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A bivariate geometric distribution allowing for positive or negative correlation

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Abstract

In this paper, we propose a new bivariate geometric model, derived by linking two univariate geometric distributions through a specific copula function, allowing for positive and negative correlations. Some properties of this joint distribution are presented and discussed, with particular reference to attainable correlations, conditional distributions, reliability concepts, and parameter estimation. A Monte Carlo simulation study empirically evaluates and compares the performance of the proposed estimators in terms of bias and standard error. Finally, in order to demonstrate its usefulness, the model is applied to a real data set.

KEYWORDS

attainable correlations; correlated counts; Farlie-Gumbel-Morgenstern copula; method of moments; two-step maximum likelihood

1. INTRODUCTION

In recent years, the construction of bivariate (and multivariate) discrete distributions has attracted much interest, since stochastic models for correlated count data find application in many fields (Kocherlakota and Kocherlkota 1992; Johnson et al. 1997; Sarabia and Gómez-Déniz 2011). For example, in marketing modeling the number of purchases of different products is of special interest for predicting sales in the future or examining the behavior of different types of buyers. In insurance risk applications, the

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numbers of claims in different customers' classes are frequently statistically dependent. In health economics, different components of health care demand, such as the number of consultations with a doctor or specialist and the number of consultations with non-doctor health professionals can be jointly modeled.

Many authors have discussed the problem of constructing a bivariate version of a given univariate distribution, although there is no universally accepted criterion for producing a unique distribution which can unequivocally be called the bivariate version of a univariate distribution. Here we focus on the geometric distribution, which along with the Poisson is perhaps the most popular distribution used for modeling counts, especially in the reliability context. The probability mass function (p.m.f.) of a geometric distribution with "success" parameter $\theta \in (0, 1)$ is $p_x(x) = \theta(1-\theta)^x$, for $x = 0, 1, \ldots$; its cumulative distribution function (c.d.f.) is $F_x(x) = P(X \leq x) = 1-(1-\theta)^{x+1}$; its survival function is $\bar{F}_x(x) = P(X > x) = 1 - F(x) = (1-\theta)^{x+1}$; its failure rate function, defined as $r_x(x) = p_x(x)/P(X \geq x)$, is constant and equal to θ.

There have been several proposals for constructing a bivariate version of the geometric distribution. Among them, Paulson and Uppuluri (1972) introduced a five-parameter geometric bivariate distribution whose p.m.f. can be computed only through recursive formulas. This distribution allows also for negative correlations and comprises the product of two independent geometrics as a special case. Phatak and Sreehari (1981) introduced a two-parameter bivariate geometric distribution with an easy expression of its p.m.f., allowing for positive correlations only; it was also studied later by Krishna and Pundir (2009). Roy (1993) extended the univariate concept of failure rate for non-negative integer valued discrete variables in two dimensions, by introducing a new definition of bivariate failure rates, and proposed a three-parameter bivariate geometric distribution enjoying an analogous property to its univariate version, i.e., locally constant bivariate failure rates. In Basu and Dhar (1995), a bivariate geometric distribution is proposed by a discrete analog of the continuous bivariate distribution of Marshall and Olkin (1967); in Dhar (1998) another bivariate geometric model, which is a discrete analog to Freund's model (Freund 1961), is introduced. In Omey and Minkova (2016), a two-parameter bivariate geometric distribution with negative correlation is studied; it assigns zero probability to the event that the two margins assume a same integer value. Gómez-Déniz et al. (2017) used an Archimedean copula to link generalized geometric margins together allowing for both positive and negative correlations. Ng et al. (2010) introduced a class of bivariate negative binomial distributions based upon an extension of the method of trivariate reduction, which have different marginal distributions.

In Barbiero (2017a), a bivariate distribution with discrete Weibull components and Farlie-Gumbel-Morgenstern (FGM) copula was proposed. A particularization of this model (Barbiero 2017b), obtained by setting the marginal shape parameters equal to 1, leads to a new bivariate geometric distribution. This model indeed belongs to a wider class of bivariate discrete distributions (the so called extended FGM class) introduced in Piperigou (2009); here we provide more specific results than those reported there, with a special focus on dependence and reliability concepts and estimation issues.

The rest of the paper is structured as follows. In Section 2, the proposed distribution is formally introduced and its properties are presented and discussed. Section 3 is devoted to parameter estimation; Section 4 presents a simulation study assessing the performance of the proposed point estimators; whereas in Section 5 an application to real data is provided. In Section 6 conclusive remarks and research perspectives are outlined.

2. THE PROPOSED BIVARIATE GEOMETRIC DISTRIBUTION

In this section, we introduce the new bivariate distribution, by specifying its c.d.f. and p.m.f., and then derive some mathematical properties.

2.1. *Definition*

A bivariate geometric distribution can be obtained by linking together two geometric distributions with parameters θ_1 and θ_2 via the Farlie-Gumbel-Morgenstern (FGM) copula (Farlie 1960) with parameter $\alpha \in [-1, +1]$. The bivariate FGM copula is given by $C(u, v) = uv\,[1 + \alpha(1 - u)(1 - v)]$, $u, v \in [0, 1]$, and can be seen as a "perturbation" of the independence copula $\Pi(u, v) = uv$, via the parameter α; if α is greater than zero, the FGM copula provides positive dependence; if α is smaller than zero, it returns negative dependence; when α is zero, it reduces to the independence copula. FGM copula is able to model only a slight level of dependence; in other terms, it is not able to cover the whole dependence spectrum, from perfect positive dependence (i.e., comonotonicity) to perfect negative dependence (i.e., countermonotonicity), as done, for example, by the Gauss copula. In particular, we have that $\rho_{uv} = \text{cor}(U, V) = \alpha/3$, implying that the linear correlation between the two uniform random components of the FGM copula, given the bounds for α, cannot exceed in absolute value 1/3. More importantly, it has been shown that Pearson's correlation coefficient between two continuous marginal distributions with FGM copula can never exceed 1/3 (Schucany et al. 1978). This feature has limited its use; however, its easy analytical expression and other nice properties still make it a basic tool for bivariate modeling.

A bivariate distribution with geometric margins $X \sim F_x(x; \theta_1)$ and $Y \sim F_y(y; \theta_2)$, linked by the FGM copula can be built by simply defining its bivariate c.d.f. as $F(x, y) = C(F_x(x), F_y(y))$, $x, y = 0, 1, 2, \dots$, from which we get:

$$\begin{aligned} F(x, y) &= F_x(x)F_y(y)[1 + \alpha(1 - F_x(x))(1 - F_y(y))] \\ &= [1 - (1 - \theta_1)^{x+1}][1 - (1 - \theta_2)^{y+1}][1 + \alpha(1 - \theta_1)^{x+1}(1 - \theta_2)^{y+1}]. \end{aligned} \tag{1}$$

The corresponding bivariate p.m.f. is then given (see also Barbiero 2017b) by recalling the relationship between bivariate p.m.f. and c.d.f.: $p(x, y) = F(x, y) - F(x - 1, y) - F(x, y - 1) + F(x - 1, y - 1)$, $x, y = 0, 1, 2, \dots$, from which it follows that

$$p(x, y) = \theta_1(1 - \theta_1)^x \theta_2(1 - \theta_2)^y \{1 + \alpha [(2 - \theta_1)(1 - \theta_1)^x - 1] [(2 - \theta_2)(1 - \theta_2)^y - 1]\} \tag{2}$$

with $0 < \theta_1, \theta_2 < 1$. Due to the discrete nature of the margins, the dependence parameter α can now take values in a wider interval than in the case of continuous margins: we have $-1 \leq \alpha \leq \min\{1/(1 - \theta_1), 1/(1 - \theta_2)\}$ (see, e.g., Cambanis 1977; Piperigou 2009). The proposed bivariate geometric model defined by Eq. (1) or (2), which is then characterized by the parameter vector $(\theta_1, \theta_2, \alpha)$, can be regarded as a particular case of the model presented in Barbiero (2017a), with the shape parameters β_1 and β_2 set equal to 1, and scale parameters $q_1 = 1 - \theta_1$ and $q_2 = 1 - \theta_2$.

