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Collage theorem-based approaches for solving inverse problems for differential equations: A review of recent developments

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Abstract. In this short survey, we review the current status of fractal-based techniques and their application to the solution of inverse problems for ordinary and partial differential equations. This involves an examination of several methods which are based on the so-called Collage Theorem, a simple consequence of Banach's Fixed Point Theorem, and its extensions.

1. Introduction

In this paper, we review the current state of the art of "fractal-based methods" of solving inverse problems for ordinary and partial differential equations. This intent of this survey paper is to summarize, and for some readers introduce, a collection of so-called "Collage Methods" to the inverse problems community. The approach is different, but not necessarily better, than those employed by the majority of people working in inverse problems.

In the context of the work reviewed in this paper, we consider the direct problem to be the determination of the solution to a completely prescribed differential equation, including known initial conditions and/or boundary conditions. On the other hand, we consider the inverse problem to be estimation of values of the parameters, or perhaps a subset thereof, in a system of differential equations, based on some information about the solution, e.g., observed data values.

The literature is rich in papers studying *ad hoc* methods to address ill-posed inverse problems by minimizing a suitable approximation error along with utilizing some regularization techniques [20, 40, 44, 45, 46, 48]. Some of the approaches, for example in [48], without any rigorous justification actually perform the same minimization that is justified by the collage theorem; the references in this survey paper additionally list some publications by others who recognize that they are using the collage theorem and/or developing new related results [2, 1, 13, 14, 15, 37, 38, 41, 43, 43, 47].

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2. Inverse problem of approximation by fixed points of contraction mappings

Most "fractal-based" methods are based on the use of contraction mappings on appropriate metric spaces. (By "appropriate", we mean appropriate to the application of concern.) It is therefore helpful if we provide some brief mathematical preliminaries.

In what follows, we shall let (X, d) denote a complete metric space. (For example, X could be a set of functions defined over an interval $[a, b] \subseteq \mathbb{R}$, e.g., the set $L^2[a, b]$ of square-integrable functions on $[a, b]$, and d the corresponding metric on that space.)

Definition 1 *(Contraction mapping) Let $T : X \to X$ be a mapping on a complete metric space (X, d). Then T is said to be contractive if there exists a constant $c \in [0, 1)$ such that $d(Tx, Ty) \leq cd(x, y)$ for all $x, y \in X$.*

Generally, the smallest such $c \in [0, 1)$ for which the above inequality holds true is known as the *contraction factor* of T. We now come to what is perhaps the most famous theorem regarding contraction maps on metric spaces and certainly central to fractal-based methods.

Theorem 1 *(Banach Fixed Point Theorem [3]) Let $T : X \to X$ be a contraction mapping on X with contraction factor $c \in [0, 1)$. mapping on X. Then,*

(i) There exists a unique element $\bar{x} \in X$, the fixed point of T, for which $T\bar{x} = \bar{x}$.

(ii) Given any $x_0 \in X$, if we form the iteration sequence $x_{n+1} = T(x_n)$, then $x_n \to \bar{x}$, i.e., $d(x_n, \bar{x}) \to 0$ as $n \to \infty$. In other words, the fixed point $\bar{x}$ is globally attractive.

We consider the following general class of inverse problems:

> Let (X, d) be a complete metric space and a "target" element $x \in X$ that we wish to approximate. Given an $\epsilon > 0$, can we find a contraction mapping $T : X \to X$ with fixed point $\bar{x} \in X$ such that $d(\bar{x}, x) < \epsilon$?

Very briefly, the original motivation for this formulation was fractal image coding [16, 6, 39]. When an image x is approximated by the fixed point $\bar{x}$ of a contractive "fractal transform" T, the amount of computer memory required to store the parameters which define T is generally much less than that required to store x. Instead of storing or transmitting x, one stores or transmit T, from which $\bar{x}$, an approximation to x can be generated via iteration. The result: *fractal image compression.*

Given the complicated nature of fractal transforms, however, the determination of optimal mappings T by minimizing the approximation error $d(\bar{x}, x)$ is intractable. An enormous simplification is achieved by means of the following simple consequence of Banach's Theorem, known in the literature as the *Collage Theorem.*

Theorem 2 *("Collage Theorem" [5, 4]) Let (X, d) be a complete metric space and $T : X \to X$ a contraction mapping with contraction factor $c \in [0, 1)$. Then for any $x \in X$,*

$$d(x, \bar{x}) \leq \frac{1}{1 - c} d(x, Tx), \tag{1}$$

where $\bar{x}$ is the fixed point of T.

This permits a reformulation of our original inverse problem as follows,

> Given an $\epsilon > 0$, can we find a contraction mapping $T : X \to X$ such that $d(Tx, x) < \delta$?

