

STABILITY FOR THE ACOUSTIC SCATTERING PROBLEM FOR SOUND-HARD SCATTERERS

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ABSTRACT. We study the stability for the direct acoustic scattering problem with sound-hard scatterers with minimal regularity assumptions on the scatterers. The main tool we use for this purpose is the convergence in the sense of Mosco.

We obtain uniform decay estimates for scattered fields and we investigate how a sound-hard screen may be approximated by thin sound-hard obstacles.

1. Introduction. We are interested in the following direct scattering problem. Let us send a time-harmonic acoustic wave across a medium in \mathbb{R}^N , $N \geq 2$. If the medium is homogeneous such an incident wave remains unperturbed, whereas the presence of a scatterer creates a perturbation by producing a scattered wave. The scattered wave is radiating and it satisfies a suitable boundary condition on the boundary of the scatterer, that depends on the nature of the scatterer.

As incident wave we use a time-harmonic plane wave, which is characterized by its field, referred to as the *incident field*. The incident field is given by $u^i(x) = e^{ikx \cdot d}$, $x \in \mathbb{R}^N$, where $k > 0$ is the wavenumber and $d \in \mathbb{S}^{N-1}$ is the direction of propagation. A *scatterer* K in \mathbb{R}^N is a compact subset of \mathbb{R}^N such that $\mathbb{R}^N \setminus K$ is connected. We say that a scatterer is an *obstacle* if K is the closure of its interior, whereas a scatterer is called a *screen* if its interior is empty. If the incident wave encounters a scatterer K , then a scattered wave is created and its corresponding field will be called the *scattered field* and denoted by u^s . The *total field* u of the acoustic wave is given by the sum of the incident field and the scattered field and

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solves the following direct scattering problem

$$(1.1) \quad \begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^N \setminus K \\ u = u^i + u^s & \text{in } \mathbb{R}^N \setminus K \\ B.C. & \text{on } \partial K \\ \lim_{r \rightarrow \infty} r^{(N-1)/2} \left(\frac{\partial u^s}{\partial r} - i k u^s \right) = 0 & r = \|x\|, \end{cases}$$

where the last limit is the *Sommerfeld radiation condition* and corresponds to the fact that the scattered wave is radiating. The boundary condition on the boundary of K depends on the character of the scatterer K . For instance, if K is *sound-soft*, then u satisfies the following homogeneous Dirichlet condition

$$(1.2) \quad u = 0 \quad \text{on } \partial K,$$

whereas if Σ is *sound-hard* we have

$$(1.3) \quad \nabla u \cdot \nu = 0 \quad \text{on } \partial K,$$

that is a homogeneous Neumann condition. Other conditions such as the impedance boundary condition or transmission conditions for penetrable scatterers may occur in the applications.

We recall that the Sommerfeld radiation condition implies that the scattered field has the asymptotic behaviour of an outgoing spherical wave, namely

$$(1.4) \quad u^s(x) = \frac{e^{ik\|x\|}}{\|x\|^{(N-1)/2}} \left\{ u_\infty^s(\hat{x}) + O\left(\frac{1}{\|x\|}\right) \right\},$$

where $\hat{x} = x/\|x\| \in \mathbb{S}^{N-1}$ and u_∞^s is the so-called *far-field pattern* of u^s . In particular, the scattered field satisfies the following decay property for some positive constants E and R

$$(1.5) \quad |u^s(x)| \leq E \|x\|^{-(N-1)/2} \quad \text{for any } x \in \mathbb{R}^N \text{ so that } \|x\| \geq R.$$

We refer to [14] for further details about the direct scattering problem (1.1). See also [5] where the corresponding inverse problems are considered.

Here we wish to find suitable conditions on sound-hard scatterers K that make the scattering problem (1.1) stable with respect to K , that is with respect to variations in the scatterer. Such a problem is of interest since in general solutions to Neumann problems for elliptic equations may not be stable under domain variations as the so-called Neumann sieve case points out, see for instance [11].

We recall that the simpler case of sound-soft scatterers has been treated in [12]. There, under minimal regularity assumptions on the scatterers, stability results for the solutions of the direct scattering problems with respect to the scatterer and uniform decay estimates for the corresponding scattered fields have been obtained.

In order to evaluate distances between scatterers and convergence of scatterers we shall use the Hausdorff distance. In particular, a sequence of bounded open sets Ω_n contained in B_R for some $R > 0$ converges to an open set Ω if their complements in $\overline{B_R}$ converges to the complement of Ω in the Hausdorff distance. The main tool we shall use is the so-called Mosco convergence of the corresponding Sobolev spaces $H^1(\Omega_n)$ to $H^1(\Omega)$. It has been proved that Mosco convergence is essentially equivalent to convergence of solutions of Neumann problems, at least for elliptic equations which are strictly coercive in H^1 . We shall show that Mosco convergence is indeed a sufficient condition also for the convergence of solutions of Neumann problems for the Helmholtz equation, and in particular for scattering problems,

provided a uniform Sobolev type inequality holds true for any Ω_n . These results are contained in Section 2.

Then we are interested in finding classes of scatterers for which we have uniform decay estimates for the scattered fields, that is estimates like (1.5) with constants E and R independent on K . This will be done in Section 3. Again the key point, besides a uniform Sobolev type inequality, is establishing sufficient conditions on scatterers that guarantee, under convergence in the Hausdorff distance, convergence of corresponding Sobolev spaces in the sense of Mosco. Such a problem, motivated by convergence of solutions of Neumann problems, has been extensively studied in the literature.

In dimension 2, the problem is fully solved since Bucur and Varchon gave a necessary and sufficient condition, [2]. The starting point of [2] was the sufficient condition proved by Chambolle and Doveri, [4], which is still a fairly convenient one to use for the applications. In dimension 2 complex analytic techniques, in particular duality arguments, are crucial for obtaining these results.

In dimension 3 and higher, there is a result by Giacomini, [8], where the admissible sets K may be, roughly speaking, described in the following way. We have a finite number of pairwise disjoint closed Lipschitz hypersurfaces, with or without boundary. Then K consists of the union of a finite number of subsets of these Lipschitz hypersurfaces, each of these subsets having in the relative topology a Lipschitz boundary.

We precisely recall these assumptions in dimension 2 and in dimension 3 and higher in Section 3. Moreover we introduce a new sufficient condition, see Theorem 3.9, which is based on a generalization of a class of sets previously introduced in [13]. As in [8], we have a finite number of closed Lipschitz hypersurfaces, with or without boundary, and K consists of the union of a finite number of subsets of these Lipschitz hypersurfaces, each of these subsets having in the relative topology a Lipschitz boundary, in a sense which is slightly less general than that used by Giacomini. On the other hand, the Lipschitz hypersurfaces need not to be pairwise disjoint that is they can intersect. In such a case, we need to control the corresponding angle of intersection. Just as an example of the main advantage we have with this new class, a propeller shape in \mathbb{R}^2 may belong to it but it can not belong to the one defined by Giacomini.

Finally, in Section 4, we show how to approximate a (Lipschitz) screen by a thin obstacle surrounding it. Such a result may ease the numerical computation of the solution to the scattering problem for a screen and it is also related to some issues arising in the so-called cloaking problem, see [10].

The plan of the paper is the following. In Section 2 we deal with the stability for the direct acoustic scattering problem with sound-hard scatterers. We first discuss Mosco convergence and the stability of Neumann problems for the Helmholtz equation, then we treat scattering problems. In Section 3 we define suitable classes of sound-hard scatterers whose corresponding scattered fields satisfy a uniform decay estimate. Finally in Section 4 we show how we can approximate sound-hard screens by thin sound-hard obstacles.

2. Stability of Neumann problems for the Helmholtz equation. For any $x \in \mathbb{R}^N$, $N \geq 2$, we denote $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and $x = (x'', x_{N-1}, x_N) \in \mathbb{R}^{N-2} \times \mathbb{R} \times \mathbb{R}$. For any $s > 0$ and any $x \in \mathbb{R}^N$, $B_s(x)$ denotes the ball contained in \mathbb{R}^N with radius s and center x , whereas $B'_s(x')$ denotes the ball contained in \mathbb{R}^{N-1}

with radius s and center x' . Moreover, $B_s = B_s(0)$ and $B'_s = B'_s(0)$. Finally, for any $E \subset \mathbb{R}^N$, we denote $B_s(E) = \bigcup_{x \in E} B_s(x)$.

2.1. Mosco convergence. We recall that, given $\{A_n\}_{n \in \mathbb{N}}$, a sequence of closed subspaces of a reflexive Banach space X , we denote

$$A' = \{x \in X : x = w\text{-}\lim_{k \rightarrow \infty} x_{n_k}, x_{n_k} \in A_{n_k}\}$$

and

$$A'' = \{x \in X : x = s\text{-}\lim_{n \rightarrow \infty} x_n, x_n \in A_n\}.$$

We note that A' and A'' are subspaces of X , that $A'' \subset A'$, and that A'' is closed. We say that A_n converges, as $n \rightarrow \infty$, to a closed subspace A in the sense of Mosco if $A = A' = A''$. In other words, the following two conditions need to be satisfied.

- i) For any $x \in X$, if there exists a subsequence A_{n_k} and a sequence x_k , $k \in \mathbb{N}$, such that x_k converges weakly to x as $k \rightarrow \infty$ and $x_k \in A_{n_k}$ for any $k \in \mathbb{N}$, then $x \in A$.
- ii) For any $x \in A$, there exists a sequence $x_n \in A_n$, $n \in \mathbb{N}$, converging strongly to x as $n \rightarrow \infty$.

Let $\Omega_1 \Subset \Omega$ be bounded open sets contained in \mathbb{R}^N , $N \geq 2$. We assume that Ω has a Lipschitz boundary. We denote

$$\mathcal{A} = \{K \subset \overline{\Omega_1} : K \text{ is compact}\}.$$

We have that \mathcal{A} is compact with respect to the Hausdorff distance, see for instance [6].

Let us notice that if $K_n \in \mathcal{A}$, $n \in \mathbb{N}$, converges in the Hausdorff distance, as $n \rightarrow \infty$, to $K \in \mathcal{A}$, then we also have $\lim_{n \rightarrow \infty} |K_n \setminus K| = 0$.

Fixed $K \in \mathcal{A}$, we consider the isometric immersion of $H^1(\Omega \setminus K)$ into $L^2(\Omega, \mathbb{R}^{N+1})$ as follows. To each $u \in H^1(\Omega \setminus K)$ we associate the vector $(u, \nabla u)$ with the convention that u and ∇u are extended to zero in K . In such a way we may consider $H^1(\Omega \setminus K)$ as a closed subspace of $L^2(\Omega, \mathbb{R}^{N+1})$.

Given a sequence $\{K_n\}_{n \in \mathbb{N}}$ contained in \mathcal{A} and $K \in \mathcal{A}$, we say that $H^1(\Omega \setminus K_n)$ converges, as $n \rightarrow \infty$, to $H^1(\Omega \setminus K)$ in the sense of Mosco if this holds considering $H^1(\Omega \setminus K_n)$, $n \in \mathbb{N}$, and $H^1(\Omega \setminus K)$ as subspaces of $L^2(\Omega, \mathbb{R}^{N+1})$.