2.2. *Survival function and failure rate*

If we define the bivariate survival function as $\bar{F}(x, y) = P(X > x, Y > y)$, we can easily derive its expression for the proposed model:

$$\begin{aligned} \bar{F}(x, y) &= 1 - F_x(x) - F_y(y) + F(x, y) \\ &= (1 - \theta_1)^{x+1}(1 - \theta_2)^{y+1}[1 + \alpha[1 - (1 - \theta_1)^{x+1}][1 - (1 - \theta_2)^{y+1}] \\ &= \bar{F}_x(x)\bar{F}_y(y)[1 + \alpha F_x(x)F_y(y)]. \end{aligned} \tag{3}$$

Comparing Eqs.(1) and (3) one can note that joint distribution and survival functions have a specular expression. This is ensured by the radial symmetry property of the FGM copula.

For a continuous r.v. (X, Y) with joint density function $f(x, y)$, the bivariate failure rate was defined by Basu (1971) as $r(x, y) = f(x, y)/\bar{F}(x, y)$. Adapting this formulation, for a bivariate discrete r.v. (X, Y) one can define the bivariate failure rate as $r(x, y) = p(x, y)/\bar{F}(x, y)$, which assumes the following expression for the bivariate geometric r.v., for $x = 0, 1, \ldots; y = 0, 1, \ldots$:

$$r(x, y) = \frac{\theta_1}{1-\theta_1}\frac{\theta_2}{1-\theta_2}\frac{1+\alpha\left[(2-\theta_1)(1-\theta_1)^x - 1\right]\left[(2-\theta_2)(1-\theta_2)^y - 1\right]}{1+\alpha[1-(1-\theta_1)^{x+1}][1-(1-\theta_2)^{y+1}]}$$

If $\alpha = 0$ (independence case), $r(x, y)$ is constant and equal to $r_0 = \frac{\theta_1}{1-\theta_1}\frac{\theta_2}{1-\theta_2}$. Moreover, if x and y are let go to ∞ with $x = y$, the failure rate $r(x, y)$ tends to r_0 for any value of $\alpha \neq -1$; it tends to $r_0 \cdot \min\left\{\frac{2-\theta_i}{1-\theta_i}; i = 1, 2\right\}$ if $\alpha = -1$. The first order partial derivative of $r(x, y)$ with respect to x is given by:

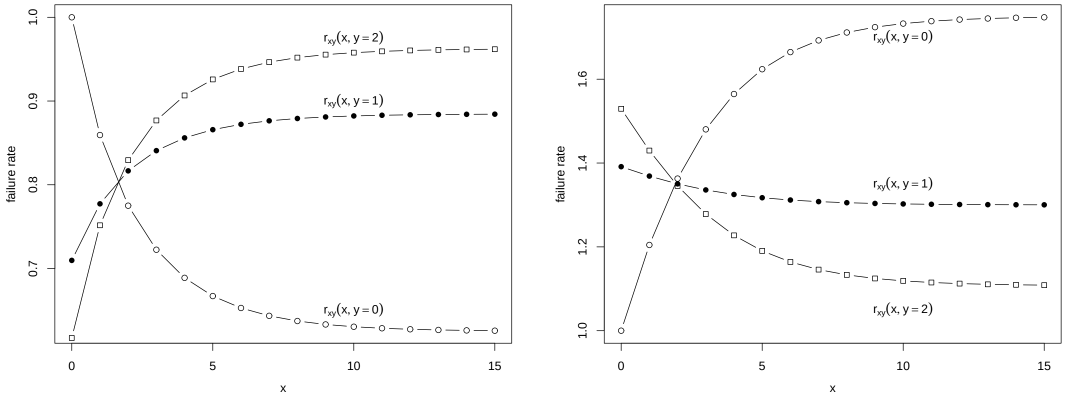
$$\frac{\partial r(x, y)}{\partial x} = \frac{r_0 \ln(1-\theta_1)\alpha(1-\theta_1)^x \cdot [a_1(\theta_2, y)(2-\theta_1) + a_2(\theta_2, y)(1-\theta_1) + \alpha a_1(\theta_2, y)a_2(\theta_2, y)]}{\{1+\alpha[1-(1-\theta_1)^{x+1}][1-(1-\theta_2)^{y+1}]\}^2}$$

where $a_1(\theta_2, y) = [(2-\theta_2)(1-\theta_2)^y - 1]$, $a_2(\theta_2, y) = [1-(1-\theta_2)^{y+1}]$. Note that $a_1(\theta_2, 0) = 1-\theta_2$, $a_2(\theta_2, 0) = \theta_2$, $\lim_{y\to\infty} a_1(\theta_2, y) = -1$, and $\lim_{y\to\infty} a_2(\theta_2, y) = 1$. Then, if $\alpha > 0$ the function $x \mapsto r(x, y)$ is decreasing in x for a fixed value of y when $y < y^*(\theta_1, \theta_2, \alpha)$; it is increasing in x for a fixed value of y when $y \geq y^*(\theta_1, \theta_2, \alpha)$, with $y^*(\theta_1, \theta_2, \alpha) \in \mathbb{N}$ a proper threshold. If $\alpha < 0$, the function $x \mapsto r(x, y)$ is increasing in x for a fixed value of y when $y < y^*(\theta_1, \theta_2, \alpha)$; it is increasing in x for a fixed value of y when $y \geq y^*(\theta_1, \theta_2, \alpha)$. If $\alpha = -1$ and $\theta_2 \geq \theta_1$, the function $x \mapsto r(x, y)$ is increasing in x for every fixed y. Analogous argument holds exchanging x with y. We cannot state that for given parameters θ_1, θ_2, α, the failure rate is increasing/decreasing in both x and y; for example, with $\theta_1 = 1/3$, $\theta_2 = 2/3$, $\alpha = 1/2$, we have $r(0, 0) = 1 > r(1, 1) = 0.7772 < r(2, 2) = 0.8294$ (see Figure 1).

2.3. *Probability Generating Function*

Due to the easy analytical expression of the joint p.m.f., it is straightforward to derive the probability generating function (p.g.f.) for the bivariate vector (X, Y) following the proposed bivariate geometric distribution. By definition, for a pair of r.v.s X and

Figure 1.: Bivariate failure rate function for some values of x and y, with $\theta_1 = 1/3$, $\theta_2 = 2/3$, $\alpha = 1/2$ (left) and $\alpha = -1/2$ (right)



Y defined each on non-negative integers only, the p.g.f. is equal to $g(t_1, t_2) = \mathbb{E}(t_1^x t_2^y)$; for the proposed model, we have then:

$$g(t_1, t_2; \theta_1, \theta_2, \alpha) = \theta_1\theta_2 \left\{ \frac{1+\alpha}{[1-t_1(1-\theta_1)][1-t_2(1-\theta_2)]} - \alpha \frac{1-\theta_1}{[1-t_1(1-\theta_1)(2-\theta_1)][1-t_2(1-\theta_2)]} \right.$$
$$\left. -\alpha \frac{1-\theta_2}{[1-t_2(1-\theta_2)(2-\theta_2)][1-t_1(1-\theta_1)]} + \alpha \frac{(1-\theta_1)(1-\theta_2)}{[1-t_1(1-\theta_1)(2-\theta_1)][1-t_2(1-\theta_2)(2-\theta_2)]} \right\}$$

2.4. *Conditional distributions*

The conditional distribution of Y given $X = x$ is

$$p_{y|x}(y; x) = \theta_2(1-\theta_2)^y \left\{1 + \alpha \left[(2-\theta_1)(1-\theta_1)^x - 1\right]\left[(2-\theta_2)(1-\theta_2)^y - 1\right]\right\}. \quad (4)$$

If α has the same sign of $[(2-\theta_1)(1-\theta_1)^x - 1]$, the p.m.f. in (4) can be regarded as a mixture of two geometric distributions with parameters θ_2 and $1-(1-\theta_2)^2$, with weights $1 - \alpha\left[(2-\theta_1)(1-\theta_1)^x - 1\right]$ and $\alpha\left[(2-\theta_1)(1-\theta_1)^x - 1\right]$, respectively. The conditional expected value of Y given $X = x$ can be then easily computed and is equal to

$$\mathbb{E}(Y|X = x) = \frac{1-\theta_2}{\theta_2}\left[1 + \alpha \cdot \frac{1-(2-\theta_1)(1-\theta_1)^x}{2-\theta_2}\right].$$

Note that $\mathbb{E}(Y|X = x)$ is greater than the unconditional moment $\mathbb{E}(Y)$ if $\alpha > 0$ and $1-(2-\theta_1)(1-\theta_1)^x > 0$, i.e., if $\alpha > 0$ and $x > -\log(2-\theta_1)/\log(1-\theta_1)$; or if $\alpha < 0$ and $x < -\log(2-\theta_1)/\log(1-\theta_1)$. From Eq.(4), the conditional c.d.f. of Y given $X = x$

can be easily derived and written as:

$$P(Y \leq y|X = x) = F_y(y) + \alpha F_y(y)[1 - F_y(y)][1 - p_x(x) - 2F_x(x-1)] \tag{5}$$

Symmetrical results hold for the conditional distribution and expected value of X given $Y = y$.