In an effort to minimize the approximation error $d(\bar{x}, x)$, we now look for contraction maps T which minimize the so-called *collage error* $d(x, Tx)$. In other words, we look for maps T which send the target x as close as possible to itself. We refer to this approach as *collage coding* [30].

Barnsley and co-workers [5, 4] were the first to see the potential of using the Collage Theorem for the purpose of image approximation and compression. Most, if not all fractal image coding methods are based on some kind of block-based collage coding method which follows the strategy originally presented by Jacquin [18].

A collage coding approach, however, may be applied in other, "nonfractal," situations where contractive mappings are encountered, as we describe below. In a practical application, we consider a family of appropriate contraction mappings T_λ, $\lambda \in \Lambda \subset \mathbb{R}^n$, and try to find the parameter $\bar{\lambda}$ which minimizes the approximation error $d(x, \bar{x}_\lambda)$. The feasible set can be defined as $\Lambda = \{\lambda \in \mathbb{R}^n : 0 \leq c_\lambda \leq c \leq C < 1\}$ which guarantees the contractivity of T_λ for any $\lambda \in \Lambda$. (Here, C is a prescribed "cutoff.") This is perhaps the main difference between our collage coding approach and Tikhonov regularization (see [45, 46]): In the former, the constraint $\lambda \in \Lambda$ guarantees that T_λ is a contraction essentially replacing the effect of the regularization term in the Tikhonov approach. The collage-based inverse problem described above can be formulated as an optimization problem as follows,

$$\min_{\lambda \in \Lambda} d(x, T_\lambda x) \,. \tag{2}$$

In general this optimization problems is nonlinear and nonsmooth. The regularity of the objective function strictly depends on the term $d(x, T_\lambda x)$. In many cases, however, the problem in (2) can be reduced to a quadratic optimization problem. A number of algorithms can then be used to solve this problem, including, for example, penalization methods, particle swarm ant colony techniques, etc..

3. Inverse problems for DEs using the Collage Theorem

The use of the Collage Theorem to solve inverse problems for ODEs was originally proposed in [21] and developed in many subsequent works including [22, 23, 25, 26, 28]. The initial value problems (IVPs) studied in these papers had the general form,

$$\begin{cases} \dot{u} = f(t, u) \\ u(0) = u_0 \,. \end{cases} \tag{3}$$

Associated with the above IVP is the following Picard integral operator,

$$(Tu)(t) = u_0 + \int_0^t f(s, u(s))\, ds. \tag{4}$$

It is well known that the solution to the IVP in (3) is a fixed point of T, i.e.,

$$Tu = u. \tag{5}$$

Consider the complete metric space $C([-\delta, \delta])$ endowed with the usual d_∞ metric and assume that $f(t, u)$ is Lipschitz in the variable u, that is there exists a $K \geq 0$ such that $|f(s, u) - f(s, v)| \leq K|u - v|$, for all $u, v \in \mathbb{R}$. For simplicity we suppose that $u \in \mathbb{R}$ but the same consideration can be developed for the case of several variables. Under these hypotheses T is Lipschitz on the space $C([-\delta, \delta] \times [-M, M])$ for some δ and $M > 0$.

In the direct or forward problem, as is well known, the above Lipschitz property of f guarantees the existence of a unique fixed point of T and therefore the solution to (3). (Standard proof: contractivity of T.) Here, however, we are concerned with the collage-based *inverse problem* associated with (3):

Given a function $u(t)$, find a Picard operator T – as defined by the function f – which maps u as close as possible to itself.

Theorem 3 *[21] The function T satisfies*

$$\|Tu - Tv\|_2 \leq c\|u - v\|_2 \tag{6}$$

for all $u, v \in C([-\delta, \delta] \times [-M, M])$ where $c = \delta K$.

Now let $\delta' > 0$ be such that $\delta' K < 1$. In order to solve the inverse problem for the Picard operator in (4) we employ the L^2 expansion of the function f. Let $\{\phi_i\}$ be a basis of functions in $L^2([-\delta', \delta'] \times [-M, M])$ and consider $f_\lambda(s, u) = \sum_{i=1}^{+\infty} \lambda_i \phi_i(s, u)$. Each sequence of coefficients $\lambda = \{\lambda_i\}_{i=1}^{+\infty}$ then defines a Picard operator T_λ. Suppose further that each function $\phi_i(s, u)$ is Lipschitz in u with constant K_i.

