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On some open problems in Banach space theory

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Summary

The main line of investigation of the present work is the study of some aspects in the analysis of the structure of the unit ball of (infinite-dimensional) Banach spaces. In particular, we analyse some questions concerning the existence of suitable renormings that allow the new unit ball to possess a specific geometric property. The main part of the thesis is, however, dedicated to results of isometric nature, in which the original norm is the one under consideration.

One of the main sources for the selection of the topics of investigation has been the recent monograph [GMZ16], entirely dedicated to collecting several open problems in Banach space theory and formulating new lines of investigation. We take this opportunity to acknowledge the authors for their effort, that offered such useful text to the mathematical community. The results to be discussed in our work actually succeed in solving a few of the problems presented in the monograph and are based on the papers [HáRu17, HKR18, HáRu19, HKR••].

Let us say now a few words on how the material is organised. The thesis is divided in four chapters (some whose contents are outlined below) which are essentially independent and can be read in whatsoever order. The unique chapter which is not completely independent from the others is Chapter 4, where we use some results from Chapter 2 and which is, in a sense, the non-separable prosecution of Chapter 3. However, cross-references are few (never implicit) and usually restricted to quoting some result; it should therefore be no problem to start reading from Chapter 4.

The single chapters all share the same arrangement. A first section is dedicated to an introduction to the subject of the chapter; occasionally, we also present the proof of known results, in most cases as an illustration of an important technique in the area. In these introductions we strove to be as self-contained as possible in order to help the novel reader to enter the field; consequently, experts in the area may find them somewhat redundant and prefer to skip most parts of them.

The first section of each chapter concludes with the statement of our most significant results and a comparison with the literature. The proofs of these results, together with additional results or generalisations, are presented in the remaining sections of the chapter. These sections usually follow closely the corresponding articles (carefully referenced) where the results were presented.

Smooth renormings

It is by now a well understood fact in Functional Analysis that the existence, on an infinite-dimensional Banach space, of an equivalent norm with good differentiability properties is a strong assumption that has profound structural consequences for the space. As a sample of this phenomenon, let us recall that if a Banach space X admits a renorming with locally uniformly continuous derivative (*e.g.*, a C^2-smooth norm) then X is either super-reflexive, or it contains a copy of c_0.

On the other hand, once one smooth norm is present in the space, it is often the case the space to admit a large supply of smooth renormings. This is true in full generality for C^1-smooth norms, in the case that the dual space admits a dual LUR renorming; it is moreover true for every separable space for C^k-smooth norms ($1 \leqslant k \leqslant \infty$).

In this chapter we proceed in the above direction and we give a sharper result for norm approximation, in the case of (separable) Banach spaces, with a Schauder basis. Our main result asserts that if a Banach space with a Schauder basis admits a C^k-smooth norm, it is possible to approximate every equivalent norm with a C^k-smooth one in a way that the approximation improves as fast as we wish on the vectors that only depend on the tail of the Schauder basis.

We also give analogous results for the case of norms locally depending on finitely many coordinates, or for polyhedral norms. It is to be noted that such 'asymptotically optimal' approximation is not always possible, for example the analogous claim for uniformly convex renormings is easily seen to be false.

Auerbach systems

One fundamental tool for the study of normed space is the investigation of systems of coordinates that a given space can be furnished with. This is, of course, true already in finite dimensions, but it is in the infinite-dimensional setting that several non-equivalent notions of a system of coordinates are available. It is therefore an important issue to understand which are the optimal systems of coordinates a given class of Banach spaces can admit.

According to a celebrated result by Kunen, there exists (under the assumption of the Continuum Hypothesis) a non-separable Banach space with virtually no (uncountable) system of coordinates, in that it admits no uncountable biorthogonal system.

In this chapter, we shall concentrate our attention to Auerbach systems. These object are particularly convenient, since of the one hand they are based on a rather weak notion of coordinates, being just biorthogonal systems, but on the other hand they have very rigid isometric properties. This allows for several construction of isometric nature, some of which are contained in the subsequent chapters.

In particular, we shall address the existence of large Auerbach systems in Banach spaces. Our first result is that every 'large' Banach space always admits a uncountable Auerbach system, therefore implying that there are no 'large' analogues of Kunen's example. We then sharpen the previous assertion for the class of WLD Banach spaces: we show that every WLD Banach space X with $\operatorname{dens} X > \omega_1$ contains an Auerbach system of the maximal possible cardinality, $\operatorname{dens} X$.

The main result of the chapter and, perhaps, the most striking result of the thesis is the fact that the above result concerning WLD spaces is sharp. To wit, we show that (under CH) there exists a renorming of $c_0(\omega_1)$ that contains no uncountable Auerbach system; this is, in a sense, analogue to Kunen's result and solves in a stronger form one problem from [GMZ16].

The second part of the chapter is dedicated to some possible uncountable extensions of a famous combinatorial lemma, due to Vlastimil Pták. To wit, we show that the validity of an analogous statement for the cardinal number ω_1 can not be decided in ZFC; we also offer sufficient conditions, for a class of larger cardinal numbers.

Symmetrically separated sequences

Kottman's theorem, asserting that the unit sphere of an infinite-dimensional normed space contains a sequence of points whose mutual distances are strictly greater than one, sparked a new insight on the non-compactness of the unit ball in infinite dimensions. Elton and Odell employed methods of infinite Ramsey theory to improve Kottman's theorem significantly by showing that the unit sphere of an infinite-dimensional normed space contains a sequence $(x_n)_{n=1}^{\infty}$ such that $\|x_n - x_k\| \geqslant 1 + \varepsilon$ $(k, n \in \mathbb{N}, k \neq n)$ for some $\varepsilon > 0$.

The main objective of the chapter is to revisit and investigate the above-mentioned results in the setting of symmetric separation: let us say that a subset A of a normed space is *symmetrically* $(\delta+)$*-separated* (respectively, *symmetrically* δ*-separated*) when $\|x \pm y\| > \delta$ (respectively, $\|x \pm y\| \geqslant \delta$) for any distinct elements $x, y \in A$. J.M.F. Castillo and P.L. Papini asked whether there is a symmetric version of the Elton–Odell theorem; however according to Castillo, it has not been known whether the unit sphere of an infinite-dimensional Banach space contains a symmetrically $(1+)$-separated sequence.

Our main results are a proof of a symmetric version of Kottman's theorem, together with a proof of the symmetric analogue to the Elton–Odell theorem, for a huge class of Banach spaces, that includes, in particular, all classical Banach spaces. We also give a good bunch of quantitative lower estimates for the ε appearing in the conclusion to the Elton–Odell theorem, for some specific classes of Banach spaces.

Uncountable separated sets

Over the last years, a renewed interest and a rapid progress in the analysis of the natural non-separable analogues of the results discussed in the previous chapter have been observed. Perhaps the first spark was lit by Mercourakis and Vassiliadis who have identified certain classes of compact Hausdorff spaces K for which the unit sphere of the Banach space $C(K)$ contains an uncountable $(1+)$-separated set. The result has been subsequently improved by Kania and Kochanek to every (non-metrisable) compact Hausdorff space K, and, very recently, sharpened by Cúth, Kurka, and Vejnar.

Our first main result for this part consists in understanding the fact that Auerbach systems can be profitably exploited to approach the above problem; this allows us to deduce that the unit sphere of every 'large' Banach space contains an uncountable $(1+)$-separated set. We also observe the–perhaps unexpected–fact that under the present assumptions the conclusion cannot be improved, in the sense that every $(1+)$-separated subset of $B_{c_0(\Gamma)}$ has cardinality at most ω_1.

We next turn our attention to some strong structural constrains on the space, which allow construction of potentially larger separated subsets of the unit sphere. For example, we strengthen considerably one result by Kania and Kochanek by proving the existence of a (1+)-separated set in the unit sphere of every (quasi-)reflexive space X that has the maximal possible cardinality, that is, equal to dens X.

In the case where the number dens X has uncountable cofinality, such set can be taken to be $(1+\varepsilon)$-separated for some $\varepsilon > 0$. When X is a super-reflexive space, we exhibit a $(1+\varepsilon)$-separated set in the unit sphere of X that also has the maximal possible cardinality—this answers a question raised by T. Kania and T. Kochanek.

To conclude, let us mention that, as a by-product of our techniques, the results discussed in this chapter actually produce symmetrically separated sets. However, the clause about symmetry is not the main issue in our results, which are sharper than the ones present in the literature, even when the symmetry assertion is removed from them.

Notation

Our notation concerning Banach space theory is standard, as in most textbooks in Functional Analysis; all undefined notations of definitions may be found, *e.g.*, in [AlKa06, FHHMZ10, LiTz77, LiTz79]. We also refer to the same textbooks for the basic results in Banach space theory, that we use implicitly, or without specific reference to the literature. For example, we do not define in the thesis the notions of reflexive, super-reflexive, or Radon–Nikodym Banach spaces; for such definitions, we refer, *e.g.*, to [AlKa06, Bea85, BeLi00, Die75, Die84, DiUh77, FHHMZ10, vDu78]. On the other hand, we have decided to introduce and briefly discuss most notions used in the thesis; this should allow the reader to read the text without jumping too frequently into the references or specialised textbooks. As a consequence, the text should be accessible to anyone having followed a course in Functional Analysis.

Let us mention here that we will restrict our considerations to normed spaces over the real field, although most of the results (Chapter 1 excluded) apply directly to complex Banach spaces. For a normed space X, we shall denote by S_X the unit sphere of X and by B_X the closed unit ball of X.

Let us then dedicate a few words on some notation concerning set theory, which follows, *e.g.*, [Cie97, Kun80b, Jec03]. We use von Neumann's definition of ordinal numbers and we regard cardinal numbers as initial ordinal numbers. In particular, we write ω for $\aleph_0$, ω_1 for $\aleph_1$, *etc.*, as we often view cardinal numbers as well-ordered sets; we also denote by $\mathfrak{c}$ the cardinality of continuum. For a cardinal number κ, we write κ^+ for the immediate successor of κ, that is, the smallest cardinal number that is strictly greater than κ.

If F and G are subsets of a certain ordinal number λ, we shall use the (perhaps self-explanatory) notation $F < G$ to mean that $\sup F < \min G$; in the case that $G = \{g\}$ is a singleton, we shall write $F < g$ instead of $F < \{g\}$. Analogous meaning is attributed to expressions such as $g < F$, $f \leqslant G$, and so on.

Finally, we mention that when $f\colon S \to Z$ is a function between sets S and Z and H is a subset of S, we shall denote by $f\restriction_H$ the restriction of the function f to the set H. In case that $S = \mathbb{N}$, *i.e.*, f is a sequence, we shall sometimes use the small abuse of notation to write $f\restriction_{[1,N]}$, instead of $f\restriction_{\{1,\dots,N\}}$.

Contents

Chapter 1

Smooth renormings

This chapter is dedicated to one contribution to the vast field of smooth renormings of Banach spaces, that stands at the intersection between the renorming theory and the study of smooth functions in normed spaces. In the first section, we shall shortly review the basic definitions and a few well-known results in this area; we will also present some of the most useful techniques, as they will be relevant for the proofs of our results. The second section contains our renorming argument and the proof of the main result of the chapter; in the last section, we present some simple modifications of the renorming result that apply to polyhedral Banach spaces.

1.1 A few smoothing techniques

The aim of this section is to introduce the topic of smooth renormings of Banach spaces, via the description of a few selected important results; let us immediately notice that the field is way too broad for a reasonably complete presentation, ever for a discussion of the main lines of investigation. For this reason, we will just describe a few results in the directions relevant to our results, in order to give a flavor of a small part of the area. For a more complete discussion and references we refer to the monographs [DGZ93] and [HáJo14], or the survey papers [God01] and [Ziz03] in the Handbook; let us also refer to the nice elementary introduction given in [FrMc02].

Our main perspective will be to motivate the fact that, unlike the finite-dimensional case, the existence of non-trivial C^k-smooth functions on an infinite-dimensional Banach space is a strong assumption, forcing several geometric constrains on the space. On the other hand, the presence of one C^k-smooth norm frequently implies the existence of a large supply of such norms.

To begin with, let us start by recalling the basic standard definitions of differentiability.

Definition 1.1.1. Let O be an open subset of a normed space X, $x \in O$ and $f\colon O \to \mathbb{R}$.

(i) f is *Gâteaux differentiable* at x if there exists $F \in X^*$ such that for every $h \in X$

$$\lim_{t\to 0} \frac{f(x+th)-f(x)}{t} = \langle F, h\rangle;$$

(ii) f is *Fréchet differentiable* if the above limit is uniform in $h \in S_X$, *i.e.*, if

$$\lim_{h\to 0} \frac{f(x+h) - f(x) - \langle F, h\rangle}{\|h\|} = 0;$$

(iii) f is *C^1-smooth* if it is Fréchet differentiable on O and the function $x \mapsto f'(x)$ is a continuous function from O to X^*.

(iv) if f is a norm on X, then f is *Gâteaux* (*Fréchet*) *differentiable* if it is a Gâteaux (Fréchet) differentiable function on $X \setminus \{0\}$.

The uniquely determined functional F is frequently denoted by $f'(x)$; we will also use the word *smooth* as a synonym to differentiable. Let us note that, in case $f = \|\cdot\|$ is a norm on X, then $\|f'(x)\| \leqslant 1$ and $\langle f'(x), x\rangle = \|x\|$; consequently, $f'(x)$ is a supporting functional at x.

Let us now pass to recall the statement of the classical Šmulyan lemma [Smu40] on Fréchet smooth norms (*cf.* [DGZ93, Theorem I.1.4]).

Lemma 1.1.2 (Šmulyan lemma)**.** *A norm $\|\cdot\|$ is Fréchet smooth at $x \in S_X$ if and only if $(f_n)_{n=1}^\infty$ is a convergent sequence, whenever $(f_n)_{n=1}^\infty \subseteq B_{X^*}$ satisfies $\langle f_n, x\rangle \to 1$.*

Although very simple, this criterion has several interesting consequences; let us give a sample of some of them (*cf.* [FHHMPZ01, Chapter 8]).

Corollary 1.1.3. *Let $(X, \|\cdot\|)$ be a Banach space.*

(i) *$\|\cdot\|$ is C^1-smooth whenever it is Fréchet differentiable;*

(ii) *If the dual norm on X^* is Fréchet smooth, X is reflexive;*

(iii) *If the dual norm on X^* is LUR, then $\|\cdot\|$ is Fréchet smooth.*

This gives a first, immediate, geometrical consequence of the existence of a dual smooth norm in the dual. In the case where the smoothness is assumed on the norm of X, rather than X^*, then one may deduce that X is an Asplund space. Let us record that it is now customary to define a Banach space X an *Asplund* space if every its separable subspace has separable dual; several characterisations of Asplund spaces may be found in [DGZ93, §I.5] or [HáJo14, Theorem 5.2].

The proof of this last claim is also very simple and we shall sketch it presently; plainly, it suffices to verify that X^* is separable, for every separable Banach space X with a Fréchet smooth norm. In fact, it is elementary to verify that a norm ν is Gâteaux differentiable at a point x if and only if x admits a unique supporting functional, which, in this case, is $\nu'(x)$. Consequently, the image of X under ν' consists exactly of those functionals in S_{X^*} that attain their norm, and it is a separable set, according to (i) of the above corollary. The Bishop–Phelps theorem leads us to the conclusion that X^* is separable.

We are next going to discuss a strengthening of this result, which gives us the possibility to introduce two fundamental notions in this area: bump functions and variational principles. A *bump* on a Banach space X is a function $f\colon X \to \mathbb{R}$ with non-empty, bounded support. Once a C^1-smooth (or, more generally, C^k) norm is present in X, it is immediate to construct a C^1- smooth bump in X; it suffices to compose the norm with a bump on the real line which is constant in a neighbourhood of the origin.

The more general statement we shall prove is that X is Asplund, whenever it admits a C^1-smooth bump (let us mention that the result can be generalised further by replacing C^1 to mere Fréchet smoothness, but with a more complicated proof, [DGZ93, Theorem II.5.3]). A main ingredient in the proof is the celebrated Ekeland variational principle, whose statement we recall.

Theorem 1.1.4 (Ekeland variational principle, [Eke74, Eke79])**.** *Let (M,d) be a complete metric space and let $\varphi\colon M \to \mathbb{R}\cup\{\infty\}$ be lower semi-continuous, bounded from below and not identically equal to ∞. Then, for every $\varepsilon > 0$, there exists a point $x_0 \in M$ with $\varphi(x) \geqslant \varphi(x_0) - \varepsilon d(x, x_0)$, whenever $x \in M$.*

Lemma 1.1.5 ([Fab87])**.** *If a Banach space X admits a C^1-smooth bump, it is Asplund.*

Proof. As above, we may assume that X is separable. Select a C^1-smooth bump φ and consider the function

$$\psi(x) = \begin{cases} \varphi^{-2}(x) & \text{if } \varphi(x) \neq 0 \\ +\infty & \text{if } \varphi(x) = 0 \end{cases};$$

let us now fix arbitrarily $f \in X^*$ and $\varepsilon > 0$. The function $\psi - f$ satisfies the assumptions of Ekeland variational principle, whence there exists a point $x_0 \in X$ with $\psi(x_0) < \infty$ and

$$\psi(x_0 + th) - \langle f, x_0 + th\rangle \geqslant \psi(x_0) - \langle f, x_0\rangle - \varepsilon t\|h\|,$$

whenever $t > 0$ and $h \in X$. ψ being Fréchet differentiable at x_0, we conclude

$$\langle \psi'(x_0), h\rangle \geqslant \langle f, h\rangle - \varepsilon\|h\|,$$

which implies $\|-2\varphi^{-3}(x_0)\cdot\varphi'(x_0) - f\| = \|\psi'(x_0) - f\| \leqslant \varepsilon$. It follows that the image of X under φ' is linearly dense in X^*, and we are done. ■

At this stage we should state one of the main open problems in the area, whether every Asplund space admits a Fréchet smooth, or even C^1, bump, *cf.* [GMZ16, Problem 144].

On the other hand, the analogous question of the existence of a C^1-norm on every Asplund space was given a negative answer by Richard Haydon after a profound study of smoothness on $C_0(T)$ spaces, where T is a tree, [Hay90], [Hay96], and [Hay99]; also see the discussion in [FrMc02, §9,10]. In particular, Haydon [Hay99] was able to prove that $C_0(T)$ admits a C^∞-smooth bump, for every tree T; it follows, in particular, the much simpler fact that $C_0(T)$ is Asplund. This result is compared with the previous example [Hay90] of a tree such that the corresponding $C_0(T)$ admits no Gâteaux differentiable norm. As a consequence, there exist Asplund spaces that admit not even a Gâteaux-smooth norm; moreover, it is in general impossible to obtain smooth norms from smooth bumps.

Let us now pass to discuss a few geometrical consequences of the existence of a C^k-smooth norm (or, more generally, bump) on a Banach space. Prior to this, let us just mention that the definition of C^k-smooth function is the expected one: $f\colon O \to \mathbb{R}$ is C^2-smooth if the map $x \mapsto f'(x)$ is a C^1-smooth map from O to X^*, and so on. Let us also mention that we shall not discuss here the basic rules of calculus; these are worked out in detail in [HáJo14, Chapter 1] or in the undergraduate-level [Col12].

A first, classical, such result is due to Meshkov [Mes78], who proved that X is isomorphic to a Hilbert space, whenever both X and X^* admit a C^2-smooth bump. This result has been later generalised replacing the assumption of C^2-smoothness by the assumption the first derivative to be locally Lipschitz, [FWZ83]; also see [FaZi99] for an alternative proof. In the same paper [FWZ83], the authors also show that the existence of a C^2-smooth norm on a Banach has profound structural consequence for the space; in some sense, such spaces are either super-reflexive, or close to c_0. More precisely, we have the following.

Theorem 1.1.6 ([FWZ83]). *If a Banach space X admits a bump with locally uniformly continuous derivative, then either X contains a copy of c_0 or it is super-reflexive.*

Moreover, in case X admits a bump with locally Lipschitz derivative and it contains no copy of c_0, then X is (super-reflexive) with type 2.

This result is specially interesting also for one ingredient in its proof, since part of the argument is an instance where it is possible to 'convexify' a smooth bump and produce a smooth norm; see the definition of the function ψ in [FWZ83, Theorem 3.2] or the discussion in [FrMc02, Theorem 25].

Deville [Dev89] succeeded in pushing the result even further and proved that the existence of a C^∞-smooth bump on a Banach space X that contain no copy of c_0 implies that X is of exact cotype 2k, for some integer k, and it contains a copy of ℓ_{2k} . In particular, it follows that every 'very smooth' Banach space contains a sequence space and, consequently, there are no Tsirelson-like very smooth Banach spaces, *cf.* [GMZ16, Problem 2], or [FHHMZ10, p. 474].

We shall now pass to a discussion of the second phenomenon we wish to emphasise in this section, namely the existence of a large supply of smooth norms on a Banach space with one such norm; more precisely, it is often the case that, if a Banach space X admits a C^k-smooth norm, then every equivalent norm can be approximated by a C^k-smooth one. Of course, this phenomenon is a particular case of one of the main classical themes in analysis, *i.e.*, the smooth approximation of continuous functions. However, the important particular case of norm approximation by smooth norms, and not merely smooth functions, has one additional difficulty, as one needs to preserve the convexity of the involved functions. In particular, the use of partitions of the unity, when available, may not be a sufficiently powerful tool.

On the other hand, broadly speaking, the construction of the smooth norm is carried out by techniques locally using only finitely many ingredients, which is, of course, an idea present already in the concept of partitions of unity. Probably the first explicit use of this technique in order to construct smooth norms is found in the work of Pechanec, Whitfield and Zizler [PWZ81]. To get an idea of the difficulty of constructing smooth norms, we refer to, *e.g.*, [MaTr91, Hay96, Hay99, HáHa07, Bib14].

Let us start with the formal definition of the relevant notion of approximation, which in short is the uniform convergence on bounded sets.

Definition 1.1.7. Let $(X, \|\cdot\|)$ be a Banach space and let P be a property of norms. We say that $\|\cdot\|$ is *approximated by norms with* P if, for every $\varepsilon > 0$, there exits a norm $|||\cdot|||$ on X, with property P and such that

$$(1-\varepsilon)\|\cdot\| \leqslant |||\cdot||| \leqslant (1+\varepsilon)\|\cdot\|.$$

Obviously, this is equivalent to $\Big| |||\cdot||| - \|\cdot\| \Big| \leqslant \varepsilon$ on the unit ball of $(X, \|\cdot\|)$. This notion of convergence turns the set of all equivalent norms on X into a Baire space, whence the use of Baire category theorem is possible. This allows, in some cases, for the approximation with norms that share two properties simultaneously; this important technique is by now called *Asplund averaging*, [DGZ93, §II.4].

With this definition at our disposal, the main problem of smooth approximation of norms can be easily formulated as follows.

Problem 1.1.8. Assume that a Banach space X admits a C^k-smooth norm. Must every equivalent norm on X be approximable by C^k-smooth norms?

To begin with, let us notice that, so far, no counterexample is known and, in its full generality, the problem is still open even in the case $k = 1$. In the separable setting, the problem has been completely solved for every separable Banach space and every k in [HáTa14], after a good bunch of partial positive results [Háj95, DFH96, DFH98]. Still in the separable context, let us mention that the analogue problem for analytic norms is open, even for ℓ_2, *cf.* [HáJo14, p. 464].

In the context of non-separable Banach spaces, the situation is way less understood.

In the case $k = 1$, clause (iii) in Corollary 1.1.3 implies that it is sufficient to approximate every dual norm on X^* with a dual LUR norm. In turn, Fabian, Zajíček, and Zizler [FZZ81] proved that the collection of dual LUR norms on X^* is residual, whenever non-empty (let us mention, in passing, that the idea of using the Baire theorem on the space of equivalent norms is introduced in this paper). Consequently, for $k = 1$, the problem has a positive solution, once a dual LUR norm is present on X^*. According to [GTWZ83], this is the case whenever X^* is WCG; this covers a wide range of Banach spaces, in particular all reflexive Banach spaces.

In the absence of a dual LUR renorming, the problem appears to be completely open. One specific Banach space for which the problem is open is the space $C([0, \omega_1])$, whose dual space does not even admit a strictly convex dual norm [Tal86], *cf.* [DGZ93, Theorem VII.5.2]. In the same paper, Talagrand proved the existence of a C^1-smooth norm on $C([0, \omega_1])$; building on this result, Haydon [Hay92, Hay96] was able to crystallise the notion of a *Talagrand operator* and prove the existence of a C^∞-smooth norm on $C([0, \omega_1])$. Nevertheless, the C^1-smooth approximation of norms in such space is still an open problem.

For $k \geqslant 2$ the problem seems to be more difficult, and no dual approach is available. In the already mentioned [PWZ81] the authors construct a particular LUR and C^1-smooth norm on $c_0(\Gamma)$ which admits C^∞-approximations. This result has later been largely generalized to include every Banach space of the form $C([0,\mu])$, μ an ordinal number, and every WLD Banach space with a C^k-smooth norm (in which case the obtained norm can be approximated by C^k-smooth norms) [HáPr14]. Moreover, it was known that every lattice norm on $c_0(\Gamma)$ admits C^∞-smooth approximations, [FHZ97]; we shall sketch part of this argument later, in Section 1.1.3.

This bunch of partial results suggests that the spaces $c_0(\Gamma)$ are very plausible candidates for a positive solution to the approximation problem and indeed, in their remarkable paper [BiSm16], Bible and Smith have succeeded in solving the C^∞-smooth approximation problem for norms on $c_0(\Gamma)$. Their approach consists in the construction of smooth norms via boundaries; for more instances of constructing smooth (or polyhedral) norms using boundaries see, *e.g.*, [Bib14, Fon80, FPST14, Háj95], or Bible's PhD thesis, [Bib16]. This approach has been very recently refined and extended in a series of papers authored by Smith and his coauthors, [AFST••, Smi••, SmTr••].

Let us then pass to the description of our contribution to the field. Our main result delves deeper into the fine behaviour of C^k-smooth approximations of norms in the separable setting. Roughly speaking, we consider approximations which are not only uniform on the unit ball, but also have a better asymptotic behaviour; the study of this notion was motivated by Problem 170 (stated somewhat imprecisely) in [GMZ16] and our main results provides a positive answer to the problem.

The result is, in some sense, analogous to the condition (ii) in [DGZ93, Theorem VIII.3.2], which claims that in a Banach space with C^k-smooth partitions of unity, the C^k-smooth approximations to continuous functions exist with a prescribed precision around each point.

Let us now present the formal statement of our main result.

Theorem 1.1.9 ([HáRu17, Theorem 1.1]). *Let X be a separable Banach space with a Schauder basis $(e_i)_{i=1}^\infty$ and assume that X admits a C^k-smooth norm. Then, for every equivalent norm $\|\cdot\|$ on X and for every sequence $(\varepsilon_N)_{N=0}^\infty$ of positive reals, there is a C^k-smooth renorming $|||\cdot|||$ of X such that for every $N \geqslant 0$*

$$\Big| |||x||| - \|x\| \Big| \leqslant \varepsilon_N \|x\| \qquad (x \in X^N),$$

where $X^N := \overline{\operatorname{span}}\{e_i\}_{i=N+1}^\infty$.

In other words, we can approximate every equivalent norm on X with a C^k-smooth one in a way that on the vectors from the 'tail' of the Schauder basis the approximation is improving (as fast as we wish). In particular, the C^k-smooth approximating norm preserves the same asymptotic structure as the approximated norm. For this reason, we expect that the above result, or analogous results involving the same notion of approximation, may have consequences in some isometric or quasi-isometric problems. One such instance could potentially be the context of metric fixed point theory (see, *e.g.*, [ADL97, GoKi90, KiSi01], or [BeLi00, Chapter 3] and the references therein), where several notions are present of properties which asymptotically improve with growing codimension. For example, let us mention the notion of *asymptotically non-expansive function* or the ones of *asymptotically isometric copy* of ℓ_1 or c_0.

It is perhaps worth crystallising such notion of approximation in a formal definition.

Definition 1.1.10. Let X be a separable Banach space with Schauder basis $(e_i)_{i=1}^{\infty}$ and let P be a property of norms. We say that a norm $\|\cdot\|$ on X admits *approximation with asymptotic improvement* by norms with property P if, for every sequence $(\varepsilon_N)_{N=0}^{\infty}$ of positive real numbers, there exists a norm $|||\cdot|||$ of X with property P and such that for every $N \geqslant 0$

$$\Big| |||x||| - \|x\| \Big| \leqslant \varepsilon_N \|x\| \qquad (x \in X^N),$$

where $X^N := \overline{\operatorname{span}}\{e_i\}_{i=N+1}^{\infty}$.

Once this notion of approximation is available, one may ask for further results concerning such approximation. Let us recall, for example, that the collection of strictly convex (resp. LUR, resp. uniformly convex) equivalent norms on X is residual, whenever non-empty ([FZZ81], see, *e.g.*, [DGZ93, Theorem II.4.1]). It follows, in particular, that every norm on a super-reflexive Banach space can be approximated by a uniformly convex one. We will now observe the simple fact that, in general, such approximation may fail to have asymptotic improvement. Consequently, the notion introduced above is not just a formal strengthening, but it leads to a truly stronger notion of approximation.

As a technical point, let us mention that we do not known if the mere assumption of the existence of a norm as above *for some* sequence $(\varepsilon_N)_{N=0}^{\infty}$, with $\varepsilon_N \to 0$ would lead to a different notion. In other words, we do not know if the speed of the approximation has a substantial rôle in the definition.

Example 1.1.11. Consider the Banach space ℓ_∞^2 with its natural basis, and then let

$$X := \left(\sum_{n=1}^{\infty} X_n \right)_{\ell_2},$$

where $X_n = \ell_\infty^2$, for every $n \in \mathbb{N}$. Let us denote by $(e_i)_{i=1}^{\infty}$ the natural Schauder basis of X, where $\{e_{2n-1}, e_{2n}\}$ is the natural basis in X_n; the original norm $\|\cdot\|$ of X can then be expressed as

$$\|x\| = \left(\sum_{n=1}^{\infty} \max \Big\{ |x(2n-1)|, |x(2n)| \Big\}^2 \right)^{1/2}.$$

We note that $\|\cdot\|$ admits no approximation with asymptotic improvement via uniformly convex norms.

In fact, for every $N \geqslant 0$,

$$X^{2N} := \overline{\operatorname{span}}\{e_i\}_{i=2N+1}^{\infty} = \left(\sum_{n=N+1}^{\infty} X_n \right).$$

Consequently, if $|||\cdot|||$ is a uniformly convex approximation of $\|\cdot\|$ with asymptotic improvement, and $\varepsilon_N \searrow 0$, then, in particular, the restriction of $|||\cdot|||$ to X_n is a better and better approximation of $\|\cdot\|_\infty$, as $n \to \infty$. As a consequence, $|||\cdot|||\restriction_{X_n}$ becomes flatter and flatter as $n \to \infty$, which contradicts its uniform convexity.

Before, we also mentioned very much in passing the approximation with polyhedral norms. Actually, the study of polyhedral norms on a (separable) Banach space is very much connected with the study of smooth norms, as polyhedral Banach spaces admit 'small' boundaries and this allows for the construction of smooth norms. Let us state the following result as a witness of this phenomenon; some undefined notions present in its statement will be introduced in subsequent sections (also see Section 1.1.4 for a proof of some its implications).

Theorem 1.1.12 ([Fon90, Háj95])**.** *For a separable Banach space X, the following are equivalent:*

(i) *X admits a polyhedral norm;*

(ii) *X admits a norm with a countable boundary;*

(iii) *X admits a norm which locally depends on finitely many coordinates;*

(iv) *X admits a C^∞-smooth norm which locally depends on finitely many coordinates.*

Moreover, in a later paper [DFH98] it was also shown that in such Banach spaces every equivalent norm can be approximated (uniformly on bounded sets) by polyhedral norms and by C^∞-smooth norms that locally depend on finitely many coordinates. This naturally suggests the question whether this result can be sharpened and the approximation can be chosen to have asymptotic improvement. As it turns out, the approximation technique used in the proof of our main result can also be adapted to these two cases; consequently, we have the following result.

Theorem 1.1.13 ([HáRu17, §3])**.** *Let X be a polyhedral Banach space with a Schauder basis and let $\|\cdot\|$ be an equivalent norm on X. Then $\|\cdot\|$ admits approximation with asymptotic improvement via polyhedral norms and via C^∞-smooth norms that locally depend on finitely many coordinates.*

In conclusion to this section, we shall describe how the remaining part of the chapter is organised. The proof of the main Theorem 1.1.9 will be presented in Section 1.2, whereas Section 1.3 is dedicated to Theorem 1.1.13 (and the definition of polyhedral Banach space). Prior to this, we shall introduce the concept of local dependence on finitely many coordinates and prove a few properties of this notion in Section 1.1.1, while Section 1.1.2 records an ubiquitous tool for constructing smooth norms, via Minkowski functionals. Moreover, in Sections 1.1.3 and 1.1.4 we will sketch some techniques to construct smooth norms.

1.1.1 Local dependence on finitely many coordinates

In this part we introduce the notion of function locally dependent on finitely many coordinates, formally introduced in [PWZ81], and we discuss some properties of Banach spaces that admit non-trivial such functions. It turns out that this notion is fundamental in the study of higher order smoothness in Banach spaces, *cf.* [HáZi06], or [HáJo14, §5.5].

Definition 1.1.14. Let X be a Banach space, Ω an open subset of X, S an arbitrary set, $f\colon \Omega \to S$ an arbitrary function, and $M \subseteq X^*$. We say that f *locally depends on finitely many coordinates from* M (is *LFC-*M, for short) if for every $x \in \Omega$ there exist an open neighbourhood O of x and a finite collection of functionals $\{x_1^*, \ldots, x_n^*\} \subseteq M$ such that $f(y) = f(z)$ for every $y, z \in O$ with $\langle x_i^*, y\rangle = \langle x_i^*, z\rangle$. We say that f is *LFC* if it is LFC-X^*.

Occasionally, we will also say that f *depends only on* $\{x_1^*, \ldots, x_n^*\}$ *on* O if $f(y) = f(z)$ for every $y, z \in O$ with $\langle x_i^*, y\rangle = \langle x_i^*, z\rangle$.

Finally, in the case where f is the norm function, we say that $\|\cdot\|$ is LFC-M if it is LFC-M on $X \setminus \{0\}$.

Equivalently, f is LFC-M if for every $x \in \Omega$ there are an open neighbourhood O of x, a finite set of functionals $\{x_1^*, \ldots, x_n^*\} \subseteq X^*$ and a function $g\colon \mathbb{R}^n \to \mathbb{R}$ such that

$$f(y) = g\left(\langle x_1^*, y\rangle, \ldots, \langle x_n^*, y\rangle\right)$$

whenever $y \in O$.

$$\begin{tikzcd} O \arrow[r, "f"] \arrow[d, "(x_1^*,\ldots,x_n^*)"'] & \mathbb{R} \\ \mathbb{R}^n \arrow[ur, dotted, "g"'] & \end{tikzcd}$$

We shall start by giving a few obvious, but important, permanence properties of the LFC condition: if functions $f_j\colon X \to S_j$ are LFC-M_j for $j = 1, \ldots, k$, then plainly $F(f_1, \ldots, f_k)$ is LFC-$(\cup M_j)$, for every function $F\colon S_1 \times \cdots \times S_k \to S$. As a particular case, sums and products of LFC functions are still LFC. One more important permanence property is the fact that the Implicit Function theorem applied to an LFC function results in an LFC implicit function.

Fact 1.1.15. *Let X, Y, and Z be normed spaces, $U \subseteq X$ and $V \subseteq Y$ be open subsets and $f\colon U \times V \to Z$ be a function. Assume that f depends only on $\{x_1^*, \ldots, x_n^*\} \subseteq (X \oplus Y)^*$ on $U \times V$. Assume further that the equation $f = 0$ defines a unique function $u : U \to V$, namely there exists a unique function $u\colon U \to V$ with $f(x, u(x)) = 0$ for $x \in U$. Then, u depends only on $\{x_1^*\restriction_X, \ldots, x_n^*\restriction_X\}$ on U.*

In particular, if f is LFC-M, then u is LFC-$(M\restriction_X)$.

Proof. Let $x, y \in U$ be such that $\langle x_i^*\restriction_X, x\rangle = \langle x_i^*\restriction_X, y\rangle$; then, of course, $\langle x_i^*, (x, u(x))\rangle = \langle x_i^*, (y, u(x))\rangle$. The LFC property of f yields us $0 = f\left(x, u(x)\right) = f\left(y, u(x)\right)$, whence $u(x) = u(y)$ follows from the uniqueness of the function u. ■

We now pass to the presentation of some examples of LFC functions: obviously, every continuous linear functional x^* on X is LFC-$\{x^*\}$; consequently, also $f\circ x^*$, where $f\colon \mathbb{R}\to S$ is an arbitrary function, is an example of an LFC function.

A less trivial, and actually important, example is the canonical norm of c_0 which is LFC-($\{e_j^*\}_{j=1}^\infty$) (on $c_0\setminus\{0\}$). To see this, fix a non-zero $x\in c_0$ and consider $O:=\{y\in c_0\colon \|x-y\|_\infty<\|x\|_\infty/4\}$. Let us then select $N\in\mathbb{N}$ such that $|x(n)|\leqslant\|x\|_\infty/2$ whenever $n>N$ and note that for every such n we have $|y(n)|<\frac{3}{4}\|x\|_\infty$, while $\|y\|_\infty>\frac{3}{4}\|x\|_\infty$, for every $y\in O$. Therefore, $\|y\|_\infty=\max_{n=1,\ldots,N}|y(n)|$ for $y\in O$ and we are done.

With only formal modifications one shows the same assertion for every $c_0(\Gamma)$ space. Moreover, we will see in Corollary 1.1.24 that $c_0(\Gamma)$ spaces even admit a C^∞-smooth LFC norm (and, consequently, a bump with the same properties).

It turns out that this example is archetypal since every infinite-dimensional Banach space with an (arbitrary) LFC bump is in fact a c_0-saturated Asplund space ([FaZi97, PWZ81]). In the remaining part of the section, we shall present (part of) the proof of this important result; in particular, we shall stress the use of variational principles in both parts of the result.

The first preparatory lemma implies in particular that the existence of an LFC bump implies the existence of an LFC upper semi-continuous bump.

Lemma 1.1.16. *Let $f\colon X\to\mathbb{R}$ be LFC-M; then $\chi_{\operatorname{supp} f}$ is LFC-M.*

Proof. Fix $x\in X$ and let O be a neighbourhood of x such that f depends only on $\{x_1^*,\ldots,x_n^*\}\subseteq M$ on O; we claim that $\chi_{\operatorname{supp} f}$ also depends only on $\{x_1^*,\ldots,x_n^*\}$ on O. Assuming by contradiction that this is false, there must exist $y,z\in O$ with $\langle x_i^*,y\rangle=\langle x_i^*,z\rangle$ but $\chi_{\operatorname{supp} f}(y)\neq\chi_{\operatorname{supp} f}(z)$; we may assume, for example, $y\in\operatorname{supp} f$ and $z\notin\operatorname{supp} f$. Then, there exists a sequence $(y_n)_{n=1}^\infty$ that converges to y and such that $f(y_n)\neq0$. Since $y_n-y+z\to z$ and O is open, we can assume that $(y_n)_{n=1}^\infty,(y_n-y+z)_{n=1}^\infty\subseteq O$; of course, we also have $\langle x_i^*,y_n\rangle=\langle x_i^*,y_n-y+z\rangle$. Consequently, the LFC property of f implies $f(y_n-y+z)=f(y_n)\neq0$. But then, $y_n-y+z\to z$ implies $z\in\operatorname{supp} f$, a contradiction. ∎

Theorem 1.1.17 ([FaZi97])**.** *If a Banach space X admits an LFC-M bump, then $\overline{\operatorname{span}}\,M=X^*$.*

Proof. Let us fix a functional $f\in X^*$ and $\varepsilon>0$. It is sufficient to find $\{f_1,\ldots,f_n\}\subseteq M$ such that $|\langle f,x\rangle|\leqslant\varepsilon\|x\|$ whenever $x\in Z:=\cap_{i=1}^n\ker f_i$. In fact, this condition means that $\|f\restriction_Z\|\leqslant\varepsilon$, so we can find an Hahn–Banach extension $g\in X^*$ of $f\restriction_Z$ such that $\|g\|\leqslant\varepsilon$. Then, of course, $f-g$ vanishes on Z, whence $f-g\in\operatorname{span}\{f_1,\ldots,f_n\}\subseteq\operatorname{span}M$; finally $\|f-(f-g)\|=\|g\|\leqslant\varepsilon$, whence the conclusion.

In order to find such $\{f_1,\ldots,f_n\}\subseteq M$, we argue as follows: find a closed and bounded set $A\neq\emptyset$ whose characteristic function is LFC-M and apply the Ekeland variational principle to find $x_0\in A$ where the functional f 'almost attains' its minimum. The LFC condition tells us that, in a neighbourhood of x_0, A contains a subspace of the form $Z=\cap_{i=1}^n\ker f_i$, $f_i\in M$; therefore, x_0 is an interior point of $A\cap Z$ in the space Z. Hence,

$-f\restriction_Z$ almost attains its maximum at an interior point and, by the maximum principle, it is almost constant. Let us give the details below.

According to the previous lemma, we may find a non-empty closed bounded set A such that χ_A is LFC-M. The Ekeland variational principle Theorem 1.1.4, applied to $(A, \|\cdot\|)$, f and ε, furnishes us with a point $x_0 \in A$ with $\langle f, x\rangle \geqslant \langle f, x_0\rangle - \varepsilon\|x - x_0\|$, *i.e.*, $\langle f, x - x_0\rangle \geqslant -\varepsilon\,\|x - x_0\|$ for every $x \in A$. From the LFC-M property of χ_A, we can find $\delta > 0$ and $\{f_1, \ldots, f_n\} \subseteq M$ such that χ_A depends only on $\{f_1, \ldots, f_n\}$ on $B^O(x_0, \delta)$. As a consequence, for every $z \in Z = \cap_{i=1}^n \ker f_i$, with $\|z\| < \delta$, we have $\langle f_i, x_0\rangle = \langle f_i, x_0 + z\rangle$, whence $1 = \chi_A(x_0) = \chi_A(x_0 + z)$. This implies that for every $z \in Z$ with $\|z\| < \delta$ we have $x_0 + z \in A$, thus $\langle f, z\rangle \geqslant -\varepsilon\|z\|$; by homogeneity, we conclude $|\langle f, z\rangle| \leqslant \varepsilon\|z\|$ for every $z \in Z$, and the proof is complete. ■

We are now ready to prove the first part of the result announced above.

Corollary 1.1.18. *If a Banach space X admits an LFC bump, then it is an Asplund space.*

Proof. Of course, the existence of an LFC bump passes to subspaces, so it is sufficient to show that X^* is separable whenever X is. Assume, therefore, that X is separable and pick an LFC bump f on X; hence, for every $x \in X$ there are an open neighbourhood O_x of x and a finite set $\Phi_x \subseteq M$ of functionals such that f depends only on Φ_x on O_x. By the Lindelöf property of X, the open cover $\{O_x\}_{x\in X}$ admits a countable subcover $\{O_n\}_{n=1}^\infty$; plainly, the LFC property of f can be witnessed using only the open sets $\{O_n\}_{n=1}^\infty$ and the corresponding set of functionals $\{\Phi_n\}_{n=1}^\infty$. In particular, f is LFC-$(\cup\Phi_n)$; hence, by the previous theorem, $X^* = \overline{\operatorname{span}}\,(\cup\Phi_n)$ is separable. ■

We shall now pass to the proof that every Banach space X with an LFC bump is c_0-saturated; as above, since the existence of an LFC bump passes to subspaces, it is sufficient to show that X contains a copy of c_0. In the argument we shall make use of the so called *compact variational principle*, [DeFa89]; we also refer to [DGZ93, §V.2] for a discussion of the result.

Theorem 1.1.19 (Compact variational principle, [DeFa89])**.** *Let X be a Banach space that contains no copy of c_0, $O \subseteq X$ be a symmetric, bounded open neighbourhood of* 0 *and let $f\colon \overline{O} \to \mathbb{R}$ be even, lower semi-continuous and such that $f(0) = 0$ and $\inf_{\partial O} f > 0$. Then there are a symmetric compact set $K \subseteq O$ and a neighbourhood V of* 0 *with $K + V \subseteq O$ such that: for every $\delta > 0$ there are $\eta > 0$ and a finite set $F \subseteq K$ such that for every $v \in V$, $\|v\| \geqslant \delta$ there is $z \in F$ satisfying $f(z + v) > f(z) + \eta$.*

In particular, for every $v \in V$, $v \neq 0$ there is $z \in K$ with $f(z + v) > f(z)$.

In order to show the necessity of the assumption that X contains no copy of c_0, let us consider the function $g\colon c_0 \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 0 & \|x\| \leqslant 1 \\ 1 & \|x\| > 1 \end{cases} = \chi_{(1,\infty)}\left(\|x\|\right);$$

clearly, g is LFC and it satisfies the assumptions of the theorem with $O = 2B^O_{c_0}$. However, let us fix a compact subset K of O and a neighbourhood V of 0; as g is LFC there are a covering $\{B^O(x_j, \delta_j)\}_{j=1}^n$ of K and a finite set $M \subseteq X^*$ such that g depends only on M on each $B^O(x_j, 2\delta_j)$. Select then a non-zero vector $v \in \cap_{x^* \in M} \ker x^* \cap V$ such that $\|v\| \leqslant \min \delta_j$. For every $x \in K$ there exists $j = 1, \dots, n$ such that $x \in B^O(x_j, \delta_j)$; therefore, $x, x + v \in B^O(x_j, 2\delta_j)$ and $\langle x^*, x\rangle = \langle x^*, x + v\rangle$ for every $x^* \in M$. Consequently, $g(x) = g(x + v)$. In other words, for every compact $K \subseteq O$ and every neighbourhood V of 0 there exists a non-zero $v \in V$ with $g(x + v) = g(x)$ for every $x \in K$, whence the conclusion of the result is false.

As it turns out, essentially the same argument applies to every Banach space with an LFC bump.

Theorem 1.1.20 ([PWZ81])**.** *Let X be an infinite-dimensional Banach space with an LFC bump. Then X contains a copy of c_0.*

Proof. According to Lemma 1.1.16, we can choose a non-empty, closed and bounded set A such that χ_A is LFC; we can assume that $0 \in A$. Moreover, $\chi_A \cdot \chi_{-A} = \chi_{A \cap (-A)}$ is also LFC, whence we can even assume that A is symmetric. Therefore, the function $g := 1 - \chi_A$ is lower semi-continuous, even, LFC and such that $g(0) = 0$ and $g = 1$ on ∂O, for some open ball O. *Verbatim* the same argument as before this proof shows that for every compact subset K of O and every neighbourhood V of 0 there exists $v \in V$, $v \neq 0$, such that $g(z + v) = g(z)$ for every $z \in K$. The compact variational principle then implies that X contains a copy of c_0. ■

To conclude, let us restate formally the main result proved in this section.

Theorem 1.1.21 ([FaZi97, PWZ81])**.** *Every infinite-dimensional Banach space with an LFC bump is a c_0-saturated Asplund space.*

1.1.2 Implicit Function theorem for Minkowski functionals

If asked to prove the existence of a smooth norm that approximates the max norm on the plane, everyone would draw a convex body with no corners and whose boundary is close to the boundary of the unit square in the plane. Then he would just claim that this is the unit ball of the desired smooth norm. What is implicit in the argument is that the smooth convex body is the unit ball of a smooth norm; in other words, the Minkowski functional of a smooth convex body is a smooth norm.

The goal of this section is to prove a technical result which exactly formalises the above idea; as it is to be expected the proof involves the Implicit Function theorem. The argument seems to appear outlined explicitly for the first time in [HáHa07], but it was already present in several arguments in the literature; the statement and proof given here are essentially [HáJo14, Lemma 5.23].

Lemma 1.1.22 (Implicit Function theorem for Minkowski functionals)**.** *Let $(X, \|\cdot\|)$ be a normed space and $D \neq \emptyset$ be an open, convex, symmetric subset of X; also let $f\colon D \to \mathbb{R}$ be even, convex and continuous. Assume the existence of $a > f(0)$ such that the set $B := \{f \leqslant a\}$ is bounded and closed in X.*

If there is an open set O with $\{f = a\} \subseteq O$ such that f is C^k-smooth on O, then the Minkowski functional μ of B is an equivalent C^k-smooth norm on X.

If f is LFC on O, then μ is LFC.

Let us notice that in the assumption of the result under consideration, the set B is assumed to be a closed subset of X, and not merely a closed subset of D; indeed, the closedness of B in X is used in the first paragraph of the argument (such paragraph actually proves a very standard property of Minkowski functionals, and its proof is included only to stress the rôle of the closedness of B in X).

As it turns out, the mere assumption that B is a closed subset of D, in fact consequence of the continuity of f, would not be sufficient to conclude the result; let us offer one relevant example here. Let us consider the space $X := \ell_\infty^2$ and let D be the open unit ball in X; let us also consider the function $f := 2 \cdot \chi_{D^\complement}$, which is evidently C^∞-smooth on D. If we set $a = 1$ and $O = \emptyset$, then the unique missing assumption is that $\{f \leqslant 1\} = D$ is not closed in X. On the other hand, the Minkowski functional of B is the norm of X, which is not differentiable.

Proof. Since f is even and convex, the set B is additionally convex and symmetric; consequently, μ is a norm equivalent to $\|\cdot\|$ and B is the unit ball of the norm μ. In order to justify this last claim, note that whenever $x \in B$, then of course $\mu(x) := \inf\{t > 0\colon x \in t \cdot B\} \leqslant 1$. Conversely, the assumption $\mu(x) \leqslant 1$, yields $t^{-1} \cdot x \in B$ for every $t > 1$; therefore, $x \in B$ follows from the closedness of B in X.

Assume now that f is C^k-smooth on O. The set

$$V := \big\{(x, \rho) \in (X \setminus \{0\}) \times (0, \infty)\colon \rho^{-1} \cdot x \in O\big\}$$

is plainly an open subset of $(X \setminus \{0\}) \times (0, \infty)$; moreover, the function $F\colon V \to \mathbb{R}$ defined by $F(x, \rho) := f(\rho^{-1} \cdot x)$ is C^k-smooth on V.

We now claim that the equation $F = a$ on V globally defines a unique implicit function from $X \setminus \{0\}$ to $(0, \infty)$; moreover, such a function is μ. In other words, for every $h \in X \setminus \{0\}$ there exists a unique $\rho > 0$ such that $F(h, \rho) = a$; in addition, $\rho = \mu(h)$. In order to prove this, let us fix $h \neq 0$. From $\mu(h)^{-1} \cdot h \in \{\mu \leqslant 1\} = B$, we first obtain $F(h, \mu(h)) = f\left(h/\mu(h)\right) \leqslant a$. In the case that the inequality were strict, then, f being continuous on the open set D, there would exist $\varepsilon > 0$ small such that $f\left(h/(\mu(h) - \varepsilon)\right) \leqslant a$; therefore we would have $h/(\mu(h) - \varepsilon) \in B$, which, however, contradicts the definition of μ. To conclude, the function $[0, \infty) \in t \mapsto f(t \cdot h)$ is convex and equals $f(0) < a$ for $t = 0$; consequently, there can not exists two distinct values $t, s > 0$ with $f(t \cdot h) = f(s \cdot h) = a$. This implies the uniqueness of ρ that satisfies the above equation and shows our assertion.

Next, from the convexity of f we also have

$$\frac{f((1 + t)h) - f(h)}{t} \geqslant f(h) - f(0)$$

for every $h \in D$ and $t > 0$ small; consequently when $f(h) = a$, we obtain

$$\langle f'(h), h\rangle = \lim_{t\to 0^+} \frac{f((1+t)h) - f(h)}{t} \geqslant f(h) - f(0) = a - f(0) > 0.$$

We may then deduce that for $h \neq 0$

$$D_2F\left(h, \mu(h)\right) = -\frac{1}{\mu(h)^2}\langle f'\left(h/\mu(h)\right), h\rangle = -\frac{1}{\mu(h)}\left\langle f'\left(\frac{h}{\mu(h)}\right), \frac{h}{\mu(h)}\right\rangle \neq 0$$

(where D_2F denotes the partial derivative of F with respect to its second variable). We are now in position to apply the Implicit Function theorem (*cf.* [HáJo14, Theorem 1.87]) and deduce that μ is C^k-smooth on $X \setminus \{0\}$.

Finally, let us assume that f is LFC on O and let us fix $(x, \rho) \in V$. By definition, we may select a neighbourhood A of $\rho^{-1} \cdot x$ such that f depends only on the finite set $\{x_1^*, \dots, x_n^*\} \subseteq X^*$ on the set A. We may now let $\tilde{A}$ be the open neighbourhood of (x, ρ) defined by $\tilde{A} := \{(y, t) \in V : t^{-1} \cdot y \in A\}$ and we may extend the functionals x_i^* to functionals $\tilde{x}_i^*$ on $X \oplus \mathbb{R}$, by letting $\langle \tilde{x}_i^*, (y, t)\rangle = \langle x_i^*, y\rangle$; we also let $x^* \in (X \oplus \mathbb{R})^*$ be defined by $\langle x^*, (y, t)\rangle = t$. We then prove that F depends only on $\{\tilde{x}_1^*, \dots, \tilde{x}_n^*, x^*\}$ on $\tilde{A}$: indeed, if this finite collection of functionals does not separate two points $(y, t), (z, s) \in \tilde{A}$, then in particular $t = s$ and $\langle x_i^*, y\rangle = \langle \tilde{x}_i^*, (y, t)\rangle = \langle \tilde{x}_i^*, (z, s)\rangle = \langle x_i^*, z\rangle$. Consequently, $\langle x_i^*, t^{-1} \cdot y\rangle = \langle x_i^*, s^{-1} \cdot z\rangle$ and we conclude $F(y, t) = f(t^{-1} \cdot y) = f(s^{-1} \cdot z) = F(z, s)$. Therefore, F is LFC on V and Fact 1.1.15 leads us to the conclusion that μ is LFC on $X \setminus \{0\}$. ■

Let us now distill a particular case of the above result, in which the technical assumptions are automatically satisfied; this particular case is already sufficient to show the existence of a C^∞-smooth and LFC norm on the Banach space $c_0(\Gamma)$.

Corollary 1.1.23. *Let $(X, \|\cdot\|)$ be a normed space and $f\colon X \to \mathbb{R}$ be an even, convex and C^k-smooth function. Assume, moreover, that there exists $a > f(0)$ such that the set $B := \{f \leqslant a\}$ is bounded. Then, the Minkowski functional μ of B is a C^k-smooth equivalent norm on X; if f is LFC, then so is μ.*

In conclusion of this part, let us present perhaps the simplest application of the above technique and construct a C^∞-smooth norm on the space $c_0(\Gamma)$. The first construction of a C^∞-smooth norm on c_0 is due to Kuiper and appeared in [BoFr66]; the argument below comes from [HáZi06]. Let us also mention that the technique of gluing together various seminorms in a single function Φ, present in the argument, is a pervasive trick in this area and will also appear in our argument.

Corollary 1.1.24. *The space $c_0(\Gamma)$ admits an LFC, C^∞-smooth equivalent norm; moreover, such norm can be chosen to approximate the original one $\|\cdot\|_\infty$.*

Proof. Let us fix $\varepsilon \in (0, 1/2)$ and select a C^∞-smooth, even and convex function $\varphi\colon \mathbb{R} \to \mathbb{R}$ with the properties that $\varphi(x) = 0$ if $|x| \leqslant 1 - \varepsilon$ and $\varphi(\pm 1) = 1$. Let us now consider the function $\Phi\colon c_0(\Gamma) \to \mathbb{R}$ defined by

$$\Phi(x) := \sum_{\gamma \in \Gamma} \varphi(x(\gamma)),$$

for $x = (x(\gamma))_{\gamma\in\Gamma} \in c_0(\Gamma)$.

We first note that the sum defining Φ is locally finite on $c_0(\Gamma) \setminus \{0\}$: in fact, for $x \neq 0$, the set $\{|x| \geqslant 1/4\}$ is finite and every $y \in c_0(\Gamma)$ with $\|x - y\|_\infty \leqslant 1/4$ satisfies $|y| \leqslant 1/2$ outside such a set. In other words, for every non-zero $x \in c_0(\Gamma)$, Φ is a finite sum on $B(x, 1/4)$; it follows that it is a C^∞-smooth and LFC function. Plainly, it is additionally even and convex.

Finally, the inclusions

$$\{\|\cdot\|_\infty \leqslant 1 - \varepsilon\} = \{\Phi = 0\} \subseteq \{\Phi \leqslant 1\} \subseteq \{\|\cdot\|_\infty \leqslant 1\}$$

allow us to apply the above corollary and deduce that the Minkowski functional μ of $\{\Phi \leqslant 1\}$ is a C^∞-smooth and LFC norm. The same list of inclusions also implies $\|\cdot\|_\infty \leqslant \mu \leqslant \frac{1}{1-\varepsilon}\|\cdot\|_\infty$, which proves the second part of the result and concludes the proof. ■

1.1.3 Smooth norms on $c_0(\Gamma)$

Here we prove the main result in the paper [FHZ97], namely the fact that every lattice norm on $c_0(\Gamma)$ can be approximated by a C^∞-smooth LFC norm. We present it because it is an instance of how to handle LFC functions and partially because we can offer a simpler argument that avoids a main technicality.

A main tool in the proof is the use of convolution operators: of course, in finite-dimensional spaces convolution is a prime tool for smooth approximation, but in the infinite-dimensional setting the absence of Lebesgue measure rules out the possibility to define the convolution in a trivial way. The main idea is to exploit the LFC property of the canonical norm of $c_0(\Gamma)$: in a neighbourhood of $x \in c_0(\Gamma)$ the norm depends only on finitely many coordinates, so one may hope that using the convolution only in those finitely many coordinates will be sufficient for the approximation.

We start by discussing the convolution for an arbitrary (convex) function $f\colon c_0(\Gamma) \to \mathbb{R}$, in order to shorten the approximation step in the proof of the result; henceforth, we will consider a fixed even, C^∞-smooth bump $b\colon \mathbb{R} \to [0, \infty)$ such that $\operatorname{supp} b \subseteq [-\delta, \delta]$ and $\int_{\mathbb{R}} b\, d\lambda = 1$ (here, λ denotes Lebesgue measure on the real line and λ_n will denote Lebesgue measure on $\mathbb{R}^n$). We have a first, immediate remark, whose proof is omitted.

Fact 1.1.25. *If $f\colon \mathbb{R} \to \mathbb{R}$ is convex, then $f * b$ is convex and $f \leqslant f * b$.*

Let now $f\colon c_0(\Gamma) \to \mathbb{R}$ be a convex function, fix a coordinate $\gamma \in \Gamma$ and consider the function $f^\gamma\colon c_0(\Gamma) \to \mathbb{R}$, defined, for $x = (x(\gamma))_{\gamma\in\Gamma}$, by

$$f^\gamma(x) := \int_{\mathbb{R}} f(x - te_\gamma) b(t)\, d\lambda(t) = \int_{\mathbb{R}} f\left(\sum_{\eta\neq\gamma} x(\eta)e_\eta + te_\gamma\right) b(x(\gamma) - t)\, d\lambda(t);$$

the integral is clearly well defined since f is continuous on $x+\mathbb{R}e_\gamma$. As in the previous fact, we check that f^γ is convex and $f \leqslant f^\gamma$. Let us also denote by I_γ the (linear) convolution operator $f \mapsto f^\gamma$. The convexity of f^γ allows us to iterate this convolution operation, in a different coordinate: fixed $\gamma' \neq \gamma$, it is immediate to verify that $I_\gamma \circ I_{\gamma'} = I_{\gamma'} \circ I_\gamma$. This allows us to unambiguously define $I_\pi := I_{\gamma_1} \circ \cdots \circ I_{\gamma_n}$, whenever $\pi = \{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$. In other words, for every finite subset $\pi = \{\gamma_1, \ldots, \gamma_n\}$ of Γ, consider the convex function $f^\pi := I_\pi(f)$, defined by

$$f^\pi(x) = \int_{\mathbb{R}^n} f\left(x - \sum_{i=1}^n t^i e_{\gamma_i}\right) \prod_{i=1}^n b(t^i)\, d\lambda_n(t^1, \ldots, t^n).$$

We already know that every $f^\pi \geqslant f$ is a convex function on $c_0(\Gamma)$ and $f^\pi \leqslant f^{\pi'}$ whenever $\pi \subseteq \pi'$. We are then in position to consider the function $F\colon c_0(\Gamma) \to \mathbb{R} \cup \{\infty\}$, defined by

$$F := \sup_{\pi \in [\Gamma]^{<\omega}} f^\pi.$$

Plainly, $F \geqslant f$ is a convex function; it is also easy to see that F is even (respectively, odd), whenever so is f.

Moreover, in case f is uniformly continuous, then F approximates f (whence, in particular, it is real valued): in fact, let ω_f be the modulus of continuity of f, fix $\varepsilon > 0$ and choose a bump b whose support is contained in $[-\delta, \delta]$, where $\omega_f(\delta) \leqslant \varepsilon$. Since $\| \sum_{i=1}^n t^i e_{\gamma_i}\|_\infty \leqslant \delta$, whenever $|t^i| \leqslant \delta$, we have

$$\begin{aligned} f^\pi(x) - f(x) &\leqslant \int_{[-\delta,\delta]^n} \left| f\left(x - \sum_{i=1}^n t^i e_{\gamma_i}\right) - f(x)\right| \prod_{i=1}^n b(t^i)\, d\lambda_n(t^1, \ldots, t^n) \\ &\leqslant \int_{[-\delta,\delta]^n} \varepsilon \prod_{i=1}^n b(t^i)\, d\lambda_n(t^1, \ldots, t^n) = \varepsilon; \end{aligned}$$

thus, $f \leqslant F \leqslant f + \varepsilon$.

Let us now pass to the statement of the main result of the section. A norm $\|\cdot\|$ on $c_0(\Gamma)$ is a *lattice norm* if $\|x\| \leqslant \|y\|$ whenever $x, y \in c_0(\Gamma)$ satisfy $|x| \leqslant |y|$ (*i.e.*, $|x(\gamma)| \leqslant |y(\gamma)|$ for every $\gamma \in \Gamma$). As an immediate consequence, note that $\|x\| = \big\||x|\big\|$.

Theorem 1.1.26 ([FHZ97, Theorem 1]). *Every equivalent lattice norm $\|\cdot\|$ on $c_0(\Gamma)$ can be approximated by C^∞-smooth LFC norms.*

We first shortly describe the strategy of the proof. A main feature of the $\|\cdot\|_\infty$ norm is that it only depends on the 'large coordinates': if we pick a vector from the unit sphere of $c_0(\Gamma)$ and set equal to 0 all its coordinates with absolute value smaller than 1/2, we do not modify its norm. A generic lattice norm may fail to have this property and the first step consists in approximating $\|\cdot\|$ with a function that depends on the large coordinates. We then use the above convolution procedure to this function and we find a smooth approximation; the Implicit Function theorem then leads us to the conclusion.

We shall now start with the first step of the argument and define this approximating function. Let us therefore fix an equivalent lattice norm $\|\cdot\|$ on $c_0(\Gamma)$; from the equivalence of $\|\cdot\|$ and $\|\cdot\|_\infty$, we may choose a constant $C > 0$ such that $\|\cdot\| \leqslant C\|\cdot\|_\infty$. Let us fix a small parameter $\Delta > 0$ and consider the vector $x \cdot \chi_{\{|x|>\Delta\}}$; this amounts exactly to neglecting the small coordinates of x. We then set

$$f_\Delta(x) := \sup\Big\{\|y\| \colon y = x \text{ on } \{|x| > \Delta\} \text{ and } |y| \leqslant \Delta \text{ on } \{|x| \leqslant \Delta\}\Big\}.$$

It will be useful to rewrite the formula in terms of an equivalence relation: let us say that $y \sim_\Delta x$ if $x \cdot \chi_{\{|x|>\Delta\}} = y \cdot \chi_{\{|y|>\Delta\}}$. We can then rewrite

$$f_\Delta(x) := \sup\{\|y\| \colon y \sim_\Delta x\}.$$

Obviously, $f_\Delta(y) = f_\Delta(x)$ whenever $y \sim_\Delta x$; since clearly $x \sim_\Delta x \cdot \chi_{\{|x|>\Delta\}}$, we deduce that $f_\Delta(x) = f_\Delta\left(x \cdot \chi_{\{|x|>\Delta\}}\right)$. This equality expresses the crucial property that f_Δ only depends on large coordinates; let us list some more properties of such function.

Lemma 1.1.27. *Let the function f_Δ be defined as above. Then:*

(i) $f_\Delta(x) = f_\Delta\left(x \cdot \chi_{\{|x|>\Delta\}}\right)$;

(ii) f_Δ *is lattice and even;*

(iii) $\|\cdot\| \leqslant f_\Delta \leqslant \|\cdot\| + 2C\Delta$;

(iv) f_Δ *is convex.*

We thus see that the approximating function f_Δ keeps the lattice property of $\|\cdot\|$, but it has the additional pleasant feature of depending on large coordinates. The proof of (i) was already observed before; (ii) and (iii) are equally immediate and therefore omitted.

Proof of (iv). The crux of the argument lies in the following claim.

Claim 1.1.28. Let $x, y \in c_0(\Gamma)$, $\lambda \in [0,1]$ and $\tilde{z} \sim_\Delta z := \lambda x + (1-\lambda)y$. Then there exist $\tilde{x}, \tilde{y} \in c_0(\Gamma)$ with $\tilde{x} \sim_\Delta x$, $\tilde{y} \sim_\Delta y$ and such that $|\tilde{z}| \leqslant \lambda|\tilde{x}| + (1-\lambda)|\tilde{y}|$.

Indeed, once this is proved, the lattice property yields

$$\|\tilde{z}\| \leqslant \|\lambda|\tilde{x}|+(1-\lambda)|\tilde{y}|\| \leqslant \lambda\Big\||\tilde{x}|\Big\|+(1-\lambda)\Big\||\tilde{y}|\Big\| = \lambda\|\tilde{x}\|+(1-\lambda)\|\tilde{y}\| \leqslant \lambda f_\Delta(x)+(1-\lambda)f_\Delta(y).$$

$\tilde{z} \sim_\Delta z$ being arbitrary, the convexity of f_Δ follows.

Proof of the Claim. Fix $x, y \in c_0(\Gamma)$, $\lambda \in [0,1]$ and $\tilde{z} \in c_0(\Gamma)$ with $\tilde{z} \sim_\Delta z := \lambda x + (1-\lambda)y$. In order to define $\tilde{x}$ and $\tilde{y}$ we consider four cases, according to the values of $x(\gamma)$ and $y(\gamma)$.

1. $|x(\gamma)|, |y(\gamma)| \leqslant \Delta$.
   In this case, $|z(\gamma)| \leqslant \Delta$ too, whence $\tilde{z}(\gamma)$ is an arbitrary element in $[-\Delta, \Delta]$. If we set $\tilde{x}(\gamma) = \tilde{y}(\gamma) := \tilde{z}(\gamma)$, the inequality is plainly satisfied for this γ.

2. $|x(\gamma)|, |y(\gamma)| > \Delta$.
   Here, we are forced to choose $\tilde{x}(\gamma) = x(\gamma)$ and $\tilde{y}(\gamma) = y(\gamma)$. In the case that $|z(\gamma)| > \Delta$ too, then $\tilde{z}(\gamma) = z(\gamma)$; hence, $\tilde{z}(\gamma) := \lambda\tilde{x}(\gamma) + (1-\lambda)\tilde{y}(\gamma)$ and the inequality is satisfied. In case $|z(\gamma)| \leqslant \Delta$, then $|\tilde{z}(\gamma)| \leqslant \Delta$, whence $|\tilde{z}(\gamma)| \leqslant \Delta \leqslant \lambda|\tilde{x}(\gamma)| + (1-\lambda)|\tilde{y}(\gamma)|$, and we are done with this case.

3. $|x(\gamma)| > \Delta$, $|y(\gamma)| \leqslant \Delta$.
   We are forced to let $\tilde{x}(\gamma) = x(\gamma)$, but we can pick any $\tilde{y}(\gamma) \in [-\Delta, \Delta]$. If $|z(\gamma)| > \Delta$, then $\tilde{z}(\gamma) = z(\gamma)$; hence, we may set $\tilde{y}(\gamma) = y(\gamma)$ and the inequality holds true. On the other hand, if $|z(\gamma)| \leqslant \Delta$, we select $\tilde{y}(\gamma)$ with $|\tilde{y}(\gamma)| = \Delta$; we have $\lambda|\tilde{x}(\gamma)| + (1-\lambda)|\tilde{y}(\gamma)| \geqslant \Delta \geqslant |\tilde{z}(\gamma)|$ and the inequality is true also in this case.

4. $|x(\gamma)| \leqslant \Delta$, $|y(\gamma)| > \Delta$.
   We argue as in the previous case, exchanging the roles of x and y.

From the construction, it is obvious that $\tilde{x} \sim_\Delta x$, $\tilde{y} \sim_\Delta y$ and $|\tilde{z}| \leqslant \lambda|\tilde{x}| + (1-\lambda)|\tilde{y}|$. We only have to check that in fact $\tilde{x}, \tilde{y} \in c_0(\Gamma)$: however, all but finitely many coordinates γ fall in the first case, whence $\tilde{x}$ and $\tilde{y}$ coincide with $\tilde{z}$ on all but finitely many coordinates. Since $\tilde{z} \in c_0(\Gamma)$, we conclude that $\tilde{x}, \tilde{y} \in c_0(\Gamma)$, and we are done. ■

■

Clauses (i)–(iii) in the above lemma are observed in [FHZ97], but the authors seem to have not realized the convexity of f_Δ. In fact, the subsequent step in their proof consists in defining a 'convexified' function, denoted C_Δ, at the beginning of p. 265. Then, much of the subsequent effort consists in proving that the properties of f_Δ are preserved when passing to C_Δ; finally the convolution technique is applied to the latter function (see from the second half of p. 269 on). Here, armed with the further information that f_Δ is convex, we apply the convolution machinery already to the function f_Δ; in particular, property (iv) above allows us to skip completely pp. 265–268.

Let us however stress that, although this convexification procedure is redundant for this argument, the technique itself is a very important one; it is, in fact, the same one used in [FWZ83] that we have mentioned when commenting on Theorem 1.1.6.

End of the proof of Theorem 1.1.26. We fix a small parameter $\Delta > 0$, in particular such that $3C\Delta$ is as small as we wish, and we consider the function f_Δ as above. We then apply the convolution procedure to this function and $\delta = \Delta/2$ (let us recall that $\operatorname{supp} b \subseteq [-\delta, \delta]$); for $\pi = \{\gamma_1, \dots, \gamma_n\} \subseteq \Gamma$ we set

$$f_\Delta^\pi(x) = \int_{\mathbb{R}^n} f_\Delta\left(x - \sum_{i=1}^n t^i e_{\gamma_i}\right) \prod_{i=1}^n b(t^i)\, d\lambda_n(t^1, \dots, t^n),$$

$$F_\Delta := \sup_{\pi \in [\Gamma]^{<\omega}} f_\Delta^\pi.$$

We already know that F_Δ is even and convex; it is moreover immediate to verify that $\|\cdot\| \leqslant F_\Delta \leqslant \|\cdot\| + 3C\Delta$. Consequently, for Δ sufficiently small, *i.e.*, $3C\Delta \leqslant \varepsilon$, the symmetric convex set $\{F_\Delta \leqslant 1\}$ satisfies

$$\{\|\cdot\| \leqslant 1-\varepsilon\} \subseteq \{F_\Delta \leqslant 1\} \subseteq \{\|\cdot\| \leqslant 1\};$$

it follows that its Minkowski functional $|||\cdot||| := \mu_{\{F_\Delta \leqslant 1\}}$ is an equivalent norm with $\|\cdot\| \leqslant |||\cdot||| \leqslant \frac{1}{1-\varepsilon}\|\cdot\|$. In order to conclude the proof it is therefore sufficient to show that F_Δ is C^∞-smooth and LFC and invoke Lemma 1.1.22.