2.5. *Simulation*

In order to simulate a sample from the bivariate geometric distribution with parameters θ_1, θ_2, and α, one can resort to the algorithm described in Barbiero (2017a), properly adapted to the geometrically distributed margins with parameters θ_1 and θ_2. The algorithm is based on the conditional c.d.f. in (5). The steps to be implemented are the following:

(1) Simulate a random pair (v_1, v_2) from two independent uniform r.v.s in $(0, 1)$, $V_1 \sim \text{Unif}(0, 1)$ and $V_2 \sim \text{Unif}(0, 1)$;
(2) Set $u = v_1$ and $x = F_x^{-1}(u)$, where F_x^{-1} denotes the quantile functions of geometric distribution with parameters θ_1, i.e., $x = \left\lceil \frac{\ln(1-u)}{\ln(1-\theta_1)} - 1 \right\rceil$, with $\lceil \cdot \rceil$ indicating the ceiling function.
(3) Set $v = 2v_2/(a + b)$, where $a = 1 + \alpha(1 - p_x(x) - 2F_x(x - 1))$ and $b = \left[a^2 - 4(a-1)v_2\right]^{1/2}$. Set $y = \left\lceil \frac{\ln(1-v)}{\ln(1-\theta_2)} - 1 \right\rceil$.
(4) (x, y) is a random pair from the proposed bivariate distribution.

2.6. *Attainable correlations*

Pearson's correlation coefficient ρ_{xy} between the two margins can be easily calculated. In fact, since for a geometric distribution with parameter θ_1 the expected value is equal to $(1 - \theta_1)/\theta_1$ and the variance is equal to $(1 - \theta_1)/\theta_1^2$, for the mixed moment $\mathbb{E}(XY)$ the following expression can be derived after some calculations:

$$\mathbb{E}(XY) = \frac{1-\theta_1}{\theta_1}\frac{1-\theta_2}{\theta_2} + \alpha \cdot \frac{1-\theta_1}{\theta_1(2-\theta_1)} \cdot \frac{1-\theta_2}{\theta_2(2-\theta_2)}$$

and then

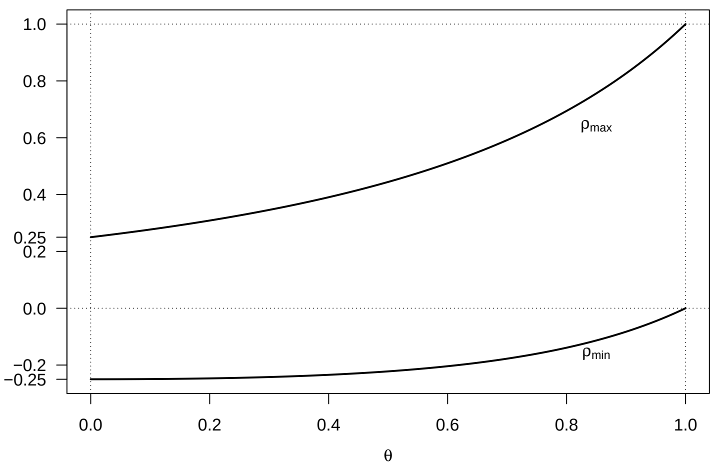
$$\rho_{xy} = \alpha \cdot \frac{\sqrt{1-\theta_1}}{2-\theta_1} \cdot \frac{\sqrt{1-\theta_2}}{2-\theta_2}.$$

If $\alpha = 0$, X and Y are clearly uncorrelated and independent (the joint p.m.f. (2) factorizes into the product of two univariate geometric p.m.f.s); if $\alpha > 0$, we have positively correlated marginal distributions, conversely, if $\alpha < 0$, we have negatively correlated marginal distributions. Since the parameter α is constrained in $[-1, \min\{1/(1-\theta_1), 1/(1-\theta_2)\}]$, its range is

$$-\frac{\sqrt{(1-\theta_1)(1-\theta_2)}}{(2-\theta_1)(2-\theta_2)} \leq \rho_{xy} \leq \min\{1/(1-\theta_1), 1/(1-\theta_2)\} \frac{\sqrt{(1-\theta_1)(1-\theta_2)}}{(2-\theta_1)(2-\theta_2)}$$

and we can observe that ρ_{xy} can never be smaller than $-1/4$. For identically distributed margins with common parameter θ, the lower bound for ρ_{xy} is $\rho_{\min} = -(1-\theta)/(2-\theta)^2$, whereas the upper bound $\rho_{\max} = 1/(2-\theta)^2$ (see also Figure 2). Although the proposed model can then yield positive as well as negative correlations, however their range is not the largest attainable and strongly depends on the marginal parameters; this may pose some limits on its application. In fact, we remark that for two identically distributed geometric margins with common parameter θ, the maximum attainable linear correlation is 1, which is achieved by linking them through the comonotonicity copula $M(u, v) = \min\{u, v\}$, whereas the minimum attainable correlation is equal to $\theta - 1$, if $\theta \geq 1/2$, whereas for $\theta < 1/2$ it is smaller than $-1/2$ and can be numerically computed by resorting to the algorithm provided in Huber and Maric (2014).

Figure 2.: Bounds of Pearson's correlation between the margins (here supposed identically distributed with common parameter θ) of the bivariate geometric model



2.7. *Reliability parameter*

The probability $R = P(X \leq Y)$ has been often investigated in the literature. If Y is the strength of a component or system which is subject to a stress X, and if the system regularly operates unless the stress exceeds the strength, then R is the probability that the system works, i.e. a measure of system performance. The majority of papers studying the computation and estimation of R deal with independent continuous probability distributions. However, in some real life situations, stress or strength can have discrete distribution. For example, when the stress is the number of the products that customers want to buy and the strength is the number of the products that factory produces (Jovanović 2017). Furthermore, X and Y may be non-independent; this may happen because a system that has to resist to higher levels of stress is designed to have higher level of strength (thus implying a positive dependence/correlation between stress and strength).

For the proposed bivariate geometric distribution, the stress-strength parameter $R = P(X \leq Y)$ can be computed as $R = P(X \leq Y) = \sum_{y=0}^{\infty} \sum_{x=0}^{y} p(x, y)$ with $p(x, y)$ given by Eq.(2), and has the following expression:

$$
\begin{aligned}
R &= \frac{\theta_1}{1-(1-\theta_1)(1-\theta_2)} + \alpha\theta_1\theta_2 \left\{ \frac{(2-\theta_1)\cdot(2-\theta_2)}{[1-(1-\theta_2)^2]\,[1-(1-\theta_1)^2\cdot(1-\theta_2)^2]} + \right. \\
&- \frac{(2-\theta_1)}{\theta_2\,[1-(1-\theta_1)^2\cdot(1-\theta_2)]} - \frac{(2-\theta_2)}{[1-(1-\theta_2)^2]\,[1-(1-\theta_1)\cdot(1-\theta_2)^2]} \\
&+ \left. \frac{1}{\theta_2\,[1-(1-\theta_1)\cdot(1-\theta_2)]} \right\}, \qquad (6)
\end{aligned}
$$