Theorem 4 *[21] Let $K, \lambda \in \ell^2(\mathbb{R})$. Then*

$$|f_\lambda(s, u) - f_\lambda(s, v)| \leq \|K\|_2 \|\lambda\|_2 |u - v| \tag{7}$$

for all $s \in [-\delta', \delta']$ and $u, v \in [-M, M]$ where $\|K\|_2 = \left(\sum_{i=1}^{+\infty} K_i^2\right)^{\frac{1}{2}}$ and $\|\lambda\|_2 = \left(\sum_{i=1}^{+\infty} \lambda_i^2\right)^{\frac{1}{2}}$

Given a target solution x, we now wish to minimize the collage distance $\| u - T_\lambda u \|_2$. The square of the collage distance becomes

$$\Delta^2(\lambda) = \|u - T_\lambda u\|_2^2 = \int_{-\delta}^{\delta} \left| u(t) - u_0 - \int_0^t \sum_{i=1}^{+\infty} \lambda_i \phi_i(s, u(s)) ds \right|^2 dt$$

and the inverse problem can be formulated as

$$\min_{\lambda \in \Lambda} \Delta(\lambda), \tag{8}$$

where $\Lambda = \{\lambda \in \ell^2(\mathbb{R}) : \|\lambda\|_2 \|K\|_2 < 1\}$. To solve this problem numerically, we consider the first n terms of the L^2 basis. In this case, the previous problem can be reduced to:

$$\min_{\lambda \in \tilde{\Lambda}} \tilde{\Delta}^2(\lambda) = \int_{-\delta}^{\delta} \left| u(t) - u_0 - \int_0^t \sum_{i=1}^{n} \lambda_i \phi_i(s, u(s)) ds \right|^2 dt, \tag{9}$$

where $\tilde{\Lambda} = \{\lambda \in \mathbb{R}^n : \|\lambda\|_2 \|K\|_2 < 1\}$. This is a classical quadratic optimization problem which can be solved by means of classical numerical methods.

Let $\tilde{\Delta}^n_{\min}$ be the minimum value of $\tilde{\Delta}$ over $\tilde{\Lambda}$. This is a nonincreasing sequence of numbers (depending on n). Following the method of [17], it can be shown that $\liminf_{n \to +\infty} \tilde{\Delta}^n_{\min} = 0$, i.e., the distance between the target element and the unknown solution of the differential equation can be made arbitrarily small.

In [9, 11, 25] the above approach was extended to consider the case of inverse problems for random and stochastic differential equations.

4. Inverse problems for PDEs using the Generalized Collage Theorem

We now review an extension of the Collage Theorem, the Generalized Collage Theorem, and show how it can be used to solve inverse problems for families of PDEs.

4.1. Elliptic equations

Consider the following variational equation,

$$a(u, v) = \phi(v), \qquad v \in H. \tag{10}$$

where $\phi(v)$ and $a(u, v)$ are linear and bilinear maps, respectively, both defined on an Hilbert space H. Let $\langle\cdot\rangle$ denote the inner product in H, $\|u\|^2 = \langle u, u\rangle$ and $d(u, v) = \|u - v\|$, for all $u, v \in H$. The existence and uniqueness of solutions to this kind of equation are provided by the classical Lax-Milgram representation theorem [12]: Let H be a Hilbert space and ϕ a bounded linear nonzero functional, i.e., $\phi : H \to \mathbb{R}$. Also suppose that $a(u, v)$ is a bilinear form on $H \times H$ which satisfies the following conditions:

(i) There exists a constant $M > 0$ s.t. $|a(u, v)| \leq M\|u\|\|v\|$ for all $u, v \in H$,

(ii) There exists a constant $m > 0$ s.t. $|a(u, u)| \geq m\|u\|^2$ for all $u \in H$.

Then there is a unique vector $u^* \in H$ such that $\phi(v) = a(u^*, v)$ for all $v \in H$.

The inverse problem may now be viewed as follows. Suppose that we have an observed solution u and a given (restricted) family of bilinear functionals $a_\lambda(u, v)$, $\lambda \in \mathbb{R}^n$. We now seek to find "optimal" values of λ.

Suppose that we have a given Hilbert space H, a "target" element $u \in H$ and a family of bilinear functionals a_λ. Then by the Lax-Milgram theorem, there exists a unique vector u_λ such that $\phi(v) = a_\lambda(u_\lambda, v)$ for all $v \in H$. We would like to determine if there exists a value of the parameter λ such that $u_\lambda = u$ or, more realistically, such that $\|u_\lambda - u\|$ is small enough. The following theorem will be useful for the solution of this problem.

