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WORKING PAPER N. 16/2

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Abstract

The reliability of risk measures of financial portfolios crucially rests on the availability of sound representations of the involved random variables. The trade-off between adherence to reality and specification parsimony can find a fitting balance in a technique that ”adjust” the moments of a density function by making use of its associated orthogonal polynomials. This approach essentially rests on the Gram-Charlier expansion of a Gaussian law which, allowing for leptokurtosis to an appreciable extent, makes the resulting random variable a tail-sensitive density function.
In this paper we determine the density of sums of leptokurtic normal variables duly adjusted for excess kurtosis via their Gram-Charlier expansions based on Hermite polynomials. The aforesaid density can be properly used to compute some risk measures such as the Value at Risk and the expected short fall. An application to a portfolio of financial returns provides evidence of the effectiveness of the proposed approach.

Keywords: Gram-Charlier expansion, Value at Risk, Expected shortfall.
JEL classification: C40, C46, C58, G10, G20.

1 Introduction

In the last decades, both the convergence of the financial and insurance markets and the evolution of the financial engineering in pure financial contracts and in new financial-linked insurance contracts, have brought to the fore the importance of an accurate evaluation of the financial risk. In this connection, the choice of the appropriate distribution function underlying the measure of financial risks is a key problem for operators and analyists.
Commonly used statistical models as well as several applications rests on the assumption that asset returns are by and large normally distributed. Empirical evidence, has highlighted by many authors like Mittnik *et al.* (2000) and Alles and Murray (2010), provides sound arguments against this hypothesis. As a matter of fact, it is a well known stylized fact that financial time series exhibit tails heavier than those of the Normal distribution. This feature turns out to be of prominent importance in modelling volatility (Shuangzhe, 2006; Curto *et al.*, 2009) and more in general in the evaluation of the risk of portfolios (Szegö, 2004). This has pushed, on the one hand, to the use of alternative distributions like the Student t, the Pearson type VII, normal inverse Gaussian, several stable distributions (see e.g., Mills and Markellos (2008); Rachev *et al.* (2010)), and, on the other hand to the development of approaches aiming at transforming the Gaussian law so as to meet the desired features (see Gallant and Tauchen (1989, 1993); Jondeau and Rockinger (2001); Zoia (2010)). This latter approach, which has the the advantage of allowing for greater flexibility in fitting empirical distributions, is the one we have followed in this paper. Recently, Zoia (2010); Bagnato *et al.* (2015)) has proposed a method to account for excess kurtosis of a density based on its polynomial transformation through its associated orthogonal polynomials. In the Gaussian case, these polynomials are the Hermite ones and the polynomially modified density is known as Gram-Charlier expansion. This approach is very interesting because it can be tailored on the specific features of the empirical distribu-

tion at hand and can be extended to other distributions besides the normal one (see Faliva *et al.* (2016)).
In this paper it is used to obtain the densities of sums of leptokurtic normal random variables with same or different kurtosis. After adjusting these latter with appropriate Hermite polynomials, the density functions of their sum is obtained. The resulting densities prove to be more tail-sensitive than an ordinary Gaussian distribution and as such suitable for measuring the well known Value at Risk. Further, being information on the magnitude of high risks extremely important, they are also used to compute a coherent risk measure like the Expected Shortfall.
An application to a portfolio of two international financial indexes with a data-set window covering the period from January 2009 to December 2014 proves the good performance of these Gram-Charlier expansions. In accordance with the regulatory framework the risk measures are evaluated at 97.5% and 99% levels to guarantee a prudential approach.
The structure of the paper is as follows. In section 2 we cast a glance at some standard risk-measures, typically used in the financial-insurance market. Section 3 explains how to obtain densities which are sums of Gram-Charlier expansions and section 4 gives closed-form expressions of the expected shortfall based on these distributions. Section 5 provides an application of these densities to a portfolio of financial returns and Section 6 concludes.

2 A glance at risk measures

As is well known, different approaches are available to measure financial and/or insurance risks (see, for all, Albrecht (2004) and Dowd and Blake (2006), and the reference quoted therein). Descriptive measures based on the moments of a probability distribution give only a partial representation of a risk. To overcome this problem, often a combination of these measures is taken into account, as it happens for the mean and standard deviation in Markowitz portfolio theory or the skewness and kurtosis when

symmetry and probability concentration in tails are of interest. Unfortunately, the estimation of the moments of a probability distribution may be quite sensitive to the sample and, when the moments are infinite, even impossible.
The standard theory for decision under risks, based on the expected utility approach, may be of difficult implementation and sensitive to individual tolerance to risk, due to the choice of the functional form of the utility function and the evaluation of the risk attitude parameter.
Risk measures based on quantiles became very popular at the end of the 1980s, because of their implementation to determine the regulatory capital requirements of the US commercial banks. Value at risk based models were introduced in the Basel II agreement and later used for the calibration of the Solvency Capital Requirement, in the Solvency II agreement.
The Value at Risk (VaR) represents the minimum loss within a certain period of time for a given probability. By denoting with $F_X(x)$ the distribution function of a variable X representing the loss and with $v_q = inf\{x : F_X(x) \geq q\}$, $q \in (0, 1)$, the quantile function, then

$$VaR_X(q) = inf\{x : F_X(x) \geq q\} = F_X^{-1}(q)$$

In view of fact that VaR is simply the threshold at a given probability q, that is

$$VaR_X(q) = v_q \tag{1}$$

it doesn't provide information about the size of the losses beyond this point of the distribution, while knowing the size of default is crucial for shareholders, management and regulators. In addition, VaR is not a coherent risk measure (see Artzner P. (1999)) because of the lack of subadditivity. Being sub-additivity very important in several financial applications like portfolio optimization, VaR can discourage diversification. Moreover, VaR estimates results improper when losses/returns are not normally distributed and this shortcoming turns out to be very critic in presence of fat tails. Furthermore VaR-models based on scenarios, typical for

discrete data series, can exhibit multiple local extrema (see Uryasev (2000)).
The interest of financial and insurance managers in tail risks justifies the introduction of risk measures offering information on the magnitude of high risks. The Tail Conditional Expectation (TCE) is defined as

$$TCE_X(q) = \mathbb{E}[X|X \geq v_q] \tag{2}$$

and can be described as the average worst possible loss. The TCE is not in general a coherent measure of risk, because it can be not sub-additive. This drawback is evident when dealing with discontinuous distributions (for example with portfolios containing derivatives) when the measure becomes very sensitive to small changes in the confidence level.
A risk measure that respects the axioms of coherence is the Expected Shortfall (ES)

$$ES_X(q) = \frac{1}{q}\Big(\mathbb{E}[X\mathbf{1}_{\{X \geq v_q\}}] + v_q(\mathbb{P}[X \geq v_q]) - (1-q)\Big) \tag{3}$$

which is in general continuous with respect to the confidence level. For real-valued random variable with continuous and strictly increasing distribution function and finite mean, the following proves true (see Acerbi and Tasche (2002))

$$TCE_X(q) = ES_X(q) \tag{4}$$

3 On the distribution of the sum of polynomially-modified Gaussian variables

In this section we tackle the issue of specifying the density function of the sum of polynomially-modified (namely Gram-Charlier expansions of) Gaussian variables assuming independence. We start by presenting classical results for i.i.d Gaussian variables and then move to Gram-Charlier expansions either with equal or different kurtosis corrections. In this work we will take advantage of the relationships between density and characteristic function. We will show in few steps the classical procedure adopted for getting the density of the sum of two independent standard normal random variables. The same procedure is extended to an arbitrary number of variables and then generalized to polynomially-modified densities of Gram-Charlier type.

