{2} + \kappa$, respectively; see, e.g. [FIKN, Chapter 4, Section 2.5]

$$\begin{aligned}
g_2 &= \frac{1}{8} (u^2 - u^4 t^2 - u^8 - u_t^4 - 2u^6 t + 2u^2 t u_t^2 + 2u^4 u_t^2) + \frac{\sigma_2}{4} (u^5 + u^3 t - u u_t^2 - u_t) \equiv \beta_1 \mathbf{1} + \beta_2 \sigma_2, \\
g_3 &= \frac{i\sigma_3}{48} (2t u_t^2 - 2u u_t - 2u^2 t^2 + u^4 t + 3u^6 - 3u^2 u_t^2 - u^6 t^3 - u^{12} + u_t^6 - 3u^{10} t - 3u^2 t u_t^4 - 3u^8 t^2 \\
&\quad + 6u^6 t u_t^2 + 3u^4 t^2 u_t^2 - 3u^4 u_t^4 + 3u^8 u_t^2) + \frac{\sigma_1}{16} (2u_t u^2 t + 2u^4 u_t - 2u_t^3 - 2u t - u^3 - u^5 t^2 - u^9 - u u_t^4 - 2u^7 t \\
&\quad + 2u^3 t u_t^2 + 2u^5 u_t^2) \equiv \gamma_1 \sigma_1 + \gamma_3 \sigma_3,
\end{aligned}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These equations lead to the following formula for the form ω ,

$$\begin{aligned}
\omega &= 8\beta_2 d\alpha_1 - 8i\beta_1 d\alpha_3 - 8i\alpha_3 d\beta_1 - 8\alpha_1 d\beta_2 + 8id\gamma_3 + \\
&\quad 8i(\alpha_1^2 + \alpha_3^2) d\alpha_3 + 8iu\alpha_3 d\alpha_1 - 8iu\alpha_1 d\alpha_3 + 8ud\beta_2 + (2it + 4iu^2) d\alpha_3 + 4u_t d\alpha_1.
\end{aligned}$$

It can be rewritten in a completely localized form, i.e. directly in terms of $u \equiv u(t; s)$ and its derivatives in the form of a linear combination of the differentials dt, ds_1, ds_2 . The result is

$$\begin{aligned}
\omega &= (u_t^2 - u^4 - tu^2) dt + \frac{2}{3} (2u_t u_{s_1} - 4u^3 t u_{s_1} - u u_{t s_1} + 2t u_t u_{t s_1} - 2ut^2 u_{s_1}) ds_1 \\
&\quad + \frac{2}{3} (2u_t u_{s_2} - 4u^3 t u_{s_2} - u u_{t s_2} + 2t u_t u_{t s_2} - 2ut^2 u_{s_2}) ds_2.
\end{aligned} \tag{4.39}$$

Simultaneously, we see that

$$\omega_{\text{JMU}} = (u_t^2 - u^4 - tu^2) dt, \tag{4.40}$$

and hence the PII tau function is given by

$$\frac{\partial \ln \tau(t; s)}{\partial t} = u_t^2 - u^4 - tu^2. \tag{4.41}$$

Remark 4.5. Equation (4.33) admits a hamiltonian reformulation with the Hamiltonian

$$H = \frac{v^2}{4} - tu^2 - u^4,$$

with canonical coordinate and momentum given by u and $v = 2u_t$. It is straightforward to verify that the Hamiltonian satisfies the following relation:

$$4H - 2tH_t - 2v u_t + uv_t = 0.$$

Using this equation, we can rewrite ω as a natural extension of the classical action (i.e. up to addition of a total differential),

$$\omega = v du - H dt + d \left(\frac{2Ht - uv}{3} \right). \tag{4.42}$$

4.3 Tau function asymptotics

We will analyze the open set in the space of solutions of the second Painlevé equation (4.33) characterized by the following conditions on monodromy data:

$$\arg(1 - s_1 s_3) \in (-\pi, \pi), \tag{4.43a}$$

$$\arg(i\sigma s_2) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad \sigma = \operatorname{sgn} \Re(is_2) = \pm 1. \tag{4.43b}$$