Theorem 5 *(Generalized Collage Theorem) [27] Suppose that $a_\lambda(u, v) : \mathcal{F} \times H \times H \to \mathbb{R}$ is a family of bilinear forms for all $\lambda \in \mathcal{F}$ and $\phi : H \to \mathbb{R}$ is a given linear functional. Let u_λ denote the solution of the equation $a_\lambda(u, v) = \phi(v)$ for all $v \in H$ as guaranteed by the Lax-Milgram theorem. Given a target element $u \in H$ then*

$$\|u - u_\lambda\| \leq \frac{1}{m_\lambda} F(\lambda), \tag{11}$$

where

$$F(\lambda) = \sup_{v \in H,\ \|v\|=1} |a_\lambda(u, v) - \phi(v)|. \tag{12}$$

In order to ensure that the approximation u_λ is close to a target element $u \in H$, we can, by the Generalized Collage Theorem, try to make the term $F(\lambda)/m_\lambda$ as close to zero as possible. The appearance of the m_λ factor complicates the procedure as does the factor $1/(1-c)$ in standard collage coding, i.e., Eq. (1). If $\inf_{\lambda \in \mathcal{F}} m_\lambda \geq m > 0$ then the inverse problem can be reduced to the minimization of the function $F(\lambda)$ on the space $\mathcal{F}$, that is,

$$\min_{\lambda \in \mathcal{F}} F(\lambda). \tag{13}$$

Next sections show that, under the condition $\inf_{\lambda \in \mathcal{F}} m_\lambda \geq m > 0$, our approach is stable. Following our earlier studies of inverse problems using fixed points of contraction mappings, we shall refer to the minimization of the functional $F(\lambda)$ as a "generalized collage method."

Now let $\langle e_i \rangle \subset H$ be a basis of the Hilbert space H, not necessarily orthogonal, so that each element $v \in H$ can be written as $v = \sum_i \alpha_i e_i$. It can easily be proved that

$$\inf_{\lambda \in \mathcal{F}} \|u - u_\lambda\| \leq \frac{1}{m} \sup_{v \in H,\ \|v\|=1} \left[\sum_i \alpha_i^2\right] \inf_{\lambda \in \mathcal{F}} \left[\sum_i |a_\lambda(u, e_i) - \phi(e_i)|^2\right]. \tag{14}$$

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Let $V_n =< e_1, e_2, \ldots, e_n >$ be the finite dimensional vector space generated by e_i, $V_n \subset H$. Given a target $u \in H$, let $\Pi_{V_n} u$ the projection of u on the space V_n and consider the following problem: find $u_\lambda \in V_n$ such that $\|\Pi_{V_n} u - u_\lambda\|$ is as small as possible. We have

$$\|\Pi_{V_n} u - u_\lambda\| \leq \frac{M}{m}\left[\sum_i |a_\lambda(u, e_i) - \phi(e_i)|^2\right], \tag{15}$$

where $M = \max_{v=\sum_{i=1}^n \alpha_i e_i \in V_h, \|v\|=1} \sum_{i=1}^n \alpha_i^2$, so that the problem has been reduced to the following minimization problem,

$$\inf_{\lambda \in \mathcal{F}} \|\Pi_{V_n} u - u_\lambda\| \leq \frac{M}{m} \inf_{\lambda \in \mathcal{F}} \sum_{i=1}^n |a_\lambda(u, e_i) - \phi(e_i)|^2 = \frac{M}{m}(F_n(\lambda))^2. \tag{16}$$

Example 1: We consider

$$\begin{aligned} -\nabla \cdot (\kappa(x,y)\nabla u(x,y)) &= f(x,y),\ (x,y) \in \Omega = \{0 < x, y < 1\} \\ u(x,y) &= 0,\ (x,y) \in \partial\Omega. \end{aligned} \tag{17}$$

Multiply (17) by a test function $v(x,y) \in H = H_0^1([0,1]^2)$, the Hilbert space built with all L^2 functions that have a weak derivative in L^2, integrate over $\bar{\Omega}$, and apply Green's first identity, with $\hat{n}$ denoting the outward unit normal to $\partial\Omega$, to get the equation $a(u,v) = \phi(v)$, with

$$a(u,v) = \iint_\Omega \kappa \nabla v \cdot \nabla u \, dA \quad \text{and} \quad \phi(v) = \iint_\Omega f v \, dA. \tag{18}$$

Now, consider the inverse problem of recovering an estimate of the diffusivity $\kappa(x,y)$ given $f(x,y)$ and a set of values of the solution u inside Ω. Using (16), we solve the inverse problem by minimizing $F_n(\lambda)$. To produce a specific example, we set

$$\kappa_{true}(x,y) = 2 + 8xy^2(1-x) \quad \text{and} \quad u(x,y) = x(1-x)\sin(\pi y) \in H_0^1([0,1]^2).$$

We determine $f(x,y)$ from (17). We use the 49 data values $u\left(\frac{i}{8}, \frac{j}{8}\right)$, $i,j = 1,..,7$. The grid of data points induces a finite element basis for $V_{7,7} \subset H$, within which we seek to recover an estimate of the 49 basis coefficients for κ. Minimization of $F_n(\lambda)$ produces a $\kappa(x,y)$ satisfying $\|\kappa(x,y) - \kappa_{true}(x,y)\|_2 = 0.0128$. Figure 1 presents the graphs of $\kappa_{true}(x,y)$ and the recovered $\kappa(x,y)$.