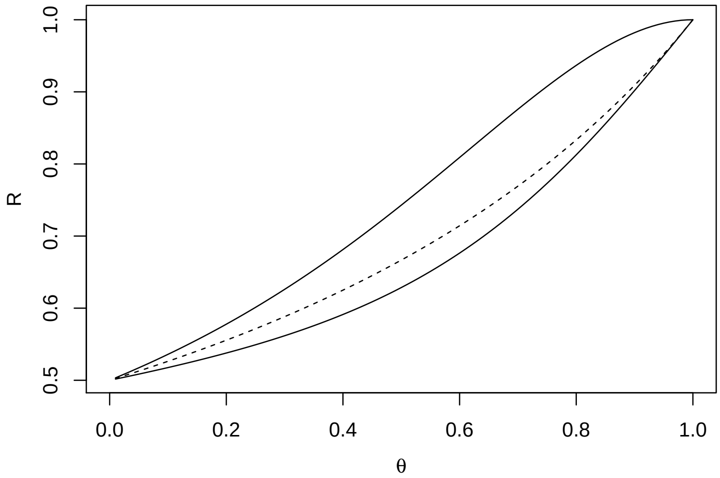
which boils down to the expression reported in Maiti (1995) for the independence case $\theta = 0$ – there, however, the parameters θ_1 and θ_2 represent the complements to 1 of the success probabilities of the two marginal geometric distributions. If we let $\theta_1 = \theta_2 = \theta$, i.e., if we consider two identically distributed geometric margins, then the expression of R in (6) becomes:

$$
\begin{aligned}
R &= \frac{1}{2-\theta} + \alpha\theta^2 \left\{ \frac{(2-\theta)}{\theta[1-(1-\theta)^4]} - \frac{2-\theta}{\theta[1-(1-\theta)^3]} - \frac{1}{\theta[1-(1-\theta)^3]} + \frac{1}{\theta^2(2-\theta)} \right\} \\
&= \frac{1}{2-\theta} + \alpha\frac{\theta(1-\theta)^2}{(\theta^2-2\theta+2)(\theta^2-3\theta+3)(2-\theta)},
\end{aligned}
$$

which shows that R is an increasing linear function of α, once θ is fixed. For example, if $\theta_1 = \theta_2 = 0.5$, then $R = 2/3$ in case of independence ($\alpha = 0$), whereas we would have

$R_{\max} = 0.7429$ and $R_{\min} = 0.6286$ if $\alpha = \alpha_{\max} = 2$ and $\alpha = \alpha_{\min} = -1$, respectively. The range of R looks quite short (0.1143); this is a direct consequence of the already mentioned fact that FGM copula does not interpolate between perfect negative and perfect positive dependence. It can be shown that the maximum value of the difference $R_{\max} - R_{\min}$ (≈ 0.1388) is obtained for $\theta \approx 0.6858$; its minimum value, 0, is obtained letting θ go to the boundary values 0 and 1, see Figure 3.

Figure 3.: Stress-strength parameter R for the bivariate geometric model with identical distributed margins with parameter θ. The two solid lines correspond to the values $\alpha = -1$ (lower curve) and $\alpha = 1/(1-\theta)$ (upper curve) and represent lower and upper bounds for R at a given value θ; the dashed line corresponds to the values of R when $\alpha = 0$.



2.8. *Convolution*

It may be interesting to compute the distribution of the sum $S = X + Y$ of the two random components of the bivariate geometric distribution. Its p.m.f. can be calculated as

$$
\begin{aligned}
p_s(s) = P(S = s) &= \sum_{x=0}^{s} P(X = x, Y = s - x) = \\
&= \sum_{x=0}^{s} \theta_1(1-\theta_1)^x \theta_2(1-\theta_2)^{s-x} \left\{1 + \alpha\left[(2-\theta_1)(1-\theta_1)^x - 1\right]\left[(2-\theta_2)(1-\theta_2)^{s-x} - 1\right]\right\}
\end{aligned} \tag{7}
$$

If $\theta_1 \neq \theta_2$, the expression of the p.m.f. of S becomes:

$$p_s(s) = \theta_1\theta_2 \left[\frac{(1-\theta_2)^{s+1} - (1-\theta_1)^{s+1}}{\theta_1 - \theta_2}(1+\alpha) + \alpha\frac{(1-\theta_2)^{2(s+1)} - (1-\theta_1)^{2(s+1)}}{(1-\theta_2)^2 - (1-\theta_1)^2}(2-\theta_1)(2-\theta_2) \right.$$
$$\left. -\alpha\frac{(1-\theta_2)^{s+1} - (1-\theta_1)^{2(s+1)}}{1-\theta_2-(1-\theta_1)^2}(2-\theta_1) - \alpha\frac{(1-\theta_2)^{2(s+1)} - (1-\theta_1)^{s+1}}{(1-\theta_2)^2-(1-\theta_1)}(2-\theta_2) \right]; \tag{8}$$

whereas if $\theta_1 = \theta_2 = \theta$, we have:

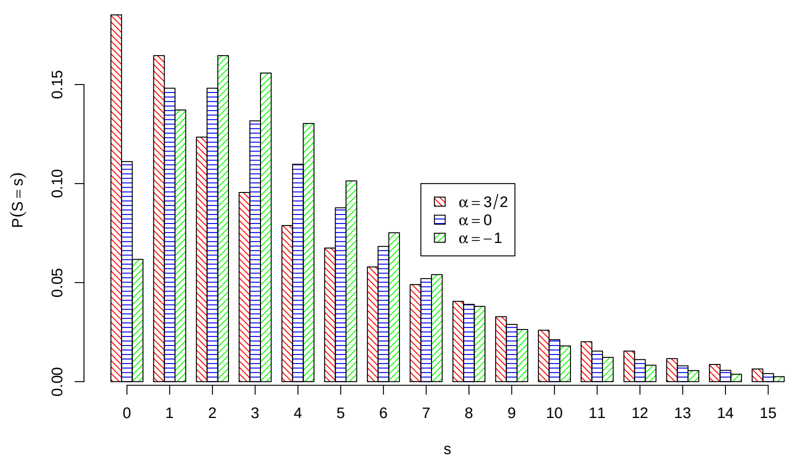
$$p_s(s) = \theta^2(1-\theta)^s \left\{ (s+1)[1+\alpha+\alpha(2-\theta)^2(1-\theta)^s] - 2\alpha\frac{2-\theta}{\theta}[1-(1-\theta)^{s+1}] \right\}$$

The expectation of S, being the two margins geometrically distributed with parameters θ_1 and θ_2, is obviously given by $\mathbb{E}(S) = \mathbb{E}(X) + \mathbb{E}(Y) = \frac{1-\theta_1}{\theta_1} + \frac{1-\theta_2}{\theta_2}$, whatever the value of α is. The variance of S is given by

$$\text{Var}(S) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X,Y) = \frac{1-\theta_1}{\theta_1^2} + \frac{1-\theta_2}{\theta_2^2} + 2\alpha\cdot\frac{1-\theta_1}{\theta_1(2-\theta_1)}\cdot\frac{1-\theta_2}{\theta_2(2-\theta_2)},$$

and thus, for fixed θ_1, θ_2 it increases with α. If $\theta_1 = \theta_2 = \theta$ and $\alpha = 0$ (independent and identical margins) then S is distributed as a negative binomial with parameters 2 and θ and p.m.f. $P(S = s) = (s+1)\theta^2(1-\theta)^s$, and then $\mathbb{E}(S) = 2\frac{1-\theta}{\theta}$, $\text{Var}(S) = 2\frac{1-\theta}{\theta^2}$. Figure 4 displays the p.m.f. of S for $\theta_1 = \theta_2 = 1/3$ and the three values of α, -1, 0, 3/2.

Figure 4.: P.m.f. of the sum $S = X + Y$ for the bivariate geometric r.v. with $\theta_1 = \theta_2 = 1/3$ and three meaningful values of α.