We are interested to find sufficient conditions on K_n , $n \in \mathbb{N}$, and K in order to have that $H^1(\Omega \setminus K_n)$ converges, as $n \rightarrow \infty$, to $H^1(\Omega \setminus K)$ in the sense of Mosco. The following results are present in the literature, under the assumption that K_n converges, as $n \rightarrow \infty$, to K in the Hausdorff distance. For any $N \geq 2$, a sufficient condition on K_n , $n \in \mathbb{N}$, is established in [8]. For $N = 2$, instead, the key assumption is that the number of connected components of K_n , $n \in \mathbb{N}$, is uniformly bounded. Under this assumption, and the convergence in the Hausdorff distance, Bucur and Varchon, [2, 3], proved that the convergence in the sense of Mosco holds if and only if $|\Omega \setminus K_n|$ converges, as $n \rightarrow \infty$, to $|\Omega \setminus K|$. A sufficient condition, and a starting point for the result of Bucur and Varchon, has been given by Chambolle and Doveri, [4]. They proved the convergence in the sense of Mosco provided we have a uniform bound on the number of connected components of ∂K_n and a uniform bound on $\mathcal{H}^1(\partial K_n)$, $n \in \mathbb{N}$.

Let us remark that the convergence in the sense of Mosco is built in such a way that is essentially equivalent to convergence of solutions of Neumann problems in

the following sense. For any $f \in L^2(\Omega)$ and for any $K \in \mathcal{A}$, let $u = u(K, f)$ be the solution to the following problem

$$(2.1) \quad \begin{cases} -\Delta u + u = f & \text{in } \Omega \setminus K \\ \nabla u \cdot \nu = 0 & \text{on } \partial(\Omega \setminus K), \end{cases}$$

that is $u \in H^1(\Omega \setminus K)$ and satisfies

$$\int_{\Omega \setminus K} \nabla u \cdot \nabla \varphi + \int_{\Omega \setminus K} u \varphi = \int_{\Omega \setminus K} f \varphi \quad \text{for any } \varphi \in H^1(\Omega \setminus K).$$

Then the following result holds.

Proposition 2.1. *Let us fix a sequence $\{K_n\}_{n \in \mathbb{N}}$ contained in \mathcal{A} and $K \in \mathcal{A}$.*

If $H^1(\Omega \setminus K_n)$ converges, as $n \rightarrow \infty$, to $H^1(\Omega \setminus K)$ in the sense of Mosco, then for any $f \in L^2(\Omega)$ we have that $u_n = u(K_n, f)$, solution to (2.1) with K replaced by K_n , converges to $u = u(K, f)$, solution to (2.1), where as before the convergence is in the sense of $L^2(\Omega, \mathbb{R}^{N+1})$.

Conversely, if K_n converges, as $n \rightarrow \infty$, to K in the Hausdorff distance and for any $f \in L^2(\Omega)$ we have that $u_n = u(K_n, f)$ converges to $u = u(K, f)$ in $L^2(\Omega, \mathbb{R}^{N+1})$, then $H^1(\Omega \setminus K_n)$ converges to $H^1(\Omega \setminus K)$ in the sense of Mosco.

Proof. See for instance Proposition 3.2 and Remark 3.3 in [3]. □

By Proposition 2.1, taking $f \equiv 1$, we easily infer that the convergence of $|\Omega \setminus K_n|$ to $|\Omega \setminus K|$, as $n \rightarrow \infty$, is a necessary condition for the convergence of $H^1(\Omega \setminus K_n)$ to $H^1(\Omega \setminus K)$ in the sense of Mosco, for any $N \geq 2$. In fact, under the convergence, as $n \rightarrow \infty$, in the sense of Mosco of $H^1(\Omega \setminus K_n)$ to $H^1(\Omega \setminus K)$, we have $\lim_{n \rightarrow \infty} |(\Omega \setminus K) \Delta (\Omega \setminus K_n)| = 0$ or, equivalently, $\lim_{n \rightarrow \infty} |K \Delta K_n| = 0$, which implies that $|\Omega \setminus K_n|$ converges, as $n \rightarrow \infty$, to $|\Omega \setminus K|$.

Moreover, the following important observation will be of use.

Proposition 2.2. *Let $\{K_n\}_{n \in \mathbb{N}}$ be a sequence in \mathcal{A} and let K and $\tilde{K} \in \mathcal{A}$. Let $\tilde{A}_n = H^1(\Omega \setminus \partial K_n)$, $n \in \mathbb{N}$, and $\tilde{A} = H^1(\Omega \setminus \tilde{K})$. Let us assume that, as $n \rightarrow \infty$, \tilde{A}_n converges to \tilde{A} in the sense of Mosco, that ∂K_n converges to \tilde{K} in the Hausdorff distance, and that K_n converges to K in the Hausdorff distance.*

Then $H^1(\Omega \setminus K_n)$ converges, as $n \rightarrow \infty$, to $H^1(\Omega \setminus K)$ in the sense of Mosco.

Proof. We begin by proving condition i) of Mosco convergence. Let K_{n_k} be a subsequence and $(u_k, \nabla u_k) \in H^1(\Omega \setminus K_{n_k})$ be a sequence such that $(u_k, \nabla u_k)$ is converging weakly to (u, V) in $L^2(\Omega, \mathbb{R}^{N+1})$. We need to show that $(u, V) \in H^1(\Omega \setminus K)$. It is easy to infer that $u \in H^1(\Omega \setminus K)$ and that $V = \nabla u$ in $\Omega \setminus K$. Therefore it remains to prove that u and V are zero almost everywhere in K . Since $\partial K \subset \tilde{K} \subset K$, it is sufficient to prove that for $K \setminus \tilde{K} = \overset{\circ}{K} \setminus \tilde{K}$, actually for any ball $B_\delta(x)$, with $\delta > 0$ and such that $B_{2\delta}(x) \subset \overset{\circ}{K} \setminus \tilde{K}$. By the Hausdorff convergence, we have that the intersection of $B_\delta(x)$ with ∂K_{n_k} is empty, for any k large enough. Since $x \in K$, we deduce that $B_\delta(x)$ is contained in K_{n_k} , again for any k large enough. Finally, since u_k is zero in K_{n_k} , we conclude that u is zero in $B_\delta(x)$. An analogous reasoning holds for V , hence $u \in H^1(\Omega \setminus K)$ and the first property of Mosco convergence is proved.

We now consider condition ii) of Mosco convergence. Let us fix $f \in L^2(\Omega)$ such that $f \equiv 0$ in a neighborhood of K . Let $u_n = u(K_n, f)$, $n \in \mathbb{N}$, solution to (2.1) with K replaced by K_n , and $u = u(K, f)$, solution to (2.1). We need to show that u_n converges to u in $L^2(\Omega, \mathbb{R}^{N+1})$.

For any $n \in \mathbb{N}$ large enough, we have that $f \equiv 0$ in K_n , therefore, since u_n is zero inside K_n , we have that $u_n \in H^1(\Omega \setminus \partial K_n)$ and u_n solves (2.1) also with K replaced by ∂K_n .

By the first part of Proposition 2.1 we have that u_n converges in $L^2(\Omega, \mathbb{R}^{N+1})$ to a function \tilde{u} solving (2.1) with K replaced by \tilde{K} . Hence we just need to show that \tilde{u} solves (2.1) also for K , that is $\tilde{u} = u$.

First of all, we show that $\tilde{u} \in H^1(\Omega \setminus K)$, that is we need to show that \tilde{u} , and its gradient, are zero almost everywhere in K . This may be done as in the previous part of this proof.

Then, take $\varphi \in H^1(\Omega \setminus K)$. We prove that φ belongs to $H^1(\Omega \setminus \tilde{K})$. Since $\tilde{K} \subset K$, we have that φ and its gradient are zero in \tilde{K} . We notice that $\Omega \setminus \tilde{K} = (\Omega \setminus K) \cup (\overset{\circ}{K} \setminus \tilde{K})$. Since $\varphi \in H^1(\Omega \setminus K)$ and φ and its gradient are zero in $\overset{\circ}{K} \setminus \tilde{K}$, we infer that $\varphi \in H^1(\Omega \setminus \tilde{K})$. So we notice that

$$\int_{\Omega \setminus K} f\varphi = \int_{\Omega \setminus \tilde{K}} f\varphi = \int_{\Omega \setminus \tilde{K}} \nabla \tilde{u} \cdot \nabla \varphi + \int_{\Omega \setminus \tilde{K}} \tilde{u}\varphi = \int_{\Omega \setminus K} \nabla \tilde{u} \cdot \nabla \varphi + \int_{\Omega \setminus K} \tilde{u}\varphi$$

and consequently $\tilde{u} = u$.

Then the second condition of Mosco convergence follows from the fact that $\{u = u(f, K) : f \in L^2(\Omega) \text{ and } f \equiv 0 \text{ in a neighborhood of } K\}$ is dense in $H^1(\Omega \setminus K)$, see for instance the argument in the proof of Corollary 1 in [4]. \square

We conclude this subsection by pointing out the following two lemmas.

Lemma 2.3. *Let $\{K_n\}_{n \in \mathbb{N}}$ be a sequence in \mathcal{A} converging, as $n \rightarrow \infty$, to $K \in \mathcal{A}$ in the Hausdorff distance. Let $A_n = H^1(\Omega \setminus K_n)$, $n \in \mathbb{N}$, and $A = H^1(\Omega \setminus K)$.*

If $|K \setminus K_n|$ goes to zero as $n \rightarrow \infty$, then we have that $A' \subset A$.

Proof. Let $\{A_{n_k}\}_{k \in \mathbb{N}}$ be a subsequence and let $(u_k, \nabla u_k) \in A_{n_k}$ for any $k \in \mathbb{N}$. We assume that $(u_k, \nabla u_k)$ weakly converges, as $k \rightarrow \infty$, to (v, V) .

For any $x \in \Omega \setminus K$ there exist $r > 0$ and $\bar{k} \in \mathbb{N}$ such that $\overline{B_r(x)} \subset \Omega \setminus K_{n_k}$ for any $k \geq \bar{k}$. It follows immediately that $v \in H^1(B_r(x))$ and that $V = \nabla v$ in $B_r(x)$. Then it is easy to conclude that $v \in H^1(\Omega \setminus K)$ and $V = \nabla v$ in $\Omega \setminus K$.

It remains to prove that v and V are 0 almost everywhere in K . Clearly $(u_k, \nabla u_k)$ are identically equal to zero on $K \cap K_{n_k}$, for any $k \in \mathbb{N}$. Therefore, for any $\varphi \in L^2(K)$ we have

$$\left| \int_K u_k \varphi \right| = \left| \int_{K \setminus K_{n_k}} u_k \varphi \right| \leq \|u_k\|_{L^2(\Omega)} \left(\int_{K \setminus K_{n_k}} \varphi^2 \right)^{1/2},$$

and clearly the right-hand side goes to zero as $k \rightarrow \infty$ since $|K \setminus K_{n_k}|$ goes to zero. Therefore u_k weakly converges in $L^2(K)$, as $k \rightarrow \infty$, to zero, that is $v = 0$ almost everywhere in K . A similar reasoning holds for ∇u_k , so the proof is concluded. \square

Lemma 2.4. *Let $\{K_n\}_{n \in \mathbb{N}}$ be a sequence in \mathcal{A} and let $K \in \mathcal{A}$. Let $A_n = H^1(\Omega \setminus K_n)$, $n \in \mathbb{N}$, and $A = H^1(\Omega \setminus K)$.*

If $K \subset K_n$ for any $n \in \mathbb{N}$ and $|K_n \setminus K|$ goes to zero as $n \rightarrow \infty$, then we have that $A \subset A''$.