Condition (4.43a) ensures that the solution $u(t)$ is smooth as $x \rightarrow -\infty$ while condition (4.43b) guarantees its smoothness as $t \rightarrow +\infty$. The respective asymptotics are given by the formulae from [Kap] (see also [IN], [DZ1], and [FIKN]),

$$u(t) = a_{00}^+ e^{\frac{2}{3}i(-t)^{\frac{3}{2}}} (-t)^{\frac{3}{2}\mu - \frac{1}{4}} + a_{00}^- e^{-i\frac{2}{3}(-t)^{\frac{3}{2}}} (-t)^{-\frac{3}{2}\mu - \frac{1}{4}} + O\left(t^{3|\Re\mu| - 1}\right), \quad t \rightarrow -\infty, \tag{4.44}$$

$$\mu = -\frac{1}{2\pi i} \ln(1 - s_1 s_3), \quad a_{00}^+ = \frac{\sqrt{\pi} e^{-i\frac{\pi}{2}\mu} 8^\mu e^{-i\frac{\pi}{4}}}{s_1 \Gamma(\mu)}, \quad a_{00}^- = \frac{\sqrt{\pi} e^{-i\frac{\pi}{2}\mu} 8^{-\mu} e^{i\frac{\pi}{4}}}{s_3 \Gamma(-\mu)}, \quad a_{00}^+ a_{00}^- = \frac{i\mu}{2}, \quad (4.45)$$

and

$$u(t) = i\sigma \sqrt{\frac{t}{2}} + b_{1,1}^+ e^{i\frac{2\sqrt{2}}{3} t^{\frac{3}{2}}} t^{-\frac{3}{2}v - \frac{1}{4}} + b_{1,1}^- e^{-i\frac{2\sqrt{2}}{3} t^{\frac{3}{2}}} t^{\frac{3}{2}v - \frac{1}{4}} + O(t^{3|\Re v| - 1}), \quad t \rightarrow +\infty, \quad (4.46)$$

$$\begin{aligned} v &= \frac{1}{\pi i} \ln(i\sigma s_2), \quad b_{1,1}^+ b_{1,1}^- = \frac{iv}{4\sqrt{2}}, \\ b_{1,1}^+ &= \frac{\sqrt{\pi} e^{i\frac{\pi}{2}v} 2^{-\frac{3}{4} - \frac{7}{2}v} e^{-i\frac{3\pi}{4}} i\sigma}{(1 + s_2 s_3) \Gamma(-v)}, \quad b_{1,1}^- = \frac{\sqrt{\pi} e^{i\frac{\pi}{2}v} 2^{-\frac{3}{4} + \frac{7}{2}v} e^{i\frac{3\pi}{4}} i\sigma}{(1 + s_1 s_2) \Gamma(v)}. \end{aligned} \quad (4.47)$$

From the previous section we already know that the 2-form $d\omega$ must be time-independent. This fact can be also established by a direct differentiation of the equation (4.39). Indeed, after straightforward though a bit tedious computation which involves using Painlevé II equation (4.33), we obtain

$$d\omega = (v_{s_2} u_{s_1} - v_{s_1} u_{s_2}) ds_2 \wedge ds_1. \quad (4.48)$$

From the equation (4.33) it also follows that

$$\frac{d}{dt} (v_{s_2} u_{s_1} - v_{s_1} u_{s_2}) = 0,$$

which implies the time independence of $d\omega$. Also, we can observe that

$$d\omega = \lim_{t \rightarrow -\infty} d\omega = 4i da_{00}^- \wedge da_{00}^+ = \lim_{t \rightarrow +\infty} d\omega = 4i\sqrt{2} db_{1,1}^+ \wedge db_{1,1}^-. \quad (4.49)$$

These relations indicate that the form ω_0 for the second Painlevé equation can be identified with the form $-4i a_{00}^+ da_{00}^-$, so that the 1-form

$$\hat{\omega} := \omega - \omega_0 = \omega + 4i a_{00}^+ da_{00}^- \quad (4.50)$$

is closed (with ω given by (4.39)). We can therefore extend the definition of the tau function as

$$\tau(t; s) := \exp \int \hat{\omega}. \quad (4.51)$$

In order to proceed with evaluation of the asymptotics of the tau function (4.51), we will need more terms in the asymptotics of $u(t)$. Denote

$$p = e^{\frac{2}{3}i(-t)^{\frac{3}{2}}} (-t)^{\frac{3}{2}\mu}, \quad \zeta = (-t)^{-\frac{1}{4}}.$$