2.9. *Regression model*

The bivariate geometric distribution proposed here easily allows for the introduction of covariates. By considering the marginal means $\mu_1 = \mathbb{E}(X)$ and $\mu_2 = \mathbb{E}(Y)$ of the bivariate geometric r.v., we have $\theta_t = (1+\mu_t)^{-1}$, $t = 1, 2$, and then by introducing two vectors of covariates $\mathbf{z}$ and $\mathbf{w}$, we can model θ_1 and θ_2 assuming a log-linear relationship between marginal means and covariates:

$$\begin{aligned} \log \mathbb{E}(X_i|\mathbf{z}_i) &= \mathbf{z}_i'\boldsymbol{\beta} = z_{i0}\beta_0 + z_{i1}\beta_1 + \cdots + z_{i,k-1}\beta_{k-1} \quad \text{for } i = 1, \ldots, n \\ \log \mathbb{E}(Y_i|\mathbf{w}_i) &= \mathbf{w}_i'\boldsymbol{\gamma} = w_{i0}\gamma_0 + w_{i1}\gamma_1 + \cdots + w_{i,k-1}\gamma_{k-1} \quad \text{for } i = 1, \ldots, n \end{aligned} \tag{9}$$

The regression functions (9) relate the logarithm of the marginal means with the explanatory variables; the correlation between X_i and Y_i is clearly specified in terms of the dependence parameter α. The values of the elements of $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$, and the value of α can be simultaneously estimated by maximizing the log-likelihood function of the model. Among others, Famoye (2010) has already proposed an analogous regression model for a bivariate negative binomial distribution allowing for positive, negative, or null correlation. To simplify the analysis, one can assume that the same covariates affect X and Y, i.e., $\mathbf{z}_i = \mathbf{w}_i \ \forall i = 1, \ldots, n$.

3. ESTIMATION

Several methods for estimating the parameters θ_1, θ_2, and α of the proposed bivariate distribution of Eq.(2) can be envisaged, given a random sample, (x_i, y_i), $i = 1, 2, \ldots, n$. Two versions of the method of moments are here considered as well as the standard maximum likelihood method and a modified version thereof.

3.1. *Method of Moments*

A method of moments (MoM) is suggested, derived by equating the two marginal moments with the mixed moment to the corresponding sample quantities. Denoting with $\hat{\mu}_{xy} = \frac{1}{n}\sum_{i=1}^{n} x_i y_i$ the sample mixed moment, it can be easily shown that the estimates of the marginal parameters are $\hat{\theta}_{1,MoM} = 1/(1+\bar{x})$ and $\hat{\theta}_{2,MoM} = 1/(1+\bar{y})$, whereas for the dependence parameter α, after some algebraic passages, we have:

$$\hat{\alpha}_{MoM} = (\hat{\mu}_{xy} - \bar{x}\bar{y}) \cdot \frac{2\bar{x}+1}{\bar{x}(\bar{x}+1)} \cdot \frac{2\bar{y}+1}{\bar{y}(\bar{y}+1)}.$$

Though very easy to derive, the method of moments cannot always ensure a feasible value for the estimate of α, i.e., $\hat{\alpha}_{MoM}$ may lie outside its natural interval $[-1, \min\left\{1/(1-\hat{\theta}_{1,MoM}), 1/(1-\hat{\theta}_{2,MoM})\right\}]$.

3.2. *Modified Method of Moments*

Alternatively, by keeping the same estimators for the marginal parameters θ_1 and θ_2 as for the original method of moments, and following similar arguments used in Li and Dhar (2013), one can resort to a different estimator of α, obtained by equating the expected value of the maximum between X and Y, $M = \max(X, Y)$, with its sample version. This option gives raise to a variant that we call "modified method of moments". We know that the c.d.f. of M can be written as $F_M(m) = P(M \leq m) = P(X \leq m, Y \leq m) = F(m, m)$ and then its p.m.f. is given by $p_M(m) = F_M(m) - F_M(m-1) = F(m, m) - F(m-1, m-1)$. After some calculations, we derive the expression of $p_M(m)$, given by

$$\begin{aligned}
p_M(m) &= [1-(1-\theta_1)^{m+1}][1-(1-\theta_2)^{m+1}][1+\alpha(1-\theta_1)^{m+1}(1-\theta_2)^{m+1}]+\\
&\quad - [1-(1-\theta_1)^m][1-(1-\theta_2)^m][1+\alpha(1-\theta_1)^m(1-\theta_2)^m]\\
&= [(1-\theta_1)(1-\theta_2)]^m[(1-\theta_1)(1-\theta_2)-1] + (1-\theta_1)^m\theta_1 + (1-\theta_2)^m\theta_2 +\\
&\quad + \alpha\left\{[(1-\theta_1)(1-\theta_2)]^m[(1-\theta_1)(1-\theta_2)-1] + (1-\theta_1)^{2m}(1-\theta_2)^m[1-(1-\theta_1)^2(1-\theta_2)]\right.\\
&\quad \left. +(1-\theta_1)^m(1-\theta_2)^{2m}[1-(1-\theta_1)(1-\theta_2)^2] - [(1-\theta_1)(1-\theta_2)]^{2m}[1-(1-\theta_1)^2(1-\theta_2)^2]\right\}
\end{aligned}$$

and then the expected moment of M, provided by:

$$\begin{aligned}
\mathbb{E}(M) &= \sum_{m=0}^{\infty} m p_M(m) = \frac{1-\theta_1}{\theta_1} + \frac{1-\theta_2}{\theta_2} - \frac{(1-\theta_1)(1-\theta_2)}{1-(1-\theta_1)(1-\theta_2)} +\\
&+ \alpha\left[\frac{(1-\theta_1)^2(1-\theta_2)}{1-(1-\theta_1)^2(1-\theta_2)} + \frac{(1-\theta_2)^2(1-\theta_1)}{1-(1-\theta_1)(1-\theta_2)^2} - \frac{(1-\theta_1)(1-\theta_2)}{1-(1-\theta_1)(1-\theta_2)} - \frac{(1-\theta_1)^2(1-\theta_2)^2}{1-(1-\theta_1)^2(1-\theta_2)^2}\right].
\end{aligned}$$

Equating it to its sample analog $\bar{m} = \sum_{i=1}^{n} \max(x_i, y_i)/n$, and substituting to the unknown θ_1 and θ_2 their MoM estimates, we can solve for α and obtain:

$$\hat{\alpha}_{MoM2} = (\bar{m} - c)/d,$$

with $c = \bar{x} + \bar{y} - \frac{\bar{x}\bar{y}}{1+\bar{x}+\bar{y}}$ and $d = -\frac{\bar{x}\bar{y}}{1+\bar{x}+\bar{y}} + \frac{\bar{x}^2\bar{y}}{(1+\bar{x})^2(1+\bar{y})-\bar{x}^2\bar{y}} + \frac{\bar{x}\bar{y}^2}{(1+\bar{x})(1+\bar{y})^2-\bar{x}\bar{y}^2} - \frac{\bar{x}^2\bar{y}^2}{(1+\bar{x})^2(1+\bar{y})^2-\bar{x}^2\bar{y}^2}$. Though very easy to derive, even this method of moments cannot ensure a feasible value for the estimate of α.

3.3. *Maximum Likelihood*

From (2), the log-likelihood function $\ell(\theta_1, \theta_2, \alpha)$ is given by

$$\begin{aligned} \ell(\theta_1, \theta_2, \alpha) &= n \ln \theta_1 + n \ln \theta_2 + \textstyle\sum_{i=1}^{n} x_i \ln(1-\theta_1) + \sum_{i=1}^{n} y_i \ln(1-\theta_2) + \\ &+ \textstyle\sum_{i=1}^{n} \ln \left\{1 + \alpha \left[(2-\theta_1)(1-\theta_1)^{x_i} - 1\right] \left[(2-\theta_2)(1-\theta_2)^{y_i} - 1\right]\right\}. \quad (10) \end{aligned}$$

Maximum likelihood estimates (MLEs) of θ_1, θ_2, and α are simultaneously obtained as the parameter values maximizing (10) over the parameter space and are denoted as $\hat{\theta}_{1,ML}$, $\hat{\theta}_{2,ML}$, $\hat{\alpha}_{ML}$. MLEs can be derived as the solutions of the normal equations, obtained by equating the partial derivatives of the log-likelihood function (10) to zero; however, they are not reported here as they cannot be solved explicitly. MLEs can be thus computed either by numerical methods or by directly maximizing the log-likelihood function. One can use the `optim`, `mle2`, or `maxLik` functions under the `R` programming environment (R Core Team 2017).