4.2. Parabolic equations

Consider the following abstract formulation of a parabolic equation,

$$\begin{cases} \langle \frac{d}{dt} u, v\rangle = \psi(v) + a(u,v) \\ u(0) = f\,, \end{cases} \tag{19}$$

where H is a Hilbert space, $\psi : H \to \mathbb{R}$ is a linear functional, $a : H \times H \to \mathbb{R}$ is a bilinear form and $f \in H$ is an initial condition. The inverse problem for the above equation consists of finding an approximation of the coefficients and parameters starting from a sample of observations of a target $u \in H$. To do this, we consider a family of bilinear functionals a_λ and let u_λ be the solution to

$$\begin{cases} \langle \frac{d}{dt} u_\lambda, v\rangle = \psi(v) + a_\lambda(u_\lambda, v) \\ u_0 = f\,. \end{cases} \tag{20}$$

We wish to determine if there exists a value of the parameter λ such that $u_\lambda = u$ or, more realistically, such that $\|u_\lambda - u\|$ is sufficiently small. To this end, Theorem 6 states that the distance between the target solution u and the solution u_λ of (20) can be reduced by minimizing a functional which depends on parameters.

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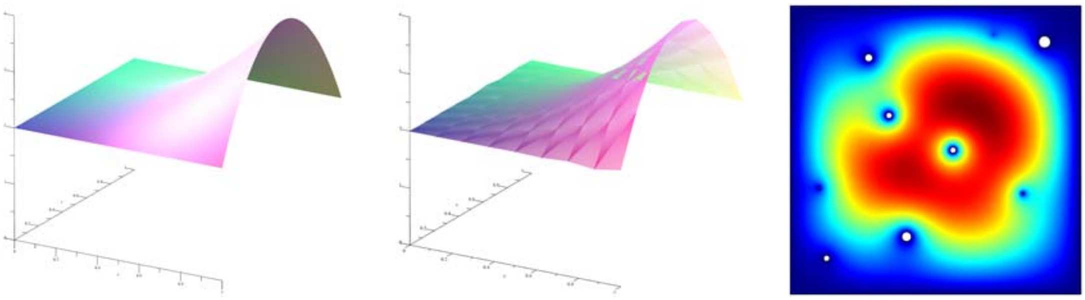

(a) $z = \kappa_{true}(x, y)$ (b) $z = \kappa(x, y)$ (c) Isotherms of $u(x, y)$

Figure 1. (a-b) The graphs of $\kappa_{true}(x, y)$ and the recovered $\kappa(x, y)$ in Example 1. (c) Isotherms of $u(x, y)$ on the perforated domain in Example 2.

Theorem 6 *[11] Let $u : [0, T] \to L^2(D)$ be the target solution which satisfies the initial condition in (19) and suppose that $\frac{d}{dt}u$ exists and belongs to H. Suppose that $a_\lambda(u, v) : \mathcal{F} \times H \times H \to \mathbb{R}$ is a family of bilinear forms for all $\lambda \in \mathcal{F}$. We have the following result:*

$$\int_0^T \|u - u_\lambda\|_H dt \leq \frac{1}{m_\lambda^2} \int_0^T \left(\sup_{\|v\|=1} \left\langle \frac{d}{dt}u, v \right\rangle - \psi(v) - a_\lambda(u, v) \right)^2 dt\,, \tag{21}$$

where u_λ is the solution of (20) s.t. $u_\lambda(0) = u(0)$ and $u_\lambda(T) = u(T)$.

Whenever $\inf_{\lambda \in \mathcal{F}} m_\lambda \geq m > 0$ then the previous result states that in order to solve the inverse problem for the parabolic equation (19) one can minimize the following functional,

$$\int_0^T \left(\sup_{\|v\|=1} \left\langle \frac{d}{dt}u, v \right\rangle - \psi(v) - a_\lambda(u, v) \right)^2 dt\,, \tag{22}$$

over all $\lambda \in \mathcal{F}$.