In preparation for this, we prove that the functions f_Δ^π (and consequently also F_Δ) depend on the large coordinates: actually, $f_\Delta^\pi(x) = f_\Delta^\pi(y)$ whenever $x \sim_{\Delta/2} y$. Note first that if $\|e\|_\infty \leqslant \Delta/2$ and $x \sim_{\Delta/2} y$, then $x + e \sim_\Delta y + e$: in fact, the two vectors coincide even on the set $\{|x| > \Delta/2\} \supseteq \{|x+e| > \Delta\}$ and on the complementary set we have $|x+e|, |y+e| \leqslant \Delta$. Therefore, if $|t^i| \leqslant \Delta/2 = \delta$ we have $x - \sum_{i=1}^n t^i e_{\gamma_i} \sim_\Delta y - \sum_{i=1}^n t^i e_{\gamma_i}$, which yields $f_\Delta\left(x - \sum_{i=1}^n t^i e_{\gamma_i}\right) = f_\Delta\left(y - \sum_{i=1}^n t^i e_{\gamma_i}\right)$; integration over $[-\delta, \delta]$ then leads us to the conclusion that $f_\Delta^\pi(x) = f_\Delta^\pi(y)$.

This property and similar calculations also prove the following two facts. For $x \in c_0(\Gamma)$, let $\pi(x) := \{|x| > \Delta/4\}$; then

(i) on the set $\{y \in c_0(\Gamma) \colon \|y - x\|_\infty < \Delta/4\}$, the function $f_\Delta^{\pi(x)}$ depends on the finitely many coordinates from $\pi(x)$;

(ii) $f_\Delta^{\pi(x)} = F_\Delta$ on the same set $\{y \in c_0(\Gamma) \colon \|y - x\|_\infty < \Delta/4\}$.

The former assertion means in particular that the function $f_\Delta^{\pi(x)}$ is LFC and it also readily implies its C^∞-smoothness; the latter implies that F_Δ locally coincides with some f_Δ^π, whence it also is C^∞-smooth and LFC. ■

As a particular case of the above result, or of the much simpler Corollary 1.1.24, we conclude the existence of a C^1-smooth norm on c_0 with locally uniformly continuous derivative. In conclusion to this section, we mention that the conclusion to this result can not be improved, in the sense that the word 'locally' can not be removed from the above sentence [Wel69]; the simple proof, based on Darboux theorem, may also be found in [HáJo14, Proposition 5.49].

Proposition 1.1.29 ([Wel69]). *There is no C^1-smooth bump on c_0 with uniformly continuous derivative.*

1.1.4 Countable boundaries

In this section we shall show how to use boundaries to construct smooth norms, proving in particular the main implication in the characterisation given in [Háj95]; we will also see the interplay between countable boundaries and polyhedral Banach spaces, in the separable setting. In particular, the subsequent results will provide a proof of some implications of Theorem 1.1.12. Let us start recording the definition of boundary for a Banach space, while the definition of polyhedral Banach space is recorded at the beginning of Section 1.3.

Definition 1.1.30. A subset B of B_{X^*} is a *boundary* for X if

$$\|x\| = \max_{x^* \in B} |\langle x^*, x\rangle|,$$

i.e., for every $x \in X$ there exists $x^* \in B$ such that $\|x\| = |\langle x^*, x\rangle|$.

The Hahn-Banach theorem obviously implies that S_{X^*} is a boundary for X; moreover, an easy argument involving the Krein–Milman theorem implies that $\operatorname{Ext} B_{X^*}$ is a boundary. Let us also observe that a Banach space admits a finite boundary if and only if it is a finite-dimensional polyhedral Banach space.

The first important result that we shall state here is due to Vladimir P. Fonf [Fon80]; one its alternative proof may be found in [Ves00].

Theorem 1.1.31 ([Fon80, Fon00])**.** *Every separable polyhedral Banach space admits a countable boundary.*

Let us notice that the converse implication does not hold, as it is possible to prove that the space c is not polyhedral, although it clearly admits a countable boundary. On the other hand, spaces with a countable boundary admit polyhedral renormings, as the next simple lemma shows. In its proof, we will need the following well-known criterion for polyhedrality, for whose proof we refer to [HáJo14, Lemma 5.101].

Proposition 1.1.32 (see, *e.g.*, [HáJo14, Lemma 5.101])**.** *Let X be a normed space and let $B \subseteq B_{X^*}$ be 1-norming for X. If B satisfies the condition*

$(*)$ *for every f, w^*-accumulation point of B, and $x \in B_X$ we have $|\langle f, x\rangle| < 1$ (i.e., no such f can satisfy $\|f\| = 1$ and attain the norm)*

then (B is a boundary for X and) X is polyhedral.

Lemma 1.1.33. *Assume that the normed space $(X, \|\cdot\|)$ admits a countable boundary. Then $\|\cdot\|$ can be approximated by polyhedral norms.*

Proof of Lemma 1.1.33. Let $(f_n)_{n=1}^\infty$ be a boundary for $\|\cdot\|$; fix $(\varepsilon_n)_{n=1}^\infty \searrow 0$ and let $g_n := (1+\varepsilon_n) f_n$. Then, the norm $\nu := \sup_n |g_n|$ approximates $\|\cdot\|$ and $(g_n)_{n=1}^\infty$ is plainly 1-norming for ν. We prove that (X, ν) is polyhedral by means of condition $(*)$. Assume that g is a w^*-accumulation point of of $(g_n)_{n=1}^\infty$; since $\varepsilon_n \to 0$, we easily see that g is also a w^*-accumulation point of $(f_n)_{n=1}^\infty$, whence $\|g\| \leqslant 1$. Pick now $x \in X$ with $\nu(x) = 1$; as $(f_n)_{n=1}^\infty$ is a boundary, there exists $k \in \mathbb{N}$ such that $|\langle f_k, x\rangle| = \|x\|$. Consequently,

$$|\langle g, x\rangle| \leqslant \|g\| \, \|x\| \leqslant \|x\| = |\langle f_k, x\rangle| < |\langle g_k, x\rangle| \leqslant \nu(x);$$

hence, $(*)$ is satisfied and Proposition 1.1.32 implies the polyhedrality of (X, ν). ■

Let us note that this simple trick of 'lifting up' the coordinates is actually quite useful and it will re-appear in the next proof. As it is apparent, it is a way to isolate a finite subset of 'larger' coordinates; its forthcoming appearance in an argument involving the LFC condition is therefore not unexpected.

We now present the main result in [Háj95], showing how to obtain a smooth norm from a countable boundary. The strategy of the proof consists in using two tools: first, the above trick of lifting up some coordinates leads to an LFC norm and, secondly, this local finiteness allows to implement the same convolution technique of Section 1.1.3.

Theorem 1.1.34 ([Háj95])**.** *Let $(X, \|\cdot\|)$ be a normed space with a countable boundary. Then $\|\cdot\|$ can be approximated by C^∞-smooth LFC norms.*

Proof. Let us fix a countable boundary $(f_n)_{n=1}^\infty \subseteq S_{X^*}$ for X; moreover, fix a strictly decreasing sequence $(\delta_n)_{n=1}^\infty \searrow 0$ and let $g_n := (1+\delta_n) f_n$. Consider then the seminorm $\nu := \sup |g_n|$; plainly, $\|\cdot\| \leqslant \nu \leqslant (1+\delta_1)\|\cdot\|$, whence ν is a norm that approximates $\|\cdot\|$. Moreover, $(g_n)_{n=1}^\infty$ is a boundary for (X,ν): to see this, fix $x \in X$ such that $\|x\| = 1$. Find n_0 such that $|\langle f_{n_0}, x\rangle| = 1$, whence $|\langle g_{n_0}, x\rangle| = 1 + \delta_{n_0}$; for every $n > n_0$ we then have $|\langle g_n, x\rangle| \leqslant 1 + \delta_n < 1 + \delta_{n_0}$. Therefore, $\nu(x) = \max_{n=1,\dots,n_0} |\langle g_n, x\rangle|$ and $(g_n)_{n=1}^\infty$ is a boundary for (X,ν). Note that a similar argument would also show that ν is LFC.

Let us then consider the isometry $T\colon (X,\nu) \to (\ell_\infty, \|\cdot\|_\infty)$ defined by $x \mapsto (\langle g_n, x\rangle)_{n=1}^\infty$ and let $Z \subseteq \ell_\infty$ be its range. It is sufficient to construct, as we will do, an equivalent norm on ℓ_∞ which approximates $\|\cdot\|_\infty$ and is C^∞-smooth and LFC when restricted to Z.

Let us fix a sequence $(\varepsilon_n)_{n=1}^\infty \searrow 0$ with $\varepsilon_n \leqslant \frac{\delta_n - \delta_{n+1}}{4}$ and pick a sequence $(b_n)_{n=1}^\infty$ of even C^∞-smooth bumps $b_n\colon \mathbb{R} \to [0,\infty)$ with $\int b_n = 1$ and $\operatorname{supp} b_n \subseteq [-\varepsilon_n, \varepsilon_n]$. For $z \in \ell_\infty$ and $n \in \mathbb{N}$ define the functions

$$F_n(z) := \int_{\mathbb{R}^n} \left\| z - \sum_{i=1}^n t^i e_i \right\|_\infty \prod_{i=1}^n b_i(t^i)\, d\lambda_n(t^1, \dots, t^n) = \int_{\mathbb{R}} F_{n-1}(z - te_n) \cdot b_n(t)\, d\lambda(t).$$

In the notation of Section 1.1.3, we have $F_n = I_{\{1,\dots,n\}}(\|\cdot\|_\infty) \geqslant \|\cdot\|_\infty$. Thus, we already know that $(F_n)_{n=1}^\infty$ is an increasing sequence of even and convex functions; moreover, $\|\sum_{i=1}^n t^i e_i\|_\infty \leqslant \varepsilon_1$ for $t^i \in \operatorname{supp} b_i$ implies that $\|\cdot\|_\infty \leqslant F_n \leqslant \|\cdot\|_\infty + \varepsilon_1$. The function $F := \sup F_n$ is therefore even, convex and it satisfies $\|\cdot\|_\infty \leqslant F \leqslant \|\cdot\|_\infty + \varepsilon_1$. Hence, the set $B := \{F \leqslant 1 + 2\delta_1\}$ satisfies

$$\{\|\cdot\|_\infty \leqslant 1 + 2\delta_1 - \varepsilon_1\} \subseteq B \subseteq \{\|\cdot\|_\infty \leqslant 1 + 2\delta_1\}$$

and its Minkowski functional μ is a norm that approximates $\|\cdot\|_\infty$. To conclude, we show that $F\restriction_Z$ is C^∞-smooth and LFC on the set $\{\|\cdot\|_\infty > 1 + \delta_1\} \supseteq \{F = 1 + 2\delta_1\}$ (this inclusion follows from $\varepsilon_1 < \delta_1$); an appeal to Lemma 1.1.22 then concludes the proof.

The rough idea is that $z \in Z$ has a finite set of coordinates which are substantially larger than the remaining ones, so this property remains true (with the same finite set of coordinates) if we pick y close to z. The convolution operator defining $F_n(y)$ only takes into account points close to y, so these points have the same finite set of large coordinates. In other words, the $\|\cdot\|_\infty$ appearing in the integral is in fact always a maximum over the same finite set, which implies the desired properties. Let us give the details below.

Fix $z = (z(n))_{n=1}^\infty \in Z$ with $\|z\|_\infty > 1 + \delta_1$; the vector $x := T^{-1}z \in X$ then satisfies $\|x\| > 1$. Therefore, there is $n_0 \in \mathbb{N}$ with $|\langle f_{n_0}, x\rangle| = \|x\| > 1$, whence $|z(n_0)| = |\langle g_{n_0}, x\rangle| = (1+\delta_{n_0})\|x\|$; on the other hand, $|z(n)| \leqslant (1+\delta_{n_0+1})\|x\|$ whenever $n \geqslant n_0 + 1$. If we select any $y \in Z$ with $\|y - z\|_\infty \leqslant \frac{\delta_{n_0} - \delta_{n_0+1}}{4} =: \varepsilon$ and $t^n \in \operatorname{supp} b_n$, where $n \geqslant n_0 + 1$, we then have:

$$\begin{aligned} |y(n_0) - t^{n_0}| - \left|y(i) - t^i\right| &\geqslant |y(n_0)| - |y(i)| - \varepsilon_{n_0} - \varepsilon_i \geqslant (\delta_{n_0} - \delta_{n_0+1})\|x\| - 2\varepsilon - 2\varepsilon_{n_0} \\ &\geqslant (\delta_{n_0} - \delta_{n_0+1}) - 2\varepsilon - 2\varepsilon_{n_0} = \frac{\delta_{n_0} - \delta_{n_0+1}}{2} - 2\varepsilon_{n_0} \geqslant 0. \end{aligned}$$

This implies that for arbitrary $n \geqslant n_0 + 1$, $y \in Z$ with $\|y - z\|_\infty \leqslant \varepsilon$ and $t^i \in \operatorname{supp} b_i$ we have

$$\left\| y - \sum_{i=1}^n t^i e_i \right\|_\infty = \max_{i=1,\dots,n_0} |y(i) - t^i|,$$

whence

$$\begin{aligned} F_n(y) &= \int_{\mathbb{R}^n} \left\| y - \sum_{i=1}^n t^i e_i \right\|_\infty \prod_{i=1}^n b_i(t^i)\, d\lambda_n(t^1, \dots, t^n) \\ &= \int_{\mathbb{R}^{n_0}} \max_{i=1,\dots,n_0} |y(i) - t^i| \prod_{i=1}^{n_0} b_i(t^i)\, d\lambda_{n_0}(t^1, \dots, t^{n_0}). \end{aligned}$$

In other words, we have shown that for every $z \in Z$ with $\|z\|_\infty > 1 + \delta_1$ there exists $n_0 \in \mathbb{N}$ such that for every $y \in B_Z\left(z, \frac{\delta_{n_0} - \delta_{n_0+1}}{4}\right)$ we have

$$F(y) = \int_{\mathbb{R}^{n_0}} \max_{i=1,\dots,n_0} |t^i| \prod_{i=1}^{n_0} b_i(y(i) - t^i)\, d\lambda_{n_0}(t^1, \dots, t^{n_0}).$$

This equality immediately implies that $F\restriction_Z$ is C^∞-smooth and LFC on the set $\{\|\cdot\|_\infty > 1 + \delta_1\}$, which finishes the proof. ■

1.2 The main renorming

This section is dedicated to the main renorming procedure and the consequent proof of Theorem 1.1.9; let us start by describing the rough idea. By the result in [HáTa14], for every N one can find a C^k-smooth norm $\|\cdot\|_N$ such that $\left| \|\cdot\|_N - \|\cdot\| \right| \leqslant \varepsilon_N \|\cdot\|$. One is then tempted to use the standard gluing together in a C^k-smooth way and hope that the resulting norm will satisfy the desired properties. Unfortunately, in this way there is no possibility to assure that on X^N only the $\|\cdot\|_n$ norms with $n \geqslant N$ will enter into the gluing procedure. To achieve this feature it is necessary that the norms $\|\cdot\|_N$ be quantitatively different on X^N and $X_N = \operatorname{span}\{e_i\}_{i=1}^N$. In particular, we need a suitable finite set of these norms to be substantially larger than the others, very much in the same spirit of the arguments in Sections 1.1.3 and 1.1.4. The first part of the argument, consisting of the geometric Lemma 1.2.1 and some easy deductions, is exactly aimed at finding new norms which are quantitatively different on tail vectors and 'front vectors'. The second step consists in iterating this renorming for every n and rescaling the norms. Finally, we suitably approximate these norms with C^k-smooth ones and we glue everything together using the standard technique.

Throughout the section, we will assume that $(X, \|\cdot\|)$ is a separable (real) Banach space that admits a Schauder basis $(e_i)_{i=1}^\infty$. We shall start by fixing some notation.

We denote by $K := \text{b.c.}(e_i)_{i=1}^\infty$ the basis constant of the Schauder basis (which of course depends on the particular norm we are using). We will also denote by P_k the usual projection defined by $P_k(\sum_{j=1}^\infty \alpha^j e_j) = \sum_{j=1}^k \alpha^j e_j$ and we set $P^k := I_X - P_k$, *i.e.*, $P^k(\sum_{j=1}^\infty \alpha^j e_j) = \sum_{j=k+1}^\infty \alpha^j e_j$. It is clear that $\|P_k\| \leqslant K$ and $\|P^k\| \leqslant K + 1$. Finally, we denote $X_k := \operatorname{span}\{e_i\}_{i=1}^k$ and $X^k = \overline{\operatorname{span}}\{e_i\}_{i=k+1}^\infty$ the ranges of the two projections respectively.

The first part of the argument to be presented below will make extensive use of convex sets. Let us recall that a convex set C in a Banach space X is said to be a *convex body* whenever its interior is not empty. Obviously, a symmetric convex body is in particular a neighbourhood of the origin and the unit ball B_X of X is a bounded, symmetric convex body (we shorthand this fact by saying that it is a *BCSB*). It is a simple classical fact that any other BCSB B in X induces an equivalent norm on X via its Minkowski functional

$$\mu_B(x) := \inf\{t > 0 \colon x \in tB\}.$$

We will also denote by $\|\cdot\|_B$ the norm induced by B, *i.e.*, $\|x\|_B := \mu_B(x)$; obviously $\|\cdot\|_{B_X}$ is the original norm of the space. Two obvious properties are that

$$B \subseteq C \implies \mu_B \geqslant \mu_C,$$

$$\mu_{\lambda B} = \frac{1}{\lambda}\mu_B$$

which, passing to the associated Minkowski functionals, yield

$$B \subseteq C \subseteq (1+\delta)B \implies \frac{1}{1+\delta}\mu_B \leqslant \mu_C \leqslant \mu_B.$$

We now start with the first part of the argument.

Lemma 1.2.1. *Assume that $(X, \|\cdot\|)$ is a Banach space with a Schauder basis $(e_i)_{i=1}^\infty$ and let K be the basis constant of $(e_i)_{i=1}^\infty$; let us also denote the unit ball of X by B. Fixed $k \in \mathbb{N}$, two parameters $\lambda > 0$ and $0 < R < 1$, consider the sets*

$$D := \left\{x \in X \colon \|P^k x\| \leqslant R\right\} \cap (1+\lambda)\cdot B,$$

$$C := \overline{\mathrm{conv}}\{D, B\}.$$

Then C is a BCSB and

$$C \cap X^k \subseteq \left(1 + \lambda \frac{K}{K+1-R}\right) \cdot B.$$

The BCSB C is the unit ball of an equivalent norm; such a new unit ball is obtained modifying the ball B in the direction of X_k. The heuristic content of the lemma is then that if we modify the unit ball in the direction of X_k, this modification results in a perturbation of the ball also in the remaining directions, but this modification is significantly smaller in a controlled way, given by the factor $\frac{K}{K+1-R} < 1$.

Proof. The fact that C is a BCSB is obvious. Let us fix a vector $x \in C \cap X^k$; a simple cone argument, based on $0 \in \mathrm{Int}C$, implies that $tx \in \mathrm{Int}C$ whenever $t \in [0, 1)$. Moreover, $\mathrm{conv}\{D, B\}$ has non-empty interior, which easily implies that its interior equals the interior of its closure; consequently, $tx \in \mathrm{Int}C = \mathrm{Int}\,(\mathrm{conv}\{D, B\}) \subseteq \mathrm{conv}\{D, B\}$. If we can show that $tx \in \left(1 + \lambda \frac{K}{K+1-R}\right) \cdot B$, we may then let $t \to 1^-$ and reach the desired conclusion. In other words, we can assume without loss of generality that $x \in X^k \cap \mathrm{conv}\{D, B\}$.

We may thus write $x = ty + (1-t)z$, where $t \in [0, 1]$, $y \in D$ and $z \in B$; in particular, we have $\|P^k y\| \leqslant R$ and $\|z\| \leqslant 1$. Moreover, $x \in X^k$ implies

$$\|x\| = \|P^k x\| \leqslant t\|P^k y\| + (1-t)\|P^k z\| \leqslant tR + (1-t)(K+1).$$

The conclusion of the lemma being evidently true if $\|x\| \leqslant 1$, we can assume $\|x\| \geqslant 1$; as a consequence, $1 \leqslant K + 1 - t(K+1-R)$ whence $t \leqslant \frac{K}{K+1-R}$.

We next consider some slight perturbations of the points y and z, in such a way that x is still their convex combination: let us fix two parameters $\tau, \eta > 0$ to be determined later and consider the vectors $u := (1-\tau)y$ and $v := (1+\eta)z$. Obviously, $x = \frac{t}{1-\tau}u + \frac{1-t}{1+\eta}v$ and we require this to be a convex combination:

$$1 = \frac{t}{1-\tau} + \frac{1-t}{1+\eta} \qquad \Longrightarrow \qquad \tau = \frac{(1-t)\eta}{t+\eta} \leqslant 1$$

(of course this choice implies $1 - \tau \geqslant 0$). Our assumption that $y \in D$ yields $\|u\| \leqslant (1-\tau)\|y\| \leqslant (1-\tau)(1+\lambda)$; moreover, we have $\|v\| \leqslant 1 + \eta$. We wish these norms to be equally small, so we require (here we use the previous choice of τ)

$$1 + \eta = (1-\tau)(1+\lambda) \qquad \Longrightarrow \qquad \eta = \lambda t.$$

With this choice for τ and η we have $\|u\|, \|v\| \leqslant 1 + \eta = 1 + \lambda t \leqslant 1 + \lambda \cdot \frac{K}{K+1-R}$; by convexity, the same assertion holds true for x, which completes the proof. ■

We now modify again the obtained BCSB in such a way that on X^k the body is an exact multiple of the original unit ball; this modification does not destroy the properties achieved before, as we prove in the corollary below. It will be useful to denote by $S := \{x \in X : \|P^k x\| \leqslant R\}$; with this notation we have $D := S \cap (1+\lambda) \cdot B$.

Corollary 1.2.2. *In the above notation, let* $\gamma := \frac{K}{K+1-R}$ *and*

$$\tilde{B} := \overline{\operatorname{conv}} \left\{ C, X^k \cap (1+\lambda\gamma)\cdot B \right\}.$$

Then $\tilde{B}$ *is a BCSB and*

$$B \subseteq \tilde{B} \subseteq (1+\lambda)\cdot B,$$
$$S \cap \tilde{B} = S \cap (1+\lambda)\cdot B,$$
$$X^k \cap \tilde{B} = X^k \cap (1+\lambda\gamma)\cdot B.$$

Proof. It is obvious that $\tilde{B}$ is a BCSB. Of course $B \subseteq C$, whence $B \subseteq \tilde{B}$, *a fortiori*. Moreover, $D \subseteq (1+\lambda)\cdot B$ implies $C \subseteq (1+\lambda)\cdot B$; this and $\gamma \leqslant 1$ assure us that $\tilde{B} \subseteq (1+\lambda)\cdot B$.

For what concerns the second assertion, the '$\subseteq$' in the second assertion follows from what we have just proved; for the converse inclusion, we just note that $S \cap (1+\lambda)\cdot B = D \subseteq \tilde{B}$.

For the last equality, obviously $X^k \cap (1+\lambda\gamma)\cdot B \subseteq \tilde{B}$, which implies the '$\supseteq$' inclusion. For the converse inclusion, let $p \in X^k \cap \tilde{B}$; exactly the same argument as in the first part of the previous proof (with C replaced by $\tilde{B}$) shows that we can assume $p \in \operatorname{conv}\{C, X^k \cap (1+\lambda\gamma)\cdot B\} \cap X^k$. We may therefore write $p = ty + (1-t)z$, for some $y \in C$ and $z \in X^k \cap (1+\lambda\gamma)\cdot B$. In the case that $t = 0$, $p = z \in X^k \cap (1+\lambda\gamma)\cdot B$, and we are done. On the other hand if $t > 0$, our assumption that $p \in X^k$ allows us to deduce that $y \in X^k$ too; in light of the previous lemma, we conclude that $y \in C \cap X^k \subseteq (1+\lambda\gamma)\cdot B$. By convexity, $p \in (1+\lambda\gamma)\cdot B$, and the proof is complete. ■

The next proposition is essentially a restatement of the above corollary in terms of norms rather than convex bodies; we write it explicitly since the remainder of the argument is better presented using equivalent norms rather than convex bodies. The general setting is the same as above: X is a separable Banach space with Schauder basis $(e_i)_{i=1}^\infty$.

Proposition 1.2.3. *Let B be a BCSB in X with induced norm $\|\cdot\|_B$; also let K be the basis constant of $(e_i)_{i=1}^\infty$ relative to $\|\cdot\|_B$. Fix $k \in \mathbb{N}$ and two parameters $\lambda > 0$ and $0 < R < 1$. Then there is a BCSB $\tilde{B}$ in X such that the induced norm $\|\cdot\|_{\tilde{B}}$ satisfies the following properties:*

(a)
$$\|\cdot\|_{\tilde{B}} \leqslant \|\cdot\|_B \leqslant (1+\lambda)\|\cdot\|_{\tilde{B}},$$

(b)
$$\|\cdot\|_B = (1+\lambda\gamma)\|\cdot\|_{\tilde{B}} \qquad \text{on } X^k,$$

(c)
$$\|x\|_B = (1+\lambda)\|x\|_{\tilde{B}} \qquad \text{whenever } \|P^k x\| \leqslant \frac{R}{1+\lambda}\|x\|,$$

where $\gamma := \frac{K}{K+1-R}$.

Proof. The desired convex body $\tilde{B}$ is the BCSB defined in the previous corollary. In fact, (a) follows immediately from the corollary and (b) is also immediate: for $x \in X^k$, we have

$$\|x\|_{\tilde{B}} = \inf\left\{t > 0\colon x \in t \cdot \tilde{B}\right\} = \inf\left\{t > 0\colon x \in t \cdot \left(\tilde{B} \cap X^k\right)\right\} =$$

$$\inf\left\{t > 0\colon x \in t \cdot \left(X^k \cap (1+\lambda\gamma) \cdot B\right)\right\} = \inf\left\{t > 0\colon x \in t\,(1+\lambda\gamma) \cdot B\right\}$$

$$= \frac{1}{1+\lambda\gamma} \inf\left\{t > 0\colon x \in t \cdot B\right\} = \frac{1}{1+\lambda\gamma} \|x\|_B.$$

The proof of clause (c) is not equally trivial since S is not a cone; so, we first modify it and we define

$$S_1 := \left\{x \in X\colon \|P^k x\| \leqslant \frac{R}{1+\lambda}\|x\|\right\}.$$

We first observe that the replacement of S with S_1 does not modify the above construction: in other words, if we set $D_1 := S_1 \cap (1+\lambda) \cdot B$, then we have $C_1 := \overline{\mathrm{conv}}\{D_1, B\} = C$. In fact, $S_1 \cap (1+\lambda) \cdot B \subseteq S$ implies $C_1 \subseteq C$ and the converse inclusion follows from $D \subseteq \mathrm{conv}\{D_1, B\}$. In order to prove this last assertion, fix $x \in D$; then, $\|P^k x\| \leqslant R < 1$ and in particular $P^k x \in B$. Now, set $x_t := P^k x + t(x - P^k x)$ and choose $t \geqslant 1$ with the property that $\|x_t\| = 1+\lambda$; with this choice of t we obtain $\|P^k x_t\| = \|P^k x\| \leqslant R = \frac{R}{1+\lambda}\|x_t\|$, whence $x_t \in D_1$. Since x is a convex combination of x_t and $P^k x$, we conclude that $D \subseteq \mathrm{conv}\{D_1, B\}$.

Next, we claim that

$$S_1 \cap \tilde{B} = S_1 \cap (1+\lambda) \cdot B.$$

In fact, '$\supseteq$' follows from the analogous relation with S, proved in the above corollary, and $S_1 \cap (1+\lambda) \cdot B \subseteq S$. The converse inclusion follows–once more–from $\tilde{B} \subseteq (1+\lambda) \cdot B$.

We are finally ready for the proof of (c): pick $x \in S_1$ and notice that

$$\left\{t > 0\colon x \in t\tilde{B}\right\} = \left\{t > 0\colon x \in t\tilde{B} \cap S_1\right\} = \left\{t > 0\colon x \in t\left(\tilde{B} \cap S_1\right)\right\}$$

$$= \{t > 0\colon x \in t(S_1 \cap (1+\lambda) \cdot B)\} = \{t > 0\colon x \in t(1+\lambda) \cdot B\};$$

consequently,

$$\inf\left\{t > 0\colon x \in t\tilde{B}\right\} = \frac{1}{1+\lambda} \inf\{t > 0\colon x \in t \cdot B\},$$

which is exactly (c). ∎

We now are in position to enter the second part of the renorming procedure, which consists in an inductive iteration of the above construction and a subsequent rescaling.

We start with the Banach space X with unit ball B and corresponding norm $\|\cdot\| := \|\cdot\|_B$ and we apply the previous proposition with $k = 1$, a certain $\lambda_1 > 0$ and $R = 1/2$. We let $B_1 := \widetilde{B}$ be the convex body constructed above and $\|\cdot\|_1 := \|\cdot\|_{B_1}$ be the induced norm. The properties proved above then imply

$$\|\cdot\|_1 \leqslant \|\cdot\| \leqslant (1+\lambda_1)\|\cdot\|_1,$$

$$\|\cdot\| = (1+\lambda_1\gamma_1)\|\cdot\|_1 \qquad \text{on } X^1,$$

$$\|x\| = (1+\lambda_1)\|x\|_1 \qquad \text{whenever } \|P^1x\| \leqslant \frac{1/2}{1+\lambda_1}\|x\|,$$

where $\gamma_1 := \frac{K}{K+1/2}$.

We proceed inductively in the obvious way: we fix a sequence $(\lambda_n)_{n=1}^\infty \subseteq (0,\infty)$ such that $\prod_{i=1}^\infty(1+\lambda_i) < \infty$ and, in order to have a more concise notation, denote by $\|\cdot\|_0 := \|\cdot\|$ the original norm of X and by $K_0 := K$. Apply inductively the previous proposition: at the step n we use the proposition with the parameters $\lambda = \lambda_n$, $R = 1/2$, $k = n$ and $B = B_{n-1}$ and we set $B_n := \widetilde{B_{n-1}}$ and $\|\cdot\|_n := \|\cdot\|_{B_n}$. This provides us with a sequence of norms $(\|\cdot\|_n)_{n=0}^\infty$ on X with the following properties (for every $n \in \mathbb{N}$):

$$\|\cdot\|_n \leqslant \|\cdot\|_{n-1} \leqslant (1+\lambda_n)\|\cdot\|_n, \tag{1.2.1}$$

$$\|\cdot\|_{n-1} = (1+\lambda_n\gamma_n)\|\cdot\|_n \qquad \text{on } X^n, \tag{1.2.2}$$

$$\|x\|_{n-1} = (1+\lambda_n)\|x\|_n \qquad \text{whenever } \|P^nx\|_{n-1} \leqslant \frac{1/2}{1+\lambda_n}\|x\|_{n-1}, \tag{1.2.3}$$

where K_n denotes the basis constant of the Schauder basis $(e_i)_{i=1}^\infty$ relative to the norm $\|\cdot\|_n$ and $\gamma_n := \frac{K_{n-1}}{K_{n-1}+1/2} \in (0,1)$.

Remark 1.2.4. The condition $\|P^nx\|_{n-1} \leqslant \frac{1/2}{1+\lambda_n}\|x\|_{n-1}$ appearing in (1.2.3) is somewhat unpleasing since the involved norms change with n; we thus replace it with the following more uniform, but weaker, condition.

$$\|x\|_{n-1} = (1+\lambda_n)\|x\|_n \qquad \text{whenever } \|P^nx\|_0 \leqslant \frac{1}{2}\prod_{i=1}^\infty(1+\lambda_i)^{-1}\cdot\|x\|_0. \tag{1.2.4}$$

The validity of (1.2.4) is immediately deduced from the validity of (1.2.1) and (1.2.3): in fact, if x satisfies $\|P^nx\|_0 \leqslant \frac{1}{2}\prod_{i=1}^\infty(1+\lambda_i)^{-1}\cdot\|x\|_0$, then by (1.2.1)

$$\|P^nx\|_{n-1} \leqslant \|P^nx\|_0 \leqslant \frac{1}{2}\prod_{i=1}^\infty(1+\lambda_i)^{-1}\cdot\|x\|_0 \leqslant \frac{1}{2}\prod_{i=1}^\infty(1+\lambda_i)^{-1}\cdot\prod_{i=1}^{n-1}(1+\lambda_i)\cdot\|x\|_{n-1}$$

$$= \frac{1/2}{1+\lambda_n}\prod_{i=n+1}^\infty(1+\lambda_i)^{-1}\cdot\|x\|_{n-1} \leqslant \frac{1/2}{1+\lambda_n}\|x\|_{n-1};$$

consequently, (1.2.3) implies that $\|x\|_{n-1} = (1+\lambda_n)\|x\|_n$.

In order to motivate the next rescaling, let us notice that for a fixed $x \in X$ the sequence $(\|x\|_n)_{n=0}^{\infty}$ has the same qualitative behavior, being a decreasing sequence; on the other hand its quantitative rate of decrease changes with n. In fact, it is clear that for a fixed $x \in X$, the condition $\|P^n x\|_0 \leqslant \frac{1}{2}\prod_{i=1}^{\infty}(1+\lambda_i)^{-1} \cdot \|x\|_0$ is eventually satisfied, so the sequence $(\|x\|_n)_{n=0}^{\infty}$ eventually decreases with rate $(1+\lambda_n)^{-1}$. On the other hand, if $x \in X^N$, then for the terms $n = 1, \dots, N$ the rate of decrease is $(1+\lambda_n\gamma_n)^{-1}$. This makes it possible to rescale the norms $\|\cdot\|_n$, obtaining norms $|||\cdot|||_n$ $(n \geqslant 0)$, in a way to have a qualitatively different behavior, increasing for $n = 1, \dots, N$ and eventually decreasing.

This property is crucial since it allows us to assure that, for $x \in X^N$, the norms $|||x|||_n$ for $n = 0, \dots, N-1$ are quantitatively smaller than $|||x|||_N$ and thus do not enter in the gluing procedure, which only takes into account 'large' coordinates. As we have hinted at in conclusion of the previous section and as it will be apparent in the proof of Lemma 1.2.7 (*cf.* Remark 1.2.9), this is exactly what we need in order the approximation on X^N to improve with N. On the other hand, the fact that the norms $|||x|||_n$ are eventually quantitatively smaller will imply that the gluing procedure locally involves finitely many ingredients, thereby preserving the smoothness.

Let us then pass to the suitable scaling.

Definition 1.2.5. Let

$$C := \prod_{i=1}^{\infty} \frac{1+\lambda_i\gamma_i}{1+\lambda_i\frac{1+\gamma_i}{2}},$$

$$|||\cdot|||_n := C \cdot \prod_{i=1}^{n}\left(1+\lambda_i\frac{1+\gamma_i}{2}\right) \cdot \|\cdot\|_n\,.$$

For later convenience, let us also set

$$|||\cdot|||_\infty = \sup_{n\geqslant 0} |||\cdot|||_n.$$

The qualitative behavior of $(|||x|||_n)_{n=0}^{\infty}$ is expressed in the following obvious, though crucial, properties of the norms $|||\cdot|||_n$. In particular, (a) will be used to show that the gluing together locally takes into account only finitely many terms; this will allow us to preserve the smoothness in Lemma 1.2.10. (b) expresses the fact that on X^N the norms $(|||\cdot|||_n)_{n=0}^{N-1}$ are smaller than $|||\cdot|||_N$ and will be used in Lemma 1.2.7 to obtain the improvement of the approximation.

Fact 1.2.6. (a) *For every $x \in X$ there is $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$*

$$|||x|||_n = \frac{1+\lambda_n\frac{1+\gamma_n}{2}}{1+\lambda_n}|||x|||_{n-1}.$$

In particular, it suffices to take any n_0 such that $\|P^n x\|_0 \leqslant \frac{1}{2}\prod_{i=1}^{\infty}(1+\lambda_i)^{-1} \cdot \|x\|_0$ for every $n \geqslant n_0$.

(b) *If $x \in X^N$, then for $n = 1, \dots, N$ we have*

$$|||x|||_n = \frac{1+\lambda_n\frac{1+\gamma_n}{2}}{1+\lambda_n\gamma_n}|||x|||_{n-1}.$$

Proof. (a) Since $P^n x \to 0$ as $n \to \infty$, condition (1.2.4) implies that there is n_0 such that for every $n \geqslant n_0$ we have $\|x\|_n = (1+\lambda_n)^{-1}\|x\|_{n-1}$. Then it suffices to translate this to the $|||\cdot|||_n$ norms:

$$|||x|||_n = \left(1+\lambda_n\frac{1+\gamma_n}{2}\right)\cdot C\cdot\prod_{i=1}^{n-1}\left(1+\lambda_i\frac{1+\gamma_i}{2}\right)\cdot\|x\|_n =$$

$$\frac{1+\lambda_n\frac{1+\gamma_n}{2}}{1+\lambda_n}\cdot C\cdot\prod_{i=1}^{n-1}\left(1+\lambda_i\frac{1+\gamma_i}{2}\right)\cdot\|x\|_{n-1} = \frac{1+\lambda_n\frac{1+\gamma_n}{2}}{1+\lambda_n}|||x|||_{n-1}.$$

(b) If $x \in X^N$ and $n = 1,\ldots,N$, then $x \in X^n$ too; thus by (1.2.2) we have $\|x\|_n = (1+\lambda_n\gamma_n)^{-1}\|x\|_{n-1}$. Now exactly the same calculation as in the other case gives the result. ■

We can now enter the third part of the renorming and conclude the renorming procedure: firstly, we smoothen up the norms $|||\cdot|||_n$ and secondly we glue together all the obtained smooth norms. Fix a decreasing sequence $(\delta_n)_{n=1}^\infty$ of positive reals with $\delta_n \to 0$ with the property that for every $n \geqslant 0$

$$(\dagger)\qquad (1+\delta_n)\frac{1+\lambda_{n+1}\gamma_{n+1}}{1+\lambda_{n+1}\frac{1+\gamma_{n+1}}{2}} \leqslant 1-\delta_n$$

(of course this is possible since $\gamma_{n+1} < 1$). We may then apply the main result in [HáTa14] (Theorem 2.10 in their paper) and deduce the existence of C^k-smooth norms $|||\cdot|||_{(s),n}$ (for $n \geqslant 0$) such that for every n

$$|||\cdot|||_n \leqslant |||\cdot|||_{(s),n} \leqslant (1+\delta_n)|||\cdot|||_n.$$

Next, we shall select functions $\varphi_n\colon [0,\infty) \to [0,\infty)$ to be C^∞-smooth, convex and such that $\varphi_n \equiv 0$ on $[0, 1-\delta_n]$ and $\varphi_n(1) = 1$; note that, of course, the φ_n's are strictly monotonically increasing on $[1-\delta_n, \infty)$.

We are finally ready to define $\Phi\colon X \to [0,\infty]$ by

$$\Phi(x) := \sum_{n=0}^{\infty}\varphi_n\left(|||x|||_{(s),n}\right)$$

and let $|||\cdot|||$ be the Minkowski functional of the set $\{\Phi \leqslant 1\}$.

The fact that $|||\cdot|||$ is the desired norm is now an obvious consequence of the next two lemmas. In the first one we show that $|||\cdot|||$ is indeed a norm and that the approximation on X^N improves with N.

Lemma 1.2.7. *$|||\cdot|||$ is a norm, equivalent to the original norm $\|\cdot\|$ of X.*

Moreover for every $N \geqslant 0$ we have

$$\prod_{i=N+1}^{\infty}(1+\lambda_i)^{-1}\cdot\|\cdot\| \leqslant |||\cdot||| \leqslant \frac{1+\delta_N}{1-\delta_N}\cdot\prod_{i=N+1}^{\infty}(1+\lambda_i)\cdot\|\cdot\| \qquad \textit{on } X^N.$$

Proof. We start by observing that for every $N \geqslant 0$

$$\left\{x \in X^N \colon |||x|||_\infty \leqslant \frac{1-\delta_N}{1+\delta_N}\right\} \subseteq \left\{x \in X^N \colon \Phi(x) \leqslant 1\right\} \subseteq \left\{x \in X^N \colon |||x|||_\infty \leqslant 1\right\}.$$

In fact, pick $x \in X^N$ such that $\Phi(x) \leqslant 1$, whence in particular $\varphi_n\left(|||x|||_{(s),n}\right) \leqslant 1$ for every n. The inequality $|||\cdot|||_n \leqslant |||\cdot|||_{(s),n}$ and the properties of φ_n then imply $|||x|||_n \leqslant 1$ for every n. This proves the right inclusion. For the first inclusion, we actually show that if $x \in X^N$ satisfies $|||x|||_\infty \leqslant \frac{1-\delta_N}{1+\delta_N}$, then $\Phi(x) = 0$. To see this, fix any $n \geqslant N$; since the function $t \mapsto \frac{1-t}{1+t}$ is decreasing on $[0,1]$ and the sequence $(\delta_n)_{n=1}^\infty$ is also decreasing, we deduce

$$|||x|||_n \leqslant |||x|||_\infty \leqslant \frac{1-\delta_N}{1+\delta_N} \leqslant \frac{1-\delta_n}{1+\delta_n}.$$

Consequently, $|||x|||_{(s),n} \leqslant 1-\delta_n$ and $\varphi_n\left(|||x|||_{(s),n}\right) = 0$ for every $n \geqslant N$. For the remaining values $n = 0, \dots, N-1$ we use (b) in Fact 1.2.6 and condition (†):

$$\begin{aligned}|||x|||_{(s),n} &\leqslant (1+\delta_n)|||x|||_n = (1+\delta_n)\frac{1+\lambda_{n+1}\gamma_{n+1}}{1+\lambda_{n+1}\frac{1+\gamma_{n+1}}{2}} \cdot |||x|||_{n+1} \\ &\leqslant (1-\delta_n)|||x|||_{n+1} \leqslant 1-\delta_n;\end{aligned}$$

hence $\varphi_n\left(|||x|||_{(s),n}\right) = 0$ for $n = 0, \dots, N-1$ too. It follows that $\Phi(x) = 0$, which proves the first inclusion.

Taking in particular $N = 0$, we see that $\{\Phi \leqslant 1\}$ is a bounded neighbourhood of the origin in $(X, |||\cdot|||_\infty)$. Since it is clearly convex and symmetric, we deduce that $\{\Phi \leqslant 1\}$ is a BCSB relative to $|||\cdot|||_\infty$. Therefore $|||\cdot|||$ is a norm on X, equivalent to $|||\cdot|||_\infty$. The fact that $|||\cdot|||$ is equivalent to the original norm $\|\cdot\|$ follows immediately from the case $N = 0$ in the second assertion, which we now prove.

Fix $N \geqslant 0$; in order to estimate the distortion between $|||\cdot|||$ and $\|\cdot\|$ on X^N, we show that, on X^N, $|||\cdot|||$ is close to $|||\cdot|||_\infty$, that $|||\cdot|||_\infty$ is close to $|||\cdot|||_N$ and finally that $|||\cdot|||_N$ is close to $\|\cdot\|$.

First, passing to the associated Minkowski functionals, the inclusions obtained in the first part of the proof yield

$$(*) \qquad |||\cdot|||_\infty \leqslant |||\cdot||| \leqslant \frac{1+\delta_N}{1-\delta_N}|||\cdot|||_\infty \qquad \text{on } X^N.$$

Secondly, we compare $|||\cdot|||_\infty$ with $|||\cdot|||_N$. Obviously, $|||\cdot|||_N \leqslant |||\cdot|||_\infty$ and by property (b) in Fact 1.2.6 already used above we also have $|||\cdot|||_n \leqslant |||\cdot|||_N$ whenever $n \leqslant N$. We thus fix $n > N$ and observe

$$|||\cdot|||_n := C\prod_{i=1}^n\left(1+\lambda_i\frac{1+\gamma_i}{2}\right) \cdot \|\cdot\|_n \leqslant$$

$$\prod_{i=N+1}^{n}\left(1+\lambda_i\frac{1+\gamma_i}{2}\right)\cdot C\cdot\prod_{i=1}^{N}\left(1+\lambda_i\frac{1+\gamma_i}{2}\right)\cdot\|\cdot\|_N =$$

$$\prod_{i=N+1}^{n}\left(1+\lambda_i\frac{1+\gamma_i}{2}\right)\cdot|||\cdot|||_N \leqslant \prod_{i=N+1}^{\infty}(1+\lambda_i)\cdot|||\cdot|||_N.$$

This yields

$$(*)\qquad |||\cdot|||_N \leqslant |||\cdot|||_\infty \leqslant \prod_{i=N+1}^{\infty}(1+\lambda_i)\cdot|||\cdot|||_N \qquad \text{on } X^N.$$

Finally, we compare $|||\cdot|||_N$ with $\|\cdot\|_0$. The subspaces X^N are decreasing with N, whence (1.2.2) implies $\|\cdot\| = \prod_{i=1}^{N}(1+\lambda_i\gamma_i)\cdot\|\cdot\|_N$ on X^N; hence

$$\|\cdot\| = \prod_{i=1}^{N}(1+\lambda_i\gamma_i)\cdot\prod_{i=1}^{\infty}\frac{1+\lambda_i\frac{1+\gamma_i}{2}}{1+\lambda_i\gamma_i}\cdot\prod_{i=1}^{N}\left(1+\lambda_i\frac{1+\gamma_i}{2}\right)^{-1}\cdot|||\cdot|||_N$$

$$= \prod_{i=N+1}^{\infty}\frac{1+\lambda_i\frac{1+\gamma_i}{2}}{1+\lambda_i\gamma_i}\cdot|||\cdot|||_N.$$

This implies in particular

$$(*)\qquad |||\cdot|||_N \leqslant \|\cdot\| \leqslant \prod_{i=N+1}^{\infty}(1+\lambda_i)\cdot|||\cdot|||_N \qquad \text{on } X^N;$$

combining the $(*)$ inequalities then leads us to the desired conclusion. ■

Remark 1.2.8. The estimate of the distortion in the particular case $N=0$ is in fact shorter than the general case given above. In fact, property (1.2.1) obviously implies $\|\cdot\|_n \leqslant \|\cdot\| \leqslant \prod_{i=1}^{n}(1+\lambda_i)\cdot\|\cdot\|_n$. It easily follows that for every n

$$\prod_{i=1}^{\infty}(1+\lambda_i)^{-1}\cdot\|\cdot\| \leqslant |||\cdot|||_n \leqslant \prod_{i=1}^{\infty}(1+\lambda_i)\cdot\|\cdot\|\,;$$

it is then sufficient to combine this with the first of the $(*)$ inequalities.

Remark 1.2.9. Inspection of the first part of the above argument shows that whenever $x\in X^N$ satisfies $|||x|||_\infty \leqslant 1$, then $\varphi_n\left(|||x|||_{(s),n}\right)=0$ for every $n=0,\dots,N-1$. In other words, on the set $X^N\cap\{|||\cdot|||_\infty\leqslant 1\}$ the sum defining Φ starts at the N-th term; let us also notice that this fact depends on Fact 1.2.6 (b). Consequently, this is the point in the argument where we see the role of the initial norms to be smaller.

The remaining part of the argument consists in checking the regularity of $|||\cdot|||$.

Lemma 1.2.10. *The norm $|||\cdot|||$ is C^k-smooth.*

Proof. We first show that for every x in the set $\{\Phi < 2\}$ there is a neighbourhood $\mathcal{U}$ of x (in X) where the function Φ is expressed by a finite sum. We have already seen in the proof of Lemma 1.2.7 that $\Phi = 0$ in a neighbourhood of 0, so the assertion is true for $x = 0$; hence we can fix $x \neq 0$ such that $\Phi(x) < 2$. Observe that clearly the properties of φ_n imply $\varphi_n(1+\delta_n) \geqslant 2$; thus x satisfies $|||x|||_n \leqslant |||x|||_{(s),n} \leqslant 1+\delta_n$ for every n.

Denote by $c := \frac{1}{2}\prod_{i=1}^{\infty}(1+\lambda_i)^{-1}$ and choose n_0 such that $\|P^n x\| \leqslant \frac{c}{2}\cdot\|x\|$ for every $n \geqslant n_0$ (this is possible since $P^n x \to 0$). Next, fix $\varepsilon > 0$ small so that $\frac{c}{2} + K\varepsilon \leqslant (1-\varepsilon)c$ and $(1+\varepsilon)(1-\delta_{n_0}) \leqslant 1$, and let $\mathcal{U}$ be the following neighbourhood of x:

$$\mathcal{U} := \left\{y \in X \colon \|y-x\| < \varepsilon\|x\| \text{ and } |||y|||_{n_0} < (1+\varepsilon)|||x|||_{n_0}\right\}.$$

Clearly for $y \in \mathcal{U}$ we have $\|x\| \leqslant \frac{1}{1-\varepsilon}\|y\|$; thus for $y \in \mathcal{U}$ and $n \geqslant n_0$ we have

$$\|P^n y\| \leqslant \|P^n y - P^n x\| + \|P^n x\| \leqslant K\varepsilon\|x\| + \frac{c}{2}\cdot\|x\| \leqslant (1-\varepsilon)c\|x\| \leqslant c\|y\|.$$

Hence (a) of Fact 1.2.6 implies that $|||y|||_n = \frac{1+\lambda_n\frac{1+\gamma_n}{2}}{1+\lambda_n}|||y|||_{n-1}$ for every $n \geqslant n_0$ and $y \in \mathcal{U}$ (let us explicitly stress the crucial fact that n_0 does not depend on $y \in \mathcal{U}$).

We have $|||y|||_{n_0} < (1+\varepsilon)|||x|||_{n_0} \leqslant (1+\varepsilon)(1+\delta_{n_0})$; using this bound and the previous choices of the parameters (in particular we use twice (†) and twice the fact that δ_n is decreasing), for every $n \geqslant n_0+2$ and $y \in \mathcal{U}$ we estimate

$$|||y|||_{(s),n} \leqslant (1+\delta_n)|||y|||_n = (1+\delta_n)\prod_{i=n_0+1}^{n}\frac{1+\lambda_i\frac{1+\gamma_i}{2}}{1+\lambda_i}\cdot|||y|||_{n_0}$$

$$\leqslant (1+\delta_n)\prod_{i=n_0+1}^{n}\frac{1+\lambda_i\frac{1+\gamma_i}{2}}{1+\lambda_i}\cdot(1+\varepsilon)(1+\delta_{n_0}) \overset{(\dagger)}{\leqslant} (1+\delta_n)\prod_{i=n_0+2}^{n}\frac{1+\lambda_i\frac{1+\gamma_i}{2}}{1+\lambda_i}\cdot(1+\varepsilon)(1-\delta_{n_0})$$

$$\leqslant (1+\delta_n)\prod_{i=n_0+2}^{n}\frac{1+\lambda_i\frac{1+\gamma_i}{2}}{1+\lambda_i} \leqslant (1+\delta_{n-1})\frac{1+\lambda_n\frac{1+\gamma_n}{2}}{1+\lambda_n}\cdot\prod_{i=n_0+2}^{n-1}\frac{1+\lambda_i\frac{1+\gamma_i}{2}}{1+\lambda_i}$$

$$\leqslant (1+\delta_{n-1})\frac{1+\lambda_n\frac{1+\gamma_n}{2}}{1+\lambda_n} \overset{(\dagger)}{\leqslant} 1-\delta_{n-1} \leqslant 1-\delta_n.$$

It follows that $\varphi_n\left(|||y|||_{(s),n}\right) = 0$ for $n \geqslant n_0+2$ and $y \in \mathcal{U}$, hence

$$\Phi = \sum_{n=0}^{n_0+2}\varphi_n \circ |||\cdot|||_{(s),n} \qquad \text{on } \mathcal{U}.$$

This obviously implies that Φ is C^k-smooth on the set $\{\Phi < 2\}$ and in particular $\{\Phi < 2\}$ is an open set. Concerning the regularity of Φ, we also observe here that Φ is lower semi-continuous on X (this follows immediately from the fact that Φ is the sum of a series of positive continuous functions).

The last step consists in applying the Implicit Function theorem for Minkowski functionals, Lemma 1.1.22, and conclude the smoothness of $|||\cdot|||$. More precisely, we apply Lemma 1.1.22 to the open, convex, symmetric set $\{\Phi < 2\}$, where the function Φ is C^k -smooth, and to the bounded, closed neighbourhood of 0 given by $\{\Phi \leqslant 1\}$. This is indeed possible, since $\{\Phi \leqslant 1\}$ is a closed subset of X, in light of the lower semi-continuity of Φ on X. Consequently, we obtain that the Minkowski functional $|||\cdot|||$ of the set $\{\Phi \leqslant 1\}$ is a C^k-smooth norm, thereby concluding the proof. ■

To conclude the section, let us formally record the, at this stage entirely obvious, proof of Theorem 1.1.9.

Proof of Theorem 1.1.9. Fix a separable Banach space as in the statement and a sequence $(\varepsilon_N)_{N=0}^{\infty}$ of positive numbers. We find a sequence $(\lambda_i)_{i=1}^{\infty} \subseteq (0, \infty)$ such that

$$\prod_{i=N+1}^{\infty} (1 + \lambda_i) < 1 + \varepsilon_N$$

for every $N \geqslant 0$; next, we find a decreasing sequence $(\delta_N)_{N=0}^{\infty}$, $\delta_N \searrow 0$, that satisfies (†) and such that

$$\frac{1 + \delta_N}{1 - \delta_N} \cdot \prod_{i=N+1}^{\infty} (1 + \lambda_i) \leqslant 1 + \varepsilon_N$$

for every $N \geqslant 0$. We then apply the renorming procedure described in this section with these parameters $(\lambda_i)_{i=1}^{\infty}$ and $(\delta_N)_{N=0}^{\infty}$ and we obtain a C^k-smooth norm $|||\cdot|||$ on X that satisfies

$$(1 - \varepsilon_N) \cdot \|\cdot\| \leqslant \prod_{i=N+1}^{\infty} (1 + \lambda_i)^{-1} \cdot \|\cdot\| \leqslant |||\cdot||| \leqslant \frac{1 + \delta_N}{1 - \delta_N} \cdot \prod_{i=N+1}^{\infty} (1 + \lambda_i) \cdot \|\cdot\| \leqslant (1 + \varepsilon_N) \cdot \|\cdot\|$$

on X^N, for every $N \geqslant 0$; since these inequalities are obviously equivalent to

$$\Big| |||x||| - \|x\| \Big| \leqslant \varepsilon_N \|x\| \qquad (x \in X^N),$$

the proof is complete. ■

1.3 Polyhedral remarks

In this short section we present some improvements of our main result in the particular case of polyhedral Banach spaces and we prove Theorem 1.1.13. Let us first start by recalling the definition of a polyhedral Banach space. Finite dimensional polyhedral Banach spaces were introduced by Klee in his paper [Kle60]; their infinite-dimensional analogue [Lin66b] has been investigated in detail, also in the isomorphic sense, by several authors, most notably Fonf, [Fon78, Fon80, Fon81, Fon90]. Let us also refer to [FLP01, §6] for an introduction to the subject.

Definition 1.3.1. A finite-dimensional Banach space X is *polyhedral* if its unit ball is a polyhedron, *i.e.*, it is finite intersection of closed half-spaces. An infinite-dimensional Banach space X is *polyhedral* if its finite-dimensional subspaces are polyhedral.

In other words, a finite dimensional Banach space X is polyhedral if and only if it admits a finite boundary. It is also elementary to see that these conditions are equivalent to the polyhedrality of X^*, or to $\mathrm{Ext}(B_X)$ being a finite set, [HáJo14, §5.6].

As we already mentioned, it is proved in [DFH98] that if X is a separable polyhedral Banach space, then every equivalent norm on X can be approximated by a polyhedral norm (see Theorem 1.1 in [DFH98], where the approximation is stated in terms of closed, convex and bounded bodies).

In analogy with our main result, it is therefore natural to ask if this result can be improved in the sense that the approximation with asymptotic improvement is possible. It is not difficult to see that if we modify the argument of the previous section, by replacing the C^k-smooth norms $|||\cdot|||_{(s),n}$ with polyhedral norms $|||\cdot|||_{(p),n}$ and the C^∞-smooth functions φ_n with piecewise linear ones, the resulting norm $|||\cdot|||$ is still polyhedral. We thus have:

Proposition 1.3.2 ([HáRu17, Proposition 3.1])**.** *Let X be a polyhedral Banach space with a Schauder basis $(e_i)_{i=1}^\infty$. Then every equivalent norm $\|\cdot\|$ on X can be approximated with asymptotic improvement by polyhedral norms.*

Proof. We are going to follow the argument present in Section 1.2 with only two differences. First, after the norms $|||\cdot|||_n$ have been constructed, we consider their approximations with polyhedral norms $|||\cdot|||_{(p),n}$, instead of smooth ones (this is possible in light of Theorem 1.1 in [DFH98], that we mentioned above). Secondly, the functions φ_n used to glue together the various norms are chosen to be piecewise linear; in particular we shall use the functions

$$\varphi_n(t) := \max\left\{0, \frac{t+\delta_n-1}{\delta_n}\right\} \qquad (t \geqslant 0).$$

When we glue together the norms $|||\cdot|||_{(p),n}$, as in the previous section, we already know from the same arguments that the function Φ is locally expressed by a finite sum on the set $\{\Phi < 2\}$. Moreover, the Minkowski functional $|||\cdot|||$ of $\{\Phi \leqslant 1\}$ is a norm that approximates with asymptotic improvement the norm $\|\cdot\|$. Consequently, the argument is complete, if we can show that $|||\cdot|||$ is polyhedral.

Let us therefore fix a finite-dimensional subspace E of X; we need to check that $B_{(E,|||\cdot|||)} = E \cap \{\Phi \leqslant 1\}$ is a polyhedron. First, we note that since Φ is locally finite on $\{\Phi < 2\}$ and $E \cap \{\Phi \leqslant 3/2\}$ is compact (let us recall that Φ is lower semi-continuous), then Φ is expressed by a finite sum on the whole $E \cap \{\Phi \leqslant 3/2\}$; so let us fix $N \in \mathbb{N}$ such that

$$\Phi(x) = \sum_{n=0}^{N} \varphi_n\left(|||x|||_{(p),n}\right) \qquad (x \in E, \Phi(x) \leqslant 3/2).$$

Moreover, the norms $|||\cdot|||_{(p),n}$ are polyhedral, whence there are functionals $\left\{f_i^{(n)}\right\}_{i\in I_n}$, where I_n is a finite set, such that

$$|||\cdot|||_{(p),n} = \max_{i\in I_n} f_i^{(n)}(\cdot) \qquad \text{on } E.$$

For every $x \in E$ with $\Phi(x) \leqslant 3/2$ we thus obtain

$$\Phi(x) = \sum_{n=0}^{N} \max\left\{0, \frac{|||x|||_{(p),n} + \delta_n - 1}{\delta_n}\right\}$$

$$= \sum_{n=0}^{N} \max\left\{0, \frac{\max_{i\in I_n} f_i^{(n)}(x) + \delta_n - 1}{\delta_n}\right\}$$

$$= \sum_{n=0}^{N} \max_{i\in I_n}\left\{0, \frac{f_i^{(n)}(x) + \delta_n - 1}{\delta_n}\right\}.$$

Let us now define affine continuous functions by $a_0^{(n)} = 0$ and $a_i^{(n)} = \frac{f_i^{(n)} + \delta_n - 1}{\delta_n}$ ($i \in I_n$, $n = 0, \ldots, N$); we thus have, for x as above,

$$\Phi(x) = \sum_{n=0}^{N} \max_{i\in I_n\cup\{0\}} a_i^{(n)}(x)$$

$$= \max_{\substack{i_j\in I_j\cup\{0\}\\ j=1,\ldots,N}} \sum_{n=0}^{N} a_{i_n}^{(n)}(x) = \max_{j\in J} A_j(x).$$

Here the functions A_j's have the form $\sum_{n=0}^{N} a_{i_n}^{(n)}$ for suitable indices $i_n \in I_n$, $n = 0, \ldots, N$; hence they are affine continuous functions. We deduce that for every $x \in E$ with $\Phi(x) \leqslant 3/2$ one has

$$|||x||| \leqslant 1 \iff \Phi(x) \leqslant 1 \iff A_j(x) \leqslant 1 \,\text{for}\, j \in J.$$

We readily conclude that $B_{(E,|||\cdot|||)}$ is a finite intersection of closed half-spaces; consequently, it is a polyhedron and the proof is concluded. ■

It is also known that on a separable polyhedral Banach space X, every equivalent norm can be approximated by a C^∞-smooth norm that depends locally on finitely many coordinates. This claim is a consequence of results from various papers, which were already mentioned before; let us record them here. We already mentioned Fonf's result [Fon80, Fon00] that every separable polyhedral Banach space admits a countable boundary. When combined with the argument in [Háj95] (*cf.* Theorem 1.1.34), we conclude that every polyhedral norm on a separable Banach space admits C^∞-smooth LFC approximations. The density of polyhedral norms in separable polyhedral Banach spaces [DFH98] then leads us to the conclusion.

By inspection of our argument it follows that if we use such approximations in our proof, the resulting C^∞-smooth norm $|||\cdot|||$ will also depend locally on finitely many coordinates. Therefore, we obtain:

Proposition 1.3.3 ([HáRu17, Proposition 3.2])**.** *Let X be a polyhedral Banach space with a Schauder basis $(e_i)_{i=1}^\infty$. Then every equivalent norm $\|\cdot\|$ on X can be approximated with asymptotic improvement by C^∞-smooth LFC norms.*

Proof. We argue as in the proof of the main theorem, with the unique difference that here the norms $|||\cdot|||_{(s),n}$ are selected to be C^∞-smooth and LFC (in particular, we use C^∞-smooth functions φ_n). We then know that the resulting norm $|||\cdot|||$ is C^∞-smooth and approximates with asymptotic improvement the norm $\|\cdot\|$. Finally, Φ is LFC on the set $\{\Phi < 2\}$, being locally a finite sum; the last clause of Lemma 1.1.22 then implies that $|||\cdot|||$ is LFC. ■

In conclusion of the chapter, we mention that we do not know whether our main result can be generalized replacing Schauder basis with Markushevich basis. The argument presented here is not directly applicable, since, for example, we have made use of the canonical projections on the basis and their uniform boundedness.

Chapter 2

Auerbach systems

The main object of the chapter is the investigation of a specific notion of systems of coordinates in some classes of non-separable Banach spaces, in particular the study of the existence of large Auerbach systems in WLD Banach space. The first three sections of the chapter are dedicated to such issues; it the first one, we shall review some relevant material from the literature and state our main results, whose proofs are postponed to the subsequent two sections. The second part of the chapter concerns some uncountable generalisations to a combinatorial lemma due to Vlastimil Pták. Though this result is not directly related to Auerbach systems, there are at least two reasons for its inclusion in the same chapter. The first one is that our interest in such result was motivated by its possible uses in the study of systems of coordinates, the second one is the common presence of combinatorial aspects and the need to consider additional set-theoretical axioms in both parts of the chapter.

2.1 Some systems of coordinates

The aim of this section is to review some material that we shall make extensive use of in the present chapter and that will also be used in some places in Chapter 4. In the first part, we shall review some basic information about systems of coordinates in (non-separable) Banach spaces, in particular biorthogonal systems and Markushevich bases; we shall also recall some properties of the class of weakly Lindelöf determined Banach spaces, that will play a crucial rôle in this chapter. The second part of the section comprises results on Auerbach systems and formally states our contributions in this area, to be discussed in detail in later sections. We conclude the section collecting in Section 2.1.2 some powerful combinatorial results, that will be important in many proofs.

Definition 2.1.1. A system $\{x_\gamma; x^*_\gamma\}_{\gamma\in\Gamma} \subseteq X \times X^*$ is a *biorthogonal system* if $\langle x^*_\alpha, x_\beta\rangle = \delta_{\alpha,\beta}$, whenever $\alpha, \beta \in \Gamma$. A biorthogonal system $\{x_\gamma; x^*_\gamma\}_{\gamma\in\Gamma}$ is a *Markushevich basis* (*M-basis*, for short) if

$$\overline{\mathrm{span}}\{x_\gamma\}_{\gamma\in\Gamma} = X \qquad \text{and} \qquad \overline{\mathrm{span}}^{w^*}\{x^*_\gamma\}_{\gamma\in\Gamma} = X^*.$$

In the context of separable Banach spaces, it is clear that every Schauder basis naturally induces an M-basis; one of the main advantages of this more general notion is the existence of an M-basis in every separable Banach space, a classical result due to Markushevich himself, [Mar43]. Among the various refinements of this result available in the literature, let us single out the following, particularly related to our purposes. A biorthogonal system $\{x_\gamma; x^*_\gamma\}_{\gamma\in\Gamma}$ is said to be λ-*bounded* ($\lambda \geqslant 1$) if $\|x_\gamma\| \cdot \|x^*_\gamma\| \leqslant \lambda$, for every $\gamma \in \Gamma$; it is said to be *bounded* in case it is λ-bounded, for some $\lambda \geqslant 1$. Pełczyński [Peł76] and Plichko [Pli77] independently proved that every separable Banach space admits, for every $\varepsilon > 0$, a $(1+\varepsilon)$-bounded M-basis (*cf.* [HMVZ08, Theorem 1.27]); one of the major problems in the area is whether ε can be taken to equal 0 in the result, *i.e.*, if every separable Banach space admits an Auerbach basis.

The situation becomes more complicated when passing to the non-separable setting. An elementary extension of Mazur's technique for constructing basic sequences allows to prove, in particular, that every Banach space X with w^*-$\operatorname{dens} X^* \geqslant \omega_1$ contains an uncountable biorthogonal system (we shall say a bit more on this point at the beginning of the proof of Theorem 2.2.3).

It is however consistent with ZFC that there exist non-separable Banach spaces that virtually contain no 'reasonable' coordinate systems, in the sense that they admit no uncountable biorthogonal systems. The first such example was obtained by Kunen in two unpublished notes [Kun75, Kun80a], under the assumption of the Continuum Hypothesis; such construction appeared later in Negrepontis' survey, [Neg84, §7]. Other published results, under $\clubsuit$ and $\diamondsuit$ respectively, are [Ost76, She85]. Let us also refer to [HMVZ08, Section 4.4] for a modification, suggested by Todorčević, of the argument in [Ost76].

As it turns out, the assumption of some additional set-theoretic axioms can not be avoided in the above results. In fact, in his fundamental work [Tod06], Todorčević has shown that Martin's Maximum (MM) allows for the existence of an uncountable biorthogonal system in every non-separable Banach space ([Tod06, Corollary 7]).

When passing to the existence problem for M-bases in non-separable Banach spaces, it is possible to give examples of classical Banach spaces that fail to admit any M-basis. As proved by Johnson [Joh70], one such example is ℓ_∞. The proof of this assertion is a nice application of the Cantor diagonal principle, and we shall sketch it below; prior to this, we need to recall one definition.

A Banach space X is a *Grothendieck space* if every w^* convergent sequence in X^* converges weakly. Every reflexive Banach space is an obvious example of a Grothendieck space; one more such example is ℓ_∞, according to a result of Grothendieck, [Gro53] (*cf.* [Die84, p. 103]).

Proposition 2.1.2 (Johnson, [Joh70])**.** *A Grothendieck space with an M-basis is reflexive. In particular, ℓ_∞ has no M-basis.*

Proof. Assume that X is a Grothendieck space with an M-basis $\{x_\gamma; x^*_\gamma\}_{\gamma\in\Gamma}$. Note that the subspace $Y := \overline{\operatorname{span}}\{x^*_\gamma\}_{\gamma\in\Gamma}$ of X^* is w^*-dense in X^* and, moreover, every element $y \in Y$

is countably supported by the set $\{x_\gamma\}_{\gamma\in\Gamma}$, in the sense that $\{\gamma \in \Gamma\colon \langle y, x_\gamma\rangle \neq 0\}$ is at most countable.

We shall now prove that Y is reflexive; by the classical Eberlein-Šmulyan theorem, this amounts to proving that the unit ball B_Y of Y is weakly sequentially compact. If $(y_n)_{n=1}^\infty$ is any sequence in B_Y, there is a countable subset N of Γ such that $\langle y_n, x_\gamma\rangle = 0$ for every $\gamma \in \Gamma\setminus N$ and $n \in \mathbb{N}$. The diagonal method allows us to obtain a subsequence $(y_{n_k})_{k=1}^\infty$ such that $(\langle y_{n_k}, x_\gamma\rangle)_{k=1}^\infty$ is a convergent sequence, for every $\gamma \in N$ and, consequently, $(y_{n_k})_{k=1}^\infty$ is a w^*-convergent sequence in X^* (according to the fact that $X = \overline{\operatorname{span}}\{x_\gamma\}_{\gamma\in\Gamma}$). The Grothendieck property of X yields that $(y_{n_k})_{k=1}^\infty$ is weakly convergent and its limit belongs to Y (as Y is weakly closed); therefore, Y is reflexive.

We may now deduce that B_Y is weakly compact and, *a fortiori*, w^*-compact in X^*; in view of the Banach–Dieudonné theorem (see, *e.g.*, [FHHMZ10, Theorem 3.92]), this implies that Y is w^* closed in X^*. Y being w^*-closed in X^*, we conclude that $X^* = Y$ is reflexive, and so is X. ■

We are now in position to recall some basic notions concerning weakly Lindelöf determined Banach spaces, a very important class of Banach spaces that can be characterised via M-bases with certain properties. The classical definition of those spaces, that we shall adhere to, however consists in requiring the dual unit ball, in the relative w^* topology, to be a Corson compact.

For the *support* of an element $x \in \mathbb{R}^\Gamma$ we understand the set $\operatorname{supp} x := \{\gamma \in \Gamma\colon x(\gamma) \neq 0\}$. We shall denote $\Sigma(\Gamma)$ the topological space consisting of all $x \in \mathbb{R}^\Gamma$ with countable support, endowed with the restriction of the product topology of $\mathbb{R}^\Gamma$. We shall refer to $\Sigma(\Gamma)$ as a Σ-*product*; clearly, $\Sigma(\Gamma)$ is a dense subset of $\mathbb{R}^\Gamma$.

Definition 2.1.3. A topological space K is a *Corson compact* whenever it is homeomorphic to a compact subset of $\Sigma(\Gamma)$, for some set Γ.

In other words, K is a *Corson compact* whenever it is homeomorphic to a compact subset C of the product space $[-1,1]^\Gamma$ for some set Γ, such that every element of C has only countably many non-zero coordinates.

Concerning stability properties of this class of compacta, closed subspaces of Corson compacta are evidently Corson. One more, non-trivial, stability property is that Hausdorff continuous images of Corson compacta are Corson; such a result was first proved in [MiRu77] and independently in [Gul77]; the proof may also be found in [KKL11, Theorem 19.12] or [Neg84, Theorem 6.26].

As very simple examples of Corson compacta, every compact metric space is easily seen to be homeomorphic to a subset of $[-1,1]^\omega$ (see, *e.g.*, [Kec95, Theorem 4.14]); therefore, every metrisable compact is Corson. It is easy to observe that these are the unique examples of separable Corson compact spaces; for later use, we shall give below a more precise statement of the result, whose formulation requires one more definition. A topological space (X,τ) is said to have *calibre* ω_1 if for every collection $(O_\alpha)_{\alpha<\omega_1}$ of non-empty open subsets of X, there exists an uncountable set $A \subseteq \omega_1$ such that

$$\bigcap_{\alpha \in A} O_\alpha \neq \emptyset.$$

An immediate verification shows that every separable topological space has calibre ω_1; the more precise statement we mentioned above includes the validity of the converse, for Corson compacta.