Let be $Y = X_1 + X_2$, where X_1 and X_2 are i.i.d. random variables. Then, the density function of Y takes the form

$$f_Y(y) = f_X(x_1) * f_X(x_2) \tag{5}$$

where the symbol $*$ denotes convolution. As is well known, the characteristic function of Y is the product of the characteristic functions of the parent variables

$$F_Y(\omega) = F_{X_1}(\omega) F_{X_2}(\omega) = F_X^2(\omega) \tag{6}$$

Bearing in mind the Fourier-transform pair

$$\sqrt{\frac{a}{\pi}} e^{-at^2} \leftrightarrow e^{-\frac{\omega^2}{4a}} \tag{7}$$

and setting $a = \frac{1}{2}$, yields the characteristic function of the standard normal distribution, that is

$$F_X(\omega) = e^{-\frac{\omega^2}{2}}. \tag{8}$$

According to (6), the characteristic function of the sum of two i.i.d. standard normal is

$$F_Y(\omega) = e^{-\omega^2}. \tag{9}$$

Likewise, by setting $a = 1/4$ in (7), the density function of the sum $f_Y(y)$ is easily obtained, that is

$$f_y(y) = (4\pi)^{-1/2} e^{-\frac{y^2}{4}} \tag{10}$$

The same procedure can be followed to obtain the density function of the sum of two Gram-Charlier expansions. In this connection we have the following

Theorem 1. *Let X_1 and X_2 be i.i.d variables with the following Gram-Charlier density*

$$f_X(x;\beta) = \left(1 + \frac{\beta}{4!}p_4(x)\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \tag{11}$$

where β is a positive parameter subject to $f_X(x;\beta)$ being non-negative definite, and

$$p_4(x) = x^4 - \binom{4}{2}x^{4-2} + 3\binom{4}{4}x^{4-4} = x^4 - 6x^2 + 3. \tag{12}$$

is the $4-th$ degree Hermite polynomial. The density function of the sum $Y = X_1 + X_2$ is

$$f_Y(x_1{+}x_2;\beta) = \left(1 + \frac{1}{2}\left(\frac{\beta}{4!}\right)p_4\left(\frac{y}{\sqrt{2}}\right) + \frac{1}{16}\left(\frac{\beta}{4!}\right)^2 p_8\left(\frac{y}{\sqrt{2}}\right)\right)\frac{1}{\sqrt{4\pi}}e^{-\frac{y^2}{4}} \tag{13}$$

where

$$p_8(x) = x^8 - \binom{8}{2}x^6 + 3\binom{8}{4}x^4 - 15\binom{8}{6}x^2 + 105\binom{8}{8}x \tag{14}$$

is the 8-th degree Hermite polynomial.

Proof. Bearing in mind the following property of Fourier transforms,

$$\frac{d^n f(x)}{dx^n} \leftrightarrow (i\omega)^n F(\omega) \tag{15}$$

together with the noteworthy property of the Gaussian law,

$$\frac{d^n \frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}}{dx^n} = (-1)^n\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}p_n(x) \tag{16}$$

the characteristic function of (11) is easily found to be

$$F_X(\omega;\beta) = \left(1 + \frac{\beta}{4!}\omega^4\right)e^{-\frac{\omega^2}{2}}. \tag{17}$$

By following the same argument, the characteristic function of the sum of two Gram-Charlier expansions proves to be

$$F_Y(\omega;\beta) = \left(1 + \frac{\beta}{4!}\omega^4\right)^2 e^{-\omega^2} = \sum_{j=0}^{2} \binom{2}{j} \left(\frac{\beta}{4!}\right)^j \omega^{4j} e^{-\omega^2}. \quad (18)$$

Now, thanks to the following property of Fourier transforms

$$|a| f(ay) \leftrightarrow F\left(\frac{\omega}{a}\right), \quad (19)$$

formula (15) can be more conveniently rewritten as

$$\frac{d^n |a| f_X(ax)}{dx^n} \leftrightarrow \left(i\frac{\omega}{a}\right)^n F\left(\frac{\omega}{a}\right) \quad (20)$$

and this, in light of (16), entails the following

$$(-1)^n \frac{|a|}{\sqrt{2\pi}} e^{-\frac{(ax)^2}{2}} p_n(ax) \leftrightarrow \left(i\frac{\omega}{a}\right)^n e^{-\frac{1}{2}\left(\frac{\omega}{a}\right)^2}. \quad (21)$$

Then, setting $a = \frac{1}{\sqrt{2}}$ in (21), yields

$$\left(\frac{1}{\sqrt{2}}\right)^{4j} \frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4}} p_{4j}\left(\frac{x}{\sqrt{2}}\right) \leftrightarrow \omega^{4j} e^{-\omega^2} \quad (22)$$

As a by-product, the density function of the sum of two Gram-Charlier expansions is given by

$$f_Y(x_1 + x_2;\beta) = \sum_{j=0}^{2} \binom{2}{j} \left(\frac{\beta}{4!}\right)^j \frac{1}{\sqrt{4\pi}} \left(\frac{1}{\sqrt{2}}\right)^{4j} e^{-\frac{y^2}{4}} p_{4j}\left(\frac{y}{\sqrt{2}}\right) \quad (23)$$

which proves the theorem. □

Let us now establish the generalization to n variables as a corollary.

Corollary 1. *Let us consider n i.i.d. random variables $X_1, \ldots, X_n$, each with density function as in formula (11). Then, the density function of the sum $Y = X_1 + \cdots + X_n$ is*

$$f_Y(x_1+\cdots+x_n;\beta) = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{\beta}{4!}\right)^j \frac{1}{\sqrt{2n\pi}} \left(\frac{1}{\sqrt{n}}\right)^{4j} e^{-\frac{y^2}{2n}} p_{4j}\left(\frac{y}{\sqrt{n}}\right). \tag{24}$$

Proof. Upon noting that the characteristic function of the sum of n variables is

$$F_Y(\omega;\beta) = \left(1 + \frac{\beta}{4!}\omega^4\right)^n e^{-\frac{n\omega^2}{2}} = \sum_{j=0}^{n} \binom{n}{j} \left(\frac{\beta}{4!}\right)^j \omega^{4j} e^{-\frac{n\omega^2}{2}} \tag{25}$$

and setting $a = \frac{1}{\sqrt{n}}$ in formula (21), we get

$$\left(\frac{1}{\sqrt{n}}\right)^{4j} \frac{1}{\sqrt{2n\pi}} e^{-\frac{y^2}{2n}} p_{4j}\left(\frac{y}{\sqrt{n}}\right) \leftrightarrow \omega^{4j} e^{-\frac{n\omega^2}{2}} \tag{26}$$

which eventually leads to the density in formula (24). □

The sum variable $Y = X_1, + \cdots + X_n$ depends on one parameter β that it is common to each X_i. In Zoia (2010) it is shown that the Gram-Charlier expansion (11) has positive density if $0 \leq \beta \leq 4$ and is unimodal if $0 \leq \beta \leq 2,4$. These constrains also hold in the case of the sum of n i.i.d variables, according to the Theorem 1.6 in Dharmadhikari (1988).
The graphs in Figure 1 depict the density functions of Gram-Charlier expansions when $n = 1$, $n = 2$ and $n = 3$. In each graph different values of β have been considered; in particular β has been set equal to 0 (its minimum value), equal to 2,4 (the maximum value which guarantees the unimodality of the Gram-Charlier density), and equal to 1 (an intermediate value in its range of variation).