3.4. *Two-Step Maximum Likelihood*

Since the "exact" maximum likelihood method discussed above can be computationally burdensome, the literature has suggested a "two-step maximum likelihood" approach. According to this method, which has been proposed in Joe and Xu (1996) and Joe (1997) and which is also termed "inference functions for margins" (IFM), univariate parameters are first estimated based on individual univariate log-likelihoods and then multivariate parameters are estimated from the multivariate log-likelihood or lower-dimensional log-likelihoods with univariate parameter estimates held fixed. While asymptotically less efficient than the one-step estimator (see again Joe (1997)), this approach has the obvious advantage of reducing the dimensionality of the problem, which is particularly useful when one has to resort to a numerical maximization. For the bivariate distribution at study, the two-step maximum likelihood method, similarly to the case presented in Barbiero (2017a), consists in computing first the maximum likelihood estimates of the parameters of the two marginal geometric distributions as if they were independent. In this case, we easily get the estimates of θ_1 and θ_2 as

$$\hat{\theta}_{1,TSML} = (1+\bar{x})^{-1}, \quad \hat{\theta}_{2,TSML} = (1+\bar{y})^{-1}$$

(they coincide with the MoM estimates). Plugging them into the expression of the log-likelihood (10), one can then maximize it with respect to the only remaining parameter α (again, numerically), obtaining the estimate $\hat{\alpha}_{TSML}$.

3.5. *Interval estimation*

Besides point estimation, asymptotic confidence intervals for each of the three parameters can be built based on Fisher information matrix, defined as

$$\mathcal{I}(\boldsymbol{\eta})_{ij} = \mathbb{E}\left(-\frac{\partial^2 \log p(x,y;\boldsymbol{\eta})}{\partial\eta_i\partial\eta_j}\right) \tag{11}$$

with $\boldsymbol{\eta} = (\theta_1, \theta_2, \alpha)$. The second order derivatives inside Eq.(11) are cumbersome but quite easy to compute. On the contrary, calculating their expected value is complicated. If a bivariate sample of size n is available from the r.v. (X,Y), one can compute the observed Fisher Information matrix, based on the MLE of $\boldsymbol{\eta}$, $\hat{\boldsymbol{\eta}}$: $\hat{\mathcal{I}}(\hat{\boldsymbol{\eta}})_{ij} = \sum_{i=1}^{n} -\frac{\partial^2 \log p(x_i,y_i;\hat{\boldsymbol{\eta}})}{\partial\eta_i\partial\eta_j}$. Asymptotic confidence intervals can be constructed by using the standard errors $\text{se}(\eta_i)$ that can be derived by computing the inverse matrix $\hat{\mathcal{I}}(\hat{\boldsymbol{\eta}})^{-1}$. Such intervals have the usual form $(\hat{\eta}_i - z_{1-\alpha/2}\text{se}(\hat{\eta}_i), \hat{\eta}_i + z_{1-\alpha/2}\text{se}(\hat{\eta}_i))$, with $\text{se}(\hat{\eta}_i) = \sqrt{[\hat{\mathcal{I}}(\hat{\boldsymbol{\eta}})^{-1}]_{ii}}$. Since the parameters η_i are bounded over a finite support, such symmetric confidence intervals may have a poor performance in terms of coverage and average length. For this reason, computing confidence intervals directly based on the log-likelihood or profile log-likelihood may represent a much better choice; they often have better small-sample properties than those based on asymptotic standard errors computed from the full likelihood (Venzon and Moolgavkar 1988). In the statistical environment `R`, the package `bbmle` (Bolker 2016) easily allows the user to compute confidence intervals based on profile likelihood.

4. MONTE CARLO STUDY

In order to study the behaviour of the parameters' estimators in terms of unbiasedness and variability, one would like to obtain their expectations and variances (or standard errors). Finding them is almost impossible, especially for methods involving maximum likelihood, which provide estimates in a numerical form only. Therefore, we study numerically the expressions for expected values and standard errors with a Monte Carlo simulation study. We consider several representative combinations of the three parameters characterizing the proposed bivariate geometric distribution; in particular, we consider all the "distinct" combinations arising from the following choice of parameters: $\theta_1, \theta_2 = 0.25; 0.5; 0.75$; $\alpha = -0.8; -0.4; 0; +0.4, +0.8, +1.2$ (thus leading to negative, null or positive dependence between the margins). "Distinct" here means that given the symmetrical nature of the problem, if the combination corresponding to the ordered parameter vector $\boldsymbol{\eta} = (\theta_1, \theta_2, \alpha)$ is considered, the combination associated to the "symmetrical" parameter vector $\boldsymbol{\eta}^* = (\theta_2, \theta_1, \alpha)$ will be skipped. This means that $6 \times 6 = 36$ combinations for $\boldsymbol{\eta}$ are considered. For each of these scenarios, the Monte Carlo simulation study is performed by generating $2,000$ samples of size $n = 100$. For each sample, the four types of estimators of θ_1, θ_2, and α, described in Section 3, are calculated as well as 95% confidence intervals. Then, over all the $2,000$ simulations, the means (expected values) and standard errors of these estimates are obtained along with coverage and average length of confidence intervals. Here we will focus on the results related to the estimates of the dependence parameter α, which are expected to be the most meaningful, since the dependence parameter is intuitively the most difficult to be estimated. The bivariate geometric model has been implemented and the whole simulation study has been carried out in the `R` statistical environment.

Results on point estimation are reported in Tables 1 to 6. We will briefly discuss the performance of the estimators comparatively, underlying how the dependence and marginal parameters' values affect it. We first remark that all four estimators are positively biased under all the scenarios examined, except for a few scenarios with $\alpha = 1.2$; the bias magnitude is however rather small. The ML and TSML estimators of α always have a very similar behaviour in terms of both bias and standard error and, quite surprisingly, for positive or null values of α, TSML should be even preferred to ML estimator, since it shows a smaller bias in absolute value and a smaller standard error. Their performance is overall better than that of MoM estimator, except when α is -0.8: in this case, both ML and TSML estimators show a very large bias in absolute value, and in this sense MoM may be preferable; however, the standard errors of the latter are always larger than those of the former and a significant proportion of MoM estimates are unfeasible (smaller than -1). MoM2 estimator is overall more biased than MoM, even if sometimes (namely, for $\alpha = 0.4; 0.8; 1.2$) it presents a smaller variability. In more detail, letting α fixed and varying the marginal parameters θ_1 and θ_2, one can note that the behaviour of the four estimators (especially in terms of standard error and to a larger extent for ML and TSML) deteriorates moving towards large values; in particular, the scenarios characterized by $\theta_1 = \theta_2 = 0.75$ are the worst ones. This is quite reasonable, since in this case the sample marginal distributions are concentrated on the first integer numbers, with an average proportion of zeros equal to 75% for both margins, and estimates of α are thus characterized by a large uncertainty. For a fixed pair of values (θ_1, θ_2), the behavior of the standard errors of the four estimators as a function of α is much more difficult to depict. For example, focusing on the ML estimator, if $(\theta_1, \theta_2) = (0.25, 0.25)$, its standard error decrease with the absolute value of α; if $(\theta_1, \theta_2) = (0.75, 0.75)$, the standard error is an increasing function of α.

Table 1.: Monte Carlo simulation study: mean and standard error of the estimates of α when $\alpha = 1.2$

		$\theta_1 = 0.25$		$\theta_1 = 0.5$		$\theta_1 = 0.75$	
		mean	se	mean	se	mean	se
$\theta_1 = 0.25$	ML	1.193	0.203	1.186	0.234	1.173	0.268
	TSML	1.184	0.215	1.180	0.234	1.169	0.268
	MoM	1.197	0.425	1.198	0.450	1.203	0.550
	MoM2	1.226	0.397	1.225	0.379	1.222	0.378
$\theta_1 = 0.5$	ML			1.217	0.334	1.221	0.428
	TSML			1.203	0.333	1.215	0.426
	MoM			1.201	0.484	1.209	0.594
	MoM2			1.229	0.431	1.235	0.485
$\theta_1 = 0.75$	ML					1.219	0.605
	TSML					1.213	0.602
	MoM					1.217	0.748
	MoM2					1.237	0.663

In figure 5, the Monte Carlo distributions of the four point estimators of α for the bivariate geometric model are displayed, when $\theta_1 = \theta_2 = 0.5$ and $\alpha = 0.8$. At a first glance, one can note that the two method of moments (MoM and MoM2) yield to estimators characterized by a larger variability with respect to the ML and TSML competitors. In particular, due also to the presence of many "outliers" (the isolated points beyond the whiskers of the boxplot; most of them are unfeasible values for α), MoM, as confirmed by the results of Table 2, presents a larger variability than MoM2.