Proof. Fixed $u \in H^1(\Omega \setminus K)$, take $u_n = u(1 - \chi_{K_n})$, $n \in \mathbb{N}$. Clearly u_n belongs to $H^1(\Omega \setminus K_n)$ and $(u_n, \nabla u_n)$ strongly converges to $(u, \nabla u)$ in $L^2(\Omega, \mathbb{R}^{N+1})$. \square

Remark 2.5. We note that, in the previous lemma, it would be sufficient to assume for instance that K_n converges, as $n \rightarrow \infty$, to K in the Hausdorff distance. In fact convergence in the Hausdorff distance implies that $|K_n \setminus K|$ goes to zero as $n \rightarrow \infty$.

Corollary 2.6. Let $\{K_n\}_{n \in \mathbb{N}}$ be a sequence in \mathcal{A} converging, as $n \rightarrow \infty$, to $K \in \mathcal{A}$ in the Hausdorff distance. Let $A_n = H^1(\Omega \setminus K_n)$, $n \in \mathbb{N}$, and $A = H^1(\Omega \setminus K)$.

If $K \subset K_n$ for any $n \in \mathbb{N}$, then we have that A_n converges to A , as $n \rightarrow \infty$, in the sense of Mosco.

2.2. Application to stability of Neumann problems. We observe that Mosco convergence is enough to pass to the limit in Neumann problems, at least in a weak sense and under a mild compactness assumption. Let us fix $k > 0$.

Proposition 2.7. Let $\{K_n\}_{n \in \mathbb{N}}$ be a sequence in \mathcal{A} and let $K \in \mathcal{A}$. Let $A_n = H^1(\Omega \setminus K_n)$, $n \in \mathbb{N}$, and $A = H^1(\Omega \setminus K)$. Let us assume that, as $n \rightarrow \infty$, A_n converges to A in the sense of Mosco.

For any $n \in \mathbb{N}$, let $u_n \in H^1(\Omega \setminus K_n)$ solve, in a weak sense,

$$(2.2) \quad \begin{cases} \Delta u_n + k^2 u_n = 0 & \text{in } \Omega \setminus K_n \\ \nabla u_n \cdot \nu = 0 & \text{on } \partial K_n, \end{cases}$$

that is

$$\int_{\Omega \setminus K_n} \nabla u_n \cdot \nabla \varphi - k^2 u_n \varphi = 0$$

for any $\varphi \in H^1(\Omega \setminus K_n)$ such that the support of φ is compactly contained in Ω .

Let us assume that, for a constant C ,

$$(2.3) \quad \|u_n\|_{L^2(\Omega \setminus K_n)} \leq C \quad \text{for any } n \in \mathbb{N}.$$

Then, up to a subsequence, we have that u_n converges weakly in $L^2(\Omega)$ to a function u solving in the same weak sense

$$(2.4) \quad \begin{cases} \Delta u + k^2 u = 0 & \text{in } \Omega \setminus K \\ \nabla u \cdot \nu = 0 & \text{on } \partial K. \end{cases}$$

Proof. Let us assume that, up to a subsequence, we have that u_n is converging weakly to u in $L^2(\Omega)$ as $n \rightarrow \infty$.

Let us fix an open subset D such that D is of class C^1 , and $\Omega_1 \Subset D \Subset \Omega$. Let $r > 0$ be such that $B_r(\partial D) \Subset \Omega \setminus \overline{\Omega_1}$.

By standard regularity estimates, we may infer that there exists a constant C_1 such that

$$\|u_n\|_{C^1(B_{r/2}(\partial D))} \leq C_1 \quad \text{for any } n \in \mathbb{N}.$$

Therefore, since

$$\int_{D \setminus K_n} |\nabla u_n|^2 = k^2 \int_{D \setminus K_n} u_n^2 + \int_{\partial D} (\nabla u_n \cdot \nu) u_n,$$

we conclude that $\{(u_n, \nabla u_n)\}_{n \in \mathbb{N}}$ is uniformly bounded in $L^2(D, \mathbb{R}^{N+1})$, so we may assume that, up to a subsequence, it converges weakly to (v, V) as $n \rightarrow \infty$. Clearly $v = u$, and, by the Mosco convergence, we have that $u \in H^1(D \setminus K)$ and $V = \nabla u$.

Then, take $\varphi \in H^1(\Omega \setminus K)$ such that $\varphi = 0$ outside D . By the Mosco convergence, we can find $\varphi_n \in H^1(\Omega \setminus K_n)$, such that $\varphi_n = 0$ outside D for any $n \in \mathbb{N}$, that converges strongly to φ as $n \rightarrow \infty$. Then, since

$$\int_{\Omega} \nabla u_n \cdot \nabla \varphi_n - k^2 u_n \varphi_n = 0,$$

we are able to pass to the limit and prove that

$$\int_{\Omega} \nabla u \cdot \nabla \varphi - k^2 u \varphi = 0.$$

The proof immediately follows by changing in a suitable way the set D . □

By Proposition 2.7 and Proposition 2.2, we immediately infer this useful modification.

Proposition 2.8. *Let $\{K_n\}_{n \in \mathbb{N}}$ be a sequence in \mathcal{A} and let K and $\tilde{K} \in \mathcal{A}$. Let $\tilde{A}_n = H^1(\Omega \setminus \partial K_n)$, $n \in \mathbb{N}$, and $\tilde{A} = H^1(\Omega \setminus \tilde{K})$. Let us assume that, as $n \rightarrow \infty$, \tilde{A}_n converges to \tilde{A} in the sense of Mosco, that ∂K_n converges to \tilde{K} in the Hausdorff distance, and that K_n converges to K in the Hausdorff distance.*

For any $n \in \mathbb{N}$, let $u_n \in H^1(\Omega \setminus K_n)$ solve (2.2) and let us assume that, for a constant C , (2.3) holds.

Then, up to a subsequence, we have that u_n converges weakly in $L^2(\Omega)$ to a function u solving (2.4).

In certain cases, weak convergence might not be enough. In order to have strong convergence in L^2 , we need to modify the previous propositions in the following way.

Proposition 2.9. *Let $\{K_n\}_{n \in \mathbb{N}}$ be a sequence in \mathcal{A} and let $K \in \mathcal{A}$. Let $A = H^1(\Omega \setminus K)$, and, for any $n \in \mathbb{N}$, $A_n = H^1(\Omega \setminus K_n)$ and $\tilde{A}_n = H^1(\Omega \setminus \partial K_n)$. Let us assume that, as $n \rightarrow \infty$, K_n converges to K in the Hausdorff distance.*

Let us assume that, as $n \rightarrow \infty$, either A_n converges to A in the sense of Mosco, or \tilde{A}_n converges to $\tilde{A} = H^1(\Omega \setminus \tilde{K})$ in the sense of Mosco and ∂K_n converges to \tilde{K} in the Hausdorff distance, for some $\tilde{K} \in \mathcal{A}$.

For any $n \in \mathbb{N}$, let $u_n \in H^1(\Omega \setminus K_n)$ solve (2.2) and assume that, for a constant C ,

$$(2.5) \quad \|(u_n, \nabla u_n)\|_{L^2(\Omega, \mathbb{R}^{N+1})} \leq C \quad \text{for any } n \in \mathbb{N}.$$

Let us further assume that there exist constants $p > 2$ and $C_1 > 0$ such that for any $n \in \mathbb{N}$ we have

$$(2.6) \quad \|v\|_{L^p(\Omega \setminus K_n)} \leq C_1 \|v\|_{H^1(\Omega \setminus K_n)} \quad \text{for any } v \in H^1(\Omega \setminus K_n).$$

Then, up to a subsequence, we have that u_n converges strongly in $L^2(\Omega)$ to a function u solving (2.4).

Proof. Either by Proposition 2.7 or by Proposition 2.8, up to a subsequence, we have that u_n converges weakly in $L^2(\Omega)$ to a function u solving (2.4). By standard regularity estimates, and by the convergence in the Hausdorff distance, we may also assume that u_n converges to u strongly in L^2 on any compact subset of $\Omega \setminus K$.

For any $\varepsilon > 0$, we can find D_ε , a compact subset of $\Omega \setminus K$, such that, denoting $E_\varepsilon = (\Omega \setminus K) \setminus D_\varepsilon$, we have $|E_\varepsilon| \leq \varepsilon$.

Let us notice that, for any $n \in \mathbb{N}$ and any $\varepsilon > 0$, fixed $p > 2$, we have

$$\|u_n\|_{L^2(E_\varepsilon)} \leq \|u_n\|_{L^p(E_\varepsilon \cap (\Omega \setminus K_n))} \varepsilon^{(p-2)/(2p)}.$$

Then, fixed $\varepsilon > 0$ and $n \in \mathbb{N}$ such that $K_n \subset K \cup E_\varepsilon$, we have that

$$\|u_n - u\|_{L^2(\Omega)} \leq \|u\|_{L^2(E_\varepsilon)} + \|u_n\|_{L^2(E_\varepsilon)} + \|u_n\|_{L^2(K \setminus K_n)} + \|u_n - u\|_{L^2(D_\varepsilon)}.$$

By (2.6) and (2.5), we infer that

$$\|u_n - u\|_{L^2(\Omega)} \leq \|u\|_{L^2(E_\varepsilon)} + CC_1(\varepsilon^{(p-2)/(2p)} + |K \setminus K_n|^{(p-2)/(2p)}) + \|u_n - u\|_{L^2(D_\varepsilon)}.$$

Fixed $\delta > 0$, we find $\varepsilon > 0$ and $\bar{n} \in \mathbb{N}$ such that for any $n \geq \bar{n}$ we have $K_n \subset K \cup E_\varepsilon$ and

$$\|u\|_{L^2(E_\varepsilon)} + CC_1(\varepsilon^{(p-2)/(2p)} + |K \setminus K_n|^{(p-2)/(2p)}) \leq \delta/2.$$

Here we have used the fact that Mosco convergence implies that $|K \Delta K_n|$ goes to 0 as $n \rightarrow \infty$. Then by the convergence on compact subsets, there exists $\tilde{n} \geq \bar{n}$ such that for every $n \geq \tilde{n}$ we have $\|u_n - u\|_{L^2(D_\varepsilon)} \leq \delta/2$, therefore for every $n \geq \tilde{n}$ we have

$$\|u_n - u\|_{L^2(\Omega)} \leq \delta$$

and the proof is concluded. □

Remark 2.10. We observe that all the results in this subsection remain valid if we let the wavenumber k depend on n , with the assumption that $k_n \geq 0$ converges to a real number $k_\infty \geq 0$.

We conclude this subsection finding sufficient conditions for the assumption (2.6) to hold. Given \mathcal{C} a fixed cone in \mathbb{R}^N , we say that an open set $D \subset \mathbb{R}^N$ satisfies the *cone condition with cone \mathcal{C}* if for every $x \in D$ there exists a cone $\mathcal{C}(x)$ with vertex in x and congruent to \mathcal{C} such that $\mathcal{C}(x) \subset D$. We remark that by a cone we always mean a bounded not empty open cone. We shall use two different kind of conditions.

Lemma 2.11. *Let \mathcal{C} be a fixed cone in \mathbb{R}^N . Let D be a bounded open set satisfying the cone condition with cone \mathcal{C} .*

Then there exist constants $p > 2$ and $C_1 > 0$ such that

$$(2.7) \quad \|v\|_{L^p(D)} \leq C_1 \|v\|_{H^1(D)} \quad \text{for any } v \in H^1(D).$$

Here p depends on N only and C_1 depends on p, N and the cone \mathcal{C} only.

Proof. See for instance the book by Adams, [1, Theorem 5.4]. □

Remark 2.12. We notice that if D is a bounded open set such that, for some constants $p > 2$ and $C_1 > 0$, (2.7) holds, then the immersion of $H^1(D)$ into $L^2(D)$ is compact.