Fact 2.1.4. *For a Corson compact K, metrisability, separability and calibre ω_1 are equivalent properties.*

Proof. We only need to prove that calibre ω_1 implies the metrisability of K. Assume that $K \subseteq \Sigma(\Gamma)$ is a Corson compact, where Γ is chosen so that $\Gamma = \cup_{x \in K} \mathrm{supp}(x)$. As a consequence, the open sets $O_\gamma := \{x \in K : x(\gamma) \neq 0\}$ are non-empty. Since for every uncountable subset A of Γ we have $\cap_{\gamma \in A} O_\gamma = \emptyset$, the calibre ω_1 property of K implies that Γ is countable. Consequently, K is a subset of the metrisable $\mathbb{R}^\Gamma$. ■

We shall next give few examples of compact spaces that are not Corson; for this, we need to recall that a topological space X is *Fréchet–Urysohn* if for every subset A of T and every $x \in \overline{A}$ there exists a sequence in A that converges to x. An elementary verification ([Kal00b, Lemma 1.6]) yields that every Σ-product is Fréchet–Urysohn. It immediately follows that the topological spaces $[0, \omega_1]$ and $[-1, 1]^\Gamma$ (for Γ uncountable) are not Corson compacta (for the second example, note that the function constantly equal to 1 belongs to the closure of $\Sigma(\Gamma)$).

In order to give more examples of Corson compacta, let us recall that a compact space K is an *Eberlein compact* whenever it is homeomorphic to a compact subset of $c_0(\Gamma)$, in its weak topology. It clearly makes no difference to consider $c_0(\Gamma)$ endowed with the weak topology or the pointwise one, which readily shows that every Eberlein compact is Corson. In their celebrated paper [AmLi68], Amir and Lindenstrauss proved that every weakly compact subset of an arbitrary Banach space is Eberlein; they also proved ([AmLi68, Corollary 2]) that (B_{X^*}, w^*) is Eberlein, whenever X is WCG (*i.e.*, weakly compactly generated). As a particular case, that is also proved by a simple direct proof, $(B_{c_0(\Gamma)^*}, w^*)$ is Eberlein. Let us also recall, in passing, that every Eberlein compact is homeomorphic to a weakly compact subset of a reflexive Banach space, [DFJP74, Corollary 2]; that reflexivity can not, in general, be replaced by super-reflexivity is the main result of [BeSt76]. Let us also refer to [FHHMZ10, §13.3] for proofs of the above results and more on Eberlein compacta.

Definition 2.1.5. A Banach space X is *weakly Lindelöf determined* (hereinafter, *WLD*) if the dual ball B_{X^*} is a Corson compact in the relative w^*-topology.

From the above considerations about Eberlein compacta it follows immediately that every WCG Banach space is WLD; in particular, every reflexive space and every $c_0(\Gamma)$ space is an example of a WLD Banach space.

It is also immediate to verify that every closed subspace Y of a WLD Banach space X is WLD. In fact, if $i\colon Y \to X$ denotes the inclusion, we have $B_{Y^*} = i^*(B_{X^*})$; hence, B_{Y^*} is a Corson compact, from the w^*-w^*-continuity of i^*. An alternative proof of this claim may be found in [HMVZ08, Corollary 5.43]. It is also immediate to verify that the class of WLD Banach spaces is stable under renormings and quotient maps.

We shall now proceed to give a characterisation of WLD Banach spaces, in terms of M-bases ([Kal00a, Kal00b, VWZ94], *cf.* [HMVZ08, Theorems 5.37 and 5.51]). A functional $x^* \in X^*$ is *countably supported* by an M-basis $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$, or $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ *countably supports* x^*, if the support of x^*,

$$\operatorname{supp} x^* := \{\gamma \in \Gamma\colon \langle x^*, x_\gamma\rangle \neq 0\}$$

is a countable subset of Γ. We shall also say that the M-basis is *countably* 1-*norming* if the set of countably supported functionals is 1-norming for X, *i.e.*, every $x \in X$ satisfies

$$\|x\| = \sup\{|\langle x^*, x\rangle|\colon x^* \in B_{X^*} \text{ is countably supported}\}.$$

Theorem 2.1.6. *Let X be a Banach space. Then the following assertions are equivalent:*

(i) *X is WLD;*

(ii) *X admits an M-basis $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ that countably supports X^*, i.e., every $x^* \in X^*$ is countably supported by $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$;*

(iii) *X admits, under any equivalent norm, a countably 1-norming M-basis.*

In this case, every M-basis countably supports X^*.

As an obvious example concerning the result, note that the canonical M-basis in $c_0(\Gamma)$ countably supports $\ell_1(\Gamma)$, which gives an alternative proof that $c_0(\Gamma)$ is WLD. On the other hand, the canonical M-basis of $\ell_1(\Gamma)$ plainly does not countably support $\ell_\infty(\Gamma)$, but it is countably 1-norming. Since the space $\ell_1(\Gamma)$, for Γ uncountable, is not even WCG, in light of its Schur property, we conclude that the existence of a countably 1-norming M-basis in the original norm is strictly weaker than (iii). Let us also notice that the implication (ii) $\Longrightarrow$ (iii) is obvious; that (ii) $\Longrightarrow$ (i) is also immediate to verify.

An immediate consequence of the theorem is the fact that $\operatorname{dens} X = w^*\text{-}\operatorname{dens} X^*$, whenever X is WLD. Indeed, fix an M-basis $\{x_\gamma; x_\gamma^*\}_{\gamma\in\Gamma}$ for X. If $\{\varphi_\alpha\}_{\alpha\in A}$ is w^* dense in X^*, then $\cup_{\alpha\in A}\operatorname{supp}\varphi_\alpha = \Gamma$; since the φ_α's are countably supported, it follows that $\operatorname{dens} X \leqslant |\Gamma| \leqslant |A|$, and we are done.

Remark 2.1.7. We conclude with one comment concerning long basic sequences in WLD spaces, to be used in a later proof; for the notion of a long basic sequence we refer, *e.g.*, to [HMVZ08, Section 4.1]. Let X be a WLD Banach space and $(e_\alpha)_{\alpha<\lambda}$ be a long basic sequence. Then $(e_\alpha)_{\alpha<\lambda}$ is a long Schauder basis of its closed linear span Y, a WLD Banach space. Since $(e_\alpha)_{\alpha<\lambda}$ is in particular an M-basis for Y, it follows from the previous theorem that it countably supports Y^*. As a consequence of this, we obtain that for every $\varphi \in X^*$ the set $\{\alpha < \lambda\colon \langle\varphi, e_\alpha\rangle \neq 0\}$ is at most countable.

For a more detailed presentation of these notions, the reader may wish to consult, *e.g.*, [DGZ93, §VI.7], [FHHMZ10, §14.5], [HMVZ08, §3.4, §5.4] [KKL11, §19.8], [Kal00b], [Ziz03] and the references therein. Let us just add here that the class of WLD Banach spaces has been introduced by Valdivia in the paper [Val88] and later given its current name after the extensive study in [ArMe93]. For additional information on Markushevich bases is some classes of non-separable Banach spaces we also refer to the very recent paper [Kal••].

2.1.1 Auerbach systems

We shall now turn our attention to the notion of Auerbach systems and Auerbach bases; these are biorthogonal systems with optimal boundedness properties, formally defined as follows.

Definition 2.1.8. An *Auerbach system* is a biorthogonal system $\{x_\gamma; x^*_\gamma\}_{\gamma\in\Gamma}$ with the property that $\|x_\gamma\| = \|x^*_\gamma\| = 1$, for $\gamma \in \Gamma$. An Auerbach system that is also an M-basis is called an *Auerbach basis*.

Such a notion originated from Auerbach's paper [Aue35], where it is shown that every finite-dimensional normed space admits an Auerbach basis. This well-known result, nowadays known as Auerbach lemma, may be found in most textbooks in Functional Analysis, and we shall refer, for example, to [LiTz77, Proposition 1.c.3]. Its proof consists in an extremal argument, based on maximizing the determinant function over vectors from the unit ball, or, in other words, in finding a parallelepiped of maximal volume generated by vectors from the unit ball.

It was recently understood by Weber and Wojciechowski [WeWo17] that critical points of the determinant function are sufficient for finding an Auerbach system; this allowed the authors to prove a conjecture, due to Pełczyński, concerning the number of distinct Auerbach bases in an n-dimensional Banach space. We refer to the paper [WeWo17] for a precise statement of the result—that, interestingly, depends on deep results from algebraic topology, such as Lyusternik–Schnirelmann category and Morse theory.

In the infinite-dimensional setting, the first general result in the positive direction is perhaps due to Day, who proved in [Day62] that every infinite-dimensional Banach space contains a closed infinite-dimensional subspace with an Auerbach basis. More precisely, the existence of an Auerbach system $\{x_n; x^*_n\}_{n\in\mathbb{N}}$ such that $(x_n)_{n=1}^\infty$ is a basic sequence is proved. Moreover, for every $\varepsilon > 0$, the sequence $(x_n)_{n=1}^\infty$ may be chosen to have basis constant at most $1+\varepsilon$. Let us mention that non-trivial topological considerations are involved in this argument too, as the proof passes through the use of Borsuk–Ulam antipodal theorem. Day's result was later generalised by Davis and Johnson in [DaJo73, Lemma 1] (also see [HMVZ08, Lemma 1.25] for the proof of this result); one more generalisation, in a different direction, appears in [CGJ92, §2, §3].

As regards Auerbach bases in the infinite-dimensional context, we already mentioned the open problem whether every infinite-dimensional separable Banach space admits an Auerbach basis; in particular, their existence is unknown for the Banach space $C[0,1]$, *cf.*

[GMZ16, Problem 103]. There are, on the other hand, several examples of classical Banach spaces that are known to admit Auerbach bases. Aside from the obvious examples of the classical sequence spaces c_0 and ℓ_p ($1 \leqslant p < \infty$), let us refer to the recent paper [Bog••], where an Auerbach basis is constructed in every finite-codimensional closed subspace of the space c_0; the case of the general (infinite-dimensional and infinite-codimensional, closed) subspace of c_0 is apparently an open problem. In the same paper it is also proved that every $C(K)$ space, where K is a countable compact, admits an Auerbach basis; as Wojciechowski pointed out to us, it is not clear whether the argument can be extended to cover the case of every scattered compact, or at least every $[0, \alpha]$ space (where α is an ordinal number).

The results discussed above imply *a fortiori* the existence of non-separable Banach spaces that admit no Auerbach bases, one example being the space ℓ_∞. On the other hand, one may wonder whether every Banach space with an M-basis also admits an Auerbach basis. A partial support for this conjecture may be found in Plichko's claim [Pli82] that every Banach space with an M-basis also admits a bounded M-basis; the proof given in [Pli82] actually proves the claim only for strong M-bases, the general case being treated in the later paper [HáMo10]. However, Plichko himself [Pli86] offered an example of a WCG Banach space with no Auerbach basis, in particular showing that every M-basis $\{x_\gamma; x^*_\gamma\}_{\gamma\in\Gamma}$ on such space satisfies

$$\sup_{\gamma\in\Gamma} \|x_\gamma\| \cdot \|x^*_\gamma\| \geqslant 2.$$

We shall now present Plichko's example.

Example 2.1.9. There exists a WCG Banach space with no λ-bounded M-basis, for $\lambda < 2$.

Proof. Let us consider the normed space

$$X := c_0[0,1] + C[0,1] = \{y + f \in \ell_\infty[0,1] \colon y \in c_0[0,1],\ f \in C[0,1]\},$$

with the canonical sup norm $\|\cdot\|$ inherited from ℓ_∞. It is immediate to verify that the decomposition of $x \in X$ as $x = y + f$, with $y \in c_0[0,1]$ and $f \in C[0,1]$, is unique and that in this case $\|f\| \leqslant \|y + f\|$. Consequently, X is isomorphic to $c_0[0,1] \oplus C[0,1]$ and it is therefore WCG. The point in the proof that explains the factor 2 is the immediate observation that, for $y \in c_0[0,1]$, one has

$$\inf_{f\in C[0,1]} \|y + f\| = \frac{1}{2}\|y\|.$$

Let now $\{x_\gamma; x^*_\gamma\}_{\gamma\in\Gamma}$ be any M-basis on X and select a sequence $(\varphi_n)_{n=1}^\infty$ in X^* that separates points on $C[0,1]$ and with the property that every φ_n vanishes on $c_0[0,1]$. Since $\{x_\gamma; x^*_\gamma\}_{\gamma\in\Gamma}$ countably supports X^*, there exists a countable subset J of Γ such that $\langle \varphi_n, x_\gamma\rangle = 0$, whenever $\gamma \notin J$ and $n \in \mathbb{N}$. Up to enlarging the set J, we may additionally assume that $C[0,1] \subseteq \overline{\operatorname{span}}\{x_\gamma\}_{\gamma\in J}$.

If we select any $\alpha \notin J$, then $\langle \varphi_n, x_\alpha\rangle = 0$ ($n \in \mathbb{N}$) and the fact that $(\varphi_n)_{n=1}^\infty$ separates points on $C[0,1]$ imply $x_\alpha \in c_0[0,1]$; moreover, x^*_α vanishes on $\overline{\operatorname{span}}\{x_\gamma\}_{\gamma\in J}$, whence

$C[0,1] \subseteq \ker x_\alpha^*$. Therefore, we conclude that

$$\|x_\alpha\| \cdot \|x_\alpha^*\| = \frac{\|x_\alpha\|}{\operatorname{dist}(x_\alpha, \ker x_\alpha^*)} \geqslant \frac{\|x_\alpha\|}{\operatorname{dist}(x_\alpha, C[0,1])} = 2.$$

■

In the same spirit as the above conjecture, one may hope that the existence of a better system of coordinates, for instance a long Schauder basis, may be sufficient for the existence of an Auerbach basis. This conjecture is once more false, as Godun proved the existence of a non-separable Banach space with an unconditional basis, but with no Auerbach bases, [God85, Theorem 2]. The same author [God90] later offered an explicit example of such a Banach space, showing that the Banach space $\ell_1([0,1])$ admits a renorming with no Auerbach bases. This result was then generalised by Godun, Lin, and Troyanski ([GLT93], also see [HMVZ08, Section 4.6]), who proved that if X is a non-separable Banach space such that B_{X^*} is w^*-separable, then there exists an equivalent norm $|||\cdot|||$ on X such that $(X, |||\cdot|||)$ admits no Auerbach basis. We shall also refer to [BDHMP05, §4.3] for some considerations on the same lines as these presented above.

These results show, in particular, that the existence of rather good systems of coordinates, like M-bases in WCG spaces or even unconditional bases, is not a sufficient assumption for a (non-separable) Banach space to admit an Auerbach basis. One may try to make one step further and investigate the existence of uncountable Auerbach systems in spaces with such 'good' systems of coordinates. Indeed, the results in [God85, God90, GLT93] motivated the authors of [GMZ16] to pose the question of whether there exists a Banach space with unconditional basis and whose no non-separable subspace admits an Auerbach basis ([GMZ16, Problem 294]).

One of the main results of the present chapter will be a solution to a stronger version of the present question, in the following sense: we shall give (under the assumption of the Continuum Hypothesis) an example of a non-separable Banach space with unconditional basis, but with no uncountable Auerbach system, in particular no non-separable subspace admits an Auerbach basis.

This statement naturally suggests the investigation of sufficient conditions for a Banach space to contain subspaces with Auerbach bases and of as large as possible density character. Therefore, a substantial part of the present chapter comprises quite general results concerning the existence of Auerbach systems in Banach spaces. To wit, we prove that a sufficiently large Banach space contains a large subspace with Auerbach basis, which is probably of interest on its own. We next improve and optimise this assertion, in the case of a WLD Banach space. Let us now give the formal statement of our result.

Theorem 2.1.10 ([HKR••]). (i) *Assume that κ is a cardinal number with $\kappa \geqslant \mathfrak{c}$. Then every Banach space X such that w^*-$\operatorname{dens} X^* > \exp_2 \kappa$ contains a subspace Y with Auerbach basis and such that $\operatorname{dens} Y = \kappa^+$.*

(ii) *Every WLD Banach space with $\operatorname{dens} X > \omega_1$ contains a subspace Y with an Auerbach basis and such that $\operatorname{dens} Y = \operatorname{dens} X$.*

Clause (ii) of the above theorem is, of course, true also in the case that $\operatorname{dens} X = \omega$, and it just reduces to Day's theorem quoted above. This naturally suggests considering the missing case that $\operatorname{dens} X = \omega_1$ and our second main result in this chapter addresses this issue by proving the following result.

Theorem 2.1.11 ([HKR••]). (CH) *There exists a renorming $|||\cdot|||$ of the space $c_0(\omega_1)$ such that the space $(c_0(\omega_1), |||\cdot|||)$ contains no uncountable Auerbach systems.*

Let us add a few comments comparing this result with the results that we discussed above. Plainly, the class of WLD Banach spaces is stable under renormings and $c_0(\omega_1)$ belongs to such class; as a consequence, Theorem 2.1.10(ii) can not be extended to the case that $\operatorname{dens} X = \omega_1$, since that assertion is consistently false in the case that $\operatorname{dens} X = \omega_1$. Let us mention here that we do not know whether this claim can be proved to be false in ZFC or if it may hold true, under some additional set theoretic axioms.

Moreover, $c_0(\omega_1)$ evidently admits an unconditional basis and therefore Theorem 2.1.11 provides a negative answer to [GMZ16, Problem 294], at least under the assumption of the Continuum Hypothesis. As we already mentioned, we actually prove a stronger result: not only our example has the additional property to be WLD, even a renorming of the space $c_0(\omega_1)$, but we are able to obtain the stronger conclusion that no uncountable Auerbach system exists.

In a sense, we may also view Theorem 2.1.11 as the counterpart to Kunen's result, for the class of WLD Banach spaces. Obviously, every non-separable WLD Banach space admits a biorthogonal system with maximal possible cardinality (any bounded M-basis is a witness of this), so there is no Kunen type example in the context of WLD spaces.

To conclude this part, let us mention that we shall prove Theorem 2.1.10 in Section 2.2, while Section 2.3 is dedicated to the proof of Theorem 2.1.11. More accurate references to the corresponding results in [HKR••] will also be given when we present the proofs of the results.

2.1.2 Infinitary combinatorics

In subsequent sections of the present chapter and in several places in Chapter 4 we shall need to exploit some results concerning infinitary combinatorics, whose statements are recalled here for convenience of the reader. More information on different aspects of this area may be found, *e.g.*, in [EHMR84, Hal17, Jec03, JuWe97, Kun80b, Wil77].

Lemma 2.1.12 (Δ-system lemma). *Consider a family $\mathcal{F} = \{F_\gamma\}_{\gamma\in\Gamma}$ of finite subsets of a set S, where $|\Gamma|$ is an uncountable regular cardinal number. Then there exist a subset Γ_0 of Γ with $|\Gamma_0| = |\Gamma|$ and a finite subset Δ of S such that*

$$F_\gamma \cap F_{\gamma'} = \Delta,$$

whenever $\gamma, \gamma' \in \Gamma_0$, $\gamma \neq \gamma'$.

This result, originally due to Shanin [Sha46], is more frequently stated in a slightly different way, *i.e.*, involving a set $\mathcal{F}$, whose cardinality is a regular cardinal number; we prefer this (equivalent) formulation, since it will apply more directly to our considerations. The difference relies in the fact that, for $\gamma \neq \gamma'$, the sets F_γ and $F_{\gamma'}$ are not necessarily distinct, so the cardinality of the set $\mathcal{F}$ may be smaller than $|\Gamma|$. We refer, *e.g.*, to [Kun80b, Lemma III.2.6] for the more usual statement of the result and to the remarks following the proof of Theorem III.2.8 for a comparison between the two formulations.

The second result that we shall use concerns partition properties for cardinal numbers and it is the counterpart for larger cardinals to the classical Ramsey's theorem. For the proof of the result to be presented and for a more complete discussion over partition properties, we refer, *e.g.*, to [HMVZ08, Theorem 5.67], [Jec03, Section 9.1], [Kun80b, pp. 237–238], or the monograph [EHMR84]. Before we state the result, we require a piece of notation.

For a cardinal number κ, one defines the iterated powers by $\exp_1 \kappa := 2^\kappa$ and then recursively $\exp_{n+1} \kappa := \exp(\exp_n \kappa)$ $(n \in \mathbb{N})$. Moreover, we shall denote by κ^+ the successor of κ, that is, the smallest cardinal number that is strictly greater than κ. If S is a set and κ is a cardinal number, then we denote by $[S]^\kappa$ the set of all subsets of S of cardinality κ, *i.e.*,

$$[S]^\kappa := \{A \subseteq S \colon |A| = \kappa\}.$$

We also need to recall the arrow notation: assume that κ, λ, and σ are cardinal numbers and n is a natural number. Then the symbol

$$\kappa \to (\lambda)^n_\sigma$$

abbreviates the following partition property: for every function $f \colon [\kappa]^n \to \sigma$ there exists a set $Z \subseteq \kappa$ with $|Z| = \lambda$ such that f is constant on $[Z]^n$; in this case, we say that Z is *homogeneous* for f. With a more suggestive notation, the function f is sometimes called a σ*-colouring* of $[\kappa]^n$ and, accordingly, the set Z is also said to be *monochromatic.*

Once this notation has been set forth, the classical Ramsey theorem [Ram29] can be shortly stated as the validity of the partition property $\omega \to (\omega)^k_n$, for every $n, k \in \mathbb{N}$. Its proof may be found in the references above or in the survey article [Gow03], where several applications of Ramsey theory to Banach space are to be found. Its non-separable counterpart appeared, together with the arrow notation, in the paper [ErRa56].

Theorem 2.1.13 (Erdős–Rado theorem)**.** *For every infinite cardinal* κ *and every* $n \in \mathbb{N}$

$$(\exp_n \kappa)^+ \to (\kappa^+)^{n+1}_\kappa.$$

The last result of combinatorial nature recorded in this section is Hajnal's theorem on free sets. Given a set S, by a *set function on* S we understand a function $f \colon S \to 2^S$. A subset H of S is a *free set for* f if $f(x) \cap H \subseteq \{x\}$ whenever $x \in H$. One may equivalently require, and such approach is frequently followed, that the function f satisfies $x \notin f(x)$ $(x \in S)$, in which case H is a free set for f if it is disjoint from $f(x)$, for every $x \in H$.

The natural question which arises is to find sufficient conditions on $f\colon \kappa \to 2^\kappa$, where κ is a cardinal number, for the existence of as large as possible free sets. It is clear that the mere assumption the cardinality of $f(x)$ to be less than κ, for $x \in \kappa$, does not even ensure the existence of a two element free set. This is simply witnessed by the function $f(\lambda) := \lambda$ $(= \{\alpha\colon \alpha < \lambda\})$ $(\lambda < \kappa)$. On the other hand, the existence of $\lambda < \kappa$ such that $|f(x)| < \lambda$ for $x \in \kappa$ turns out to be sufficient for the existence of free sets of the maximal possible cardinality. This result was first conjectured by Ruziewicz [Ruz36] and finally proved by Hajnal [Haj61]; partial previous results were obtained in [Láz36, Pic37a, Pic37b, Sie37, Erd50, BrEr51, Fod52].

Let us now formally state Hajnal's theorem; we shall refer, *e.g.*, to [EHMR84, §44] or [Wil77, §3.1] for the proof of the result and for further information on the subject.

Theorem 2.1.14 (Hajnal)**.** *Let λ and κ be cardinal numbers with $\lambda < \kappa$ and κ infinite. Then for every function $f\colon \kappa \to [\kappa]^{<\lambda}$ there exists a set of cardinality κ that is free for f.*

2.2 Existence of Auerbach systems

Our main goal for the present section is to prove some general results concerning the existence of uncountable Auerbach systems in some classes of Banach spaces; we also aim at finding as large as possible such systems. As a matter of fact, our arguments will actually produce subspaces with Auerbach bases, not merely Auerbach systems. In the first part of the section, we shall give results that only depend on density character assumptions, while in the second part, we specialise our results to WLD Banach spaces.

Theorem 2.2.1 ([HKR••, Theorem 3.1])**.** *Let $\kappa \geqslant \mathfrak{c}$ be a cardinal number and let X be a Banach space with w^*-$\operatorname{dens} X^* > \exp_2 \kappa$. Then X contains a subspace Y with Auerbach basis and such that $\operatorname{dens} Y = \kappa^+$.*

Proof. Let $\kappa \geqslant \mathfrak{c}$ be a cardinal number. Suppose that X is a Banach space with the property that $\lambda := w^*$-$\operatorname{dens} X^* > \exp_2 \kappa$. We may then find a long basic sequence $(e_\alpha)_{\alpha<\lambda}$ of unit vectors in X and a long sequence $(\varphi_{\alpha,\beta})_{\alpha<\beta<\lambda}$ of unit functionals in X^* with the following properties:

(i) $\varphi_{\alpha,\beta}$ is a norming functional for the molecule $e_\alpha - e_\beta$ for each $\alpha < \beta < \lambda$;

(ii) $e_\gamma \in \ker \varphi_{\alpha,\beta}$ for every $\alpha < \beta < \gamma < \lambda$.

The existence of such sequences is proved by a very simple transfinite induction argument, which is a minor modification over the Mazur technique, [HMVZ08, Corollary 4.11]. Assuming to have already constructed elements $(e_\alpha)_{\alpha<\gamma}$ and $(\varphi_{\alpha,\beta})_{\alpha<\beta<\gamma}$ satisfying the two properties above (for some $\gamma < \lambda$), the unique difference is that we additionally require $e_\gamma \in \cap_{\alpha<\beta<\gamma} \ker \varphi_{\alpha,\beta}$. This is indeed possible, as $|\{\varphi_{\alpha,\beta}\}_{\alpha<\beta<\gamma}| \leqslant |\gamma| < \lambda$, whence the family $\{\varphi_{\alpha,\beta}\}_{\alpha<\beta<\gamma}$ does not separate points on X. In order to conclude the inductive argument, it is then sufficient to choose, for each $\alpha < \gamma$, a norming functional $\varphi_{\alpha,\gamma}$ for the vector $e_\alpha - e_\gamma$.

We are now in position to invoke the Erdős–Rado theorem; let us consider the following colouring $c\colon [\lambda]^3 \to [-1,1]$ of $[\lambda]^3$. Given any $p \in [\lambda]^3$, we may uniquely write $p = \{\alpha, \beta, \gamma\}$ with $\alpha < \beta < \gamma$; we can therefore unambiguously set $c(p) := \langle \varphi_{\beta,\gamma}, e_\alpha \rangle \in [-1,1]$. According to the Erdős–Rado theorem, we have $(\exp_2 \kappa)^+ \to (\kappa^+)^3_\kappa$, whence from our assumptions we deduce *a fortiori* $\lambda \to (\kappa^+)^3_{\mathfrak{c}}$. Consequently, there exists a subset Λ of λ with $|\Lambda| = \kappa^+$ which is monochromatic for the colouring c. In other words, there exists $t \in [-1,1]$ such that $\langle \varphi_{\beta,\gamma}, e_\alpha \rangle = t$ for every triple $\alpha, \beta, \gamma \in \Lambda$ with $\alpha < \beta < \gamma$; of course, we immediately deduce that $e_\alpha - e_\beta \in \ker \varphi_{\gamma,\eta}$ whenever $\alpha, \beta, \gamma, \eta \in \Lambda$ satisfy $\alpha < \beta < \gamma < \eta$.

In order to conclude the argument, consider the unit vectors $u_{\alpha,\beta} := \frac{e_\alpha - e_\beta}{\|e_\alpha - e_\beta\|}$ ($\alpha, \beta \in \Lambda$, $\alpha < \beta$); those are actually well defined, since $e_\alpha \neq e_\beta$ for $\alpha \neq \beta$. From our argument above, we have $\langle \varphi_{\gamma,\eta}, u_{\alpha,\beta} \rangle = 0$ whenever $\alpha < \beta < \gamma < \eta$ are in Λ. Moreover, condition (i) clearly gives $\langle \varphi_{\alpha,\beta}, u_{\alpha,\beta} \rangle = 1$, while (ii) assures us that $\langle \varphi_{\alpha,\beta}, u_{\gamma,\eta} \rangle = 0$ ($\alpha, \beta, \gamma, \eta \in \Lambda$, $\alpha < \beta < \gamma < \eta$). At this stage it is, of course, immediate to construct an Auerbach system of length κ^+. To wit, find ordinal numbers $(\alpha_\theta)_{\theta < \kappa^+}$ and $(\beta_\theta)_{\theta < \kappa^+}$ in Λ such that:

(i) $\alpha_\theta < \beta_\theta < \alpha_\eta$ whenever $\theta < \eta < \kappa^+$;

(ii) $\Lambda = \{\alpha_\theta\}_{\theta < \kappa^+} \cup \{\beta_\theta\}_{\theta < \kappa^+}$.

Then, the unit vectors $u_\theta := u_{\alpha_\theta, \beta_\theta}$ ($\theta < \kappa^+$) with the corresponding biorthogonal functionals $\varphi_\theta := \varphi_{\alpha_\theta, \beta_\theta}$ ($\theta < \kappa^+$) clearly constitute an Auerbach system. Finally, $\{u_\theta; \varphi_\theta\}_{\theta < \kappa^+}$ is an M-basis for the subspace $Y := \overline{\mathrm{span}}\{u_\theta\}_{\theta < \kappa^+}$, which concludes the proof. ■

It is a standard fact that the weak* density of the dual of X, w^*-dens X^* does not exceed some cardinal number λ if and only if there exists a linear continuous injection of X into $\ell_\infty(\lambda)$. Consequently, when $w^*\text{-dens}\, X^* \leqslant \lambda$ we deduce that $\operatorname{dens} X \leqslant |\ell_\infty(\lambda)| = \exp \lambda$, or, in other words, $\operatorname{dens} X \leqslant \exp(w^*\text{-dens}\, X^*)$. Combining this inequality with the content of the previous theorem leads us to the following corollary.

Corollary 2.2.2. *Let $\kappa \geqslant \mathfrak{c}$ be a cardinal number. Suppose that X is a Banach space with* $\operatorname{dens} X > \exp_3 \kappa$*. Then X contains a subspace Y with Auerbach basis and such that* $\operatorname{dens} Y = \kappa^+$*.*

In the case where X is a WLD Banach space, we can improve the previous result and obtain, under some cardinality assumptions on $\operatorname{dens} X$, the existence of subspaces with Auerbach basis and of the maximal possible density character, namely equal to the density character of the Banach space X. We also point out that the restrictions on $\operatorname{dens} X$ are in fact necessary, in view of the main result of Section 2.3.

Theorem 2.2.3 ([HKR••, Theorem 3.3])**.** *Suppose that X is a WLD Banach space with* $\operatorname{dens} X > \omega_1$ *and let κ be a regular cardinal number such that* $\omega_1 < \kappa \leqslant \operatorname{dens} X$*. Then X contains an Auerbach system with cardinality κ.*

Proof. It is easy to construct in X a normalised, monotone long Schauder basic sequence $(e_\alpha)_{\alpha < \kappa}$ of length k. This is achieved via the uncountable analogue to the classical Mazur technique (*cf.* [HMVZ08, Corollary 4.11]) combined with the equally elementary fact (which we noted immediately after Theorem 2.1.6) that, X being WLD, $w^*\text{-dens}\, X^* =$

$\operatorname{dens} X$. Note that we may actually obtain one such sequence with length $\operatorname{dens} X$, but we shall not need it in what follows. Also observe that, as pointed out in Remark 2.1.7, for every $\varphi \in X^*$ the set $\{\alpha < \kappa \colon \langle \varphi, e_\alpha \rangle \neq 0\}$ is at most countable. We shall exploit this property at the very end of the argument.

Being a monotone long basic sequence, the family $(e_\alpha)_{\alpha<\kappa}$ is in particular *right-monotone*, in the sense that for every finite set $F \subseteq \kappa$ and for every $\beta < \kappa$ with $F < \beta$ [1] one has

$$\left\| \sum_{\alpha \in F} c_\alpha e_\alpha \right\| \leqslant \left\| \sum_{\alpha \in F} c_\alpha e_\alpha + c_\beta e_\beta \right\|,$$

for every choice of scalars $(c_\alpha)_{\alpha \in F \cup \{\beta\}}$. This condition is clearly equivalent to the requirement that, for every pair of finite sets $F, G \subseteq \kappa$ with $F < G$, one has

$$\left\| \sum_{\alpha \in F} c_\alpha e_\alpha \right\| \leqslant \left\| \sum_{\alpha \in F} c_\alpha e_\alpha + \sum_{\alpha \in G} c_\alpha e_\alpha \right\|.$$

Given a subfamily $(e_\alpha)_{\alpha\in\Gamma}$ of the basic sequence $(e_\alpha)_{\alpha<\kappa}$, we shall say that the family $(e_\alpha)_{\alpha\in\Gamma}$ is *left-monotone* if for every finite set $F \subseteq \Gamma$ and for every $\beta \in \Gamma$ with $F < \beta$ one has

$$\|c_\beta e_\beta\| \leqslant \left\| \sum_{\alpha \in F} c_\alpha e_\alpha + c_\beta e_\beta \right\|,$$

for every choice of scalars $(c_\alpha)_{\alpha \in F \cup \{\beta\}}$. We shall also say that a family is *bi-monotone* if it is both left-monotone and right-monotone. Since every subfamily of the right-monotone family $(e_\alpha)_{\alpha<\kappa}$ is right-monotone too, the left-monotonicity of $(e_\alpha)_{\alpha\in\Gamma}$ is equivalent to it being bi-monotone.

We are interested in such bi-monotone families, since it is immediate to construct an Auerbach system out of a bi-monotone family. Assume indeed that $(e_\alpha)_{\alpha\in\Gamma}$ is a bi-monotone family; the vectors $\{e_\alpha\}_{\alpha\in\Gamma}$ plainly constitute a linearly independent set, whence we can define linear functionals φ_α $(\alpha \in \Gamma)$ on $\operatorname{span}\{e_\alpha\}_{\alpha\in\Gamma}$ to be biorthogonal to the system $(e_\alpha)_{\alpha\in\Gamma}$. Moreover, for every finite set $F \subseteq \Gamma$ and for every $\beta \in F$, we may express $F = F_- \cup \{\beta\} \cup F_+$ with $F_- < \beta < F_+$. Consequently, an appeal to the left-monotonicity and then to the right-monotonicity yields us

$$\begin{aligned} \left| \left\langle \varphi_\beta, \sum_{\alpha \in F} c_\alpha e_\alpha \right\rangle \right| &= |c_\beta| = \|c_\beta e_\beta\| \leqslant \left\| \sum_{\alpha \in F_-} c_\alpha e_\alpha + c_\beta e_\beta \right\| \\ &\leqslant \left\| \sum_{\alpha \in F_-} c_\alpha e_\alpha + c_\beta e_\beta + \sum_{\alpha \in F_+} c_\alpha e_\alpha \right\| = \left\| \sum_{\alpha \in F} c_\alpha e_\alpha \right\|. \end{aligned}$$

[1] Let us remind the reader that this notation, like the analogous $F < G$ used below, was defined in the section Notation at the beginning of the dissertation.

Therefore, φ_β is a bounded linear functional of norm 1. By the Hahn–Banach theorem, we can extend it to a functional, still denoted φ_β, defined on the whole X; the system $\{e_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma}$ is then an Auerbach system. Moreover, $\{e_\alpha; \varphi_\alpha\}_{\alpha\in\Gamma}$ is clearly an M-basis for the subspace $Y := \overline{\operatorname{span}}\{e_\alpha\}_{\alpha\in\Gamma}$, which is then the desired subspace.

In other words, in order to prove the result it will be sufficient to find a left-monotone subfamily $(e_\alpha)_{\alpha\in\Gamma}$ of $(e_\alpha)_{\alpha<\kappa}$ such that $|\Gamma| = \kappa$. Therefore, we now turn our attention to the construction of such a family, and we start with some more notation.

Given subsets Γ_1 and Γ_2 of κ, we say that Γ_2 is an *extension* of Γ_1 if $\Gamma_2 = \Gamma_1 \cup G$, for some set $G \subseteq \kappa$ such that $G > \Gamma_1$, *i.e.*, we add some indices 'after the end' of Γ_1. In such a case, we also say that the family $(e_\alpha)_{\alpha\in\Gamma_2}$ is an *extension* of the family $(e_\alpha)_{\alpha\in\Gamma_1}$; clearly, such extension relation defines a partial ordering on the collection of all subfamilies of the family $(e_\alpha)_{\alpha<\kappa}$.

Let us now fix an element $\gamma < \kappa$ and observe that the family $\{e_\gamma\}$ is trivially bi-monotone. We can therefore consider the non-empty partially ordered set $\mathscr{P}$ consisting of all left-monotone extensions of the family $\{e_\gamma\}$, endowed with the partial order induced by the extension relation described above. It is immediate to verify that every chain in $\mathscr{P}$ admits an upper bound, given by the union of the elements of the chain; therefore an appeal to Zorn's lemma provides us with the existence of a maximal element in $\mathscr{P}$. In other words, there exists a subset $\Gamma(\gamma)$ of κ such that the corresponding family $(e_\alpha)_{\alpha\in\Gamma(\gamma)}$ is a maximal left-monotone extension of $\{e_\gamma\}$; in particular, every proper extension of $(e_\alpha)_{\alpha\in\Gamma(\gamma)}$ fails to be left-monotone. We also notice that, according to our definition of extendability, $\gamma = \min\Gamma(\gamma)$.

We are now in position to consider the following dichotomy: either $|\Gamma(\gamma)| < \kappa$ for every $\gamma < \kappa$, or there exists an ordinal $\tilde\gamma < \kappa$ with the property that $|\Gamma(\tilde\gamma)| = \kappa$. In the latter case, the family $(e_\alpha)_{\alpha\in\Gamma(\tilde\gamma)}$ is by definition a bi-monotone family of length κ; consequently, it induces an Auerbach system of the desired cardinality, as described above, and the proof is complete. Our plan is to show that the former case actually leads us to a contradiction, therefore only the latter case could occur, which in turn would conclude the proof.

Start with $\gamma_0 := 0$ and set $\Gamma_0 := \Gamma(\gamma_0)$. Since $|\Gamma_0| < \kappa$ and κ is regular, it is possible to select an ordinal $\gamma_1 < \kappa$ such that $\Gamma_0 < \gamma_1$; we may also set $\Gamma_1 := \Gamma(\gamma_1)$. We next proceed analogously by transfinite induction: assuming to have already found $(\gamma_\beta)_{\beta<\delta}$, with the corresponding set $\Gamma_\beta := \Gamma(\gamma_\beta)$, for some $\delta < \omega_1$, then $|\Gamma_\beta| < \kappa$ and the properties of κ imply $|\cup_{\beta<\delta}\Gamma_\beta| < \kappa$. Exploiting the regularity once again, we may thus find an ordinal $\gamma_\delta < \kappa$ such that $\cup_{\beta<\delta}\Gamma_\beta < \gamma_\delta$. Consequently, we have found an increasing ω_1-sequence $(\gamma_\beta)_{\beta<\omega_1}$, together with sets $\Gamma_\beta := \Gamma(\gamma_\beta)$, with the property that $\Gamma_\beta < \gamma_\delta$ whenever $\beta < \delta < \omega_1$. Moreover, the assumption that $\kappa > \omega_1$ allows us to conclude that we can find an ordinal $\overline\gamma < \kappa$ such that $\cup_{\beta<\omega_1}\Gamma_\beta < \overline\gamma$.

Consider now an arbitrarily fixed $\beta < \omega_1$; the family $(e_\alpha)_{\alpha\in\Gamma_\beta}$ is a left-monotone family which, by the very definition, admits no left-monotone extension. Since $\Gamma_\beta \cup \{\overline\gamma\}$ plainly is an extension of Γ_β, the family $(e_\alpha)_{\alpha\in\Gamma_\beta\cup\{\overline\gamma\}}$ fails to be left-monotone. The only way for this to be possible, from the left-monotonicity of $(e_\alpha)_{\alpha\in\Gamma_\beta}$, is that there are a finite set $F \subseteq \Gamma_\beta$

and scalars $(c_\alpha)_{\alpha\in\Gamma_\beta\cup\{\overline{\gamma}\}}$ such that

$$\left\|\sum_{\alpha\in F} c_\alpha e_\alpha + c_{\overline{\gamma}} e_{\overline{\gamma}}\right\| < |c_{\overline{\gamma}}|.$$

Up to a scaling, we can also assume that $c_{\overline{\gamma}} = 1$. Therefore, setting $b_\beta := \sum_{\alpha\in F} c_\alpha e_\alpha$, we have found (for every $\beta < \omega_1$) a block $b_\beta \in \operatorname{span}\{e_\alpha\}_{\alpha\in\Gamma_\beta}$ satisfying

$$\|b_\beta + e_{\overline{\gamma}}\| < 1.$$

The uncountable cofinality of ω_1 therefore provides us with an uncountable subset Ω of ω_1 and a real $\delta > 0$ such that $\|b_\beta + e_{\overline{\gamma}}\| \leqslant 1-\delta$, for every $\beta \in \Omega$. If b is any convex combination of the b_β's ($\beta \in \Omega$), then of course $\|b + e_{\overline{\gamma}}\| \leqslant 1-\delta$ too. Therefore, for every such b we have $\|b\| \geqslant \delta$ and, setting $C := \overline{\operatorname{conv}}\{b_\beta\}_{\beta\in\Omega}$, we readily conclude that $\operatorname{dist}(C, 0) \geqslant \delta$. Consequently, an appeal to the Hahn–Banach separation theorem furnishes us with a normalised functional $\varphi \in X^*$ such that $\langle \varphi, b\rangle \geqslant \delta$ for every $b \in C$, in particular $\langle \varphi, b_\beta\rangle \geqslant \delta$ for every $\beta \in \Omega$.

This is, however, in contradiction with the fact that φ is countably supported by the basic sequence $(e_\alpha)_{\alpha<\kappa}$. In fact, the vectors b_β ($\beta \in \Omega$) are evidently disjointly supported, whence only countably many of the supports of the b_β's can intersect the support of φ. Finally, if β is any index in Ω with the property that $\operatorname{supp}(b_\beta) \cap \operatorname{supp}(\varphi) = \emptyset$, then clearly $\langle \varphi, b_\beta\rangle = 0$ and this is in contradiction with the conclusion of the previous paragraph. This ultimately shows that the former possibility in the above dichotomy cannot indeed occur, thereby concluding the proof. ■

We shall now present an (unpublished) alternative approach to the above theorem, that allows us to obtain a much shorter proof of the existence of Auerbach systems, together with an improvement of the result. On the other hand, the argument to be presented depends on the rather heavy Hajnal's theorem (*cf.* Theorem 2.1.14), while the above proof was essentially self-contained.

Theorem 2.2.4. *Every WLD Banach space X with* $\operatorname{dens} X > \omega_1$ *contains a subspace Y with Auerbach basis and such that* $\operatorname{dens} Y = \operatorname{dens} X$.

Proof. Let us denote by $\kappa = \operatorname{dens} X$ and select an M-basis $\{e_\alpha; e^*_\alpha\}_{\alpha<\kappa}$ for X; we may assume that $\|e_\alpha\| = 1$ ($\alpha < \kappa$). We may also find, for each $\alpha < \kappa$, a functional $x^*_\alpha \in S_{X^*}$ such that $\langle x^*_\alpha, e_\alpha\rangle = 1$. According to the fact that $\{e_\alpha\}_{\alpha<\kappa}$ countably supports X^*, the sets

$$N_\alpha := \operatorname{supp} x^*_\alpha = \{\beta < \kappa \colon \langle x^*_\alpha, e_\beta\rangle \neq 0\} \qquad (\alpha < \kappa)$$

are at most countable. We may therefore apply Hajnal's theorem to the function $f\colon \kappa \to [\kappa]^{<\omega_1}$ defined by $\alpha \mapsto N_\alpha$; this yields the existence of a set H, with $|H| = \kappa$, that is free for f. Given distinct $\alpha, \beta \in H$, the condition $\beta \notin f(\alpha) \cap H$ translates to $\beta \notin N_\alpha$, *i.e.*, $\langle x^*_\alpha, e_\beta\rangle = 0$. Consequently, the system $\{e_\alpha; x^*_\alpha\}_{\alpha\in H}$ is biorthogonal, and we are done. ■

2.3 A renorming of $c_0(\omega_1)$ with no uncountable Auerbach systems

The main result of this section is the construction of a WLD Banach space with density character ω_1 that contains no uncountable Auerbach systems. More specifically, we wish to construct a renorming of the space $c_0(\omega_1)$ such that in this new norm the space fails to contain uncountable Auerbach systems. This is indeed possible, at least under the assumption of the Continuum Hypothesis. The formal statement of our result is as follows.

Theorem 2.3.1 ([HKR••, Theorem 5.1] (CH))**.** *There exists a renorming $|||\cdot|||$ of the space $c_0(\omega_1)$ such that the space $(c_0(\omega_1), |||\cdot|||)$ contains no uncountable Auerbach systems.*

The subsequent results present in this section (all from [HKR••, §5]) are entirely devoted to the construction of the desired norm and the verification of the asserted property. The section is therefore naturally divided into two parts: in the former one, we introduce a family of equivalent norms (depending on some parameters) on $c_0(\omega_1)$ and prove some properties of every such norm; in the latter, we prove that careful choice of the parameters implies that the resulting space contains no uncountable Auerbach system.

We shall denote by $\|\cdot\|_\infty$, or just $\|\cdot\|$ if no confusion may arise, the canonical norm on $c_0(\omega_1)$ and by $(e_\alpha)_{\alpha<\omega_1}$ its canonical long Schauder basis; the corresponding set of biorthogonal functionals, in $\ell_1(\omega_1) = c_0(\omega_1)^*$, will be denoted by $(e^*_\alpha)_{\alpha<\omega_1}$.

Fix a parameter $\delta > 0$ so small that $\Delta := \frac{\delta}{1-\delta} \leqslant 1/5$ (for example, we could choose $\delta = 1/6$). We also select an injective long sequence $(\lambda_\alpha)_{\alpha<\omega_1} \subseteq (0, \delta)$. Moreover, for every $\alpha < \omega_1$ there exists an enumeration σ_α of the set $[0, \alpha)$, *i.e.*, a bijection $\sigma_\alpha : |\alpha| \to \alpha$ (observing that $|\alpha|$ is either ω or a finite cardinal). We may therefore assume to have selected, for every $\alpha < \omega_1$, a fixed bijection σ_α. Having fixed such notation, we are now in position to define elements $\varphi_\alpha \in \ell_1(\omega_1)$ $(\alpha < \omega_1)$ as follows:

$$\varphi_\alpha(\eta) = \begin{cases} 1 & \text{if } \eta = \alpha \\ 0 & \text{if } \eta > \alpha \\ (\lambda_\alpha)^k & \text{if } \eta < \alpha,\ \eta = \sigma_\alpha(k). \end{cases}$$

The above enumerations may be chosen arbitrarily and the subsequent argument will not depend on any specific such choice. On the other hand, a substantial part of the argument to be presented will consist in explaining how to properly choose the coefficients λ_α.

We start with a few elementary properties of the functionals $(\varphi_\alpha)_{\alpha<\omega_1}$ and their use in the definition of the renorming. Plainly, $\|\varphi_\alpha - e^*_\alpha\|_1 = \sum_{k=1}^{|\alpha|}(\lambda_\alpha)^k \leqslant \sum_{k=1}^{\infty}\delta^k = \Delta$, whence it follows that $\varphi_\alpha \in c_0(\omega_1)^*$ and $\|\varphi_\alpha\|_1 \leqslant 1 + \Delta$. Moreover, the canonical basis of $\ell_1(\Gamma)$ is stable under more drastic perturbation than the ones allowed in the general Small Perturbation lemma (see, *e.g.*, [Jam74, Example 30.12]) and, in particular, the inequality $\|\varphi_\alpha - e^*_\alpha\|_1 \leqslant \Delta < 1$ is sufficient to imply that $(\varphi_\alpha)_{\alpha<\omega_1}$ is a Schauder basis for $\ell_1(\omega_1)$

Fact 2.3.2. *$(\varphi_\alpha)_{\alpha<\omega_1}$ is a long Schauder basis for $\ell_1(\omega_1)$, equivalent to the canonical Schauder basis $(e^*_\alpha)_{\alpha<\omega_1}$ of $\ell_1(\omega_1)$.*

Proof. From the inequality $\|\varphi_\alpha - e^*_\alpha\|_1 \leqslant \Delta$ observed above, it follows that

$$\left\|\sum_{i=1}^{n} d_i\varphi_{\alpha_i}\right\|_1 \geqslant \left\|\sum_{i=1}^{n} d_i e^*_{\alpha_i}\right\|_1 - \left\|\sum_{i=1}^{n} d_i(e^*_{\alpha_i} - \varphi_{\alpha_i})\right\|_1 \geqslant$$

$$\geqslant \sum_{i=1}^{n} |d_i| - \sum_{i=1}^{n} |d_i| \|\varphi_{\alpha_i} - e^*_{\alpha_i}\|_1 \geqslant (1-\Delta)\sum_{i=1}^{n} |d_i|$$

for every choice of scalars $(d_i)_{i=1}^n$. Combining this inequality with $\|\varphi_\alpha\|_1 \leqslant 1+\Delta$ results in

$$(1-\Delta)\sum_{\alpha<\omega_1} |d_\alpha| \leqslant \left\|\sum_{\alpha<\omega_1} d_\alpha\varphi_\alpha\right\|_1 \leqslant (1+\Delta)\sum_{\alpha<\omega_1} |d_\alpha|.$$

In order to prove that $\overline{\mathrm{span}}\{\varphi_\alpha\}_{\alpha<\omega_1} = \ell_1(\omega_1)$, we shall consider the bounded linear operator $T\colon \ell_1(\omega_1) \to \ell_1(\omega_1)$ such that $T(e^*_\alpha) = \varphi_\alpha$. The first part of the argument shows that T is, indeed, a bounded linear operator and that $\|T - I\| \leqslant \Delta < 1$. Consequently, T is an isomorphism of $\ell_1(\omega_1)$ into itself, whence the closed linear span of $(\varphi_\alpha)_{\alpha<\omega_1}$ equals $\ell_1(\omega_1)$. ■

We may now exploit the functionals $(\varphi_\alpha)_{\alpha<\omega_1}$ to define a renorming of $c_0(\omega_1)$.

Definition 2.3.3.

$$|||x||| := \sup_{\alpha<\omega_1} |\langle\varphi_\alpha, x\rangle| \qquad (x \in c_0(\omega_1)).$$

Moreover, we also denote by X the space $X := (c_0(\omega_1), |||\cdot|||)$.

Let us preliminarily note that if $\alpha < \beta$, then there exists $k \in \mathbb{N}$ such that $\alpha = \sigma_\beta(k)$ and consequently $|\langle\varphi_\beta, e_\alpha\rangle| = (\lambda_\beta)^k \leqslant 1$; it immediately follows that $|||e_\alpha||| = 1$. We may then readily check that $|||\cdot|||$ is a norm, equivalent to $\|\cdot\|_\infty$. In fact, the inequality $|||\cdot||| \leqslant (1+\Delta)\|\cdot\|_\infty$ is obvious and the lower estimate follows from a familiar pattern: if $\gamma < \omega_1$ is such that $\|x\|_\infty = |x(\gamma)|$, we then have

$$|||x||| = |||2x(\gamma)e_\gamma + x - 2x(\gamma)e_\gamma||| \geqslant 2|x(\gamma)| - |||x - 2x(\gamma)e_\gamma|||$$

$$\geqslant 2\|x\|_\infty - (1+\Delta)\|x - 2x(\gamma)e_\gamma\|_\infty = (1-\Delta)\|x\|_\infty.$$

We shall also denote by $|||\cdot|||$ the dual norm on $\ell_1(\omega_1)$; needless to say, such a norm satisfies $(1+\Delta)^{-1}\|\cdot\|_1 \leqslant |||\cdot||| \leqslant (1-\Delta)^{-1}\|\cdot\|_1$. The definition of $|||\cdot|||$ clearly implies that $|||\varphi_\alpha||| \leqslant 1$ and $\langle\varphi_\alpha, e_\alpha\rangle = 1$ actually forces $|||\varphi_\alpha||| = 1$.

By the very definition, $\{\varphi_\alpha\}_{\alpha<\omega_1}$ is a 1-norming set for X; we now note the important fact that such a collection of functionals is actually a boundary for X. A similar argument also shows that X is a polyhedral Banach space and we also record such information in the next lemma, even if we shall not need it in what follows.

Lemma 2.3.4. *$\{\varphi_\alpha\}_{\alpha<\omega_1}$ is a boundary for X. Moreover, X is a polyhedral Banach space.*

Proof. In the proof of the first assertion, by homogeneity, it is clearly sufficient to consider $x \in (c_0(\omega_1), |||\cdot|||)$ with $\|x\|_\infty = 1$; in particular, there is $\alpha < \omega_1$ with $|x(\alpha)| = 1$. For such an α we thus have:

$$|\langle\varphi_\alpha, x\rangle| \geqslant |\langle e^*_\alpha, x\rangle| - |\langle\varphi_\alpha - e^*_\alpha, x\rangle| \geqslant 1 - \|\varphi_\alpha - e^*_\alpha\|_1\|x\|_\infty \geqslant 1 - \Delta.$$

On the other hand, consider the finite set $N_x := \{\gamma < \omega_1 \colon |x(\gamma)| > \Delta\}$. Then for each $\alpha \notin N_x$ we obtain:

$$|\langle\varphi_\alpha, x\rangle| \leqslant |\langle e^*_\alpha, x\rangle| + |\langle\varphi_\alpha - e^*_\alpha, x\rangle| \leqslant |x(\alpha)| + \Delta \leqslant 2\Delta \leqslant 1 - \Delta.$$

Consequently, the supremum appearing in the definition of $|||x|||$ is actually over the finite set N_x and it is therefore attained.

We then turn to the polyhedrality of $(c_0(\omega_1), |||\cdot|||)$. Let E be any finite-dimensional subspace of $(c_0(\omega_1), |||\cdot|||)$ and let $x_1, \dots, x_n$ be a finite $\Delta/2$-net (relative to the $\|\cdot\|_\infty$ norm) for the set $\{x \in E \colon \|x\|_\infty = 1\}$. Let us consider the finite set $N_E := \cup_{i=1}^n N_{x_i}$, where $N_x := \{\gamma < \omega_1 \colon |x(\gamma)| > \Delta/2\}$; then for every $x \in E$ with $\|x\|_\infty = 1$ there clearly holds

$$\{\gamma < \omega_1 \colon |x(\gamma)| > \Delta\} \subseteq N_E.$$

The same calculations as before then demonstrate that for all such x we have

$$\sup_{\alpha<\omega_1} |\langle\varphi_\alpha, x\rangle| = \max_{\alpha\in N_E} |\langle\varphi_\alpha, x\rangle|.$$

Consequently, $|||\cdot||| = \max_{\alpha\in N_E} |\langle\varphi_\alpha, \cdot\rangle|$ on E and $\{\varphi_\alpha\}_{\alpha\in N_E}$ is a finite boundary for E, which is thus polyhedral. ■

We now turn to the first crucial result for what follows, namely the fact that every norm-attaining functional on X is finitely supported with respect to the basis $(\varphi_\alpha)_{\alpha<\omega_1}$.

Theorem 2.3.5. *Let $g \in S_{X^*}$ be a norm-attaining functional and let $u \in S_X$ be such that $\langle g, u\rangle = 1$. Also, denote by F the finite set $F := \{\alpha < \omega_1 \colon |u(\alpha)| > \frac{\Delta}{1-\Delta}\}$. Then*

$$g = \sum_{\alpha\in F} g_\alpha\varphi_\alpha \qquad \textit{and} \qquad \sum_{\alpha\in F} |g_\alpha| \leqslant 1.$$

Proof. We begin with two very simple remarks, that we shall use in the course of the argument. Combining the estimate in the proof of Fact 2.3.2 with $|||\varphi_\alpha||| = 1$, we readily deduce that

$$\frac{1-\Delta}{1+\Delta}\sum_{\alpha<\omega_1} |d_\alpha| \leqslant \left|\left|\left|\sum_{\alpha<\omega_1} d_\alpha\varphi_\alpha\right|\right|\right| \leqslant \sum_{\alpha<\omega_1} |d_\alpha|. \tag{2.3.1}$$

Moreover, we have

$$\overline{\mathrm{conv}}^{w^*}\{\pm\varphi_\alpha\}_{\alpha<\omega_1} = B_{X^*}. \tag{2.3.2}$$

This is an immediate consequence of the Hahn–Banach separation theorem and the fact that $\{\pm\varphi_\alpha\}_{\alpha<\omega_1}$ is 1-norming for X.

We now start with the argument: let $g \in S_{X^*}$ be a norm-attaining functional and choose $u \in S_X$ such that $\langle g, u\rangle = 1$. We may express g as $g = \sum_{\alpha<\omega_1} g_\alpha \varphi_\alpha$, and we also set $F := \{\alpha < \omega_1 \colon |u(\alpha)| > \frac{\Delta}{1-\Delta}\}$. Moreover, let us choose arbitrarily

$$f = \sum_{\alpha<\omega_1} d_\alpha \varphi_\alpha \in \operatorname{conv}\{\pm\varphi_\alpha\}_{\alpha<\omega_1}$$

(*i.e.*, only finitely many d_α's are non-zero and $\sum_{\alpha<\omega_1} |d_\alpha| \leqslant 1$).
Claim. If $\langle f, u\rangle \geqslant 1-\eta$, then

$$\sum_{\alpha\notin F} |d_\alpha| \leqslant 12\eta.$$

Proof of the Claim. Recall that $(1+\Delta)^{-1} \leqslant \|u\|_\infty \leqslant (1-\Delta)^{-1}$; hence for each $\alpha \notin F$ we have

$$|\langle\varphi_\alpha, u\rangle| \leqslant |\langle e^*_\alpha, u\rangle| + |\langle\varphi_\alpha - e^*_\alpha, u\rangle| \leqslant \frac{\Delta}{1-\Delta} + \Delta\|u\|_\infty \leqslant 2\frac{\Delta}{1-\Delta}.$$

Consequently, setting $\rho := \sum_{\alpha\notin F} |d_\alpha|$, we have

$$1-\eta \leqslant \langle f, u\rangle \leqslant \sum_{\alpha\notin F} |d_\alpha||\langle\varphi_\alpha, u\rangle| + \sum_{\alpha\in F} d_\alpha\langle\varphi_\alpha, u\rangle \leqslant 2\rho\frac{\Delta}{1-\Delta} + \sum_{\alpha\in F} d_\alpha\langle\varphi_\alpha, u\rangle,$$

whence

$$(\dagger) \qquad \sum_{\alpha\in F} d_\alpha\langle\varphi_\alpha, u\rangle \geqslant 1-\eta-2\rho\frac{\Delta}{1-\Delta}.$$

On the other hand, there exists $\overline{\alpha} < \omega_1$ with $|u(\overline{\alpha})| \geqslant (1+\Delta)^{-1}$ and for such $\overline{\alpha}$ we have

$$(\ddagger) \qquad |\langle\varphi_{\overline{\alpha}}, u\rangle| \geqslant |\langle e^*_{\overline{\alpha}}, u\rangle| - |\langle\varphi_{\overline{\alpha}} - e^*_{\overline{\alpha}}, u\rangle| \geqslant \frac{1}{1+\Delta} - \frac{\Delta}{1-\Delta}.$$

Let us now consider the functional

$$\psi := \sum_{\alpha\in F} d_\alpha\varphi_\alpha + \rho\cdot\operatorname{sgn}\langle\varphi_{\overline{\alpha}}, u\rangle\varphi_{\overline{\alpha}};$$

clearly, $\psi \in \operatorname{conv}\{\pm\varphi_\alpha\}_{\alpha<\omega_1}$, so $|||\psi||| \leqslant 1$. We can therefore combine this information with ($\dagger$) and ($\ddagger$) and conclude that

$$\begin{aligned} 1 \geqslant \langle\psi, u\rangle &= \sum_{\alpha\in F} d_\alpha\langle\varphi_\alpha, u\rangle + \rho\cdot|\langle\varphi_{\overline{\alpha}}, u\rangle| \\ &\geqslant 1-\eta-2\rho\frac{\Delta}{1-\Delta} + \rho\cdot\left(\frac{1}{1+\Delta} - \frac{\Delta}{1-\Delta}\right) \\ &= 1-\eta+\rho\cdot\left(\frac{1}{1+\Delta} - 3\frac{\Delta}{1-\Delta}\right) \geqslant 1-\eta+\frac{\rho}{12}; \end{aligned}$$

in the last inequality we used the fact that $\Delta \mapsto \left(\frac{1}{1+\Delta} - 3\frac{\Delta}{1-\Delta}\right)$ is a decreasing function on $(0,1)$ and the assumption that $\Delta \leqslant 1/5$. It follows that $\rho/12 \leqslant \eta$, whence the proof of the claim is concluded. ■

We now amalgamate those facts together. According to (2.3.2), we may find a net $(f_\tau)_{\tau\in I}$ in $\operatorname{conv}\{\pm\varphi_\alpha\}_{\alpha<\omega_1}$ such that $f_\tau \to g$ in the w^*- topology; the elements f_τ have the form $f_\tau = \sum_{\alpha<\omega_1} d_\tau^\alpha \varphi_\alpha$, where, for each $\tau \in I$, $\sum_{\alpha<\omega_1} |d_\tau^\alpha| \leqslant 1$ and only finitely many d_τ^α's are different from zero.

The net $\left(\sum_{\alpha\in F} d_\tau^\alpha \varphi_\alpha\right)_{\tau\in I}$ is plainly a bounded net in a finite-dimensional Banach space (for what concerns the boundedness, observe that (2.3.1) implies $\left|\!\left|\!\left|\sum_{\alpha\in F} d_\tau^\alpha \varphi_\alpha\right|\!\right|\!\right| \leqslant 1$). Hence, up to passing to a subnet, we may safely assume that it converges in $|\!|\!|\cdot|\!|\!|$, and *a fortiori* in the w^*-topology of X^*, to an element, say $\sum_{\alpha\in F} \tilde{d}_\alpha \varphi_\alpha$. The basis equivalence contained in the inequalities (2.3.1) now allows us to deduce in particular that $\sum_{\alpha\in F} |\tilde{d}_\alpha| \leqslant 1$.

As a consequence of this currently obtained norm convergence, we see that

$$\sum_{\alpha\notin F} d_\tau^\alpha \varphi_\alpha = f_\tau - \sum_{\alpha\in F} d_\tau^\alpha \varphi_\alpha \xrightarrow{w^*} g - \sum_{\alpha\in F} \tilde{d}_\alpha \varphi_\alpha = \sum_{\alpha\notin F} g_\alpha \varphi_\alpha + \sum_{\alpha\in F} \left(g_\alpha - \tilde{d}_\alpha\right) \varphi_\alpha$$

and our present goal is to estimate the $|\!|\!|\cdot|\!|\!|$-norm of the right hand side, making use of this w^* convergence and the above claim.

Let us fix temporarily $\eta > 0$; from the w^* convergence we obtain $\langle f_\tau, u\rangle \to \langle g, u\rangle = 1$, whence the existence of $\tau_0 \in I$ such that $\langle f_\tau, u\rangle \geqslant 1 - \eta$ for every $\tau \geqslant \tau_0$ [2]. Consequently, according to the above claim we deduce

$$\left|\!\left|\!\left|\sum_{\alpha\notin F} d_\tau^\alpha \varphi_\alpha\right|\!\right|\!\right| \leqslant \sum_{\alpha\notin F} |d_\tau^\alpha| \leqslant 12\eta \qquad (\tau \geqslant \tau_0).$$

This and the w^* lower semi-continuity of the $|\!|\!|\cdot|\!|\!|$-norm on X^* then reassure us that

$$\left|\!\left|\!\left|\sum_{\alpha\notin F} g_\alpha \varphi_\alpha + \sum_{\alpha\in F} \left(g_\alpha - \tilde{d}_\alpha\right) \varphi_\alpha\right|\!\right|\!\right| \leqslant 12\eta.$$

Since $\eta > 0$ was fixed arbitrarily, we may let $\eta \to 0^+$ in the above inequality and, also exploiting the fact that $(\varphi_\alpha)_{\alpha<\omega_1}$ is a Schauder basis, we conclude that $g_\alpha = 0$ whenever $\alpha \notin F$, while $g_\alpha = \tilde{d}_\alpha$ for every $\alpha \in F$. Therefore,

$$g = \sum_{\alpha<\omega_1} g_\alpha \varphi_\alpha = \sum_{\alpha\in F} g_\alpha \varphi_\alpha,$$

where $\sum_{\alpha\in F} |g_\alpha| = \sum_{\alpha\in F} |\tilde{d}_\alpha| \leqslant 1$. ■

Note that in the conclusion of the theorem we necessarily have $\sum_{\alpha\in F} |g_\alpha| = 1$, since $1 = |\!|\!|g|\!|\!| \leqslant \sum_{\alpha\in F} |g_\alpha|$. Consequently, if $g = \sum_{\alpha\in F} g_\alpha \varphi_\alpha$ is any norm-attaining functional we have $\left|\!\left|\!\left|\sum_{\alpha\in F} g_\alpha \varphi_\alpha\right|\!\right|\!\right| = \sum_{\alpha\in F} |g_\alpha|$. Since the set of norm-attaining functionals is dense in X^*, according to the Bishop–Phelps theorem, this equality holds true for every functional in X^*. We thus have the following immediate corollary.

[2] we are denoting by $\geqslant$ both the order on the real line and the preorder in the directed set I—this should cause no confusion, hopefully.

Corollary 2.3.6. *$(\varphi_\alpha)_{\alpha<\omega_1}$ is isometrically equivalent to the canonical basis of $\ell_1(\omega_1)$.*

In turn, this also implies (in the same notation as in the theorem)

$$\{\alpha < \omega_1 \colon g_\alpha \neq 0\} \subseteq \{\alpha < \omega_1 \colon |\langle \varphi_\alpha, u\rangle| = 1\}.$$

In fact, $1 = |||g||| = \sum_{\alpha\in F} |g_\alpha|$ and $1 = |||u||| \geqslant |\langle\varphi_\alpha, u\rangle|$ imply

$$1 = \langle g, u\rangle = \sum_{\alpha<\omega_1} g_\alpha \langle\varphi_\alpha, u\rangle \leqslant \sum_{\alpha<\omega_1} |g_\alpha| \cdot |\langle\varphi_\alpha, u\rangle| \leqslant \sum_{\alpha<\omega_1} |g_\alpha| = 1.$$

Consequently, all inequalities are, in fact, equalities and it follows that $|\langle\varphi_\alpha, u\rangle| = 1$ whenever $g_\alpha \neq 0$. It also follows that for every $\alpha < \omega_1$ we have $|g_\alpha| = g_\alpha \cdot \langle\varphi_\alpha, u\rangle$ and in particular $\langle\varphi_\alpha, u\rangle = \mathrm{sgn}(g_\alpha)$ whenever $g_\alpha \neq 0$.

As a piece of notation, it will be convenient to denote by $\mathrm{supp}(g)$ the set $\mathrm{supp}(g) := \{\alpha < \omega_1 \colon g_\alpha \neq 0\}$ for a functional $g = \sum_{\alpha<\omega_1} g_\alpha \varphi_\alpha \in X^*$.

We can now approach the investigation of uncountable Auerbach systems in the space X. Our last observation for this first part is the fact that if X contains an uncountable Auerbach system, then it also contains an uncountable Auerbach system such that the supports of the functionals are in a very specific position: either the supports are mutually disjoint and consecutive or the collection of the supports has an initial common root with cardinality 1, followed by consecutive blocks. As it is to be expected, the Δ-system lemma will play a prominent rôle in the proof; we shall also exploit again the same 'transfer of mass' principle already present in the proof of the claim in Theorem 2.3.5.

Lemma 2.3.7. *Assume that X contains an uncountable Auerbach system. Then X also contains an Auerbach system $\{\tilde{u}_\alpha; \tilde{g}_\alpha\}_{\alpha<\omega_1}$ such that one of the following two conditions is satisfied:*

1. *either there exists $\gamma < \omega_1$ with the properties that $\gamma = \min(\mathrm{supp}(\tilde{g}_\alpha))$ for every $\alpha < \omega_1$ and $\mathrm{supp}(\tilde{g}_\alpha) \setminus \{\gamma\} < \mathrm{supp}(\tilde{g}_\beta) \setminus \{\gamma\}$ for every $\alpha < \beta < \omega_1$;*
2. *or $\mathrm{supp}(\tilde{g}_\alpha) < \mathrm{supp}(\tilde{g}_\beta)$ for every $\alpha < \beta < \omega_1$.*

Needless to say, the above conditions are mutually exclusive. Let us also point out explicitly that (1) implies in particular $\mathrm{supp}(\tilde{g}_\alpha) \cap \mathrm{supp}(\tilde{g}_\beta) = \{\gamma\}$ for $\alpha < \beta < \omega_1$, while (2) implies that the sets $\mathrm{supp}(\tilde{g}_\alpha)$ are mutually disjoint.

Proof. [3] Assume that X contains an uncountable Auerbach system and let us fix one, say $\{u_\alpha; g_\alpha\}_{\alpha<\omega_1}$; according to Theorem 2.3.5, we know that the sets $\mathrm{supp}(g_\alpha)$ are finite sets. In view of the Δ-system lemma we can therefore assume (up to passing to an uncountable subset of ω_1 and relabeling) that there exists a finite set $\Delta \subseteq \omega_1$ such that $\mathrm{supp}(g_\alpha) \cap$

[3] In this proof we shall denote by Δ a finite set, obtained via the Δ-system lemma. This will not create conflicts with the parameter $\Delta \leqslant 1/5$, fixed at the very beginning of the section, since such parameter will play no rôle here.

$\operatorname{supp}(g_\beta) = \Delta$ for distinct $\alpha, \beta < \omega_1$. In the case that Δ is the empty set we have obtained an uncountable Auerbach system with mutually disjointly supported functionals and a simple transfinite induction argument yields the existence of an uncountable subcollection of functionals whose supports are consecutive. In this case, (2) holds and we are done.

Alternatively, if $\Delta \neq \emptyset$ we first have to concentrate all the mass present in Δ in a single coordinate. Fixed any $\gamma \in \Delta$, we have $\gamma \in \operatorname{supp}(g_\alpha)$, whence $|\langle \varphi_\gamma, u_\alpha\rangle| = 1$ for every $\alpha < \omega_1$. Consequently, we can pass to an uncountable subset of ω_1 and assume that $\langle \varphi_\gamma, u_\alpha\rangle$ has the same sign for every $\alpha < \omega_1$. If we repeat the same procedure for all the finitely many γ's in Δ we see that we can assume without loss of generality that there are signs $\varepsilon_\gamma = \pm 1$ $(\gamma \in \Delta)$ such that $\langle \varphi_\gamma, u_\alpha\rangle = \varepsilon_\gamma$ for each $\alpha < \omega_1$ and $\gamma \in \Delta$. Let us also fix arbitrarily an element $\overline{\gamma} \in \Delta$ and assume, for simplicity, that $\varepsilon_{\overline{\gamma}} = 1$ (if this is not the case, we can just achieve it by replacing the Auerbach system $\{u_\alpha; g_\alpha\}_{\alpha<\omega_1}$ by $\{-u_\alpha; -g_\alpha\}_{\alpha<\omega_1}$).

We are now in position to define the functionals $\tilde{g}_\alpha$ $(\alpha < \omega_1)$. Assume that

$$g_\alpha = \sum_{\gamma<\omega_1} g_\alpha^\gamma \varphi_\gamma = \sum_{\gamma\in\Delta} g_\alpha^\gamma \varphi_\gamma + \sum_{\gamma\in\Delta^\complement} g_\alpha^\gamma \varphi_\gamma,$$

where $\sum_{\gamma<\omega_1} |g_\alpha^\gamma| = 1$, for every $\alpha < \omega_1$. We may then define

$$\tilde{g}_\alpha = \sum_{\gamma\in\Delta} |g_\alpha^\gamma| \cdot \varphi_{\overline{\gamma}} + \sum_{\gamma\in\Delta^\complement} g_\alpha^\gamma \varphi_\gamma;$$

plainly, $|||\tilde{g}_\alpha||| = 1$ too (this follows from Corollary 2.3.6).