We have the following formal asymptotic expansion at $t = -\infty$:

$$u(t) = \sum_{l \geq k \geq 0, \epsilon = \pm} a_{kl}^\epsilon p^{\epsilon(2k+1)} \zeta^{6l+1}.$$

A few first terms are

$$u(t) = a_{00}^+ p \zeta + a_{00}^- p^{-1} \zeta + a_{11}^+ p^3 \zeta^7 + a_{11}^- p^{-3} \zeta^7 + a_{01}^+ p \zeta^7 + a_{01}^- p^{-1} \zeta^7 + \dots,$$

where

$$a_{11}^\pm = -\frac{1}{4} (a_{00}^\pm)^3, \quad a_{01}^\pm = i a_{00}^\pm \left(\mp \frac{17}{8} \mu^2 + \frac{3}{4} \mu \mp \frac{5}{48} \right).$$

Similarly, denoting

$$q = e^{\frac{2}{3}i\sqrt{2}t^{\frac{3}{2}}} t^{-\frac{3}{2}v}, \quad \xi = t^{-\frac{1}{4}},$$

we have a formal asymptotic expansion at $t = +\infty$:

$$u(t) = \sum_{l \geq k \geq 0, \epsilon = \pm} b_{2k+1, 6l+1}^\epsilon q^{\epsilon(2k+1)} \xi^{6l+1} + \sum_{l \geq k \geq 0, \epsilon = \pm} b_{2k, 6l-2}^\epsilon q^{2\epsilon k} \xi^{6l-2}.$$

Let us record its several first terms:

$$u(t) = \sigma \frac{i}{\sqrt{2}} \xi^{-2} + b_{1,1}^+ q \xi + b_{1,1}^- q^{-1} \xi + b_{3,7}^+ q^3 \xi^7 + b_{3,7}^- q^{-3} \xi^7 + b_{0,4} \xi^4 + b_{2,4}^+ q^2 \xi^4 + b_{2,4}^- q^{-2} \xi^4 + b_{0,10} \xi^{10} + b_{2,10}^+ q^2 \xi^{10} \\ + b_{2,10}^- q^{-2} \xi^{10} + b_{4,10}^+ q^4 \xi^{10} + b_{4,10}^- q^{-4} \xi^{10} + b_{1,7}^+ q \xi^7 + b_{1,7}^- q^{-1} \xi^7 + \dots,$$

where

$$b_{0,4} = -\sigma \frac{3}{4} v, \quad b_{2,4}^\pm = -\sigma \frac{i\sqrt{2}}{2} (b_{1,1}^\pm)^2, \quad b_{3,7}^\pm = -\frac{1}{2} (b_{1,1}^\pm)^3, \quad b_{0,10} = \frac{i\sqrt{2}}{2} \left(\frac{51}{32} v^2 + \frac{1}{8} \right), \\ b_{2,10}^\pm = \sigma \frac{(b_{1,1}^\pm)^2}{2} \left(\mp \frac{17}{8} v^2 - \frac{11}{8} v \mp \frac{41}{48} \right), \quad b_{4,10}^\pm = \sigma \frac{i\sqrt{2}}{4} (b_{1,1}^\pm)^4, \quad b_{1,7}^\pm = b_{1,1}^\pm \frac{i\sqrt{2}}{6} \left(\mp \frac{51}{8} v^2 - \frac{3}{2} v \mp \frac{17}{16} \right).$$

Justification of this asymptotics can be done using the Riemann-Hilbert approach, cf [DZ2]. Plugging it into (4.39), we get for sufficiently small $|\Re \mu|$ and $|\Re v|$ the following behaviors:

$$\omega = d \left(-\frac{4i\mu}{3} (-t)^{\frac{3}{2}} - \frac{3}{2} \mu^2 \ln(-t) - \mu^2 \right) + 2i (a_{00}^- da_{00}^+ - a_{00}^+ da_{00}^-) + o(1), \quad t \rightarrow -\infty, \quad (4.52a)$$

$$\omega = d \left(\frac{t^3}{12} - \frac{6v^2+1}{8} \ln t + \frac{2i\sqrt{2}}{3} v t^{\frac{3}{2}} - \frac{v^2}{2} \right) + 2i\sqrt{2} (b_{1,1}^+ db_{1,1}^- - b_{1,1}^- db_{1,1}^+) + o(1), \quad t \rightarrow +\infty. \quad (4.52b)$$