Lemma 2.13. *Let D be a bounded open set satisfying the following condition. Fix constants $p > 2$ and $\tilde{C} > 0$ and an integer M . We assume that there exist M open subsets D_1, \dots, D_M of D such that $|D \setminus \bigcup_{i=1}^M D_i| = 0$ and*

$$\|v\|_{L^p(D_i)} \leq \tilde{C} \|v\|_{H^1(D_i)} \quad \text{for any } v \in H^1(D_i) \text{ and any } i = 1, \dots, M.$$

Then there exists a constant $C_1 > 0$ such that

$$\|v\|_{L^p(D)} \leq C_1 \|v\|_{H^1(D)} \quad \text{for any } v \in H^1(D).$$

Here C_1 depends on \tilde{C} and M only.

Proof. Let $v \in H^1(D)$. We have that

$$\|v\|_{L^p(D)}^p \leq \sum_{i=1}^M \|v\|_{L^p(D_i)}^p \leq \sum_{i=1}^M \tilde{C}^p \|v\|_{H^1(D_i)}^p \leq M \tilde{C}^p \|v\|_{H^1(D)}^p,$$

therefore it is enough to choose $C_1 = M^{1/p} \tilde{C}$. □

2.3. The scattering case. Let us consider the following scattering problem. Let $\{K_n\}_{n \in \mathbb{N}}$ be a sequence of compact sets contained in $\overline{B_R}$ for some $R > 0$. Let, for any $n \in \mathbb{N}$, $A_n = H^1(B_{R+1} \setminus K_n)$ and $\tilde{A}_n = H^1(B_{R+1} \setminus \partial K_n)$.

Let us assume that $\mathbb{R}^N \setminus K_n$ is connected and that the immersion of $H^1(B_{R+1} \setminus K_n)$ into $L^2(B_{R+1} \setminus K_n)$ is compact for any $n \in \mathbb{N}$. Let u^i be an entire solution of the Helmholtz equation in \mathbb{R}^N . Let us remark that here and in what follows we may replace, with the obvious modifications, u^i with a solution of the Helmholtz equation in $\mathbb{R}^N \setminus \{x_0\}$, with $\|x_0\| > R + 1$, and we may also replace $R + 1$ with any $R_1 > R$. Then for any $n \in \mathbb{N}$ there exists a unique weak solution to the following scattering problem

$$(2.8) \quad \begin{cases} \Delta u_n + k^2 u_n = 0 & \text{in } \mathbb{R}^N \setminus K_n \\ u_n = u^i + u_n^s & \text{in } \mathbb{R}^N \setminus K_n \\ \nabla u_n \cdot \nu = 0 & \text{on } \partial K_n \\ \lim_{r \rightarrow +\infty} r^{(N-1)/2} \left(\frac{\partial u_n^s}{\partial r} - i k u_n^s \right) = 0 & r = \|x\|. \end{cases}$$

For existence and uniqueness see for instance [14].

We assume that K_n converges, as $n \rightarrow \infty$, to $K \subset \overline{B_R}$ in the Hausdorff distance, where K is compact and such that $\mathbb{R}^N \setminus K$ is connected.

We also assume that, as $n \rightarrow \infty$, either A_n converges to $A = H^1(B_{R+1} \setminus K)$ in the sense of Mosco, or \tilde{A}_n converges to $\tilde{A} = H^1(B_{R+1} \setminus \tilde{K})$ in the sense of Mosco and ∂K_n converges to \tilde{K} in the Hausdorff distance, for some $\tilde{K} \subset \overline{B_R}$, \tilde{K} compact.

Let us further assume that there exist constants $p > 2$ and $C_1 > 0$ such that for any $n \in \mathbb{N}$ we have

$$\|v\|_{L^p(B_{R+1} \setminus K_n)} \leq C_1 \|v\|_{H^1(B_{R+1} \setminus K_n)} \quad \text{for any } v \in H^1(B_{R+1} \setminus K_n).$$

We begin with the following lemma.

Lemma 2.14. *Let us assume that, for some positive constant C , we have*

$$(2.9) \quad \|u_n\|_{L^2(B_{R+1})} \leq C \quad \text{for any } n \in \mathbb{N}.$$

Then u_n converges to a function u strongly in $L^2(B_r)$ for any $r > 0$, with u solving

$$(2.10) \quad \begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^N \setminus K \\ u = u^i + u^s & \text{in } \mathbb{R}^N \setminus K \\ \nabla u \cdot \nu = 0 & \text{on } \partial K \\ \lim_{r \rightarrow +\infty} r^{(N-1)/2} \left(\frac{\partial u^s}{\partial r} - i k u^s \right) = 0 & r = \|x\|. \end{cases}$$

Proof. By Lemma 3.1 in [12], we have that, up to a subsequence, u_n converges to a function u uniformly on compact subsets of $\mathbb{R}^N \setminus K$, with u solving

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^N \setminus K \\ u = u^i + u^s & \text{in } \mathbb{R}^N \setminus K \\ \lim_{r \rightarrow +\infty} r^{(N-1)/2} \left(\frac{\partial u^s}{\partial r} - i k u^s \right) = 0 & r = \|x\|. \end{cases}$$

By Proposition 2.7 or Proposition 2.8, without loss of generality we may assume that u_n converges to u weakly in $L^2(B_{R+1})$ and u also satisfies the boundary condition

$$\nabla u \cdot \nu = 0 \quad \text{on } \partial K,$$

that is u solves (2.10). Since (2.10) has at most one solution, we have that the whole sequence u_n converges to u uniformly on compact subsets of $\mathbb{R}^N \setminus K$.

It remains to prove that u_n converges to u in $L^2(B_r)$ for a fixed r , $R < r < R + 1$. By the reasonings used in the proof of Proposition 2.7, we have that there exists a constant $\tilde{C} > 0$ such that

$$(2.11) \quad \|(u_n, \nabla u_n)\|_{L^2(B_r, \mathbb{R}^{N+1})} \leq \tilde{C} \quad \text{for any } n \in \mathbb{N}.$$

Then we can conclude the proof by using Proposition 2.9. □

In the next proposition we wish to drop the assumption (2.9).

Proposition 2.15. *Let $\{K_n\}_{n \in \mathbb{N}}$ be a sequence of compact sets contained in $\overline{B_R}$ for some $R > 0$. Let us assume that $\mathbb{R}^N \setminus K_n$ is connected.*

We assume that K_n converges, as $n \rightarrow \infty$, to $K \subset \overline{B_R}$ in the Hausdorff distance, where K is compact and such that $\mathbb{R}^N \setminus K$ is connected.

We also assume that, as $n \rightarrow \infty$, either A_n converges to $A = H^1(B_{R+1} \setminus K)$ in the sense of Mosco, or \tilde{A}_n converges to $\tilde{A} = H^1(B_{R+1} \setminus \tilde{K})$ in the sense of Mosco and ∂K_n converges to \tilde{K} in the Hausdorff distance, for some $\tilde{K} \subset \overline{B_R}$, \tilde{K} compact.

Let us further assume that there exist constants $p > 2$ and $C_1 > 0$ such that for any $n \in \mathbb{N}$ we have

$$\|v\|_{L^p(B_{R+1} \setminus K_n)} \leq C_1 \|v\|_{H^1(B_{R+1} \setminus K_n)} \quad \text{for any } v \in H^1(B_{R+1} \setminus K_n).$$

Let u_n solve (2.8), for any $n \in \mathbb{N}$. Then u_n converges to a function u strongly in $L^2(B_r)$ for any $r > 0$, with u solving (2.10).

Proof. Let $a_n = \|u_n\|_{L^2(B_{R+1})}$. If $\{a_n\}_{n \in \mathbb{N}}$ is bounded, then the conclusion follows from the previous lemma.

By contradiction, let us assume that $\lim_{n \rightarrow \infty} a_n = +\infty$, possibly by passing to a subsequence. Let us consider $v_n = u_n/a_n$. We have that

$$\|v_n\|_{L^2(B_{R+1})} = 1.$$

Therefore v_n , up to a subsequence, converges to a function v strongly in L^2 on any compact subset of \mathbb{R}^N . The function v satisfies

$$(2.12) \quad \begin{cases} \Delta v + k^2 v = 0 & \text{in } \mathbb{R}^N \setminus K \\ \nabla v \cdot \nu = 0 & \text{on } \partial K. \end{cases}$$

Clearly we also have that $\|v\|_{L^2(B_{R+1})} = 1$.

On the other hand, we have that $\|u_n^s/a_n\|_{L^2(B_{R+1})}$, $n \in \mathbb{N}$, is uniformly bounded. Therefore, again up to a subsequence, u_n^s/a_n converges, as $n \rightarrow \infty$, to a function w strongly in L^2 on any compact subset of $\mathbb{R}^N \setminus K$. Such a function w satisfies

$$(2.13) \quad \begin{cases} \Delta w + k^2 w = 0 & \text{in } \mathbb{R}^N \setminus K \\ \lim_{r \rightarrow +\infty} r^{(N-1)/2} \left(\frac{\partial w}{\partial r} - ikw \right) = 0 & r = \|x\|. \end{cases}$$

Since $v_n = u^i/a_n + u_n^s/a_n$, we may immediately conclude that, outside K , we have $v = w$. That is v solves

$$(2.14) \quad \begin{cases} \Delta v + k^2 v = 0 & \text{in } \mathbb{R}^N \setminus K \\ \nabla v \cdot \nu = 0 & \text{on } \partial K \\ \lim_{r \rightarrow +\infty} r^{(N-1)/2} \left(\frac{\partial v}{\partial r} - ikv \right) = 0 & r = \|x\|. \end{cases}$$

By uniqueness, and the fact that $\mathbb{R}^N \setminus K$ is connected, we may conclude that v is identically zero, which leads to a contradiction since $\|v\|_{L^2(B_{R+1})}$ should be equal to 1. □

Remark 2.16. We observe that all the results in this subsection remain valid if we let the incident field u^i depend on n . In particular we may have that the wavenumber k and the direction of propagation d depend on n , with the assumption that $k_n > 0$ converges to a real number $k_\infty > 0$ and d_n converges to a direction d_∞ , therefore u_n^i converges to the incident field $u_\infty^i(x) = e^{ik_\infty x \cdot d_\infty}$, $x \in \mathbb{R}^N$.

We further notice that, if $N \geq 3$, then we may allow k_∞ to be even equal to zero. In this case we need to replace the Sommerfeld radiation condition with the following condition at infinity for the scattered field

$$u^s = o(1) \quad \text{as } r = \|x\| \rightarrow +\infty.$$

We refer to [12, Lemma 3.1] for further details. For $N = 2$, instead, we need to require $k_\infty > 0$, see for instance the discussion in [9] for the low wavenumber asymptotics of scattering problems in dimension 2.

3. Uniform decay property for scattered fields. We begin by defining suitable classes of admissible scatterers.

Definition 3.1. We say that $\tilde{\mathcal{A}}$ is an *admissible class of scatterers* in \mathbb{R}^N if the following properties holds.