Finally, in order to evaluate $\langle \tilde{g}_\alpha, u_\beta\rangle$, note preliminarily that if $\gamma \in \Delta$ then $\gamma \in \operatorname{supp}(g_\alpha)$ for each $\alpha < \omega_1$; since g_α attains its norm at u_α, the remarks preceding the present lemma imply that

$$|g_\alpha^\gamma| = g_\alpha^\gamma \langle \varphi_\gamma, u_\alpha\rangle = g_\alpha^\gamma \varepsilon_\gamma = g_\alpha^\gamma \langle \varphi_\gamma, u_\beta\rangle,$$

whenever $\alpha, \beta < \omega_1$. Consequently,

$$\begin{aligned}
\langle \tilde{g}_\alpha, u_\beta\rangle &:= \sum_{\gamma\in\Delta} |g_\alpha^\gamma| \cdot \langle \varphi_{\overline{\gamma}}, u_\beta\rangle + \sum_{\gamma\in\Delta^\complement} g_\alpha^\gamma \langle \varphi_\gamma, u_\beta\rangle \\
&= \sum_{\gamma\in\Delta} |g_\alpha^\gamma| + \sum_{\gamma\in\Delta^\complement} g_\alpha^\gamma \langle \varphi_\gamma, u_\beta\rangle \\
&= \sum_{\gamma\in\Delta} g_\alpha^\gamma \langle \varphi_\gamma, u_\beta\rangle + \sum_{\gamma\in\Delta^\complement} g_\alpha^\gamma \langle \varphi_\gamma, u_\beta\rangle \\
&= \sum_{\gamma<\omega_1} g_\alpha^\gamma \langle \varphi_\gamma, u_\beta\rangle = \langle g_\alpha, u_\beta\rangle.
\end{aligned}$$

It follows that $\{u_\alpha; \tilde{g}_\alpha\}_{\alpha<\omega_1}$ is also an Auerbach system and the mutual intersections of the supports of the functionals $\tilde{g}_\alpha$ all reduce to the singleton $\{\overline{\gamma}\}$. Finally, a transfinite induction argument analogous to the one needed above proves the existence of an Auerbach system satisfying (1). ∎

We can finally pass to the second part of our considerations and prove the main result of the section. Prior to this, the following remark is dedicated to the presentation, in a simplified setting, of one of the main ingredients in the proof of the result, concerning the choice of the parameters λ_α.

Remark 2.3.8. Let $u \in c_0(\omega_1)$ be any non-zero vector and fix an ordinal $\gamma < \omega_1$ with $\mathrm{supp}(u) < \gamma$. We note that there are possible many choices of λ_γ such that the corresponding functional φ_γ satisfies $\langle \varphi_\gamma, u\rangle \neq 0$. Since the definition of φ_γ depends on the choice of the parameter λ_γ, we shall also denote by $\varphi_\gamma(\lambda)$ the functional obtained choosing $\lambda_\gamma = \lambda$. Let us then observe that the function

$$\lambda \mapsto \langle \varphi_\gamma(\lambda), u\rangle := \sum_{\alpha<\gamma} u(\alpha)\langle \varphi_\gamma(\lambda), e_\alpha\rangle = \sum_{k=1}^{|\gamma|} u(\sigma_\gamma(k))\lambda^k$$

is expressed by a power series with bounded coefficients, not all of which equal zero. Therefore $\lambda \mapsto \langle \varphi_\gamma(\lambda), u\rangle$ is a nontrivial real-analytic function on $(-1,1)$ and, in view of the identity principle for real-analytic functions, it necessarily has finitely many zeros in $(0,\delta)$. This consideration allows us for many choices of a parameter λ_γ such that $\langle \varphi_\gamma, u\rangle \neq 0$. In the course of the proof of the result to follow, we shall need to exploit the same argument involving analyticity in a more complicated setting.

Theorem 2.3.9 (CH). *There exists a choice of the parameters $(\lambda_\alpha)_{\alpha<\omega_1}$ such that the corresponding space $X = (c_0(\omega_1), |\!|\!|\cdot|\!|\!|)$ does not contain any uncountable Auerbach system.*

Proof. Let us denote by $c_0(\alpha)$ ($\alpha < \omega_1$) the subspace of $c_0(\omega_1)$ consisting of all vectors $u \in c_0(\omega_1)$ such that $\mathrm{supp}(u) \subseteq \alpha$. Since every element of $c_0(\omega_1)$ is countably supported, we have $c_0(\omega_1) = \cup_{\alpha<\omega_1} c_0(\alpha)$; moreover, every space $c_0(\alpha)$ is isometric to c_0, hence it has cardinality the continuum. As a consequence, $|c_0(\omega_1)| = \mathfrak{c}$ too. Assuming (CH), we may therefore well order the non-zero vectors of $c_0(\omega_1)$ in an ω_1-sequence $(v_\alpha)_{\alpha<\omega_1}$, *i.e.*, $c_0(\omega_1) \setminus \{0\} = \{v_\alpha\}_{\alpha<\omega_1}$.

As in the previous remark, we shall occasionally denote by $\varphi_\gamma(\lambda_\alpha)$ the functional φ_γ, whenever it will be desirable to stress the dependence of φ_γ on the parameter λ_γ. Those parameters $(\lambda_\alpha)_{\alpha<\omega_1}$ will be chosen as to satisfy the conclusion of the following claim.

Claim. *It is possible to choose the parameters $(\lambda_\alpha)_{\alpha<\omega_1}$ in such a way that the following two assertions* **(A)** *and* **(B)** *are satisfied.*

(A) *For every $N \in \mathbb{N}$, $N \geqslant 2$, for every choice of ordinal numbers $\alpha_1, \dots, \alpha_N < \omega_1$ and $\beta_1, \dots, \beta_N < \omega_1$ with the properties that:*

- (i) $\{v_{\alpha_1}, \dots, v_{\alpha_N}\}$ *is a linearly independent set;*
- (ii) $\beta_1 < \beta_2 < \dots < \beta_N$*;*
- (iii) $\alpha_1, \dots, \alpha_N < \beta_2$*;*
- (iv) $\mathrm{supp}(v_{\alpha_1}), \dots, \mathrm{supp}(v_{\alpha_N}) < \beta_2$*;*
- (v) $\langle \varphi_{\beta_1}, v_{\alpha_i}\rangle \neq 0$ *for every $i = 1, \dots, N$;*

one has:

$$\det\left(\left(\langle \varphi_{\beta_i}, v_{\alpha_j}\rangle\right)_{i,j=1}^{N}\right) \neq 0.$$

(B) *For every $N \in \mathbb{N}$, for every choice of ordinal numbers $\alpha_1, \ldots, \alpha_N < \omega_1$ and $\beta_1, \ldots, \beta_N < \omega_1$ with the properties that:*

(i) $\{v_{\alpha_1}, \ldots, v_{\alpha_N}\}$ *is a linearly independent set;*
(ii) $\beta_1 < \beta_2 < \cdots < \beta_N$*;*
(iii) $\alpha_1, \ldots, \alpha_N < \beta_1$*;*
(iv) $\mathrm{supp}(v_{\alpha_1}), \ldots, \mathrm{supp}(v_{\alpha_N}) < \beta_1$*;*

one has:

$$\det\left(\left(\langle \varphi_{\beta_i}, v_{\alpha_j}\rangle\right)_{i,j=1}^{N}\right) \neq 0.$$

Proof of the claim. We shall argue by transfinite induction on $\gamma := \beta_N < \omega_1$, observing that for $\gamma = 0$ both conditions **(A)** and **(B)** are trivially satisfied, while $\varphi_0 = e_0^*$ regardless of the choice of λ_0. We may therefore assume by the transfinite induction assumption that, for a certain $\gamma < \omega_1$, we have already chosen parameters $(\lambda_\alpha)_{\alpha<\gamma}$ with the corresponding functionals $\varphi_\alpha = \varphi_\alpha(\lambda_\alpha)$ in such a way that conditions **(A)** and **(B)** are satisfied for every choice of the ordinals as above, subject to the condition $\beta_N < \gamma$. In order to verify the claim for γ, we only need to define λ_γ and therefore φ_γ; we shall be considering the function $\lambda \mapsto \varphi_\gamma(\lambda)$ and show that a suitable choice of $\lambda = \lambda_\gamma$ is possible.

We shall first focus on choosing λ in such a way to achieve condition **(A)**. Let us select arbitrarily $N \geqslant 2$ and ordinal numbers $\alpha_1, \ldots, \alpha_N$ and $\beta_1, \ldots, \beta_N$ satisfying conditions (i)–(v) and such that $\beta_N = \gamma$; note that in particular the functionals $\varphi_{\beta_1}, \ldots, \varphi_{\beta_{N-1}}$ have already been defined, and we only need to choose a suitable parameter λ in $\varphi_{\beta_N}(\lambda) = \varphi_\gamma(\lambda)$. Observe that, by using the Laplace expansion for the determinant on the last column,

$$\det\left(\left(\langle \varphi_{\beta_i}, v_{\alpha_j}\rangle\right)_{i,j=1}^{N}\right) = \sum_{j=1}^{N} (-)^{N+j} \langle \varphi_{\beta_N}(\lambda), v_{\alpha_j}\rangle d_j = \left\langle \varphi_{\beta_N}(\lambda), \sum_{j=1}^{N} (-)^{N+j} d_j v_{\alpha_j} \right\rangle,$$

where d_j is the determinant of the $(N-1)\times(N-1)$ matrix obtained removing the j-th row and the N-th column from the original matrix $\left(\langle \varphi_{\beta_i}, v_{\alpha_j}\rangle\right)_{i,j=1}^{N}$. In the case that $N = 2$, then actually $d_j = \langle \varphi_{\beta_1}, v_{\alpha_{2-j}}\rangle$, whence $d_j \neq 0$, according to (v). On the other hand, if $N \geqslant 3$, then d_j is the determinant of the matrix obtained from the action of the functionals $\{\varphi_{\beta_1}, \ldots, \varphi_{\beta_{N-1}}\}$ on the set of vectors $\{v_{\alpha_1}, \ldots, v_{\alpha_N}\} \setminus \{v_{\alpha_j}\}$. Those vectors and functionals plainly satisfy conditions (i)–(v) with $\beta_{N-1} < \gamma$ instead of γ; thereby, $d_j \neq 0$ follows from the transfinite induction assumption.

Consequently, in each case we may conclude that $d_j \neq 0$ ($j = 1, \ldots, N$) and condition (i) then forces the vector $\sum_{j=1}^{N} (-)^{N+j} d_j v_{\alpha_j}$ to be non-zero. According to the remark preceding the proof, we may now deduce that the function

$$\lambda \mapsto \left\langle \varphi_{\beta_N}(\lambda), \sum_{j=1}^{N} (-)^{N+j} d_j v_{\alpha_j} \right\rangle$$

is a non-trivial real-analytic function on $(-1,1)$ and consequently it has only finitely many zeros on the set $(0,\delta)$. However, due to conditions (ii) and (iii), there are only countably many choices for $N \in \mathbb{N}$, $\alpha_1, \dots, \alpha_N$ and $\beta_1, \dots, \beta_N$ as in **(A)** and satisfying $\beta_N = \gamma$. Therefore, there exists a countable set $\Lambda_{(\mathbf{A})} \subseteq (0,\delta)$ such that all the determinants appearing in **(A)** and with $\beta_N = \gamma$ are different from zero, for every choice of $\lambda \in (0,\delta) \setminus \Lambda_{(\mathbf{A})}$. With any such choice, condition **(A)** is verified for γ.

A very similar consideration also applies to condition **(B)**. The argument is even simpler, and we therefore omit the straightforward modifications, since we don't need to distinguish between the cases $N = 2$ and $N \geqslant 3$, the above argument for $N \geqslant 3$ now being applicable to every $N \geqslant 1$; incidentally, this is the reason why we do not need condition (v) in this case. Therefore, we obtain a countable subset $\Lambda_{(\mathbf{B})}$ of $(0,\delta)$ such that condition **(B)** is verified for γ, whenever $\lambda \in (0,\delta) \setminus \Lambda_{(\mathbf{B})}$.

Finally, choosing any $\lambda_\gamma \in (0,\delta) \setminus (\Lambda_{(\mathbf{A})} \cup \Lambda_{(\mathbf{B})})$ and such that $\lambda_\gamma \neq \lambda_\alpha$ for $\alpha < \gamma$ then provides us with a functional φ_γ for which both assertions **(A)** and **(B)** are satisfied for γ; therefore, the transfinite induction step is complete and so is the proof of the claim. ■

Having the claim proved, we may choose parameters $(\lambda_\alpha)_{\alpha<\omega_1}$ satisfying conditions **(A)** and **(B)** above; we then denote by $X := (c_0(\omega_1), |||\cdot|||)$ the space obtained as described in the first part of the section, where the functionals $(\varphi_\alpha)_{\alpha<\omega_1}$ are obtained from the presently chosen sequence $(\lambda_\alpha)_{\alpha<\omega_1}$. We may now conclude the proof, by showing that the space X does not contain any uncountable Auerbach system.

Assume by contradiction that such systems do exist. Then we may find one, say $\{u_\alpha; g_\alpha\}_{\alpha<\omega_1}$, that satisfies the conclusion to Lemma 2.3.7. Moreover, according to Theorem 2.3.5 the sets $\mathrm{supp}(g_\alpha)$ $(\alpha < \omega_1)$ are finite sets; therefore, we can also assume that they all have the same finite cardinality, say N. On the other hand, we shall presently show that both the cases contained in the conclusion to Lemma 2.3.7 are in contradiction with conditions **(A)** and **(B)**; this ultimately leads us to the desired contradiction and concludes the proof.

Firstly, we show that the validity of **(A)** rules out the possibility (1) in Lemma 2.3.7 to hold true. Assume by contradiction that the supports of $(g_\alpha)_{\alpha<\omega_1}$ satisfy condition (1) and let us write $\mathrm{supp}(g_\alpha) := \{\beta_1^\alpha, \dots \beta_N^\alpha\}$, where $\beta_1^\alpha < \beta_2^\alpha < \dots < \beta_N^\alpha < \omega_1$. According to assumption (1), we have $\beta_1^\alpha = \beta_1^\eta$ and $\{\beta_2^\alpha, \dots \beta_N^\alpha\} < \{\beta_2^\eta, \dots \beta_N^\eta\}$ for $\alpha < \eta < \omega_1$; it follows in particular that $\sup_{\alpha<\omega_1} \beta_2^\alpha = \omega_1$. Moreover, the non-zero vectors $\{u_1, \dots, u_N\}$ have been enumerated in the ω_1-sequence $(v_\alpha)_{\alpha<\omega_1}$, so we can find indices $\alpha_1, \dots, \alpha_N$ such that $u_j = v_{\alpha_j}$ for every $j = 1, \dots, N$. We may also fix an ordinal number $\overline{\alpha} < \omega_1$ such that $\alpha_j < \overline{\alpha}$ and $\mathrm{supp}(v_{\alpha_j}) < \overline{\alpha}$ for $j = 1, \dots, N$.

Since $\sup_{\alpha<\omega_1} \beta_2^\alpha = \omega_1$, it is possible to choose an ordinal $\alpha < \omega_1$ with the property that $\beta_2^\alpha > \overline{\alpha}$; needless to say, we may also assume that $\alpha > N$. Let us set $\beta_j := \beta_j^\alpha$, for such a choice of α. With such a choice of the indices $\{\alpha_1, \dots \alpha_N\}$ and $\{\beta_1, \dots \beta_N\}$ it is apparent that requirements (ii)–(iv) in condition **(A)** are satisfied; also (i) is undoubtedly valid. Therefore, we only need to check the validity of (v): in order to achieve this, note preliminarily that $\beta_1 \in \mathrm{supp}(g_\alpha)$ for every $\alpha < \omega_1$, in particular for $\alpha = \alpha_j$. The comments following Corollary 2.3.6 and the fact that g_{α_j} attains its norm at v_{α_j} then assure us that actually $|\langle\varphi_{\beta_1}, v_{\alpha_j}\rangle| = 1$.

Consequently, all the assumptions in condition **(A)** have been verified and the validity of **(A)** leads us to the conclusion that

$$\det\left(\left(\langle\varphi_{\beta_i}, v_{\alpha_j}\rangle\right)_{i,j=1}^N\right) \neq 0.$$

On the other hand, we may write $g_\alpha := \sum_{i=1}^N c_i\varphi_{\beta_i}$ for some choice of scalars c_i ($i = 1,\dots,N$) with $\sum_{i=1}^N |c_i| = 1$. Since $\alpha > N$ we have

$$0 = \langle g_\alpha, u_j\rangle = \langle g_\alpha, v_{\alpha_j}\rangle = \sum_{i=1}^N c_i\langle\varphi_{\beta_i}, v_{\alpha_j}\rangle \qquad j = 1,\dots,N.$$

In matrix form, the present equations read

$$\begin{pmatrix} \langle\varphi_{\beta_1}, v_{\alpha_1}\rangle & \dots & \langle\varphi_{\beta_N}, v_{\alpha_1}\rangle \\ \vdots & & \vdots \\ \langle\varphi_{\beta_1}, v_{\alpha_N}\rangle & \dots & \langle\varphi_{\beta_N}, v_{\alpha_N}\rangle \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ \vdots \\ c_N \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix};$$

this obviously contradicts $\det\left(\left(\langle\varphi_{\beta_i}, v_{\alpha_j}\rangle\right)_{i,j=1}^N\right) \neq 0$ and therefore clause (1) in Lemma 2.3.7 cannot occur.

A very similar argument, which we only sketch, proves that (2) is in contradiction with **(B)**. In fact, under the assumption of the validity of (2), and keeping the above notation for $\text{supp}(g_\alpha) := \{\beta_1^\alpha, \dots \beta_N^\alpha\}$, we conclude that $\sup_{\alpha<\omega_1} \beta_1^\alpha = \omega_1$; we also select $\alpha_1, \dots, \alpha_N$ and $\overline{\alpha}$, proceeding in the same way as above. We are now in position to choose $\alpha < \omega_1$ such that $\beta_1^\alpha > \overline{\alpha}$ and $\alpha > N$ and define $\beta_j := \beta_j^\alpha$, for such a choice of α. Having chosen $\{\alpha_1, \dots \alpha_N\}$ and $\{\beta_1, \dots \beta_N\}$ in such a way, requirements (i)–(iv) in condition **(B)** are satisfied. Therefore **(B)** assures us that

$$\det\left(\left(\langle\varphi_{\beta_i}, v_{\alpha_j}\rangle\right)_{i,j=1}^N\right) \neq 0$$

and a contradiction follows from *verbatim* the same argument as in the previous case.

Consequently, both clauses (1) and (2) in the conclusion to Lemma 2.3.7 fail to hold and the contrapositive form to Lemma 2.3.7 itself implies that X contains no uncountable Auerbach system, thereby concluding the argument. ■

Remark 2.3.10. In the above claim one could have replaced condition **(B)** above with the following condition **(C)**.

(C) For every choice of a vector v_α such that:

(i) $\alpha < \gamma$;

(ii) $\text{supp}(v_\alpha) < \gamma$;

one has

$$\langle \varphi_\gamma, v_\alpha \rangle \neq 0.$$

The main motivation why one may consider the present condition **(C)** is clearly that **(B)** is way more complicated than **(C)**, even though **(A)** and **(B)** are actually quite similar. On the other hand, the drawback of **(C)** is that **(A)**&**(B)** are exactly the conditions needed to face the two possible cases contained in the conclusion to Lemma 2.3.7, while the argument exploiting **(A)**&**(C)** would be somewhat more involved.

2.4 Pták's combinatorial lemma

In his 1959 paper [Ptá59], Vlastimil Pták distilled a combinatorial lemma aimed at the investigation of weak compactness in Banach spaces. An interesting application of the lemma, and partial motivation for the result itself, was an elementary proof of the fact that if a uniformly bounded sequence of continuous functions $(f_n)_{n=1}^\infty \subseteq C(K)$ converges pointwise to a continuous function f, then f may be uniformly approximated by convex combinations of the f_n ([Ptá63b, §2.1]). We shall record such an elementary proof below, in Lemma 2.4.8. It is actually a standard exercise in Functional Analysis to understand this assertion as a particular case of Mazur's theorem that a closed and convex subset of a Banach space is weakly closed. However, this approach requires the Riesz representation theorem for $C(K)^*$, Lebesgue's dominated convergence theorem and the Hahn–Banach separation theorem; therefore, it relies on much deeper principles than the assertion itself, which, in particular, involves no measure theory whatsoever.

In later papers, Pták also applied and extended the same combinatorial ideas to the study of separately continuous functions, *cf.* [Ptá63a, Ptá64], and interchangeability of double limits [Ptá63b], aimed, in particular, at the study of weak compactness in Banach spaces. When combined with previous results of Grothendieck [Gro52], this approach leads to a proof of Krein's theorem; such a proof is presented in the monograph [Köt69, §24.6] and we shall say a bit more on it at the very end of the section.

This interesting lemma attracted the attention of the mathematical community, as witnessed by several papers dedicated to its different proofs or extensions; let us mention, among them, [Kin83, Sim67, Sim72a, Sim72b]. More recently, it was also used—and given a different, Banach space theoretic, proof—in the paper [BHO89]. This proof is also included in [FHHMZ10, Exercise 14.28], where Pták's elementary proof of Mazur theorem is also outlined (*cf.* [FHHMZ10, Exercise 14.29]). One further introduction to this result may be found in [Tod97, §I.3], or in the systematic survey [Ptá01] by Pták himself.

Let us now proceed to recall the statement of the result under investigation; we shall require a piece of terminology, and we follow Pták's notation from [Ptá59]. Given a set S and a function $\lambda\colon S \to \mathbb{R}$ by the *support* of λ we understand the set $\mathrm{supp}(\lambda) := \{s \in S\colon \lambda(s) \neq 0\}$; in the case that $\mathrm{supp}(\lambda)$ is a finite set, we shall say that λ is *finitely supported.*

A *convex mean* is a finitely supported function $\lambda\colon S \to [0,\infty)$ such that

$$\sum_{s\in S}\lambda(s)=1.$$

Plainly, a convex mean can also be naturally interpreted as a finitely supported probability measure on $\left(S, 2^S\right)$ via the definition $\lambda(A) := \sum_{s\in A}\lambda(s)$, for $A\subseteq S$. In what follows, we shall profit from this notation, whenever convenient.

All the necessary notation being set forth, we are now in position to recall the original statement of Pták's lemma.

Lemma 2.4.1 (Pták's combinatorial lemma, [Ptá59])**.** *Let S be an infinite set and $\mathcal{F}\subseteq [S]^{<\omega}$ be a collection of finite subsets of S. Then the following conditions are equivalent:*

(i) *there exist an infinite subset H of S and $\delta > 0$ such that for every convex mean λ with* $\operatorname{supp}(\lambda)\subseteq H$ *one has*
$$\sup_{F\in\mathcal{F}}\lambda(F)\geqslant\delta;$$

(ii) *there exist a strictly increasing sequence of finite sets $(B_n)_{n=1}^{\infty}\subseteq [S]^{<\omega}$ and a sequence $(F_n)_{n=1}^{\infty}\subseteq\mathcal{F}$ such that $B_n\subseteq F_n$, for every $n\in\mathbb{N}$.*

Let us observe that the proof of the implication (ii) $\implies$ (i) is immediate, as witnessed by the choice of the set $H := \cup_{n=1}^{\infty} B_n$. Therefore, the actual content of the lemma lies in the validity of the implication (i) $\implies$ (ii) and it is precisely this implication to appear in the result as devised in [BHO89].

In order to state this second formulation, we need one more definition. A family $\mathcal{F}\subseteq 2^S$ is said to be *hereditary* if whenever $F\in\mathcal{F}$ and $G\subseteq F$, then $G\in\mathcal{F}$ too.

Lemma 2.4.2 (Pták's lemma, second formulation, [BHO89])**.** *Let S be an infinite set and let $\mathcal{F}\subseteq [S]^{<\omega}$ be an hereditary family. Assume that there exists $\delta>0$ such that for every convex mean λ on S one has*
$$\sup_{F\in\mathcal{F}}\lambda(F)\geqslant\delta.$$
Then there exists an infinite subset M of S such that every finite subset of M is in $\mathcal{F}$.

Observe that, for an hereditary family $\mathcal{F}$, condition (ii) of Lemma 2.4.1 is equivalent to the conclusion of Lemma 2.4.2, as a simple verification shows. More precisely, the condition $\mathcal{F}$ being hereditary is only used in the verification that (ii) implies the conclusion of Lemma 2.4.2. Moreover, the assumption in Lemma 2.4.2 immediately implies (i) with $H=S$ and, conversely, under the validity of (i), the assumption of Lemma 2.4.2 is satisfied for the infinite set H and the hereditary family $\{F\cap H\colon F\in\mathcal{F}\}=\{F\in\mathcal{F}\colon F\subseteq H\}$.

Consequently, the two statements are formally equivalent. The advantage of the second formulation, from our perspective, is that it immediately suggests its possible generalisations to larger cardinalities, that we shall consider in what follows.

In order to have a more succinct formulation of our results, the following definition seems appropriate.

Definition 2.4.3. Let κ be an infinite cardinal number. We say that *Pták's lemma holds true for* κ if for every set S with $|S| \geqslant \kappa$ and every hereditary family $\mathcal{F} \subseteq [S]^{<\omega}$ such that

$$(\dagger) \qquad \delta := \inf \left\{ \sup_{F \in \mathcal{F}} \lambda(F) \colon \lambda \text{ is a convex mean on } S \right\} > 0,$$

there exists a subset M of S, with $|M| = \kappa$, such that every finite subset of M belongs to $\mathcal{F}$.

Let us now proceed to give the formal statements of our main results. Prior to this, we only mention that when the notation λ^ω is used, it will be cardinal exponentiation that is intended.

Theorem 2.4.4 ([HáRu19, Theorem A])**.** *The validity of Pták's lemma for ω_1 is independent of ZFC. More precisely:*

(i) (MA_{ω_1}) *Pták's lemma holds true for ω_1;*

(ii) (CH) *Pták's lemma fails to hold for ω_1.*

Theorem 2.4.5 ([HáRu19, Theorem B])**.** *Let κ be a regular cardinal number such that $\lambda^\omega < \kappa$ whenever $\lambda < \kappa$. Then Pták's lemma is true for κ.*

Let us now single out a few simple particular cases of Theorem 2.4.5. In the case when $\kappa = \tau^+$ is a successor cardinal (hence, in particular, regular), the condition $\lambda^\omega < \kappa$ whenever $\lambda < \kappa$ is satisfied exactly when $\tau^\omega = \tau$. Moreover, if μ is any infinite cardinal number, then $\tau := 2^\mu$ satisfies $\tau^\omega = \tau$; we therefore arrive at the following corollary.

Corollary 2.4.6. *If τ is an infinite cardinal number such that $\tau^\omega = \tau$, then Pták's lemma is true for τ^+. In particular, Pták's lemma is true for $(2^\mu)^+$, whenever μ is an infinite cardinal number.*

Consequently, there are in ZFC arbitrarily large cardinal numbers for which Pták's lemma is true. Moreover, the smallest cardinal the above corollary applies to is $\mathfrak{c}^+$. The assumption of additional set-theoretical axioms allows us to deduce one more corollary, whose first clause is immediate.

Corollary 2.4.7.

(i) (CH) *Pták's lemma is true for ω_2;*

(ii) (GCH) *If τ is a cardinal number with $\mathrm{cf}(\tau) > \omega$, then Pták's lemma is true for τ^+.*

Proof. We only need to prove the second assertion; in light of the previous corollary, this amounts to proving that $\tau^\omega = \tau$, whenever $\mathrm{cf}(\tau) > \omega$. If $\tau = \alpha^+$ is a successor cardinal, then by GCH we have $\tau = 2^\alpha$, whence $\tau^\omega = \tau$.

In the case that τ is a limit ordinal, we may write

$$\tau = \bigcup \{\alpha \colon \alpha < \tau,\ \alpha \text{ successor}\}.$$

If $s \in \tau^\omega$ is any sequence in τ, the assumption on the cofinality of τ then implies the existence of a successor ordinal $\alpha < \tau$ such that the image of the sequence s is contained in α. In other words,

$$\tau^\omega = \bigcup\{\alpha^\omega \colon \alpha < \tau,\ \alpha \text{ successor}\},$$

where τ^ω denotes here the collection of all functions from ω to τ. Since $|\alpha^\omega| = \alpha$, from the previous case, we conclude that $|\tau^\omega| = \tau$ too. ■

Our presentation of the above theorems is organised as follows: in Section 2.5.1 we shall present some general observations concerning the condition appearing in Pták's lemma. These considerations will, in particular, allow us to present the proof of Pták's original result and are based on the proof given in [BHO89]. In Section 2.5.2 we shall prove Theorem 2.4.4, while Section 2.5.3 is dedicated to the proof of Theorem 2.4.5. Prior to this, we shall conclude this section with two instances of uses of Pták's lemma, proving the claim about convex combinations of continuous functions and describing Pták's proof of Krein theorem. Moreover, we shall also dedicate Section 2.4.1 below to a self-contained presentation of Martin's Axiom, that will be needed for Theorem 2.4.4.

Lemma 2.4.8 ([Ptá63b, §2.1])**.** *Let $(f_n)_{n=1}^\infty \subseteq C(K)$ be a bounded sequence of continuous functions that converge pointwise to a continuous function f. Then f can be uniformly approximated by convex combinations of the f_n.*

Proof. We may assume without loss of generality that $(f_n)_{n=1}^\infty$ converges pointwise to 0 and that $\|f_n\| \leqslant 1$, for every $n \in \mathbb{N}$. Let us now fix $\varepsilon > 0$ and consider, for $x \in K$, the finite set $F_x := \{n \in \mathbb{N} \colon |f_n(x)| \geqslant \varepsilon/2\}$; we shall apply Lemma 2.4.1 to the family $\mathcal{F} := \{F_x \colon x \in K\}$.

If condition (ii) in Lemma 2.4.1 is satisfied there exist a strictly increasing sequence $(B_n)_{n=1}^\infty$ of finite subsets of $\mathbb{N}$ and a sequence $(x_n)_{n=1}^\infty \in K$ such that $B_n \subseteq F_{x_n}$ ($n \in \mathbb{N}$); as a consequence, $|f_k(x_n)| \geqslant \varepsilon/2$, whenever $n \in \mathbb{N}$ and $k \in B_n$. Let also x_0 be an accumulation point of the sequence $(x_n)_{n=1}^\infty$. If $k \in \cup B_n$, then k belongs to B_n for every n sufficiently large, since $(B_n)_{n=1}^\infty$ is an increasing sequence; consequently, $|f_k(x_n)| \geqslant \varepsilon/2$ for n large and the continuity of f_k implies $|f_k(x_0)| \geqslant \varepsilon/2$. We conclude that $k \in F_{x_0}$, whence $F_{x_0} \supseteq \cup B_n$ is an infinite set, a contradiction.

Pták's lemma then implies the existence of a convex mean λ on $\mathbb{N}$ such that $\lambda(F_x) < \varepsilon/2$ whenever $x \in K$; we shall prove that $\sum_{i=1}^\infty \lambda(i) f_i$ has norm at most ε, whence it is the desired convex combination. In fact, if $x \in K$, we have

$$\left|\sum_{i=1}^\infty \lambda(i) f_i(x)\right| \leqslant \sum_{i \in F_x} \lambda(i)|f_i(x)| + \sum_{i \notin F_x} \lambda(i)|f_i(x)|$$

$$\leqslant \sum_{i \in F_x} \lambda(i) + \varepsilon/2 \sum_{i \notin F_x} \lambda(i) \leqslant \lambda(F_x) + \varepsilon/2 < \varepsilon.$$

■

It is elementary to give examples that show the necessity to assume the sequence $(f_n)_{n=1}^{\infty}$ to be bounded in $C(K)$; one such simple example is outlined below.

Example 2.4.9. Consider a sequence of functions $(f_n)_{n=1}^{\infty} \in C([0,1])$ with the properties that $\operatorname{supp} f_n \subseteq [\frac{1}{n+1}, \frac{1}{n}]$, $f_n \geqslant 0$ and $\|f_n\| = 2^n$. Of course, the sequence $(f_n)_{n=1}^{\infty}$ converges pointwise to 0 and it consists of disjointly supported functions. Therefore, for every finite sequence $(\lambda_n)_{n=1}^{N}$ of non-negative reals, we have

$$\left\| \sum_{n=1}^{N} \lambda_n f_n \right\| = \max\{2^n \lambda_n \colon n =, \dots, N\}.$$

As a consequence, if $\| \sum_{n=1}^{N} \lambda_n f_n \| < 1$, then $\lambda_n < 2^{-n}$, whence $\sum \lambda_n < 1$. We conclude that no convex combination of the functions f_n can have norm smaller than 1.

To conclude the section, we shall outline a proof, via Pták's lemma, of Krein's theorem on the weak compactness of the closed convex hull of a weakly compact set. The main building block of the argument is Grothendieck's characterisation of weak compactness in terms of interchangeability of iterated limits [Gro52], inspired from earlier work by Eberlein [Ebe47], where the approach through iterated limits was only implicit. The argument may be found, *e.g.*, in [Köt69, §24.6(1)], [Die84, p. 20], or [Woj91, p. 50].

Definition 2.4.10. A subset A of a Banach space X is said to *interchange limits* if for every pair of sequences $(x_n)_{n=1}^{\infty} \subseteq A$ and $(x_k^*)_{k=1}^{\infty} \subseteq B_{X^*}$ the existence of both iterated limits

$$\lim_{n\to\infty} \lim_{k\to\infty} \langle x_k^*, x_n \rangle \qquad \lim_{k\to\infty} \lim_{n\to\infty} \langle x_k^*, x_n \rangle$$

forces their equality.

Theorem 2.4.11 (Grothendieck, [Gro52]). *A bounded subset A of a Banach space is relatively weakly compact if and only if it interchanges limits.*

The necessity of the condition being essentially obvious, the real content of the result lies in the fact that interchangeability of limits yields a sufficient condition for (relative) weak compactness. Let us also observe that such implication contains the hard implication of the Eberlein–Šmulian theorem as a particular case, with essentially the same proof.

Proof. Assume that A is relatively weakly compact and let $(x_n)_{n=1}^{\infty} \subseteq A$ and $(x_k^*)_{k=1}^{\infty} \subseteq B_{X^*}$ be two sequences such that both iterated limits exist. If x is a weak cluster point of $(x_n)_{n=1}^{\infty} \subseteq A$ and x^* is w^* cluster point of $(x_k^*)_{k=1}^{\infty} \subseteq B_{X^*}$, it is clear that both double limits equal $\langle x^*, x \rangle$.

The strategy for the proof of the converse implication consists in showing that $\overline{A}^{w^*} \subseteq X^{**}$ is actually contained in X; since the restriction of the w^* topology of X^{**} to X is obviously the weak topology of X, this implies that A is relatively weakly compact.

We shall therefore argue by contraposition and pick $\psi \in \overline{A}^{w^*}$ such that $\operatorname{dist}(\psi, X) := d > 0$. We then have the following simple fact.

Fact 2.4.12. *For every finite subset F of X and $\varepsilon > 0$, there exists $x^* \in B_{X^*}$ such that*

$$|\langle f, x^* \rangle| < \varepsilon \ \ (f \in F) \qquad \text{and} \qquad |\langle \psi, x^* \rangle - d| < \varepsilon.$$

Indeed, the Hahn–Banach theorem yields a functional $\Lambda \in B_{X^{***}}$ such that $\Lambda\restriction_X = 0$ and $\langle \Lambda, \psi \rangle = d$; Goldstine theorem then reassures us of the validity of the assertion.

We shall next argue by induction as follows: we first fix arbitrarily $a_1 \in A$ and, according to the above fact, we find $x_1^* \in B_{X^*}$ such that

$$|\langle a_1, x_1^* \rangle| < 1 \qquad \text{and} \qquad |\langle \psi, x_1^* \rangle - d| < 1.$$

The assumption that $\psi \in \overline{A}^{w^*}$ then yields $a_2 \in A$ such that $|\langle \psi - a_2, x_1^* \rangle| < 1/2$. One more application of the fact provides us with $x_2^* \in B_{X^*}$ such that

$$|\langle a_1, x_2^* \rangle| < 1/2, \ |\langle a_2, x_2^* \rangle| < 1/2, \qquad \text{and} \qquad |\langle \psi, x_2^* \rangle - d| < 1/2.$$

One more step: the w^* neighbourhood of ψ induced by x_1^*, x_2^* yields $a_3 \in A$ such that

$$|\langle \psi - a_3, x_1^* \rangle| < 1/3, \ |\langle \psi - a_3, x_2^* \rangle| < 1/3;$$

application of the fact to $F = \{a_1, a_2, a_3\}$ then yields $x_3^* \in B_{X^*}$ such that

$$|\langle a_1, x_3^* \rangle| < 1/3, \ |\langle a_2, x_3^* \rangle| < 1/3, \ |\langle a_3, x_3^* \rangle| < 1/3, \ |\langle \psi, x_3^* \rangle - d| < 1/3.$$

If we proceed by induction in the same way, we obtain sequences $(a_n)_{n=1}^\infty \subseteq A$ and $(x_k^*)_{k=1}^\infty \subseteq B_{X^*}$ such that

(i) $|\langle \psi - a_n, x_k^* \rangle| < 1/n$, for $k \leqslant n-1$;

(ii) $|\langle a_n, x_k^* \rangle| < 1/k$, for $n \leqslant k$;

(iii) $|\langle \psi, x_k^* \rangle - d| < 1/k$.

Finally, (ii) implies $\lim_k \langle a_n, x_k^* \rangle = 0$, whence $\lim_n \lim_k \langle a_n, x_k^* \rangle = 0$. On the other hand, by (i) and (iii), we conclude

$$\lim_k \lim_n \langle a_n, x_k^* \rangle = \lim_k \langle \psi, x_k^* \rangle = d \neq 0.$$

■

Pták's lemma is then exploited to show that conv$A$ interchanges limits whenever A does, [Ptá01, Theorem 4.4], [Ptá63b, Theorem 3.3], or [Köt69, §24.6(4)]. Krein's theorem is an immediate consequence of the two above results. Let us also mention that the same method of proof, still exploiting Pták's lemma, can be extended to prove a quantitative version to Krein's theorem, [FHMZ05]; also see [HMVZ08, §3.6]. Let us conclude the section with the proof of such further application of Pták's lemma.

Theorem 2.4.13 (Pták, [Ptá63b])**.** *Assume that a bounded subset A of a Banach space X interchanges limits. Then* conv A *interchanges limits.*

Proof. We assume, without loss of generality, that $A \subseteq B_X$, and we argue by contraposition. We may therefore select two sequences $(u_n)_{n=1}^{\infty} \subseteq \operatorname{conv} A$ and $(x_k^*)_{k=1}^{\infty} \subseteq B_{X^*}$ such that both iterated limits exist and

$$\left| \lim_{k\to\infty} \lim_{n\to\infty} \langle x_k^*, u_n\rangle - \lim_{n\to\infty} \lim_{k\to\infty} \langle x_k^*, u_n\rangle \right| := \varepsilon > 0;$$

up to replacing $(x_k^*)_{k=1}^{\infty}$ with $(-x_k^*)_{k=1}^{\infty}$, we may actually assume

$$\lim_{k\to\infty} \lim_{n\to\infty} \langle x_k^*, u_n\rangle - \lim_{n\to\infty} \lim_{k\to\infty} \langle x_k^*, u_n\rangle = \varepsilon.$$

If we let $x^* \in B_{X^*}$ be a w^* cluster point of the sequence $(x_k^*)_{k=1}^{\infty}$, we plainly have $\lim_k \langle x_k^*, u_n\rangle = \langle x^*, u_n\rangle$; moreover, up to discarding finitely many indices k, we can also assume that

$$\lim_{n\to\infty} \langle x_k^* - x^*, u_n\rangle = \lim_{n\to\infty} \langle x_k^*, u_n\rangle - \lim_{n\to\infty} \langle x^*, u_n\rangle \geqslant 3\varepsilon/4 \qquad (k \in \mathbb{N}). \tag{2.4.1}$$

Let us now fix a countable set $T \subseteq A$ such that $(u_n)_{n=1}^{\infty} \subseteq \operatorname{conv} T$; by diagonalisation, up to passing to a subsequence, we can assume that the sequence $(x_k^*)_{k=1}^{\infty}$ converges pointwise on T, hence on $\operatorname{conv} T$. As a consequence, we have $\lim_k \langle x_k^*, t\rangle = \langle x^*, t\rangle$, whenever $t \in T$. This allows us to define finite sets F_t, $t \in T$, as follows:

$$F_t := \{k \in \mathbb{N} \colon |\langle x_k^* - x^*, t\rangle| \geqslant \varepsilon/4\}.$$

Claim 2.4.14. The family $\{F_t : t \in T\}$ satisfies (†).

Proof of the claim. Assume, by contradiction, the existence of a convex mean λ on $\mathbb{N}$ such that $\lambda(F_t) \leqslant \varepsilon/8$, for every $t \in T$. Let us then consider the functional $\varphi := \sum \lambda(k) \cdot (x_k^* - x^*)$ and evaluate its action on $t \in T$:

$$|\langle \varphi, t\rangle| \leqslant \sum_{k\in F_t} \lambda(k) |\langle x_k^* - x^*, t\rangle| + \sum_{k\notin F_t} \lambda(k)\varepsilon/4 \leqslant \sum_{k\in F_t} 2\lambda(k) + \varepsilon/4 = 2\lambda(F_t) + \varepsilon/4 \leqslant \varepsilon/2.$$

Consequently, the above estimate holds true for $t \in \operatorname{conv} T$, whence $|\langle \varphi, u_n\rangle| \leqslant \varepsilon/2$, for every $n \in \mathbb{N}$. However, this is a contradiction, since (2.4.1) yields

$$\lim_n \langle \varphi, u_n\rangle = \sum_k \lambda(k) \lim_n \langle x_k^* - x^*, u_n\rangle \geqslant \sum \lambda(k) \cdot 3\varepsilon/4 = 3\varepsilon/4.$$

■

It then follows, *a fortiori*, that (†) is satisfied by the hereditary family

$$\mathcal{F} := \{F \subseteq \mathbb{N} \colon F \subseteq F_t, \text{ for some } t \in T\},$$

to which we apply Pták's lemma. We may therefore conclude that there exists an infinite subset of $\mathbb{N}$ every whose finite subset is contained in some F_t. Up to passing to one more subsequence of the sequence $(x_k^*)_{k=1}^\infty$, we may assume that such infinite set equals $\mathbb{N}$; in other words, for every $n \in \mathbb{N}$ there exists $t_n \in T$ such that $\{1, \dots, n\} \subseteq F_{t_n}$, *i.e.*,

$$|\langle x_k^* - x^*, t_n \rangle| \geqslant \varepsilon/4 \qquad (k = 1, \dots, n).$$

Up to passing to one more subsequence (both of $(x_k^*)_{k=1}^\infty$ and of $(t_n)_{n=1}^\infty$), this inequality implies

$$\lim_k \lim_n |\langle x_k^* - x^*, t_n \rangle| \geqslant \varepsilon/4.$$

On the other hand, the fact that $(x_k^*)_{k=1}^\infty$ converges pointwise to x^* on T yields

$$\lim_n \lim_k \langle x_k^* - x^*, t_n \rangle = 0,$$

which ultimately shows that the two sequences $(t_n)_{n=1}^\infty \subseteq A$ and $(\frac{x_k^* - x^*}{2})_{k=1}^\infty \subseteq B_{X^*}$ do not interchange limits and concludes the proof. ■

2.4.1 A few words on Martin's Axiom

This section is dedicated to one additional set-theoretical axiom, consistent with ZFC, Martin's Axiom (MA, for short). Since we shall need to assume the validity of such axiom twice in this thesis, once in the present chapter and once in Chapter 4, we are going to give here a reasonably short self-contained presentation of the axiom. In this discussion, we shall follow mostly [Cie97, Chapter 8] and [JuWe97, Chapter 18]; more extensive introductions can also be found in [Jec03, Chapter 16] and [Kun80b, §III.3]. Let us also mention here the very elementary introduction given in the Monthly article [Sho75].

We will state the axiom in its partial order formulation; therefore, we shall start reviewing some notions on partially ordered sets, or *posets* for brevity.

Definition 2.4.15. Let $(\mathbb{P}, \leqslant)$ be a partially ordered set. Two elements $p, q \in \mathbb{P}$ are *compatible* ($p \not\perp q$) if there exists $r \in \mathbb{P}$ such that $r \leqslant p$ and $r \leqslant q$. If p and q are not compatible, we say that they are *incompatible* and we write $p \perp q$.

An *antichain* is a subset A of $\mathbb{P}$ whose elements are pairwise incompatible.

$(\mathbb{P}, \leqslant)$ has the *countable chain condition* (*ccc*, for short) if every antichain in $\mathbb{P}$ is countable.

As a piece of notation, when $p \leqslant q$ it is frequently said that p *extends* q; with this notation, p and q are compatible iff they have a common extension.

Let us now give two examples that will guide our intuition throughout the section.

Example 2.4.16. (a) Given a topological space (X, τ), consider the poset $(\tau \setminus \{\emptyset\}, \subseteq)$. Two non-empty open sets U and V are compatible iff there is a non-empty open set W with $W \subseteq U$ and $W \subseteq V$, *i.e.*, iff $U \cap V \neq \emptyset$. Consequently, an antichain in $(\tau \setminus \{\emptyset\}, \subseteq)$ is a collection of non-empty mutually disjoint open sets and $(\tau \setminus \{\emptyset\}, \subseteq)$ has the ccc if and only if the topological space (X, τ) is ccc.

(b) Given sets I and J, $Fn(I, J)$ denotes the collection of *finite partial functions* from I to J, namely, $p \in Fn(I, J)$ if p is a function with domain $\mathrm{dom}(p)$ a finite subset of I and with values in J. Given p and q in $Fn(I, J)$, we say that $p \leqslant q$ if p extends q as a function. As a consequence, functions p and q are compatible iff they have a common extension, that is iff they agree on the set $\mathrm{dom}(p) \cap \mathrm{dom}(q)$.

Observe that if $I \neq \emptyset$ and J is uncountable, then the poset $(\mathbb{P}, \leqslant) := (Fn(I, J), \leqslant)$ is not ccc. To see this, fix $i_0 \in I$ and consider, for $j \in J$, the singleton function f_j such that $\mathrm{dom}(f_j) = \{i_0\}$ and $f_j(i_0) = j$. It is clear that the collection $\{f_j\}_{j \in J}$ consists of mutually incompatible functions, hence $(Fn(I, J), \leqslant)$ does not have the ccc.

As it turns out, the above poset is ccc if and only if $I = \emptyset$ or J is countable. The proof of the reverse implication is a simple consequence of the Δ-system lemma, so short that we will include it below (*cf.* [Kun80b, Lemma III.3.7]). Let us mention that the Δ-system lemma is frequently used in arguments involving ccc.

Lemma 2.4.17. *$(Fn(I, J), \leqslant)$ has the ccc iff $I = \emptyset$ or J is countable.*

Proof. Note that if $I = \emptyset$ or $J = \emptyset$, then $(Fn(I, J), \leqslant) = \{\emptyset\}$, so we can assume I and J to be non-empty in what follows. Therefore, we only need to prove that $(Fn(I, J), \leqslant)$ is ccc, whenever J is countable.

Let $p_\alpha \in (Fn(I, J), \leqslant)$ $(\alpha < \omega_1)$; by the Δ-system lemma, we may pass to an uncountable subset B of ω_1 such that the sets $\{\mathrm{dom}(p_\alpha)\}_{\alpha \in B}$ have a common root Δ. The functions $p_\alpha \restriction_\Delta$ $(\alpha \in B)$ are elements of the countable set J^Δ, so we may find distinct $\alpha, \beta \in B$ with $p_\alpha \restriction_\Delta = p_\beta \restriction_\Delta$. Therefore, p_α and p_β are compatible and $(p_\alpha)_{\alpha < \omega_1}$ is not an antichain. ■

Definition 2.4.18. A subset D af a partially ordered set $(\mathbb{P}, \leqslant)$ is *dense* (or *cofinal*) if for every $p \in \mathbb{P}$ there exists $d \in D$ with $d \leqslant p$.

In other words, every element of $\mathbb{P}$ admits an extension that belongs to D. Let us describe some dense subsets in the posets introduced above.

Example 2.4.19. (a) If $\mathcal{B}$ is a basis for a topological space (X, τ), then by definition $\mathcal{B} \setminus \{\emptyset\}$ is a dense subset of $(\tau \setminus \{\emptyset\}, \subseteq)$. Moreover, given an open subset O of X, consider the set
$$D_{\leqslant O} := \{U \in \tau \setminus \{\emptyset\} \colon U \subseteq O\}.$$
The density of $D_{\leqslant O}$ in $(\tau \setminus \{\emptyset\}, \subseteq)$ is equivalent to the requirement that for every $V \in \tau \setminus \{\emptyset\}$ the set $O \cap V \neq \emptyset$; in other words, $D_{\leqslant O}$ is dense in $(\tau \setminus \{\emptyset\}, \subseteq)$ iff O is dense in (X, τ).

(b) We shall also use the following variation of part (a): if X is a compact Hausdorff topological space and O is a dense open subset of X, then
$$\overline{D}_{\leqslant O} := \{U \in \tau \setminus \{\emptyset\} \colon \overline{U} \subseteq O\}$$
is dense in $(\tau \setminus \{\emptyset\}, \subseteq)$. In fact, if V is a non-empty open set in X, $V \cap O \neq \emptyset$ and the regularity of X [Rud87, Theorem 2.7] allows us to find a non-empty open set U with $\overline{U} \subseteq V \cap O$. Consequently, $U \subseteq V$, $U \in \overline{D}_{\leqslant O}$, and we are done.

(c) Assume that I and J are non-empty sets with I infinite and consider the poset $(Fn(I,J),\leqslant)$. Examples of dense subsets of $Fn(I,J)$ are (for $i\in I$ and $j\in J$)

$$\{p\in Fn(I,J)\colon i\in\operatorname{dom}(p)\}\qquad\text{and}\qquad\{p\in Fn(I,J)\colon j\in\operatorname{ran}(p)\}.$$

In fact, every finite function admits an extension whose range contains j and whose domain contains i.

Before we proceed, let us give an heuristic explanation of a common use of Martin's Axiom, which also motivates the terminology introduced so far. The rough idea is that we wish to construct a certain object with some prescribed properties; we are then invited to find a partially ordered set $\mathbb{P}$, whose elements are 'approximations' of the desired object and with $p\leqslant q$ to be interpreted as the claim that p is a better approximation than q. A dense subset D of $\mathbb{P}$ (or, more generally, a collection $\mathcal{D}$ of such dense sets) represents a list of conditions we wish our object to satisfy. Martin's Axiom claims the existence of a 'good collection' of approximations having the properties encoded in $\mathcal{D}$. Once this collection of approximations is available, one may try to glue them together (respecting the $\leqslant$) and prove that the obtained object is as desired.

We shall give below a few proofs where this heuristic recipe is implemented. Prior to this, and before the formal statement of Martin's Axiom, we shall prove (in ZFC) a countable version of MA, as a motivation for MA itself.

Lemma 2.4.20 (Rasiowa–Sikorski)**.** *Let $(\mathbb{P},\leqslant)$ be a poset and $\mathcal{D}=(D_n)_{n=1}^{\infty}$ be a countable collection of dense subsets of $\mathbb{P}$. Then there exists a decreasing sequence $(p_n)_{n=1}^{\infty}$ in $\mathbb{P}$ that intersects every element of $\mathcal{D}$.*

Proof. Assuming, inductively, to have already found $p_1\geqslant p_2\geqslant\ldots\geqslant p_n$, the density of D_{n+1} allows us to find $p_{n+1}\leqslant p_n$ with $p_{n+1}\in D_{n+1}$. ■

The decreasing sequence $(p_n)_{n=1}^{\infty}$ in the conclusion of the lemma is the 'good collection' of approximations we were hinting at before. As an example where the collection of approximations given by the Rasiowa–Sikorski lemma is used to construct an object with specific properties, consider the construction of the Gurariĭ space $\mathbb{G}$ described in [GaKu11].

We wish to have the same statement as in the Rasiowa–Sikorski lemma, but admitting sets $\mathcal{D}$ with larger cardinality; in other words, we wish to admit the possibility to insert more than countably many conditions in the construction. In this case, the existence of a decreasing sequence in $\mathbb{P}$ that intersects every element of $\mathcal{D}$ is a too strong requirement and it is therefore replaced by the existence of a subset with the following monotonicity property.

Definition 2.4.21. A subset A of a poset $(\mathbb{P},\leqslant)$ is *directed* if every pair of elements in A admits a common extension in A; *i.e.*, for every $p,q\in A$ there exists $r\in A$ with $r\leqslant p$ and $r\leqslant q$.

As a consequence, every finite subset of a directed set $A \subseteq \mathbb{P}$ admits a common extension in A. Even replacing the existence of a decreasing sequence with the existence of a directed subset that intersects every element of $\mathcal{D}$, the conclusion of the Rasiowa–Sikorski lemma fails to hold if we allow uncountable sets $\mathcal{D}$. As a simple example for this phenomenon, consider:

Example 2.4.22. Consider the poset $Fn(\omega, \omega_1)$ and, for $\alpha < \omega_1$, the set

$$D_\alpha := \{p \in Fn(\omega, \omega_1) \colon \alpha \in \operatorname{ran}(p)\};$$

according to Example 2.4.19 (c), we know that such sets are dense. Assume now that there exists a directed set $A \subseteq Fn(\omega, \omega_1)$ that intersects every D_α, $\alpha < \omega_1$; since every two functions in A are compatible, there exists a function f_A, defined on some subset of ω and with values in ω_1, that extends every element of A. If $p \in A \cap D_\alpha$, by definition $\alpha \in \operatorname{ran}(p) \subseteq \operatorname{ran}(f_A)$; consequently, $A \cap D_\alpha \neq \emptyset$ for every $\alpha < \omega_1$ implies that f_A is surjective, which is obviously impossible.

Of course, for the above example we needed to consider $Fn(I, J)$ with J uncountable, in which case Example 2.4.16 (b) tells us that $Fn(I, J)$ is not ccc. This suggests that we add the ccc assumption on the poset $\mathbb{P}$ and finally leads us to the statement of Martin's Axiom.

Definition 2.4.23. *MA_κ* is the statement that for every ccc poset $\mathbb{P}$ and every family $\mathcal{D}$ of dense subsets of $\mathbb{P}$, with $|\mathcal{D}| \leqslant \kappa$, there exists a dense subset of $\mathbb{P}$ that intersects every element of $\mathcal{D}$.

MA is the statement that MA_κ holds whenever $\kappa < \mathfrak{c}$.

Finally, $\mathfrak{m}$ is the least cardinal number κ for which MA_κ fails to hold.

It is obvious from the definition that MA_κ implies the validity of MA_λ whenever $\lambda < \kappa$; moreover, MA_ω is true in ZFC, as a particular case of the Rasiowa–Sikorski lemma. In order to justify the requirement that $\kappa < \mathfrak{c}$ in the statement MA, let us now prove that MA_κ implies that $\kappa < \mathfrak{c}$. This will also be a consequence of Theorems 2.4.25 or 2.4.27 applied to $[0, 1]$, but we are going to present a direct proof here (*cf.* [Kun80b, Lemma III.3.13]), as an illustration of how to construct objects with ideal properties under MA.

Lemma 2.4.24. *If MA_κ holds, then $\kappa < \mathfrak{c}$.*

Let us first give the rough idea of the argument, which is an elaboration over Example 2.4.22. We shall consider the poset $Fn(I, J)$, where I and J are countably infinite sets; if A is any directed set in $Fn(I, J)$, then we may find a function f_A that simultaneously extends every element of A. By using countably many conditions, we can easily force f_A to be defined on the whole I; moreover, if $h \colon I \to J$, we can also force f_A to be distinct from h. Since there are continuum many such functions h, the validity of $\mathrm{MA}_\mathfrak{c}$ would imply that f_A is distinct from every $h \colon I \to J$, a contradiction.

Proof. Let us fix sets I and J with $|I| = |J| = \omega$ and consider the ccc poset $Fn(I, J)$ (the ccc follows from the fact that $Fn(I, J)$ is countable). Let us now consider, for a function $h\colon I \to J$, the dense set $D_h := \{p \in Fn(I, J)\colon p \not\leqslant h\}$, *i.e.*, the collection of all finite functions that are *not* extended by h. The verification that D_h is dense is very easy: if $q \in \mathbb{P}$, just find $i \notin \mathrm{dom}(q)$ and let $p \in \mathbb{P}$ be an extension of q such that $p(i) \neq h(i)$.

Since the collection of dense sets

$$\{D_h\colon h \in J^I\} \cup \{\{p \in \mathbb{P}\colon i \in \mathrm{dom}(p)\}\colon i \in I\}$$

has cardinality the continuum, the validity of $\mathrm{MA}_{\mathfrak{c}}$ would imply the existence of a directed set A that meets each of the above dense sets. In particular, the common extension f_A of the functions in A has domain I. However, if $p \in D_h \cap A$, by definition h does not extend p, while f_A is an extension of it; as a consequence, f_A is distinct from every $h \in J^I$, a contradiction. ■

As a consequence of the lemma, we have that $\omega_1 \leqslant \mathfrak{m} \leqslant \mathfrak{c}$ and MA can be restated as the equality $\mathfrak{m} = \mathfrak{c}$. Of course, the Continuum Hypothesis implies Martin's Axiom, but, remarkably, many consequences of CH can actually be derived directly from MA (let us refer to the article [MaSo70] for this perspective). On the other hand, Martin's Axiom plus the negation of the Continuum Hypothesis MA+¬CH is relatively consistent with ZFC, as proved by Solovay and Tennenbaum [SoTe71]; also see [Cie97, §9.5] or [Kun80b, §V.4]. It is also very frequent to be able to obtain results under the assumption of MA+¬CH, or even of the weaker MA_{ω_1}, as it will be the case for the uses in the present work. (Let us note here that MA_{ω_1} is strictly weaker than MA+¬CH, [Wei84, p. 835].)

We shall now prove two consequences of MA, the former being a version of Baire Category theorem and the latter its analogue for Lebesgue null sets. These proofs are just a sample of how to use Martin's Axiom to construct objects with prescribed properties, as hinted at before. Many more (mostly, topological) consequences can be found, *e.g.*, in [Kun80b], [Rud75], or [Rud77]; an entire monograph deserves being dedicated to the subject, [Fre84].

Theorem 2.4.25 (MA_κ)**.** *Let K be a ccc compact Hausdorff topological space and let O_α ($\alpha < \kappa$) be open dense subsets of K. Then $\cap_{\alpha<\kappa} O_\alpha$ is dense in K.*

Proof. Fix a non-empty open subset O of K; we shall prove that $(\cap_{\alpha<\kappa} O_\alpha) \cap O \neq \emptyset$. Consider the poset $(\mathbb{P}, \leqslant)$ comprising all non-empty open subsets p of K with $\overline{p} \subseteq O$, with $p \leqslant q$ iff $p \subseteq q$. Like in Example 2.4.16 (a), it is immediate to verify that $(\mathbb{P}, \leqslant)$ is ccc; moreover, for $\alpha < \kappa$ the sets

$$\overline{D}_{\leqslant O_\alpha} := \{p \in \mathbb{P}\colon \overline{p} \subseteq O_\alpha\}$$

are dense subsets of $(\mathbb{P}, \leqslant)$ (the argument in Example 2.4.19 (b) shows this).

Consequently, MA_κ implies the existence of a directed subset A of $\mathbb{P}$ that meets every dense set $\overline{D}_{\leqslant O_\alpha}$; in particular A is a collection of open sets with the finite intersection property. The compactness of K then implies

$$p_A := \bigcap_{p \in A} \overline{p} \neq \emptyset;$$

moreover, $p_A \subseteq O$ since every $p \in A$ is an element of the poset $\mathbb{P}$. Finally, for every $\alpha < \kappa$ select $p \in A \cap \overline{D}_{\leqslant O_\alpha}$, whence $p_A \subseteq \overline{p} \subseteq O_\alpha$. Consequently, $p_A \subseteq (\cap_{\alpha<\kappa} O_\alpha) \cap O$. ∎

Remark 2.4.26. In the argument, the ideal object we wish to construct is the set p_A, to witness that $\cap O_\alpha$ intersects O. Elements of $\mathbb{P}$ are outer approximations of it, with the information to be contained in O; a 'better' approximation is more likely to be contained in $\cap O_\alpha$, since $\leqslant$ is $\subseteq$. The additional properties of our object are encoded in the sets $\overline{D}_{\leqslant O_\alpha}$.

As an immediate consequence of the result, if H_α ($\alpha < \kappa$) are nowhere dense subsets of K, then $\cup H_\alpha \neq K$; when applied to the singletons of the ccc compact space $[0, 1]$, this yields one more proof that $\mathrm{MA}_{\mathfrak{c}}$ is false. The above theorem then leads us consider the topological version of MA to be the statement

No ccc compact Hausdorff topological space is the union of less than $\mathfrak{c}$ its nowhere dense subsets.

As it turns out, such axiom can be proved equivalent (in ZFC) to MA itself, *cf.* [Kun80b, Theorem III.4.7], or [Wei84, Theorem 1.7]. Other similar equivalents of MA are also given in [Fre84, Theorem 13.A].

In the classical statement of Baire Category theorem the compactness assumption can be replaced by completeness and, perhaps not surprisingly, the above result admits more general variants; let us refer *e.g.*, to [Rud77, Theorem 14], [Kun80b, p. 194], or [Juh77, Theorem 1.2] for three such instances. On the other hand, there is not a complete analogy with the Baire Category theorem: a very simple (not ccc) example is based on the fact that the complete metric space $\ell_2(\omega_1)$ is the union of ω_1 many its closed proper subspaces, [Kun80b, Exercise III.3.86]. Let us also refer to [Wei84, Example 1.10] for a more striking example of a regular Baire space with ccc, which is union of ω_1 nowhere dense subsets.

In the opposite direction, let us also mention here the result by Baumgartner [Bau85] of the relative consistency with ZFC that $\mathfrak{c} > \omega_1$ and *every* ccc compact Hausdorff topological space without isolated points is the union of ω_1 nowhere dense subsets. In other words, it is relatively consistent with ZFC+$\neg$CH that the conclusion of the topological version of MA is false *for every* ccc compact Hausdorff topological space without isolated points (note that, trivially, a topological space with at least one isolated point is never union of nowhere dense subsets).

We then give the measure theoretic analogue to the above result. The poset to be used in the next proof is sometimes called *amoeba order*, for reasons explained by Kunen after the proof of the result, [Kun80b, Lemma III.3.28].

Theorem 2.4.27 (MA_κ)**.** *The union of at most κ Lebesgue null subsets of $\mathbb{R}^d$ is Lebesgue null.*

Proof. Let E_α $(\alpha < \kappa)$ be Lebesgue null subsets of $\mathbb{R}^d$ and set $E := \cup_{\alpha<\kappa} E_\alpha$. Let us also fix $\varepsilon > 0$; our aim is to find an open subset of $\mathbb{R}^d$ that contains E and whose measure does not exceed ε (in this proof, we shall denote m Lebesgue measure on $\mathbb{R}^d$).

Consider the poset $\mathbb{P}$ consisting of all open sets p of $\mathbb{R}^d$ with $m(p) < \varepsilon$ endowed with the partial order $p \leqslant q$ iff $p \supseteq q$ [4]. Our first duty is the verification of the ccc; in order to achieve this, let us preliminarily note that $p, q \in \mathbb{P}$ are incompatible if and only if $m(p \cup q) \geqslant \varepsilon$. Let therefore $Q \subseteq \mathbb{P}$ be an antichain and consider the sets

$$Q_n := \{q \in Q \colon m(q) < (1 - 2^{-n}) \cdot \varepsilon\};$$

it is sufficient to verify that every Q_n is a countable set. For $q \in Q_n$, select an open subset $\tilde{q}$ of q that is finite union of open rectangles with rational endpoints and such that $m(q \setminus \tilde{q}) < 2^{-n} \cdot \varepsilon$. Observe that if $\tilde{p} = \tilde{q}$ for distinct $p, q \in Q_n$, then $p \cup q = p \cup (q \setminus \tilde{q})$, whence $m(p \cup q) < \varepsilon$; a contradiction with $p \perp q$. Consequently, the correspondence $q \mapsto \tilde{q}$ is injective and, the collection of all $\tilde{q}$ being countable, we conclude that Q_n is countable.

Let us now consider the dense sets $D_\alpha := \{p \in \mathbb{P} \colon E_\alpha \subseteq p\}$ $(\alpha < \kappa)$. The verification of the density is a much simpler task: given an open set $p \in \mathbb{P}$, $m(p) < \varepsilon$ and the fact that E_α is Lebesgue null yield the existence of an open set q such that $E_\alpha \subseteq q$ and $m(q) < \varepsilon - m(p)$. Then $p \cup q \in D_\alpha$ witnesses the density of D_α.

Let, finally, A be a directed subset of $\mathbb{P}$ that meets every D_α and consider the open set $G_A := \cup_{p \in A} p$; clearly, $E = \cup E_\alpha \subseteq G_A$. Finally, the (hereditary) Lindelöf property of $\mathbb{R}^d$ implies the existence of a countable subset $(p_n)_{n=1}^\infty$ of A such that $G_A = \cup_{n=1}^\infty p_n$. Moreover, for every $N \in \mathbb{N}$, $m(p_1 \cup \cdots \cup p_N) < \varepsilon$, since A is directed; hence, $m(G_A) \leqslant \varepsilon$. ■

In conclusion to this section we shall state one more consequence of Martin's Axiom, to be used in the proof of Theorem 4.1.15. Before we state the result we need one more definition (*cf.* [Wei84, §3]).

Definition 2.4.28. A subset F of a partially ordered set $(\mathbb{P}, \leqslant)$ is *centred* if every its finite subset admits a common extension in $\mathbb{P}$, namely, for every $p_1, \dots, p_k \in F$ there exists $r \in \mathbb{P}$ with $r \leqslant p_i$ $(i = 1, \dots, k)$.

A poset $(\mathbb{P}, \leqslant)$ is *σ-centred* whenever it can be expressed as countable union of centred subsets.

Every directed subset of $\mathbb{P}$ is an example of a centred set, *a fortiori*; in the case of the poset $(\tau \setminus \{\emptyset\}, \subseteq)$, a collection $A \subseteq \tau \setminus \{\emptyset\}$ is centred if and only if it has the finite intersection property.

One more immediate observation is that every σ-centred poset is ccc; we may therefore understand this condition as a chain condition. Further examples of such chain conditions can be found in the above mentioned [Wei84, §3], or in the monograph [CoNe82]. The consequence of MA to be stated below yields the validity of the converse implication for a certain class of posets; for its proof, we refer to [Wei84, Theorem 4.5].

[4] Our ideal object is an open set with small measure that contains every E_α; consequently, a larger open set in $\mathbb{P}$ is a better approximation. Accordingly, our ideal object will be obtained as the *union* of a certain directed collection of elements of $\mathbb{P}$.

Theorem 2.4.29 (MA_κ). *Every ccc poset of cardinality at most κ is also σ-centred.*

2.5 Uncountable extensions of Pták's lemma

This section is dedicated to the proof of the uncountable versions of Pták's combinatorial lemma that we stated above. As we already mentioned, we shall divide our arguments in three parts: in the first one, we shall present some general observations that will be of use in all the proofs. The second section is dedicated to the first uncountable cardinal, while in the last one we are concerned with larger cardinal numbers.

2.5.1 General remarks

If S is any set, a subset A of S can be naturally identified, via the correspondence $A \mapsto \chi_A$, with an element of the compact topological space $\{0,1\}^S$, endowed with the canonical product topology. Let us recall that, under this identification, if $A \in 2^S$, a basis of neighbourhoods of A is given by the collection of sets

$$\left\{B \in 2^S \colon F \subseteq B \subseteq S \setminus G\right\},$$

where F and G are finite subsets of A and $S \setminus A$ respectively. Throughout this section, we shall make this identification and we shall not distinguish between the set A and its characteristic function χ_A. Therefore, when $\mathcal{F} \subseteq 2^S$, we may consider the closure $\overline{\mathcal{F}}$ of $\mathcal{F}$, in the product topology of $\{0,1\}^S$; henceforth, whenever we use the notation $\overline{\mathcal{F}}$ it will be the product topology the one under consideration.

In the case that $\mathcal{F}$ is an hereditary family, it is easily seen that $\overline{\mathcal{F}}$ is an *adequate compact*, in the sense of the following definition, first introduced by Talagrand [Tal79, Tal84]. A family $\mathcal{G} \subseteq 2^S$ is said to be *adequate* if:

(i) whenever $G \subseteq F$ and $F \in \mathcal{G}$, then $G \in \mathcal{G}$, *i.e.*, $\mathcal{G}$ is hereditary;

(ii) if every finite subset of G belongs to $\mathcal{G}$, then $G \in \mathcal{G}$ too.

Conversely, every adequate family $\mathcal{G}$ can be expressed as $\overline{\mathcal{F}}$, for some hereditary family of finite sets, namely $\mathcal{F} = \{F \in \mathcal{G} \colon |F| < \omega\}$; in particular, every adequate family is a closed subset of $\{0,1\}^S$. As it turns out, compact sets that originate from adequate families of sets are very fascinating objects in Functional Analysis and have been exploited in several important examples; let us refer, *e.g.*, to [AAM09, AMN88, Ark92, BeSt76, Lei88, LeSo84, Ple95, Tal79, Tal84] for a sample of some of these constructions.

Our interest in adequate families originates from the following fact, a particular case of the observation that $\overline{\mathcal{F}}$ is adequate, whenever $\mathcal{F}$ is hereditary.

Fact 2.5.1. *Let $\mathcal{F}$ be an hereditary family and $M \in \overline{\mathcal{F}}$. Then every finite subset of M belongs to $\mathcal{F}$.*

In particular, if a finite set $M \in \overline{\mathcal{F}}$, then actually M belongs to $\mathcal{F}$.

Proof. Assume that $M \in \overline{\mathcal{F}}$, where $\mathcal{F}$ is an hereditary family, and let F be a finite subset of M. Since the set
$$\left\{B \in 2^S \colon F \subseteq B\right\}$$
is a neighbourhood of M, some element B of such neighbourhood belongs to $\mathcal{F}$. $\mathcal{F}$ being hereditary, $F \subseteq B \in \mathcal{F}$ yields $F \in B$, and we are done. ■

The next proposition is the non-separable counterpart to the argument in [BHO89, Lemma 3.1], with the same proof, which we include for the sake of completeness.

Proposition 2.5.2. *Let S be an infinite set and $\mathcal{F} \subseteq [S]^{<\omega}$ be an hereditary family such that (†) holds. Then $C(\overline{\mathcal{F}})$ contains an isomorphic copy of $\ell_1(S)$.*

Proof. Let us preliminarily note that if λ is any convex mean on S, then $\sup_{F\in\mathcal{F}} \lambda(F) \geqslant \delta$; since this supremum is actually over the finite set consisting of all $F \subseteq \mathrm{supp}(\lambda)$, it follows that there exists $F \in \mathcal{F}$ with $\lambda(F) \geqslant \delta$. Consequently, for every finitely supported function $\lambda \colon S \to [0, \infty)$ there exists $F \in \mathcal{F}$ such that
$$\sum_{s\in F} \lambda(s) \geqslant \delta \cdot \sum_{s\in S} \lambda(s).$$

For an element $x = (x(s))_{s\in S} \in c_{00}(S)$, let us define
$$\|x\| := \sup \left\{ \left| \sum_{s\in F} x(s) \right| : F \in \mathcal{F} \right\};$$
we claim that $\|\cdot\|$ is a norm on $c_{00}(S)$, equivalent to the $\|\cdot\|_1$ norm. In order to prove this, fix $x \in c_{00}(S)$ and let P be the finite set $P := \{s \in S \colon x(s) > 0\}$; up to replacing x with $-x$, we may assume without loss of generality that
$$\sum_{s\in P} x(s) \geqslant \frac{1}{2} \sum_{s\in S} |x(s)|.$$
Moreover, our assumption implies the existence of $F \in \mathcal{F}$, with $F \subseteq P$, such that
$$\delta \cdot \sum_{s\in P} x(s) \leqslant \sum_{s\in F} x(s).$$
Consequently, we obtain
$$\frac{\delta}{2} \cdot \sum_{s\in S} |x(s)| \leqslant \delta \cdot \sum_{s\in P} x(s) \leqslant \sum_{s\in F} x(s) \leqslant \|x\|,$$
which proves our claim. In particular, the completion X of $(c_{00}(S), \|\cdot\|)$ is isomorphic to $\ell_1(S)$.