This in turn yields the asymptotics of $\tau(t)$ defined by (4.51),

$$\ln \tau(t) = -\frac{4}{3} i\mu (-t)^{\frac{3}{2}} - \frac{3}{2} \mu^2 \ln(-t) - \mu^2 - \mu + c_1 + o(1), \quad t \rightarrow -\infty, \quad (4.53a)$$

$$\ln \tau(t) = \frac{t^3}{12} + \frac{2i\sqrt{2}v}{3} t^{\frac{3}{2}} - \frac{6v^2+1}{8} \ln t + \frac{v(1-v)}{2} + 4i \int (a_{00}^+ da_{00}^- + \sqrt{2} b_{1,1}^+ db_{1,1}^-) + c_2 + o(1), \quad t \rightarrow +\infty, \quad (4.53b)$$

up to numerical constants c_1, c_2 independent of s_1 and s_2 .

Remark 4.6. As follows from (4.48) and also from (4.42), the form $\Omega = d\omega$ coincides with the symplectic form for the second Painlevé equation (4.33). From (4.49) one can obtain an expression of this form in terms of either the asymptotic data at $t = -\infty$ or at $t = +\infty$. For instance, in the former case we have

$$\Omega = 4i da_{00}^- \wedge da_{00}^+. \quad (4.54)$$

The last equation can in turn be transformed into an expression of the symplectic form Ω in terms of monodromy data $s \in \mathcal{M}_{\text{PII}}$. Indeed, from (4.45) we see that the amplitudes a_{00}^\pm can be written in the form

$$a_{00}^\pm = \frac{1}{s_{1,3}} f_\pm(\mu), \quad (4.55)$$

where $f_\pm(\mu)$ are functions of a single argument which can of course be written explicitly using (4.45). From these explicit formulae we will only need one relation

$$\frac{1}{s_1 s_3} f_+(\mu) f_-(\mu) = \frac{i\mu}{2}. \quad (4.56)$$

Let us also notice that (see again (4.45)),

$$s_1 s_3 = 1 - e^{-2\pi i \mu}. \quad (4.57)$$

Now, differentiating (4.55), we get

$$da_{00}^+ = \left(-\frac{1}{s_1^2} f_+ + \frac{1}{s_1} f_+' \frac{\partial \mu}{\partial s_1} \right) ds_1 + \frac{1}{s_1} f_+' \frac{\partial \mu}{\partial s_3} ds_3, \\ da_{00}^- = \left(-\frac{1}{s_3^2} f_- + \frac{1}{s_3} f_-' \frac{\partial \mu}{\partial s_3} \right) ds_3 + \frac{1}{s_3} f_-' \frac{\partial \mu}{\partial s_1} ds_1.$$

Substituting these two equations into (4.54) and using (4.56), we arrive at the equation

$$\Omega = 4i \left(-\frac{i\mu}{2s_1s_3} + \frac{1}{s_1s_3} f_- f_+ \frac{1}{s_3} \frac{\partial\mu}{\partial s_1} + \frac{1}{s_1s_3} f_- f_+ \frac{1}{s_1} \frac{\partial\mu}{\partial s_3} \right) ds_1 \wedge ds_3. \quad (4.58)$$

Noticing that

$$\frac{1}{s_3} \frac{\partial\mu}{\partial s_1} = \frac{1}{s_1} \frac{\partial\mu}{\partial s_3} = \frac{1}{2\pi i (1 - s_1s_3)},$$

the equation (4.58) is transformed into

$$\Omega = 4i \left(-\frac{i\mu}{2s_1s_3} + \frac{1}{2\pi i s_1s_3(1-s_1s_3)} (f_- f_+)' \right) ds_1 \wedge ds_3. \quad (4.59)$$

Finally, differentiating (4.56) with respect to μ , taking into account that $(s_1s_3)' = 2\pi i (1 - s_1s_3)$ in view of (4.57), we conclude that

$$-\frac{i\mu}{2s_1s_3} + \frac{1}{2\pi i s_1s_3(1-s_1s_3)} (f_- f_+)' = \frac{1}{4\pi(1-s_1s_3)},$$

which means that we end up with the following expression of the symplectic form Ω in terms of monodromy parameters:

$$d\omega = \Omega = \frac{i}{\pi} \frac{ds_1 \wedge ds_3}{1 - s_1s_3}. \quad (4.60)$$

This expression coincides with the one obtained for the curvature $d\omega_{\text{MB}}$ in [Ber1] where Painlevé II was also used to exemplify the general Malgrange-Bertola form (4.21). It should be mentioned that the first derivation of this formula was done by H. Flaschka and A. Newell in [FN2]. Also note that in our asymptotics-based derivation of (4.60) we mimic the methodology used in [BFT] for the evaluation of the KdV symplectic form in terms of the relevant scattering data — the PDE analog of the monodromy data.