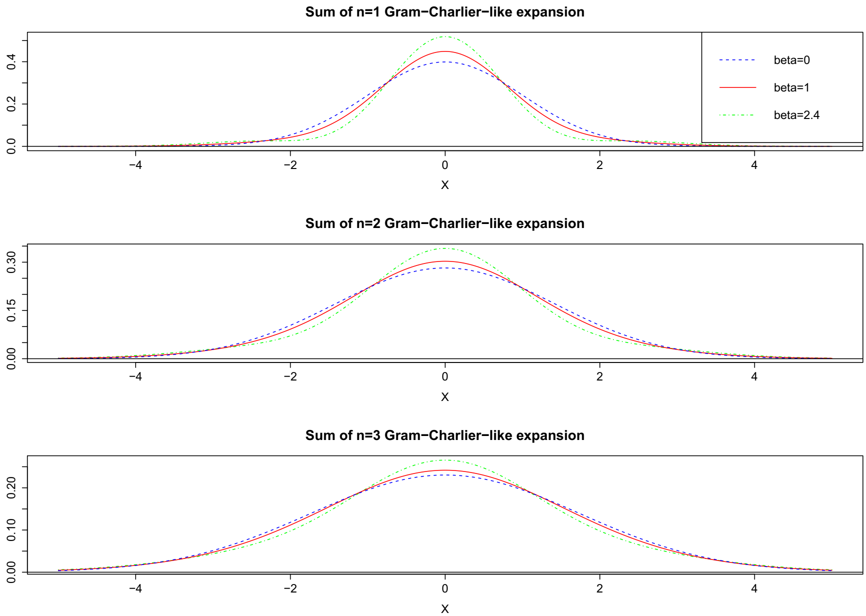

Figure 1: Gram-Charlier-like expansions for $n = 1, 2, 3$, and $\beta = 0, 1, 2.4$.

As a further extension of the Theorem 1, we prove the following corollary which covers the case of Gram-Charlier expansions of sum of variables characterized by different parameters $\beta's$.

Corollary 2. *Let us consider two independent Gram-Charlier expansions* X_1 *and* X_2*, characterized by the parameters* β_1 *and* β_2*, respectively. Then, the density function of the sum* $Y = X_1 + X_2$*, denoted with* $f_Y(x_1 + x_2; \beta_1, \beta_2)$*, is*

$$\begin{aligned} & f_Y(x_1 + x_2; \beta_1, \beta_2) = \\ & \left(1 + \tfrac{1}{4}\left(\tfrac{\beta_1+\beta_2}{4!}\right) p_4\left(\tfrac{y}{\sqrt{2}}\right) + \tfrac{1}{16}\tfrac{\beta_1\beta_2}{(4!)^2} p_8\left(\tfrac{y}{\sqrt{2}}\right)\right) \tfrac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}}. \end{aligned} \tag{27}$$

Proof. In light of (6), the characteristic function of Y is

$$\begin{aligned} F_{Y=X_1+X_2}(\omega; \beta_1, \beta_2) &= e^{-\omega^2} \prod_{j=1}^{2} \left(1 + \tfrac{\beta_j}{4!}\omega^4\right) = \\ &= e^{-\omega^2} \left(1 + \tfrac{\beta_1+\beta_2}{4!}\omega^4 + \tfrac{\beta_1\beta_2}{(4!)^2}\omega^8\right) \end{aligned} \tag{28}$$

and can be written as

$$F_{Y=X_1+X_2}(\omega; \beta_1, \beta_2) = \sum_{j=0}^{2} \frac{b_{2,j}}{(4!)^j} \omega^{4j} e^{-\omega^2} \tag{29}$$

where the coefficients $b_2 j$ are the sum of the combinations of the two parameters β_j taken j at a time, namely $b_{20} = 1$, $b_{21} = \sum_{j=1}^{2} \beta_j$ and $b_{22} = \prod_{j=1}^{2} \beta_j$.

Hence, by setting $a = \frac{1}{\sqrt{2}}$ in formula (21), with some computations we obtain (27). □

Finally, we state the following corollary which generalizes the statement of Corollary 2 to n variables with different excess kurtosis.

Corollary 3. *Let us consider* n *independent Gram-Charlier expansions of the random variables* $X_1, \ldots, X_n$*, characterized by parameters* $\beta_1 \ldots, \beta_n$*, respectively. Then, the density function of the sum* $Y = X_1 + \cdots + X_n$*, denoted with* $f_Y(x_1 + \cdots + x_n; \beta_1, \ldots, \beta_n)$*, is*

$$f_Y(x_1 + \cdots + x_n; \beta_1, \ldots, \beta_n) = \sum_{j=0}^{n} \left(\frac{b_{n,j}}{(4!)^j}\right) \frac{1}{\sqrt{2n\pi}} \left(\frac{1}{\sqrt{n}}\right)^{4j} e^{-\frac{y^2}{2n}} p_{4j}\left(\frac{y}{\sqrt{n}}\right) \tag{30}$$

where $b_{n,j}$ is the sum of the combinations of the n parameters β_j taken j at a time without repetition.

Proof. As in (6), the characteristic function, $F_{Y=X_1+\cdots+X_n}(\omega;\beta_1,\ldots,\beta_n)$ of the sum of the n Gram-Charlier expansions with different parameter is

$$e^{-\omega^2}\prod_{j=1}^{n}\left(1+\frac{\beta_j}{4!}\omega^4\right)= \\ e^{-\omega^2}\left(1+\frac{\beta_1+\cdots+\beta_n}{4!}\omega^4+\frac{\beta_1\beta_2+\cdots+\beta_{n-1}\beta_n}{(4!)^2}\omega^8+\cdots+\frac{\prod_{j=1}^{n}\beta_j}{(4!)^n}\omega^{n4}\right)= \\ \sum_{j=0}^{n}\frac{b_nj}{(4!)^j}\omega^{4j}e^{-\omega^2}. \tag{31}$$

Then, by setting $a=\frac{1}{\sqrt{n}}$ in formula (21), with some computations we get (30). □

This approach can be extended to other densities, besides the normal. However, when other distributions are considered, the density of the sum may be more conveniently obtained by making the convolution of the densities of the variables involved in the sum.

4 Expected Shortfall for sum of Gram- Charlier expansions

Gram-Charlier expansions (GC) prove able to catch the excess of kurtosis and the asymmetry of a random variable (rv) better than the usual normal density. and this property is true not only for a singular rv but also for densities which are sums of rvs.
Hence, the next step is to use GC to measure risks related to portfolios of insurance or financial assets. In this section, following the analysis of Landsman and Valdez (2003) on TCE for sums of elliptic distributions and bearing in mind the studies of Acerbi and Tasche (2002), we show how to compute the expected short fall, ES, to evaluate the right-tail risk of a sum of GC expansions. First we will consider the case of r.v with same excess kurtosis, then with different excess kurtosis.

Assuming that the loss is likely to exceed a certain value υ_q (referred to as the q-th-quantile), the ES is defined as follows

$$ES_Y(\upsilon_q) = E(Y|Y > \upsilon_q) = \frac{\int_{\upsilon_q}^{\infty} yf(y)dy}{\int_{\upsilon_q}^{\infty} f(y)dy} \tag{32}$$

where, for our purpose, $f(y) = f(x_1 + x_2)$.
The following theorem shows how the integrals in (32) can be evaluated by making use of the definition and properties of the error function and of the Hermite polynomials

Theorem 2. *Let $f(y, \beta)$ be defined as in (13).Then the ES of y takes the value*

$$ES_Y(\upsilon_q)) =$$
$$= \frac{\frac{1}{\sqrt{\pi}} e^{-\frac{\upsilon_q^2}{4}} \left[1 + \frac{1}{2}\left(\frac{\beta}{4!}\right)\left(p_4\left(\frac{\upsilon_q}{\sqrt{2}}\right) + 4p_2\left(\frac{\upsilon_q}{\sqrt{2}}\right)\right) + \frac{1}{16}\left(\frac{\beta}{4!}\right)^2 \left(p_8\left(\frac{\upsilon_q}{\sqrt{2}}\right) + 8p_6\left(\frac{\upsilon_q}{\sqrt{2}}\right)\right)\right]}{\frac{1}{2} erfc\left(\frac{\upsilon_q}{2}\right) + \frac{1}{\sqrt{2\pi}} e^{-\frac{\upsilon_q^2}{4}} \left[\frac{1}{2}\left(\frac{\beta}{4!}\right) p_3\left(\frac{\upsilon_q}{\sqrt{2}}\right) + \frac{1}{16}\left(\frac{\beta}{4!}\right)^2 p_7\left(\frac{\upsilon_q}{\sqrt{2}}\right)\right]} \tag{33}$$