- i) There exists a constant $R > 0$ such that any $K \in \tilde{\mathcal{A}}$ is a compact set contained in $\overline{B_R} \subset \mathbb{R}^N$.
- ii) For any $K \in \tilde{\mathcal{A}}$, we have that $\mathbb{R}^N \setminus K$ is connected. Furthermore, if K is the limit in the Hausdorff distance of a sequence of elements of the class $\tilde{\mathcal{A}}$, then $\mathbb{R}^N \setminus K$ is connected.
- iii) $\tilde{\mathcal{A}}$ satisfies one of the two following conditions. Either for any sequence $K_n \in \tilde{\mathcal{A}}$, $n \in \mathbb{N}$, converging in the Hausdorff distance to K , we have that $H^1(B_{R+1} \setminus K_n)$ converges to $H^1(B_{R+1} \setminus K)$ in the sense of Mosco. Or for any sequence $K_n \in \tilde{\mathcal{A}}$, $n \in \mathbb{N}$, such that ∂K_n converges in the Hausdorff distance to \tilde{K} , we have that $H^1(B_{R+1} \setminus \partial K)$ converges to $H^1(B_{R+1} \setminus \tilde{K})$ in the sense of Mosco.
- iv) There exist constants $p > 2$ and $C_1 > 0$ such that for any $K \in \tilde{\mathcal{A}}$ we have

$$\|v\|_{L^p(B_{R+1} \setminus K)} \leq C_1 \|v\|_{H^1(B_{R+1} \setminus K)} \quad \text{for any } v \in H^1(B_{R+1} \setminus K).$$

Let us then fix an admissible class of scatterers $\tilde{\mathcal{A}}$. By condition ii), $\mathbb{R}^N \setminus K$ is connected, for any $K \in \tilde{\mathcal{A}}$. We also recall that condition iv) above implies that the immersion of $H^1(B_{R+1} \setminus K)$ into $L^2(B_{R+1} \setminus K)$ is compact, for any $K \in \tilde{\mathcal{A}}$.

Let us fix constants $0 < \underline{k} < \bar{k}$ and let us denote, for any $N \geq 2$,

$$(3.1) \quad I_N = \begin{cases} [\underline{k}, \bar{k}] & \text{if } N = 2, \\ (0, \bar{k}] & \text{if } N \geq 3. \end{cases}$$

Fixed $K \in \tilde{\mathcal{A}}$, for a fixed wavenumber $k \in I_N$ and a fixed direction of propagation $d \in \mathbb{S}^{N-1}$, let the incident field u^i be the corresponding plane wave, that is $u^i(x) = e^{ikx \cdot d}$, $x \in \mathbb{R}^N$. Then, let $u_{K,k,d}$ be the solution to (2.10) and $u_{K,k,d}^s$ be its corresponding scattered field.

We begin by stating the following uniform boundedness of solutions.

Proposition 3.2. *Let us fix constants $0 < \underline{k} < \bar{k}$ and let I_N be defined as in (3.1). Let $\tilde{\mathcal{A}}$ be an admissible class of scatterers in \mathbb{R}^N .*

Fixed $K \in \tilde{\mathcal{A}}$, $k \in I_N$, and $d \in \mathbb{S}^{N-1}$, let $u^i(x) = e^{ikx \cdot d}$, $x \in \mathbb{R}^N$, $u_{K,k,d}$ be the solution to (2.10) and $u_{K,k,d}^s$ be its corresponding scattered field.

Then there exists a constant E , depending on $\tilde{\mathcal{A}}$ and I_N only, such that

$$(3.2) \quad \|u_{K,k,d}\|_{L^2(B_{R+1})} \leq E \quad \text{for any } K \in \tilde{\mathcal{A}}, \text{ any } k \in I_N, \text{ and any } d \in \mathbb{S}^{N-1}.$$

Proof. We sketch the proof. We argue by contradiction. Let us assume that there exist, for any $n \in \mathbb{N}$, $K_n \in \tilde{\mathcal{A}}$, $k_n \in I_N$, and $d_n \in \mathbb{S}^{N-1}$ such that

$$\|u_{K_n,k_n,d_n}\|_{L^2(B_{R+1})} \geq n.$$

Then we define $v_n = u_{K_n,k_n,d_n} / \|u_{K_n,k_n,d_n}\|_{L^2(B_{R+1})}$, $n \in \mathbb{N}$.

Then we obtain a contradiction arguing in an analogous manner as in Proposition 2.15 and using Remark 2.16. \square

As a corollary, as in Lemma 3.2 in [12], we may prove the following uniform decay property.

Corollary 3.3. *Under the same assumptions and notation of Proposition 3.2, there exists a constant E_1 , depending on the constant E in (3.2), I_N , R and N only, such that for any $K \in \tilde{\mathcal{A}}$, any $k \in I_N$, and any $d \in \mathbb{S}^{N-1}$ we have*

$$|u_{K,k,d}^s(x)| \leq E_1 \|x\|^{-(N-1)/2} \quad \text{for any } x \in \mathbb{R}^N \text{ so that } \|x\| \geq R + 2.$$

In the remaining part of this section we wish to discuss Assumptions i)-ii)-iii)-iv) defined above.

Assumption i) is self-explanatory. We notice however that it guarantees that $\tilde{\mathcal{A}}$ is relatively compact with respect to the Hausdorff distance.

A sufficient condition for Assumption ii) to hold is the following, [12, Lemma 2.5].

Assumption A (uniform exterior connectedness). *Let $\delta : (0, +\infty) \rightarrow (0, +\infty)$ be a nondecreasing left-continuous function. We assume that for any $K \in \tilde{\mathcal{A}}$, for any $t > 0$, for any two points $x_1, x_2 \in \mathbb{R}^N$ so that $B_t(x_1)$ and $B_t(x_2)$ are contained in $\mathbb{R}^N \setminus K$, and for any $s, 0 < s < \delta(t)$, then we can find a smooth (for instance C^1) curve γ connecting x_1 to x_2 so that $B_s(\gamma)$ is contained in $\mathbb{R}^N \setminus K$ as well.*

Let us notice that such an Assumption A is closed under convergence in the Hausdorff distance.

Assumption iv) has been already discussed in Lemma 2.11 and Lemma 2.13. Let us further notice that such an assumption is closed under convergence in the sense of Mosco of the corresponding Sobolev spaces. In particular, it holds for any K which is the limit in the Hausdorff distance of a sequence of elements of the class $\tilde{\mathcal{A}}$, provided Assumptions i) and iii) hold true, possibly by using Proposition 2.2.

About Assumption iii) we state three sufficient conditions. The first one is proved in [4] and it holds only for $N = 2$, the second is taken from [8] and it is valid in any dimension $N \geq 2$. The third one is new, and it will be proven here, and it uses a class which is a generalization of one developed in [13]. Let us finally remark that in what follows Assumption i) will be always tacitly assumed.

Assumption B [4]. *Let us assume that $N = 2$. Let us assume that there exist constants $M \in \mathbb{N}$ and $C > 0$ such that for any $K \in \tilde{\mathcal{A}}$ we have that the number of connected components of ∂K is bounded by M and $\mathcal{H}^1(\partial K) \leq C$.*

We notice that, under Assumption B, for any $K \in \tilde{\mathcal{A}}$ we have that the number of connected components of K is bounded by M . Moreover, if K is the limit in the Hausdorff distance of a sequence K_n , $n \in \mathbb{N}$, of sets belonging to $\tilde{\mathcal{A}}$, we have that the number of connected components of K is bounded by M . Furthermore, without loss of generality, we may assume that ∂K_n converges, in the Hausdorff distance, to

a compact set \tilde{K} . Since $\partial K \subset \tilde{K} \subset K$, we deduce by a general version of Golab's Theorem, see for instance Corollary 3.3 in [7], that $H^1(\partial K) \leq H^1(\tilde{K}) \leq C$ as well. We finally point out that the number of connected components of \tilde{K} is bounded by M , whereas the same may not be true for ∂K . However, if Assumption ii) holds, then $\mathbb{R}^2 \setminus K$ is connected, therefore, by the next Lemma 3.4, K and ∂K have the same number of connected components, so also the number of connected components of ∂K is bounded by M . In other words, if Assumption ii) holds, then Assumption B is closed under convergence in the Hausdorff distance.

Lemma 3.4. *Let us assume that K is a compact subset of \mathbb{R}^2 such that $\mathbb{R}^2 \setminus K$ is connected. Then K and ∂K have the same number of connected components,*

Proof. Without loss of generality we can assume that K is connected and we prove that ∂K is connected as well. The result is trivial if K is smooth enough, for instance it is the closure of a smooth connected open set. We sketch the approximation argument which is needed for a nonsmooth set K .

By contradiction, let ∂K be the union of A_1 and A_2 , two closed, not empty, disjoint sets. Let $d_0 > 0$ be such that $B_{d_0}(A_1) \cap B_{d_0}(A_2) = \emptyset$.

We can construct a sequence of sets $K_n, n \in \mathbb{N}$, such that for any $n \in \mathbb{N}$ K_n is the closure of a C^∞ open sets and

$$K \subset \overset{\circ}{K}_{n+1} \subset K_{n+1} \subset \overset{\circ}{K}_n \subset K_n \subset B_{1/n}(K).$$

We have that $\partial K_n = \bigcup_{j=1}^{m_n} \gamma_j^n$ where γ_j^n are smooth pairwise disjoint Jordan curves. By the Jordan curve theorem, we set D_j^n the open set bounded by γ_j^n .

We have that for any j there exists $\delta > 0$ such that one between $B_\delta(\gamma_j^n) \cap D_j^n$ and $B_\delta(\gamma_j^n) \setminus \overline{D_j^n}$ has empty intersection with K_n . Without loss of generality we can assume that $K \setminus \overline{D_j^n}$ or, respectively, $K \cap D_j^n$ is not empty.

Let now fix x_1 and x_2 and a smooth curve α connecting them such that for any $i = 1, 2$, $\text{dist}(x_i, A_i) < d_0/4$ and $\gamma \subset \mathbb{R}^2 \setminus K$. For any n large enough and any $i = 1, 2$, $x_i \notin K_n$ and we can find $\gamma_{j_i}^n$ such that $\text{dist}(x_i, \gamma_{j_i}^n) < d_0/4$ and $\gamma_{j_i}^n \subset B_{1/n}(A_i)$. Also, $\alpha \cap K_n = \emptyset$.

It remains to analyse the following cases. For each $i = 1, 2$ we can have that either x_i belongs to $D_{j_i}^n$ or to its complement. Furthermore, either $D_{j_1}^n$ and $D_{j_2}^n$ are disjoint or one of them is contained in the other. We thus obtain 8 different cases, each of them implying that either α intersects K_n or K is not connected, thus a contradiction. □

Assumption C [8]. *There exist a fixed cone \mathcal{C} in \mathbb{R}^{N-1} and positive constants δ, L_1 and L_2 such that any $K \in \tilde{\mathcal{A}}$ satisfies the following condition.*

For any $x \in \partial K$ there exists a bi-Lipschitz function $\Phi_x : B_\delta(x) \rightarrow \mathbb{R}^N$ such that

C1) *for any $z_1, z_2 \in B_\delta(x)$ we have*

$$L_1 \|z_1 - z_2\| \leq \|\Phi_x(z_1) - \Phi_x(z_2)\| \leq L_2 \|z_1 - z_2\|;$$

C2) $\Phi_x(x) = 0$ and $\Phi_x(\partial K \cap B_\delta(x)) \subset \pi = \{y \in \mathbb{R}^N : y_N = 0\}$;

C3) *for any $y \in \partial K \cap B_{\delta/2}(x)$ we have*

$$\Phi_x(y) \in \overline{\mathcal{C}_y} \subset \Phi_x(\partial K \cap B_\delta(x)),$$

where \mathcal{C}_y is obtained by the cone \mathcal{C} through a rigid change of coordinates.

It is not difficult to see that Assumption C is closed under convergence in the Hausdorff distance. With the following lemma we show that Assumption C guarantees that not only Assumption iii) is satisfied but also Assumption iv) is.