4.4 Connection constant

The asymptotic equations (4.53a) and (4.53b) can be rewritten in the form

$$\tau(t) \simeq \begin{cases} C_- e^{-\frac{4}{3}i\mu(-t)^{\frac{3}{2}}} (-t)^{-\frac{3}{2}\mu^2} & \text{as } t \rightarrow -\infty, \\ C_+ e^{\frac{t^3}{12} + \frac{2i\sqrt{2}}{3}vt^{\frac{3}{2}}} t^{-\frac{3}{4}v^2 - \frac{1}{8}} & \text{as } t \rightarrow +\infty. \end{cases}$$

Our goal is to find the ratio C_+/C_- in terms of monodromy data $s \in \mathcal{M}_{\text{PII}}$.

From (4.53a), (4.53b) we have

$$\ln \frac{C_+}{C_-} = -\frac{v^2}{2} + \frac{v}{2} + \mu^2 + \mu + 4i \int \left(a_{00}^+ da_{00}^- + \sqrt{2} b_{1,1}^+ db_{1,1}^- \right). \quad (4.61)$$

Hence, our task is to evaluate the integral in the right hand side of (4.61). To this end, it is convenient to introduce new monodromy parameters ρ and η by the equations

$$(1 + s_1s_2)^{-1} = e^{i\pi\rho}, \quad s_3^{-1} = e^{i\pi\eta}. \quad (4.62)$$

This transforms the integral in question into

$$4i \int a_{00}^+ da_{00}^- + \sqrt{2} b_{1,1}^+ db_{1,1}^- = -2 \int \mu (\ln a_{00}^-)'_{\mu} d\mu - \int v (\ln b_{1,1}^-)'_{v} dv - i\pi \int (vd\rho + 2\mu d\eta). \quad (4.63)$$

The first two integrals on the right can be rewritten as

$$\int \mu (\ln a_{00}^-)'_{\mu} d\mu = -\frac{6\ln 2 + i\pi}{4} \mu^2 - \int \mu d \ln \Gamma(-\mu), \quad (4.64a)$$

$$\int v (\ln b_{1,1}^-)'_{v} dv = \frac{7\ln 2 + i\pi}{4} v^2 - \int v d \ln \Gamma(v). \quad (4.64b)$$

In order to simplify the third integral, we first notice that due to cyclic relation $s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0$ satisfied by the Stokes parameters we may write

$$\mu = -\frac{1}{2\pi i} \ln \left(\frac{1 - e^{-2i\pi\eta + i\pi\sigma}}{1 - e^{i\pi\nu - i\pi\eta + i\pi\frac{\sigma}{2}}} \right), \quad \rho = \frac{i}{\pi} \ln \left(\frac{1 - e^{2\pi i\nu}}{1 - e^{i\pi\nu - i\pi\eta + i\pi\frac{\sigma}{2}}} \right). \quad (4.65)$$

These formulae allow one to express the third integral in (4.63) in terms of dilogarithms. We have (cf. similar calculations in [ILT14])

$$\begin{aligned} & \int 2\mu d\eta + \nu d\rho - \nu\rho = \int 2\mu d\eta - \rho d\nu = \\ & = \frac{1}{2\pi^2} \left[\text{Li}_2 \left(e^{-2i\pi\eta + i\pi\sigma} \right) + \text{Li}_2 \left(e^{2\pi i\nu} \right) - 2\text{Li}_2 \left(e^{i\pi\nu - i\pi\eta + i\pi\frac{\sigma}{2}} \right) \right], \end{aligned} \quad (4.66)$$

where $\text{Li}_2(z)$ denotes the dilogarithm function

$$\text{Li}_2(z) = -\int_0^z \frac{\ln(1-x)}{x} dx.$$

The formulae (4.64a), (4.64b), and (4.66) enable us to complete the evaluation of the ratio C_+/C_- in terms of Barnes G -functions. Indeed, it suffices to use the classical formula (3.80) for the integrals (4.64a), (4.64b), and another classical formula

$$\text{Li}_2 \left(e^{2\pi iz} \right) = -2\pi i \ln \frac{G(1+z)}{G(1-z)} - 2\pi iz \ln \frac{\sin \pi z}{\pi} - \pi^2 z(1-z) + \frac{\pi^2}{6},$$

for the integral (4.66). Skipping some straightforward though tedious calculations, we arrive at the final result.