Proof. Let us proceed by considering separately the numerator and the denominator of formula (32) which, in the following, will be denoted by A and B, respectively.
By replacing in the numerator A the density function $f(y, \beta)$ defined as in (13) we obtain

$$A = \int_{\upsilon_q}^{\infty} \left(y + \frac{y}{2}\left(\frac{\beta}{4!}\right) p_4\left(\frac{y}{\sqrt{2}}\right) + \frac{y}{16}\left(\frac{\beta}{4!}\right)^2 p_8\left(\frac{y}{\sqrt{2}}\right)\right) \frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}} dy =$$
$$= \underbrace{\int_{\upsilon_q}^{\infty} \frac{y}{\sqrt{4\pi}} e^{-\frac{y^2}{4}} dy}_{A1} + \underbrace{\int_{\upsilon_q}^{\infty} \frac{y}{2}\left(\frac{\beta}{4!}\right) p_4\left(\frac{y}{\sqrt{2}}\right) \frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}} dy}_{A2} +$$
$$\underbrace{\int_{\upsilon_q}^{\infty} \frac{y}{16}\left(\frac{\beta}{4!}\right)^2 p_8\left(\frac{y}{\sqrt{2}}\right) \frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}} dy}_{A3} \tag{34}$$

By setting $t = \frac{y}{\sqrt{2}}$ in $A1$ and bearing in mind that $p_1(t) = t$, we get

$$\begin{aligned} A_1 =& \frac{1}{\sqrt{\pi}} \int_{\frac{\upsilon_q}{\sqrt{2}}}^{\infty} t e^{-\frac{1}{2}t^2} dt \\ =& \frac{1}{\sqrt{\pi}} \int_{\frac{\upsilon_q}{\sqrt{2}}}^{\infty} p_1(t) e^{-\frac{1}{2}t^2} dt \end{aligned} \tag{35}$$

Now, observe that in light of (16), the following

$$\frac{d}{dy}\left[\frac{d^n}{dy^n} e^{-\frac{y^2}{2}}\right] = \frac{d^{n+1}}{dy^{n+1}} e^{-\frac{y^2}{2}} = (-1)^{n+1} e^{-\frac{y^2}{2}} p_{n+1}(y) \tag{36}$$

holds true.
This entails that

$$\begin{aligned} \int (-1)^{n+1} e^{-\frac{y^2}{2}} p_{n+1}(y) dy = \int \frac{d^{n+1}}{dy^{n+1}} e^{-\frac{y^2}{2}} = \\ = \frac{d^n}{dy^n} e^{-\frac{y^2}{2}} = \\ = (-1)^n e^{-\frac{y^2}{2}} p_n(y) \end{aligned} \tag{37}$$

By using this result, formula (35) becomes

$$\begin{aligned} A_1 &= -\frac{1}{\sqrt{\pi}} \int_{\frac{\upsilon_q}{\sqrt{2}}}^{\infty} (-1)^1 p_1(t) e^{-\frac{t^2}{2}} dt \Big|_{\frac{\upsilon_q}{\sqrt{2}}}^{\infty} = \\ &= -\frac{1}{\sqrt{\pi}} e^{-\frac{y^2}{2}} \Big|_{\frac{\upsilon_q}{\sqrt{2}}}^{\infty} \\ &= \frac{1}{\sqrt{\pi}} e^{-\frac{\upsilon_q^2}{4}} \end{aligned} \tag{38}$$

Following the same procedure, we can compute the integrals in A_2 and A_3.
In particular, by setting $t = \frac{y}{\sqrt{2}}$ in A_2 yields

$$A_2 = \frac{1}{2\sqrt{\pi}} \left(\frac{\beta}{4!}\right) \int_{\frac{\upsilon_q}{\sqrt{2}}}^{\infty} t p_4(t) e^{-\frac{1}{2}t^2} dt. \tag{39}$$

Now, observe that the integral (39) can be rewritten as

$$A_2 = \frac{1}{2\sqrt{\pi}} \left(\frac{\beta}{4!}\right) \left(\int_{\frac{\upsilon_q}{\sqrt{2}}}^{\infty} p_5(t) e^{-\frac{1}{2}t^2} dt + 4 \int_{\frac{\upsilon_q}{\sqrt{2}}}^{\infty} p_3(t) e^{-\frac{1}{2}t^2} dt \right). \quad (40)$$

in light of the following property of Hermite polynomials

$$p_{n+1}(y) = y p_n(y) - n p_{n-1}(y) \quad (41)$$

Now, by applying again formula (37) to the integral in (40), with simple computation we get

$$A_2 = \frac{1}{2\sqrt{\pi}} \left(\frac{\beta}{4!}\right) e^{-\frac{\upsilon_q^2}{4}} \left(p_4\left(\frac{\upsilon_q}{\sqrt{2}}\right) + 4 p_2\left(\frac{\upsilon_q}{\sqrt{2}}\right) \right). \quad (42)$$

Following the same approach, the integral A_3 can be written as

$$A_3 = \frac{1}{16\sqrt{\pi}} \left(\frac{\beta}{4!}\right)^2 e^{-\frac{\upsilon_q^2}{4}} \left(p_8\left(\frac{\upsilon_q}{\sqrt{2}}\right) + 8 p_6\left(\frac{\upsilon_q}{\sqrt{2}}\right) \right). \quad (43)$$

Hence, the numerator A turns out to be

$$\begin{aligned} A =& \frac{1}{\sqrt{\pi}} e^{-\frac{\upsilon_q^2}{4}} + \frac{1}{2\sqrt{\pi}} \left(\frac{\beta}{4!}\right) e^{-\frac{\upsilon_q^2}{4}} \left(p_4\left(\frac{\upsilon_q}{\sqrt{2}}\right) + 4 p_2\left(\frac{\upsilon_q}{\sqrt{2}}\right) \right) + \\ & \frac{1}{16\sqrt{\pi}} \left(\frac{\beta}{4!}\right)^2 e^{-\frac{\upsilon_q^2}{4}} \left(p_9\left(\frac{\upsilon_q}{\sqrt{2}}\right) + 8 p_6\left(\frac{\upsilon_q}{\sqrt{2}}\right) \right) \end{aligned} \quad (44)$$

We proceed similarly splitting the denominator B of formula (32) into three parts

$$\begin{aligned} B = & \underbrace{\int_{\upsilon_q}^{\infty} \frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}} dy}_{B1} + \underbrace{\int_{\upsilon_q}^{\infty} \frac{1}{2} \left(\frac{\beta}{4!}\right) p_4\left(\frac{y}{\sqrt{2}}\right) \frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}} dy}_{B2} + \\ & \underbrace{+ \int_{\upsilon_q}^{\infty} \frac{1}{16} \left(\frac{\beta}{4!}\right)^2 p_8\left(\frac{y}{\sqrt{2}}\right) \frac{1}{\sqrt{4\pi}} e^{-\frac{y^2}{4}} dy}_{B3}. \end{aligned} \quad (45)$$