Associated with $F \in 2^S$ there is a naturally defined functional $F^* \in X^*$, given by $F^*x := \sum_{s\in F} x(s)$; note that F^* is well defined for every $F \subseteq S$ in light of the fact that X

is isomorphic to $\ell_1(S)$. It is also clear from the definition of $\|\cdot\|$ that $F^* \in B_{X^*}$, whenever $F \in \mathcal{F}$. Moreover, the correspondence $F \mapsto F^*$ defines a function $\Phi\colon \{0,1\}^S \to (X^*, w^*)$, which is easily seen to be continuous and, of course, injective. It readily follows that Φ establishes an homeomorphism between $\overline{\mathcal{F}} \subseteq \{0,1\}^S$ and $\overline{\mathcal{F}^*}^{w^*} \subseteq B_{X^*}$, where $\mathcal{F}^* := \Phi(\mathcal{F})$.

Finally, it is a standard fact that X isometrically embeds into $C\left(\overline{\mathcal{F}^*}^{w^*}\right) = C\left(\overline{\mathcal{F}}\right)$, as a consequence of $\mathcal{F}^*$ clearly being 1-norming for X. The fact that X is isomorphic to $\ell_1(S)$ then allows us to conclude the proof. ■

Remark 2.5.3. Since the argument is completely direct, it is actually possible to keep track of the various embeddings and localise precisely the position of $\ell_1(S)$ into $C\left(\overline{\mathcal{F}}\right)$; more precisely, it is possible to describe the vectors in $C\left(\overline{\mathcal{F}}\right)$ that correspond to the canonical basis of $\ell_1(S)$.

For $s \in S$, let us denote by $\pi_s : \{0,1\}^S \to \{0,1\}$ the canonical projection and let V_s be the clopen set

$$V_s := \pi_s^{-1}(\{1\}) \cap \overline{\mathcal{F}} = \left\{F \in \overline{\mathcal{F}}\colon s \in F\right\}.$$

Inspection of the proof of the previous proposition shows that, assuming (†), the collection $(\chi_{V_s})_{s\in S} \subseteq C\left(\overline{\mathcal{F}}\right)$ is equivalent to the canonical basis of $\ell_1(S)$.

A simple modification of an argument given in the course of the proof of Theorem 2.4.4(ii) will also prove the validity of the converse implication.

Remark 2.5.4. From the appearance of Rosenthal's celebrated paper [Ros74], a well known criterion to prove that a family $(f_\alpha)_{\alpha<\tau} \subseteq B_{C(K)}$ is equivalent to the canonical basis of $\ell_1(\tau)$ consists in showing that, for some reals r and $\delta > 0$, the collection of sets

$$(\{f_\alpha \leqslant r\}, \{f_\alpha \geqslant r + \delta\})_{\alpha<\tau}$$

is independent. (We refer to [Ros74, Proposition 4] for the definition of the notion of independence and for the simple proof of this claim.) It is perhaps of interest to note that the copy of $\ell_1(S)$ obtained in Proposition 2.5.2 does not originate from such criterion, unless we are in the trivial case that $\mathcal{F} = [S]^{<\omega}$.

In fact, if there were reals r and $\delta > 0$ such that

$$(\{\chi_{V_s} \leqslant r\}, \{\chi_{V_s} \geqslant r + \delta\})_{s\in S}$$

is independent, then this would imply that $(V_s^{\complement}, V_s)_{s\in S}$ is an independent family. As a consequence, for distinct $s_1, \dots, s_n \in S$ we would have

$$\emptyset \neq V_{s_1} \cap \cdots \cap V_{s_n} = \left\{F \in \overline{\mathcal{F}}\colon \{s_1, \dots, s_n\} \subseteq F\right\};$$

it would follow from this and $\overline{\mathcal{F}}$ being hereditary that $\{s_1, \dots, s_n\} \in \mathcal{F}$, hence $\mathcal{F} = [S]^{<\omega}$.

In conclusion to this section, let us record how the results presented so far imply the validity of the original statement of Pták's lemma.

Proof of Lemma 2.4.2. We start observing that, without loss of generality, we can assume $|S| = \omega$. Let us, in fact, consider a subset S_1 of S such that $|S_1| = \omega$ and the hereditary family $\mathcal{F} \cap S_1 := \{F \in \mathcal{F} \colon F \subseteq S_1\}$. If λ is any convex mean on S_1, we may extend it to S in the obvious (and unique) way; plainly, if $F \in \mathcal{F}$, $\lambda(F) = \lambda(F \cap S_1)$, where $F \cap S_1 \subseteq \mathcal{F} \cap S_1$. Consequently, up to replacing S with S_1 and $\mathcal{F}$ with $\mathcal{F} \cap S_1$, we may assume that $|S| = \omega$.

Now, Proposition 2.5.2 yields that $C(\overline{\mathcal{F}})$ contains a copy of ℓ_1, which in turn implies that $C(\overline{\mathcal{F}})$ is not an Asplund space. As a consequence of this, $\overline{\mathcal{F}}$ is necessarily uncountable and it can not be a subset of the countable set $[S]^{<\omega}$. Fact 2.5.1 leads us to the desired conclusion. ■

2.5.2 Pták's lemma for ω_1

This section is dedicated to the proof of Theorem 2.4.4; both clauses will heavily depend on results from [AMN88]. The proof of claim (i) is essentially the same argument as in the proof of Lemma 2.4.2 given above, but with the Asplund property being replaced by the WLD one. Let us recall a bit of terminology, in order to explain this.

A compact space K has property (M) if every regular Borel measure on K has separable support. Here, for the *support* of a Borel measure μ on K, we understand the closed set

$$\mathrm{supp}(\mu) := \{x \in K \colon |\mu|(U) > 0 \text{ for every neighbourhood } U \text{ of } x\}.$$

An elementary verification, based on the regularity of μ, readily shows that $\mu(K\backslash\mathrm{supp}(\mu)) = 0$; it follows, in particular, that the support of μ is a singleton set precisely when μ is multiple of a Dirac delta. One more elementary property is that if a sequence $(\mu_n)_{n=1}^{\infty} \subseteq M(K)$ converges to a measure μ in the w^* topology of $M(K) = C(K)^*$, then

$$\mathrm{supp}(\mu) \subseteq \overline{\cup_{n=1}^{\infty}\mathrm{supp}(\mu_n)}$$

We shall need the following topological characterisation of WLD Banach spaces of continuous functions, due to Argyros, Mercourakis, and Negrepontis [AMN88, Theorem 3.5].

Theorem 2.5.5 ([AMN88])**.** *Let K be a compact topological space. Then $C(K)$ is WLD if and only if K is a Corson compact with property (M).*

Proof. We shall only discuss the necessity of the condition; we shall therefore assume that the dual unit ball, in the w^* topology, $(B_{M(K)}, w^*)$ is a Corson compact. As it is well known, K is homeomorphic to a subset of $(B_{M(K)}, w^*)$, which immediately implies that K is a Corson compact.

In order to prove property (M), we will use the equally well known fact that

$$B_{M(K)} = \overline{\mathrm{conv}}^{w^*}\{\pm\delta_x \colon x \in K\}$$

(this can be seen as a consequence of the Hahn–Banach theorem, or of the Krein–Milman theorem, or given a simple direct proof). So, fix a measure $\mu \in B_{M(K)}$; the Fréchet–Urysohn property of $(B_{M(K)}, w^*)$ yields the existence of a sequence $(\mu_n)_{n=1}^{\infty} \subseteq \operatorname{conv}\{\pm\delta_x \colon x \in K\}$ that w^* converges to μ. As a consequence, $\operatorname{supp}(\mu) \subseteq \overline{\cup \operatorname{supp}(\mu_n)}$, where $\overline{\cup \operatorname{supp}(\mu_n)}$ is evidently a separable Corson compact. It follows that $\operatorname{supp}(\mu)$ is also separable, according to Fact 2.1.4.

For the proof of the converse implication we shall refer to [AMN88, Theorem 3.5], [Kal00b, Theorem 5.4], or [KKL11, Theorem 19.21]. ■

The rôle of Martin's axiom MA_{ω_1} in connection with the above result is that, under MA_{ω_1}, every Corson compact has property (M) (*cf.* [AMN88, Remark 3.2.3] or [HMVZ08, Theorem 5.62]). More precisely, recall that a compact space K satisfies the *countable chain condition* (*ccc*, for short) if every collection of non-empty disjoint open sets in K is at most countable. It is clear that if μ is a regular Borel measure on K, then the support of μ is ccc.

Consequently, the previous claim follows once we show that, under MA_{ω_1}, every ccc Corson compact K is separable ([CoNe82, p. 201, Theorem (b)], or [Fre84, p. 207, Exercise (i)]); according to Fact 2.1.4, it is equivalent to prove that K has calibre ω_1 . Let us also mention that it is possible to give a notion of *pre-calibre* for a partially ordered set and prove a version of the result in the context of posets, from which the topological form follows; we refer to [Wei84, Theorem 4.2] and [Kun80b, Exercise III.3.45].

Theorem 2.5.6 (MA_{ω_1}). *Every ccc compact Hausdorff topological space K has calibre ω_1.*

Proof. We shall first observe that if U is an open subset of K, then $\overline{U}$ is also ccc. In fact, assume that $\{O_\gamma\}_{\gamma\in\Gamma}$ is a collection of mutually disjoint non-empty open subsets of $\overline{U}$; by definition, $O_\gamma \cap U$ $(\gamma \in \Gamma)$, are non-empty open subsets of U, hence of K. Since they are also pairwise disjoint, the ccc of K implies $|\Gamma| \leqslant \omega$.

Let now $(O_\alpha)_{\alpha<\omega_1}$ be a collection of non-empty open subsets of K and consider the open sets $U_\alpha := \cup_{\alpha\leqslant\beta<\omega_1} O_\beta$ $(\alpha < \omega_1)$. We claim that for all but countably many α's, we have $U_\alpha \subseteq \overline{U_{\alpha+1}}$; in fact, if not, the sets $U_\alpha \setminus \overline{U_{\alpha+1}}$, for α in this uncountable set, would be uncountably many disjoint non-empty open sets, contradicting ccc. As $U_{\alpha+1} \subseteq U_\alpha$, it follows the existence of an ordinal α_0 such that $\overline{U_\alpha} = \overline{U_{\alpha_0}}$ whenever $\alpha_0 \leqslant \alpha < \omega_1$.

We therefore have a collection $\{U_\alpha\}_{\alpha_0\leqslant\alpha<\omega_1}$ of dense open subsets of the ccc compact space $\overline{U_{\alpha_0}}$ (the ccc follows from the observation opening the proof). Consequently, the topological version of Martin's Axiom, Theorem 2.4.25, yields the existence of $x \in \cap_{\alpha_0\leqslant\alpha<\omega_1} U_\alpha$; it immediately follows that such x belongs to uncountably many O_α's, and we are done. ■

The combination of the above considerations assures us that, assuming MA_{ω_1}, a compact space K is Corson if and only if $C(K)$ is WLD; by means of this equivalence, we may now readily prove the first part of Theorem 2.4.4.

Proof of Theorem 2.4.4(i). According to Proposition 2.5.2, $C(\overline{\mathcal{F}})$ contains an isomorphic copy of $\ell_1(\omega_1)$ and, therefore, it fails to be WLD. Consequently, $\overline{\mathcal{F}}$ is not Corson and it follows immediately that there exists $M \in \overline{\mathcal{F}}$ with $|M| \geqslant \omega_1$; in fact, if this were false, then the inclusion map $\overline{\mathcal{F}} \subseteq [0,1]^{\omega_1}$ would witness the fact that $\overline{\mathcal{F}}$ is Corson. We may therefore apply Fact 2.5.1 and conclude the proof. ■

As it turns out, assuming some additional set-theoretic axioms is necessary for the validity of the results described above. In particular, the Continuum Hypothesis allows for the construction of Corson compacta failing property (M). The first such example was constructed by Kunen in [Kun81] and one its generalisation, the Kunen–Haydon–Talagrand example, is described in [Neg84, §5], combining Kunen's construction with Haydon's and Talagrand's examples, [Hay78, Tal80]. Such compact K also has the property that $C(K)$ fails to contain an isomorphic copy of $\ell_1(\omega_1)$. One simpler example, still under CH, is based upon the Erdős space and may be found in [AMN88, Theorem 3.12] or [HMVZ08, Theorem 5.60]. Interestingly, if the Corson compact K is an adequate compact, then K fails to have property (M) if and only if $C(K)$ contains an isomorphic copy of $\ell_1(\omega_1)$ [AMN88, Theorem 3.13]; in other words, $C(K)$ contains $\ell_1(\omega_1)$, whenever it fails to be WLD.

The proof of claim (ii) in Theorem 2.4.4 that we shall give presently will also implicitly depend on the Erdős space and is based on a combination of arguments from [AMN88, Theorems 3.12, 3.13].

Proof of Theorem 2.4.4(ii). The assumption of the validity of the Continuum Hypothesis allows us to enumerate in an ω_1-sequence $(K_\alpha)_{\alpha<\omega_1}$ the collection of all compact subsets of $[0,1]$ with positive Lebesgue measure (which, in what follows, we shall denote $\mathscr{L}$). We may also let $(x_\alpha)_{\alpha<\omega_1}$ be a well ordering of the interval $[0,1]$. The set $K_\alpha \cap \{x_\beta\}_{\alpha\leqslant\beta<\omega_1}$ having positive measure, the regularity of $\mathscr{L}$ allows us to select a compact subset C_α of $K_\alpha \cap \{x_\beta\}_{\alpha\leqslant\beta<\omega_1}$ such that $\mathscr{L}(C_\alpha) > 0$, for each $\alpha < \omega_1$. Note that if $A \subseteq \omega_1$ is any uncountable set, then $\sup A = \omega_1$ and it follows that

$$\bigcap_{\alpha\in A} C_\alpha \subseteq \bigcap_{\alpha\in A} \{x_\beta\}_{\alpha\leqslant\beta<\omega_1} = \emptyset.$$

We are now in position to define a Corson compact that fails property (M). Consider the set

$$\mathcal{A} := \left\{ A \subseteq \omega_1 \colon \bigcap_{\alpha\in A} C_\alpha \neq \emptyset \right\};$$

if every finite subset of a given set A belongs to $\mathcal{A}$, then the collection of closed sets $\{C_\alpha\}_{\alpha\in A}$ has the finite intersection property and $A \in \mathcal{A}$ follows by compactness. Consequently, $\mathcal{A}$ is an adequate compact. Moreover, the previous consideration shows that every $A \in \mathcal{A}$ is a countable subset of ω_1, whence $\mathcal{A}$ is a Corson compact. The proof that $\mathcal{A}$ fails to have property (M) may be found in [AMN88, Theorem 3.12], or [HMVZ08, Theorem 5.60] and we shall not reproduce it here. Therefore, we may fix a positive regular Borel measure μ on $\mathcal{A}$, whose support is not separable.

We now consider again the clopen subsets of $\mathcal{A}$ (*cf.* Remark 2.5.3)

$$V_\alpha := \pi_\alpha^{-1}(\{1\}) \cap \mathcal{A} = \{A \in \mathcal{A} \colon \alpha \in \mathcal{A}\} \qquad (\alpha < \omega_1)$$

and we shall consider the set $I := \{\alpha < \omega_1 \colon \mu(V_\alpha) > 0\}$. Plainly, for $A \in \operatorname{supp}(\mu)$, we have $\mu(V_\alpha) > 0$ whenever $\alpha \in A$; consequently, $A \subseteq I$ and we obtain that $\operatorname{supp}(\mu) \subseteq 2^I$. In light of the fact that the support of μ is not separable, it follows that I is uncountable. (Here, we are using again the fact that every subspace of a separable Corson compact is separable, *cf.* Fact 2.1.4.) In turn, we also obtain the existence of an uncountable subset S of I and a real $\delta > 0$ such that $\mu(V_\alpha) > \delta$ for $\alpha \in S$.

We may now define the desired hereditary family of finite sets: let us consider $\mathcal{F}_0 := \{F \in \mathcal{A} \colon F \text{ is a finite set}\}$ and set $\mathcal{F} := \{F \in \mathcal{F}_0 \colon F \subseteq S\}$. Clearly, $\overline{\mathcal{F}} \subseteq \overline{\mathcal{F}_0} = \mathcal{A}$; we infer, in particular, that $\overline{\mathcal{F}}$ contains no uncountable set and the conclusion of Pták's lemma for the cardinal number ω_1 fails to hold for $\mathcal{F}$.

On the other hand, for every convex mean λ on S we have

$$\left\| \sum_{s\in S} \lambda(s)\chi_{V_s} \right\|_{C(\mathcal{A})} \geqslant \mu\left(\sum_{s\in S} \lambda(s)\chi_{V_s}\right) = \sum_{s\in S} \lambda(s)\mu(V_s) > \delta \cdot \sum_{s\in S} \lambda(s) = \delta.$$

From this strict inequality and $\overline{\mathcal{F}_0} = \mathcal{A}$, we conclude the existence of $F \in \mathcal{F}_0$ such that

$$\sum_{s\in S} \lambda(s)\chi_{V_s}(F) > \delta;$$

therefore,

$$\delta < \sum_{s\in S} \lambda(s)\chi_{V_s}(F) = \sum_{s\in F\cap S} \lambda(s) = \lambda(F\cap S) \leqslant \sup_{G\in\mathcal{F}} \lambda(G)$$

and we see that $\mathcal{F}$ satisfies (†). ■

2.5.3 Larger cardinals

In this section we are going to prove Theorem 2.4.5; before entering into the core of the proof, it will be convenient to recall some results that we shall make use of in the course of the argument.

A topological space is *totally disconnected* if every non-empty connected subset is a singleton. Clearly, topological products and subspaces of totally disconnected spaces are totally disconnected.

For a topological space (X, τ) and a point $x \in X$, a *local π-basis* for x (*cf.* [Juh80, §1.15]) is a family $\mathcal{B}$ of non-empty open subsets of X such that for every neighbourhood V of x there exists $B \in \mathcal{B}$ with $B \subseteq V$ (note that B is not required to contain x). Every local basis is a local π-basis, *a fortiori*. The *pseudo-weight* of (X, τ) at x is the minimal cardinality of a local π-basis for x.

The first ingredient we need is the following result, due to Šapirovskiĭ, [Sap75] (see, *e.g.*, [Juh80, §3.18] or [Neg84, Theorem 2.11]).

Theorem 2.5.7 (Šapirovskiĭ). *Let K be a totally disconnected compact topological space and κ be an infinite cardinal number. Then there exists a continuous function from K*

onto $\{0,1\}^\kappa$ if and only if there exists a non-empty closed subset F of K such that the pseudo-weight of F at x is at least κ, for every $x \in F$.

The second building block for our proof is a characterisation, due to Richard Haydon, of those compact spaces whose associated Banach space of continuous functions contains an isomorphic copy of $\ell_1(\kappa)$, for a certain cardinal number κ. Let us, preliminarily, briefly review some results in this area.

Pełczyński [Peł68] and Hagler [Hag73] proved that a Banach space X contains an isomorphic copy of ℓ_1 (let us write $\ell_1 \hookrightarrow X$, for short) if and only if $L_1[0,1] \hookrightarrow X^*$. Pełczyński also demonstrated that, for an infinite cardinal κ, $L_1\{0,1\}^\kappa \hookrightarrow X^*$ whenever $\ell_1(\kappa) \hookrightarrow X$ and he conjectured the validity of the converse implication. (Here, by $L_1\{0,1\}^\kappa$ we understand the Lebesgue space corresponding to $\{0,1\}^\kappa$, the Borel σ-algebra and the Haar measure on the compact group $\{0,1\}^\kappa$.)

The complete solution to Pełczyński's conjecture follows from a combination of results due to Argyros and Haydon: Haydon [Hay78] proved that the conjecture is false for $\kappa = \omega_1$ under the assumption of the Continuum Hypothesis. On the other hand, Argyros [Arg82] proved the correctness of the conjecture for $\kappa \geqslant \omega_2$ (in ZFC) and for $\kappa = \omega_1$, assuming MA_{ω_1}. A different proof of Argyros' result can be obtained from [ABZ84]; let us also refer to [AHLO86, Hay77, Hay79, Neg84] for a discussion of these and related results.

In a related direction, Talagrand [Tal81] (also see [Arg83], for a simplified proof) proved that, for a cardinal number κ with $\mathrm{cf}(\kappa) \geqslant \omega_1$, $\ell_1(\kappa) \hookrightarrow X$ if and only if there exists a continuous function from (B_{X^*}, w^*) onto $[0,1]^\kappa$. The result we shall need is a similar statement in the case that X is a $C(K)$ space (*cf.* [Hay77, Remark 2.5]).

Theorem 2.5.8 (Haydon)**.** *Let κ be a regular cardinal number such that $\lambda^\omega < \kappa$ whenever $\lambda < \kappa$ and let K be a compact topological space. Then $\ell_1(\kappa) \hookrightarrow C(K)$ if and only if there exists a continuous function from K onto $[0,1]^\kappa$.*

Having recorded all the results we shall build on, we can now approach the proof of Theorem 2.4.5.

Proof of Theorem 2.4.5. Let κ be a regular cardinal number such that $\lambda^\omega < \kappa$ whenever $\lambda < \kappa$, let S be a set with $|S|$
kappa and $\mathcal{F} \subseteq [S]^{<\omega}$ be an hereditary family such that (†) holds. When we combine Proposition 2.5.2 with Haydon's result, we obtain the existence of a continuous surjection from $\overline{\mathcal{F}}$ to $[0,1]^\kappa$. Consequently, there exists a closed subset $\mathcal{K}_1$ of $\overline{\mathcal{F}}$ that continuously maps onto $\{0,1\}^\kappa$; note that, being a subspace of $\{0,1\}^\kappa$, $\mathcal{K}_1$ is totally disconnected. In light of Šapirovskiĭ's theorem, we conclude that there exists a closed subspace $\mathcal{K}$ of ($\mathcal{K}_1$, hence of) $\overline{\mathcal{F}}$ such that the pseudo-weight of $\mathcal{K}$ at x is at least κ, for every $x \in \mathcal{K}$. In particular, if $\mathcal{B}$ is any local basis for the topology of $\mathcal{K}$ at any $x \in \mathcal{K}$, then $|\mathcal{B}| \geqslant \kappa$.

Before we proceed, introducing a bit of notation is in order. If $A \in \mathcal{K}$ and F and G are finite subsets of A and $S \setminus A$ respectively, then we shall denote by

$$\mathcal{U}_A(F,G) := \{B \in \mathcal{K} \colon F \subseteq B \subseteq S \setminus G\},$$

a neighbourhood of A in $\mathcal{K}$. A particular case of this piece of notation is that

$$\mathcal{U}_A(F,\emptyset) := \{B \in \mathcal{K} \colon F \subseteq B\}.$$

Plainly,

$$\{\mathcal{U}_A(F,G) \colon F \in [A]^{<\omega}, G \in [S \setminus A]^{<\omega}\}$$

is a local basis for the topology of $\mathcal{K}$ at A.

We now note that $\mathcal{K}$ may also be considered as a partially ordered set, with respect to set inclusion. When such partial order is considered, then every chain in $\mathcal{K}$ admits an upper bound in $\mathcal{K}$; in fact, the union of the chain belongs to $\mathcal{K}$, $\mathcal{K}$ being closed in the pointwise topology. An appeal to Zorn's lemma allows us to deduce that there exist in $\mathcal{K}$ maximal elements with respect to inclusion.

Claim. If $M \in \mathcal{K}$ is any maximal element, then a local basis for the topology of $\mathcal{K}$ at M is given by

$$\mathcal{B} := \{\mathcal{U}_M(F,\emptyset) \colon F \in [M]^{<\omega}\}.$$

Since clearly $|\mathcal{B}| \leqslant |M|$, our previous considerations allow us to conclude that, if $M \in \mathcal{K}$ is any maximal element, then $|M| \geqslant \kappa$. If we select any such maximal element–whose existence we noted above–then Fact 2.5.1 assures us that M is the set we were looking for. Therefore, in order to conclude the proof, we only need to establish the claim.

Proof of the claim. Assume by contradiction that $\mathcal{B}$ is not a local basis. Then there exist finite sets $\tilde{F}$ and $\tilde{G}$ with $\tilde{F} \subseteq M$ and $\tilde{G} \subseteq S \setminus M$ such that no element of $\mathcal{B}$ is contained in $\mathcal{U}_M(\tilde{F},\tilde{G})$. In particular, for every finite set I with $\tilde{F} \subseteq I \subseteq M$ there exists an element $A_I \in \mathcal{K}$ with $A_I \in \mathcal{U}_M(I,\emptyset) \setminus \mathcal{U}_M(\tilde{F},\tilde{G})$. Being $\tilde{F} \subseteq I$, this is equivalent to the fact that $I \subseteq A_I$ and $A_I \cap \tilde{G} \neq \emptyset$.

Let us denote by $\mathcal{I}$ the directed set $\mathcal{I} := \{I \in [M]^{<\omega} \colon \tilde{F} \subseteq I\}$; we therefore have a net $(A_I)_{I\in\mathcal{I}}$ in $\mathcal{K}$ such that $I \subseteq A_I$ and $A_I \cap \tilde{G} \neq \emptyset$, for every $I \in \mathcal{I}$. By compactness of $\mathcal{K}$, such a net clusters at some $\tilde{A} \in \mathcal{K}$. A simple argument, whose details we include below for the sake of completeness, then implies that $M \subseteq \tilde{A}$ and $\tilde{A} \cap \tilde{G} \neq \emptyset$. Consequently, $\tilde{A}$ is a proper extension of M (recall that $M \cap \tilde{G} = \emptyset$), thereby contradicting the maximality of M and thus concluding the proof.

In order to check that $M \subseteq \tilde{A}$, fix any finite set F with $\tilde{F} \subseteq F \subseteq M$ and consider the neighbourhood $\mathcal{U}_{\tilde{A}}(F \cap \tilde{A}, F \setminus \tilde{A})$ of $\tilde{A}$. By definition, there must exist $I \in \mathcal{I}$ with $F \subseteq I$ such that $A_I \in \mathcal{U}_{\tilde{A}}(F \cap \tilde{A}, F \setminus \tilde{A})$; it follows, in particular, that $A_I \subseteq (F \setminus \tilde{A})^{\complement} = \tilde{A} \cup F^{\complement}$. Therefore, $F \subseteq I \subseteq A_I \subseteq \tilde{A} \cup F^{\complement}$ yields $F \subseteq \tilde{A}$ and $M \subseteq \tilde{A}$ follows.

Finally, for the second assertion, we consider the neighbourhood $\mathcal{U}_{\tilde{A}}(\tilde{G} \cap \tilde{A}, \tilde{G} \setminus \tilde{A})$ of $\tilde{A}$. By definition, some A_I belongs to such neighbourhood and it follows that $A_I \subseteq \tilde{A} \cup \tilde{G}^{\complement}$. Consequently, $\emptyset \neq A_I \cap \tilde{G} \subseteq (\tilde{A} \cup \tilde{G}^{\complement}) \cap \tilde{G} = \tilde{A} \cap \tilde{G}$, and we are done. ∎

∎

In conclusion to the chapter, we shall add a few comments on Haydon's result, Theorem 2.5.8. At the appearance of [Hay77] it was unknown whether the equivalence stated in Theorem 2.5.8 (or, more generally, the equivalence between the assertions in [Hay77, Remark 2.5]) could possibly hold under more general assumptions on κ. The sufficient condition holding true for every cardinal κ, Haydon himself (unpublished) later noted that the necessary condition fails to hold for $\kappa = \omega_1$, under the Continuum Hypothesis. One such example was also obtained by N. Kalamidas, in his Doctoral dissertation (*cf.* [Neg84, Example 1.3]).

Incidentally, this is also a consequence of our results about Pták's lemma, since the unique point where the proof of Theorem 2.4.5 depends on some cardinality assumption is the appeal to Theorem 2.5.8; in particular, Pták's lemma actually holds true for every cardinal number for which the equivalence in Theorem 2.5.8 holds. Theorem 2.4.4(ii) then yields the desired counterexample.

In accordance with Argyros' results on Pełczyński's conjecture that we mentioned above, it is natural to conjecture that Haydon's equivalence may actually be valid for every cardinal number $\kappa \geqslant \omega_2$. This would, of course, imply the validity of Pták's lemma for every $\kappa \geqslant \omega_2$.

In case that the conjecture were true, it would also lead to a negative answer to the following question.

Problem 2.5.9. Is the existence of a Corson compact K such that $\ell_1(\omega_2) \hookrightarrow C(K)$ consistent with ZFC?

Let us just note that, under CH, such a compact space can not exist, in light of Theorem 2.5.8 and the fact that continuous images of Corson compacta are Corson compacta ([Gul77, MiRu77]), while $[0,1]^{\omega_2}$ is not Corson. Such a compact space also fails to exist under MA_{ω_1}, according to the results we recorded at the beginning of Section 2.5.2.

Chapter 3

Symmetrically separated sequences

The chapter is dedicated to the study of some results on distances between unit vectors, in particular to the construction of symmetrically separated sequences in the unit ball of a Banach space. This is a variant of a well-known field of investigation that originated from the classical Riesz lemma and which has been studied in detail in the last century. In the first section of the chapter we will describe a few results in this area, introduce the notion of symmetric separation and state our main results. The remaining sections contain the proofs of these results as well as of some their consequences.

3.1 Kottman's constant

The study of distances between unit vectors is a main topic in Banach space theory, that originates perhaps with the classical Riesz' lemma, [Rie16]. This celebrated result is proved in almost every book in Functional Analysis, as a witness that the closed unit ball of an infinite-dimensional normed space is never compact. In this sense, the lemma can be considered at the origin of infinite-dimensional Analysis; being such a seminal result, it has been extended and improved in various directions, a few of which are the content of the present chapter. Let us start with the following formal definition.

Definition 3.1.1. A subset A of a normed space X is said to be δ-*separated* (respectively, $(\delta+)$-*separated*) if $\|x-y\| \geqslant \delta$ (respectively, $\|x-y\| > \delta$) for distinct $x, y \in A$.

With a very minor abuse of notation, it is also said that a sequence $(x_n)_{n=1}^{\infty}$ is δ-separated if $\|x_n - x_k\| \geqslant \delta$ for $n \neq k$. Following this notation, we may state Riesz' lemma as the claim that the unit sphere of every infinite-dimensional normed space contains a 1-separated sequence. For example, in the Banach space ℓ_p $(1 \leqslant p < \infty)$, the canonical basis is an example of a $2^{1/p}$-separated sequence, thereby giving an example where it is possible to find a $(1+\varepsilon)$-separated sequence, for some $\varepsilon > 0$. Interestingly, in the case of ℓ_p-spaces, this choice is, in a sense to be made more precise in a moment, the optimal one; actually, it is frequently the case that 'disjointly supported' sequences are already $(1+\varepsilon)$-separated sets. On the other hand, this is far from being true in general, as witnessed by the space c_0:

even though every two disjointly supported unit vectors in c_0 have distance 1, it is easy to find 2-separated sequences in the unit ball of c_0. This shows that, in general, combinatorial arguments involving the intersection of the supports play a crucial rôle; this point will be evident in this chapter and even more in Chapter 4.

Kottman's theorem [Kot75], asserting that the unit sphere of an infinite-dimensional normed space contains a (1+)-separated sequence, then sparked a new insight on the non-compactness of the unit ball in infinite dimensions. Even though the original argument of Kottman already contained some combinatorial features, in the form of an appeal to Ramsey theorem, the argument has been greatly simplified over time and simple Riesz-type proofs are available. We will offer two of them, together with a discussion of Kottman's argument, at the beginning of Section 3.2.

In their famous paper [ElOd81], Elton and Odell employed methods of infinite Ramsey theory to improve Kottman's theorem significantly by showing that the unit sphere of an infinite-dimensional normed space contains a $(1+\varepsilon)$-separated sequence, for some $\varepsilon > 0$. Unlike the proof of Kottman's theorem, it was only recently when a second proof of the Elton–Odell theorem was obtained ([FOSZ18]); however, this argument is still Ramsey-theoretic and it is by no way a simpler proof. One more elaboration over the same ideas, that uses in particular results from [FOSZ18], allows to obtain a stronger version of the Elton–Odell theorem, [GlMe••].

It is perhaps no surprise that the ε appearing in the statement of the Elton–Odell theorem is intimately related to the geometry of the underlying space. For example, in the case of the space ℓ_p $(1 \leqslant p < \infty)$ it cannot be greater than the attained bound $2^{1/p} - 1$ (*cf.* Section 3.1.2 for a discussion of this). Thus, studying geometric or structural properties of the space will often help in identifying possible lower bounds for separation constants of sequences in the unit sphere of the space. In order to achieve this it is natural to consider a constant which keeps track of the best possible ε in the conclusion of the Elton–Odell theorem. Such constant, nowadays known as Kottman constant, was probably first considered explicitly in [Kot75].

Definition 3.1.2 ([Kot75])**.** The *Kottman constant* $K(X)$ of a normed space X is

$$K(X) := \sup\Big\{\sigma > 0\colon \exists (x_n)_{n=1}^{\infty} \subset S_X\colon\ \|x_n - x_k\| \geqslant \sigma\ \forall n \neq k\Big\}.$$

Obviously, a normed space X is finite-dimensional if and only if $K(X) = 0$; moreover, the Elton–Odell theorem may now be restated as the claim that $K(X) > 1$, whenever X is an infinite-dimensional normed space. Moreover, an easy but important observation is that the definition of $K(X)$ would not be altered if one replaced S_X by B_X; for a proof, see Section 3.1.1. As a sample of results for classical Banach spaces, we already observed that $K(\ell_p) = 2^{1/p}$, whenever $1 \leqslant p < \infty$; moreover, $K(c_0) = 2$, as witnessed by the sequence $-e_{n+1} + \sum_{k=1}^{n} e_k$ $(n \in \mathbb{N})$.

It is a quite remarkable result due to Kryczka and Prus [KrPr00] that for every non-reflexive Banach space X one has $K(X) \geqslant \sqrt[5]{4}$. Their result is an ingenious and surprisingly simple combination of a result of James [Jam64b], with a spreading model technique. In the same paper, the authors also conjecture that the constant $K(X) \geqslant \sqrt[5]{4}$ may even be improved and offer an upper bound for such possible improvements. More precisely, they show that for the generalised James space J_p ([Jam51]) one has $K(J_p) = (1 + 2^{p-1})^{1/p}$ $(1 < p < \infty)$.

This latter argument has later been generalised in [CGP17, §5] to construct the first example of a (obviously, non-reflexive) Banach space whose Kottman constant is different from that of its bidual, thus answering a problem from their previous paper [CaPa11]. More precisely, they show that for a certain J-sum $J_p(\ell_1^n)$ of finite dimensional Banach spaces, introduced in [Bel82], one has $K(J_p(\ell_1^n)) = (1 + 2^{p-1})^{1/p}$, while $K(J_p(\ell_1^n)^{**}) = 2$. Observe that this also offers an example of a non-reflexive Banach space X for which $K(X) < 2$ (a completely opposite phenomenon, a reflexive Banach space every whose renorming has Kottman constant equal to 2, will be discussed in Section 3.4.2).

Obviously, there is no analogue of the result by Kryczka and Prus in the class of reflexive Banach spaces, as witnessed by the space ℓ_p $(1 < p < \infty)$. What's more, it happens that some geometric properties can be used in upper bounds for the Kottman constant; this is already present in Kottman's work, where it is shown that $K(X) < 2$, provided X is uniformly convex or uniformly smooth ([Kot70, Theorems 3.6, 3.7]). An alternative and shorter proof may be found in [NaSa78, §1]. It was later noted by Maluta and Papini that these results have a quantitative counterpart, more precisely the following estimates hold true, [MaPa09, Theorem 2.6 and Corollary 2.10]:

$$K(X) \leqslant 2(1 - \delta_X(1)) \qquad \text{and} \qquad K(X) \leqslant 1 + 2\rho_X(1).$$

In the other direction, the moduli of uniform convexity and uniform smoothness can also be used for lower estimates of the Kottman constant. The first result in this direction is perhaps due to van Neerven [vNe05], who proved the inequality

$$K(X) \geqslant 1 + \frac{1}{2}\delta_X(2/3).$$

As it turns out, this estimate is rather weak and several improvements are possible; let us just mention here two of them and refer to [MaPa09] for a more complete discussion over some those inequalities. Delpech [Del10] gave a much simpler proof of a result, which implies in particular the estimate

$$K(X) \geqslant 1 + \delta_X(1);$$

we will say more on his result in a moment. The other estimate we wish to record, also due to [MaPa09, Corollary 2.15] is that

$$K(X) \geqslant 1 + \sqrt{2} \cdot \delta_X(\sqrt{2}).$$

Further quantitative estimates of Kottman's constant expressed in terms of various moduli of convexity and related results may be found, *e.g.*, in [CGP17, DrOl06, Pru10];

let us also refer to [CaPa11] for an extensive bibliography where several additional results can be found.

We will now conclude this introduction to the Kottman constant by briefly mentioning two of its applications in different areas. The first area we wish to mention is the topic of extension of Lipschitz mappings between metric spaces. One seminal result here is due to Kirszbraun [Kir34] who proved that if A is any subset of a Hilbert space H_1 and f is a Lipschitz function from A to a Hilbert space H_2, then f admits an extension $\tilde{f}\colon H_1 \to H_2$ with $Lip(f) = Lip(\tilde{f})$. This problem has been investigated by several authors in the last years and we shall refer to [Kal08, §3.3, 3.4] for a presentation of some results in the area. The case where the range space is a C(K) space has been investigated by Kalton [Kal07a, Kal07b], and previously in [LaRa05]. Kalton, in particular, related the Kottman constant of a Banach space with the extendability of Lipschitz functions with values in c_0; we need one definition to explain this.

Definition 3.1.3. Let X and Y be metric spaces. $e(X,Y)$ denotes the infimum of all constants λ such that every Lipschitz function f from a subset of X into Y admits a Lipschitz extension $\tilde{f}\colon X \to Y$ with $Lip(\tilde{f}) \leqslant \lambda \cdot Lip(f)$.

Kalton's result [Kal07a, Proposition 5.8] can be now stated as follows; a direct proof of one inequality may be found in [CaPa11, §4].

Theorem 3.1.4 ([Kal07a])**.** *For an infinite-dimensional Banach space X one has $K(X) = e(X, c_0)$.*

The second application we briefly mention is the notion of a *measure of noncompactness*, a very important tool in metric fixed point theory; for information, consult, *e.g.*, [AKPRS92, ADL97, BaGo80]. The very rough idea is that one wishes to have a notion to measure 'how far' a set is from being compact and then consider maps that increase the compactness of sets for fixed point results. Let us just give the definition of measure of noncompactness and a few examples; for a more complete discussion and some uses, we refer to [ADL97, Chapter II].

Definition 3.1.5. Let (X,d) be a complete metric space and $\mathcal{B}$ be the family of bounded subsets of X. A *measure of noncompactness* on X is a map $\phi\colon \mathcal{B} \to [0,\infty)$ with the following properties:

(i) $\phi(B) = 0$ iff B is relatively compact;

(ii) $\phi(B) = \phi(\overline{B})$;

(iii) $\phi(A \cup B) = \max\{\phi(A), \phi(B)\}$.

For example, it is simple to prove the following interesting generalisation of the Cantor intersection theorem: if $(B_n)_{n=1}^{\infty}$ is a decreasing sequence of non-empty, closed and bounded subsets of X, then $\phi(B_n) \to 0$ implies that $\cap B_n$ is non-empty and compact.

Example 3.1.6. Let us present the main examples of measures of noncompactness; in all them $\mathcal{B}$ denotes the family of bounded subsets of a metric space (X,d).

1. The *discrete measure of noncompactness*:

$$\phi(B) := \begin{cases} 0 & \text{if } B \text{ is relatively compact} \\ 1 & \text{otherwise.} \end{cases}$$

2. The *Kuratowski measure of noncompactness*:

$$\alpha(B) := \inf\{\varepsilon > 0\colon B \text{ is covered by finitely many sets of diameter at most } \varepsilon\}.$$

3. The *Hausdorff measure of noncompactness*:

$$\chi(B) := \inf\{\varepsilon > 0\colon B \text{ has a finite } \varepsilon\text{-net}\}.$$

4. The *separation measure of noncompactness*:

$$\beta(B) := \sup\{\sigma > 0\colon B \text{ contains an infinite } \sigma\text{-separated subset}\}.$$

The fact that all these examples constitute measures of non-compactness is immediate; equally immediate is that, for a Banach space X, $\beta(B_X)$ is the Kottman constant. In the context of Banach space theory, the separation measure is perhaps even more natural than the Hausdorff and Kuratowski ones, since α and χ do not distinguish the unit balls of different Banach spaces. More precisely, for every infinite-dimensional Banach space X one has $\alpha(B_X) = 2$ and $\chi(B_X) = 1$, *cf.* [ADL97, Theorem II.2.5].

The main objective of the chapter is to revisit and investigate the above-mentioned results in the setting of symmetric separation; let us start with the definition of such notion.

Definition 3.1.7. A subset A of a normed space is *symmetrically δ-separated* (respectively, *symmetrically $(\delta+)$-separated*) when $\|x \pm y\| \geqslant \delta$ (respectively, $\|x \pm y\| > \delta$) for any distinct elements $x, y \in A$.

The study of this notion was probably undertaken explicitly for the first time by J. M. F. Castillo and P. L. Papini in [CaPa11] and prosecuted in the paper [CGP17]. In the former work, the authors asked whether there is a symmetric version of the Elton–Odell theorem ([CaPa11, Problem 1]); however according to Castillo ([Cas17]) prior to the research in [HKR18], it has not been known whether the unit sphere of an infinite-dimensional Banach space contains a symmetrically $(1+)$-separated sequence. This question was the main motivation of our investigation in [HKR18], that we will describe in this chapter.

Castillo and Papini [CaPa11] gave some partial answers to the problem in their paper; in particular, they proved that the answer is affirmative for uniformly non-square spaces and for $\mathscr{L}_\infty$-spaces. We shall discuss later their result concerning $\mathscr{L}_\infty$-spaces, in Section 3.4, while we will largely extend the result concerning uniformly non square to reflexive Banach spaces, with a completely different proof, *cf.* Section 3.3. Their results also contain some quantitative features, that we discuss in some detail here; these results require the introduction of the symmetric variation of Kottman constant.

Definition 3.1.8 ([CaPa11])**.** The *symmetric Kottman constant* of a Banach space X is

$$K^s(X) := \sup\Big\{\sigma > 0 \colon \exists (x_n)_{n=1}^\infty \subset S_X \colon \ \|x_n \pm x_k\| \geqslant \sigma \ \forall n \neq k\Big\}.$$

Let us notice here that, just as for $K(X)$, the definition of $K^s(X)$ would be the same replacing S_X with B_X, as we note in Section 3.1.1.

Let us now discuss in some detail some results on $K^s(X)$, for uniformly non square Banach spaces X; we start with the definition of uniform non squareness, introduced by James in [Jam64a], also see [Bea85, §4.III.1]. The constant $J(X)$ was not explicitly defined in James' paper and it was studied, *e.g.*, in [Cas86].

Definition 3.1.9. The *James constant* $J(X)$ of a normed space X is defined to be

$$J(X) := \sup_{x,y \in B_X} \min\big\{\|x-y\|, \|x+y\|\big\}.$$

A normed space is *uniformly non square* (*UNS*, in short) if $J(X) < 2$.

With this definition, we may now state the result by Castillo and Papini [CaPa11, Lemma 2.2] that

$$\frac{2}{J(X)} \leqslant K^s(X) \leqslant J(X),$$

for every infinite-dimensional Banach space. (Let us notice that the right-hand-side inequality is an immediate consequence of the definitions.) One immediate consequence is that every uniformly non square space X satisfies $1 < K^s(X) < 2$, hence it offers a sufficient condition for the validity of the symmetric analogue to the Elton–Odell theorem.

This result is also very interesting since it allows us to answer a very natural question, by offering an example of a Banach space X for which the symmetric Kottman constant differs from $K(X)$, [NaSa78, Example 3.2].

Example 3.1.10. There exists a (uniformly non square) Banach space X for which $K(X) = 2$ and $K^s(X) < 2$.

Let us consider the equivalent norm on $(\ell_p, \|\cdot\|_p)$ defined by

$$\|x\| := \max\left\{\|x\|_p, \max_{i,j} |x(i) - x(j)|\right\}.$$

Obviously, the canonical basis is a 2-separated sequence (which is *not* symmetrically 2-separated). It is not hard to see that for $\frac{\log 3}{\log 2} < p < 2$, the space $X := (\ell_p, \|\cdot\|)$ is uniformly non square, whence $K^s(X) < 2$. An heuristic motivation for this may be found in the fact that the unit ball of a 2-dimensional subspace generated by two vectors of the canonical basis is very far from a square; however, the case of the general subspace is not equally obvious, as it depends on Clarkson's inequality.

Moreover, although not stated explicitly, it follows from the proof of the main result of [Del10] that $K^s(X) > 1$ for asymptotically uniformly convex spaces X in which case the lower bound for the symmetric separation constant is expressed in terms of the so-called modulus of asymptotic uniform convexity (see Section 3.4 for more information). Certainly the unit spheres of both ℓ_1 and c_0 contain symmetrically 2-separated sequences (in the former case plainly the standard vector basis is an example of such sequence, in the latter case one may take $x_n = -e_{n+1} + \sum_{k=1}^{n} e_k$ $(n \in \mathbb{N})$). Consequently, if X contains an isomorphic copy of either space, by the James distortion theorem one concludes that $K^s(X) = 2$. (This simple observation will be formally recorded in Section 3.1.1.)

Let us then pass to state and discuss our contributions in the area of symmetric separation and refer to later sections in this chapter for the proofs of the stated results and further information. Our first main result is a positive answer to Castillo's problem, namely, we obtain an extension of Kottman's theorem to symmetrically separated sequences. The formal statement of our first result therefore reads as follows.

Theorem 3.1.11 (Symmetric version of Kottman's theorem, [HKR18, Theorem A])**.** *Let X be an infinite-dimensional Banach space. Then the unit sphere of X contains a symmetrically* (1+)*-separated sequence.*

The proof of this result will be presented in Section 3.2. It is important to observe that our argument is not of combinatorial nature, in the spirit of Kottman original proof, but it involves induction and a dichotomy concerning a geometric property. Moreover, such approach also permits, to some extent, to continue the induction argument beyond the countable setting and construct uncountable symmetrically (1+)-separated sets, when the underlying Banach space is non-separable; *cf.* Theorem 4.3.5. Let us also mention here two other instances where a version of Kottman theorem was pushed to the non-separable setting, namely [KaKo16, Theorems 3.1 and 3.8].

Subsequently, we identify several classes of Banach spaces for which a symmetric version of the Elton–Odell theorem holds true. In particular, we prove that spaces containing boundedly complete basic sequences satisfy a symmetric version of the Elton–Odell theorem; this theorem will be the main result presented and proved in Section 3.3.

Theorem 3.1.12 ([HKR18, Theorem 1.1])**.** *Let X be a Banach space that contains a boundedly complete basic sequence. Then for some $\varepsilon > 0$, the unit sphere of X contains a symmetrically $(1+\varepsilon)$-separated sequence.*

Of course, this result applies when the Banach space is an infinite-dimensional reflexive Banach space; in the same section we will also combine our condition with various results from the literature in order to extend this assertion to more classes of Banach spaces. For example, we are able to obtain the same conclusion if the Banach space has the Radon–Nikodym property, or it contains an unconditional basic sequence.

Subsequently, we turn our attention to classes of spaces where a lower bound for the ε appearing in the statement of the Theorem may be computed explicitly. To wit, we dedicate Section 3.4, to some quantitative estimates on the symmetric Kottman constant. Perhaps the main result we prove is the following theorem.

Theorem 3.1.13 ([HKR18, Theorem C]). *Let X be an infinite-dimensional Banach space. Suppose that either*

(i) *X contains a normalised basic sequence satisfying a lower q-estimate for some $q < \infty$,*

(ii) *or X has finite cotype q.*

Then for every $\varepsilon > 0$ the unit sphere of X contains a symmetrically $(2^{1/q} - \varepsilon)$-separated sequence.

Section 3.4 also contains further quantitative results, in particular involving spreading models and constructions, via biorthogonal systems, of renormings whose unit balls have optimal separation properties. We will also collect some further estimates present in the literature, with a description of some their proofs. A few more such results from the literature are proved in detail in the next two parts, Sections 3.1.1 and 3.1.2. The former comprises basically obvious comments, that are however useful in several places in the chapter, while the latter is dedicated to the Kottman constant of ℓ_p direct sums.

3.1.1 A few useful observations

This part is dedicated to a few simple inequalities in normed spaces and their consequences involving the Kottman constant. The results recorded here are all essentially obvious, but we state them here since we will make frequent use of them (tacitly, in most cases) throughout the chapter.

The first lemma that we present is an upper estimate for the mutual distance of distinct elements of a sequence, in term of the Kottman constant. It implies, in particular, that for every infinite-dimensional Banach space X we may find a sequence in the unit ball the mutual distances of whose elements are almost equal to $K(X)$. Not surprisingly, this stabilisation result depends on Ramsey theorem.

Lemma 3.1.14. *Let X be a normed space and $(x_n)_{n=1}^{\infty}$ be a sequence in B_X. Then, for every $\varepsilon > 0$, there exists a subsequence $(x_{n_k})_{k=1}^{\infty}$ such that*

$$\|x_{n_k} - x_{n_j}\| \leqslant K(X) + \varepsilon \qquad (k, j \in \mathbb{N}).$$

Proof. Let us consider the colouring

$$\{n, k\} \mapsto \begin{cases} (>) & \|x_n - x_k\| > K(X) + \varepsilon \\ (\leqslant) & \|x_n - x_k\| \leqslant K(X) + \varepsilon \end{cases}$$

of $[\mathbb{N}]^2$. An appeal to Ramsey's theorem [Ram29] yields an infinite monochromatic subset M of $\mathbb{N}$. If the colour of such a set were $(>)$, then $(x_n)_{n\in M}$ would be an infinite $(K(X) + \varepsilon)$-separated set, which is obviously impossible; consequently, $(x_n)_{n\in M}$ is the desired subsequence. ■

We shall make frequent use of the following well-known folklore inequality, which can be found for instance in [KaKo16, Lemma 2.2] or (in a slightly weaker formulation) in [MSW01, Lemma 6]. On the other hand, the present formulation or similar estimates can surely be found in older papers scattered throughout the literature. As a sample, let us just mention [MaPa93, Lemma 3.1], where a very similar statement (actually, under slightly more general assumptions) can be found.

Lemma 3.1.15. *Let X be a normed space. Suppose that x, y are non-zero vectors in the unit ball of X. If $\|x - y\| \geqslant 1$, then*

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \geqslant \|x - y\|.$$

Proof. Without loss of generality, we may assume that $\|x\| \geqslant \|y\|$; we then consider the convex function $\mathbb{R} \ni t \mapsto g(t) := \|x - ty\|$. Since $g(1) = \|x - y\| \geqslant 1 \geqslant \|x\| = g(0)$, the convexity of g yields $g(\|x\|/\|y\|) \geqslant g(1)$. Therefore,

$$\|x - y\| \leqslant g\left(\frac{\|x\|}{\|y\|}\right) = \|x\| \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leqslant \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|.$$

■

The lemma implies, in particular, that, if $\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leqslant 1$, then $\|x - y\| \leqslant 1$ too.

One further important consequence is that in order to find a (symmetrically) $(1+)$-separated (or $(1+\varepsilon)$-separated) sequence of unit vectors it is in fact sufficient to find such a sequence in the unit ball, with no need to insist that all vectors are normalised. This property is particularly convenient and we will use it tacitly throughout this and the next chapter. A particular case of it is found the following equivalent definition of the (symmetric) Kottman constant.

Corollary 3.1.16. *For every infinite-dimensional Banach space,*

$$K(X) := \sup \big\{ \sigma > 0 \colon \exists (x_n)_{n=1}^{\infty} \subset B_X \colon \ \|x_n - x_k\| \geqslant \sigma \ \forall n \neq k \big\}.$$

Analogously for $K^s(X)$.

Let us mention that the above corollary is a particular case of a general result pertaining to the separation measure of non-compactness, namely, the equality $\beta(B) = \beta(\operatorname{conv} B)$, for every bounded subset B of a Banach space X, [Ari91]. The proof of this general case is, however, much more complicated; one alternative, still technical, such proof may be found in [ADL97, Theorem II.3.6].

One more consequence we may record is an estimate of the (symmetric) Kottman constant of a Banach space in terms of the corresponding constant of its quotients; this was formally recorded, *e.g.*, in [KaKo16, Proposition 2.3].

Corollary 3.1.17. *Let X be a Banach space and let Y be isometric to a quotient of X. Then $K(X) \geqslant K(Y)$ and $K^s(X) \geqslant K^s(Y)$.*

Proof. Let us fix $\delta > 0$ and select a $(K(Y) - \delta)$-separated sequence $(y_n)_{n=1}^{\infty}$ in the unit sphere of Y (we can assume that Y is infinite-dimensional, the conclusion being otherwise trivial). We also choose a representative x_n for y_n with $\|x_n\| \leqslant 1 + \delta$. Therefore,

$$\left\| \frac{x_n}{1+\delta} - \frac{x_k}{1+\delta} \right\| \geqslant \frac{1}{1+\delta} \|y_n - y_k\| \geqslant \frac{K(Y) - \delta}{1+\delta}$$

and the previous corollary then yields $K(X) \geqslant \frac{K(Y)-\delta}{1+\delta}$, whence the first part of the conclusion follows. In the case of the symmetric separation, the proof is the same. ■

The last observation in this section is devoted to the justification of the observation made in the introduction asserting that if X contains an isomorphic copy of either c_0 or ℓ_1, then for every $\varepsilon \in (0, 1)$ the unit sphere of X contains a symmetrically $(2 - \varepsilon)$-separated subset.

Lemma 3.1.18. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces and let $A \subseteq S_X$ be a (symmetrically) $(1 + \varepsilon)$-separated set ($\varepsilon > 0$). Suppose that $T\colon X \to Y$ is an isomorphic embedding such that $\|T\| \cdot \|T^{-1}\| \leqslant 1 + \delta$ ($\delta > 0$). If $\delta \leqslant \varepsilon$, then the set*

$$\tilde{A} := \left\{ \frac{Tx}{\|Tx\|_Y} : x \in A \right\} \subseteq S_Y$$

is (symmetrically) $\frac{1+\varepsilon}{1+\delta}$-separated.

Proof. Up to a scaling, we may assume without the loss of generality that for all $x \in X$ we have $\|x\|_X \leqslant \|Tx\|_Y \leqslant (1 + \delta)\|x\|_X$. Consequently, $(1 + \delta)^{-1} \cdot Tx \in B_Y$ for $x \in A$. Moreover, for distinct $x, y \in A$ we have

$$\left\| \frac{Tx}{1+\delta} - \frac{Ty}{1+\delta} \right\|_Y \geqslant \frac{1}{1+\delta} \cdot \|x - y\|_X \geqslant \frac{1+\varepsilon}{1+\delta} \geqslant 1.$$

Therefore, Lemma 3.1.15 applied to the vectors $(1 + \delta)^{-1}Tx$ and $(1 + \delta)^{-1}Ty$ gives

$$\left\| \frac{Tx}{\|Tx\|_Y} - \frac{Ty}{\|Ty\|_Y} \right\|_Y \geqslant \left\| \frac{Tx}{1+\delta} - \frac{Ty}{1+\delta} \right\|_Y \geqslant \frac{1+\varepsilon}{1+\delta}.$$

The symmetric assertion is proved in the same way. ■

Let us mention that the unique motivation to employ Lemma 3.1.15 in the above proof was to obtain a slightly better estimate for the separation. The, even more trivial, proof only involving the triangle inequality would have led to separation $1 + \varepsilon - 2\delta$.

A direct consequence of the result and of James' non distortion theorem is the first clause of the following corollary. Its second part, in turn, follows from the first one and Corollary 3.1.17.

Corollary 3.1.19. *If a Banach space X contains an isomorphic copy of c_0 or ℓ_1, then $K^s(X) = 2$. Therefore, $K^s(X) = 2$ whenever X admits a quotient isomorphic to c_0 or ℓ_1.*

3.1.2 ℓ_p-spaces and direct sums

In this part, we present the elementary proof of some classical exact computations of the Kottman constant, more precisely, we prove the fact mentioned above that the Kottman constant of the space ℓ_p is $2^{1/p}$ $(1 \leqslant p < \infty)$. We also extend this result (and its method of proof) to discuss ℓ_p-sums of Banach spaces.

The computation of $K(\ell_p)$ can be traced back at least to Kottman's paper [Kot70, Lemma 1.5]; analogous previous results in terms of packing spheres are present, *e.g.*, in [BRR58, Spe70]. Since then the result appears in many textbooks, for example [ADL97, Theorem 3.13] or [WeWi75, Theorem 16.9].

Proposition 3.1.20. *$K(\ell_p) = 2^{1/p}$, for every $p \in [1, \infty)$.*

Prior to the proof, let us recall the following well-known fact, which is proved via a very simple *sliding hump* argument. If $(y_n)_{n=1}^{\infty}$ is any weakly null sequence in ℓ_p $(1 \leqslant p < \infty)$ and $y \in \ell_p$, then $\|y\|^p + \limsup_{n\to\infty} \|y_n\|^p = \limsup_{n\to\infty} \|y_n + y\|^p$.

Proof. As we already noted, the canonical unit vector basis of ℓ_p is $2^{1/p}$-separated; consequently, we only need to prove the upper bound $K(\ell_p) \leqslant 2^{1/p}$. This estimate being trivial for $p = 1$, we may additionally assume that $p \in (1, \infty)$.

Let us therefore assume that $(x_n)_{n=1}^{\infty}$ is an r-separated sequence in the unit ball of ℓ_p. Up to passing to a subsequence, we can assume that $(x_n)_{n=1}^{\infty}$ is weakly convergent to a vector, say x. Let us now fix arbitrarily $\varepsilon > 0$ and find $N \in \mathbb{N}$ such that $\|x\restriction_{[N+1,\infty)}\| \leqslant \varepsilon$; since $x_n \to x$ weakly, we also have $\|(x_n - x_k)\restriction_{[1,N]}\| \leqslant \varepsilon$, whenever n, k are sufficiently large. For all such n we thus have:

$$\begin{aligned} r^p \leqslant \limsup_{k\to\infty} \|x_n - x_k\|^p &= \limsup_{k\to\infty} \left(\|(x_n - x_k)\restriction_{[1,N]}\|^p + \|(x_n - x_k)\restriction_{[N+1,\infty)}\|^p \right) \\ &\leqslant \varepsilon^p + \limsup_{k\to\infty} \|(x_n - x)\restriction_{[N+1,\infty)} + (x - x_k)\restriction_{[N+1,\infty)}\|^p \\ &= \varepsilon^p + \|(x_n - x)\restriction_{[N+1,\infty)}\|^p + \limsup_{k\to\infty} \|(x - x_k)\restriction_{[N+1,\infty)}\|^p \leqslant \varepsilon^p + 2(1+\varepsilon)^p. \end{aligned}$$

Letting $\varepsilon \to 0$ concludes the proof. ∎

Of course, this result implies the same conclusion for the non-separable spaces $\ell_p(\Gamma)$, *i.e.*, $K(\ell_p(\Gamma)) = 2^{1/p}$, whenever Γ is an infinite set.

Moreover, essentially the same proof shows that if $(X_n)_{n=1}^{\infty}$ are finite-dimensional normed spaces, then $(\sum X_n)_{\ell_p}$ also has Kottman's constant equal to $2^{1/p}$. This is formally noted in [Kot70, Remark 1.5] and motivated the study of the Kottman constant of ℓ_p-direct sums of arbitrary Banach spaces; such issue was then undertaken in [Kot75, Lemma 8].

A much shorter proof, together with a slight improvement, was later given in [CaPa11, Proposition 1.1]. This argument is however slightly flawed, since it is not clear why the two cases given there cover all the possibilities. On the other hand, these two cases illustrate the two parts of the argument: a stabilisation argument on a finite initial part plus a sliding hump to control the tail part. The proof in [CaPa11] is therefore a clear indication of what a detailed proof would consist of; such detailed proof follows.

Theorem 3.1.21. *For a sequence $(X_n)_{n=1}^\infty$ of Banach spaces one has:*

$$K\left(\left(\sum_{n=1}^N X_n\right)_{\ell_p}\right) = \max\{K(X_n)\}_{n=1}^N$$

$$K\left(\left(\sum_{n=1}^\infty X_n\right)_{\ell_p}\right) = \sup\{K(X_n), 2^{1/p}\}.$$

Proof. Clearly, both the right-hand sides are lower or equal to the corresponding left-hand sides; consequently, we only need to prove the upper estimates.

For what concerns the first assertion, it is obviously sufficient to prove it for $N = 2$, namely to prove that $K(X \oplus_p Y) \leqslant \max\{K(X), K(Y)\}$. Let therefore $(x_n, y_n)_{n=1}^\infty$ be an r-separated sequence in the unit ball of $X \oplus_p Y$; fix $\varepsilon > 0$ and assume, up to passing to a subsequence and relabeling, that $\left|\|x_n\|^p - \|x_k\|^p\right| \leqslant \varepsilon$ and $\left|\|y_n\|^p - \|y_k\|^p\right| \leqslant \varepsilon$ for every $n, k \in \mathbb{N}$. Setting $\alpha := \sup \|x_n\|$ and $\beta := \sup \|y_n\|$, we consequently obtain $\alpha^p \leqslant \|x_n\|^p + \varepsilon$ and $\beta^p \leqslant \|y_n\|^p + \varepsilon$, for every $n \in \mathbb{N}$; this in turn yields $\alpha^p + \beta^p \leqslant 1 + 2\varepsilon$. Up to one more subsequence, Lemma 3.1.14 allows us to assume that $\|x_n - x_k\|^p \leqslant \alpha^p \cdot K(X)^p + \varepsilon$ and $\|y_n - y_k\|^p \leqslant \alpha^p \cdot K(Y)^p + \varepsilon$, for every $n, k \in \mathbb{N}$. We may then conclude that

$$\begin{aligned} r^p &\leqslant \|(x_n, y_n) - (x_k, y_k)\|^p = \|x_n - x_k\|^p + \|y_n - y_k\|^p \leqslant \alpha^p \cdot K(X)^p + \varepsilon + \beta^p \cdot K(Y)^p + \varepsilon \\ &\leqslant (\alpha^p + \beta^p) \cdot \max\{K(X), K(Y)\}^p + 2\varepsilon \leqslant (1 + 2\varepsilon) \cdot \max\{K(X), K(Y)\}^p + 2\varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$ concludes the proof of the first assertion.

For the proof of the second claim, we shall start by fixing one piece of notation. For a vector $x \in (\sum X_n)_{\ell_p}$, we shall write $x = (x(n))_{n=1}^\infty$ and we understand that $x(n) \in X_n$ for every $n \in \mathbb{N}$. In this case, we will, as usual, refer to the set $\{n \in \mathbb{N} \colon x(n) \neq 0\}$ as the *support* of x. We shall also denote by $x\restriction_{[1,N]}$ the vector whose first N components are equal to those of x and the remaining ones are equal to 0, in other words, $x\restriction_{[1,N]} = (x(1), \dots, x(N), 0, \dots)$. ($x\restriction_{[N,\infty)}$ is defined similarly.) Finally, we shall use the shorthand notation $K := K((\sum X_n)_{\ell_p})$ and $R := \sup\{K(X_n), 2^{1/p}\}$.

Let us fix arbitrarily $\varepsilon > 0$ and select a $(K - \varepsilon)$-separated sequence $(x_k)_{k=1}^\infty$ in the unit ball of $(\sum X_n)_{\ell_p}$; up to a small perturbation, we may assume without loss of generality that the support of every x_k is a finite set. Since, for fixed $N \in \mathbb{N}$, $\|x_k\restriction_{[1,N]}\| \leqslant 1$, we may pass to a subsequence and assume that $(\|x_k\restriction_{[1,N]}\|)_{k=1}^\infty$ converges to a limit α_N. By diagonalisation and relabeling, we may assume that the above limit exists for every N, namely

$$\lim_{k\to\infty} \|x_k\restriction_{[1,N]}\| = \alpha_N \qquad (N \in \mathbb{N}).$$

The sequence $(\alpha_N)_{N=1}^\infty$ is plainly non decreasing and, therefore, it admits a limit, $\alpha_\infty \in [0, 1]$. Let us select a natural number $\overline{N}$ so large that $\alpha_{\overline{N}}^p > \alpha_\infty^p - \varepsilon$. As $\|x_k\restriction_{[1,\overline{N}]}\|^p \to \alpha_{\overline{N}}^p$ when $k \to \infty$, up to discarding finitely many terms, we may additionally assume that

$$\alpha_\infty^p - \varepsilon \leqslant \|x_k\restriction_{[1,\overline{N}]}\|^p \leqslant \alpha_\infty^p + \varepsilon \qquad (k \in \mathbb{N}). \tag{3.1.1}$$

As a consequence of this inequality and the first part of the result, Lemma 3.1.14 assures us that (up to passing to a further subsequence)

$$\|x_k\restriction_{[1,\overline{N}]}-x_n\restriction_{[1,\overline{N}]}\|^p \leqslant (\alpha^p_\infty+\varepsilon)\cdot\max\{K(X_1),\dots,K(X_{\overline{N}})\}^p+\varepsilon \leqslant (\alpha^p_\infty+\varepsilon)\cdot R^p+\varepsilon. \quad (3.1.2)$$

We shall now treat the tail part of the vectors, and we shall exploit once more a *sliding hump* argument. Let us, for notational simplicity, denote $\tilde{x}_k := x_k\restriction_{[N+1,\infty)}$; as a consequence of the above inequality (3.1.1) one has $\|\tilde{x}_k\|^p \leqslant 1+\varepsilon-\alpha^p_\infty$. Moreover, for every $N \geqslant \overline{N}+1$, one has

$$\|\tilde{x}_k\restriction_{[1,N]}\|^p = \|x_k\restriction_{[1,N]}\|^p - \|x_k\restriction_{[1,\overline{N}]}\|^p \to \alpha^p_N - \alpha^p_{\overline{N}} < \varepsilon.$$

Let now $N_1 \in \mathbb{N}$ be such that $\operatorname{supp}(\tilde{x}_1) \subseteq [1, N_1]$; let, moreover, $k_1 := 1$ and $y_1 := \tilde{x}_1$. Since $\lim_k \|\tilde{x}_k\restriction_{[1,N_1]}\|^p < \varepsilon$, we may select an index k_2 such that $\|\tilde{x}_{k_2}\restriction_{[1,N_1]}\|^p < \varepsilon$ and we shall set $y_2 := \tilde{x}_{k_2} - \tilde{x}_{k_2}\restriction_{[1,N_1]}$; note that $\|y_2\|^p \leqslant \|\tilde{x}_{k_2}\|^p \leqslant 1+\varepsilon-\alpha^p_\infty$ and $\|y_2-\tilde{x}_{k_2}\|^p < \varepsilon$. If we continue by induction in the same way, we then obtain a disjointly supported sequence $(y_j)_{j=1}^\infty$ and a subsequence $(\tilde{x}_{k_j})_{j=1}^\infty$ of $(\tilde{x}_k)_{k=1}^\infty$ with the properties that $\|y_j\|^p \leqslant 1+\varepsilon-\alpha^p_\infty$ and $\|y_j - \tilde{x}_{k_j}\|^p < \varepsilon$ for every $j \in \mathbb{N}$.

In light of the fact that $(y_j)_{j=1}^\infty$ is a disjointly supported sequence, we immediately obtain that $\|y_j - y_n\|^p \leqslant 2\cdot(1+\varepsilon-\alpha^p_\infty)$; in turn, this yields

$$\|\tilde{x}_{k_j} - \tilde{x}_{k_n}\| \leqslant 2^{1/p}\cdot(1+\varepsilon-\alpha^p_\infty)^{1/p} + 2\varepsilon^{1/p} \leqslant R\cdot(1+\varepsilon-\alpha^p_\infty)^{1/p} + 2\varepsilon^{1/p}.$$

Finally, when we combine this inequality with (3.1.2), we obtain

$$\begin{aligned}(K-\varepsilon)^p &\leqslant \|x_{k_j}-x_{k_n}\|^p = \|x_{k_j}\restriction_{[1,\overline{N}]} - x_{k_n}\restriction_{[1,\overline{N}]}\|^p + \|\tilde{x}_{k_j}-\tilde{x}_{k_n}\|^p \\ &\leqslant (\alpha^p_\infty+\varepsilon)\cdot R^p + \varepsilon + \left(R\cdot(1+\varepsilon-\alpha^p_\infty)^{1/p} + 2\varepsilon^{1/p}\right)^p;\end{aligned}$$

letting $\varepsilon \to 0$ then gives

$$K^p \leqslant \alpha^p_\infty \cdot R^p + R^p\cdot(1-\alpha^p_\infty) = R^p.$$

■

3.2 A symmetric version of Kottman's theorem

In this section, we present our solution to Castillo's question of the validity of a symmetric analogue of Kottman's theorem; to wit, we prove Theorem 3.1.11, asserting that the unit ball of every infinite-dimensional Banach space contains a symmetrically (1+)-separated sequence. Let us observe in passing that the argument actually applies to every infinite-dimensional normed space.

Prior to the presentation of our result, we shall open the section with a discussion of the proof of Kottman's result itself; as we already mentioned, we will only describe Kottman's proof and we will then present more recent and simpler proofs.

The main step in Kottman's proof, from [Kot75], consists is a combinatorial lemma [Kot75, Lemma 1] , nowadays known as Kottman's lemma, which depends on Ramsey theorem. As we already hinted at, simpler and non-combinatorial proofs of Kottman's theorem are now available, but such combinatorial features are present in the Elton–Odell theorem and their presence will be even more pervasive when passing to the non-separable setting, in Chapter 4.

For the statement of Kottman's lemma, we need to fix one notation. Let $\mathcal{U}$ comprise all sequences in c_{00} with values in $\{0, \pm 1\}$; moreover, we shall denote by $(e_i)_{i=1}^\infty$ the canonical basis of c_{00}.

Lemma 3.2.1 (Kottman's lemma, [Kot75])**.** *Let A be a subset of $\mathcal{U}$ such that A is symmetric and $(e_i)_{i=1}^\infty \subseteq A$. Then there exists an infinite subset B of A such that for distinct $x, y \in B$ one has $x - y \notin A$.*

Let us now show how to deduce Kottman's theorem from this lemma.

First proof of Kottman's theorem, [Kot75]. Let X be an infinite dimensional Banach space and select an infinite Auerbach system $\{e_i; e_i^*\}_{i=1}^\infty$ in X ([Day62]). We then consider the set

$$E := \left\{ x = \sum_{i=1}^n a_i e_i \in X \colon a_i \in \{0, \pm 1\} \text{ and } \|x\| = 1 \right\};$$

it is immediate to see that if distinct $x, y \in E$ satisfy $\|x - y\| \leqslant 1$, then $x - y \in E$. Let us then consider the non-expansive linear operator $T \colon \operatorname{span} E \to c_{00}$ defined by $T(x) := (\langle e_i^*, x\rangle)_{i=1}^\infty$; the set $A := T(E) \subseteq \mathcal{U}$ plainly satisfies the assumptions of Kottman's lemma.