Table 2.: Monte Carlo simulation study: mean and standard error of the estimates of α when $\alpha = 0.8$

		$\theta_1 = 0.25$		$\theta_1 = 0.5$		$\theta_1 = 0.75$	
		mean	se	mean	se	mean	se
$\theta_1 = 0.25$	ML	0.812	0.290	0.813	0.321	0.813	0.382
	TSML	0.806	0.288	0.808	0.319	0.810	0.381
	MoM	0.801	0.428	0.803	0.450	0.809	0.544
	MoM2	0.832	0.417	0.830	0.420	0.832	0.473
$\theta_1 = 0.5$	ML			0.811	0.352	0.815	0.446
	TSML			0.806	0.350	0.811	0.444
	MoM			0.805	0.479	0.811	0.580
	MoM2			0.832	0.445	0.834	0.507
$\theta_1 = 0.75$	ML					0.818	0.594
	TSML					0.814	0.591
	MoM					0.817	0.714
	MoM2					0.836	0.650

Table 3.: Monte Carlo simulation study: mean and standard error of the estimates of α when $\alpha = 0.4$

		$\theta_1 = 0.25$		$\theta_1 = 0.5$		$\theta_1 = 0.75$	
		mean	se	mean	se	mean	se
$\theta_1 = 0.25$	ML	0.410	0.319	0.409	0.341	0.411	0.421
	TSML	0.407	0.317	0.406	0.339	0.409	0.419
	MoM	0.406	0.421	0.404	0.446	0.410	0.537
	MoM2	0.434	0.429	0.433	0.447	0.432	0.534
$\theta_1 = 0.5$	ML			0.406	0.365	0.408	0.457
	TSML			0.404	0.362	0.407	0.455
	MoM			0.405	0.472	0.411	0.570
	MoM2			0.432	0.453	0.430	0.516
$\theta_1 = 0.75$	ML					0.410	0.585
	TSML					0.409	0.583
	MoM					0.414	0.692
	MoM2					0.428	0.638

Table 4.: Monte Carlo simulation study: mean and standard error of the estimates of α when $\alpha = 0$

		$\theta_1 = 0.25$		$\theta_1 = 0.5$		$\theta_1 = 0.75$	
		mean	se	mean	se	mean	se
$\theta_1 = 0.25$	ML	0.008	0.326	0.006	0.346	0.010	0.422
	TSML	0.008	0.324	0.006	0.344	0.010	0.420
	MoM	0.010	0.406	0.009	0.425	0.010	0.502
	MoM2	0.037	0.432	0.034	0.464	0.036	0.570
$\theta_1 = 0.5$	ML			0.005	0.368	0.007	0.447
	TSML			0.005	0.366	0.007	0.445
	MoM			0.010	0.448	0.010	0.531
	MoM2			0.035	0.448	0.034	0.512
$\theta_1 = 0.75$	ML					0.009	0.546
	TSML					0.009	0.543
	MoM					0.011	0.636
	MoM2					0.024	0.597

Table 5.: Monte Carlo simulation study: mean and standard error of the estimates of α when $\alpha = -0.4$

		$\theta_1 = 0.25$		$\theta_1 = 0.5$		$\theta_1 = 0.75$	
		mean	se	mean	se	mean	se
$\theta_1 = 0.25$	ML	-0.393	0.308	-0.389	0.324	-0.380	0.393
	TSML	-0.390	0.306	-0.387	0.322	-0.378	0.391
	MoM	-0.385	0.376	-0.383	0.392	-0.385	0.460
	MoM2	-0.364	0.427	-0.360	0.467	-0.354	0.601
$\theta_1 = 0.5$	ML			-0.390	0.342	-0.380	0.410
	TSML			-0.388	0.340	-0.378	0.408
	MoM			-0.383	0.412	-0.385	0.484
	MoM2			-0.362	0.431	-0.363	0.502
$\theta_1 = 0.75$	ML					-0.371	0.481
	TSML					-0.370	0.479
	MoM					-0.386	0.568
	MoM2					-0.374	0.546

Table 6.: Monte Carlo simulation study: mean and standard error of the estimates of α when $\alpha = -0.8$

		$\theta_1 = 0.25$		$\theta_1 = 0.5$		$\theta_1 = 0.75$	
		mean	se	mean	se	mean	se
$\theta_1 = 0.25$	ML	-0.758	0.230	-0.753	0.246	-0.730	0.291
	TSML	-0.754	0.231	-0.750	0.246	-0.727	0.290
	MoM	-0.780	0.329	-0.780	0.339	-0.781	0.384
	MoM2	-0.762	0.413	-0.760	0.463	-0.752	0.615
$\theta_1 = 0.5$	ML			-0.748	0.258	-0.726	0.302
	TSML			-0.745	0.258	-0.724	0.302
	MoM			-0.780	0.353	-0.781	0.400
	MoM2			-0.764	0.407	-0.764	0.467
$\theta_1 = 0.75$	ML					-0.709	0.344
	TSML					-0.706	0.358
	MoM					-0.785	0.454
	MoM2					-0.772	0.470

Figure 5.: Boxplots of the Monte Carlo distributions of the four point estimators of α for the bivariate geometric model, with $\theta_1 = \theta_2 = 0.5$, and $\alpha = 0.8$.

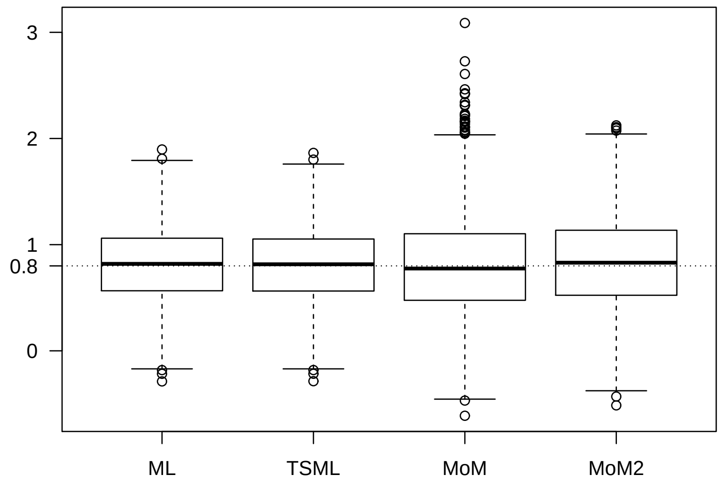


Table 7 shows how coverages and average length of confidence intervals are affected by the values of the marginal parameters θ_1, θ_2, and by the dependence parameter α. As to the latter, the extreme negative value examined here ($\alpha = -0.8$) always leads to over-coverage and often, holding fixed θ_1 and θ_2, to the shortest average length; whereas the other values (with a few exceptions for $\alpha = 0.8; 1.2$) overall produce under-coverage and wider intervals. Keeping α fixed, larger values of θ_1 and θ_2 (in our study, 0.75), i.e., parameter values leading to larger proportions of zeros in both sample components, always yield wider intervals (which turn out to be very little

Table 7.: Coverage and average length of 95% confidence intervals for α

		$\theta_2 = 0.25$		$\theta_2 = 0.5$		$\theta_2 = 0.75$	
	α	coverage	length	coverage	length	coverage	length
$\theta_1 = 0.25$	-0.8	0.978	0.784	0.974	0.829	0.972	0.981
	-0.4	0.939	1.063	0.944	1.119	0.941	1.278
	0	0.936	1.159	0.938	1.214	0.937	1.389
	0.4	0.942	1.132	0.938	1.232	0.936	1.442
	0.8	0.932	0.883	0.936	0.924	0.928	1.047
	1.2	0.974	0.684	0.972	0.770	0.974	0.924
$\theta_1 = 0.5$	-0.8			0.978	0.872	0.976	1.028
	-0.4			0.943	1.174	0.946	1.354
	0			0.943	1.312	0.946	1.550
	0.4			0.946	1.332	0.934	1.618
	0.8			0.945	1.266	0.934	1.503
	1.2			0.945	1.194	0.940	1.514
$\theta_1 = 0.75$	-0.8					0.975	1.219
	-0.4					0.959	1.593
	0					0.941	1.869
	0.4					0.930	2.044
	0.8					0.940	2.142
	1.2					0.940	2.193

informative), whereas the effect on the coverage is not sharp.