Then, by replace $\frac{y}{2}$ with t in the integral B_1 and taking into account formula 7.1.2 in Abramowitz and Stegun, we get

$$\begin{aligned} B_1 &= \frac{1}{\sqrt{\pi}} \int_{\frac{\upsilon_q}{\sqrt{2}}}^{\infty} e^{-t^2} dt = \\ &= \frac{1}{2}\text{erfc}\left(\frac{\upsilon_q}{2}\right). \end{aligned} \tag{46}$$

where erfc(x) is the complementary Gauss error function. The second and third integral in (45) can be evaluated by using the same approach followed before for the integral A_1. This leads to the following results

$$B_2 = \frac{1}{2\sqrt{2\pi}} \left(\frac{\beta}{4!}\right) e^{-\frac{\upsilon_q^2}{4}} p_3 \left(\frac{\upsilon_q}{\sqrt{2}}\right) \tag{47}$$

$$B_3 = \frac{1}{16\sqrt{2\pi}} \left(\frac{\beta}{4!}\right)^2 e^{-\frac{\upsilon_q^2}{4}} p_7 \left(\frac{\upsilon_q}{\sqrt{2}}\right). \tag{48}$$

Accordingly, the denominator B turns out to be

$$B = \frac{1}{2}\text{erfc}\left(\frac{\upsilon_q}{2}\right) + \frac{1}{2\sqrt{2\pi}} \left(\frac{\beta}{4!}\right) e^{-\frac{\upsilon_q^2}{4}} p_3 \left(\frac{\upsilon_q}{\sqrt{2}}\right) + \frac{1}{16\sqrt{2\pi}} \left(\frac{\beta}{4!}\right)^2 e^{-\frac{\upsilon_q^2}{4}} p_7 \left(\frac{\upsilon_q}{\sqrt{2}}\right) \tag{49}$$

Finally, by replacing the numerator of formula (32) with (44) and the denominator of the same formula with (49), respectively, we eventually obtain formula (33). □

The ES_Y can be easily extended to the sum of n independent variables. In particular, the following corollary shows the expression of ES_Y for the sum of n i.i.d Gram-Charlier expansions.

Corollary 4. *Let us consider the sum of n i.i.d Gram-Charlier expansions $Y = X_1 + \cdots + X_n$. Then, the $ES_Y(\upsilon_q)$ has the following form*

$$ES_Y(\upsilon_q) = \frac{\sqrt{\frac{n}{2\pi}} e^{-\frac{\upsilon_q^2}{2n}} \left[1 + \sum_{j=1}^{n} \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^j \left(p_{4j}\left(\frac{\upsilon_q}{\sqrt{n}}\right) + 4j p_{4j-2}\left(\frac{\upsilon_q}{\sqrt{n}}\right)\right)\right]}{\frac{1}{2} erfc\left(\frac{\upsilon_q}{\sqrt{2n}}\right) + \frac{1}{\sqrt{2\pi}} e^{-\frac{\upsilon_q^2}{2n}} \left[\sum_{j=1}^{n} \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^j p_{4j-1}\left(\frac{\upsilon_q}{\sqrt{n}}\right)\right]} \tag{50}$$

Proof. The proof follows the same lines of Theorem 2. When $f(y)$ is as in (24), the numerator of formula (32), henceforth denoted with A, can be written as

$$A=\sum_{j=0}^{n}\left(\frac{1}{\sqrt{n}}\right)^{4j}\binom{n}{j}\left(\frac{\beta}{4!}\right)^{j}\frac{1}{\sqrt{2\pi}}\int_{\upsilon_q}^{\infty}\frac{y}{\sqrt{n}}e^{-\frac{y^2}{2n}}p_{4j}\left(\frac{y}{\sqrt{n}}\right)dy=$$
$$=\underbrace{\frac{1}{\sqrt{2\pi}}\int_{\upsilon_q}^{\infty}\frac{y}{\sqrt{n}}e^{-\frac{y^2}{2n}}}_{A_1}+\underbrace{\sum_{j=1}^{n}\left(\frac{1}{\sqrt{n}}\right)^{4j}\binom{n}{j}\left(\frac{\beta}{4!}\right)^{j}\frac{1}{\sqrt{2\pi}}\int_{\upsilon_q}^{\infty}\frac{y}{\sqrt{n}}e^{-\frac{y^2}{2n}}p_{4j}\left(\frac{y}{\sqrt{n}}\right)dy}_{A_2} \tag{51}$$

Then setting $t=\frac{y}{\sqrt{n}}$ in A_1 and using formula (37) we get

$$A_1=\sqrt{\frac{n}{2\pi}}e^{-\frac{\upsilon_q^2}{2n}} \tag{52}$$

while setting $t=\frac{y}{\sqrt{n}}$ in the integral A_2,and making use of both (41) and (37), yields

$$A_2=\sqrt{\frac{n}{2\pi}}e^{-\frac{\upsilon_q^2}{2n}}\sum_{j=1}^{n}\left(\frac{1}{\sqrt{n}}\right)^{4j}\binom{n}{j}\left(\frac{\beta}{4!}\right)^{j}\left(p_{4j}\left(\frac{\upsilon_q}{\sqrt{n}}\right)+4jp_{4j-2}\left(\frac{\upsilon_q}{\sqrt{n}}\right)\right). \tag{53}$$

Accordingly the integral A becomes

$$\begin{aligned}A=\;&\sqrt{\tfrac{n}{2\pi}}e^{-\frac{\upsilon_q^2}{2n}}+\\&+\sqrt{\tfrac{n}{2\pi}}e^{-\frac{\upsilon_q^2}{2n}}\textstyle\sum_{j=1}^{n}\left(\frac{1}{\sqrt{n}}\right)^{4j}\binom{n}{j}\left(\frac{\beta}{4!}\right)^{j}\left(p_{4j}\left(\frac{\upsilon_q}{\sqrt{n}}\right)+4jp_{4j-2}\left(\frac{\upsilon_q}{\sqrt{n}}\right)\right)\end{aligned} \tag{54}$$

Similarly, after replacing $f(y)$, in (24), in the denominator of (32), we get

$$B=\sum_{j=0}^{n}\left(\frac{1}{\sqrt{n}}\right)^{4j}\binom{n}{j}\left(\frac{\beta}{4!}\right)^{j}\frac{1}{\sqrt{2\pi}}\int_{\upsilon_q}^{\infty}\frac{1}{\sqrt{n}}e^{-\frac{y^2}{2n}}p_{4j}\left(\frac{y}{\sqrt{n}}\right)dy=$$
$$=\underbrace{\frac{1}{\sqrt{2\pi}}\int_{\upsilon_q}^{\infty}\frac{1}{\sqrt{n}}e^{-\frac{y^2}{2n}}}_{B_1}+\underbrace{\sum_{j=1}^{n}\left(\frac{1}{\sqrt{n}}\right)^{4j}\binom{n}{j}\left(\frac{\beta}{4!}\right)^{j}\frac{1}{\sqrt{2\pi}}\int_{\upsilon_q}^{\infty}\frac{1}{\sqrt{n}}e^{-\frac{y^2}{2n}}p_{4j}\left(\frac{y}{\sqrt{n}}\right)dy}_{B_2}. \tag{55}$$

Now, setting $t = \frac{y}{\sqrt{2n}}$ in the integral B_1 yields

$$B_1 = \frac{1}{2}\text{erfc}\left(\frac{\upsilon_q}{\sqrt{2n}}\right). \tag{56}$$

and setting $t = \frac{y}{\sqrt{n}}$ in the integral B_2 and using the result (37), yields

$$B_2 = \frac{1}{\sqrt{2\pi}} e^{-\frac{\upsilon_q^2}{2n}} \left[\sum_{j=1}^{n} \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^{j} p_{4j-1}\left(\frac{\upsilon_q}{\sqrt{n}}\right)\right]. \tag{57}$$