Lemma 3.5. *Let us assume that \tilde{A} satisfies Assumption C. Then there exist constants $p > 2$ and $C_1 > 0$ such that for any $K \in \tilde{A}$ we have*

$$\|v\|_{L^p(B_{R+1} \setminus \partial K)} \leq C_1 \|v\|_{H^1(B_{R+1} \setminus \partial K)} \quad \text{for any } v \in H^1(B_{R+1} \setminus \partial K),$$

hence the same property is satisfied by $B_{R+1} \setminus K$.

Proof. We sketch the proof. Without loss of generality, let δ be such that $0 < \delta \leq 1/2$. We can find positive constants δ_1, δ_2 such that for any $x \in \partial K$ we have that $B_{\delta_2} \subset \Phi_x(B_\delta(x))$ and $B_{\delta_1}(x) \subset \Phi_x^{-1}(B_{\delta_2})$. Clearly δ_1 and δ_2 depend on δ, L_1 and L_2 only.

Then we obtain that $\Phi_x^{-1}(B_{\delta_2}) \setminus \partial K$ is contained, up to a set of measure zero, in the set $U_x = \Phi_x^{-1}(B_{\delta_2} \setminus \pi)$. Since U_x is the image through a bi-Lipschitz map of $B_{\delta_2} \setminus \pi$, we have that U_x satisfies, for some $p > 2$ and $C > 0$ depending on δ, L_1 and L_2 only, the following Sobolev inequality

$$\|v\|_{L^p(U_x)} \leq C \|v\|_{H^1(U_x)} \quad \text{for any } v \in H^1(U_x).$$

We have that $\overline{B_{\delta_1/4}(\partial K)}$ is contained in $\bigcup_{x \in \overline{B_{\delta_1/4}(\partial K)}} B_{\delta_1/4}(x)$. We can find a finite number of points $z_i \in \overline{B_{\delta_1/4}(\partial K)}$, $i = 1, \dots, m_1$, such that $\overline{B_{\delta_1/4}(\partial K)} \subset \bigcup_{i=1}^{m_1} B_{\delta_1/4}(z_i)$. With a simple construction, it is possible to choose m_1 depending on δ_1 and R only, for instance by taking points such that $B_{\delta_1/8}(z_i) \cap B_{\delta_1/8}(z_j)$ is empty for $i \neq j$. Then, for any $i = 1, \dots, m_1$ we can find $x_i \in \partial K$ such that $B_{\delta_1/4}(z_i) \subset B_{\delta_1}(x_i)$, therefore $\overline{B_{\delta_1/4}(\partial K)} \subset \bigcup_{i=1}^{m_1} B_{\delta_1}(x_i)$.

Then consider the set $A = \overline{B_{R+1/2} \setminus B_{\delta_1/4}(\partial K)}$. Again, we can find points $y_j \in A$, $j = 1, \dots, m_2$, such that $A \subset \bigcup_{j=1}^{m_2} B_{\delta_1/4}(y_j)$. Again, with the same kind of construction, it is possible to choose m_2 depending on δ_1 and R only. We notice that $B_{\delta_1/4}(y_j) \subset B_{R+1} \setminus \partial K$ for any $j = 1, \dots, m_2$.

Therefore, $B_{R+1} \setminus K$ is contained, up to a set of measure zero, in the following union

$$(B_{R+1} \setminus \overline{B_{R+1/2}}) \cup \left(\bigcup_{i=1}^{m_1} U_{x_i} \right) \cup \left(\bigcup_{j=1}^{m_2} B_{\delta_1/4}(y_j) \right).$$

The conclusion immediately follows by Lemma 2.13. □

The third sufficient condition is a generalization of arguments developed in [13]. We need to fix some preliminary notation and prove some lemmas.

Let us fix a bounded open set Ω . Let $K \subset \overline{\Omega}$ be a compact subset of \mathbb{R}^N . We say that K is a *Lipschitz hypersurface*, with or without boundary, with positive constants r and L if the following holds.

For any $x \in K$ there exists a bi-Lipschitz function $\Phi_x : B_r(x) \rightarrow \mathbb{R}^N$ such that

a) for any $z_1, z_2 \in B_r(x)$ we have

$$L^{-1} \|z_1 - z_2\| \leq \|\Phi_x(z_1) - \Phi_x(z_2)\| \leq L \|z_1 - z_2\|;$$

b) $\Phi_x(x) = 0$ and $\Phi_x(K \cap B_r(x)) \subset \pi = \{y \in \mathbb{R}^N : y_N = 0\}$.

We say that $x \in K$ belongs to the interior of K if there exists δ , $0 < \delta \leq r$, such that $B_\delta(0) \cap \pi \subset \Phi_x(K \cap B_r(x))$. Otherwise we say that x belongs to the boundary of K . We remark that the boundary of K might be empty. Further we assume that

c) for any x belonging to the boundary of K , we have that

$$\Phi_x(K \cap B_r(x)) = \Phi_x(B_r(x)) \cap \pi^+$$

where $\pi^+ = \{y \in \mathbb{R}^N : y_N = 0, y_{N-1} \geq 0\}$.

Let us notice that, by compactness, such an assumption is enough to guarantee that $\mathcal{H}^{N-1}(K)$ is bounded, hence $|K| = 0$. In particular, if $\Omega \subset \overline{B_R}$ for some $R > 0$, then $\mathcal{H}^{N-1}(K)$ is bounded by a constant depending on R , r and L only. Furthermore, the boundary of K has \mathcal{H}^{N-2} measure bounded by a constant again depending on R , r and L only.

Moreover, K has a finite number of connected components, again bounded by a constant depending on R , r and L only, and the distance between two different connected components of K is bounded from below by a positive constant depending on r and L only.

We begin with the following lemma.

Lemma 3.6. *Let us fix a bounded open set Ω and positive constants r and L . Let $\mathcal{B} = \mathcal{B}(r, L)$ be the class of compact sets $K \subset \overline{\Omega}$ such that K is a Lipschitz hypersurface with constants r and L .*

Then \mathcal{B} is closed under the convergence in the Hausdorff distance, that is if $\{K_n\}_{n \in \mathbb{N}}$ is a sequence in \mathcal{B} , then up to a subsequence K_n converges, as $n \rightarrow \infty$, to a set $K \in \mathcal{B}$ in the Hausdorff distance. Furthermore, the boundary of K_n converges to the boundary of K in the Hausdorff distance.

Proof. Without loss of generality, up to a subsequence, we may assume that K_n and their boundaries converge to a compact set K and a compact set H , respectively.

Let $x \in K$. Then there exists a sequence $x_n \in K_n$ such that $\lim_{n \rightarrow \infty} x_n = x$. Again up to subsequences, we have that $\Phi_{x_n}^n$ converges to a function $\Phi_x : B_r(x) \rightarrow \mathbb{R}^N$ satisfying Condition a) above. Clearly, since $\Phi_{x_n}^n(x_n) = 0$, we have that $\Phi_x(x) = 0$ as well. It is not difficult to show that Condition b) is also satisfied.

As far as Condition c) is concerned, we first prove that H coincide with the boundary of K . Let $x \in H$, then there exists x_n belonging to the boundary of K_n such that $\lim_{n \rightarrow \infty} x_n = x$. Again up to subsequences, we have that $\Phi_{x_n}^n$ converges to a function $\Phi_x : B_r(x) \rightarrow \mathbb{R}^N$ satisfying Condition a) above. Again it is not difficult to prove that Φ_x satisfies both Condition b) and Condition c). Therefore, H is contained in the boundary of K . We need to prove that the boundary of K is contained in H . By contradiction, let us assume that there exists x belonging to the boundary of K such that x does not belong to H . For some positive constant c and for any n large enough, we have that the distance of x from the boundary of K_n is greater than c . If $x_n \in K_n$ converges to x , we have that x_n has a distance from the boundary of K_n greater than $c/2$, for any n large enough. Therefore there exists a positive constant c_1 such that for any n large enough we have $B_{c_1}(0) \cap \pi \subset \Phi_{x_n}^n(K_n \cap B_r(x_n))$. Passing to the limit, we obtain that $B_{c_1}(0) \cap \pi \subset \Phi_x(K \cap B_r(x))$ as well, therefore x belongs to the interior of K . Such a contradiction concludes the proof. \square

Definition 3.7. Let us fix a bounded open set Ω and positive constants r and L . We assume that $\partial\Omega$ consists of a finite number of hypersurfaces without boundary belonging to $\mathcal{B}(r, L)$.

Let us also fix $\omega : (0, +\infty) \rightarrow (0, +\infty)$ a nondecreasing left-continuous function.

We say that a compact set $K \subset \bar{\Omega}$ belongs to the class $\tilde{\mathcal{B}} = \tilde{\mathcal{B}}(r, L, \omega)$ if $\hat{K} = K \cup \partial\Omega$ satisfies the following conditions

- 1) $\hat{K} = \bigcup_{i=1}^M K^i$ where $K^i \in \mathcal{B}(r, L)$ for any $i = 1, \dots, M$;
- 2) for any $i \in \{1, \dots, M\}$, and any $x \in K^i$, if its distance from the boundary of K^i is $\delta > 0$, then the distance of x from the union of K^j , with $j \neq i$, is greater than or equal to $\omega(\delta)$.

Let us notice that in the previous definition the number M may depend on K . However, there exists an integer M_0 , depending on the diameter of Ω , r , L and ω only, such that $M \leq M_0$ for any $K \in \tilde{\mathcal{B}}$. As before, we obtain that $\mathcal{H}^{N-1}(\hat{K})$ is bounded, hence $|\hat{K}| = 0$. In particular, if $\Omega \subset \bar{B}_R$ for some $R > 0$, then $\mathcal{H}^{N-1}(\hat{K})$ is bounded by a constant depending on R , r , L and M_0 only. Furthermore, if we set as the boundary of \hat{K} the union of the boundaries of K^i , $i = 1, \dots, M$, then the boundary of \hat{K} has \mathcal{H}^{N-2} measure bounded by a constant again depending on R , r , L and M_0 only.

We also remark that, by Condition 2), we have that $K^i \cap K^j$ is contained in the intersection of the boundaries of K^i and K^j , for any $i \neq j$.

We prove the analogous of Lemma 3.6 for the class $\tilde{\mathcal{B}}$.

Lemma 3.8. *Under the previous notation and assumptions, we have that $\tilde{\mathcal{B}} = \tilde{\mathcal{B}}(r, L, \omega)$ as in Definition 3.7 is closed under the convergence in the Hausdorff distance, that is if $\{K_n\}_{n \in \mathbb{N}}$ is a sequence in $\tilde{\mathcal{B}}$, then up to a subsequence K_n converges, as $n \rightarrow \infty$, to a set $K \in \tilde{\mathcal{B}}$ in the Hausdorff distance. Clearly also \hat{K}_n converges to \hat{K} in the Hausdorff distance. Furthermore, if \tilde{K}_n , $n \in \mathbb{N}$, is the corresponding sequence of boundaries of \hat{K}_n and \tilde{K} is the boundary of \hat{K} , then \tilde{K}_n converges to \tilde{K} in the Hausdorff distance.*

Proof. Up to a subsequence, we may assume that as $n \rightarrow \infty$ K_n converges to a set K , hence \hat{K}_n converges to $\hat{K} = K \cup \partial\Omega$, and \tilde{K}_n converges to a set H . Moreover, we may assume that $M(n) = M$ for any $n \in \mathbb{N}$ and, by Lemma 3.6, that for any $i = 1, \dots, M$ K_n^i converges to a set $K^i \in \mathcal{B} = \mathcal{B}(r, L)$ and the boundary of K_n^i converges to the boundary of K^i . It is not difficult then to show that $\hat{K} = \bigcup_{i=1}^M K^i$ and that H is the union of the boundaries of K^i , $i = 1, \dots, M$. Therefore K satisfies Condition 1) above.