If every infinite subset of E contains two distinct points x, y with mutual distance $\|x - y\| \leqslant 1$, then such points satisfy $x - y \in E$; consequently, every infinite subset of A contains two distinct points whose difference belongs to A, which is in contradiction with Kottman's lemma. Consequently, E contains a (1+)-separated sequence. ■

This argument has been carefully investigated and improved by Glakousakis and Mercourakis, [GlMe15]. In particular, the authors have found a finite-dimensional version of Kottman's lemma, whence the same argument as above leads them to the following result: every finite-dimensional normed space of dimension n contains a (1+)-separated subset of its unit sphere, with cardinality $n + 1$ [GlMe15, Theorem 0.2]. In the case of complex Banach spaces of (complex) dimension n, it is also possible to improve the result and obtain a (1+)-separated collection of unit vectors of cardinality $2n + 2$, [GlMe15, Theorem 2.10].

Let us state here this finite-dimensional analogue of Kottman's lemma.

Lemma 3.2.2 ([GlMe15, Theorem 0.1])**.** *Let C_n be the cube $C_n := \{0, \pm 1\}^n$ and let $A \subseteq C_n$ be a symmetric subset which contains $e_1, \dots, e_n$. Then there exists a subset B of A, with cardinality $n + 1$ such that for distinct $x, y \in B$, one has $x - y \notin A$.*

It is interesting to observe that this finite-dimensional counterpart actually subsumes the original Kottman's statement, [GlMe15, Proposition 1.10]; on the other hand, part of the proof of this lemma may be, in turn, deduced from Kottman's lemma, see Proposition 1.9 there.

We shall next present a simpler and non-combinatorial proof of Kottman's theorem; this argument may be found in [Die84, pp. 7–8], where it is given credit to Tom Starbird.

Second proof of Kottman's theorem, [Die84]. We are going to construct by induction two normalised sequences $(x_n)_{n=1}^{\infty} \subseteq S_X$ and $(\varphi_n)_{n=1}^{\infty} \subseteq S_{X^*}$ with the following properties:

(i) $\langle \varphi_n, x_n \rangle = 1$;

(ii) $\langle \varphi_i, x_n \rangle < 0$ for $i < n$;

(iii) $\{\varphi_n\}_{n=1}^{\infty}$ is a linearly independent set.

Once this is achieved, for $i < n$ we clearly have

$$\|x_i - x_n\| \geqslant \langle \varphi_i, x_i - x_n \rangle = 1 - \langle \varphi_i, x_n \rangle > 1,$$

and we are done. Assume that, for some $n \geqslant 1$, we have already found unit vectors $x_1, \ldots, x_n$ and norm-one functionals $\varphi_1, \ldots, \varphi_n$ with the above properties (for $n = 1$ this is trivially possible). Let us then recall that $y^* \in \operatorname{span}\{y_1^*, \ldots, y_n^*\}$ if and only if $\ker y^* \subseteq \cap \ker y_i^*$, whence (iii) implies that for every $i = 1, \ldots, n$ there exists $y_i \in \cap_{j \neq i} \ker \varphi_j$ such that $\langle \varphi_i, y_i \rangle < 0$. Therefore, the vector $y := \sum y_i$ satisfies $\langle \varphi_i, y \rangle < 0$ for $i = 1, \ldots, n$.

This vector y is not yet what we are looking for, so we first choose $z \in \cap \ker \varphi_i$ such that $\|y\| < \|y + z\|$. We may finally set $x_{n+1} := \frac{y+z}{\|y+z\|}$ and pick a norming functional φ_{n+1} for x_{n+1}; in this way, the first two properties are trivially satisfied. Finally, if it were that φ_{n+1} is a linear combination of the previous functionals φ_i's, then we would have $\langle \varphi_{n+1}, z \rangle = 0$ too. However, this results in a contradiction since

$$1 = \langle \varphi_{n+1}, x_{n+1} \rangle = \frac{\langle \varphi_{n+1}, y \rangle}{\|y + z\|} \leqslant \frac{\|y\|}{\|y + z\|} < 1.$$

■

The third, and last, proof of Kottman's theorem that we shall give, a modification of the proof given in [SWWY15, Theorem 1.1], is perhaps less short than Starbird's argument above, but in a sense more natural. The rough idea is that we follow Riesz' inductive argument and at each step we check whether the initial string actually is (1+)-separated. In case it is, we pass to the subsequent step, if not, we suitably modify the last selected vector.

Third proof of Kottman's theorem, [SWWY15]. It suffices to prove the following lemma, a variation of Riesz' lemma, and argue by induction.

Lemma 3.2.3. *Let X be an infinite-dimensional normed space and let $(x_i)_{i=1}^{n} \subseteq S_X$ be a finite sequence, which is a (1+)-separated set and such that*

$$\operatorname{dist}(x_j, \operatorname{span}\{x_1, \ldots, x_{j-1}\}) = 1 \qquad j = 1, \ldots, n.$$

Then there exists a unit vector $x \in X$ with

$$\mathrm{dist}(x, \mathrm{span}\{x_1, \ldots, x_n\}) = 1$$

and such that $(x_i)_{i=1}^n \cup \{x\}$ is $(1+)$-separated.

Proof of the Lemma. Let us assume, by contradiction, that no such x exists. According to Riesz' lemma, we may find a unit vector $y \in S_X$ such that

$$\mathrm{dist}(y, \mathrm{span}\{x_1, \ldots, x_n\}) = 1.$$

By our assumption, there exists an index $i_1 \in \{1, \ldots, n\}$ such that $\|y - x_{i_1}\| \leqslant 1$; evidently,

$$\mathrm{dist}(y - x_{i_1}, \mathrm{span}\{x_1, \ldots, x_n\}) = 1,$$

whence in particular $y_1 := y - x_{i_1}$ is a unit vector.

Once more, our assumption implies that $(x_i)_{i=1}^n \cup \{y_1\}$ is not $(1+)$-separated; hence, there exists $i_2 \in \{1, \ldots, n\}$ such that $\|y_1 - x_{i_2}\| \leqslant 1$; we set $y_2 := y_1 - x_{i_2} = y - x_{i_1} - x_{i_2}$ and we continue.

We have thus found a sequence $(y_k)_{k=1}^\infty$ consisting of unit vectors, of the form

$$y_k = y - \sum_{i=1}^n a_k^i x_i,$$

where the numbers a_k^i are natural numbers and $\sum_{i=1}^n a_k^i = k$, for every k. This is, however, impossible. In fact, $(x_i)_{i=1}^n \cup \{y\}$ is a basis for its linear span, whence there is $\delta > 0$ such that

$$\left\| \alpha y + \sum_{i=1}^n \alpha_i x_i \right\| \geqslant \delta \cdot \left(|\alpha| + \sum_{i=1}^n |\alpha_i| \right),$$

for every choice of the scalars $\alpha, (\alpha_i)_{i=1}^n$. Since y_k belongs to such linear span for every k, we then conclude $\|y_k\| \geqslant \delta(1 + k)$, for every $k \in \mathbb{N}$, a contradiction. ■

■

Let us then pass to the proof of Theorem 3.1.11.

Proof of Theorem 3.1.11. Let X be an infinite-dimensional Banach space. We consider the following property of an infinite-dimensional subspace $\tilde{X}$ of X: $\tilde{X}$ has property (▭) if there exist a unit vector $x \in S_{\tilde{X}}$ and an infinite-dimensional subspace Y of $\tilde{X}$ such that $\|x + y\| > 1$ for every unit vector $y \in S_Y$. In symbols,

$\tilde{X}$ has (▭) if: $\exists x \in S_{\tilde{X}}, \exists Y \subseteq \tilde{X}$ infinite-dimensional subspace: $\forall y \in S_Y\ \|x + y\| > 1$.

Then we have the following dichotomy: either every infinite-dimensional subspace of X has (▭) or some infinite-dimensional subspace has (¬▭).

The proof of the result in the first alternative of the dichotomy is very simple: in fact, the assumption that X has ($\square$) yields a unit vector $x_1 \in X$ and an infinite-dimensional subspace X_1 of X such that $\|x_1 + y\| > 1$ for every $y \in S_{X_1}$. Since X_1 has ($\square$) too, we can find a unit vector x_2 in X_1 and an infinite-dimensional subspace X_2 of X_1 such that $\|x_2 + y\| > 1$ for every $y \in S_{X_2}$. We proceed by induction in the obvious way and we find a sequence $(x_n)_{n=1}^{\infty}$ of unit vectors in X and a decreasing sequence $(X_n)_{n=1}^{\infty}$ of infinite-dimensional subspaces of X such that, for every $n \in \mathbb{N}$

(i) $x_{n+1} \in X_n$ and

(ii) $\|x_n + y\| > 1$ for every $y \in S_{X_n}$.

The sequence $(x_n)_{n=1}^{\infty} \subseteq S_X$ is then the desired symmetrically (1+)-separated sequence since for $1 \leqslant k < n$ we have $\pm x_n \in X_{n-1} \subseteq X_k$; hence $\|x_k \pm x_n\| > 1$.

In the second alternative, there exists an infinite-dimensional subspace $\tilde{X}$ of X with property ($\neg\square$); since we shall construct the desired sequence in the subspace $\tilde{X}$, we can assume without loss of generality that $\tilde{X} = X$. We first note that the assumption X to admit property ($\neg\square$) is equivalent to the formally stronger property

$$(\blacksquare) \qquad \forall x \in B_X, \forall Y \subseteq X \text{ infinite-dimensional subspace } \exists y \in S_Y \colon \|x + y\| \leqslant 1.$$

In fact, for $x \in S_X$, ($\blacksquare$) is exactly the negation of ($\square$), while for $x = 0$ it is trivially true. Given a non-zero $x \in B_X$ and an infinite-dimensional subspace Y of X, ($\neg\square$) provides us with a vector $y \in S_Y$ with $\left\|\frac{x}{\|x\|} + y\right\| \leqslant 1$; consequently $\|x + y\| \leqslant 1$, by Lemma 3.1.15.

We finally prove the result under the additional assumption that X has property ($\blacksquare$). Fix a decreasing sequence $(\delta_n)_{n=1}^{\infty}$ of positive reals with $\sum_{n=1}^{\infty} \delta_n \leqslant 1/4$, say $\delta_n = 2^{-(n+2)}$. Also, choose any $z \in X$ with $\|z\| = 3/4$ and find a norming functional $\psi \in S_{X^*}$ for z.

We now construct by induction two sequences $(y_n)_{n=1}^{\infty}$ in S_X and $(\varphi_n)_{n=1}^{\infty}$ in S_{X^*} such that:

(i) $\langle \varphi_n, y_n \rangle = 1$ $(n \in \mathbb{N})$;

(ii) $y_1 \in \ker \psi$ and $y_{n+1} \in \ker \psi \cap \bigcap_{i=1}^{n} \ker \varphi_i$ $(n \in \mathbb{N})$;

(iii) $\|z + y_1\| \leqslant 1$ and $\|z - \delta_1 y_1 - \ldots - \delta_n y_n + y_{n+1}\| \leqslant 1$ $(n \in \mathbb{N})$.

In fact, by ($\blacksquare$), there exists a unit vector $y_1 \in \ker \psi$ such that $\|z + y_1\| \leqslant 1$; we also find a norming functional φ_1 for y_1. Assume that we have already found $y_1, \ldots, y_n$ and $\varphi_1, \ldots, \varphi_n$ for some $n \geqslant 1$. Of course, the triangle inequality and our choice of $(\delta_n)_{n=1}^{\infty}$ imply

$$\|z - \delta_1 y_1 - \ldots - \delta_n y_n\| \leqslant 1,$$

thus ($\blacksquare$) ensures us of the existence of a unit vector y_{n+1} in $\ker \psi \cap \bigcap_{i=1}^{n} \ker \varphi_i$ such that

$$\|z - \delta_1 y_1 - \ldots - \delta_n y_n + y_{n+1}\| \leqslant 1.$$

To complete the induction step it is then sufficient to take a norming functional φ_{n+1} for y_{n+1}.

We now define $x_1 := z + y_1$ and $x_{n+1} := z - \delta_1 y_1 - \ldots - \delta_n y_n + y_{n+1}$ $(n \in \mathbb{N})$. Fix two natural numbers $k < n$. By the very construction, each y_i lies in $\ker \psi$, so we have

$$\|x_n + x_k\| \geqslant \langle \psi, x_n + x_k \rangle = \langle \psi, 2z \rangle = 2\|z\| > 1.$$

Moreover, $y_i \in \ker \varphi_k$ for every $i > k$, whence

$$\|x_k - x_n\| \geqslant \langle \varphi_k, x_k - x_n \rangle = \langle \varphi_k, y_k + \delta_k y_k + \ldots + \delta_{n-1} y_{n-1} - y_n \rangle = \langle \varphi_k, (1+\delta_k) y_k \rangle = 1 + \delta_k > 1.$$

Consequently, $(x_n)_{n=1}^\infty$ is a symmetrically $(1+)$-separated sequence and the vectors x_n are contained in B_X, due to (iii). It thus follows from Lemma 3.1.15 that the unit sphere of X contains a symmetrically $(1+)$-separated sequence. ■

3.3 Symmetrically $(1+\varepsilon)$-separated sequences

The aim of this section is to present the proof of Theorem 3.1.12, concerning the existence of symmetrically $(1+\varepsilon)$-separated sequences in the unit ball of Banach spaces with boundedly complete basic sequences. In the second part of the section, we shall combine this result with known results from the literature and derive the same conclusion for a quite large class of Banach spaces, which includes, in particular, all classical Banach spaces.

For convenience of the reader, we start recalling the relevant definitions and a few properties of boundedly complete sequences. A basic sequence $(e_j)_{j=1}^\infty$ in a Banach space X is *boundedly complete* if the series $\sum_{j=1}^\infty a^j e_j$ converges in X for every choice of the scalars $(a^j)_{j=1}^\infty$ such that

$$\sup_{k \in \mathbb{N}} \left\| \sum_{j=1}^k a^j e_j \right\| < \infty.$$

It is very simple to verify that if $(e_j)_{j=1}^\infty$ is a boundedly complete basic sequence, then so is every block basic sequence of $(e_j)_{j=1}^\infty$.

We shall require a small refinement of the classical Mazur technique of constructing basic sequences (see, *e.g.*, [LiTz77, Theorem 1.a.5]). In particular, we shall exploit along the way the following well-known lemma due to Mazur.

Lemma 3.3.1 (Mazur's lemma). *Let E be a finite-dimensional subspace of a Banach space X and $\varepsilon > 0$. Then there exists a finite-codimensional subspace F of X such that for every $x \in E$ and $v \in F$*

$$\|x\| \leqslant (1+\varepsilon)\|x + v\|.$$

The formulation given here is not formally identical to the more usual statement of the lemma, [LiTz77, Lemma 1.a.6], and it can be found, *e.g.*, in [HáJo14, Lemma 4.66]. On the other hand, the proof is verbatim the same: fixed a finite $\varepsilon/2$-net $\{y_i\}_{i=1}^n$ for the unit ball of E and norming functionals $y_i^* \in X^*$ for y_i $(i = 1, \ldots, n)$, the finite-codimensional subspace $F := \cap \ker y_i^*$ is as desired.

Lemma 3.3.2. *Let X be an infinite-dimensional Banach space and let $(e_j)_{j=1}^\infty$ be a basic sequence in X. Suppose that $(\varepsilon_j)_{j=1}^\infty$ is a sequence of positive real numbers that converges to 0. Then there exists a block basic sequence $(x_j)_{j=1}^\infty$ of $(e_j)_{j=1}^\infty$, such that*

$$\|P_j\| \leqslant 1 + \varepsilon_j \quad (j \in \mathbb{N}),$$

where $P_j \colon \overline{\operatorname{span}}\{x_j\}_{j=1}^\infty \to \overline{\operatorname{span}}\{x_j\}_{j=1}^\infty$ denotes the j-th canonical projection associated to the basic sequence $(x_j)_{j=1}^\infty$.

In particular, if $(e_j)_{j=1}^\infty$ is boundedly complete, then so is $(x_j)_{j=1}^\infty$.

Proof. Fix a sequence $(\delta_j)_{j=1}^\infty \searrow 0$ such that $\prod_{j=n}^\infty (1+\delta_j) \leqslant 1 + \varepsilon_n$ for every n. We start choosing a unit vector $x_1 \in \operatorname{span}\{e_j\}_{j=1}^\infty$. We then find a finite-codimensional subspace F_1 of X, obtained applying Mazur's lemma to $\operatorname{span}\{x_1\}$ and δ_1. For n_1 sufficiently large, we have $x_1 \in \operatorname{span}\{e_j\}_{j=1}^{n_1-1}$; since F_1 is finite-codimensional, we can choose a unit vector x_2 in $F_1 \cap \operatorname{span}\{e_j\}_{j=n_1}^\infty$. By Mazur's lemma, for all scalars α^1, α^2 such x_2 satisfies

$$\|\alpha^1 x_1\| \leqslant (1+\delta_1)\|\alpha^1 x_1 + \alpha^2 x_2\|.$$

We proceed analogously by induction: assume that we have already found a finite block sequence $(x_j)_{j=1}^n$ of $(e_j)_{j=1}^\infty$ such that

$$\left\| \sum_{j=1}^{k} \alpha^j x_j \right\| \leqslant (1+\delta_k) \left\| \sum_{j=1}^{k+1} \alpha^j x_j \right\|$$

for every $k = 1, \dots, n-1$ and scalars $\alpha^1, \dots, \alpha^n$. Let F_n be a finite-codimensional subspace of X as in the conclusion of Mazur's lemma, applied to $\operatorname{span}\{x_1, \dots, x_n\}$ and δ_n. Moreover let $N \in \mathbb{N}$ be so large that $x_1, \dots, x_n \in \operatorname{span}\{e_j\}_{j=1}^{N-1}$. We can then choose a unit vector x_{n+1} in $F_n \cap \operatorname{span}\{e_j\}_{j=N}^\infty$ and such a choice ensures us that

$$\left\| \sum_{j=1}^{n} \alpha^j x_j \right\| \leqslant (1+\delta_n) \left\| \sum_{j=1}^{n+1} \alpha^j x_j \right\| \tag{3.3.1}$$

for every choice of the scalars $\alpha^1, \dots, \alpha^{n+1}$. This concludes the inductive procedure.

From (3.3.1) it is clear that for every $n, k \in \mathbb{N}$

$$\left\| \sum_{j=1}^{n} \alpha^j x_j \right\| \leqslant \prod_{j=n}^{\infty}(1+\delta_j) \left\| \sum_{j=1}^{n+k} \alpha^j x_j \right\| \leqslant (1+\varepsilon_n) \left\| \sum_{j=1}^{n+k} \alpha^j x_j \right\|,$$

hence $\|P_n\| \leqslant 1 + \varepsilon_n$. It is also clear from the construction that $(x_j)_{j=1}^\infty$ is a block basic sequence of $(e_j)_{j=1}^\infty$. Finally, the last assertion of the lemma follows from the already mentioned observation that block sequences of boundedly complete basic sequences are boundedly complete. ∎

We are ready to enter the proof of the main theorem and we start introducing a bit of terminology. Given a basic sequence $(e_j)_{j=1}^{\infty}$, by a *block* we mean a vector in $\operatorname{span}\{e_j\}_{j=1}^{\infty}$, and we also say that a block is a *finitely supported* vector. A *unit block* is of course a block which is also a norm one vector. Two blocks b_1, b_2 are *consecutive* if $b_1 \in \operatorname{span}\{e_j\}_{j=1}^{N}$ and $b_2 \in \operatorname{span}\{e_j\}_{j=N+1}^{\infty}$; in this case we write $b_1 < b_2$. We also write $N < b$, where $N \in \mathbb{N}$, if $b \in \operatorname{span}\{e_j\}_{j=N+1}^{\infty}$, namely 'the support of b begins after N' (and analogously for $N \leqslant b$, $b < N$ or $b \leqslant N$). An *extension* of a finite set of blocks $b_1 < b_2 < \cdots < b_n$ is the choice of a block b with $b_n < b$.

Proof of Theorem 3.1.12. Fix a boundedly complete basic sequence $(e_j)_{j=1}^{\infty}$ in X and a decreasing sequence $(\varepsilon_j)_{j=1}^{\infty}$ of numbers in the interval $(0,1)$ with $\sum_{j=1}^{\infty} \varepsilon_j < \infty$. According to Lemma 3.3.2 we can assume that the canonical projections $(P_j)_{j=1}^{\infty}$ associated to $(e_j)_{j=1}^{\infty}$ satisfy $\|P_j\| \leqslant 1 + \varepsilon_j$. We are going to construct the desired symmetrically separated sequence as a block basic sequence of $(e_j)_{j=1}^{\infty}$, so we can safely assume without loss of generality that $X = \operatorname{span}\{e_j\}_{j=1}^{\infty}$. In other words, our actual assumptions are that X admits a boundedly complete Schauder basis $(e_j)_{j=1}^{\infty}$, whose associated canonical projections satisfy $\|P_j\| \leqslant 1 + \varepsilon_j$.

We now begin with the construction. Either every symmetrically $(1+\varepsilon_1)$-separated finite family of unit blocks $b_1 < b_2 < \cdots < b_n$ admits a symmetrically $(1+\varepsilon_1)$-separated extension $b_1 < b_2 < \cdots < b_n < b$, with b a unit block, or there exists a symmetrically $(1+\varepsilon_1)$-separated finite family of unit blocks $b_1 < b_2 < \cdots < b_n$ that admits no such extension. In the first case we start with a family with cardinality 1 and we can easily produce by induction a symmetrically $(1+\varepsilon_1)$-separated sequence $b_1 < b_2 < \cdots < b_n < \dots$ consisting of unit blocks. In this case the proof is complete. Alternatively, we have found a finite family of unit blocks $\mathcal{B}_1 := (b_i^{(1)})_{i=1}^{N_1}$ which is symmetrically $(1+\varepsilon_1)$-separated and admits no extension with the same property. In other words, the family $\mathcal{B}_1$ satisfies:

$$b_1^{(1)} < b_2^{(1)} < \cdots < b_{N_1}^{(1)},\ \|b_i^{(1)}\| = 1,\ \|b_i^{(1)} \pm b_j^{(1)}\| \geqslant 1 + \varepsilon_1 \quad (i,j \in \{1,\dots,N_1\}, i \neq j)$$

and for every unit block $b > b_{N_1}^{(1)}$ there are $i = 1,\dots,N_1$ and $\sigma = \pm 1$ with $\|\sigma b_i^{(1)} + b\| < 1 + \varepsilon_1$.

We next repeat the same alternative, but now we are in search of symmetrically $(1+\varepsilon_2)$-separated families of unit blocks and we only look for blocks $b > b_{N_1}^{(1)}$. Hence, either every symmetrically $(1+\varepsilon_2)$-separated finite family of unit blocks $b_1 < b_2 < \cdots < b_n$ with $b_{N_1}^{(1)} < b_1$ admits a symmetrically $(1+\varepsilon_2)$-separated extension $b_1 < b_2 < \cdots < b_n < b$, with b a unit block, or there exists a symmetrically $(1+\varepsilon_2)$-separated finite family of unit blocks $b_1 < b_2 < \cdots < b_n$ that admits no such extension. In the first case, the proof is completed by the simple induction argument, while in the second one we have obtained a family $\mathcal{B}_2 := (b_i^{(2)})_{i=1}^{N_2}$ such that

$$b_{N_1}^{(1)} < b_1^{(2)} < b_2^{(2)} < \cdots < b_{N_2}^{(2)},\ \|b_i^{(2)}\| = 1\ \|b_i^{(2)} \pm b_j^{(2)}\| \geqslant 1 + \varepsilon_2 \quad (i,j \in \{1,\dots,N_2\}, i \neq j)$$

and for every unit block $b > b_{N_2}^{(2)}$ there are $i = 1,\dots,N_2$ and $\sigma = \pm 1$ with $\|\sigma b_i^{(2)} + b\| < 1 + \varepsilon_2$.

We proceed by induction in the obvious way: if at some step, say step n, we fall in the first of the two alternatives, then we easily conclude the existence of a symmetrically $(1+\varepsilon_n)$-separated sequence of unit vectors. In this case the proof is concluded and, of course, we stop our construction. In the other case, we tenaciously proceed for every n and we consequently find families $\mathcal{B}_n := (b_i^{(n)})_{i=1}^{N_n}$ such that for every $n \in \mathbb{N}$:

(i) $\|b_i^{(n)}\| = 1$ $(i = 1, \dots, N_n)$;

(ii) $b_1^{(n)} < b_2^{(n)} < \cdots < b_{N_n}^{(n)} < b_1^{(n+1)}$;

(iii) $\|b_i^{(n)} \pm b_j^{(n)}\| \geqslant 1 + \varepsilon_n$ $(i, j \in \{1, \dots, N_n\}, i \neq j)$;

(iv) for any unit block $b > b_{N_n}^{(n)}$ there are $i = 1, \dots, N_n$ and $\sigma = \pm 1$ with $\|\sigma b_i^{(n)} + b\| < 1 + \varepsilon_n$.

Our plan now is to show that the existence of such families $(\mathcal{B}_n)_{n=1}^{\infty}$ is in contradiction with the assumption that $(e_j)_{j=1}^{\infty}$ is a boundedly complete Schauder basis. This implies that at some step we actually fall in the first alternative, and in turn concludes the proof. The basic idea we exploit to implement our plan is to use elements of $\mathcal{B}_{n+1}$ to witness the non-extendability of $\mathcal{B}_n$. We will also use the following obvious inequality[1]: if a, b are vectors in a normed space X and $1 - \varepsilon \leqslant \|b\| \leqslant 1 + \varepsilon$, then

$$\|a + b\| \leqslant \left\| a + \frac{b}{\|b\|} \right\| + \varepsilon. \tag{3.3.2}$$

Fix any natural number $k \geqslant 2$ and choose arbitrarily one index $n_k(k) \in \{1, \dots, N_k\}$; by condition (iv) there exist an index $n_{k-1}(k) \in \{1, \dots, N_{k-1}\}$ and a sign $\sigma_{k-1}(k) = \pm 1$ such that

$$\left\| \sigma_{k-1}(k) b_{n_{k-1}(k)}^{(k-1)} + b_{n_k(k)}^{(k)} \right\| < 1 + \varepsilon_{k-1}.$$

Moreover, we can find an index n with $b_{n_{k-1}(k)}^{(k-1)} \leqslant n < b_{n_k(k)}^{(k)}$ and clearly such n satisfies $n \geqslant k - 1$. Hence

$$1 = \left\| b_{n_{k-1}(k)}^{(k-1)} \right\| = \left\| P_n \left(\sigma_{k-1}(k) b_{n_{k-1}(k)}^{(k-1)} + b_{n_k(k)}^{(k)} \right) \right\| \leqslant (1 + \varepsilon_{k-1}) \left\| \sigma_{k-1}(k) b_{n_{k-1}(k)}^{(k-1)} + b_{n_k(k)}^{(k)} \right\|.$$

Consequently,

$$1 - \varepsilon_{k-1} \leqslant \left\| \sigma_{k-1}(k) b_{n_{k-1}(k)}^{(k-1)} + b_{n_k(k)}^{(k)} \right\| < 1 + \varepsilon_{k-1}.$$

The vector

$$b := \frac{\sigma_{k-1}(k) b_{n_{k-1}(k)}^{(k-1)} + b_{n_k(k)}^{(k)}}{\left\| \sigma_{k-1}(k) b_{n_{k-1}(k)}^{(k-1)} + b_{n_k(k)}^{(k)} \right\|}$$

is, of course, a unit block with $b > b_{N_{k-2}}^{(k-2)}$ and we can now use it to witness the maximality of $\mathcal{B}_{k-2}$. By condition (iv), there must exist an index $n_{k-2}(k) \in \{1, \dots, N_{k-2}\}$ and a sign $\sigma_{k-2}(k) = \pm 1$ such that

$$\left\| \sigma_{k-2}(k) b_{n_{k-2}(k)}^{(k-2)} + b \right\| < 1 + \varepsilon_{k-2}.$$

[1] We record it explicitly here only because below we will use it with some slightly complicated expressions and it would perhaps be not immediate to see how trivial the estimate is.

By the inequality (3.3.2) it then follows

$$1 - \varepsilon_{k-2} \leqslant \left\| \sigma_{k-2}(k) b^{(k-2)}_{n_{k-2}(k)} + \sigma_{k-1}(k) b^{(k-1)}_{n_{k-1}(k)} + b^{(k)}_{n_k(k)} \right\| < 1 + \varepsilon_{k-2} + \varepsilon_{k-1}.$$

(where the lower bound is obtained applying a suitable projection P_n, as we have already done above). We proceed by going backwards in a similar way: the normalisation of the vector $\sigma_{k-2}(k) b^{(k-2)}_{n_{k-2}(k)} + \sigma_{k-1}(k) b^{(k-1)}_{n_{k-1}(k)} + b^{(k)}_{n_k(k)}$ is a unit block, which we use to witness the maximality of $\mathcal{B}_{k-3}$, and so on. In particular, we have proved the existence of a string of indices and signs $\{n_1(k), \sigma_1(k), \ldots, n_k(k), \sigma_k(k)\}$, where $\sigma_k(k) = +1$, such that

$$\left\| \sigma_1(k) b^{(1)}_{n_1(k)} + \cdots + \sigma_k(k) b^{(k)}_{n_k(k)} \right\| < 1 + \varepsilon_1 + \cdots + \varepsilon_{k-1}.$$

If we apply again a suitable projection P_n, we also deduce the validity of the following stronger assertion: for every $k \in \mathbb{N}$ there exists a string of indices and signs $I_k = \{n_i(k), \sigma_i(k)\}_{i=1}^k$, where $\sigma_i(k) = \pm 1$ and $n_i(k) \in \{1, \ldots, N_i\}$ for $i = 1, \ldots, k$, such that for every $\ell \in \mathbb{N}$, $\ell \leqslant k$ we have

$$\left\| \sum_{i=1}^{\ell} \sigma_i(k) b^{(i)}_{n_i(k)} \right\| \leqslant (1 + \varepsilon_1) \cdot \left(1 + \sum_{j=1}^{\infty} \varepsilon_j \right) =: C < \infty. \tag{3.3.3}$$

Of course, there are only finitely many possibilities for the first two items of the strings $(I_k)_{k=1}^{\infty}$, so by the pigeonhole principle we may find an index $n_1 \in \{1, \ldots, N_1\}$ and a sign $\sigma_1 = \pm 1$ such that infinitely many strings begin with the pattern $\{n_1, \sigma_1\}$. Analogously, there are n_2, σ_2 such that an infinite subset of those strings begins with the pattern $\{n_1, \sigma_1, n_2, \sigma_2\}$. Continuing inductively, we find an infinite string $\{n_i, \sigma_i\}_{i=1}^{\infty}$ such that every its initial substring $\{n_i, \sigma_i\}_{i=1}^{\ell}$ is the initial part of infinitely many I_k's. In particular, equation (3.3.3) then implies that for every $\ell \in \mathbb{N}$

$$\left\| \sum_{i=1}^{\ell} \sigma_i b^{(i)}_{n_i} \right\| \leqslant C.$$

Finally, if we set $b_i := \sigma_i b^{(i)}_{n_i}$, the sequence $(b_i)_{i=1}^{\infty}$ is a block basic sequence of $(e_j)_{j=1}^{\infty}$, hence it is boundedly complete. Moreover, the last inequality now reads

$$\sup_{\ell} \left\| \sum_{i=1}^{l} b_i \right\| \leqslant C.$$

It follows that the series $\sum_{i=1}^{\infty} b_i$ converges in X, which is a blatant contradiction with the fact that the b_i's are unit vectors. ∎

We now pass to the second part of the section and we conclude the validity of a symmetric version of the Elton–Odell theorem, for a large class of Banach spaces. Our first, immediate, deduction is based on the well-known fact that, in a reflexive Banach space, every basic sequence is boundedly complete ([Jam50, Theorem 1]; see also [AlKa06, Theorem 3.2.13]). We therefore arrive at the following corollary.

Corollary 3.3.3 ([HKR18, Corollary 1.2])**.** *Let X be an infinite-dimensional reflexive Banach space. Then for some $\varepsilon > 0$ the unit sphere of X contains a symmetrically $(1+\varepsilon)$-separated sequence.*

The above observation may be extended to more general spaces, as Johnson and Rosenthal proved that if X is isomorphic to an infinite-dimensional subspace of a separable dual space, then it contains a boundedly complete basic sequence ([JoRo72, Theorem IV.1.(ii)]). As a consequence, the unit sphere of a Banach space X contains a symmetrically $(1+\varepsilon)$-separated sequence, whenever the Banach space X contains an infinite-dimensional subspace isomorphic to a dual Banach space.

Notably, spaces with the Radon–Nikodym property, or more generally, spaces with the so-called *point-of-continuity property* (in short, *PCP*) contain separable dual Banach spaces. Let us record a few definitions here to explain this point. A Banach space X has the *point-of-continuity property* if every of its weakly closed and bounded subset C admits a point y of weak-to-norm continuity, *i.e.*, the identity is continuous at y as a map from C with its weak topology into C with the norm topology. The proof that Banach spaces with PCP contain infinite-dimensional dual space may be found in [GhMa85, Corollary II.1].

Moreover, let us recall that a point y is a *denting point* for a bounded subset C of a Banach space, if x is contained in slices of C of arbitrarily small diameter. In this case, it is immediate that x is a point of weak-to-norm continuity for C. This observation combines with well-known results on the existence of denting points in Banach spaces with the RNP [BeLi00, §5.2], to conclude that every Banach space with the RNP admits the PCP.

We may record then the following corollary to Theorem 3.1.12.

Corollary 3.3.4 ([HKR18, Corollary 1.3])**.** *Suppose that X contains a subspace isomorphic to a subspace of a separable dual space. Then for some $\varepsilon > 0$ the unit sphere of X contains a symmetrically $(1+\varepsilon)$-separated sequence.*

Consequently, the assertion holds true in the case where X has the Radon–Nikodym property (or more generally, PCP).

Of course the prototypical example of Banach space which is not included in the previous considerations is the space c_0; on the other hand, we already discussed in the first section how to use James non distortion theorem and obtain a symmetrically $(1+\varepsilon)$-separated sequence in the unit sphere of every Banach space that contains a copy of c_0.

This observation combines with James' dichotomy [Jam50] for Banach spaces with unconditional basis to give a symmetric version of the Elton–Odell theorem for every Banach space with unconditional basis. This argument can be extended to the class of Banach lattices, since for (a space isomorphic to) a Banach lattice X we have the following three, not necessarily exclusive, possibilities: X is reflexive, X contains a subspace isomorphic to c_0 or X contains a subspace isomorphic to ℓ_1 (*cf.* [LiTz79, Theorem 1.c.5]). By Theorem 3.1.12 we thus obtain the following result.

Theorem 3.3.5 ([HKR18, Theorem B])**.** *Suppose that X is a Banach space that contains an infinite-dimensional subspace isomorphic to a Banach lattice (for example, a space with* *an unconditional basis). Then for some $\varepsilon > 0$ the unit sphere of X contains a symmetrically $(1+\varepsilon)$-separated sequence.*

At this stage, we came across a famous problem which has been open for decades, namely whether every infinite-dimensional Banach space contains an unconditional basic sequence; this problem seems to have been explicitly stated in [BePe58, Problem 5.1] for the first time, but most likely it was known before. Of course, a positive answer to this problem would then imply the validity of a symmetric analogue to the Elton–Odell theorem; on the other hand, the negative solution to the unconditional basic sequence problem, due to Gowers and Maurey [GoMa93], is by now a very famous result.

At the appearance of such counterexample X_{GM}, Johnson observed that the space X_{GM} has the stronger property to be hereditarily indecomposable. An infinite-dimensional Banach space is *decomposable* if it can be decomposed as a topological sum of two its closed, infinite-dimensional subspaces; it is *indecomposable* if it is not decomposable. Moreover, an infinite-dimensional Banach space is *hereditarily indecomposable* (for short, *HI*) when every its infinite-dimensional subspace is indecomposable. Certainly, hereditarily indecomposable spaces do not contain unconditional basic sequences; in a sense, a converse of this assertion is also valid. More precisely, the celebrated Gowers' Dichotomy Theorem ([Gow96]) asserts that an infinite-dimensional Banach space contains an infinite-dimensional subspace with an unconditional basis or a hereditarily indecomposable subspace.

For a presentation of some parts of this theory, we refer to the Handbook articles, authored by the Gowers and Maurey themselves, [Gow03], [Mau03]; let us mention that one ancestor of the first HI space, Schlumprecht's space [Sch91], is also at the basis of the solution of the distortion problem [OdSc93, OdSc94]. Moreover, HI spaces were also used for a solution to the scalar-plus-compact problem, [ArHa11].

On the other hand, the original example of a hereditarily indecomposable space X_{GM} was reflexive, which immediately implies the existence of a symmetrically $(1+\varepsilon)$-separated sequence of unit vectors in X_{GM}, in light of Corollary 3.3.3. In fact, there exist hereditarily indecomposable spaces without reflexive subspaces; the first example is due to Gowers [Gow94]. However, Gowers' space admits an equivalent uniformly Kadets–Klee norm ([DGK94, Corollary 10]), so it has PCP since it does not contain ℓ_1 ([DGK94, Proposition 2]). Consequently, Corollary 3.3.4 applies to any renorming of Gowers' space.

More recently, Argyros and Motakis ([ArMo••]) constructed a $\mathscr{L}_\infty$-space X_{AM} without reflexive subspaces whose dual is isomorphic to ℓ_1. In particular, X_{AM} is an Asplund space containing weakly Cauchy sequences that do not converge weakly, so the unit ball of X_{AM} is not completely metrisable in the relative weak topology. By [EdWh84, Theorem A], X_{AM} fails PCP; the same reasoning applies to any closed subspace of X_{AM}. (We are indebted to Pavlos Motakis for having explained this to us.) Nevertheless, X_{AM} being a $\mathscr{L}_\infty$-space, by a result of Castillo and Papini, contains a symmetrically $(1+\varepsilon)$-separated sequence in the unit sphere for some $\varepsilon > 0$.

As a consequence of this discussion, even in the absence of unconditional bases, the conjunction of the corollaries to our main result seems to apply to a wide range of HI

Banach spaces. Actually, we do not know of any explicit example of Banach space none of the above corollaries applies to. However, it its full generality, the following problem is still open to us.

Problem 3.3.6. Is the symmetric version of the Elton–Odell theorem valid for every Banach space? Namely, is it true that for every Banach space there are $\varepsilon > 0$ and a symmetrically $(1+\varepsilon)$-separated sequence of unit vectors?

From our results it follows that it would be sufficient to prove the result under the additional assumption that X is hereditarily indecomposable or non-reflexive. In particular, a way to solve Problem 3.3.6 would be to find a symmetric version of the result by Kryczka and Prus. For this reason, we can also ask the following:

Problem 3.3.7. Is there a constant $c > 1$ such that the unit ball of every non-reflexive Banach space contains a symmetrically c-separated sequence?

One immediate property that we noted in Corollary 3.1.17 is that if Z is an isometric quotient of a Banach space X, then $K^s(X) \geqslant K^s(Z)$. In particular, every Banach space with an infinite-dimensional reflexive quotient contains a symmetrically $(1+\varepsilon)$-separated sequence of unit vectors, for some $\varepsilon > 0$. If we combine this with the, already mentioned more than once, fact that every infinite-dimensional $\mathscr{L}_\infty$-space X satisfies $K^s(X) = 2$, we infer that a positive answer to the following problem would solve in the positive the main Problem 3.3.6.

Problem 3.3.8. Does every infinite-dimensional Banach space either contain an infinite-dimensional $\mathscr{L}_\infty$-space or admit an infinite-dimensional reflexive quotient?

3.4 Estimates for the symmetric Kottman constant

In this last section of the chapter, we shall present quantitative results in which it is possible to provide explicit estimates on the symmetric separation constant. Let us record formally here the definition of the constant subject of the investigation of this section (which was actually already defined in the first section).

Definition 3.4.1 ([CaPa11])**.** The *symmetric Kottman constant* of a Banach space X is

$$K^s(X) := \sup\Big\{\sigma > 0\colon \exists (x_n)_{n=1}^\infty \subset B_X\colon\ \|x_n \pm x_k\| \geqslant \sigma\ \forall n \neq k\Big\}.$$

Let us start, for the sake of completeness, restating a few results already present in the literature concerning this constant. First, we restate a few immediate consequences of the elementary observations in Section 3.1.1: a Banach space X satisfies $K^s(X) = 2$, whenever it contains an isomorphic copy of c_0 or ℓ_1, or it admits a quotient isomorphic to one of these spaces.

The first non entirely obvious claim is probably a well known folklore fact, but we were not able to find it explicitly stated in the literature: if a Banach space X admits a spreading model isomorphic to ℓ_1, then $K^s(X) = 2$. We shall say more on this in Section 3.4.2, where we will in particular briefly recall the notion of a spreading model and give a proof of such result.

As we have already hinted at in the introduction, Castillo and Papini [CaPa11, Proposition 3.4] proved that if X is a $\mathscr{L}_\infty$-space, then $K^s(X) = 2$; below, we will record this result and shortly discuss its proof.

Delpech [Del10] proved that every asymptotically uniformly convex Banach space X satisfies $K^s(X) \geqslant 1 + \overline{\delta}_X(1)$, where $\overline{\delta}_X$ is the modulus of asymptotic uniform convexity (as we already mentioned, the symmetry assertion is not contained in the statement, but it follows immediately from inspection of the proof). Let us refer to Section 4.4.1 for more information on the modulus of asymptotic uniform convexity and on this result. In particular, in that section we will present a generalisation of Delpech's argument to non-separable Banach spaces, which subsumes the result in [Del10] as a particular case.

Prus [Pru10, Corollary 5] proved, among other things, that if X has cotype $q < \infty$, then $K(X) \geqslant 2^{1/q}$; it is not apparent from the argument whether it should also follow that $K^s(X) \geqslant 2^{1/q}$. Therefore, we offer an alternative shorter proof of Prus' result which also provides an estimate for the symmetric Kottman constant (*cf.* Section 3.4.1).

Let us then record formally the result by Castillo and Papini; we also outline its proof. For information on $\mathscr{L}_p$-spaces, we shall refer to [LiPe68, LiRo69], or [LiTz73, Chapter 5].

Theorem 3.4.2 ([CaPa11]). *Let X be an $\mathscr{L}_\infty$-space; then $K^s(X) = 2$.*

Outline of the proof. It follows from a standard 'closing off' argument ([LiPe68, Proposition 7.2]) that every $\mathscr{L}_\infty$-space contains a separable subspace which is also an $\mathscr{L}_\infty$-space; consequently, we may assume that X is a separable Banach space. Moreover, from the elementary properties of the symmetric Kottman constant that we recorded above, we see that it is sufficient to verify that X admits a quotient isomorphic to c_0. Moreover, for a separable Banach space X, this condition is equivalent to the requirement that X^* contains a copy of ℓ_1 ([JoRo72], *cf.* [LiTz77, Proposition 2.e.9]). Finally, that the dual of every separable $\mathscr{L}_\infty$-space contains a copy of ℓ_1 follows from well-known results on $\mathscr{L}_p$-spaces: in fact, if X is an $\mathscr{L}_\infty$-space, then X^* is an $\mathscr{L}_1$-space ([LiRo69, Theorem III(a)]) and every $\mathscr{L}_1$-space contains a (complemented) copy of ℓ_1 ([LiPe68, Proposition 7.3]). ■

Our results in this area will be presented in the subsequent sections; in particular, the first one is dedicated to Banach spaces with finite cotype, in the second one we shall exploit spreading models and the last subsection depends on the use of biorthogonal systems.

3.4.1 Cotype and symmetric separation

The goal of this part is to prove a counterpart of Prus' result and relate the symmetric Kottman constant with the cotype of the Banach space; part of the argument to be presented is based on an idea from [KaKo16].

A normalised basic sequence $(x_n)_{n=1}^{\infty}$ *satisfies a lower q-estimate* if there is a constant $c > 0$ such that

$$c \cdot \left(\sum_{i=n}^{N} |a_n|^q \right)^{1/q} \leqslant \left\| \sum_{n=1}^{N} a_n x_n \right\|$$

for every choice of scalars $(a_n)_{n=1}^{N}$ and every $N \in \mathbb{N}$.

Let X be a Banach space with a Schauder basis $(x_n)_{n=1}^{\infty}$ and let us denote $X_n := \overline{\operatorname{span}}\{x_i\}_{i=n}^{\infty}$ $(n \in \mathbb{N})$. We say that an operator $T\colon X \to Y$ is *bounded by a pair* (γ, ϱ), where $0 < \gamma \leqslant \varrho < \infty$, if $\|T\| \leqslant \varrho$ and $\|T|_{X_n}\| \geqslant \gamma$ for every $n \in \mathbb{N}$.

Theorem 3.4.3 ([HKR18, Proposition 4.1])**.** *Let X be a Banach space that contains a normalised basic sequence satisfying a lower q-estimate. Then $K^s(X) \geqslant 2^{1/q}$.*

Proof. Let $(x_n)_{n=1}^{\infty}$ be a normalised basic sequence with a lower q-estimate. We are going to construct the separated sequence as a block sequence of the basic sequence, so we can assume without loss of generality that $X = \overline{\operatorname{span}}\{x_i\}_{i=1}^{\infty}$. Then the assignment $Tx_n := e_n$ $(n \in \mathbb{N})$ defines an injective, bounded linear operator $T\colon X \to \ell_q$.

Set $\varrho_n = \|T|_{X_n}\|$ $(n \in \mathbb{N})$. Clearly, $(\varrho_n)_{n=1}^{\infty}$ is a decreasing sequence with $\varrho_n \geqslant 1$ for every $n \in \mathbb{N}$. Moreover, $T|_{X_k}$ $(k \in \mathbb{N})$ is bounded by the pair $(\inf_{n\geqslant 1} \varrho_n, \varrho_k)$ and, of course, $\varrho_k \to \inf_{n\geqslant 1} \varrho_n$ as $k \to \infty$. In other words, up to replacing X with X_k, for k sufficiently large, we can (and do) assume that $T\colon X \to \ell_q$ is bounded by a pair (γ, ϱ) with $\frac{\gamma}{\varrho}$ as close to 1 as we wish (of course with $\frac{\gamma}{\varrho} < 1$).

Armed with this further information, we may now conclude the proof: let $\tilde{\gamma} < \gamma$ be such that $\frac{\tilde{\gamma}}{\varrho}$ is still as close to 1 as we wish. Since $\|T\| > \tilde{\gamma}$, we can find a unit vector y_1 in $\operatorname{span}\{x_i\}_{i=1}^{\infty}$ such that $\|Ty_1\| > \tilde{\gamma}$. Assume now that we have already found unit vectors $y_1, \dots, y_n$ in $\operatorname{span}\{x_i\}_{i=1}^{\infty}$ such that $\|Ty_k\| > \tilde{\gamma}$ and the Ty_k have mutually disjoint supports. Then there is N such that $y_1, \dots, y_n \in \operatorname{span}\{x_i\}_{i=1}^{N}$ and the fact that $\|T|_{X_{N+1}}\| > \tilde{\gamma}$ allows us to find a unit vector $y_{n+1} \in \operatorname{span}\{x_i\}_{i=N+1}^{\infty}$ such that $\|Ty_{n+1}\| > \tilde{\gamma}$.

Consequently, we have found a sequence $(y_n)_{n=1}^{\infty}$ in S_X such that $\|Ty_n\| > \tilde{\gamma}$ and the supports of Ty_n are finite and mutually disjoint. Hence for $n \neq k$ we have

$$\varrho \cdot \|y_n \pm y_k\| \geqslant \|Ty_n \pm Ty_k\| = \left(\|Ty_n\|^q + \|Ty_k\|^q\right)^{1/q} \geqslant \tilde{\gamma} \cdot 2^{1/q}.$$

So

$$K^s(X) \geqslant \frac{\tilde{\gamma}}{\varrho} \cdot 2^{1/q}$$

and, since $\frac{\tilde{\gamma}}{\varrho}$ could be chosen to be as close to 1 as we wish, the proof is complete. ∎

Recall that for a Banach space X one sets

$$q_X := \inf \big\{ q \in [2, \infty] : X \text{ has cotype } q \big\}.$$

Corollary 3.4.4. *Let X be an infinite-dimensional Banach space. Then $K^s(X) \geqslant 2^{1/q_X}$.*

Proof. In the case that $q_X = \infty$, the assertion just reduces to a consequence of Riesz' lemma; consequently, we shall assume that $q_X < \infty$. Moreover, if X is a Schur space, then by Rosental's ℓ_1-theorem X contains a copy of ℓ_1 and the James' non-distortion theorem even implies $K^s(X) = 2$. In the other case, there exists a weakly null normalised basic sequence in X; it then follows that for every $r > q_X$ such a sequence admits a subsequence with a lower r-estimate (see, *e.g.*, [HáJo14, Proposition 4.36]). The result now follows from the previous proposition. ■

3.4.2 Spreading models

The goal of this section is to obtain estimates on the symmetric Kottman constant of a Banach space by looking at its spreading models. In particular, the idea will be to find suitable separated sequences in a spreading model of X and then transfer them back to a separated sequence in the target space X. One instance when we are able to implement this technique is the case of spreading models isomorphic to ℓ_1, which case applies in particular to Tsirelson's space.

Let us start by mentioning the motivation behind these results. It was already known to Kottman that it is possible to enlarge the Kottman constant of a Banach space, via a suitable renorming; see Section 3.4.3 for more information on this point, in particular for a symmetric counterpart to the result.

It is therefore natural to try to renorm a Banach space and decrease its Kottman constant. One example where this is possible is [MaPa09, Theorem 2.6], where the authors show that for every Banach space X one has $K(X) \leqslant 2 \cdot (1 - \delta_X(1))$, δ_X denoting the modulus of uniform convexity of X. It follows in particular that every super-reflexive Banach space X admits a renorming $|||\cdot|||$ such that $K((X, |||\cdot|||)) < 2$. This motivated the authors to ask whether every space which fails to contain c_0 or ℓ_1, or at least every reflexive space, admits a renorming with the Kottman constant smaller than 2. To the best of our knowledge, and also according to the authors themselves, before the appearance of [HKR18, §5.2] there seemed to be no published solution to these questions. We will explicitly record the answers to this question in the present section.

An example of a Banach space which does not contain isomorphic copies of either c_0 or ℓ_1 and still has the Kottman constant equal to 2 under every renorming is the Bourgain–Delbaen space Y_{BD} ([BoDe80, Section 5]). Y_{BD} is the first example of a $\mathscr{L}_\infty$-space that is saturated by reflexive subspaces; in particular, it contains no copy of c_0 or ℓ_1. Still, every renorming of Y_{BD} has the Kottman constant (even K^s) equal to 2 by [CaPa11, Proposition 3.4] already quoted above. More generally, every predual of ℓ_1 is another example of space for which the symmetric Kottman constant is equal to 2 under every renorming; we note that the space constructed by Argyros and Motakis is such an example, which does not contain c_0 either.

In order to offer an example of a reflexive Banach space every whose renorming has (symmetric) Kottman constant equal to 2, thereby answering in the negative the question by Maluta and Papini, we next observe that if $\|\cdot\|$ is any renorming of the Tsirelson space T, then $K^s((T, \|\cdot\|)) = 2$. Let us mention that, as it is now customary, the space T we consider is the one constructed by Figiel and Johnson [FiJo74], see also [LiTz77, Example 2.e.1], and it is the isometric dual to the original Tsirelson's space T^* [Tsi74].

As we already mentioned, the argument will exploit the construction of spreading models, which we now pass to briefly discuss (we refer, *e.g.*, to [BeLa84, Ode02] for detailed discussions of spreading models). The starting point is the following important result, due to Brunel and Sucheston, [BrSu74, Proposition 1].

Proposition 3.4.5 ([BrSu74])**.** *Let $(x_n)_{n=1}^\infty$ be a bounded sequence in a Banach space X. Then there exists a subsequence $(y_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that for every $k \in \mathbb{N}$ and scalars $\alpha^1, \dots, \alpha^k$ the following limit exists:*

$$\lim_{\substack{n_1 < \cdots < n_k \\ n_1 \to \infty}} \left\| \sum_{i=1}^{k} \alpha^i y_{n_i} \right\|.$$

The proof of this result is actually an elementary argument, based on repeated use of Ramsey theorem and diagonalisation; let us shortly describe the scheme of the argument. Fixed scalars $\alpha^1, \dots, \alpha^k$, the range of the function $[\mathbb{N}]^k \ni \{n_1, \dots, n_k\} \mapsto \| \sum_{i=1}^k \alpha^i x_{n_i} \|$, where $n_1 < n_2 < \cdots < n_k$, is contained in a bounded interval I. If we partition such interval in two disjoint sub-intervals with equal length, an appeal to Ramsey theorem yields an infinite subset M of $\mathbb{N}$ such that $\| \sum_{i=1}^k \alpha^i x_{n_i} \|$ belongs to the same sub-interval, whenever $\{n_1, \dots, n_k\} \in [M]^k$. We may now repeat the same argument inductively and the diagonal subsequence $(y_n)_{n=1}^\infty$ has the property that

$$\lim_{\substack{n_1 < \cdots < n_k \\ n_1 \to \infty}} \left\| \sum_{i=1}^{k} \alpha^i y_{n_i} \right\|,$$

for this specific choice of $\alpha^1, \dots, \alpha^k$. We then repeat the same argument for every k-tuple $\alpha^1, \dots, \alpha^k$, consisting of rational scalars, and for every k; the diagonal subsequence obtained from this procedure has then the desired property, by an immediate density argument.

We now apply such result for the construction of spreading models of a Banach space X. Let us fix a Banach space X and a bounded sequence $(x_n)_{n=1}^\infty$ in X; let also $(y_n)_{n=1}^\infty$ be a subsequence of $(x_n)_{n=1}^\infty$ with the property described in the above proposition. For a vector $(\alpha(i))_{i=1}^\infty \in c_{00}$ with $\operatorname{supp}(\alpha(i))_{i=1}^\infty \leqslant k$ denote the above limit by $L\left((\alpha(i))_{i=1}^\infty\right)$; it is immediate to check that L defines a seminorm on c_{00} and that such a seminorm is actually a norm provided that the sequence $(y_n)_{n=1}^\infty$ is not convergent in X. In such a case, the completion of c_{00} under the norm L (which we will henceforth denote $\|\cdot\|$) is called a *spreading model* of X. The canonical basis $(e_n)_{n=1}^\infty$ of c_{00}, will be called the *fundamental sequence* of the spreading model. One its fundamental property, which is actually an obvious consequence of the definitions, is that the fundamental sequence is *invariant under spreading*, *i.e.*, for every choice of natural numbers $n_1 < \cdots < n_k$

$$\left\| \sum_{i=1}^{k} \alpha^i e_i \right\| = \left\| \sum_{i=1}^{k} \alpha^i e_{n_i} \right\|.$$

We will be interested in the question when a Banach space X admits a spreading model isomorphic to ℓ_1. A first very simple consequence of Rosenthal's ℓ_1-theorem and the invariance under spreading is the following (see, *e.g.*, [BeLa84, Lemme II.2.1]): if F is a spreading model of a Banach space X, then F is isomorphic to ℓ_1 if and only if the fundamental sequence of F is equivalent to the canonical basis of ℓ_1. The next characterization is due to Beauzamy, [Bea79, Theorem II.2] (it may also be found in [BeLa84, Théorème II.2.3]).

Theorem 3.4.6 ([Bea79])**.** *Let X be a Banach space. Then the following are equivalent:*

(i) *X admits a spreading model isomorphic to ℓ_1;*

(ii) *there are $\delta > 0$ and a bounded sequence $(x_n)_{n=1}^\infty$ in X such that, for every $k \in \mathbb{N}$, $\varepsilon_1, \dots, \varepsilon_k = \pm 1$ and $n_1 < \cdots < n_k$ one has*

$$\frac{1}{k} \left\| \sum_{i=1}^{k} \varepsilon_i x_{n_i} \right\| \geqslant \delta;$$

(iii) *for every $\eta > 0$ there is a bounded sequence $(x_n)_{n=1}^\infty$ in X such that, for every $k \in \mathbb{N}$, $\varepsilon_1, \dots, \varepsilon_k = \pm 1$ and $n_1 < \cdots < n_k$ one has*

$$1 - \eta \leqslant \frac{1}{k} \left\| \sum_{i=1}^{k} \varepsilon_i x_{n_i} \right\| \leqslant 1 + \eta.$$

The equivalence between conditions (ii) and (iii) above may be understood as a counterpart of James non distortion theorem for spreading models; this is true not only of the statement, but also of the method of proof, *cf.* [BeLa84, Proposition II.2.4].

As simple deduction from the above characterisation, we may now conclude sharp estimates on the symmetric Kottman constant of Banach spaces with ℓ_1 spreading models.

Corollary 3.4.7 ([HKR18, Corollary 5.6])**.** *Suppose that a Banach space X admits a spreading model isomorphic to ℓ_1. Then for every renorming $|||\cdot|||$ of X one has $K^s((X, |||\cdot|||)) = 2$.*

Proof. From the equivalence between (i) and (ii) in the previous theorem, it is obvious that if X admits a spreading model isomorphic to ℓ_1, then the same occurs to $(X, |||\cdot|||)$. Hence, we only need to show that $K^s(X) = 2$. Applying now (iii) of the same theorem yields, for every $\eta > 0$, a sequence $(x_n)_{n=1}^\infty$ such that

$$1 - \eta \leqslant \|x_n\| \leqslant 1 + \eta \quad \text{and} \quad \frac{1}{2}\|x_n \pm x_k\| \geqslant 1 - \eta \quad (n, k \in \mathbb{N}, n \neq k).$$

Consequently, the sequence $(\frac{x_n}{1+\eta})_{n=1}^\infty \subseteq B_X$ is symmetrically $2 \cdot \frac{1-\eta}{1+\eta}$-separated. Lemma 3.1.15 then yieds $K^s(X) \geqslant 2 \cdot \frac{1-\eta}{1+\eta}$ and letting $\eta \to 0^+$ concludes the proof. ∎

As we already mentioned, we shall apply the above corollary to Tsirelson's space, whose construction is recorded here. We follow the construction due to [FiJo74], as described in [AlKa06] or in the monograph [CaSh89]; the latter contains a throughout investigation of Tsirelson's space and its variations.

We shall start by fixing some notation: if E and F are finite subsets of $\mathbb{N}$, we keep the notation $E < F$ to mean that $\max E < \min F$; and analogously for $E \leqslant F$. In the case that $E = \{n\}$ is a singleton, we write $n < E$ in place of $\{n\} < E$. For a vector $x \in c_{00}$ and a finite subset E of $\mathbb{N}$, we consider the vector $Ex := (\chi_E(j) \cdot x(j))_{j=1}^{\infty}$, *i.e.*, we also denote by E the associated linear projection[2].

Definition 3.4.8. A collection $(E_j)_{j=1}^{k}$ of finite subsets of $\mathbb{N}$ is *admissible* if

$$k < E_1 < E_2 < \cdots < E_k.$$

In other words, we are allowed to select k consecutive blocks, provide that we select them to 'start after' k. As a shorthand, especially in mathematical formulas, we will write $(E_j)_{j=1}^{k}$ *adm* to mean that $(E_j)_{j=1}^{k}$ is an admissible family.

We may now define a sequence of norms on c_{00}.

Definition 3.4.9. For a vector $x \in c_{00}$, consider

$$\|x\|_0 := \|x\|_{c_0} := \max_{j=1,\dots,\infty} |x(j)|,$$

$$\|x\|_1 := \max \left\{ \|x\|_0, \sup_{(E_j)_{j=1}^{k} \text{adm}} \frac{1}{2} \cdot \sum_{j=1}^{k} \|E_j x\|_0 \right\}$$

$$\vdots$$

$$\|x\|_{n+1} := \max \left\{ \|x\|_n, \sup_{(E_j)_{j=1}^{k} \text{adm}} \frac{1}{2} \cdot \sum_{j=1}^{k} \|E_j x\|_n \right\}$$

Obviously, $(\|\cdot\|_n)_{n=0}^{\infty}$ is an increasing sequence of norms on c_{00}; it is moreover immediate to prove by induction that $\|\cdot\|_n \leqslant \|\cdot\|_{\ell_1}$ for every n. The following definition is therefore well posed.

Definition 3.4.10. Let $\|\cdot\|_T$ be the norm on c_{00} defined by

$$\|x\|_T := \lim_{n\to\infty} \|x\|_n = \sup_n \|x\|_n \qquad (x \in c_{00}).$$

Tsirelson's space T is the completion of $(c_{00}, \|\cdot\|_T)$.

[2]In other parts of the thesis such object would have been denoted by $x\restriction_E$, but here we prefer to keep the notation of [CaSh89]

One also immediately sees by induction that $(e_j)_{j=1}^{\infty}$ is a 1-unconditional basis for T. The following elementary fact contains an alternative description of the norm of T; its simple proof may be found in [CaSh89, Chapter I].

Fact 3.4.11. *For every $x \in c_{00}$ and $n \in \mathbb{N}$*

$$\|x\|_{n+1} := \max \left\{ \|x\|_0, \sup_{(E_j)_{j=1}^k \text{adm}} \frac{1}{2} \cdot \sum_{j=1}^k \|E_j x\|_n \right\}.$$

Consequently, $\|\cdot\|_T$ is the unique norm on c_{00} that satisfies the implicit equation

$$(T) \qquad \|x\| := \max \left\{ \|x\|_{c_0}, \sup_{(E_j)_{j=1}^k \text{adm}} \frac{1}{2} \cdot \sum_{j=1}^k \|E_j x\| \right\} \qquad (x \in c_{00}).$$

Let us then pass to discuss the first fundamental property of T, namely the fact that T contains no isomorphic copy of c_0, or ℓ_p $(1 \leqslant p < \infty)$. The starting point is an immediate, but important, consequence of the definition of $\|\cdot\|_T$. Assume that $(u_j)_{j=1}^k$ is a normalised block sequence in T such that $k < \operatorname{supp} u_1$. If we set $E_j := \operatorname{supp} u_j$, it is obvious that $(E_j)_{j=1}^k$ is admissible, whence (T) yields

$$\left\| \sum_{j=1}^k a_j u_j \right\|_T \geqslant \frac{1}{2} \sum_{i=1}^k \left\| E_i \left(\sum_{j=1}^k a_j u_j \right) \right\|_T = \frac{1}{2} \sum_{i=1}^k \|a_i u_i\|_T = \frac{1}{2} \sum_{i=1}^k |a_i|.$$

Consequently, the finite sequence $(u_j)_{j=1}^k$ is 2-equivalent to the canonical basis of ℓ_1^k. An immediate consequence of this assertion is that, if $(u_j)_{j=1}^{\infty}$ is any normalised block basis sequence in T, then for every $k \in \mathbb{N}$ there is a finite subsequence $u_{j_1}, \ldots, u_{j_k}$ which is 2-equivalent to the canonical basis of ℓ_1^k.

When combined with the Bessaga–Pełczyński selection principle [AlKa06, Proposition 1.3.10], this readily implies that T contains no copy of c_0 or ℓ_p $(1 < p < \infty)$. The proof that ℓ_1 does not embed in T is somewhat more difficult, as it depends on a finer analysis of the norm of T and on James' non distortion theorem, *cf.* [CaSh89, Proposition I.3]. Let us then state the main property of T that follows from a combination of the above observations (see [CaSh89, Theorem I.8]).

Theorem 3.4.12 ([Tsi74]). *T is a reflexive Banach space with 1-unconditional basis. Moreover, it contains no copy of c_0 or ℓ_p $(1 \leqslant p < \infty)$, nor an infinite-dimensional super-reflexive subspace.*

We shall now turn our attention to the spreading models of T, starting with those generated by block sequence of the canonical basis. Let F be a spreading model of T generated by a normalised block sequence $(u_j)_{j=1}^{\infty}$ of the basis of T. Up to passing to a subsequence, we may assume that the sequence $(u_j)_{j=1}^{\infty}$ satisfies the conclusion of Proposition 3.4.5 (in this discussion, let us call any sequence that satisfies the conclusion of Proposition 3.4.5

a *spreading sequence*). Given scalars $\alpha_1, \dots, \alpha_k$, the norm in F of the vector $\sum_{j=1}^k \alpha_j e_j$ is defined to be equal to

$$\lim_{\substack{n_1 < \dots < n_k \\ n_1 \to \infty}} \left\| \sum_{j=1}^k \alpha_j u_{n_j} \right\|_T .$$

Evidently, for n_1 sufficiently large, we have $k < \operatorname{supp} u_{n_1}$; therefore, the above observation yields that

$$\frac{1}{2} \sum_{j=1}^k |\alpha_j| \leqslant \left\| \sum_{j=1}^k \alpha_j u_{n_j} \right\|_T \leqslant \sum_{j=1}^k |\alpha_j|.$$

It immediately follows that the fundamental sequence $(e_j)_{j=1}^\infty$ of F is 2-equivalent to the ℓ_1 basis; in particular, T admits a spreading model isomorphic to ℓ_1. For the sake of completeness, we shall also presently sketch a proof of the fact that *every* spreading model of T is isomorphic to ℓ_1 (see, *e.g.*, [BeLa84, Proposition IV.2.F.2]). Further results on spreading models of T may be found in [OdSc98, §4].

Theorem 3.4.13 ([BeLa84]). *Every spreading model of T is isomorphic to ℓ_1.*

Proof. Let us preliminarily note that if $(u_j)_{j=1}^\infty$ is any seminormalised block basis sequence in T, then it generates a spreading model isomorphic to ℓ_1; the proof of this is the same as above. Moreover, it is obvious that if $(u_j)_{j=1}^\infty$ is any spreading sequence and $(v_j)_{j=1}^\infty$ is any sequence such that $\|u_j - v_j\| \to 0$, then $(v_j)_{j=1}^\infty$ is a spreading sequence that generates (isometrically) the same spreading model.

Assume now that $(x_j)_{j=1}^\infty$ is any (bounded) spreading sequence, that generates a spreading model F with fundamental sequence $(e_j)_{j=1}^\infty$; obviously, every subsequence of $(x_j)_{j=1}^\infty$ generates the same spreading model. Consequently, the reflexivity of T allows us to assume that $(x_j)_{j=1}^\infty$ admits a weak limit x. By the Bessaga–Pełczyński selection principle, we can also assume that there exists a seminormalised block sequence $(u_j)_{j=1}^\infty$ of the basis of T such that $\|(x_j - x) - u_j\| \to 0$. The comments at the beginning of the argument imply that $(x_j - x)_{j=1}^\infty$ is a spreading sequence with spreading model isomorphic to ℓ_1 and fundamental sequence $(f_j)_{j=1}^\infty$ equivalent to the ℓ_1 basis.

To conclude, we now deduce that also $(e_j)_{j=1}^\infty$ is equivalent to the ℓ_1 basis (*cf.* [BeLa84, Proposition I.5.5]). In fact, for scalars $\alpha_1, \dots, \alpha_k$ we have

$$\left\| \sum_{i=1}^k \alpha_i e_i - \sum_{i=1}^k \alpha_i e_{k+i} \right\| = \lim \left\| \sum_{i=1}^k \alpha_i x_{n_i} - \sum_{i=1}^k \alpha_i x_{n_{k+i}} \right\| =$$

$$\lim \left\| \sum_{i=1}^k \alpha_i (x_{n_i} - x) - \sum_{i=1}^k \alpha_i (x_{n_{k+i}} - x) \right\| = \left\| \sum_{i=1}^k \alpha_i f_i - \sum_{i=1}^k \alpha_i f_{k+i} \right\| \geqslant 2\delta \sum_{i=1}^k |\alpha_i|.$$

On the other hand, the invariance under spreading of $(e_j)_{j=1}^\infty$ yields

$$\left\| \sum_{i=1}^k \alpha_i e_i - \sum_{i=1}^k \alpha_i e_{k+i} \right\| \leqslant \left\| \sum_{i=1}^k \alpha_i e_i \right\| + \left\| \sum_{i=1}^k \alpha_i e_{k+i} \right\| = 2 \left\| \sum_{i=1}^k \alpha_i e_i \right\|,$$

whence the conclusion follows from the boundedness of $(e_j)_{j=1}^{\infty}$. ∎

When we conbine the above result with Corollary 3.4.7, we immediately infer the following corollary.

Corollary 3.4.14 ([HKR18, Corollary 5.7])**.** *For every renorming $|||\cdot|||$ of Tsirelson's space T we have $K^s((T, |||\cdot|||)) = 2$.*

In particular, we have an example of a reflexive Banach space every whose renorming has symmetric Kottman constant equal to 2; this is the desired counterexample to the question in [MaPa09].

The natural counterpart of the above results concerning ℓ_1 spreading models would be that every Banach space with a spreading model isomorphic to c_0 has symmetric Kottman constant equal to 2. However, we do not know whether this analogue result holds true, namely, the following problem is open to us.

Problem 3.4.15. Suppose that a Banach space X admits a spreading model isomorphic to c_0. Does it follow that $K^s(X) = 2$?

More in general, one go even further and ask whether the Kottman's constant of a Banach space is lower bounded by that of its spreading models, *i.e.*, the following:

Problem 3.4.16. Let X be a Banach space and let Z be a spreading model of X. Is it true that $K^s(X) \geqslant K^s(Z)$? Of course, the same question may be posed for $K(\cdot)$.

A positive answer to the above problems would, in particular, yield information on the symmetric Kottman constant of the original Tsirelson's space T^*, [Tsi74]; the proof that the space constructed by Tsirelson is indeed isometric to the dual of T defined above may be found in [CaSh89, p. 17]. In particular, we do not know if Corollary 3.4.14 can be extended also to T^*:

Problem 3.4.17. Let $\|\cdot\|$ be an equivalent norm on the original Tsirelson's space T^*. Is $K^s((T^*, \|\cdot\|)) = 2$?

On the other hand, concerning the original norm $\|\cdot\|_{T^*}$ of T^* we have the following simple observation, which was noted during a conversation with Pavlos Motakis.

Proposition 3.4.18. *The unit sphere of T^* contains a symmetrically 2-separated sequence.*

Proof. Let us denote by $(e_j)_{j=1}^{\infty}$ the canonical 1-unconditional basis of T and let $(e_j^*)_{j=1}^{\infty}$ be the associated biorthogonal functionals. Notice that $\|e_j^*\|_{T^*} = 1$, due to $(e_j)_{j=1}^{\infty}$ being 1-unconditional. Notice, moreover, that for distinct indices $i, j \in \mathbb{N}$ one has $\|e_i \pm e_j\|_T = 1$. In fact, obviously, $\|e_i \pm e_j\|_T \leqslant 2$, whence (T) gives

$$\|e_i \pm e_j\|_T = \max\left\{\|e_i \pm e_j\|_{c_0}, \frac{1}{2}\|e_i \pm e_j\|_T\right\} = \max\left\{1, \frac{1}{2}\|e_i \pm e_j\|_T\right\} = 1.$$

Consequently,

$$\|e_i^* \pm e_j^*\|_{T^*} \geqslant \langle e_i^* \pm e_j^*, e_i \pm e_j \rangle = 2$$

and $(e_j^*)_{j=1}^{\infty}$ is the desired symmetrically 2-separated sequence. ∎

As it was apparent, the results concerning Tsirelson's space T and its dual T^* presented in this section are actually easy consequences of known results; on the other hand Tsirelson's space is a very important object in Banach space theory, which is therefore natural to investigate. It is therefore quite surprising that these results were apparently not recorded before in the literature, which motivated the project proposed in [GMZ16, Problem 292].

3.4.3 Renormings and biorthogonal systems

In this short part, we observe that the problem of finding symmetrically $(1+\varepsilon)$-separated sequences of unit vectors is a much easier task if we allow renormings of the spaces under investigation.