Accordingly, the integral B becomes

$$B = \frac{1}{2}\text{erfc}\left(\frac{\upsilon_q}{\sqrt{2n}}\right) + \frac{1}{\sqrt{2\pi}} e^{-\frac{\upsilon_q^2}{2n}} \left[\sum_{j=1}^{n} \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{\beta}{4!}\right)^{j} p_{4j-1}\left(\frac{\upsilon_q}{\sqrt{n}}\right)\right] \tag{58}$$

Finally, formula (50) is obtained by substituting the numerator and the denominator of formula (32) with A and B given in (54) and (58), respectively. □

Corollary 5. *Let us consider the sum of two independent Gram-Charlier expansions $Y = X_1 + X_2$ with extra-kurtosis β_1 and β_2, respectively. Then, the $ES_Y(\upsilon_q)$ has the following form*

$$ES_Y(\upsilon_q) = \\ = \frac{\frac{1}{\sqrt{\pi}} e^{-\frac{\upsilon_q^2}{4}} \left[1 + \frac{1}{4}\left(\frac{\beta_1+\beta_2}{4!}\right)\left(p_4\left(\frac{\upsilon_q}{\sqrt{2}}\right) + 4p_2\left(\frac{\upsilon_q}{\sqrt{2}}\right)\right) + \frac{1}{16}\left(\frac{\beta_1\beta_2}{(4!)^2}\right)\left(p_8\left(\frac{\upsilon_q}{\sqrt{2}}\right) + 8p_6\left(\frac{\upsilon_q}{\sqrt{2}}\right)\right)\right]}{\frac{1}{2} erfc\left(\frac{\upsilon_q}{2}\right) + \frac{1}{\sqrt{2\pi}} e^{-\frac{\upsilon_q^2}{4}} \left[\frac{1}{4}\left(\frac{\beta_1+\beta_2}{4!}\right) p_3\left(\frac{\upsilon_q}{\sqrt{2}}\right) + \frac{1}{16}\left(\frac{\beta_1\beta_2}{(4!)^2}\right) p_7\left(\frac{\upsilon_q}{\sqrt{2}}\right)\right]} \tag{59}$$

Proof. Observe that the density of the sum of two Gram-Charlier expansions with different parameters, given by (27), differs from that of two Gram-Charlier expansions with equal parameters, given by (13), just for the coefficients of the Hermite polynomials $p_4\left(\frac{y}{\sqrt{2}}\right)$ and $p_8\left(\frac{y}{\sqrt{2}}\right)$. Hence, replacing in (33) the coefficients of the density (13) with those of the density (27), yields (59). □

The same procedure can be simply generalized to the case of n random variables with different extra-kurtosis parameters β_i.

Corollary 6. *Let us consider the sum of n independent Gram-Charlier expansions $Y = X_1 + \cdots + X_n$ with extra-kurtosis $\beta_1, \ldots, \beta_n$ respectively. Then, the $ES_Y(\upsilon_q)$ has the following form*

$$ES_Y(\upsilon_q) = \\ = \frac{\sqrt{\frac{n}{2\pi}} e^{-\frac{\upsilon_q^2}{2n}} \left[1 + \sum_{j=1}^{n} \left(\frac{1}{\sqrt{n}}\right)^{4j} \binom{n}{j} \left(\frac{b_{n,j}}{(4!)^j}\right) \left(p_{4j}\left(\frac{\upsilon_q}{\sqrt{n}}\right) + 4j p_{4j-2}\left(\frac{\upsilon_q}{\sqrt{n}}\right)\right)\right]}{\frac{1}{2} erfc\left(\frac{\upsilon_q}{\sqrt{2n}}\right) + \frac{1}{\sqrt{2\pi}} e^{-\frac{\upsilon_q^2}{2n}} \left[\sum_{j=1}^{n} \left(\frac{1}{\sqrt{n}}\right)^{4j} \left(\frac{b_{n,j}}{(4!)^j}\right) p_{4j-1}\left(\frac{\upsilon_q}{\sqrt{n}}\right)\right]} \tag{60}$$

Proof. The proof follows the same lines of Theorem 2 with density (30) replacing density (13) in formula (33). □

5 An application to financial asset indexes

In this section the good performance of GC expansions of sums of r.v in dealing with financial asset indexes is proved. To this end, we have considered a set of 4 european (UK, Germany, Italy, France) and 2 asian (China, Japan) stock exchange indexes and 2 arbitrary indexes of the pharmaceutical and alimentary industries.The preliminary statistics about these data are reported in Tables 1 and 2.
Table 1 shows the mean (μ), the standard deviation (sd), the skewness (sk) and the kurtosis index (k) of the mentioned series. As the analysis carried out in the previous section is valid for independent rvs, we propose seven couples of indexes for which we tested a low correlation as reported in Table 2.

Being interested in measuring losses, the returns from data have been computed as minus the logarithm of the ratio between the prices at time t and $t-1$.
The sample size has been divided into two periods. The data of the first period (from 01/01/2009 to 17/09/2013) have been used to estimate the Gram-Charlier (GC) densities and compute the

Table 1: Summary statistics of losses

	^FTSE	^GDAXI	FTSEMIB.MI	^FCHI	^HSI	^N225	SXDP.Z	KO
μ	-0,0260	-0,0493	0,0046	-0,0170	-0,0300	-0,0478	-0,0537	-0,0590
sd	1,1252	1,4278	1,8222	1,4813	1,4350	1,5138	0,8937	1,1148
sk	0,0781	0,0476	0,2106	-0,0242	-0,1662	0,5063	0,3305	-0,1975
k	6,4554	5,7822	5,7534	6,3576	7,0437	6,8020	4,9275	8,1970

The table reports for each loss the mean (μ), the standard deviation (sd), the skewness index (sk) and the kurtosis index (k).

Table 2: Correlation coefficient of the losses

ρ	^FTSE	^GDAXI	FTSEMIB.MI	^FCHI	^HSI	SXDP.Z	KO
^N225	0.3036	0.2921	0.2490	0.2917	0.5729	0.1776	0.0954

corresponding risk functions. The data of the second period (from 18/09/2013 to 31/12/2014) have been used to evaluate the goodness of the risk measure forecasts.
The GC expansions of the sum (GCS) for each couple of indexes have been estimated as in (27).
Table 3 reports the values of the extra-kurtosis β for each couple of series under consideration.
In order to assess the goodness of fit of GCS to data, the Hellinger's entropy distance (Granger *et al.*, 2004; Maasoumi and Racine, 2002) between the empirical and the estimated distributions have been computed. Low values of this index denote a good fit of GCS to data. The last column of Table 3 shows the values of this index for the GCS densities.

Figure 2 shows the tails of the estimated GCS densities superimposed on those of the corresponding empirical distributions. Both the values of the Hellinger's entropy index and the graphs highlight the good fit of GCS to empirical data, especially in the tail areas which are the loci involved in the risk measure estimates.
Figure 3 compares the VaR estimated via GCS in the first period of the sample at the $97,5\%$ and 99% levels with the corresponding

Table 3: Parameter estimates of the GCS distribution on the first 1000 days with the relative Hellinger's entropy distance S_ρ.

Index 1	Index 2	$\hat{\beta}_1$	$\hat{\beta}_2$	S_ρ
^N225	^FTSE	3.9666	2.9189	0.0203
^N225	^GDAXI	3.9666	2.9189	0.0213
^N225	FTSEMIB.MI	3.9666	2.4250	0.0200
^N225	^FCHI	3.9666	2.8433	0.0185
^N225	^HSI	3.9666	3.5847	0.0215
^N225	SXDP.Z	3.9666	1.6780	0.0232
^N225	KO	3.9666	4.000	0.0173

empirical quantile. As all the VaR estimates exceed the corresponding empirical values, the conclusion that the GCS provide precautionary VaR estimates against potential losses can be easily drawn. Notice that in the case of normal distribution, theoretical $VaR_{\alpha=0.025}$ it is always equal to $1,9599$ while $Var_{\alpha=0.001}$ it is always equal to $2,3263$. In both cases we underestimate this risk measure, what follows can be very dangerous for the risk management, but also in stark contrast to the regulatory philosophy.