We now deal with Condition 2). Let us take x belonging to the interior of K^i and let $\delta > 0$ be its distance from the boundary of K^i . Let $x_n \in K_n^i$ converge to x . For any ε , $0 < \varepsilon < \delta/2$, there exists $\bar{n} = \bar{n}(\varepsilon)$ such that for any $n \geq \bar{n}$ we have that the distance of x_n from the boundary of K_n^i is greater than $\delta - \varepsilon$. We obtain that the distance of x_n from the union of K_n^j , with $j \neq i$, is greater than or equal to $\omega(\delta - \varepsilon)$, for any $n \geq \bar{n}$. Provided $0 < \varepsilon < \omega(\delta - \varepsilon)$, we can find a further $\hat{n} = \hat{n}(\varepsilon)$ such that for any $n \geq \hat{n}$ we have that the distance of x_n from the union of K^j , with $j \neq i$, is greater than or equal to $\omega(\delta - \varepsilon) - \varepsilon$. Passing to the limit as $n \rightarrow \infty$ we obtain that the distance of x from the union of K^j , with $j \neq i$, is greater than or equal to $\omega(\delta - \varepsilon) - \varepsilon$ for any $0 < \varepsilon < \omega(\delta - \varepsilon)$. We then let $\varepsilon \rightarrow 0^+$ and we conclude by the left-continuity of ω . □

We are now in the position of stating and proving the following Mosco convergence result.

Theorem 3.9. *Let Ω and $\tilde{\mathcal{B}} = \tilde{\mathcal{B}}(r, L, \omega)$ be as in Definition 3.7.*

Let $\{K_n\}_{n \in \mathbb{N}}$ be a sequence in $\tilde{\mathcal{B}}$ converging, as $n \rightarrow \infty$, to $K \in \tilde{\mathcal{B}}$ in the Hausdorff distance.

Then $H^1(\Omega \setminus K_n)$ converges, as $n \rightarrow \infty$, to $H^1(\Omega \setminus K)$ in the sense of Mosco.

Proof. Let us denote $A_n = H^1(\Omega \setminus K_n)$, $n \in \mathbb{N}$, and $A = H^1(\Omega \setminus K)$.

Since $|K| = 0$, then by Lemma 2.3, we immediately have that $A' \subset A$. Therefore it is enough to prove that $A \subset A''$ or, in other words, that for every $\varphi \in A$ there exists $\varphi_n \in A_n$ such that φ_n converges as $n \rightarrow \infty$ to φ in $L^2(\Omega, \mathbb{R}^{N+1})$. We notice that it is enough to prove that for any subsequence A_{n_k} there exists a further subsequence $A_{n_{k_j}}$ and $\varphi_j \in A_{n_{k_j}}$ such that φ_j converges as $j \rightarrow \infty$ to φ in $L^2(\Omega, \mathbb{R}^{N+1})$. Therefore during our proof we can always pass to subsequences, without loss of generality.

We may assume that \hat{K}_n is converging to \hat{K} and \tilde{K}_n is converging to \tilde{K} in the Hausdorff distance.

Since A'' is closed, it is enough to prove the result for any φ in a dense subset of A . For instance, let us consider the following subset of A

$$\tilde{A} = \{\varphi \in H^1(\Omega \setminus K) : \varphi \text{ is bounded and } \varphi = 0 \text{ in a neighborhood of } \tilde{K}\}.$$

We wish to show that \tilde{A} is dense in A . By an easy truncation argument we can show that $\{\varphi \in H^1(\Omega \setminus K) : \varphi \text{ is bounded}\}$ is dense in A . It is enough to show that \tilde{A} is dense in this last set. Since $\mathcal{H}^{N-2}(\tilde{K})$ is finite, then \tilde{K} has zero capacity. Hence for any neighborhood U of \tilde{K} and for any $\varepsilon > 0$ there exists a function χ_ε such that $\chi_\varepsilon \in H^1(\Omega)$, $0 \leq \chi_\varepsilon \leq 1$ almost everywhere in Ω , $\chi_\varepsilon = 1$ almost everywhere outside U , $\chi_\varepsilon = 0$ almost everywhere in a neighborhood of \tilde{K} , and

$$\int_{\Omega} \|\nabla \chi_\varepsilon\|^2 \leq \varepsilon.$$

Take $\varphi \in H^1(\Omega \setminus K)$ such that φ is bounded. Clearly we have that $\chi_\varepsilon \varphi \in \tilde{A}$ and

$$\|\chi_\varepsilon \varphi - \varphi\|_{L^2(\Omega)} \leq \|\varphi\|_{L^2(U)}, \quad \|\nabla(\chi_\varepsilon \varphi) - \nabla \varphi\|_{L^2(\Omega)} \leq \|\nabla \varphi\|_{L^2(U)} + \|\varphi\|_{L^\infty(\Omega)} \sqrt{\varepsilon}.$$

Since U and ε are arbitrary, we conclude that \tilde{A} is dense in A .

Take $\varphi \in \tilde{A}$ and let \tilde{U} be an open neighborhood of \tilde{K} on which φ is zero. We can find an open subset D compactly contained in $\Omega \setminus K$, a finite number of points $x_j \in \hat{K}$ and positive numbers δ_j , $j = 1, \dots, m$, such that $B_{2\delta_j}(x_j) \cap \tilde{K} = \emptyset$ for any $j = 1, \dots, m$ and

$$\bar{\Omega} \subset (\tilde{U} \cap \Omega) \cup D \cup \left(\bigcup_{j=1}^m (B_{\delta_j}(x_j) \cap \Omega) \right).$$

Moreover, for any $j = 1, \dots, m$, we have that x_j belongs to the interior of $K^{i(j)}$ for some $i = i(j) \in \{1, \dots, M\}$ and we may assume that $2\delta_j \leq r$ and that $B_{2\delta_j}(x_j) \cap \hat{K} = B_{2\delta_j}(x_j) \cap K^{i(j)}$.

By using a partition of unity, we may therefore reduce ourselves to the following cases. It is sufficient to consider a function $\varphi \in \tilde{A}$ that is compactly supported either in $B_{\delta_j}(x_j)$, for some $j \in \{1, \dots, m\}$, or in D . In the latter case, we have that $D \subset (\Omega \setminus K_n)$, hence $\varphi \in H^1(\Omega \setminus K_n)$, for any n large enough, so the convergence is trivially proved.

It remains to prove the convergence for $\varphi \in \tilde{A}$ that is compactly supported in $B_\delta(x)$ for some $0 < \delta \leq r/2$ and $x \in K^i \subset \tilde{K}$ such that $B_{2\delta}(x) \cap \tilde{K} = \emptyset$ and $B_{2\delta}(x) \cap \hat{K} = B_{2\delta}(x) \cap K^i$.

We use the reasoning developed in the proof of Theorem 4.2 in [8]. For the convenience of the reader we repeat the construction. We suppose that $x \in \Omega$, the case when $x \in \partial\Omega$ requires just a little modification. Possibly passing to a subsequence, let $x_n \in K_n$ converge to x and $\Phi_{x_n}^n$ converge to a function $\Phi_x : B_r(x) \rightarrow \mathbb{R}^N$. Without loss of generality, we may assume that $B_\delta(x) \subset \Phi_x^{-1}(B_{r_1})$ for some positive r_1 such that $B_{r_1} \subset \Phi_x(B_r(x))$. Moreover, we may also assume that $B_{r_1} \cap \pi \subset \Phi_x(B_r(x) \cap K)$ and $B_{r_1} \cap \pi \subset \Phi_{x_n}^n(B_r(x_n) \cap K_n)$ for any n large enough. Let $\psi = \varphi \circ \Phi_x^{-1}$. Then $\psi \in H^1(B_{r_1} \setminus \pi)$. We denote by ψ^\pm the function ψ defined above or below π , that is in the halfspaces $T^\pm = \{y \in \mathbb{R}^N : \pm y_n > 0\}$, respectively. Then, by an odd reflection, we may define two $H_0^1(B_{r_1})$ functions, $\tilde{\psi}^\pm$ such that $\tilde{\psi}^\pm = \psi^\pm$ on T^\pm . Let $\tilde{\varphi}^\pm = \tilde{\psi}^\pm \circ \Phi_x \in H_0^1(B_\delta(x))$.

Finally we define

$$\varphi_n = \begin{cases} \tilde{\varphi}^+(x) & \text{if } \Phi_{x_n}^n(x) \in T^+ \\ \tilde{\varphi}^-(x) & \text{if } \Phi_{x_n}^n(x) \in T^- \end{cases}$$

By construction we have that $\varphi_n \in H^1(\Omega \setminus K_n)$. Furthermore, φ_n converges almost everywhere to φ . Since $|\varphi_n| \leq \max\{|\tilde{\varphi}^+|, |\tilde{\varphi}^-|\}$, by the Dominated Convergence Theorem we have that φ_n converges to φ in L^2 . The same argument holds true for the gradients, so the proof is concluded. \square

As an immediate corollary to Theorem 3.9, we infer that the following assumption on \tilde{A} is a sufficient condition for Assumption iii) to hold.

Assumption D. *Fixed $R > 0$, let $\Omega = B_{R+1}$. Let us fix positive constants r and L and let $\omega : (0, +\infty) \rightarrow (0, +\infty)$ be a nondecreasing left-continuous function.*

We assume that \tilde{A} satisfies Assumption i) and that for any $K \in \tilde{A}$ we have that ∂K belongs to $\tilde{\mathcal{B}}(r, L, \omega)$.

We conclude the section simply by pointing out that, by Lemma 3.8, we have that Assumption D is closed under convergence in the Hausdorff distance.

4. Approximation of sound-hard screens. Let K be a scatterer in \mathbb{R}^N , that is K is a compact set such that $\mathbb{R}^N \setminus K$ is connected. Let us assume that $K \subset \overline{B_R}$ for some $R > 0$.

We define a class of scatterers K satisfying some minimal Lipschitz type regularity assumptions.

Definition 4.1. A scatterer K is said to be *Lipschitz* if, for some positive constants r and L , the following assumptions hold.

For any $x \in \partial K$, there exists a function $\varphi : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$, such that $\varphi(0) = 0$ and which is Lipschitz with Lipschitz constant bounded by L , such that, up to a rigid change of coordinates, we have $x = 0$ and

$$B_r(x) \cap \partial K \subset \{y \in B_r(x) : y_N = \varphi(y')\}.$$

We say that $x \in \partial K$ belongs to the interior of ∂K if there exists $\delta, 0 < \delta \leq r$, such that $B_\delta(x) \cap \partial K = \{y \in B_\delta(x) : y_N = \varphi(y')\}$. Otherwise we say that x belongs to the boundary of ∂K . We remark that the boundary of ∂K might be empty and that, if $x \in \partial K$ belongs to the interior of ∂K , then K may lie at most on one side

of ∂K , that is $B_\delta(x) \cap K = B_\delta(x) \cap \partial K$, or $B_\delta(x) \cap K = \{y \in B_\delta(x) : y_N \geq \varphi(y')\}$, or $B_\delta(x) \cap K = \{y \in B_\delta(x) : y_N \leq \varphi(y')\}$.