4.3. Hyperbolic equations

Let us now consider the following weakly-formulated hyperbolic equation,

$$\begin{cases} \langle \frac{d^2}{dt^2}u, v \rangle = \psi(v) + a(u, v) \\ u(0) = f \\ \frac{d}{dt}u(0) = g\,, \end{cases} \tag{23}$$

where $\psi : H \to \mathbb{R}$ is a linear functional, $a : H \times H \to \mathbb{R}$ is a bilinear form, and $f, g \in H$ are the initial conditions. As in previous sections, the inverse problem for the above system of equations is to reconstruct the coefficients starting from a sample of observations of a target $u \in H$. We consider a family of bilinear functionals a_λ and let u_λ be the solution to the following problem,

$$\begin{cases} \langle \frac{d}{dt}u_\lambda, v \rangle = \psi(v) + a_\lambda(u_\lambda, v) \\ u_0 = f \\ \frac{d}{dt}u(0) = g\,. \end{cases} \tag{24}$$

We wish to determine if there exists a value of the parameter λ such that $u_\lambda = u$ or, more realistically, such that $\|u_\lambda - u\|$ is sufficiently small. Theorem 7 states that the distance between the target solution u and the solution u_λ of (24) can be reduced by minimizing a functional which depends on parameters.

Theorem 7 *Let $u : [0,T] \to L^2(D)$ be the target solution which satisfies the initial condition in (23) and suppose that $\frac{d^2}{dt^2}u$ exists and belongs to H. Suppose that there exists a family of $m_\lambda > 0$ such that $a_\lambda(v,v) \geq m_\lambda \|v\|^2$ for all $v \in H$. We have the following result:*

$$\int_0^T \|u_t - (u_\lambda)_t\|^2 dt \leq \frac{1}{m_\lambda^2} \int_0^T \left(\sup_{\|v\|=1} \left\langle \frac{d^2}{dt^2} u_t, v \right\rangle - \psi(v) - a(u_t, v) \right)^2 dt \,, \tag{25}$$

where $(u_\lambda)_t$ is the solution of (24) s.t. $u(0) = (u_\lambda)(0)$ and $u(T) = (u_\lambda)(T)$.

The proof of the theorem follows the same path as that of Theorem 6.

5. Inverse Problems for DEs using a Collage Theorem for Banach spaces

The results presented in the previous two sections have been extended to a wider class of elliptic equations problems by considering not only Hilbert spaces but also reflexive Banach spaces. Let us mention that this kind of formulation arises, for instance, when the boundary constraints are weakly imposed. Details can be found in [7, 8, 31]. The following result presents an extended version of the Lax–Milgram theorem.

Let $N \geq 1$, E, $F_1, \ldots, F_N$ are real vector spaces, $a_1 : E \times F_1 \longrightarrow \mathbb{R}, \ldots, a_N : E \times F_N \longrightarrow \mathbb{R}$ are bilinear forms and $y_1^* : F_1 \longrightarrow \mathbb{R}, \ldots, y_N^* : F_N \longrightarrow \mathbb{R}$ and consider the system,

$$x \in E \text{ such that} \begin{cases} y_1^* = a_1(x, \cdot) \\ \vdots \\ y_N^* = a_N(x, \cdot) \,. \end{cases}$$

If this system admits a solution, then such a solution is unique if and only if, the corresponding homogeneous problem has one and only one solution. Given a real normed space G, we write G^* for its topological dual space.

Theorem 8 *[7, 8, 31] Suppose that E is a real reflexive Banach space, $N \geq 1$, $F_1, \ldots, F_N$ are Banach spaces and that $a_1 : E \times F_1 \longrightarrow \mathbb{R}, \ldots, a_N : E \times F_N \longrightarrow \mathbb{R}$ are continuous bilinear forms. Then, for all $y_1^* \in F_1^*, \ldots, y_N^* \in F_N^*$ there exists a unique $x_0 \in E$ such that*

$$\begin{cases} y_1^* = a_1(x_0, \cdot) \\ \vdots \\ y_N^* = a_N(x_0, \cdot) \end{cases}$$

if and only if

$$x \in E \text{ and } \left. \begin{array}{c} y_1^* = a_1(x_0, \cdot) \\ \vdots \\ y_N^* = a_N(x_0, \cdot) \end{array} \right\} \Rightarrow x = 0$$

and there exists $\rho > 0$ satisfying

$$(y_1, \ldots, y_N) \in F_1 \times \cdots \times F_N \;\Rightarrow\; \rho \sum_{k=1}^N \|y_k\| \leq \left\| \sum_{k=1}^N a_k(\cdot, y_k) \right\| .$$

Moreover, if these equivalent conditions hold and $x_0 \in E$ is the unique solution, then

$$\|x_0\| \leq \frac{1}{\rho} \max_{k=1,\ldots,N} \|y_k^*\|.$$

The above result implies the following corollary which represents an extended version of the collage theorem.