This phenomenon was already observed by Kottman ([Kot75, Theorem 7]), who showed that every infinite-dimensional Banach space admits a renorming such that the new unit sphere contains a 2-separated sequence. An inspection of his argument shows that actually the resulting sequence is symmetrically 2-separated. We note in passing that van Dulst and Pach ([vDPa81]) proved a stronger renorming result (with a significantly more difficult proof) which also implies the same conclusion; however, we shall not require this stronger result here. Let us present here a simple proof of this result.

Proposition 3.4.19 ([HKR18, Proposition 5.1])**.** *Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space. Then X admits an equivalent norm $|||\cdot|||$ such that $S_{(X,|||\cdot|||)}$ contains a symmetrically 2-separated sequence.*

Proof. By the main result in [Day62], X contains an Auerbach system $\{x_i, f_i\}_{i=1}^{\infty}$. Set

$$\nu(x) := \sup_{i\neq k\in\mathbb{N}} \left(|\langle f_i, x\rangle| + |\langle f_k, x\rangle|\right)$$

and let us define

$$|||x||| = \max\{\|x\|, \nu(x)\} \quad (x \in X).$$

Then $|||\cdot|||$ is an equivalent norm on X as $\|x\| \leqslant |||x||| \leqslant 2\|x\|$ $(x \in X)$. From the biorthogonality we deduce that $\nu(x_i) = 1$, so $|||x_i||| = 1$ $(i \in \mathbb{N})$. Moreover,

$$|||x_i \pm x_j||| \geqslant \nu(x_i \pm x_j) = 2 \quad (i, j \in \mathbb{N}, i \neq j).$$

Hence, $(x_i)_{i=1}^{\infty}$ is a symmetrically 2-separated sequence in the unit sphere of $(X, |||\cdot|||)$. ■

A modification of the above renorming yields a new norm $|||\cdot|||$ that approximates $\|\cdot\|$ and such that the unit sphere of $|||\cdot|||$ contains a symmetrically $(1+\varepsilon)$-separated sequence. This shows how simple the symmetric version of the Elton–Odell theorem would be if we were allowed to consider arbitrarily small perturbations of the original norm.

Proposition 3.4.20 ([HKR18, Proposition 5.2])**.** *Let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space. Then, for every $\varepsilon > 0$, X admits an equivalent norm $|||\cdot|||$ such that $\|\cdot\| \leqslant |||\cdot||| \leqslant (1+\varepsilon)\|\cdot\|$ and $S_{(X,|||\cdot|||)}$ contains an infinite symmetrically $(1+\delta)$-separated subset, for some $\delta > 0$.*

In other words, for every infinite-dimensional Banach space, the set of all equivalent norms for which the symmetric version of the Elton–Odell theorem is true is dense in the set of all equivalent norms.

Proof. The very basic idea is that in the definition of ν we replace the sum of the two terms by an approximation of their maximum. Clearly, we may assume that $\varepsilon \in (0,1)$ (in which case we could actually choose $\delta = \varepsilon$); we may then select a norm Φ on $\mathbb{R}^2$ with the following properties:

(i) $\|\cdot\|_\infty \leqslant \Phi \leqslant (1+\varepsilon) \cdot \|\cdot\|_\infty$;

(ii) $\Phi((1,0)) = \Phi((0,1)) = 1$;

(iii) $\Phi((1,1)) = 1+\varepsilon$.

For example, one can choose

$$\Phi((\alpha,\beta)) := \max\left\{\|(\alpha,\beta)\|_\infty, (1+\varepsilon)\cdot\frac{|\alpha+\beta|}{2}\right\}.$$

We also fix an Auerbach system $\{x_i, f_i\}_{i\in\mathbb{N}}$ in X. Then we set

$$\nu(x) := \sup_{i\neq k\in\mathbb{N}} \Phi(|\langle f_i, x\rangle|, |\langle f_k, x\rangle|) \quad (x \in X)$$

and, exactly as above,

$$|||x||| = \max\{\|x\|, \nu(x)\} \quad (x \in X).$$

Note that

$$\nu(x) \leqslant (1+\varepsilon) \sup_{i\neq k\in\mathbb{N}} \max\{|\langle f_i, x\rangle|, |\langle f_k, x\rangle|\} \leqslant (1+\varepsilon)\|x\|,$$

which immediately implies $\|\cdot\| \leqslant |||\cdot||| \leqslant (1+\varepsilon)\,\|\cdot\|$.

Finally, from the biorthogonality we deduce that $\nu(x_i) = 1$ ($i \in \mathbb{N}$) and $\nu(x_i \pm x_j) = 1+\varepsilon$ ($i,j \in \mathbb{N}$, $i \neq j$). Hence, $|||x_i||| = 1$ and $|||x_i \pm x_j||| \geqslant 1+\varepsilon$ for $i \neq j$. Consequently, $(x_i)_{i=1}^\infty$ is a symmetrically $(1+\varepsilon)$-separated sequence in the unit sphere of $(X, |||\cdot|||)$. ■

We conclude this part with the following remark, in sharp contrast with Proposition 3.4.19. It belongs to obvious mathematical folklore, but it fits so well here, that we could not resist the temptation of including it.

Remark 3.4.21. Every separable Banach space admits a strictly convex renorming (even a locally uniformly rotund one; see, *e.g.*, [FHHMZ10, Theorem 8.1]), so in particular the unit sphere under such renorming contains no 2-separated sequences. Indeed, let X be any Banach space and let $x, y \in S_X$ be linearly independent and 2-separated vectors. Then $\frac{x-y}{2}$, the midpoint of the non-trivial segment joining x and $-y$, is a point on the unit sphere of X; hence it is a witness that X is not strictly convex.

The previous assertion is no longer true if the separability assumption is dropped. In fact, Partington ([Par80, Theorem 1]) showed that, when Γ is uncountable, every renorming of $\ell_\infty(\Gamma)$ contains an isometric copy of ℓ_∞. In particular, the unit sphere of every renorming of $\ell_\infty(\Gamma)$ contains a 2-separated sequence.

A consequence of Partington's result is obviously that, for Γ uncountable, $\ell_\infty(\Gamma)$ admits no strictly convex renorming. There actually exist examples of spaces with potentially smaller density character, for example ℓ_∞/c_0 ([Bou80]), that admit no strictly convex renorming. (One has to bear in mind that the space $\ell_\infty(\Gamma)$ has density character equal to $2^{|\Gamma|}$ as long as Γ is infinite, however it may happen that in some models of set theory $2^{\aleph_0} = 2^{|\Gamma|}$ for all uncountable sets of cardinality less than the continuum.) This is related to a question of A. Aviles ([GaKu11, Question 7.7]) of whether there exists, without extra set-theoretic assumptions, a Banach space with density character $\aleph_1$ which has no strictly convex renorming. We may then ask the following related question.

Problem 3.4.22. Does there exist in ZFC a Banach space X with density character $\aleph_1$ such that the unit sphere of every renorming of X contains a 2-separated sequence?

In conclusion to the chapter, we shall state one more question, concerning the extendability of the results of the present chapter to the context of complex Banach spaces. If X is a complex normed space, we may naturally adjust the definition of symmetric separation to encompass complex number of modulus 1. Thus, let us call a set $A \subset X$ ($\delta+$)-*toroidally separated* (respectively, δ-*toroidally separated*) when for all distinct $x, y \in A$ and complex numbers θ with $|\theta| = 1$ we have $\|x - \theta y\| > \delta$ (respectively, $\|x - \theta y\| \geqslant \delta$). A quick inspection of Delpech's proof of the main theorem in [Del10] reveals that the unit sphere of a complex asymptotically uniformly convex space contains a toroidally $(1+\varepsilon)$-separated sequence, for some $\varepsilon > 0$. Similarly, Theorem 3.4.3 has a natural counterpart in the complex case for toroidally separated sequences. It is then reasonable to ask whether the theorems of Kottman and Elton–Odell have such counterparts too.

Chapter 4

Uncountable separated sets

In the present chapter we shall investigate to what extent the results presented in Chapter 3 may be improved in the context of non-separable Banach spaces, where it is natural to investigate the existence of uncountable separated subsets of the unit ball. (Let us recall also here that finding such sets in the unit ball is equivalent to finding them in the unit sphere, *cf.* Section 3.1.1.) Such a field of research has been subject of extensive study in the last few years, most notably in the context of $C(K)$-spaces. In the first section of the chapter we shall review the results present in the literature and state our main contributions, comparing them with the status of the art. In the subsequent sections we shall discuss the proofs of our results.

4.1 Overview

Over the last years, a renewed interest and a rapid progress in delineating the structure of both qualitative and quantitative properties of well-separated subsets of the unit sphere of a Banach space have been observed. Perhaps the first spark was lit by Mercourakis and Vassiliadis [MeVa15] who have identified certain classes of compact Hausdorff spaces K for which the unit sphere of the Banach space $C(K)$ of all scalar-valued continuous functions on K contains an uncountable (1+)-separated set.

They also asked whether an 'uncountable' analogue of Kottman's theorem, or even the Elton–Odell theorem, is valid for every non-separable $C(K)$-space. One may thus extrapolate Mercourakis' and Vassiliadis' question to the class of all non-separable Banach spaces and ask for the following 'uncountable' version of Kottman's theorem.

Must the unit sphere of a non-separable Banach space contain an uncountable (1+)-separated subset?

A partial motivation to support the validity of the above conjecture may be obtained from Riesz' lemma: in fact, transfinite iteration of Riesz' argument immediately implies that, for every $\varepsilon > 0$, the unit ball of a non-separable Banach space X contains a $(1-\varepsilon)$-

separated subset of the maximal possible cardinality, namely $\operatorname{dens} X$. In the case that X is reflexive, the same argument produces a 1-separated subset of cardinality $\operatorname{dens} X$.

A slight modification of the above argument, based on the Mazur technique of norming functionals instead of the Riesz one, also serves to produce a 1-separated subset of the unit ball—in general, with a cardinality smaller than $\operatorname{dens} X$. Since we shall make extensive use of similar arguments in the chapter, let us record the very simple proof here.

Lemma 4.1.1. *The unit ball of every infinite-dimensional Banach space X contains a 1-separated subset of cardinality w^*-$\operatorname{dens} X^*$.*

Proof. Let $\lambda := w^*$-$\operatorname{dens} X^*$; we are going to find a 1-separated transfinite sequence $(x_\alpha)_{\alpha<\lambda} \subseteq S_X$. Assume, by transfinite induction, to have already found $(x_\alpha)_{\alpha<\beta}$ for some $\beta < \lambda$; select, for every $\alpha < \beta$, a norming functional $f_\alpha \in S_{X^*}$ with $\langle f_\alpha, x_\alpha \rangle = 1$. The subspace $Y_\beta = \overline{\operatorname{span}}^{w^*}\{f_\alpha\}_{\alpha<\beta} \subseteq X^*$, having w^*-density at most $|\beta| < \lambda$, is a proper subspace. Consequently, $\operatorname{span}\{f_\alpha\}_{\alpha<\beta}$ is not w^*-dense in X^*, hence it does not separate points; this yields a vector $x_\beta \in S_X$ with $\langle f_\alpha, x_\beta \rangle = 0$ for $\alpha < \beta$. Finally,

$$\|x_\alpha - x_\beta\| \geqslant \langle f_\alpha, x_\alpha - x_\beta \rangle = 1.$$

■

Unfortunately, the answer to the above question in its full generality remains unknown; in this section we shall discuss several attempts to solve it, leading to fairly general results. Quite surprisingly, it is also unknown, at least to the best of our knowledge, whether the unit ball of every Banach space contains a 1-separated subset whose cardinality equals the density character of the underlying Banach space. Concerning the latter problem, we are only aware of the above lemma and of the equally easy [CKV••, Proposition 21], giving a positive answer for the class of $C(K)$-spaces (*cf.* Proposition 4.1.6).

On the other hand, it was already observed by Elton and Odell themselves [ElOd81] that an 'uncountable' version of the Elton–Odell theorem is in general false, as witnessed by the space $c_0(\omega_1)$. The argument is a surprisingly simple application of the Δ-system lemma. We present it below, partially because of its simplicity, and more notably since it is the simplest instance of a combinatorial argument in the study of separated sets.

Proposition 4.1.2 ([ElOd81, p. 109])**.** *Every $(1+\varepsilon)$-separated subset of the unit ball of $c_0(\Gamma)$ is at most countable.*

Proof. Let us assume by contradiction that, for some $\varepsilon > 0$, the unit ball of $c_0(\Gamma)$ contains an uncountable $(1+\varepsilon)$-separated subset, say $\mathcal{F}$. Of course, for every $x \in \mathcal{F}$ the set $N(x) := \{\gamma \in \Gamma \colon |x(\gamma)| \geqslant \varepsilon/2\}$ is a finite set; according to the Δ-system lemma, Lemma 2.1.12, we may deduce the existence of a finite set $\Delta \subseteq \Gamma$ and of an uncountable subfamily $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $N(x) \cap N(y) = \Delta$ for distinct $x, y \in \mathcal{F}_0$.

Now, fix $x \neq y \in \mathcal{F}_0$ and $\gamma \in \Gamma \setminus \Delta$; since $\gamma \notin N(x) \cap N(y)$, we can assume for example that $|x(\gamma)| < \varepsilon/2$. We infer that $|y(\gamma) - x(\gamma)| \leqslant 1 + \varepsilon/2$, whence $\|(x-y)\restriction_{\Gamma\setminus\Delta}\| < 1 + \varepsilon$. Of course, this implies $\|x\restriction_\Delta - y\restriction_\Delta\| \geqslant 1 + \varepsilon$ for every $x \neq y \in \mathcal{F}_0$, which means that the family $\{x\restriction_\Delta\}_{x\in\mathcal{F}_0}$ is a $(1+\varepsilon)$-separated subset of the unit ball of $\ell_\infty(\Delta)$. However, this is blatantly impossible, since $\ell_\infty(\Delta)$ is finite-dimensional. ■

In the light of the results of Chapter 3 concerning separation in separable Banach spaces, the above-discussed problems also gained new natural 'symmetric' counterparts, where separation, that is the distance $\|x-y\|$, is replaced with the symmetric distance, *i.e.*, $\|x \pm y\|$.

The primary aim of this chapter is therefore to develop results concerning constructions of as large separated subsets of the unit sphere of a Banach space as possible that are moreover symmetrically separated sets, wherever possible. Let us mention that the clause about symmetry is not the main issue in our results to be presented, since they are sharper than the ones present in the literature, even when the symmetry assertion is removed from them.

Before entering into the discussion of the results contained in the chapter, we shall discuss what was already known concerning the main problem quoted above. Kania and Kochanek answered this question affirmatively ([KaKo16, Theorem B]) for the class of all non-separable $C(K)$-spaces as well as for all non-separable (quasi-)reflexive Banach spaces ([KaKo16, Theorem A(i)]). In the case that the Banach space X is additionally super-reflexive, they are even able to produce an uncountable $(1+\varepsilon)$-separated subset of B_X, for some $\varepsilon > 0$, [KaKo16, Theorem A(ii)].

In the context of $C(K)$-spaces, Koszmider proved that an 'uncountable' version of the Elton–Odell theorem for non-separable $C(K)$-spaces is independent of ZFC, [Kos18]. Interestingly, if the unit sphere of a $C(K)$-space contains a $(1+\varepsilon)$-separated subset for some $\varepsilon > 0$, then it also contains a 2-separated subset of the same cardinality, [MeVa15, Theorem 1]. Very recently, Cúth, Kurka, and Vejnar [CKV••] improved significantly [KaKo16, Theorem B] by identifying a very broad class of $C(K)$-spaces whose unit spheres contain (1+)-separated (or even 2-separated) subsets of the maximal possible cardinality, thereby giving many sufficient conditions for the solution to the 'tantalising problem' whether the unit ball of every $C(K)$ space contains a (1+)-separated subset of the maximal possible cardinality, [KaKo16, p. 40].

The results of this chapter will not proceed in the direction of a more complete understanding of separated subsets of the unit ball of a $C(K)$-space; however, since many results appeared in the recent years, we shall give a more detailed survey of this area in Section 4.1.1.

Our results will be of two different, and in a sense antipodal, natures. In the former part we shall exploit combinatorial methods and obtain results under minimal assumptions on the underlying Banach space. In the second part, on the other hand, we shall present a geometric approach and give several sufficient conditions, both of geometric and topological nature, for carrying over such approach.

Our first main result consists in understanding the fact that Auerbach systems can be profitably exploited to approach the problem of the existence of (1+)-separated subsets of the unit sphere; this allows us to deduce that, for large enough Banach spaces, the main question has a positive answer indeed. In the particular case where the underlying Banach space X is moreover weakly Lindelöf determined (WLD), we are able to obtain a better relation between the density character of the space X and the cardinality of the

(1+)-separated subset of the unit sphere.

It turns out–perhaps surprisingly–that under the present assumptions it is not possible to strengthen the results and obtain either (1+)-separated families of unit vectors, with cardinality larger than ω_1, or uncountable $(1+\varepsilon)$-separated families. We have already mentioned that the unit sphere of $c_0(\omega_1)$ does not contain uncountable $(1+\varepsilon)$-separated subsets. Koszmider noticed that every (1+)-separated subset in the unit sphere of $c_0(\Gamma)$ has cardinality at most continuum ([KaKo16, Proposition 4.13]). We optimise this result by actually decreasing the continuum to ω_1, which shows that even for rather well-behaved and very large spaces the size of (1+)-separated subsets of the sphere may be relatively small.

Let us then present our first main result formally.

Theorem 4.1.3 ([HKR••, Theorem A])**.**

(i) *Let X be a Banach space with w^*-dens $X^* > \exp_2 \mathfrak{c}$. Then both X and X^* contain uncountable symmetrically* (1+)*-separated families of unit vectors.*

(ii) *Let X be a WLD Banach space with* dens $X > \mathfrak{c}$*. Then the unit spheres of X and of X^* contain uncountable symmetrically* (1+)*-separated subsets.*

(iii) *Furthermore, if $X = c_0(\Gamma)$, then every* (1+)*-separated subset of $S_{c_0(\Gamma)}$ has cardinality at most ω_1.*

The proof of Theorem 4.1.3 will be presented in Section 4.2; in particular the first two parts are obtained as Corollary 4.2.2 and 4.2.3, while the last assertion is Theorem 4.2.8. In the same section we shall also present some results of isomorphic nature, concerning the existence of equivalent norms whose unit balls contain uncountable (1+)-separated subsets.

Clause (ii) in the above theorem generalises [KaKo16, Theorem A(iii)], where the result is only proved for X^*, under the same assumptions; part of the argument given there consists in showing that X contains an Auerbach system of cardinality dens X. However it was brought to the attention of the authors of [KaKo16] by Marek Cúth quite rightly that the proof contains a gap, namely that it is not clear why the system constructed in the proof of [KaKo16, Theorem 3.8] is biorthogonal—the existence of a gap in that claim is also a consequence of our Theorem 2.1.11. Therefore, we can also understand the second clause of Theorem 4.1.3 as a remedy to this problem, together with an improvement of the result, by exhibiting the sought (1+)-separated subset both in S_X, and in S_{X^*}.

In the second part of the chapter, we turn our attention to some strong structural constrains on the space, which allow construction of potentially larger separated subsets of the unit sphere. For example, we strengthen considerably [KaKo16, Theorem A(i)] by proving the existence of a symmetrically (1+)-separated set in the unit sphere of every (quasi-)reflexive space X that has the maximal possible cardinality, that is, the cardinality equal to dens X, the density of the underlying Banach space. Moreover, we also show the existence, for some $\varepsilon > 0$, of an uncountable symmetrically $(1+\varepsilon)$-separated subset of B_X.

When X is a super-reflexive space, we improve [KaKo16, Theorem A(ii)] by exhibiting a symmetrically $(1+\varepsilon)$-separated set in the unit sphere of X that also has the maximal possible cardinality—this answers a question raised by T. Kochanek and T. Kania, [KaKo16, Remark 3.7]. Let us then present our result formally.

Theorem 4.1.4 ([HKR••, Theorem B]). *Let X be an infinite-dimensional, (quasi-)reflexive Banach space. Then,*

(i) *S_X contains a symmetrically* $(1+)$*-separated subset with cardinality* $\operatorname{dens} X$*;*

(ii) *for every cardinal number $\kappa \leqslant \operatorname{dens} X$ with uncountable cofinality there exist $\varepsilon > 0$ and a symmetrically $(1+\varepsilon)$-separated subset of S_X of cardinality κ;*

(iii) *if X is super-reflexive, there exist $\varepsilon > 0$ and a symmetrically $(1+\varepsilon)$-separated subset of S_X of cardinality* $\operatorname{dens} X$*.*

Quite remarkably, Theorem 4.1.4 involves properties that are preserved by isomorphisms, not only isometries, of Banach spaces. Let us also immediately note that clause (ii) is optimal, as it was observed in [KaKo16, Remark 3.7]; we shall prove this in detail in Proposition 4.5.1.

Let us also recall that a Banach space X is *quasi-reflexive* whenever the canonical image of X in its bidual X^{**} has finite codimension. The proof of Theorem 4.1.4 will be presented in Sections 4.3, 4.4, 4.5; in particular, the first clause is Theorem 4.3.1, the second one is contained in Corollary 4.4.3 and the last part will be discussed in Section 4.5. In the same sections, we shall also present several further results whose proofs follow similar patterns; in particular, we shall also prove a generalisation of the first two parts of the theorem to the class of Banach spaces with the RNP.

In conclusion to this part, let us give a more precise comparison between Theorem 4.1.4 and the existing literature, in particular [KaKo16, Theorem A]. The first improvement contained in our result consists in finding a $(1+)$-separated set in the unit ball of every non-separable reflexive Banach space of cardinality $\operatorname{dens} X$, instead of merely an uncountable set. Moreover, Theorem 4.1.4(ii) is precisely [KaKo16, Theorem A(ii)], with reflexivity replacing super-reflexivity; notice, in particular, that [KaKo16, Theorem A] gives no information whatsoever on the existence of uncountable $(1+\varepsilon)$-separated sets in the unit ball of a reflexive Banach space. A third improvement is given by the technique of the proof: the proofs of (i) and (ii) in [KaKo16, Theorem A] are based on completely different ideas and are both quite non-trivial. On the other hand, we are able to deduce both (i) and (ii) of our result from the same method of proof, which is moreover based on a simpler idea. The last improvement, aside from the clause concerning symmetric separation, is the optimal result for super-reflexive spaces, that we already mentioned above.

4.1.1 $C(K)$-spaces

In this section we are going to review some results concerning separated sets and equilateral sets in Banach spaces of the form $C(K)$; we shall also give selected proofs to illustrate a few techniques. All the results have been obtained in one of the aforementioned papers [MeVa15, KaKo16, Kos18, CKV••].

In the context of separable $C(K)$ spaces, namely for metrisable K, it is well known and easy that $C(K)$ contains an isometric copy of c_0, whenever K is an infinite compact (see, *e.g.*, [AlKa06, Proposition 4.3.11]). As a consequence, the unit ball of $C(K)$ contains a 2-separated, hence equilateral, sequence. Therefore, we shall only consider non-metrisable (Hausdorff) compacta in this section, even with no explicit reference.

We shall start with a well-known topological description of the density character of a $C(K)$-space; we also sketch its proof, for the sake of completeness. Let us recall that the *weight* of a compact space K, denoted $w(K)$, is the minimal cardinality of a basis for the topology of K.

Fact 4.1.5.

$$\operatorname{dens} C(K) = w(K) = \min\{|\mathcal{F}|\colon \mathcal{F} \subseteq C(K) \text{ separates points on } K\}.$$

Proof. If the family $\mathcal{F}$ separates points on K, then the algebra it generates has density character at most $|\mathcal{F}|$ and it is dense in $C(K)$, in light of the Stone–Weierstrass theorem (*cf.* [Con90, Theorem V.8.1]). Conversely, if $\mathcal{F}$ is dense in $C(K)$, it is easily seen by Urysohn's lemma ([Rud87, Lemma 2.12]) that $\mathcal{F}$ separates points on K; this shows the equality between the first and third cardinal numbers.

Moreover, if $\mathcal{F}$ is dense in $C(K)$ and $\mathcal{O}$ is a countable basis for the topology of $\mathbb{R}$, then the collection of open sets $\mathcal{B} := \{f^{-1}(O)\colon f \in \mathcal{F}, O \in \mathcal{O}\}$ is a basis for the topology of K and $|\mathcal{B}| \leqslant |\mathcal{F}|$. In fact, let us select a non-empty open set V in K and $x \in V$ and choose $x \in V$; we may also find a function $\varphi \in C(K)$ such that $\varphi(x) = 1$ and $\varphi(y) = -1$ on $V^{\complement}$. If $O \in \mathcal{O}$ is a small neighbourhood of 1 in $\mathbb{R}$ and $f \in \mathcal{F}$ is sufficiently close to φ, then $f^{-1}(O) \in \mathcal{B}$ satisfies $x \in f^{-1}(O) \subseteq V$. It follows that $w(K) \leqslant \operatorname{dens} C(K)$.

For the converse inequality, assume that $\mathcal{B}$ is a basis for the topology of K; for every finite covering $\{U_1, \dots, U_n\} \subseteq \mathcal{B}$ of K, select a partition of the unity $\{\varphi_1, \dots, \varphi_n\}$ subordinated to the covering. It is then standard to verify that the collection comprising all the functions φ obtained in this way is linearly dense in $C(K)$ and its cardinality does not exceed $|\mathcal{B}|$. This implies $\operatorname{dens} C(K) \leqslant w(K)$, thereby concluding the proof. ■

As an immediate application, let us show that the unit ball of every $C(K)$-space contains a 1-separated subset of the maximal possible cardinality; the argument is the simplest instance of a maximality argument which turns out to be very profitable in this context.

Proposition 4.1.6 ([CKV••, Proposition 21])**.** *For every infinite compact K, the unit ball of $C(K)$ contains a 1-separated set of cardinality $w(K)$.*

Proof. Let $\mathcal{F}$ be a maximal (with respect to inclusion) 1-separated subset of $B_{C(K)}$; if $|\mathcal{F}| < w(K)$, then $\mathcal{F}$ does not separate points and therefore there exist distinct $x, y \in K$ such that $f(x) = f(y)$ whenever $f \in \mathcal{F}$. If $\varphi \in C(K)$ satisfies $\varphi(x) = 1$, $\varphi(y) = -1$, then $\mathcal{F} \cup \{\varphi\}$ is 1-separated, a contradiction. ■

We shall now turn our attention to the existence of $(1+\varepsilon)$ separated subsets of the unit ball and we first show their surprising correlation with equilateral sets; this equivalence passes through the notion of a *linked family*, which was also introduced in [MeVa15]. Let us recall that a subset A of a metric space (M,d) is λ-*equilateral* if $d(x,y)=\lambda$ for distinct $x,y\in A$.

Definition 4.1.7 ([MeVa15, Definition 2.1])**.** Let S be a set and $\mathcal{F}:=\{(A_\alpha,B_\alpha)\}_{\alpha\in I}$ be a collection of pairs of subsets of S. $\mathcal{F}$ is *linked* (or *intersecting*) if:

(i) $A_\alpha\cap B_\alpha=\emptyset$ for $\alpha\in I$;

(ii) for distinct $\alpha,\beta\in I$, either $A_\alpha\cap B_\beta\neq\emptyset$ or $A_\beta\cap B_\alpha\neq\emptyset$;

Note that (ii) implies $A_\alpha\cup B_\alpha\neq\emptyset$; moreover, the conditions easily imply $A_\alpha\neq A_\beta$ and $B_\alpha\neq B_\beta$ whenever $\alpha\neq\beta$. In turn, it follows that at most one set A_α (and at most one B_α) is empty; as an example where this happens, the family $\{(\emptyset,S),(S,\emptyset)\}$ is clearly linked.

The following trivial fact illustrates the relation with equilateral sets in $C(K)$; we shall denote $B^+_{C(K)}$ the positive part of the unit ball, *i.e.*, $B^+_{C(K)}=\{f\in B_{C(K)}\colon f\geqslant 0\}$.

Fact 4.1.8. *Let $S\subseteq B^+_{C(K)}$ and consider the sets $A_f:=\{f=0\}$ and $B_f:=\{f=1\}$, for $f\in S$. Then the family $\{(A_f,B_f)\}_{f\in S}$ is linked if and only if S is 1-equilateral.*

Proof. For distinct $f,g\in S$, the condition $A_f\cap B_g\neq\emptyset$ or $A_g\cap B_f\neq\emptyset$ means that there is a point $x\in K$ such that either $f(x)=0$ and $g(x)=1$ or $f(x)=1$ and $g(x)=0$. Since $0\leqslant f(x),g(x)\leqslant 1$, this condition is equivalent to $\|f-g\|=1$. ■

We are now ready for the aforementioned equivalence between equilateral sets and $(1+\varepsilon)$-separated sets; we shall also offer its proof.

Theorem 4.1.9 ([MeVa15, Theorem 2.6])**.** *For a compact topological space K and a cardinal number λ, the following are equivalent:*

(i) *$B^+_{C(K)}$ contains a 1-equilateral set of cardinality λ;*

(ii) *$B_{C(K)}$ contains a 2-equilateral set of cardinality λ;*

(iii) *$B_{C(K)}$ contains a $(1+\varepsilon)$-separated set of cardinality λ;*

(iv) *there exists a linked family of closed subsets of K, with size λ.*

Proof. The implication (i) $\Longrightarrow$ (iv) is part of the previous fact, since the sets A_f and B_f are closed.

Let us now assume (iv) and select a linked family $\{(A_\alpha,B_\alpha)\}_{\alpha\in I}$ consisting of closed sets. For every $\alpha\in I$, an appeal to Urysohn's lemma allows us to select a continuous function $f_\alpha\in B^+_{C(K)}$ such that $f_\alpha=1$ on B_α and $f_\alpha=0$ on A_α (in the case $A_\alpha=\emptyset$, we may just consider $f_\alpha=1$ and analogously for $B_\alpha=\emptyset$). The family $\{(A_\alpha,B_\alpha)\}_{\alpha\in I}$ being linked, for every distinct $\alpha,\beta\in I$ we conclude the existence of $x\in K$ such that $x\in(A_\alpha\cap B_\beta)\cup(A_\beta\cap B_\alpha)$. In either cases, we conclude $|f_\alpha(x)-f_\beta(x)|=1$, whence the collection $\{f_\alpha\}_{\alpha\in I}$ is 1-equilateral. This proves (iv) $\Longrightarrow$ (i).

If we proceed analogously with functions g_α such that $g_\alpha = -1$ on A_α, we also see that (iv) $\Longrightarrow$ (ii). (ii) $\Longrightarrow$ (iii) being trivial, we only need to show that (iii) $\Longrightarrow$ (iv).

Assume that the set $S \subseteq B_{C(K)}$ is $(1+\varepsilon)$-separated and consider the mutually disjoint and closed sets $A_f := \{f \leqslant -\varepsilon\}$ and $B_f := \{f \geqslant \varepsilon\}$. Since $\|f - g\| \geqslant 1 + \varepsilon$ for distinct $f, g \in S$, we may find $x \in K$ such that $|f(x) - g(x)| \geqslant 1 + \varepsilon$ and we may assume without loss of generality that $f(x) > g(x)$. Consequently, we have $1 + \varepsilon \leqslant f(x) - g(x) \leqslant 1 - g(x)$, whence $g(x) \leqslant -\varepsilon$. Analogously, $f(x) \geqslant \varepsilon$, which implies that $\{(A_f, B_f)\}_{f \in S}$ is a linked family of closed sets and concludes the proof. ■

We may now proceed to give several sufficient conditions on K for the Banach space $C(K)$ to contain an uncountable 2-equilateral subset in its unit ball. Those conditions have been obtained in the papers [MeVa15, KaKo16] independently. We start with a few topological notions.

Definition 4.1.10. A topological space T is *Lindelöf* if every open covering of T admits a countable subcovering. T is *hereditarily Lindelöf* if every its subspace is Lindelöf. Finally, a topological space T is *perfectly normal* if every its closed subset is a G_δ set.

Note that if every open subset of a topological space T is Lindelöf, then T is hereditarily Lindelöf. In case of compact topological spaces, we have the following standard characterisation of hereditary Lindelöf spaces.

Fact 4.1.11. *For a compact topological space K, the following are equivalent:*

(i) *K is perfectly normal;*

(ii) *for every open set $U \subseteq K$ there exists (a non-negative) $f \in C(K)$ with $U = \{f > 0\}$;*

(iii) *K is hereditarily Lindelöf;*

(iv) *there exists no uncountable right separated family, that is, a family $(x_\alpha)_{\alpha<\omega_1} \subseteq K$ such that $x_\alpha \notin \overline{\{x_\beta\}_{\alpha<\beta<\omega_1}}$ $(\alpha < \omega_1)$.*

Proof. Assume that K is perfectly normal and let U be an open subset of K; by assumption we may write $U = \cup F_n$, where each F_n is a closed subset of K. Let $\varphi_n \colon K \to [0,1]$ be a continuous function such that $\varphi_n = 1$ on F_n and $\varphi_n = 0$ on $U^{\complement}$. The function $\sum 2^{-n}\varphi_n$ then witnesses that (i) $\Longrightarrow$ (ii).

For the converse implication, we just have to note that $\{f > 0\} = \cup\{f \geqslant 1/n\}$ is F_σ.

Assume now that K is hereditarily Lindelöf and let U be an open subset of K. By the regularity of K ([Rud87, Theorem 2.7]), for every $x \in U$ there exists an open set U_x with $x \in U_x$ and $\overline{U_x} \subseteq U$. U being Lindelöf, we may find a countable subcover for the covering $\{U_x\}_{x\in U}$; in other words, we have a countable collection of open sets $(U_n)_{n=1}^\infty$ such that $U = \cup U_n$ and $\overline{U_n} \subseteq U$. Consequently, $U = \cup \overline{U_n}$ is F_σ and (i) follows.

Conversely, for (i) $\Longrightarrow$ (iii), let A be any subset of K and $\{O_\alpha\}_{\alpha\in I}$ be a collection of open subsets of K that covers A. By assumption, $\cup O_\alpha$ is F_σ, whence we may find closed subsets $\{F_n\}_{n=1}^\infty$ of K such that $\cup O_\alpha = \cup F_n$. By compactness of F_n, we may find a countable subset I_0 of I such that $\cup F_n = \cup_{\alpha\in I_0} O_\alpha$. Therefore, $\{O_\alpha\}_{\alpha\in I_0}$ is the desired countable subcover of A.

Assume, next, that K is not hereditarily Lindelöf and choose a subspace Z of K and an open covering $\mathcal{O}$ of Z with no countable subcover. By transfinite induction, it is immediate to find open sets $(O_\alpha)_{\alpha<\omega_1} \subseteq \mathcal{O}$ and points $(z_\alpha)_{\alpha<\omega_1} \subseteq Z$ such that $z_\alpha \in O_\alpha$ and $z_\beta \notin O_\alpha$ whenever $\alpha < \beta < \omega_1$. We infer that $(z_\beta)_{\alpha<\beta<\omega_1} \cap O_\alpha = \emptyset$, whence $z_\alpha \notin \overline{\{z_\beta\}_{\alpha<\beta<\omega_1}}$.

Finally, if $(z_\alpha)_{\alpha<\omega_1} \subseteq K$ is an uncountable right-separated family, by the very definition $(z_\beta)_{\alpha<\beta<\omega_1}$ is a closed subset of $Z := \{z_\alpha\}_{\alpha<\omega_1}$; therefore, $U_\alpha := \{z_\beta\}_{\beta\leqslant\alpha}$ is an open subset of Z. Evidently, $\{U_\alpha\}_{\alpha<\omega_1}$ is an open covering of Z that admits no countable subcover. ■

Remark 4.1.12. Let us note that the equivalences (i) $\iff$ (ii) and (iii) $\iff$ (iv) are actually true for every topological space X, with no compactness assumption. The above fact should be understood as the claim that hereditary Lindelöf and perfectly normal are equivalent notions for compact topological spaces, the other clauses being reformulations of the definitions. A similar argument also proves that a topological space T is hereditarily separable if and only if it admits no uncountable *left-separated family*, *i.e.*, a family $(x_\alpha)_{\alpha<\omega_1} \subseteq T$ such that $x_\beta \notin \overline{\{x_\alpha\}_{\alpha<\beta}}$ $(\beta < \omega_1)$.

Theorem 4.1.13 ([MeVa15], [KaKo16])**.** *The unit ball of the Banach space $C(K)$ contains an uncountable 2-equilateral set, provided the compact K satisfies at least one of the following conditions:*

1. *K is not hereditarily separable;*
2. *K is not hereditarily Lindelöf;*
3. *K contains a non-metrisable, totally disconnected subspace;*
4. *$|K| > \mathfrak{c}$;*
5. *K carries a measure μ such that $L_1(\mu)$ is non-separable;*
6. *K is a Rosenthal compact.*

We shall not give a complete proof of this result, we restrict ourselves to indicate the simplest arguments. Assume that K is not hereditarily separable, and let $(x_\alpha)_{\alpha<\omega_1} \subseteq K$ be an uncountable left-separated family. In light of Urysohn's lemma, for every $\beta < \omega_1$, we may find a continuous function $\varphi_\beta \colon K \to [-1,1]$ such that $\varphi_\beta(x_\beta) = 1$ and $\varphi_\beta(x_\alpha) = -1$, whenever $\alpha < \omega_1$. The family $\{\varphi_\beta\}_{\beta<\omega_1}$ is then 2-equilateral; essentially the same argument also proves 2. These two parts are proved both in [MeVa15, Theorem 2.9] and [KaKo16, Proposition 4.3], with the above argument.

3. is also immediate: assume that $L \subseteq K$ is non-metrisable and totally disconnected and let $\{U_\alpha\}_{\alpha\in I}$ be a basis for the topology of L consisting of clopen sets. Then the family of continuous functions $\{\chi_{U_\alpha} - \chi_{L\setminus U_\alpha}\}_{\alpha\in I}$ in an uncountable 2-equilateral set in the unit ball of $C(L)$ (note that $w(L) \leqslant |I|$ is uncountable, as L is non-metrisable). According to the Tietze–Urysohn extension theorem ([Eng89, Theorem 2.1.8]), we obtain the desired 2-equilateral set in the unit ball of $C(K)$.

4. follows from 2. and the classical Arkhangel'skiĭ's inequality [Ark69] (see also [Rud75, p. 7]), according to which every compact first countable topological space has cardinality at most continuum. In fact, one can show that every hereditarily Lindelöf compact is first countable; for further details, we refer to [MeVa15, Theorem 2.9(iv)], or the proof of [CKV••, Theorem 7(ii)].

The last clauses are [MeVa15, Theorem 2.9(v)] and [KaKo16, Proposition 4.7] respectively.

Remark 4.1.14. Very recently, Cúth, Kurka and Vejnar [CKV••] have substantially improved the above results in the direction of finding sufficient conditions on K for the existence of a 2-equilateral subset of $B_{C(K)}$ of the maximal possible cardinality $w(K)$. We shall not enter the discussion of all the results here, but we shall restrict ourselves to give the flavor of some their results.

By generalising one above argument, it is easy to see that if there exists a subset A of K with $\operatorname{dens} A \geqslant w(K)$, then $B_{C(K)}$ contains a 2-equilateral subset of cardinality $w(K)$. This is shown to be the case [CKV••, Corollary 8] if either K is a Valdivia compact or $w(K)$ is a strong limit cardinal.

Further sufficient conditions are shown to be that K is homogeneous, continuous image of $\{0,1\}^\kappa$ for some cardinal κ (*i.e.*, dyadic), or homeomorphic to $L\times L$, to a compact convex subset in a locally convex space, or to a compact line ([CKV••, §3]). Some of these results follow from the interesting result ([CKV••, Theorem 9]) that the unit ball of $C(K\times\{0,1\})$ always contains a 2-equilateral subset of cardinality $w(K)$.

The extensive amount of sufficient conditions for the existence of an uncountable 2-equilateral set in the unit ball of $C(K)$ may lead to the conjecture that such sets ought to exist in every non-separable $C(K)$-space. However, as we already mentioned, Koszmider [Kos18] proved that the answer to the above problem is undecidable in ZFC. In particular, one his result consists in a positive answer to the conjecture, subject to the assumption of Martin's Axiom and the negation of the Continuum Hypothesis.

We shall present this surprisingly simple argument below; the main observation it is built on is that linked families of closed pairs are antichains in a natural partially ordered set, which was already considered before, *e.g.*, in [Kos99]. Therefore, the existence of no uncountable linked family of closed pairs implies that such partially ordered set has the countable chain condition, which allows for the use of Martin's Axiom; for a review of Martin's Axiom, we refer to Section 2.4.1 and the references therein.

Theorem 4.1.15 ([Kos18, Theorem 5.2], MA_{ω_1})**.** *The unit ball of every non-separable $C(K)$-space contains an uncountable 2-equilateral subset.*

Proof. Fix a non-metrisable compact Hausdorff topological space K; we first note that we may assume $w(K)=\omega_1$. In fact, if X is a closed sub-$*$-algebra of $C(K)$, with $\operatorname{dens} X=\omega_1$, then X is a $C(L)$-space, in light of the Gelfand–Naimark theorem (*cf.* [Rud91, Theorem 11.18]). Alternatively, L can be identified with a quotient of K, see [Con90, Exercise 4, p. 148]. According to 2. of Theorem 4.1.13, we may also assume that K is perfectly normal.

Let us now fix a basis $\mathcal{B}$ for the topology of K such that $|\mathcal{B}| = \omega_1$; up to enlarging it (but keeping its cardinality fixed), we can assume that $\mathcal{B}$ is closed under finite unions, *i.e.*, $U \cup V \in \mathcal{B}$ whenever $U, V \in \mathcal{B}$. Let us now consider the poset

$$\mathbb{P} := \left\{p = (U_p, V_p) \colon U_p, V_p \in \mathcal{B}, \overline{U_p} \cap \overline{V_p} = \emptyset\right\}$$

endowed with the partial order $p \leqslant q$ iff $U_p \supseteq U_q$ and $V_p \supseteq V_q$. Since $\mathcal{B}$ is closed under finite unions, elements $p, q \in \mathbb{P}$ are compatible iff $(U_p \cup U_q, V_p \cup V_q) \in \mathbb{P}$, *i.e.*, $\overline{U_p \cup U_q} \cap \overline{V_p \cup V_q} = \emptyset$, that is, $\overline{U_p} \cap \overline{V_q} = \emptyset = \overline{U_q} \cap \overline{V_p}$. Consequently, $p, q \in \mathbb{P}$ are incompatible if and only iff $\{(\overline{U_p}, \overline{V_p}), (\overline{U_q}, \overline{V_q})\}$ is a linked family of closed pairs; we conclude from Theorem 4.1.9 that the existence of an uncountable 2-equilateral set in $B_{C(K)}$ is equivalent to the existence of an uncountable antichain in $\mathbb{P}$. Therefore, we only need to show that $\mathbb{P}$ is not ccc.

To conclude, assume by contradiction that $\mathbb{P}$ is ccc; MA_{ω_1} and Theorem 2.4.29 yield that $\mathbb{P}$ is also σ-centred. We may therefore write $\mathbb{P} = \cup \mathbb{P}_n$, where each $\mathbb{P}_n$ is a centred set; in particular, $U_p \cap V_q = \emptyset = U_q \cap V_p$ whenever $p, q \in \mathbb{P}_n$. We deduce that (for every $n \in \mathbb{N}$) the sets $U_n := \cup_{p \in \mathbb{P}_n} U_p$ and $V_n := \cup_{p \in \mathbb{P}_n} V_p$ are disjoint open sets and the complete regularity of K implies the existence of continuous functions $\varphi_n \colon K \to [-1, 1]$ such that $U_n = \{\varphi_n > 0\}$ and $V_n = \{\varphi_n < 0\}$. Finally, for distinct $x, y \in K$, there exists $(U, V) \in \mathbb{P}$ such that $x \in U$ and $y \in V$; if $n \in \mathbb{N}$ is such that $(U, V) \in \mathbb{P}_n$, we infer that $\varphi_n(x) > 0 > \varphi_n(y)$. Therefore, the sequence $(\varphi_n)_{n=1}^{\infty}$ separates points on K, a contradiction. ■

On the other hand, the main result in [Kos18] consists in showing the relative consistency with ZFC of the existence of a non-separable $C(K)$ space whose unit ball contains no uncountable equilateral set. We shall not discuss the techniques involved in the proof, and we shall refer to the very clearly explained argument in [Kos18]; in particular, the forcing argument is postponed to the last part of the proof §4 thereby allowing readers not familiar with forcing to grasp most of the proof.

Let us now highlight some open problems which arise from Koszmider's work, most of which are raised by the author himself. Firstly, the main result of the paper provides the first example of a non-separable Banach space which contains no uncountable equilateral set. It is not known if, outside the class of $C(K)$ spaces, it is possible to find an absolute such example; this important problem has also been recorded in [GMZ16, Problem 293]. In this connection, let us also mention Terenzi's wonderful example of a renorming of ℓ_1 without infinite equilateral sets, [Ter89]; we shall take a small detour and prove such a gem below, in Theorem 4.1.17.

It follows from results of Todorcevic [Tod06] and Mercourakis and Vassiliadis [MeVa14] that consistently every non-separable Banach space admits a renorming that contains an uncountable equilateral set; also see the discussion preceding Corollary 4.2.5. It is an open problem whether there consistently exist a non-separable Banach space whose no renorming has uncountable equilateral sets; in particular, it is not known if such phenomenon is possible for a $C(K)$-space.

Let us close with one more problem, in the class of $C(K)$-spaces formulated in [MeVa15, Remark 2.10]; its negative answer would be a result in the same line of Theorem 4.1.9.

Problem 4.1.16. Does there exist a consistent example of a $C(K)$-space that contains uncountable equilateral subsets, but whose unit ball contains no uncountable 2-equilateral set?

We shall next digress a bit and prove Terenzi's result.

Theorem 4.1.17 ([Ter89]). *There exists a renorming of ℓ_1 that contains no infinite equilateral set.*

Proof. Let $|||\cdot|||$ be a renorming of ℓ_1 such that $(\ell_1, |||\cdot|||)$ is strictly convex and

$$(*) \qquad \limsup_{n\to\infty} |||x_n + x||| = \limsup_{n\to\infty} |||x_n||| + |||x|||$$

whenever $x \in \ell_1$ and $(x_n)_{n=1}^{\infty} \subseteq \ell_1$ converges to 0 coordinatewise. We claim that $(\ell_1, |||\cdot|||)$ contains no infinite equilateral set. In fact, if by contradiction $(x_n)_{n=1}^{\infty}$ is a 1-equilateral sequence, up to a subsequence we may assume that $(x_n)_{n=1}^{\infty}$ admits a coordinatewise limit; up to subtracting this limit, we can additionally assume $(x_n)_{n=1}^{\infty}$ to be coordinatewise null. Therefore, for every $n \in \mathbb{N}$ we have

$$1 = \limsup_{k\to\infty} |||x_k - x_n||| = \limsup_{k\to\infty} |||x_k||| + |||x_n|||;$$

this implies that $(|||x_n|||)_{n=1}^{\infty}$ is a constant sequence and in turn yields $|||x_n||| = 1/2$ ($n \in \mathbb{N}$). The strict convexity of $|||\cdot|||$ and $|||x_n - x_k||| = 1$ for distinct $n, k \in \mathbb{N}$ then imply $x_n = -x_k$ ($n \neq k$), an obvious contradiction.

In order to conclude the argument, we thus only need to give one example of a norm with the above two properties. For $x = (x(n))_{n=1}^{\infty} \in \ell_1$, consider

$$|||x||| := \sum_{n=1}^{\infty} |x(n)| + \left(\sum_{n=1}^{\infty} 2^{-n} \cdot |x(n)|^2\right)^{1/2};$$

the strict convexity is an obvious consequence of Day's lemma, [Day55, (3) p. 518]. Finally, the verification of $(*)$ is an immediate application of a sliding hump argument, combined with the fact—due to the 2^{-n} factor—that the second summand in $|||x_n|||$ tends to 0, whenever $(x_n)_{n=1}^{\infty}$ is a bounded coordinatewise null sequence. ■

In the last part to this section, we shall focus our attention to the existence of an uncountable (1+)-separated subset of the unit ball of $C(K)$, the results discussed so far giving a strong evidence for a positive answer to the conjecture (formulated in [MeVa15, Question 2.7]) whether the unit ball of every non-separable $C(K)$-space contains an uncountable (1+)-separated subset.

The conjecture has indeed been given a positive answer in [KaKo16, Theorem B]; in combination with clause 2. of Theorem 4.1.13, it is sufficient to consider the case where K is perfectly normal. The simpler proof we shall present below–one more instance of a maximality argument–is based on the elaboration given in the proof of [CKV••, Theorem 6].

Theorem 4.1.18 ([KaKo16, Theorem 4.11])**.** *If K is a perfectly normal compact, the unit ball of $C(K)$ contains a* $(1+)$*-separated subset of cardinality* $w(K)$*.*

Proof. Given a function $f \in C(K)$, we shall say that a real value t is a *local maximum* for f if there exists a point $x \in K$ such that $f(x) = t$ and x is a point of local maximum for f. We shall use the following version of Urysohn's lemma for perfectly normal compacta.

Claim 4.1.19 (*cf.* [KaKo16, Lemma 4.10]). If x, y are distinct points of K, there exists a continuous function $f\colon K \to [-1, 1]$ such that $f(x) = 1$, $f = -1$ in a neighbourhood of y and 0 is not a local maximum for f.

Proof of the claim. According to Urysohn's lemma and the regularity of K, we may find a continuous $\varphi\colon K \to [-1, 1]$ such that $\varphi(x) = 1$ and $\varphi = -1$ in some open neighbourhood of y. If $x \in K$ is any point of local maximum for φ with $\varphi(x) = 0$, select an open neighbourhood U_x of x such that $-1/2 \leqslant \varphi \leqslant 0$ on U_x. Denote by U the union of all such sets U_x and find a continuous function $\psi\colon K \to [-1/2, 0]$ which is strictly negative on U and vanishes elsewhere.

We shall prove that $f := \varphi + \psi$ is the desired function; the fact that f keeps the same properties as φ is clear, since $-1/2 \leqslant \varphi \leqslant 0$ on U and ψ vanishes elsewhere. Assume now that $x \in K$ satisfies $f(x) = 0$; note that $f < 0$ on U, so $x \notin U$. As a consequence of this, we also obtain $\varphi(x) = 0$. In light of the fact that x is not a point of local maximum for φ, for every neighbourhood V of x there exists a point $t \in V$ with $\varphi(t) > 0$. Necessarily, $t \notin U$, which implies $\psi(t) = 0$, whence $f(t) > 0$; this shows that x is not a point of local maximum for f and proves the claim. ■

Let us now take a maximal $(1+)$-separated subset $\mathcal{F}$ of the unit ball of $C(K)$ such that 0 is a local maximum of no $f \in \mathcal{F}$. We shall prove that $|\mathcal{F}| = w(K)$; if this were not the case, then $\mathcal{F}$ would not separate points, hence there would exist distinct $x, y \in K$ with $f(x) = f(y)$ whenever $f \in \mathcal{F}$. Letting φ be any function as in the above claim, the set $\mathcal{F} \cup \{\varphi\}$ would also be $(1+)$-separated, a contradiction.

Finally, let us check that $\mathcal{F} \cup \{\varphi\}$ is $(1+)$-separated. If $f \in \mathcal{F}$ satisfies $f(x) = f(y) \neq 0$, then clearly $\|f - \varphi\| > 1$. Otherwise, let U be an open neighbourhood of y where $\varphi = -1$; since y is not a point of local maximum for f, there exists $t \in U$ with $f(t) > 0$. Consequently, $\|f - \varphi\| \geqslant |f(t) - \varphi(t)| = f(t) + 1 > 1$, and we are done. ■

As we hinted at above, this argument is taken from [CKV••, Theorem 6], where a more general result is also proved; this more general statement reads as follows.

Theorem 4.1.20 ([CKV••, Theorem 6])**.** *For every compact space K the unit ball of $C(K)$ contains either a* $(1+)$*-separated subset of cardinality* $w(K)$ *or a* 2*-equilateral set of cardinality* $\mathfrak{c}$*.*

In the particular case where $w(K) \leqslant \mathfrak{c}$, we conclude that the unit ball of $C(K)$ contains a $(1+)$-separated subset of the maximal possible cardinality. This is, indeed, a generalisation of [KaKo16, Theorem 4.11], since perfectly normal compacta are fist countable, hence they have cardinality at most continuum, as we already mentioned above.

Let us also mention that it is still not known whether the above results can be extended to cover all non-metrisable compacta, *i.e.*, whether the unit ball of every $C(K)$-space contains a $(1+)$-separated subset of cardinality $w(K)$. Of course, several further sufficient conditions are the results in [CKV••], some of which we described in Remark 4.1.14.

Sketch of the proof. Consider the following property (which was inspired by the argument in [KaKo16]):
(KK): For distinct $x, y \in K$, there exists a continuous function $f\colon K \to [-1,1]$ such that $f(x) = 1$, $f = -1$ in a neighbourhood of y and 0 is not a local maximum for f.

In the case that (KK) holds, then we argue exactly as in the proof of the previous theorem—note that, in this notation, *Claim* 4.1.19 just means that every perfectly normal compact satisfies (KK).

In the other case, select two distinct points $x, y \in K$ that witness the failure of (KK) and let $f \in B_{C(K)}$ be any function such that $f(x) = 1$ and $f = -1$ in a neighbourhood of y. By the negation of (KK), 0 is a local maximum for f; moreover, every $t \in (-1,1)$ is easily seen to be a local maximum for f—just consider $\phi \circ f$, where $\phi\colon [-1,1] \to [-1,1]$ is a strictly increasing function such that $\phi(-1) = -1$, $\phi(1) = 1$ and $\phi(t) = 0$.

Consequently, for every $t \in (-1,1)$ there exists a point of local maximum x_t for f, with $f(x_t) = t$. It is easy to deduce that the family $(x_t)_{-1<t<1}$ is right-separated, whence the existence of a 2-equilateral subset of cardinality continuum follows. ■

4.2 Combinatorial analysis

In this section we shall start our investigation of uncountable separated families of unit vectors by means of a combinatorial approach, in particular we wish to obtain results that depend only on the density character of a given Banach space and, possibly, on no its specific geometric property. In its first part, we shall give quite general sufficient conditions that depend on the existence of suitable coordinate systems, while in the second part we analise the spaces of the form $c_0(\Gamma)$. In all the section, we shall frequently use some of the combinatorial results that were recalled in Section 2.1.2.

4.2.1 The rôle of Auerbach systems

The main goal of the section is to see how to use Auerbach systems or, more generally, biorthogonal systems for the construction of separated families of unit vectors. Therefore, in the first part of the section we shall also make use of the general results about the existence of Auerbach systems that were proved in Chapter 2.

Having those general results at our disposal, we first give a basic proposition showing how to obtain separated families of unit vectors starting from a long Auerbach system.

Proposition 4.2.1 ([HKR••, Proposition 3.4])**.** *Suppose that the Banach space X contains an Auerbach system of cardinality $\mathfrak{c}^+$. Then both S_X and S_{X^*} contain an uncountable symmetrically $(1+)$-separated subset.*

Proof. Clearly, if $\{e_\gamma;\varphi_\gamma\}_{\gamma\in\Gamma}$ is an Auerbach system in X, then we can consider $\{\varphi_\gamma;e_\gamma\}_{\gamma\in\Gamma}$ as an Auerbach system in X^*; consequently, it suffices to prove the result for X. Let therefore $\{e_\alpha;\varphi_\alpha\}_{\alpha<\mathfrak{c}^+}$ be an Auerbach system in X and consider the following colouring $c\colon [\mathfrak{c}^+]^2 \to \{(>),(\leqslant)\}$:

$$\{\alpha,\beta\} \mapsto \begin{cases} (>) & \|e_\alpha - e_\beta\| > 1 \\ (\leqslant) & \|e_\alpha - e_\beta\| \leqslant 1. \end{cases}$$

The Erdős–Rado theorem assures us of the validity of $\mathfrak{c}^+ \to (\omega_1)^2_2$, whence the colouring c admits a monochromatic set Λ with cardinality ω_1. In other words, we have found a family $\{e_\alpha\}_{\alpha\in\Lambda}$, where $|\Lambda| = \omega_1$, such that either $\|e_\alpha - e_\beta\| > 1$ for every $\alpha,\beta\in\Lambda$ with $\alpha\neq\beta$ or $\|e_\alpha - e_\beta\| \leqslant 1$ for every such α,β. In the first case, the family $\{e_\alpha\}_{\alpha\in\Lambda}$ is obviously $(1+)$-separated, and we are done; we thus only have to consider the case of the colour $(\leqslant)$.

In such a case, let us, for notational simplicity, well order the set Λ, thereby obtaining a family $\{e_\alpha\}_{\alpha<\omega_1}$ of unit vectors (with the corresponding norm-one biorthogonal functionals $\{\varphi_\alpha\}_{\alpha<\omega_1}$) with the property that $\|e_\alpha - e_\beta\| \leqslant 1$ whenever $\alpha,\beta<\omega_1$. We may now modify the vectors e_α as follows: for $1\leqslant\alpha<\omega_1$, we set

$$\tilde{e}_\alpha := e_\alpha - \sum_{\gamma<\alpha} c_\alpha^\gamma e_\gamma,$$

where $\{c_\alpha^\gamma\}_{\gamma<\alpha}$ is a collection of positive real numbers such that, for $1\leqslant\alpha<\omega_1$, $c_\alpha^0 \geqslant 3/4$ and $\sum_{\gamma<\alpha} c_\alpha^\gamma = 1$ (such collections do exist since the set $\{\gamma\colon \gamma<\alpha\}$ is countable).

Let us first observe that this modification does not change the norm of the vectors: indeed, on the one hand

$$\|\tilde{e}_\alpha\| = \left\|\sum_{\gamma<\alpha} c_\alpha^\gamma e_\alpha - \sum_{\gamma<\alpha} c_\alpha^\gamma e_\gamma\right\| \leqslant \sum_{\gamma<\alpha} c_\alpha^\gamma \|e_\alpha - e_\gamma\| \leqslant 1;$$

on the other hand, $\langle\varphi_\alpha,\tilde{e}_\alpha\rangle = \langle\varphi_\alpha,e_\alpha\rangle = 1$. Consequently, $\tilde{e}_\alpha\in S_X$.

Finally, the family $\{\tilde{e}_\alpha\}_{1\leqslant\alpha<\omega_1}$ is symmetrically $(1+)$-separated. Indeed, for any choice $1\leqslant\alpha<\beta<\omega_1$, we have

$$\|\tilde{e}_\alpha - \tilde{e}_\beta\| \geqslant \langle\varphi_\alpha,\tilde{e}_\alpha - \tilde{e}_\beta\rangle = \langle\varphi_\alpha,e_\alpha\rangle - \left\langle \varphi_\alpha, e_\beta - \sum_{\gamma<\beta} c_\beta^\gamma e_\gamma \right\rangle = 1 + c_\beta^\alpha > 1,$$

$$\|\tilde{e}_\alpha + \tilde{e}_\beta\| \geqslant |\langle\varphi_0,\tilde{e}_\alpha + \tilde{e}_\beta\rangle| = |c_\alpha^0 + c_\beta^0| \geqslant 3/2.$$

■

Let us note explicitly that, though the above argument is definitely not hard, it is not possible to find more clever proofs that suffice to provide stronger results; in particular, it is not possible to obtain a larger $(1+)$-separated set, just assuming the existence of a larger Auerbach system. This is in light of the result–that we already mentioned–that every $(1+)$-separated family in $c_0(\Gamma)$ has cardinality at most ω_1, regardless of the set Γ (*cf.* Theorem 4.2.8).

If we combine this proposition with Theorem 2.2.1 and Theorem 2.2.3 respectively, concerning the existence of Auerbach systems, we immediately arrive at the following results. For the proof of the second one, we just need to observe that $\mathfrak{c}^+$, being a successor cardinal, is regular (*cf.* [Jec03, Corollary 5.3]).

Corollary 4.2.2 ([HKR••, Corollary 3.5])**.** *Let X be a Banach space with w^*-dens $X^* > \exp_2 \mathfrak{c}$. Then both X and X^* contain uncountable symmetrically $(1+)$-separated families of unit vectors.*

In particular, the unit sphere of every Banach space X with $\operatorname{dens} X > \exp_3 \mathfrak{c}$ contains an uncountable symmetrically $(1+)$-separated subset.

Corollary 4.2.3 ([HKR••, Corollary 3.6])**.** *Let X be a WLD Banach space with $\operatorname{dens} X > \mathfrak{c}$. Then the unit spheres of X and of X^* contain uncountable symmetrically $(1+)$-separated subsets.*

In the second part of this section, we shall present a few, much simpler, renorming results, which also depend on the existence of Auerbach systems (or, more generally, biorthogonal systems). Those results are the non-separable counterparts to Section 3.4.3, with essentially the same proofs.

In this context, we should also mention that Mercourakis and Vassiliadis have been able to show that the existence of an uncountable biorthogonal system allows for the existence of an uncountable equilateral set, under a renorming of the space, [MeVa14, Theorem 3]. It is to be noted that the notion of an equilateral set is a finer notion than the one of a separated set and the corresponding existence results are typically more difficult; to grasp an idea of some such difficulties, let us refer, among others, to the papers [FOSS14, Kos18, MeVa14, MeVa15, Ter87, Ter89]. The results that we shall present below–though analogous and somewhat simpler–are not direct consequences of those in [MeVa14]. On the other hand, our first result to be presented below also proves the existence of a 2-equilateral set, with a simpler renorming than [MeVa14, Corollary 1].

Recall that a biorthogonal system $\{x_i; f_i\}_{i\in I}$ in X is said to be *bounded* if

$$\sup_{i\in I} \|x_i\| \cdot \|f_i\| < \infty.$$

Of course, up to a scaling, we can always assume that the system is *normalised*, *i.e.*, such that $\|x_i\| = 1$ $(i \in I)$.

Proposition 4.2.4 ([HKR••, Proposition 3.7])**.** *Let $(X, \|\cdot\|)$ be a Banach space that contains a bounded biorthogonal system $\{x_i; f_i\}_{i\in I}$. Then there exists an equivalent norm $|||\cdot|||$ on X such that $S_{(X,|||\cdot|||)}$ contains a symmetrically 2-separated subset with cardinality $|I|$. In particular, $\{x_i\}_{i\in I}$ is such a set.*

The proof is the same as in Proposition 3.4.19, and it is therefore omitted. Let us just mention that the norm $|||\cdot|||$ can be explicitly defined as

$$|||x||| = \max\left\{\sup_{i\neq k\in I}\left(\left|\langle f_i, x\rangle\right| + \left|\langle f_k, x\rangle\right|\right), \|x\|\right\} \quad (x\in X)$$

and that the above renorming was already present in the proof of [Kot75, Theorem 7].

Note that, if $\{x_i; f_i\}_{i\in I}$ is a biorthogonal system and $|I|$ has uncountable cofinality, we can pass to a subsystem with the same cardinality and which is bounded. In fact the sets $I_n := \{i\in I\colon \|x_i\|\cdot\|f_i\|\leqslant n\}$ satisfy $\cup_{n=1}^{\infty} I_n = I$; hence, for some $n\in\mathbb{N}$, we have $|I_n| = |I|$ and, of course, $\{x_i; f_i\}_{i\in I_n}$ is a bounded biorthogonal system.

We can now combine these simple observations with deep results concerning the existence of uncountable biorthogonal systems in non-separable Banach spaces. In his fundamental work [Tod06], Todorčević has proved the consistency with ZFC of the claim that every non-separable Banach space admits an uncountable biorthogonal system, proving in particular the result under Martin's Maximum (MM) ([Tod06, Corollary 7]).

Let us mention–even if not strictly needed–that under suitable additional set theoretic assumptions there exist non-separable Banach spaces with no uncountable Auerbach systems. The first such example, under (CH), is due to Kunen (unpublished) and appeared later in the survey [Neg84]; other published results, under ♣ and $\diamondsuit$ respectively, are [Ost76, She85]. We also refer to [HMVZ08, Section 4.4] for a modification, suggested by Todorčević, of the argument in [Ost76]. Let us also refer to [FaGo88, FiGo89, GoTa82, Laz81, Ste75, Tod06], or [HMVZ08, Section 4.3], for some absolute (ZFC) results. These results combined with the above renorming argument give, in particular, the existence of a renorming whose unit sphere contains an uncountable symmetrically 2-separated subset for quite large classes of Banach spaces. Moreover, Todorčević's result yields that such a class may consistently consist of every non-separable Banach space.

Corollary 4.2.5. *It is consistent with ZFC that every non-separable Banach space X admits an equivalent norm $|||\cdot|||$ such that $S_{(X,|||\cdot|||)}$ contains an uncountable symmetrically 2-separated subset.*

In the particular case that the biorthogonal system is an Auerbach system, we can specialise the above renorming and obtain an approximation of the original norm.

Proposition 4.2.6 ([HKR••, Proposition 3.9])**.** *Assume that a Banach space $(X,\|\cdot\|)$ contains an Auerbach system $\{x_i, f_i\}_{i\in I}$. Then, for every $\varepsilon>0$, X admits an equivalent norm $|||\cdot|||$ such that $\|\cdot\|\leqslant|||\cdot|||\leqslant(1+\varepsilon)\|\cdot\|$ and (for some $\delta>0$) $S_{(X,|||\cdot|||)}$ contains a symmetrically $(1+\delta)$-separated subset with cardinality $|I|$.*

4.2.2 $c_0(\Gamma)$ spaces

In this part we shall investigate the existence of separated families of unit vectors in spaces of the form $c_0(\Gamma)$. Such spaces are natural candidates for investigation since they constitute the archetypal examples of spaces which fail to contain uncountable $(1+\varepsilon)$-separated families of unit vectors. This was already pointed out by Elton and Odell ([ElOd81, Remark (2)]); the very simple proof is also recorded in [KaKo16, Proposition 2.1] (*cf.* Proposition 4.1.2). On the other hand, it was noted in [GlMe15, Remark (2), p. 558] that, for uncountable Γ, the unit sphere of $c_0(\Gamma)$ contains an uncountable $(1+)$-separated subset; it even contains an uncountable symmetrically $(1+)$-separated subset. The very quick argument is also included here.

Example 4.2.7. For every uncountable set Γ, $c_0(\Gamma)$ contains an uncountable symmetrically $(1+)$-separated family of unit vectors.

Of course, it suffices to prove the claim for $\Gamma=\omega_1$. For $1\leqslant\alpha<\omega_1$, we choose a unit vector $x_\alpha\in c_0(\omega_1)$ such that

$$x_\alpha(\lambda)=\begin{cases}1 & \lambda=0,\alpha\\ <0 & 1\leqslant\lambda<\alpha\\ 0 & \alpha<\lambda<\omega_1;\end{cases}$$

such a choice is indeed possible since, for $\alpha<\omega_1$, the set $\{\lambda\colon 1\leqslant\lambda<\alpha\}$ is at most countable. It is obvious that the family $(x_\alpha)_{1\leqslant\alpha<\omega_1}$ is symmetrically $(1+)$-separated, since, for $1\leqslant\alpha<\beta<\omega_1$, we have $\|x_\alpha-x_\beta\|\geqslant|x_\alpha(\alpha)-x_\beta(\alpha)|=1-x_\beta(\alpha)>1$ and $\|x_\alpha+x_\beta\|\geqslant|x_\alpha(0)+x_\beta(0)|=2$.

Our next result shows that the above obvious construction can not be improved, as every $(1+)$-separated family of unit vectors in a $c_0(\Gamma)$ space has cardinality at most ω_1. This result improves an observation due to P. Koszmider (see [KaKo16, Proposition 4.13]). Just as the assertion about $(1+\varepsilon)$-separation, the proof exploits the Δ-system lemma.

Theorem 4.2.8 ([HKR••, Theorem 3.11])**.** *Let $A\subseteq S_{c_0(\Gamma)}$ be a $(1+)$-separated set. Then $|A|\leqslant\omega_1$.*

Proof. The assertion is trivially true for $|\Gamma|\leqslant\omega_1$; on the other hand, if $|\Gamma|>\omega_2$ and the unit sphere of $c_0(\Gamma)$ contains a $(1+)$-separated family of cardinality ω_2, then the union of the supports of those vectors is a set with cardinality at most ω_2; therefore, we would find a $(1+)$-separated family with cardinality ω_2 in the unit sphere of $c_0(\omega_2)$. Consequently, we may without loss of generality restrict our attention to the case $\Gamma=\omega_2$.

Assume, in search for a contradiction, that the unit sphere of $c_0(\omega_2)$ contains a subset $\{x_\alpha\}_{\alpha<\omega_2}$ such that $\|x_\alpha-x_\beta\|>1$ for every choice of distinct $\alpha,\beta<\omega_2$. For $\alpha<\omega_2$, consider the finite sets $N_\alpha:=\{|x_\alpha|\geqslant 1/2\}$, where $\{|x_\alpha|\geqslant 1/2\}$ is a shorthand for the set $\{\gamma<\omega_2\colon |x_\alpha(\gamma)|\geqslant 1/2\}$. The Δ-system lemma allows us to assume (up to passing to a subset that still has cardinality ω_2) that there is a finite subset Δ of ω_2 such that $N_\alpha\cap N_\beta=\Delta$ whenever $\alpha\neq\beta$; moreover, using again the regularity of ω_2, we can also assume that all the N_α's have the same (finite) cardinality. Since Δ is a finite set, the unit ball of $\ell_\infty(\Delta)\subseteq c_0(\omega_2)$ is compact and it can be covered by finitely many balls of radius $1/2$; also, $x_\alpha\restriction_\Delta\in B_{\ell_\infty(\Delta)}$ for every α. These two facts imply that there is a subset Ω of

ω_2, still with cardinality ω_2, such that all $x_\alpha\restriction_\Delta$'s ($\alpha \in \Omega$) lie in the same ball; in other words, up to passing to a further subset, we can assume that, for every $\alpha, \beta < \omega_2$, we have $\|x_\alpha\restriction_\Delta - x_\beta\restriction_\Delta\| \leqslant 1$.

Let us summarise what we have obtained so far. If, by contradiction, the conclusion of the theorem is false, then there is a (1+)-separated subset $\{x_\alpha\}_{\alpha<\omega_2}$ of $S_{c_0(\omega_2)}$ such that the following hold true:

(i) there is a finite set $\Delta \subseteq \omega_2$ with $N_\alpha \cap N_\beta = \Delta$ for distinct $\alpha, \beta < \omega_2$;

(ii) the sets N_α have the same finite cardinality;

(iii) $\|x_\alpha\restriction_\Delta - x_\beta\restriction_\Delta\| \leqslant 1$ for every $\alpha, \beta < \omega_2$.