For any x belonging to the boundary of ∂K , we assume that there exists another function $\varphi_1 : \mathbb{R}^{N-2} \rightarrow \mathbb{R}$, such that $\varphi_1(0) = 0$ and which is Lipschitz with Lipschitz constant bounded by L , such that, up to the previous rigid change of coordinates, we have $x = 0$ and

$$B_r(x) \cap \partial K = \{y \in B_r(x) : y_N = \varphi(y'), y_{N-1} \leq \varphi_1(y'')\}.$$

Finally, for any $x \in \partial K$, let $e_1(x), \dots, e_N(x)$ be the unit vectors representing the orthonormal base of the coordinate system for which the previous representations hold. Then we assume that $e_N(x)$ is a Lipschitz function of $x \in \partial K$, with Lipschitz constant bounded by L , and $e_{N-1}(x)$ is a Lipschitz function of x , as x varies in the boundary of ∂K , with Lipschitz constant bounded by L .

The following properties hold for a Lipschitz scatterer. Let us notice that, by compactness, $\mathcal{H}^{N-1}(\partial K)$ is bounded, hence $|\partial K| = 0$. In particular, if $K \subset \overline{B_R}$ for some $R > 0$, then $\mathcal{H}^{N-1}(\partial K)$ is bounded by a constant depending on R, r and L only.

We also have that ∂K is a Lipschitz hypersurface as defined in Section 3. We notice that K has a finite number of connected components, again bounded by a constant depending on R, r and L only, and the distance between two different connected components of K is bounded from below by a positive constant depending on r and L only. More precisely, we have that each connected component of K is either the closure of a Lipschitz domain or a Lipschitz hypersurface with boundary. Obviously, the exterior of any connected component of K is connected, since the exterior of K is. Furthermore, the numbers of connected components of K and ∂K coincide, that is if a connected component of K is the closure of a Lipschitz domain then its boundary is connected. Here we have made use again of the fact that the exterior of K is connected. We may conclude that the exterior of K is connected if and only if the exteriors of its connected components are connected.

Finally any connected component of ∂K is an oriented Lipschitz hypersurface with or without boundary. In the case of a hypersurface without boundary, then this is the boundary of a Lipschitz domain contained in K .

Let us now fix a scatterer K and let us denote $d : \mathbb{R}^N \rightarrow [0, +\infty)$ the function defined as follows

$$d(x) = \text{dist}(x, K) \quad \text{for any } x \in \mathbb{R}^N.$$

We say that a Lipschitz function $\tilde{d} : \mathbb{R}^N \rightarrow [0, +\infty)$ is a *regularized distance* for K if the following properties are satisfied.

First, there exist constants a and b , $0 < a \leq 1 \leq b$, such that

$$ad(x) \leq \tilde{d}(x) \leq bd(x) \quad \text{for any } x \in \mathbb{R}^N.$$

For any $h > 0$, let us call $K_h = \{x \in \mathbb{R}^N : \tilde{d}(x) \leq h\}$. Second, we require that, for some constants $h_0 > 0, p > 2$ and $C_1 > 0$, for any $h, 0 < h \leq h_0, \mathbb{R}^N \setminus K_h$ is connected and

$$\|v\|_{L^p(B_{R+1} \setminus K_h)} \leq C_1 \|v\|_{H^1(B_{R+1} \setminus K_h)} \quad \text{for any } v \in H^1(B_{R+1} \setminus K_h).$$

The interest in such a regularized distance is given by the following remark. If we take a decreasing sequence of positive numbers $\{\varepsilon_n\}_{n \in \mathbb{N}}$ such that $\lim_n \varepsilon_n = 0$ and define $K_n = K_{\varepsilon_n}$ for any $n \in \mathbb{N}$, we immediately have that K_n converges to

K in the Hausdorff distance and that $H^1(B_{R+1} \setminus K_n)$ converges to $H^1(B_{R+1} \setminus K)$ in the sense of Mosco. Finally, all the assumptions of Proposition 2.15 are satisfied.

We wish to find sufficient conditions on K so that a regularized distance \tilde{d} exists.

A simple sufficient condition is that K is a compact convex set. In fact, clearly we have that $\mathbb{R}^N \setminus K$ is connected. Then, we can take $\tilde{d} = d$ or the distance from K with respect to any norm on \mathbb{R}^N , not only with respect to the Euclidean one. Then, for any $h > 0$, $K_h = \overline{B_h(K)}$, clearly with respect to the chosen norm. For any $h > 0$, we have that $\overline{B_h(K)}$ is still a convex set, therefore $\mathbb{R}^N \setminus \overline{B_h(K)}$ is connected and also the other required properties are satisfied, for example by using Lemma 2.11. Clearly we can extend this property to a set K which is the union of a finite number of compact convex sets which are pairwise disjoint.

The main result of this section is the following.

Proposition 4.2. *Let K be a Lipschitz scatterer. The K admits a regularized distance.*

We finally remark that if K is the union of a finite number of pairwise disjoint compact convex sets and Lipschitz scatterers, then it also admits a regularized distance.

Proof. The main difficulty in the proof of Proposition 4.2 lies in the Lipschitz character of K . If K would be smoother then the distance d itself would be enough. Another difficulty is the fact that we need a global construction, locally it would be enough to raise (or lower) ∂K by a translation along the direction e_N .

Fixed $h > 0$, let

$$(4.1) \quad K_h = K \cup \{x + te_N(x) : x \in \partial K, t \in [-h, h]\} \cup \{x + se_{N-1}(x) + te_N(x) : x \text{ in the boundary of } \partial K, t \in [-h, h], s \in [0, h]\}.$$

Notice that $K_h \subset B_{\tilde{h}}(K)$ for any $\tilde{h} > \sqrt{2}h$. In order to construct the function \tilde{d} and prove the required properties, without loss of generality we may assume that K has only one connected component.

We begin with the simpler case of $K = \overline{D}$, D being a Lipschitz domain. In such a case, at least for $h \leq r/2$,

$$K_h = K \cup \{x + te_N(x) : x \in \partial K, t \in [0, h]\}.$$

By a standard use of the contraction mapping principle, it is not difficult to prove that there exist positive constants $h_0, c, 0 < c < 1, r_1$ and L_1 , depending on r and L only, such that for any $h, 0 < h \leq h_0$, the following holds.

For any $x \in \partial K_h$ there exists a function $\varphi^h : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$, such that $\varphi(0) = 0$ and which is Lipschitz with Lipschitz constant bounded by L_1 , such that, up to a rigid change of coordinates, we have $x = 0$ and

$$B_{r_1}(x) \cap K_h = \{y \in B_{r_1}(x) : y_N \leq \varphi^h(y')\}.$$

Furthermore, for any $x \in \partial K$ we have that $B_{ch}(x) \subset K_h$, that is $B_{ch}(K) \subset K_h$. Finally K_h and its boundary are connected and its exterior is connected as well.

We notice that such a Lipschitz condition implies that $B_{R+1} \setminus K_h$ satisfies a cone condition with a cone not depending on h but only on r and L . Therefore, by Lemma 2.11, the required properties are all satisfied.

A sketch of the proof is the following. Let us fix $x \in \partial K$. Without loss of generality, we take $x = 0$ and assume that

$$K \cap B_r = \{y \in B_r : y_N \leq \varphi(y')\},$$

with $\varphi : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ a Lipschitz function with Lipschitz constant bounded by L and such that $\varphi(0) = 0$. There exists a constant $c_1, 0 < c_1 \leq 1/4$, depending on L only such that the graph of φ with $y' \in B'_{c_1 r}$ is contained in $B_{r/4}$ and, consequently, it is a connected set.

For any $y' \in B'_{c_1 r}$, let $\nu(y') = e_N(y', \varphi(y'))$, which is a Lipschitz function of y' with Lipschitz constant bounded by $L(L + 1)$. Notice that $\nu(0) = e_N(0) = e_N$ and that, as usual, $\nu = (\nu', \nu_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$. Fixed h , let $G_h : B'_{c_1 r} \rightarrow \mathbb{R}^{N-1}$ be defined as follows. For any $y' \in B'_{c_1 r}$, $G_h(y') = y' + h\nu'(y')$. Provided $|h|L(L + 1) \leq 1/2$, we have that G_h is Lipschitz with Lipschitz constant bounded by $3/2$. Moreover, G_h is injective and, by the contraction mapping principle, we have that $G_h(B'_{c_1 r})$ contains $B'_{c_1 r/2}$. Finally, G_h^{-1} is Lipschitz on $B'_{c_1 r/2}$ with a Lipschitz constant bounded by 2. For any $z' \in B'_{c_1 r/2}$, let $\varphi^h(z') = h\nu_N(G_h^{-1}(z')) + \varphi(G_h^{-1}(z'))$. Such a function φ^h is Lipschitz with a Lipschitz constant bounded by L_1, L_1 depending on L only. Again we notice that $\varphi^h(0) = h$.

Next, we can find constants $h_0, 0 < h_0 \leq 1/(2L(L + 1))$, and $r_1, 0 < r_1 \leq c_1 r/2$, depending on r and L only, such that for any $z' \in B'_{r_1}$ we have that $\varphi^h(z')$ is a strictly increasing function with respect to $h, -h_0 \leq h \leq h_0$. Therefore, possibly after reducing r_1 , for any $h > 0$ and any $z' \in B'_{r_1}$ we have that

$$B_{r_1}(z', \varphi^h(z')) \cap \partial K_h = \{y \in B_{r_1}(z', \varphi^h(z')) : y_N = \varphi^h(y')\}.$$

Thus we have obtained that ∂K_h satisfies a Lipschitz condition, with constants depending on r and L only, and hence $\mathbb{R}^N \setminus K_h$ and K_h satisfy a cone condition with a cone depending on r and L only. Finally, taking $z' = 0$, it is easy to show that there exists $c, 0 < c < 1$ depending on such a cone only, such that $B_{ch}(0) \subset K_h$, that is $B_{ch}(K) \subset K_h$.

Then, for any $x \in K_{h_0}$, we define $\tilde{d}(x) = \min\{h \geq 0 : x \in K_h\}$, where $K_0 = K$. For any $x \in \mathbb{R}^N \setminus K_{h_0}$, we define $\tilde{d}(x) = h_0 + \text{dist}(x, K_{h_0})$. Clearly the definition of K_h with respect to the distance \tilde{d} is consistent with (4.1) for any $h, 0 < h \leq h_0$. Finally, it is now easy to show that the function \tilde{d} satisfies all the required properties.

In the case when K is equal to ∂K and consists of one connected component given by a Lipschitz hypersurface with boundary, we can prove that $B_{R+1} \setminus K_h$ satisfies a cone condition with a cone not depending on h , for any h small enough. The arguments and the construction of the function \tilde{d} are similar to the one exposed above. Some care along the boundary of the hypersurface is needed. The basic idea is the following. We can enlarge ∂K into another Lipschitz hypersurface \tilde{K} by taking, for some $h_0 > 0$ small enough,

$$\tilde{K} = \{x + se_{N-1}(x) : x \text{ in the boundary of } \partial K, s \in [0, h_0]\}.$$

We have that \tilde{K} satisfies the same kind of Lipschitz properties and, for any h small enough, the two surfaces $\gamma_{h\pm} = \{x \pm he_N(x) : x \in \tilde{K}\}$ are Lipschitz as well. On the other hand, also $\alpha = \{x + te_N(x) : x \text{ in the boundary of } \partial K, t \in [-h_0, h_0]\}$ is a Lipschitz hypersurface and thus also $\alpha_h = \{x + he_{N-1}(x) : x \in \alpha\}$, for any h small enough. Then to have the desired properties of K_h it is enough to check the way α_h intersects γ_{h+} and γ_{h-} . We leave the lengthy but straightforward details to the reader. □

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