Theorem 4.7. *Under the assumptions (4.43a), (4.43b) on the monodromy data, the connection coefficient for the Painlevé II tau function is given, up to a numerical constant, by*

$$\begin{aligned} \frac{C_+}{C_-} &= \text{const} \cdot 2^{3\mu^2 - \frac{7}{4}\nu^2} (2\pi)^{-\mu - \frac{\nu}{2}} e^{\frac{\pi i}{4}(\eta^2 + 2\mu^2 + 2\eta\nu - 8\mu\eta - \sigma\eta - \sigma\nu + 4\sigma\mu)} \times \\ & \times \frac{G(1 + \eta - \frac{\sigma}{2}) G(1 - \nu) G^2(1 + \frac{\sigma}{4} + \frac{\nu}{2} - \frac{\eta}{2})}{G(1 + \frac{\sigma}{2} - \eta) G^2(1 - \frac{\sigma}{4} - \frac{\nu}{2} + \frac{\eta}{2}) G^2(1 - \mu)}, \end{aligned} \quad (4.67)$$

where μ, ν, η, σ are related to Stokes parameters s_1, s_2, s_3 by

$$e^{-2\pi i\mu} = 1 - s_1 s_3, \quad e^{i\pi\nu} = i\sigma s_2, \quad e^{i\pi\eta} = s_3^{-1}, \quad \sigma = \text{sgn } \Re(i s_2).$$

Remark 4.8. In order to bring the final answer to the compact form (4.67), one has to use the relations

$$e^{i\pi(-2\mu + \frac{\eta}{2} + \frac{\nu}{2} - \frac{\sigma}{4})} = \frac{\sin \pi(\frac{\sigma}{2} - \eta)}{\sin \pi(\frac{\nu}{2} - \frac{\eta}{2} + \frac{\sigma}{4})}, \quad e^{i\pi(-\rho - \frac{\eta}{2} - \frac{\nu}{2} + \frac{\sigma}{4})} = \frac{\sin \pi\nu}{\sin \pi(\frac{\nu}{2} - \frac{\eta}{2} + \frac{\sigma}{4})}.$$

which can be verified with the help of (4.65). This allows to get rid of all gamma functions in the final formula.

Remark 4.9. Strictly speaking, we have derived (4.67) under assumption that $|\Re\mu|$ and $|\Re\nu|$ are sufficiently small. However, in the final result we can lift this restriction by noticing that both sides of (4.67) are analytic functions of monodromy/Riemann-Hilbert data. (For the ratio C_+/C_- this is a corollary of the general Birkhoff-Grothendieck-Malgrange theory).

We are finishing by a short discussion of the numerical constant in (4.67). By now, it remains undetermined. Here is how we plan to address this issue in a later work. Consider Hastings-McLeod solution $u_{\text{HM}}(t)$ corresponding to the monodromy data

$$s_1 = -i, \quad s_2 = 0, \quad s_3 = i.$$

Then $\tilde{u}_{\text{HM}}(t) = e^{\frac{2\pi i}{3}} u_{\text{HM}}\left(e^{\frac{2\pi i}{3}} t\right)$ corresponds to the monodromy data

$$s_1 = 0, \quad s_2 = -i, \quad s_3 = -i \quad \Rightarrow \quad \sigma = 1, \quad \eta = \frac{1}{2}, \quad \nu = 0.$$

These data satisfy the conditions (4.43a), (4.43b). Therefore, to find the numerical constant in (4.67) it would be enough to compute the connection constant for the tau function corresponding to the Hastings-McLeod solution on the ray $e^{\frac{2\pi i}{3}} \mathbb{R}$. A similar constant in the case of real line was evaluated in [DIK]. The asymptotic behavior of the Hastings-McLeod solution itself preserves its character in the relevant sectors around the real line. Hence our expectation is that the numerical constant in (4.67) can be simply borrowed from [DIK].

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