The analysis of simulation results about standard errors and confidence intervals' length highlights how estimation of the dependence parameter for the proposed bivariate geometric model is critical; it is characterized by a high level of uncertainty even if the chosen sample size is moderately large. This may be partially explained by the discrete nature of the data, but more importantly by the meaning of the dependence parameter itself: although taking both negative or positive values, it does not encompass the whole range of dependence, from perfect negative to positive negative dependence and this fact may translate into an "inflation" of the variability of the estimates. Large asymptotic variance of estimators of parameter α for the FGM copula, though limited to the continuous case, has been already acknowledged in Genest (2013).

5. APPLICATION TO REAL DATA

In this section, a numerical example is provided to illustrate the application of the proposed bivariate geometric distribution. The data, considered in Mitchell and Paulson (1981), consist of the number of aborts by 109 aircrafts in two consecutive semesters

Table 8.: Bivariate distribution of the data taken from Mitchell and Paulson (1981): number of flight aborts by 109 aircrafts in two consecutive periods. Theoretical frequencies for the bivariate geometric model fitted to these data are reported between brackets in italic. Cell borders highlight a possible aggregation of cells into 9 groupings with frequencies' sum greater than 5, where the chi-square goodness-of-fit statistic is computed.

x, y	0	1	2	3	≥ 4	total
0	34 *(34.82)*	20 *(17.98)*	4 *(8.22)*	6 *(3.60)*	4 *(2.70)*	68 *(67.32)*
1	17 *(16.63)*	7 *(5.57)*	0 *(2.10)*	0 *(0.84)*	0 *(0.60)*	24 *(25.74)*
2	6 *(6.84)*	4 *(1.94)*	1 *(0.65)*	0 *(0.24)*	0 *(0.17)*	11 *(9.84)*
3	0 *(2.69)*	4 *(0.71)*	0 *(0.23)*	0 *(0.08)*	0 *(0.06)*	4 *(3.76)*
4	0 *(1.04)*	0 *(0.27)*	0 *(0.08)*	0 *(0.03)*	0 *(0.02)*	0 *(1.44)*
≥ 5	2 *(0.65)*	0 *(0.17)*	0 *(0.05)*	0 *(0.02)*	0 *(0.01)*	2 *(0.89)*
total	59 *(62.67)*	35 *(26.64)*	5 *(11.32)*	6 *(4.81)*	4 *(3.56)*	109 *(109)*

(number of aborts in the first period = x, in the second period = y), see Table 8. Summary statistics for the dataset are $\bar{x} = 0.624$, $\bar{y} = 0.725$, $\text{var}(x) = 1.024$, $\text{var}(y) = 1.062$, $\hat{\rho}_{xy} = -0.161$. Looking at the empirical marginal distributions of x and y, one can assume they come from two geometric r.v.s. Through a proper univariate goodness-of-fit test one can check this hypothesis; by resorting to Kolmogorov-Smirnov test, see Bracquemond et al. (2002), we accept the null hypotheses that x and y come from two geometric distributions (bootstrapped p-values 0.7989 and 0.2984, respectively). Then, we can further assume and check whether the bivariate sample (x, y) comes from the bivariate geometric model with FGM copula. Fitting it to these data and estimating its parameters through each method presented in section 3 lead to the results reported in Table 9. Note that the estimates of θ_1 and θ_2 computed through ML are very close to the corresponding estimates of TSML, MoM, and MoM2. The estimates of α computed through MoM and MoM2 are quite sensibly different (in opposite directions) from the ML and TSML estimates, which are instead quite close to each other.

Table 9.: Parameter estimates for the bivariate geometric model applied to Mitchell and Paulson (1981) data.

method	θ_1	θ_2	α
ML	0.6176	0.5749	-0.6174
TSML	0.6158	0.5798	-0.6091
MoM	0.6158	0.5798	-0.7293
MoM2	0.6158	0.5798	-0.4548

The observed Fisher Information Matrix is given by:

$$\hat{\mathcal{I}}(\hat{\boldsymbol{\eta}}) = \begin{pmatrix} 0.001341 & -0.000183 & -0.000737 \\ -0.000183 & 0.001313 & 0.002217 \\ -0.000737 & 0.002217 & 0.158698 \end{pmatrix}$$

from which the asymptotic standard errors of the three MLEs can be easily derived: $\text{se}(\hat{\theta}_1) = 0.036618$, $\text{se}(\hat{\theta}_2) = 0.036238$, $\text{se}(\hat{\alpha}) = 0.398370$. Note the large uncertainty associated to the dependence parameter α. 95% confidence intervals for the three parameters based on profile log-likelihood are provided as $(\theta_{1L}, \theta_{1U}) = (0.5445, 0.6873)$, $(\theta_{2L}, \theta_{2U}) = (0.5034, 0.6446)$, $(\alpha_{1L}, \alpha_{1U}) = (-1, 0.1730)$.

The value of the log-likelihood function computed at the MLEs is $\ell_{\max} = -244.63$ and the corresponding value of the Akaike Information Criterion ($\text{AIC} = 2k - 2\ell_{\max}$, with $k = 3$ being the number of parameters) is 495.26. The bivariate negative binomial distribution proposed in Mitchell and Paulson (1981) showed an AIC equal to 498.54, thus indicating a worse fit to the data; the bivariate discrete Weibull distribution proposed in Barbiero (2017a), though obviously providing a greater value of the log-likelihood function (-243.966), provides a greater value of the AIC (497.93), thus denoting again a worse fit than the proposed bivariate geometric model. The bivariate geometric distribution proposed by Roy (1993) yields a better fit: the AIC is equal to 494.0382, being the maximum value of the log-likelihood function -244.0191. In order to obtain an "absolute" measure of fit of the proposed bivariate model, we can resort to the standard chi-square goodness-of-fit test. We computed the theoretical absolute joint frequencies, by using the p.m.f. in (2) with the MLEs of the parameters θ_1, θ_2 and α; they are displayed between brackets in Table 8. Then we aggregated cells in order to obtain for each grouping an aggregate frequency larger than 5; we computed the chi-square statistic as $\chi^2 = \sum_{g=1}^{G} (\hat{n}_g - n_g)^2/\hat{n}_g$, where n_g is the observed count for grouping g, $\hat{n}_g$ is its theoretical analog, G is the number of groupings (in this case

$G = 9$). Under the null hypothesis that the bivariate sample comes from the proposed distribution, χ^2 is approximately distributed as a chi-square r.v. with $9 - 3 - 1 = 5$ degrees of freedom. The empirical value of χ^2 is 5.434; its p-value is 0.365 and being far larger than 0 it denotes a satisfactory fit of the model to the data.

Plugging in the MLEs of the three parameters into (6), one derives the MLE of R as $\hat{R} = 0.7123997$, which represents the estimated probability that the number of aborts in the second period is not smaller than the number of aborts in the first one. By the way, the MLE of R is very close to the standard non-parametric estimate $\tilde{R} = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{x_i \leq y_i} = 76/109 = 0.6972477$.

6. CONCLUSION

The bivariate geometric model proposed in this work is able to handle correlated geometrically distributed counts with a moderate level of dependence, regulated by the unique parameter of the Farlie-Gumbel-Morgenstern (FGM) copula linking its univariate margins. It can be employed in several fields where discrete data arise, such as industrial quality control, insurance, health economics, marketing, an so on. The easy form of the joint probability mass function allows to derive interesting analytical results for the bivariate failure rate, attainable correlations, conditional distributions and moments, reliability parameter, and, partially, estimation. Moreover, the parameter components have a clear interpretation. An application to a real dataset, reporting the number of failures occurred to a group of aircrafts in two consecutive periods, has shown how the model can be easily fitted. However, we are aware that the use of this bivariate discrete model may lead to some problems when modeling real data (the natural space of the dependence parameter depends on the values of the marginal parameters; the model can allow only for a moderate linear correlation, whose lower and upper bounds depend again on the values of the marginal parameters) and in the estimation step (even for moderately large sample size, estimators of the dependence parameter can be little precise). Further research will investigate possible extensions of the model accommodating a wider range of dependence, by resorting to generalized FGM copulas, and thus possibly tackling also drawbacks in estimation. We hope that the proposed model will be a viable alternative to the existing models dealing with the kind of data sets considered here.

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