Corollary 1 *[7] Let E be a real reflexive Banach space, let $N \geq 1$, let $F_1, \ldots, F_N$ be Banach spaces, let $y_1^* \in F_1^*, \ldots, y_N^* \in F_N^*$ and let Λ be a nonempty set such that for all $\lambda \in \Lambda$ there exist N continuous bilinear forms $a_{1\lambda} : E \times F_1 \longrightarrow \mathbb{R}, \ldots, a_{N\lambda} : E \times F_N \longrightarrow \mathbb{R}$ and $\rho_\lambda > 0$ with*

$$x \in E \text{ and } \left. \begin{array}{c} y_1^* = a_{1\lambda}(x_0, \cdot) \\ \vdots \\ y_N^* = a_{N\lambda}(x_0, \cdot) \end{array} \right\} \Rightarrow x = 0$$

and

$$(y_1, \ldots, y_N) \in F_1 \times \cdots \times F_N \Rightarrow \rho_\lambda \sum_{k=1}^{N} \|y_k\| \leq \left\| \sum_{k=1}^{N} a_{k\lambda}(\cdot, y_k) \right\|.$$

Let us also suppose that for all $\lambda \in \Lambda$, $x_\lambda \in E$ is the unique solution of the variational system,

$$x \in E \text{ and } \left\{ \begin{array}{c} y_1^* = a_{1\lambda}(x, \cdot) \\ \vdots \\ y_N^* = a_{N\lambda}(x, \cdot) \end{array} \right. .$$

Then for each $x_0 \in E$ and for all $\lambda \in \Lambda$ the inequality,

$$\|x_\lambda - x_0\| \leq \frac{1}{\rho_\lambda} \max_{k=1,\ldots,N} \|y_k^* - a_{k\lambda}(x_0, \cdot)\|,$$

is valid.

Let us observe that if one wants to approximate the solution x_0 in the sense of the collage distance, that is, minimize $\{\|x_\lambda - x_0\| : \lambda \in \Lambda\}$, then according to Corollary 1, it suffices to minimize

$$\left\{ \frac{1}{\rho_\lambda} \max_{k=1,\ldots,N} \|y_k^* - a_{k\lambda}(x_0, \cdot)\| : \lambda \in \Lambda \right\}.$$

If $\rho := \inf_{\lambda \in \Lambda} \rho_\lambda > 0$, then the approximation problem is reduced to

$$\left\{ \max_{k=1,\ldots,N} \|y_k^* - a_{k\lambda}(x_0, \cdot)\| : \lambda \in \Lambda \right\}.$$

Some more details about the implementation of the numerical scheme and more numerical examples that demonstrate the validity of this approach can be found in [7, 8, 31].

6. Inverse Problems for DEs on perforated domains using the Collage Theorem

In this section we review one of the latest applications of the Collage Theorem to the solution of inverse problems, namely, on perforated or porous media. The results recalled in this section can be found with more details and applications in [32, 33]. In addition, [34] provides a numerical exploration for systems of DEs on perforated domains. When a differential equation is formulated over a porous medium, the term "porous" implies that the state equation is written only in the solid portion (or matrix) while boundary conditions should be imposed on the entire boundary of the matrix, including the boundary of the pore space (or holes).

Given a compact and convex set Ω, we denote by Ω_B the collection of circular holes $\cup_{j=1}^m B(x_j, \varepsilon_j)$ where $x_j \in \Omega$, ε_j are strictly positive numbers, and the holes $B(x_j, \varepsilon_j)$ are assumed to be nonoverlapping and to lie strictly in the interior of Ω. Let $\varepsilon = \max_j \varepsilon_j$. and denote by Ω_ε the closure of the set $\Omega \backslash \Omega_B$. In this section, we set $H = H_0^1(\Omega)$ and $H_\varepsilon = H_0^1(\Omega_\varepsilon)$.

In [32, 33] we have considered the linear system

$$\text{Find } u \in H \text{ that satisfies } a_\lambda(u,v) = \varphi(v) \quad \forall v \in H. \tag{P}$$

where $\lambda \in \Lambda$ denotes some parameters of the functionals and the corresponding system on the domain with holes,

$$\text{Find } u \in H_\epsilon \text{ that satisfies } a_\lambda(u,v) = \varphi(v) \;\; \forall v \in H_\epsilon. \tag{P_ε}$$

Our goal is to address the following inverse problem: Given observational data for a solution to (P_ε), estimate λ. Our approach is to use the data in (P) to estimate λ by establishing connections between the parameters λ in (P) and (P_ε) for ε small.