We will show that these properties lead to a contradiction. According to (i), we may write $N_\alpha = \Delta \cup \tilde{N}_\alpha$, where $\tilde{N}_\alpha \subseteq \omega_2 \setminus \Delta$ and the sets $\tilde{N}_\alpha$ are mutually disjoint. Moreover, by (ii), they also have the same finite cardinality, say k; let us then write $\tilde{N}_\alpha = \{\lambda_1^\alpha, \dots, \lambda_k^\alpha\}$ with $\lambda_1^\alpha < \lambda_2^\alpha < \cdots < \lambda_k^\alpha < \omega_2$. The disjointness of the $\tilde{N}_\alpha$'s forces in particular $\lambda_1^\alpha \neq \lambda_1^\beta$ for $\alpha \neq \beta$ and this in turn implies

$$\sup_{\alpha<\omega_2} \lambda_1^\alpha = \omega_2. \tag{4.2.1}$$

We then consider the set $\cup_{\alpha<\omega_1}\mathrm{supp} x_\alpha$; such a set has cardinality at most ω_1, so its supremum is necessarily strictly smaller than ω_2. Therefore, combining this information and (4.2.1), we infer that there exists an ordinal $\beta < \omega_2$ such that

$$\sup\left(\bigcup_{\alpha<\omega_1} \mathrm{supp} x_\alpha\right) < \lambda_1^\beta;$$

in particular, this implies that $\tilde{N}_\beta \cap \mathrm{supp} x_\alpha = \emptyset$ for every $\alpha < \omega_1$. Moreover, the set $\{0 < |x_\beta| < 1/2\}$ is of course countable, whence it can intersect at most countably many of the disjoint sets $\{\tilde{N}_\alpha\}_{\alpha<\omega_1}$. Consequently, we can find an ordinal $\alpha < \omega_1$ such that $\{0 < |x_\beta| < 1/2\} \cap \tilde{N}_\alpha = \emptyset$ too. In order to understand where the supports of those x_α and x_β could possibly intersect, we note that for every γ we have the disjoint union

$$\mathrm{supp}\, x_\gamma = \Delta \cup \left(\tilde{N}_\gamma \cup \{0 < |x_\gamma| < 1/2\}\right).$$

Therefore, using our previous choices of β and α, we obtain:

$$\begin{aligned} \mathrm{supp} x_\alpha \cap \mathrm{supp} x_\beta &= \Delta \cup \left(\left(\tilde{N}_\alpha \cup \{0 < |x_\alpha| < 1/2\}\right) \cap \left(\tilde{N}_\beta \cup \{0 < |x_\beta| < 1/2\}\right)\right) \\ &= \Delta \cup \left(\{0 < |x_\alpha| < 1/2\} \cap \{0 < |x_\beta| < 1/2\}\right). \end{aligned}$$

Finally, for every $\gamma \in \{0 < |x_\alpha| < 1/2\} \cap \{0 < |x_\beta| < 1/2\}$ it is obvious that we have $|x_\alpha(\gamma) - x_\beta(\gamma)| \leqslant 1$; consequently, the condition $\|x_\alpha - x_\beta\| > 1$ can only be witnessed by coordinates from Δ, *i.e.*, $\|x_\alpha\restriction_\Delta - x_\beta\restriction_\Delta\| > 1$; however, this readily contradicts (iii). ■

Remark 4.2.9. Let us notice in passing that if we consider the spaces $c_{00}(\Gamma)$, then every (1+)-separated family of unit vectors is actually at most countable; this is also pointed out in [GlMe15, Remark (2), p. 558] and it immediately follows from the Δ-system lemma exactly as in [ElOd81, Remark (2)].

The results mentioned so far in this section draw a complete picture about separated families of unit vectors in $c_0(\Gamma)$ spaces; we therefore conclude this section presenting some results concerning renormings of those spaces.

Proposition 4.2.6 in the previous section applies in particular to the canonical basis of $c_0(\Gamma)$; consequently, the canonical norm $\|\cdot\|_\infty$ on $c_0(\Gamma)$ can be approximated by norms whose unit spheres contain (for some $\varepsilon > 0$) symmetrically $(1+\varepsilon)$-separated subsets of cardinality $|\Gamma|$. If we combine this result with James' non distortion theorem, we obtain the approximation of every equivalent norm on $c_0(\Gamma)$. Before we proceed, a historical remark about the non-separable counterparts of James' theorems is in order.

Remark 4.2.10. It was communicated to us by W. B. Johnson that the non-separable analogue of both James' non-distortion theorems were known to the experts immediately after the paper of James [Jam64a] had been published. At the beginning of this century, A. S. Granero being unaware of this situation, circulated a note containing the proofs of these theorems ([Gra]). It seems that the first published proof of non-separable versions of James's non-distortion theorems may be found in [HáNo18, Theorem 3].

Proposition 4.2.11 ([HKR••, Proposition 3.14])**.** *Every equivalent norm on $c_0(\Gamma)$ can be approximated by norms whose unit spheres contain (for some $\delta > 0$) symmetrically $(1+\delta)$-separated subsets of cardinality $|\Gamma|$.*

Proof. Let $\|\cdot\|$ be any equivalent norm on $c_0(\Gamma)$ and fix $\varepsilon > 0$. By the non-separable version of James' non-distortion theorem, there exists a subspace Y of $c_0(\Gamma)$, with $\operatorname{dens}(Y) = |\Gamma|$, such that Y is ε-isometric to $(c_0(\Gamma), \|\cdot\|_\infty)$. Let $T\colon (Y, \|\cdot\|) \to (c_0(\Gamma), \|\cdot\|_\infty)$ be a linear isomorphism witnessing this fact; in particular, we may assume that $\|T\| \leqslant 1+\varepsilon$ and $\|T^{-1}\| \leqslant 1$. According to Proposition 4.2.6, we can choose a norm ν on $(c_0(\Gamma), \|\cdot\|_\infty)$ with $\|\cdot\|_\infty \leqslant \nu \leqslant (1+\varepsilon)\|\cdot\|_\infty$ and such that $S_{(c_0(\Gamma),\nu)}$ contains (for some $\delta > 0$) a symmetrically $(1+\delta)$-separated family of cardinality $|\Gamma|$.

Consider a new norm on Y given by $y \mapsto \nu(Ty)$ $(y \in Y)$; since

$$\|y\| \leqslant \|Ty\|_\infty \leqslant \nu(Ty) \leqslant (1+\varepsilon)\|Ty\|_\infty \leqslant (1+\varepsilon)^2\|y\|,$$

it is well known that we can extend the norm $\nu \circ T$ to a norm $|\!|\!|\cdot|\!|\!|$ defined on $(c_0(\Gamma), \|\cdot\|)$ and still satisfying $\|\cdot\| \leqslant |\!|\!|\cdot|\!|\!| \leqslant (1+\varepsilon)^2\|\cdot\|$. Finally, T is an isometry from $(Y, |\!|\!|\cdot|\!|\!|)$ onto $(c_0(\Gamma), \nu)$, whence the unit sphere of $(c_0(\Gamma), |\!|\!|\cdot|\!|\!|)$ contains a symmetrically $(1+\delta)$-separated family of cardinality $|\Gamma|$. ■

Remark 4.2.12. Let us note in passing that, by a very similar argument, the unit sphere of every equivalent norm on $\ell_1(\Gamma)$ contains (for some $\varepsilon > 0$) a symmetrically $(1+\varepsilon)$-separated family of cardinality $|\Gamma|$.

In the last result for this section we provide a sufficient condition for a renorming of $c_0(\Gamma)$ to contain an uncountable (1+)-separated family of unit vectors; the argument elaborates over the proof of Proposition 4.2.1. Further sufficient conditions will follow from some results from Section 4.3.

Let Γ be any set and $\|\cdot\|$ be a norm on $c_0(\Gamma)$ (not necessarily equivalent to the canonical $\|\cdot\|_\infty$ norm of $c_0(\Gamma)$). We say that $\|\cdot\|$ is a *lattice norm* if $\|x\| \leqslant \|y\|$ for every pair of vectors $x, y \in c_0(\Gamma)$ such that $|x(\gamma)| \leqslant |y(\gamma)|$ for each $\gamma \in \Gamma$. In other words, $\|\cdot\|$ is a lattice norm if $(c_0(\Gamma), \|\cdot\|)$ is a normed lattice when endowed with the canonical coordinate-wise partial ordering. Accordingly, in what follows we shall denote by $|x|$ the element defined by $|x|(\gamma) := |x(\gamma)|$ $(\gamma \in \Gamma)$.

Proposition 4.2.13 ([HKR••, Proposition 3.16])**.** *Let Γ be an uncountable set and $\|\cdot\|$ be a (not necessarily equivalent) lattice norm on $c_0(\Gamma)$. Then the unit sphere of $(c_0(\Gamma), \|\cdot\|)$ contains an uncountable (1+)-separated subset.*

Proof. It is sufficient to prove the result for $\Gamma = \omega_1$. Let us denote by e_α $(\alpha < \omega_1)$ the α-th element of the canonical basis, *i.e.*, $e_\alpha(\gamma) := \delta_{\alpha,\gamma}$; we also denote by $\tilde{e}_\alpha$ the unit vector $\tilde{e}_\alpha := e_\alpha / \|e_\alpha\|$. We may now choose, for every $\beta < \omega_1$, real numbers $(c^\alpha_\beta)_{\alpha<\beta}$ subject to the following three conditions:

(i) $c^\alpha_\beta \geqslant 0$ for each $\alpha < \beta$;

(ii) $c^\alpha_\beta > 0$ if and only if $\|\tilde{e}_\alpha - \tilde{e}_\beta\| \leqslant 1$;

(iii) if $c^\alpha_\beta > 0$ for some $\alpha < \beta$, then $\sum_{\alpha<\beta} c^\alpha_\beta = 1$.

Observe that condition (iii) could be equivalently stated as the requirement $\sum_{\alpha<\beta} c^\alpha_\beta$ to equal either 0 or 1; note, further, that (ii) implies $c^\alpha_\beta \cdot \|\tilde{e}_\alpha - \tilde{e}_\beta\| \leqslant c^\alpha_\beta$ for each $\alpha < \beta$.

We are now in position to define vectors $f_\beta \in c_0(\omega_1)$ $(\beta < \omega_1)$ as follows:

$$f_\beta := \tilde{e}_\beta - \sum_{\alpha<\beta} c^\alpha_\beta \, \tilde{e}_\alpha.$$

We readily verify that $\|f_\beta\| = 1$. In fact, if $\sum_{\alpha<\beta} c^\alpha_\beta = 0$, then $f_\beta = \tilde{e}_\beta$ and there is nothing to prove. In the other case, *i.e.*, $\sum_{\alpha<\beta} c^\alpha_\beta = 1$, we have, according to (ii),

$$\|f_\beta\| = \left\| \sum_{\alpha<\beta} c^\alpha_\beta \, \tilde{e}_\beta - \sum_{\alpha<\beta} c^\alpha_\beta \, \tilde{e}_\alpha \right\| \leqslant \sum_{\alpha<\beta} c^\alpha_\beta \|\tilde{e}_\beta - \tilde{e}_\alpha\| \leqslant \sum_{\alpha<\beta} c^\alpha_\beta = 1.$$

On the other hand, $|f_\beta| \geqslant |\tilde{e}_\beta|$, whence $\|f_\beta\| = 1$ follows from the lattice property.

To conclude, we prove that the vectors $(f_\alpha)_{\alpha<\omega_1}$ are (1+)-separated. Given $\alpha < \beta < \omega_1$, we distinguish two cases. If $c^\alpha_\beta = 0$, then by our previous choice we have $\|\tilde{e}_\alpha - \tilde{e}_\beta\| > 1$. Moreover, $|f_\alpha - f_\beta| \geqslant |\tilde{e}_\alpha - \tilde{e}_\beta|$ and the lattice property imply $\|f_\alpha - f_\beta\| \geqslant \|\tilde{e}_\alpha - \tilde{e}_\beta\| > 1$. On the other hand, if $c^\alpha_\beta > 0$, we note that $|f_\alpha - f_\beta| \geqslant |(1 + c^\alpha_\beta)\tilde{e}_\alpha|$; consequently, exploiting once more the lattice property, we conclude that $\|f_\alpha - f_\beta\| \geqslant \|(1 + c^\alpha_\beta)\tilde{e}_\alpha\| = 1 + c^\alpha_\beta > 1$. ■

Remark 4.2.14. Concerning lattice norms, we would like to point out here the validity of the analogous result for lattice norms on the space $C([0,\omega_1])$. Inspection of the proof of [FHZ97, Proposition 2] shows that if $\|\cdot\|$ is any (not necessarily equivalent) lattice norm on $C([0,\omega_1])$, then $(C([0,\omega_1]),\|\cdot\|)$ contains an isometric copy of $(c_0(\omega_1),\|\cdot\|_\infty)$. Consequently, the unit sphere of $(C([0,\omega_1]),\|\cdot\|)$ contains an uncountable $(1+)$-separated subset.

4.3 Exposed points and $(1+)$-separation

In the present section we shall be concerned with the proof of clause (i) in Theorem 4.1.4; the argument will depend on rather powerful results concerning the notion of an exposed point (which we shall record below), whence the title of this section. We will also present an abstract and more general result, whose proof follows an analogous pattern, and show some its consequences. The material also allows for natural analogues concerning $(1+\varepsilon)$-separation, that will be discussed in the next section.

Let X be a Banach space and let $C \subseteq X$ be a non-empty, closed, convex and bounded set. A point $x \in C$ is an *exposed point* for C if there is a functional $\varphi \in X^*$ such that $\langle \varphi, y\rangle < \langle \varphi, x\rangle$ for every $y \in C$, $y \neq x$. In other words, φ attains its supremum over C at the point x and only at that point. In such a case, we also say that the functional φ *exposes* the point x. $x \in C$ is a *strongly exposed point* for C if there is a functional $\varphi \in X^*$ that exposes x and with the property that $y_n \to x$ for every sequence $(y_n)_{n=1}^\infty$ in C such that $\langle \varphi, y_n\rangle \to \langle \varphi, x\rangle$. In such a case, we shall say that φ *strongly exposes* the point x.

Of course, every strongly exposed point is an exposed point and it is immediate to check that every exposed point is an extreme point. By a result of Lindenstrauss and Troyanski ([Lin63, Tro71] see, *e.g.*, [FHHMZ10, Theorem 8.13]), every weakly compact set in a Banach space is the closed convex hull of its strongly exposed points. We shall use the immediate consequence that every non-empty weakly compact set in a Banach space admits an exposed point.

Theorem 4.3.1 ([HKR••, Theorem 4.1])**.** *Let X be an infinite-dimensional, reflexive Banach space. Then the unit sphere of X contains a symmetrically $(1+)$-separated subset of cardinality* $\operatorname{dens} X$*.*

An important result for the structure of quasi-reflexive Banach spaces is the fact that every quasi-reflexive Banach space X contains a reflexive subspace Y with the same density character as X. For non-separable X this was proved in [CiHo57, Theorem 4.6], while the assertion in the separable case follows from some results by Johnson and Rosenthal (*cf.* [JoRo72, Corollary IV.1]). Consequently, the above result implies assertion (i) in Theorem 4.1.4.

Proof. Let X be an infinite-dimensional, reflexive Banach space and set $\lambda = \operatorname{dens} X$. According to the result by Lindenstrauss and Troyanski quoted above, the weakly compact set B_X contains an exposed point x_1; we can then choose a functional $\varphi_1 \in X^*$ that exposes x_1 (and we let $X_1 := X$ for notational consistency). Needless to say, the subspace

$X_2 := \ker \varphi_1 \subseteq X_1$ is a reflexive Banach space, whence B_{X_2} is a weakly compact subset of X. Consequently, we can choose an exposed point $x_2 \in B_{X_2}$ and a functional $\varphi_2 \in X^*$ that exposes x_2. We now proceed by transfinite induction. Assume, for some $\beta < \lambda$, to have already found closed subspaces $(X_\alpha)_{\alpha<\beta}$ of X, unit vectors $(x_\alpha)_{\alpha<\beta}$ and functionals $(\varphi_\alpha)_{\alpha<\beta}$ such that for each $\alpha < \beta$:

(i) $X_\alpha = \bigcap_{\gamma<\alpha} \ker \varphi_\gamma$;

(ii) $x_\alpha \in X_\alpha$ is an exposed point for B_{X_α} in X and the functional $\varphi_\alpha \in X^*$ exposes x_α.

Consider the closed subspace $X_\beta := \bigcap_{\alpha<\beta} \ker \varphi_\alpha$. The reflexivity of X implies that $w^*\text{-dens}\, X^* = \operatorname{dens} X = \lambda > |\beta|$, whence the linear span of the family $\{\varphi_\alpha\}_{\alpha<\beta}$ can not be w^*-dense in X^*. Consequently, $\{\varphi_\alpha\}_{\alpha<\beta}$ does not separate points on X and X_β does not reduce to the zero vector. Moreover, B_{X_β} is a weakly compact subset of X, hence it admits an exposed point x_β, exposed by $\varphi_\beta \in X^*$. The fact that $X_\beta \neq \{0\}$ ensures us that x_β is a unit vector. This completes the inductive step and shows the existence of families of closed subspaces $(X_\alpha)_{\alpha<\lambda}$ of X, unit vectors $(x_\alpha)_{\alpha<\lambda}$ and functionals $(\varphi_\alpha)_{\alpha<\lambda}$ with the two properties above.

In conclusion, we show that the family $(x_\alpha)_{\alpha<\lambda}$ is symmetrically (1+)-separated: if $\alpha < \beta < \lambda$, by construction $x_\beta \in X_\beta \subseteq \ker \varphi_\alpha$, so $\langle \varphi_\alpha, x_\alpha \pm x_\beta \rangle = \langle \varphi_\alpha, x_\alpha \rangle$. Since φ_α exposes x_α, this last equality implies that $x_\alpha \pm x_\beta \notin B_{X_\alpha}$. But of course $x_\alpha \pm x_\beta \in X_\alpha$ and consequently $\|x_\alpha \pm x_\beta\| > 1$, thus concluding the proof. ■

It is perhaps clear that the above reasoning can be adapted to more general spaces, notably spaces with the RNP. However, instead of giving various related results with similar proofs, we prefer to present an abstract version of the present reasoning, which subsumes many concrete examples. We then present concrete consequences of this general theorem.

Definition 4.3.2. An infinite-dimensional Banach space $(X, \|\cdot\|)$ is said to admit the *point with small flatness property* (shortly, *PSF*) if there exist $x \in S_X$ and a closed subspace Y of X, with $\dim(X/Y) < \infty$, such that $\|x + y\| > 1$ for every unit vector $y \in S_Y$. In symbols,

$$X \text{ has PSF whenever: } \quad \begin{cases} \exists x \in S_X, \exists Y \subseteq X \text{ closed subspace, with} \dim(X/Y) < \infty : \\ \forall y \in S_Y \; \|x + y\| > 1. \end{cases}$$

This notion is inspired from condition (▭) in the proof of Theorem 3.1.11. We are going to see in a moment various examples of spaces with property (PSF), therefore it is perhaps worth including here an example of a Banach space failing this property.

Example 4.3.3. The space $X = c_0(\omega_1)$ is an easy example of a space failing (PSF). Indeed, fix arbitrarily $x \in S_X$ and a finite-codimensional closed subspace Y of X. For $\alpha < \omega_1$, we denote by $X_{>\alpha}$ the closed subspace of X consisting of all vectors whose support is contained in $[\alpha + 1, \omega_1)$. Of course, there exists an ordinal $\alpha < \omega_1$ such that $\operatorname{supp} x \leqslant \alpha$ and, moreover, $Y \cap X_{>\alpha}$ is a finite-codimensional subspace of $X_{>\alpha}$. In particular, if we choose $y \in Y \cap X_{>\alpha}$ with $\|y\| = 1$, x and y are disjointly supported, whence $\|x + y\| = 1$.

Let us also mention that there exists examples of separable Banach spaces that fail (PSF). In particular, it is possible to show that the space c_0 does not satisfy property (PSF). The proof of this claim is similar to the one above, together with some linear algebra calculations, similar to the argument in [Bog••, Theorem 2.1]; for the sake of completeness, we shall present it below.

Fact 4.3.4. *The space c_0 does not have property (PSF).*

Proof. Fix $x \in c_0$, $\|x\| = 1$ and let Y be a finite-codimensional subspace of c_0. Let us select a finite collection of linear functionals $z_1, \dots, z_n \in \ell_1$ such that $Y = \ker z_1 \cap \dots \cap \ker z_k$; we can clearly assume that such functionals are linearly independent. Let us fix $N \in \mathbb{N}$ such that $|x(j)| \leqslant 1/2$ for every $j \geqslant N$; since we shall construct a vector $y \in S_Y$ with $\mathrm{supp}(y) \geqslant N$, we can also assume that the vectors $z_1\restriction_{[N,\infty)}, \dots, z_n\restriction_{[N,\infty)}$ are linearly independent. It is then easy, *e.g.*, by the Gauß reduction method, to find n columns, say with indices $N \leqslant k_1 < \dots < k_n$, such that the matrix

$$\begin{pmatrix} z_1(k_1) & \dots & z_1(k_n) \\ \vdots & & \vdots \\ z_n(k_1) & \dots & z_n(k_n) \end{pmatrix}$$

is non singular. Since the vectors $z_1, \dots, z_n$ are in ℓ_1, it is possible to find $k_{n+1} \in \mathbb{N}$ with $k_{n+1} > k_n$ such that the unique solution to the linear system

$$\begin{pmatrix} z_1(k_1) & \dots & z_1(k_n) \\ \vdots & & \vdots \\ z_n(k_1) & \dots & z_n(k_n) \end{pmatrix} \cdot \begin{pmatrix} y(k_1) \\ \vdots \\ y(k_n) \end{pmatrix} = \mathrm{sgn}(x(k_{n+1})) \begin{pmatrix} z_1(k_{n+1}) \\ \vdots \\ z_n(k_{n+1}) \end{pmatrix}$$

has all coordinates bounded by $1/2$ in absolute value. Here, we use the convention that $\mathrm{sgn}(0) = 1$. Therefore, setting $y(k_{n+1}) := -\mathrm{sgn}(x(k_{n+1}))$, it is clear that the vector

$$y := \sum_{j=1}^{n+1} y(k_j) e_{k_j} \in c_0$$

satisfies $\langle z_j, y \rangle = 0$ $(j = 1, \dots, n)$, hence $y \in Y$. Our construction also assures that $|y(k_j)| \leqslant 1/2$ $(j = 1, \dots, n)$ and $|y(k_{n+1})| = 1$, whence $\|y\| = 1$. Finally, $|y(k_j) + x(k_j)| \leqslant |y(k_j)| + |x(k_j)| \leqslant 1$ $(j = 1, \dots, n)$ and also $|y(k_{n+1}) + x(k_{n+1})| \leqslant 1$ follows from our previous choice. Consequently, $\|x + y\| \leqslant 1$ and we are done. ∎

Our general result now reads as follows.

Theorem 4.3.5 ([HKR••, Theorem 4.4])**.** *Let X be an infinite-dimensional Banach space such that every infinite-dimensional subspace $\tilde{X}$ of X has (PSF). Then S_X contains a symmetrically $(1+)$-separated subset with cardinality w^*-$\mathrm{dens}\, X^*$.*

Proof. Let $\lambda := w^*\text{-dens}\, X^*$. If $\lambda = \omega$, then the result is contained in [HKR18, Theorem A], so we assume $\lambda > \omega$. We shall construct by transfinite induction a family of unit vectors $(x_\alpha)_{\alpha<\lambda} \subseteq S_X$ and a decreasing family of closed subspaces $(H_\alpha)_{\alpha<\lambda}$ of X such that, for every $\alpha < \lambda$:

(i) $\|x_\alpha + y\| > 1$ for every $y \in S_{H_\alpha}$;

(ii) $x_\beta \in H_\alpha$ for $\alpha < \beta < \lambda$;

(iii) $\dim((\cap_{\beta<\alpha} H_\beta)/H_\alpha) < \infty$.

Condition (iii) is a technical condition that we need for the inductive procedure to continue until λ; it implies in particular that $\dim(X/H_1) < \infty$ and $\dim(H_\alpha/H_{\alpha+1}) < \infty$ for each $\alpha < \lambda$. Once such a family is constructed, for $\alpha < \beta < \lambda$ we have $\pm x_\beta \in H_\alpha$, by (ii). From (i) we conclude that $\|x_\alpha \pm x_\beta\| > 1$, whence the family $(x_\alpha)_{\alpha<\lambda}$ is symmetrically (1+)-separated.

Before entering the construction of those families, we note the following: if condition (iii) is satisfied for every $\alpha < \gamma$, then $\cap_{\alpha<\gamma} H_\alpha$ is the intersection of at most $\max\{|\gamma|, \omega\}$ kernels of functionals from X, *i.e.*, $w^*\text{-dens}((X/\cap_{\alpha<\gamma} H_\alpha)^*) \leqslant \max\{|\gamma|, \omega\}$.

The proof of this is also based on a simple transfinite induction argument: assuming the statement to be true for every $\alpha < \delta$ (where $\delta < \gamma$), if $\delta = \delta' + 1$ is a successor ordinal, then of course $\cap_{\alpha<\delta} H_\alpha = H_{\delta'}$. Consequently, the facts that $\cap_{\alpha<\delta'} H_\alpha$ is the intersection of at most $\max\{|\delta'|, \omega\}$ kernels of functionals and that $\dim((\cap_{\alpha<\delta'} H_\alpha)/H_{\delta'}) < \infty$ imply the desired assertion for $\cap_{\alpha<\delta} H_\alpha$. If δ is a limit ordinal, then $\cap_{\alpha<\delta} H_\alpha = \cap_{\beta<\delta}(\cap_{\alpha<\beta} H_\alpha)$ and each $\cap_{\alpha<\beta} H_\alpha$ is the intersection of at most $\max\{|\beta|, \omega\} \leqslant |\delta|$ kernels of functionals; hence the same is true for $\cap_{\alpha<\delta} H_\alpha$.

We now turn to the construction of the families $(x_\alpha)_{\alpha<\lambda}$ and $(H_\alpha)_{\alpha<\lambda}$ with the desired properties. Assume by transfinite induction to have already found such elements for every $\alpha < \gamma$, where $\gamma < \lambda$. From the above observation, we infer that

$$w^*\text{-dens}\,(X/\bigcap_{\alpha<\gamma} H_\alpha)^* \leqslant \max\{|\gamma|, \omega\} < \lambda$$

and this readily implies that $w^*\text{-dens}((\cap_{\alpha<\gamma} H_\alpha)^*) = \lambda$. In particular, $\cap_{\alpha<\gamma} H_\alpha$ (is infinite-dimensional, hence it) has property (PSF) and we can thus find a unit vector $x_\gamma \in \cap_{\alpha<\gamma} H_\alpha$ and a finite-codimensional subspace H_γ of $\cap_{\alpha<\gamma} H_\alpha$ such that $\|x_\gamma + y\| > 1$ for every $y \in S_{H_\gamma}$. It is then clear that properties (i)–(iii) are satisfied with such a choice; therefore, the transfinite induction step is complete and we are done. ∎

We shall now pass to discuss some concrete situations where the above theorem applies.

Definition 4.3.6. Let X be a normed space and F be a non-empty subset of S_X. We say that F is a *face* of B_X if there exists a functional $\varphi \in S_{X^*}$ such that $F = B_X \cap \{\langle \varphi, \cdot \rangle = 1\}$.

Note that if S_X contains a face F with $\operatorname{diam}(F) < 1$, then X has (PSF). In fact, let $\varphi \in S_{X^*}$ be such that $F = B_X \cap \{\langle \varphi, \cdot \rangle = 1\}$ and choose any $x \in F$. Then, for every $y \in Y := \ker \varphi$ with $\|y\| = 1$ we have $\|x + y\| > 1$ (otherwise, $x + y \in F$, whence

$\operatorname{diam}(F) \geqslant \|(x+y)-x\| = 1)$. Consequently, the following proposition is an immediate consequence of the previous result and the simple fact that if Y is a subspace of X, then every face of B_Y is contained in a face of B_X.

Proposition 4.3.7 ([HKR••, Proposition 4.6]). *Let X be an infinite-dimensional Banach space. Suppose that, for every infinite-dimensional subspace Y of X, the unit ball B_Y contains a face with diameter strictly smaller than 1. Then the unit sphere of X contains a symmetrically $(1+)$-separated subset with cardinality $w^*\text{-dens}\, X^*$.*

In particular, the conclusion holds true if every face of X has diameter strictly smaller than 1.

Clearly, if X is strictly convex, then every face of B_X is a singleton; we can therefore record the following immediate consequence to the present proposition.

Corollary 4.3.8. *If X is an infinite-dimensional strictly convex Banach space, then the unit sphere of X contains a symmetrically $(1+)$-separated subset of cardinality $w^*\text{-dens}\, X^*$.*

Our next application of Theorem 4.3.5 leads us back to the dawning of this section, *i.e.*, to exposed points. In fact, if $x \in S_X$ is an exposed point for B_X and $\varphi \in S_{X^*}$ exposes x, then, for every unit vector $y \in Y := \ker \varphi$, we have $\langle \varphi, x+y \rangle = 1$, whence $\|x+y\| > 1$. Consequently, if the unit ball of X admits an exposed point, X has (PSF).

We also need to recall that the result by Lindenstrauss and Troyanski the proof of Theorem 4.3.1 heavily relied on is in fact consequence of a more general result, due to Phelps ([Phe74], *cf.* [BeLi00, Theorem 5.17]): if C is a closed, convex and bounded set with the Radon-Nikodym property (RNP) in a Banach space X, then C is the closed convex hull of its strongly exposed points. This assertion is truly more general, since every weakly compact convex set has the RNP ([BeLi00, Theorem 5.11.(i)]), while the unit ball of ℓ_1 is a simple example of a set with the RNP that fails to be weakly compact. Therefore, if a Banach space X has the RNP, the unit ball of every its subspace is the closed convex hull of its strongly exposed points. As a consequence, every infinite-dimensional subspace of X has (PSF) and we infer the validity of the following result. Note that, of course, it contains Theorem 4.3.1 as a particular case.

Theorem 4.3.9 ([HKR••, Theorem 4.8]). *Let X be an infinite-dimensional Banach space with the Radon–Nikodym property. Then there exists a symmetrically $(1+)$-separated family of unit vectors in X, with cardinality $w^*\text{-dens}\, X^*$.*

Our last result in this chapter is devoted to duals to Gâteaux differentiability spaces and exploits w^*-exposed points; let us proceed to recall a few definitions. A functional $\varphi \in S_{X^*}$ is a *w^*-exposed point* of B_{X^*}, whenever there exists a unit vector $x \in S_X$ such that $\langle \varphi, x \rangle = 1$ and $\operatorname{Re} \langle \psi, x \rangle < 1$ for every $\psi \in B_{X^*}$, $\psi \neq \varphi$; in other words, φ is the unique supporting functional at x. A Banach space X is called a *Gâteaux differentiability space* if every convex continuous function defined on a non-empty open convex subset D of X is Gâteaux differentiable at densely many points of D. This notion differs from the notion of *weak Asplund space* only by virtue of the fact that the set of differentiability points is not required to contain a dense G_δ, but merely to be dense in D (for information concerning those spaces, consult [Fab97, Phe93]). On the other hand, let us stress the fact that the two notions are actually distinct, as it was first proved in [MoSo06]. Let us also refer to [Moo05] for a simplified proof of the example and to [KaSt99, Kal02, KMS01, MoSo02] for related results.

The interplay between w^*-exposed points and Gâteaux differentiability spaces stems from the fact that points of Gâteaux differentiability of the norm correspond to w^*-exposed points in the dual space. More precisely, the norm $\|\cdot\|$ of a Banach space X is Gâteaux differentiable at $x \in S_X$ if and only if there exists a unique $\varphi \in B_{X^*}$ with $\langle \varphi, x \rangle = 1$ (see, *e.g.*, [FHHMZ10, Corollary 7.22]); in which case, φ is w^*-exposed by x.

Note that, if X is a Gâteaux differentiability space, there exists a functional $f_1 \in S_{X^*}$ which is w^*-exposed by some $x_1 \in S_X$; in particular, for every $g \in \{x_1\}^\perp$ we have $\|f_1 \pm g\| > 1$. The w^*-closed subspace $\{x_1\}^\perp$ of X^* is the dual to $X/\operatorname{span} x_1$, which is a Gâteaux differentiability space, according to [Phe93, Proposition 6.8]; therefore, we may repeat the argument to the subspace $\{x_1\}^\perp$. When we proceed by a transfinite induction argument completely analogous to those already presented in this section, we reach the following result.

Proposition 4.3.10 ([HKR••, Proposition 4.9])**.** *Let X be a Banach space dual to a Gâteaux differentiability space. Then the unit sphere of X contains a symmetrically* (1+)*-separated subset with cardinality w^*-$\operatorname{dens} X^*$.*

4.4 $(1+\varepsilon)$-separation

In this part we shall proceed to present results parallel to those of the previous section, but concerning the existence of large symmetrically $(1+\varepsilon)$-separated sets. It is clear that one could formulate a uniform analogue to condition (PSF) and adapt the arguments to be presented in this section to deduce an analogue to Theorem 4.3.5. However, we shall not pursue this direction and we shall restrict ourselves to the consideration of some classes of Banach spaces.

In the first result we shall exploit the full power of the notion of strongly exposed point in order to treat spaces with the RNP. We start with the following simple observation.

Lemma 4.4.1. *Let X be a Banach space and $x \in B_X$ be a strongly exposed point of B_X; also let $\varphi \in X^*$ be a strongly exposing functional for x. Then*

$$\inf\{\|x+v\| : v \in \ker\varphi, \|v\| = 1\} > 1.$$

Proof. Note preliminarily that the above infimum is necessarily greater or equal to 1. In fact, φ exposes x, so for every non-zero $v \in \ker\varphi$ we have $\|x+v\| > 1$. If by contradiction the conclusion of the lemma is false, we may find a sequence of unit vectors $(v_n)_{n=1}^\infty$ in $\ker\varphi$ such that $\|x+v_n\| \to 1$. The vectors $r_n := \frac{x+v_n}{\|x+v_n\|} \in B_X$ then satisfy $\langle \varphi, r_n \rangle = \frac{\langle \varphi, x\rangle}{\|x+v_n\|} \to$

$\langle \varphi, x\rangle$; consequently, our assumption that x is strongly exposed by φ allows us to conclude that $r_n \to x$. This is however an absurdity, since

$$\|r_n - x\| \geqslant \|x + v_n - x\| - \|r_n - (x + v_n)\| = 1 - \|x + v_n\| \cdot \left| \frac{1}{\|x + v_n\|} - 1 \right| \to 1.$$

■

Theorem 4.4.2 ([HKR••, Theorem 4.11])**.** *Let X be an infinite-dimensional Banach space with the RNP and let $\kappa \leqslant w^*$-dens X^* be a cardinal number with uncountable cofinality. Then, for some $\varepsilon > 0$, the unit sphere of X contains a symmetrically $(1+\varepsilon)$-separated subset, with cardinality κ.*

Proof. The argument follows a pattern similar to the proofs of the previous section, therefore we only sketch it. Let $\lambda := w^*$-dens X^*; a transfinite induction argument as in the proof of Theorem 4.3.1 shows the existence of families of closed subspaces $(X_\alpha)_{\alpha<\lambda}$ of X, unit vectors $(x_\alpha)_{\alpha<\lambda}$ and functionals $(\varphi_\alpha)_{\alpha<\lambda}$ with the following properties, for every $\alpha < \lambda$:

(i) $X_\alpha = \bigcap_{\gamma<\alpha} \ker \varphi_\gamma$;

(ii) $x_\alpha \in X_\alpha$ is a strongly exposed point for B_{X_α} in X_α, strongly exposed by φ_α.

According to Lemma 4.4.1, we can also find, for each $\alpha < \lambda$, a real $\varepsilon_\alpha > 0$ such that $\|x_\alpha + v\| \geqslant 1 + \varepsilon_\alpha$ for every unit vector $v \in \ker \varphi_\alpha \cap X_\alpha$. In particular, for every $\alpha < \beta < \lambda$ we have $\pm x_\beta \in \ker \varphi_\alpha \cap X_\alpha$, whence $\|x_\alpha \pm x_\beta\| \geqslant 1 + \varepsilon_\alpha$.

We finally exploit the cofinality of κ to conclude the proof. Of course, the union of the sets $\Gamma_n := \{\alpha < \kappa \colon \varepsilon_\alpha \geqslant 1/n\}$ covers κ, whence the uncountable cofinality of κ implies the existence of n_0 such that $|\Gamma_{n_0}| = \kappa$. Consequently, for any $\alpha, \beta \in \Gamma_{n_0}, \alpha < \beta$, we have

$$\|x_\alpha \pm x_\beta\| \geqslant 1 + \varepsilon_\alpha \geqslant 1 + 1/n_0.$$

Therefore, the family $\{x_\alpha\}_{\alpha\in\Gamma_{n_0}}$ has cardinality κ and it is symmetrically $(1 + 1/n_0)$-separated, which concludes the proof. ■

Plainly, reflexive Banach spaces have the RNP and satisfy dens $X = w^*$-dens X^*. Therefore, the following corollary is a particular case of the previous theorem; note that, by the considerations at the beginning of Section 4.3 concerning quasi-reflexive Banach spaces, it implies (ii) in Theorem 4.1.4.

Corollary 4.4.3 ([HKR••, Corollary 4.12])**.** *Suppose that X is an infinite-dimensional reflexive Banach space. Let $\kappa \leqslant$ dens X be a cardinal number with uncountable cofinality. Then there exists a symmetrically $(1+\varepsilon)$-separated family of unit vectors in S_X, with cardinality κ.*

We next give the $(1+\varepsilon)$-separation analogue to Corollary 4.3.8; as it is to be expected, we need to assume a uniform analogue to strict convexity. Let us recall that a norm $\|\cdot\|$ on a Banach space X is *locally uniformly rotund* (hereinafter, *LUR*) if, for every $x \in S_X$ and every sequence $(x_n)_{n=1}^\infty$ in S_X, the condition $\|x_n + x\| \to 2$ implies $x_n \to x$. It is very easy to verify the standard fact that if the norm of X is LUR, then every point of S_X is a strongly exposed point for B_X.

Consequently, by following the same pattern as in the proof of Theorem 4.4.2, we may conclude the following result.

Proposition 4.4.4 ([HKR••, Proposition 4.13]). *Let X be an infinite-dimensional Banach space and let $\kappa \leqslant w^*\text{-dens}\, X^*$ be a cardinal number with $\mathrm{cf}(\kappa)$ uncountable. If X is LUR, then for some $\varepsilon > 0$ the unit sphere of X contains a symmetrically $(1+\varepsilon)$-separated subset, with cardinality κ.*

Remark 4.4.5. It follows from a well-known renorming result, due to Troyanski and Zizler ([Tro71, Ziz84], also see [DGZ93, Section 7.1]), that every Banach space with a projectional skeleton admits a LUR renorming; we shall not recall the definition of a projectional skeleton here and we refer the reader to [Kub09] or [KKL11, Corollary 17.5] for information on this topic. As a consequence, when we combine this result with our previous proposition, we obtain the existence of a renorming whose unit sphere contains a large symmetrically $(1+\varepsilon)$-separated family, with no need to exploit biorthogonal systems, as we made in Section 4.2.1.

4.4.1 Asymptotically uniformly convex spaces

The last result for this section is dedicated to asymptotically uniformly convex Banach spaces; some basic facts concerning this notion will be discussed in the first part of the section and we shall refer, *e.g.*, to [GKL01, JLPS02, KOS99, Mil71] for more information. The study of separated sequences in those spaces was undertaken by Delpech, [Del10]; our result will, in particular, provide the non-separable counterpart to Deplech's contribution.

Definition 4.4.6. Let X be an infinite-dimensional Banach space. The *modulus of asymptotic uniform convexity* $\overline{\delta}_X$ is given, for $t \geqslant 0$, by

$$\overline{\delta}_X(t) := \inf_{\|x\|=1} \sup_{\dim(X/H)<\infty} \inf_{\substack{h\in H \\ \|h\|\geqslant t}} (\|x+h\|-1).$$

We shall also denote by $\overline{\delta}_X(\cdot, x)$ the *modulus of asymptotic uniform convexity at* x:

$$\overline{\delta}_X(t,x) := \sup_{\dim(X/H)<\infty} \inf_{\substack{h\in H \\ \|h\|\geqslant t}} (\|x+h\|-1).$$

An infinite-dimensional Banach space X is *asymptotically uniformly convex* if $\overline{\delta}_X(t) > 0$ for every $t > 0$.

Let us first note that $\overline{\delta}_X(t,x) \geqslant 0$ for every $t \geqslant 0$ and $x \in S_X$. In fact, we can find a norming functional x^* for x and consider $H = \ker x^*$; of course, $\|x+h\| \geqslant 1$ for each $h \in H$. Hence

$$\overline{\delta}_X(t,x) \geqslant \inf_{\substack{h\in \ker x^* \\ \|h\|\geqslant t}} (\|x+h\|-1) \geqslant 0.$$

Choosing, for each H with $\dim(X/H) < \infty$, $h = 0 \in H$, shows that $\overline{\delta}_X(0, x) = 0$, so $\overline{\delta}_X(0) = 0$ too. It is also obvious that $\overline{\delta}_X$ is a non-decreasing function, and that $\overline{\delta}_X(\cdot, x)$ is non-decreasing for each fixed x; in fact, $\inf_{h \in H, \|h\| \geqslant t}(\|x + h\| - 1)$ clearly increases with t.

One more property which is easily verified (*cf.* [JLPS02, Proposition 2.3.(3)]) is the fact that, for each $t \in [0, 1]$, $\delta_X(t) \leqslant \overline{\delta}_X(t)$, where δ_X denotes the modulus of uniform convexity—which we recall to be given by

$$\delta_X(t) := \inf \left\{ 1 - \frac{\|x + y\|}{2} : x, y \in B_X,\ \|x - y\| \geqslant t \right\}.$$

Lemma 4.4.7. *$\delta_X(t) \leqslant \overline{\delta}_X(t)$ for each $t \in [0, 1]$.*

Proof. Fix $\varepsilon > 0$ and let $x \in S_X$ be such that $\overline{\delta}_X(t, x) < \overline{\delta}_X(t) + \varepsilon$; let us then select a norming functional x^* for x and consider the hyperplane $H = \ker x^*$. It follows that $\inf_{h \in \ker x^*, \|h\| \geqslant t}(\|x + h\| - 1) < \overline{\delta}_X(t) + \varepsilon$, so we can find $y \in \ker x^*$ with $\|y\| \geqslant t$ such that $\|x + y\| < 1 + \overline{\delta}_X(t) + \varepsilon$. Of course, $\|x + \lambda y\| \geqslant \langle x^*, x + \lambda y \rangle = 1$ ($\lambda \in \mathbb{R}$); consequently, the convex function $\mathbb{R} \ni \lambda \mapsto \|x + \lambda y\|$ attains its minimum for $\lambda = 0$ and it is necessarily non-decreasing on $[0, \infty)$. In other words, we can assume without loss of generality that $\|y\| = t$; in particular, $y \in B_X$. Consider now vectors

$$u := \frac{x + y}{\|x + y\|} \quad \text{and} \quad v := u - y = \frac{1}{\|x + y\|} x + \left(1 - \frac{1}{\|x + y\|}\right)(-y);$$

as v is a convex combination of x and $-y$, we have $u, v \in B_X$. Moreover, $\|u - v\| = \|y\| = t$. We infer that

$$\delta_X(t) \leqslant 1 - \frac{1}{2}\|u + v\| \leqslant 1 - \frac{1}{2}\langle x^*, u + v \rangle = 1 - \langle x^*, u \rangle = 1 - \frac{1}{\|x + y\|}$$

$$\leqslant 1 - \frac{1}{1 + \overline{\delta}_X(t) + \varepsilon} = \frac{\overline{\delta}_X(t) + \varepsilon}{1 + \overline{\delta}_X(t) + \varepsilon} \leqslant \overline{\delta}_X(t) + \varepsilon;$$

letting $\varepsilon \to 0^+$ concludes the proof. ∎

The non-unexpected fact that uniformly convex Banach spaces are asymptotically uniformly convex immediately follows. The two notions are however non equivalent, as it is easy to see that ℓ_1 is asymptotically uniformly convex; let us show this by computing the modulus $\overline{\delta}_X$ for the spaces c_0 and ℓ_p, $1 \leqslant p < \infty$.

Example 4.4.8. Let X be either ℓ_p or c_0. Then for every $x \in S_X$ we have

$$\overline{\delta}_X(t, x) := \begin{cases} \sqrt[p]{1 + t^p} - 1 & \text{if } X = \ell_p \\ \max\{1, t\} - 1 & \text{if } X = c_0 \end{cases}$$

Proof. Fix $\varepsilon > 0$ and find an index n such that $\|x\restriction_{[n+1,\infty)}\| < \varepsilon$. Given any finite-codimensional subspace H of X, there is $h \in H$ such that $h\restriction_{[1,n]} = 0$ (since the map $h \mapsto h\restriction_{[1,n]}$ is, of course, not injective); of course we can assume that $\|h\| = t$. For such h we have $\|(x+h)\restriction_{[1,n]}\| = \|x\restriction_{[1,n]}\| \leqslant 1$ and $\|(x+h)\restriction_{[n+1,\infty)}\| \leqslant t+\varepsilon$. Using the behavior of $\|\cdot\|_X$ on disjointly supported vectors (which is the heart of the present argument) we immediately infer that

$$\inf_{\substack{\tilde{h}\in H\\ \|\tilde{h}\|\geqslant t}} (\|x+\tilde{h}\|-1) \leqslant \|x+h\|-1 = \begin{cases} \sqrt[p]{\|(x+h)\restriction_{[1,n]}\|^p + \|(x+h)\restriction_{[n+1,\infty)}\|^p} - 1 & \text{if } X = \ell_p \\ \max\{\|(x+h)\restriction_{[1,n]}\|, \|(x+h)\restriction_{[n+1,\infty)}\|\} - 1 & \text{if } X = c_0 \end{cases}$$

$$\leqslant \begin{cases} \sqrt[p]{1+(t+\varepsilon)^p} - 1 & \text{if } X = \ell_p \\ \max\{1, t+\varepsilon\} - 1 & \text{if } X = c_0. \end{cases}$$

Passing to the supremum over H then gives

$$\overline{\delta}_X(t,x) \leqslant \begin{cases} \sqrt[p]{1+(t+\varepsilon)^p} - 1 & \text{if } X = \ell_p \\ \max\{1, t+\varepsilon\} - 1 & \text{if } X = c_0. \end{cases}$$

and letting $\varepsilon \to 0$ proves the upper bound.

For the lower bound, fix $\varepsilon > 0$ and find n such that, as above, $\|x\restriction_{[n+1,\infty)}\| < \varepsilon$; considering $H := \overline{\operatorname{span}}\{e_j\}_{j=n+1}^{\infty}$ we have that $\overline{\delta}_X(t,x) \geqslant \inf_{h\in H, \|h\|\geqslant t}(\|x+h\|-1)$. But for every $h \in H$ with $\|h\| \geqslant t$ we have $\|(x+h)\restriction_{[1,n]}\| \geqslant 1-\varepsilon$ and $\|(x+h)\restriction_{[n+1,\infty)}\| \geqslant t-\varepsilon$, so that

$$\overline{\delta}_X(t,x) \geqslant \begin{cases} \sqrt[p]{(1-\varepsilon)^p + (t-\varepsilon)^p} - 1 & \text{if } X = \ell_p \\ \max\{1-\varepsilon, t-\varepsilon\} - 1 & \text{if } X = c_0. \end{cases}$$

Finally, letting $\varepsilon \to 0^+$ proves the lower bound and concludes the proof. ■

Remark 4.4.9. Note that the same argument applies to the spaces $c_0(\Gamma)$ and $\ell_p(\Gamma)$ and, more in general, to any sum $\left(\sum_{\gamma\in\Gamma} X_\gamma\right)_{c_0(\Gamma)}$ or $\left(\sum_{\gamma\in\Gamma} X_\gamma\right)_{\ell_p(\Gamma)}$, provided that the spaces X_γ are finite-dimensional.

In the proof of our result we shall need one more property of this modulus, namely that passage to a subspace improves the modulus $\overline{\delta}$ (*cf.* [JLPS02, Proposition 2.3.(2)]). We also present its very simple proof, for the sake of completeness.

Fact 4.4.10. *Let Y be a closed infinite-dimensional subspace of a Banach space X. Then $\overline{\delta}_Y \geqslant \overline{\delta}_X$.*

Proof. Fix any $t \geqslant 0$ and $y \in Y$. If $H \subseteq X$ is such that $\dim(X/H) < \infty$, then also $\dim(Y/(Y\cap H)) < \infty$ (the inclusion $Y \hookrightarrow X$ induces an injection $Y/(Y\cap H) \hookrightarrow X/H$); consequently,

$$\inf_{\substack{h\in H\\ \|h\|\geqslant t}} (\|y+h\|-1) \leqslant \inf_{\substack{h\in H\cap Y\\ \|h\|\geqslant t}} (\|y+h\|-1) \leqslant \overline{\delta}_Y(t,y).$$

Passage to the supremum over H gives $\overline{\delta}_X(t) \leqslant \overline{\delta}_X(t,y) \leqslant \overline{\delta}_Y(t,y)$. We now pass to the infimum over $y \in Y$ and conclude the proof. ■

Theorem 4.4.11 ([HKR••, Theorem 4.15]). *Let X be an infinite-dimensional Banach space and let $d < 1 + \overline{\delta}_X(1)$. Then the unit sphere of X contains a symmetrically d-separated family with cardinality equal to w^*-dens X^*.*

We should perhaps mention that the result is of interest only for asymptotically uniformly convex Banach spaces, or, more generally, whenever $\overline{\delta}_X(1) > 0$. In fact, for $d < 1$, the existence of a symmetrically d-separated subset of S_X, with cardinality $\operatorname{dens} X$, is an immediate consequence of Riesz' lemma.

Proof. Let $\lambda := w^*$-dens X^*. We construct by transfinite induction a long sequence of unit vectors $(x_\alpha)_{\alpha<\lambda} \subseteq S_X$ and a decreasing long sequence $(H_\alpha)_{\alpha<\lambda}$ of infinite-dimensional subspaces of X with $\dim(H_\alpha/H_{\alpha+1}) < \infty$, $\dim(X/H_1) < \infty$ and such that:

(i) $\|x_\alpha + h\| \geqslant d$, for each $h \in S_{H_\alpha}$ and $\alpha < \lambda$;

(ii) $x_\beta \in H_\alpha$, for $\alpha < \beta < \lambda$.

It immediately follows that, for $\alpha < \beta$, $\pm x_\beta \in H_\alpha$, whence $\|x_\alpha \pm x_\beta\| \geqslant d$ and we are done.

We start with an arbitrary $x_1 \in S_X$; since $d - 1 < \overline{\delta}_X(1, x_1)$ there exists a finite-codimensional subspace H_1 of X such that $\inf_{h\in H_1, \|h\|\geqslant 1} \|x_1 + h\| \geqslant d$. In particular, $\|x_1 + h\| \geqslant d$ for every $h \in S_{H_1}$. We now choose arbitrarily $x_2 \in S_{H_1}$ and we proceed by transfinite induction. Assuming to have already found $(x_\alpha)_{\alpha<\gamma}$ and $(H_\alpha)_{\alpha<\gamma}$ for some $\gamma < \lambda$ we consider two cases: if $\gamma = \tilde{\gamma}+1$ is a successor ordinal, we can choose arbitrarily $x_\gamma \in S_{H_{\tilde{\gamma}}}$. According to Fact 4.4.10, we have $d - 1 < \overline{\delta}_X(1) \leqslant \overline{\delta}_{H_{\tilde{\gamma}}}(1) \leqslant \overline{\delta}_{H_{\tilde{\gamma}}}(1, x_\gamma)$; consequently, we may find a finite-codimensional subspace H_γ of $H_{\tilde{\gamma}}$ such that $\|x_\gamma + h\| \geqslant d$, for every $h \in S_{H_\gamma}$.

In the case that γ is a limit ordinal, we first note that $\cap_{\alpha<\gamma} H_\alpha$ is infinite-dimensional. In fact, each $H_{\alpha+1}$ is the intersection of H_α with the kernels of finitely many functionals; therefore, $\cap_{\alpha<\gamma} H_\alpha$ is the intersection of the kernels of at most $|\gamma| < \lambda$ functionals. Consequently, the minimal cardinality of a family of functionals that separates points on $\cap_{\alpha<\gamma} H_\alpha$ is λ and, in particular, $\cap_{\alpha<\gamma} H_\alpha$ is infinite-dimensional. We can therefore choose a norm-one $x_\gamma \in \cap_{\alpha<\gamma} H_\alpha$ and, arguing as above, we find $H_\gamma \subseteq \cap_{\alpha<\gamma} H_\alpha$ such that $\dim(\cap_{\alpha<\gamma} H_\alpha / H_\gamma) < \infty$ and $\|x_\alpha + h\| \geqslant d$ for $h \in S_{H_\gamma}$. This completes the inductive step and consequently the proof. ■

If we combine the above theorem with the inequality $\delta_X(1) \leqslant \overline{\delta}_X(1)$, we arrive at the following particular case to claim (iii) in Theorem 4.1.4.

Corollary 4.4.12. *Let X be a uniformly convex Banach space. Then for every $\varepsilon > 0$, the unit sphere of X contains a symmetrically $(1 + \delta_X(1) - \varepsilon)$-separated family with cardinality* $\operatorname{dens} X$.

4.5 Super-reflexive spaces

This section is dedicated to the investigation of the question under which conditions the unit sphere of a non-separable Banach space contains a $(1 + \varepsilon)$-separated set of cardinality dens X. In light of the results of Section 4.4 it is natural to consider the class of reflexive Banach spaces; in particular Corollary 4.4.3 implies that one sufficient condition is that X is a reflexive Banach space, whose density is a cardinal number with uncountable cofinality.

We shall however note that the cofinality assumption can not be dispensed with, meaning in particular that the conclusion of Corollary 4.4.3 can not be improved. This fact was already observed in [KaKo16, Remark 3.7] with an indication of the proof and we shall present a detailed argument below.

In the same remark the authors also conjecture that the situation may be different under the assumption of super-reflexivity, this conjecture being partially motivated also by our previous Corollary 4.4.12. In the main part of the present section, we shall provide a positive answer to this conjecture, proving part (iii) of Theorem 4.1.4.

Proposition 4.5.1 ([KaKo16, Remark 3.7])**.** *Let $(p_n)_{n=1}^{\infty} \subseteq (1,\infty)$ be an increasing sequence of reals such that $p_n \to \infty$ and consider the Banach space*

$$X := \left(\sum_{n=1}^{\infty} \ell_{p_n}(\omega_n) \right)_{\ell_2}.$$

Then X is a reflexive Banach space with density character ω_ω, whose unit sphere does not contain a $(1+\varepsilon)$-separated subset of cardinality ω_ω ($\varepsilon > 0$).

Proof. The fact that X is reflexive is obvious, X being the ℓ_2 sum of reflexive Banach spaces; equally obvious is that dens $X = \omega_\omega$. As in the proof of Theorem 3.1.21, for $x \in X$, we shall write $x\restriction_{[1,N]}$ for the vector whose first N components are equal to those of x and the successive ones are equal to 0; if $x = (x(n))_{n=1}^{\infty} \in X$, with $x(n) \in \ell_{p_n}(\omega_n)$, we shall also keep the notation $\operatorname{supp}(x) := \{n \in \mathbb{N} \colon x(n) \neq 0\}$.

Assume now, by contradiction, that there exist $\varepsilon > 0$ and a subset $\{x_\alpha\}_{\alpha\in\Gamma}$ of B_X such that $|\Gamma| = \omega_\omega$ and $\|x_\alpha - x_\beta\| \geqslant 1+\varepsilon$ for distinct $\alpha, \beta \in \Gamma$. Up to a small perturbation and, if necessary, replacing ε with $\varepsilon/2$, we may assume without loss of generality that the support of every x_α is a finite set.

Since $p_n \nearrow \infty$, we may find $N \in \mathbb{N}$ such that $2^{1/p_N} < 1+\varepsilon/2$. Moreover, the vectors $x_\alpha\restriction_{[1,N]}$ ($\alpha \in \Gamma$) belong to the Banach space $\left(\sum_{n=1}^{N} \ell_{p_n}(\omega_n)\right)_{\ell_2}$, whose density character equals ω_N; consequently, there exists an uncountable subset Λ of Γ such that $\|x_\alpha\restriction_{[1,N]} - x_\beta\restriction_{[1,N]}\| \leqslant \varepsilon/2$ whenever $\alpha, \beta \in \Lambda$. Therefore, when we consider the vectors $\tilde{x}_\alpha := x_\alpha - x_\alpha\restriction_{[1,N]}$ ($\alpha \in \Lambda$), we obtain $\|\tilde{x}_\alpha - \tilde{x}_\beta\| \geqslant 1+\varepsilon/2$ for distinct $\alpha, \beta \in \Lambda$. Also observe that the vectors $\tilde{x}_\alpha$ satisfy $\|\tilde{x}_\alpha\| \leqslant 1$ and $\operatorname{supp}(\tilde{x}_\alpha) > N$.

Finally, observe that $\{\operatorname{supp}(\tilde{x}_\alpha)\}_{\alpha\in\Lambda}$ is an uncountable collection of finite subsets of $\mathbb{N}$ and consequently, there exists an uncountable subset Λ_0 of Λ such that the supports of every $\tilde{x}_\alpha$ ($\alpha \in \Lambda_0$) are the same finite subset, say F, of $\mathbb{N}$. We therefore conclude that $(\tilde{x}_\alpha)_{\alpha\in\Lambda_0}$ is an uncountable $(1+\varepsilon/2)$-separated subset of the unit ball of $\left(\sum_{n\in F} \ell_{p_n}(\omega_n)\right)_{\ell_2}$; let us also note that $F > N$. On the other hand, Theorem 3.1.21 assures us that

$$K\left(\left(\sum_{n\in F} \ell_{p_n}(\omega_n) \right)_{\ell_2} \right) = \max\{2^{1/p_n} \colon n \in F\} \leqslant 2^{1/p_N} < 1+\varepsilon/2,$$

a contradiction. ■

As a consequence, the mere assumption of reflexivity on a Banach space X is not sufficient for the unit ball of X to contain (for some $\varepsilon > 0$) a $(1+\varepsilon)$-separated subset whose cardinality equals the density character of X. Therefore in the second (and main) part of the present section we shall assume that the Banach space X is super-reflexive and we shall present the proof of Theorem 4.1.4(iii); let us also record formally here the statement to be proved, for convenience of the reader.

Theorem 4.5.2 ([HKR••, Theorem B(iii)])**.** *Let X be a super-reflexive Banach space. Then, for some $\varepsilon > 0$, the unit sphere of X contains a symmetrically $(1+\varepsilon)$-separated set of cardinality* $\operatorname{dens} X$.

Loosely speaking, the argument to be presented is divided in two parts: in the first one, using an idea already present in [KaKo16, Theorem 3.5], we shall exploit the Gurariĭ–James inequality and inject a suitable subspace of the super-reflexive Banach space X into some $\ell_p(\Gamma)$, in a way to map some collection of unit vectors onto the canonical unit vector basis of $\ell_p(\Gamma)$. Once this is achieved, the second part of the argument is a sharpening of the argument in [KaKo16] and consists of a stabilisation argument, similar to the one present in Theorem 3.4.3.

Let us now start the first part of the argument by recalling the Gurariĭ–James inequality, in the formulation given in [Jam72, Theorem 4]. Let us mention that basically the same inequality was proved in [GuGu71] for Banach spaces that are uniformly convex and uniformly smooth; however, at the time of these papers it was not yet known that every super-reflexive Banach space admits a uniformly convex and uniformly smooth renorming.

Theorem 4.5.3 (Gurariĭ–James inequality)**.** *Let X be a super-reflexive Banach space. Then for every constants $K \geqslant 1$, $0 < c < 1/2K$, and $C > 1$ there are exponents $1 < p, q < \infty$ such that for every normalised basic sequence $(e_n)_{n=1}^\infty$ with basis constant at most K*

$$c\left(\sum_{n=1}^{N}|a_n|^p\right)^{1/p} \leqslant \left\|\sum_{n=1}^{N}a_n e_n\right\| \leqslant C\left(\sum_{n=1}^{N}|a_n|^q\right)^{1/q},$$

for every choice of scalars $(a_n)_{n=1}^N$ and $n \in \mathbb{N}$.

For our purposes it will not be important that the constants c and C can be selected to be arbitrarily close to $1/2K$ and 1 respectively; on the other hand, it is going to be crucial their independence from the basic sequence. Let us also observe here the interesting fact that the validity of the above inequality actually characterises super-reflexive Banach spaces, [Jam72, Theorem 6].

As a particular case of the inequality, for every $c < 1/2$ there is an exponent $p \in (1, \infty)$ such that for every monotone basic sequence $(e_n)_{n=1}^\infty$ in X one has, for every $N \in \mathbb{N}$,

$$c\left(\sum_{n=1}^{N}\|e_n\|^p\right)^{1/p} \leqslant \left\|\sum_{n=1}^{N}e_n\right\|.$$

Of course, the Gurariĭ–James inequality is a 'separable' result. The next idea, due to Benyamini and Starbird ([BeSt76, p. 139]), consists in exploiting the present particular case of the inequality to the case where the monotone basic sequence is obtained from different blocks of a projectional resolution of the identity. In this way one obtains a formulation of the inequality, more suited to the non-separable setting.

We therefore proceed to recall a few basic notions concerning projectional resolutions of the identity.

Definition 4.5.4. Let X be a Banach space and denote by $\lambda := \mathrm{dens}(X)$. A *projectional resolution of the identity* (*PRI*, in short) in X is a family $(P_\alpha)_{\alpha\leqslant\lambda}$ of norm-one projections $P_\alpha\colon X\to X$ such that:

(i) $P_0 = 0$ and $P_\lambda = Id_X$;

(ii) $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ for every $\alpha\leqslant\beta\leqslant\lambda$;

(iii) $\mathrm{dens}(P_\alpha X)\leqslant\max\{|\alpha|,\omega\}$ for $\alpha\leqslant\lambda$;

(iv) $P_\beta X = \overline{\cup_{\alpha<\beta}P_\alpha X}$ for limit ordinals $\beta<\lambda$.

According to (ii), $(P_\alpha X)_{\alpha\leqslant\lambda}$ is a well ordered chain of subspaces of X. Observe, moreover, that condition (iv) is equivalent to the requirement that $\lim_{\alpha\to\beta}P_\alpha x = P_\beta x$ ($x\in X$) whenever β is a limit ordinal: in fact, the above equality is obvious for $x\in\cup_{\alpha<\beta}P_\alpha X$, whence (iv) implies it for $x\in P_\beta X$. The result for the generic $x\in X$ follows from this case applied to $P_\beta x$; the converse implication is clear.

The notion of PRI has been introduced by Lindenstrauss [Lin66a], where it is shown that every non-separable reflexive Banach space admits a PRI; in particular, PRI's do exist in non-separable super-reflexive spaces. For more on PRI, we may refer, for example, to [DGZ93, Chapter VI], [FHHMZ10, Chapter 13], [HMVZ08, §3.4], or [KKL11, Chapter 17].

Lemma 4.5.5 ([BeSt76])**.** *Let X be a non-separable, super-reflexive Banach space and $\lambda=\mathrm{dens}(X)$. Then for any $\varepsilon>0$ there exists $p\in(1,\infty)$ such that for every PRI $(P_\alpha)_{\alpha\leqslant\lambda}$*

$$\left(\sum_{\alpha<\lambda}\|(P_{\alpha+1}-P_\alpha)x\|^p\right)^{1/p}\leqslant(2+\varepsilon)\|x\|\qquad(x\in X).$$

In other words, the operator

$$T\colon X\longrightarrow\left(\bigoplus_{\alpha<\lambda}(P_{\alpha+1}-P_\alpha)(X)\right)_{\ell_p(\lambda)}\tag{4.5.1}$$

given by

$$Tx := (P_{\alpha+1}x-P_\alpha x)_{\alpha<\lambda}\qquad(x\in X)$$

has norm at most $2+\varepsilon$.

Proof. We shall start observing that, for every $x \in X$, $(\|P_{\alpha+1}x - P_\alpha x\|)_{\alpha<\lambda} \in c_0(\lambda)$. Indeed, fix $\varepsilon > 0$; if there exists infinitely many α's such that $\|P_{\alpha+1}x - P_\alpha x\| \geqslant \varepsilon$, we may then choose an increasing sequence $(\alpha_n)_{n=1}^\infty$ such that $\|P_{\alpha_n+1}x - P_{\alpha_n}x\| \geqslant \varepsilon$ for every $n \in \mathbb{N}$. On the other hand, letting $\alpha_\infty = \sup \alpha_n = \lim \alpha_n$, we see that $P_{\alpha_n}x \to P_{\alpha_\infty}x$, a contradiction.

It follows that if we let $(\alpha_n)_{n=1}^\infty$ be an increasing enumeration of the indices α with $P_{\alpha+1}x - P_\alpha x \neq 0$ and $\alpha_\infty = \sup \alpha_n$, then $P_{\alpha_1}x = 0$, $P_{\alpha_n+1}x = P_{\alpha_{n+1}}x$ and $P_{\alpha_\infty}x = x$, due to (i). Therefore, as $N \to \infty$,

$$\sum_{n=1}^{N}(P_{\alpha_n+1} - P_\alpha)x = \sum_{n=1}^{N}(P_{\alpha_{n+1}} - P_\alpha)x = P_{\alpha_{N+1}}x - P_{\alpha_1}x \longrightarrow x.$$

Finally, we just observe that $((P_{\alpha_n+1} - P_{\alpha_n})x)_{n=1}^\infty$ is a monotone basic sequence, whence for every $c < 1/2$ there exists $p \in (1,\infty)$ (depending only on c, and not on x) such that for every $N \in \mathbb{N}$

$$c\left(\sum_{n=1}^{N}\|(P_{\alpha_n+1} - P_{\alpha_n})x\|^p\right)^{1/p} \leqslant \left\|\sum_{n=1}^{N}(P_{\alpha_n+1} - P_{\alpha_n})x\right\|,$$

and letting $N \to \infty$ concludes the proof. ■

We are now ready to record in the following proposition–which was already implicitly present in [KaKo16, p. 10]–the conclusion to the first part of the argument.

Proposition 4.5.6 ([HKR••, Proposition 4.17]). *Let X be a super-reflexive space X and let $\lambda = \mathrm{dens}(X)$. Then for every $\varepsilon > 0$ there exist $p \in (1,\infty)$, a closed subspace Y of X with* $\mathrm{dens}(Y) = \lambda$*, and a linear injection $S\colon Y \to \ell_p(\lambda)$ with $\|S\| \leqslant 2+\varepsilon$ that maps some family of unit vectors $(y_\alpha)_{\alpha<\lambda}$ in Y onto the unit vector basis of $\ell_p(\lambda)$.*

Proof. The case where X is separable follows directly from the Gurariĭ–James inequality, applied to a normalised basic sequence with basis constant sufficiently close to 1, with Y being the closed linear span of the basic sequence. Let us consider the case where X is non-separable.

Let us fix $\varepsilon > 0$ and a PRI $(P_\alpha)_{\alpha\leqslant\lambda}$ in X and take the corresponding operator T given by (4.5.1). For every $\alpha < \lambda$ let us select a unit vector y_α in $(P_{\alpha+1} - P_\alpha)(X)$; plainly, $\|Ty_\alpha\| = 1$. Moreover, the closed linear span in the codomain of T of the set $\{Ty_\alpha\colon \alpha < \lambda\}$ is isometric to $\ell_p(\lambda)$, $\{Ty_\alpha\colon \alpha < \lambda\}$ being isometrically equivalent to its canonical basis. Consequently, Y being the closed linear span of $\{y_\alpha\colon \alpha < \lambda\}$ is the desired subspace of X and for S we simply take the restriction of T to Y. ■

Remark 4.5.7. As proved by Hájek, [Háj94], the hypothesis of super-reflexivity cannot be relaxed to mere reflexivity of the space X. In fact, the main result of [Háj94] consists in the construction of a 'long' Tsirelson-like space X with $\mathrm{dens}\, X = \omega_1$ and such that no its non-separable subspace admits a bounded injection into a super-reflexive Banach space.

We shall now enter the second part of the argument. Let us mention that the definition of operator *bounded by a pair*, given in the proof below, is inspired from the analogous definition given in [KaKo16], but the two definitions are actually different. It is precisely this difference that allows to drop the cofinality assumption from the argument (compare with [KaKo16, Lemma 3.4]).

Proposition 4.5.8 ([HKR••, Proposition 4.19]). *Let X be a Banach space and let $\lambda = \operatorname{dens}(X)$. Suppose that there exists a bounded linear injection $S\colon X \to \ell_p(\lambda)$ that maps some collection $(y_\alpha)_{\alpha<\lambda}$ of unit vectors in X onto the standard unit vector basis of $\ell_p(\lambda)$. Then for every $\varepsilon > 0$, S_X contains a symmetrically $(2^{1/p} - \varepsilon)$-separated subset of cardinality λ.*

Proof. We shall start the argument by fixing some notation. For a subset Λ of λ, let us denote X_Λ to be the closed linear span of the set $\{y_\beta\colon \beta \in \Lambda\}$; we shall also set $\varrho_\Lambda = \|S\!\restriction_{X_\Lambda}\|$. As $\|Sy_\alpha\| = \|y_\alpha\| = 1$ whenever $\alpha < \lambda$, we have $\varrho_\Lambda \geqslant 1$.

Consider now the directed set $\mathscr{L}$ comprising subsets of λ whose complements have cardinality less than λ, ordered with the reversed inclusion. We then say that S is *bounded by a pair* (ϖ, ϱ), when $\|S\| \leqslant \varrho$ and $\|S\!\restriction_{X_\Lambda}\| \geqslant \varpi$ for every $\Lambda \in \mathscr{L}$; for example, we plainly have that S is bounded by the pair $(1, \|S\|)$.

Moreover, for every $\Lambda \in \mathscr{L}$, the restriction operator $S\!\restriction_{X_\Lambda}$ is bounded by the pair

$$\left(\inf_{\Upsilon\in\mathscr{L}} \varrho_\Upsilon, \varrho_\Lambda\right).$$

The net $(\varrho_\Lambda)_{\Lambda\in\mathscr{L}}$ is evidently non-increasing, whence $\lim_{\Lambda\in\mathscr{L}} \varrho_\Lambda = \inf_{\Lambda\in\mathscr{L}} \varrho_\Lambda \geqslant 1$. Consequently, we may always replace X with X_Λ for a small enough set $\Lambda \in \mathscr{L}$ and assume that $S\colon X \to \ell_p(\lambda)$ is bounded by a pair (ϖ, ϱ) with $\frac{\varpi}{\varrho}$ as close to 1 as we wish (in which case $\frac{\varpi}{\varrho} < 1$).

Let us now fix arbitrarily $\tilde{\varpi} \in (0, \varpi)$. Since $\|S\| > \tilde{\varpi}$, we may find a unit vector y_1 in X such that $\|Sy_1\| > \tilde{\varpi}$ and Sy_1 is finitely supported. Take a non-zero $\alpha < \lambda$ and assume that we have already found unit vectors y_β ($\beta < \alpha$) in X such that

(i) $\|Sy_\beta\| > \tilde{\varpi}$ ($\beta < \alpha$),

(ii) Sy_β is finitely supported ($\beta < \alpha$),

(iii) $\operatorname{supp} Sy_{\beta_1} \cap \operatorname{supp} Sy_{\beta_2} = \emptyset$ for distinct $\beta_1, \beta_2 < \alpha$.

As the vectors Sy_β are finitely supported, the set

$$\Lambda_\alpha = \bigcup_{\beta<\alpha} \operatorname{supp} Sy_\beta$$

has cardinality $|\Lambda_\alpha| < \lambda$, that is, $\lambda \setminus \Lambda_\alpha \in \mathscr{L}$. Consequently, $\|S\!\restriction_{X_{\lambda\setminus\Lambda_\alpha}}\| \geqslant \varpi > \tilde{\varpi}$ and we may find a unit vector $y_\alpha \in X_{\lambda\setminus\Lambda_\alpha}$ such that $\|Sy_\alpha\| > \tilde{\varpi}$; note that, by construction, $\operatorname{supp} Sy_\alpha$ is disjoint from Λ_α.

It follows that there exists a family $(y_\alpha)_{\alpha<\lambda}$ of unit vectors in X that satisfy conditions (i)–(iii) above. The vectors $(Sy_\alpha)_{\alpha<\lambda}$ being pairwise disjointly supported, for distinct $\alpha, \beta < \lambda$ we have

$$\varrho \cdot \|y_\beta \pm y_\alpha\| \geqslant \|Sy_\beta \pm Sy_\alpha\| = (\|Sy_\beta\|^p + \|Sy_\alpha\|^p)^{1/p} \geqslant \tilde{\varpi} \cdot 2^{1/p}.$$

Since $\frac{\tilde{\varpi}}{\varrho}$ may be chosen to be as close to 1 as we wish, the proof is complete. ■

To conclude, it is plain that the conjunction of the two propositions presented in the section implies Theorem 4.5.2. It is just sufficient to apply Proposition 4.5.8 to the subspace Y obtained in Proposition 4.5.6.

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