To evaluate the out-of-sample performance of the GCS densities, we have computed the VaR for $\alpha = 0.025$ and $\alpha = 0.01$ on the second part of sample (the last 374 days) which has not been used in the estimation process of the GCS densities.
Further, some punctual measures of losses in this period have been computed. These are the ABLF (average binary loss function), the AQLF (average quadratic loss function) and the UL (unexpected loss).
The values of these indexes as well as VaR values are displayed in Table 4. As it happens in the sample (first 1000 days), the VaR values for GC distribution give a quite precautionary perspective respect to the Normal one. As a matter of fact all the loss mea-

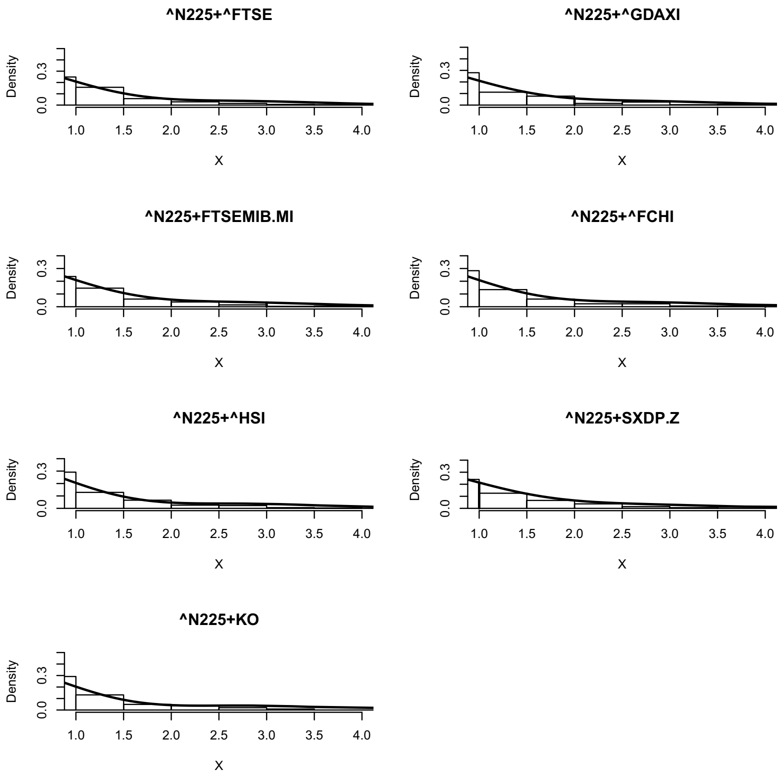

Figure 2: Histograms of the portfolio losses with the estimate GCS densities.

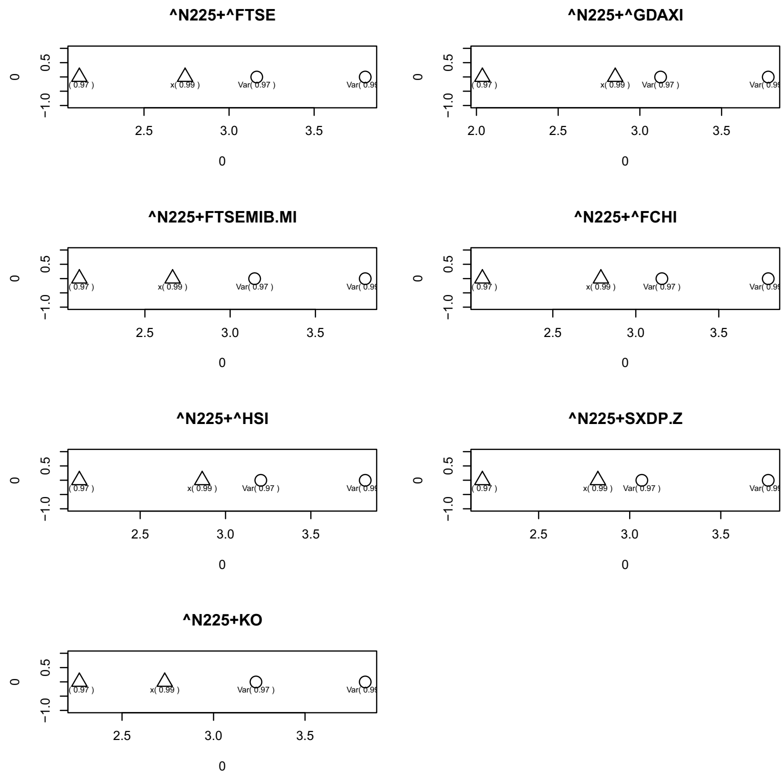

Figure 3: Empirical vs theoretical VaR of the portfolio losses. Triangles denote empirical VaR at $1-\alpha = 0,975$, $1-\alpha = 0,99$ while circles denote estimated VaR with GCS at the same levels.

sures proposed confirm that the GC distribution offers the best out of sample performance.

To forecast performance of VaR values estimated via GCS, at a chosen significance level, has been evaluated by implementing two tests, the likelihood-ratio test and the binomial two-sided test, see Table 5. The null hypothesis of both tests assumes that the percentage of forecast losses are coherent with the effective ones against the bi-lateral alternative which assumes that the VaR values overestimate or underestimate this percentage. A p-value lower or equal to 0.01 can be interpreted as evidence against the correct model (for more details see (Kupiec, 1995; Christoffersen *et al.*, 1998)).
According to the likelihood ratio test 11 out of 14 GCS engender forecasts which are coherent at the chosen α level. As shown in Figure 3, the rejection happens for GCS densities whose VaR estimates are most distant from the corresponding empirical quantiles. These results are in accordance with the results of the binomial tests.
Furthermore, a lecture of the likelihood-ratio test of the $VaR_{\alpha=0.01}$, inspired on the "traffic light" approach suggested by the Basel Committee, seems to place the GCS results in the "green zone".

Also the less debatable expected shortfall (ES) has been computed as risk measure. The ES has been computed in the first period of the sample (the first 1000 days) using the VaR estimated via GCS as quantile. This procedure has been carried out for different α levels and more precisely for $\alpha = 0.05$, $\alpha = 0.025$ and $\alpha = 0.01$. The estimates of the expected shortfall for these α values, ES_α from now on, are shown in Table 6.
In order to evaluate the out-of sample performance of this risk measure, the ES have been computed also in the second part of the sample (last 374 days). These values, denoted with ES_{emp} and reported in Table 6, have been obtained using VaR from GCS estimated in the first sample period for different α values ($\alpha = 0.05$, $\alpha = 0.025$ and $\alpha = 0.01$). The goodness of ES_α estimates has been evaluated by implementing two tests based on bootstrap proce-

Table 4: Descriptive analysis of VaR.