Since any function in $H_0^1(\Omega_\varepsilon)$ can be extended to be zero-valued over the holes, $H_0^1(\Omega_\varepsilon)$ can be embedded in $H_0^1(\Omega)$. Given u, letting $P_\varepsilon u$ be the projection of $u \in H_0^1(\Omega_\varepsilon)$ onto $H_0^1(\Omega)$, one can prove

$$\|u - P_\varepsilon u\|_{H_0^1(\Omega)} \to 0 \text{ whenever } \varepsilon \to 0.$$

When Neumann boundary conditions are considered, it is still possible to extend a function in $H_0^1(\Omega_\varepsilon)$ to a function of $H_0^1(\Omega)$.

Now supposing that there exist positive constants m, M, and μ such that

$$\begin{aligned} a_{\lambda,\epsilon}(u,u) &\geq m\|u\|^2 \;\; \forall u \in H_\varepsilon \;\; \text{(coercivity)} \\ a_{\lambda,\epsilon}(u,v) &\leq M\|u\|\|v\| \;\; \forall u,v \in H_\varepsilon \\ \phi_{\lambda,\epsilon}(u) &\leq \mu\|u\| \;\; \forall u \in H_\varepsilon \end{aligned}$$

means that problem (P) has a unique solution u^λ for each $\lambda \in \Lambda$ and problem (P_ε) has a unique solution u_ε^λ and for each positive ε and each $\lambda \in \Lambda$, by the Lax-Milgram type theorem in [31]. Now, we can establish relationships between (P) and (P_ε). For each $u \in (H_0^1(\Omega_\varepsilon))$, define

$$F_\varepsilon(u,\lambda) = \sup_{v \in H_\varepsilon, \|v\|=1} \|a_\lambda^\varepsilon(u,v) - \phi_\lambda^\varepsilon(v)\|\,. \tag{26}$$

The following four results were proved in [32].

Proposition 1 *The function $F(u,\lambda)$ is Lipschitz with Lipschitz constant equal to M.*

Proposition 2 *The following inequality holds:*

$$\|P_\varepsilon u - u_\varepsilon^\lambda\|_{H_\varepsilon} \leq \frac{F(u,\lambda)}{m} + \frac{M}{m}\|P_\varepsilon u - u\|_H\,.$$

Proposition 3 *The exists a constant $C(\varepsilon,u)$, which depends only on ε and u, such that the following inequality holds:*

$$F(P_\varepsilon u,\lambda) \leq F_\varepsilon(P_\varepsilon u,\lambda) + C(\varepsilon,u) \sup_{\|v\|=1} \|P_\varepsilon v - v\|_{H_\varepsilon}\,.$$

The nature of C in Proposition 3 is changed slightly compared to [32], but the same proof works. Note that the constant $C(\epsilon,u) = M\|P_\varepsilon u\|_{H_\varepsilon} + \mu$ converges to $C(u) = M\|u\| + \mu$ whenever $\varepsilon \to 0$. We have also corrected an imprecision in the statement of Proposition 4 from [32], with the same proof working.

Proposition 4 *Let us suppose that, for each fixed* $u \in H_0^1(\Omega_\varepsilon)$*,* F *is lower continuous w.r.t.* $\lambda \in \Lambda$*. If* $\lambda_\varepsilon = \arg\min_{\lambda\in\Lambda} F_\varepsilon(P_\varepsilon u, \lambda)$*, and* $\lambda_\varepsilon \to \lambda^* \in \Lambda$ *then* $\lambda^* = \arg\min_{\lambda\in\Lambda} F(u, \lambda)$*.*

Example 2: We consider the model problem (17), replacing the domain Ω by a domain Ω_ε that has a number of holes. Using the same $\kappa_{true}(x, y) = 2 + 8x^2y - 8x^2y^2$ and $f(x, y)$ as in Example 1, we solve the problem numerically on Ω_ε, using homogeneous Dirichlet boundary conditions on the interior holes. The isotherms of the solution are depicted in Figure 1(c). As in the earlier example, we sample this solution at 49 uniformly-distributed points inside Ω_ε; if a point lies inside a hole, we obtain no data for that point. Using this data points, we then solve the inverse problem on the region with no holes, Ω, appealing to Proposition 4. When we seek a κ of the form $\kappa(x, y) = \lambda_0 + \lambda_1 x^2y + \lambda_2 x^2y^2$, we obtain $(\lambda_0, \lambda_1, \lambda_2) = (2.704, 7.301, -7.934)$, with L^2 error 0.479. If we shrink the holes, use Neumann boundary conditions on them, and/or use more data points, the estimation improves.

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