				GC				**Normal**			
Index 1	Index 2	$1-\alpha$	VaR_{emp}	VaR	ABLF	AQLF	UL	Var	ABLF	AQLF	UL
ˆN225	ˆFTSE	0,975	2,1199	3,1628	0,0053	0,0093	0,0042	1,96	0,0213	0,05	0,019
ˆN225	ˆFTSE	0,990	2,7415	3,8004	0,0027	0,0033	0,0014	2,3263	0,016	0,0331	0,0127
ˆN225	ˆGDAXI	0,975	2,0362	3,1269	0,0080	0,0130	0,0059	1,96	0,0186	0,0487	0,017
ˆN225	ˆGDAXI	0,990	2,8491	3,7856	0,0053	0,0057	0,0014	2,3263	0,008	0,0276	0,0123
ˆN225	FTSEMIB.MI	0,975	2,1155	3,1437	0,0080	0,0120	0,0044	1,96	0,0186	0,0443	0,0148
ˆN225	FTSEMIB.MI	0,990	2,6624	3,7924	0,0027	0,0033	0,0013	2,3263	0,008	0,0245	0,0109
ˆN225	ˆFCHI	0,975	2,0755	3,1575	0,0053	0,0076	0,0032	1,96	0,0213	0,0424	0,0147
ˆN225	ˆFCHI	0,990	2,7897	3,7982	0,0027	0,0028	0,0006	2,3263	0,0106	0,023	0,0095
ˆN225	ˆHSI	0,975	2,1437	3,2068	0,0080	0,0101	0,0038	1,96	0,016	0,0401	0,0149
ˆN225	ˆHSI	0,990	2,8635	3,8186	0,0027	0,0027	0,0001	2,3263	0,008	0,0229	0,0108
ˆN225	SXDP.Z	0,975	2,1910	3,0666	0,0053	0,0141	0,0066	1,96	0,0426	0,083	0,0289
ˆN225	SXDP.Z	0,990	2,8253	3,7606	0,0053	0,0075	0,0029	2,3263	0,0293	0,0532	0,0167
ˆN225	KO	0,975	2,2667	3,2319	0,0106	0,0206	0,0088	1,96	0,0319	0,0908	0,0327
ˆN225	KO	0,990	2,7333	3,8290	0,0080	0,0107	0,0037	2,3263	0,0213	0,0597	0,0237

For each couple of indexes (first two columns) at each level α (third column) there are displayed the empirical VaR evaluated on the first sample (fourth column), the theoretical VaR for GC distribution and Normal distribution (fifth and ninth columns), the three statistical indexes ABLF, AQLF and UL for GC distribution and Normal distribution (sixth-eighth columns and tenth-twelfth columns).

dure. Both of them consider the performance of the GCS density under examination inadequate if the ES_α systematically underestimates the effective losses mean (ES_{emp}), this implying great damage.
The first test proposed by McNeil and Frey (2000) is based on the following statistic

$$Z_1 = \frac{1}{N}\sum_{t=1}^{N}\left(\frac{X_t I_{X_t>VaR_\alpha}}{ES_\alpha} - 1\right) \tag{61}$$

where N is the number of losses X_t in the second part of the sample (the last 374 days) lying over the VaR_α, $I_{X_t>VaR_\alpha}$ is an indicative variable which assumes values equal to 1 if $X_t > VaR_\alpha$ and 0 otherwise. ES_α is the expected shortfall estimated by using the GCS density. Under the null hypothesis, assuming the correctness of the GCS densities or equivalently the goodness of he $ES\alpha$ estimates, Z_1 takes low values.

Table 5: Analysis of VaR: test.

Index 1	Index 2	$1-\alpha$	LRuc	p-val(LUrc)	p-val(VaR)
ˆN225	ˆFTSE	0,975	8,7581	0,0031	0,0076
ˆN225	ˆFTSE	0,99	2,8916	**0,0890**	**0,1958**
ˆN225	ˆGDAXI	0,975	6,0585	**0,0138**	**0,0301**
ˆN225	ˆGDAXI	0,99	1,0032	**0,3165**	**0,5981**
ˆN225	FTSEMIB.MI	0,975	6,0585	**0,0138**	**0,0301**
ˆN225	FTSEMIB.MI	0,99	2,8916	**0,0890**	**0,1958**
ˆN225	ˆFCHI	0,975	8,7581	0,0031	0,0076
ˆN225	ˆFCHI	0,99	2,8916	**0,0890**	**0,1958**
ˆN225	ˆHSI	0,975	6,0585	**0,0138**	**0,0301**
ˆN225	ˆHSI	0,99	2,8916	**0,0890**	**0,1958**
ˆN225	SXDP.Z	0,975	8,7581	0,0031	0,0076
ˆN225	SXDP.Z	0,99	1,0032	**0,3165**	**0,5981**
ˆN225	KO	0,975	4,0438	**0,0443**	**0,0947**
ˆN225	KO	0,99	0,1667	**0,6831**	**1,0000**

For each couple of indexes (first two columns) at each level α (third column) there are displayed the statistic test of likelihood ratio test LR_{uc} and the associated p-value p-val(LRuc) and the p-value of the binomial two-sided test p-val(VaR) for the GC distribution (fourth-sixth columns). The significance level is fixed at 1%.

The second test, proposed by Acerbi and Szekely (2014), is quite similar to the previous one. The statistic test is

$$Z_2 = \frac{1}{T} \sum_{t=1}^{T} \frac{X_t I_{X_t > VaR_\alpha}}{\alpha ES_\alpha} - 1 \qquad (62)$$

where T denotes the sample size. The null hypothesis of this test is the same as that of the Z_1 test and, similarly to this latter, the Z_2 statistic assumes low values under the null hypothesis. Then a bootstrap simulation has been implemented. In both cases, 999 bootstrap samples have been selected from the out-of-sample dataset without making any assumption on the the underlying data distribution and the statistics Z_1 and Z_2 have been computed by using these 999 bootstrap samples. The p-values of both tests have been computed as percentages of the Z_1 and Z_2 statistics obtained from bootstrap samples exceeding the corresponding statistics Z_1 and Z_2, respectively, computed on the second part of the data (last 374 days). Looking at these p-values, reported in Table 6, we can conclude that the out of the sample performance of the GCS densities is quite good in most of the cases.
All the analysis have been carried out by using software R (R Core Team, 2015). In particular, basic financial operations have been worked out by using *tseries* (Trapletti and Hornik, 2015) package, computations involving Hermite's polynomials with *EQL* (Thorn Thaler, 2009) package and tests for the evaluation of goodness of fitting have been implemented by using *np* (Hayfield and Racine, 2008) package.

6 Conclusion

In this paper, we propose an approach to model the sums of leptokurtic Gaussian variables. This approach rests on the polynomial transformation of the Gaussian variables by means of their associated Hermite polynomials. The resulting distributions are known as Gram-Charlier expansions. The sum of these Gram Charlier expansions (GCS) proves to be a tail sensitive density

Table 6: Out-of-sample ES performance

Index 1	Index 2	$1-\alpha$	VaR	ES_{emp}	ES_α	Z_1	$pval(Z_1)$	Z_2	$pval(Z_2)$
^N225	^FTSE	0,950	2,4725	3,2659	3,3266	-0,0182	**0,5165**	-0,9863	**0,4675**
^N225	^FTSE	0,975	3,1628	3,9579	3,8500	0,0280	**0,3443**	-0,9944	**0,5395**
^N225	^FTSE	0,990	3,8004	4,3089	4,4539	-0,0326	**0,3744**	-0,9974	**0,6647**
^N225	^GDAXI	0,950	2,4342	3,8722	3,2944	0,1754	**0,4494**	-0,9901	**0,4975**
^N225	^GDAXI	0,975	3,1269	3,8722	3,8269	0,0118	**0,4675**	-0,9917	**0,4535**
^N225	^GDAXI	0,990	3,7856	4,0568	4,4359	-0,0855	**0,4935**	-0,9951	**0,5355**
^N225	FTSEMIB.MI	0,950	2,4513	3,6975	3,3092	0,1173	**0,4184**	-0,9906	**0,4905**
^N225	FTSEMIB.MI	0,975	3,1437	3,6975	3,8376	-0,0365	**0,5005**	-0,9921	**0,4745**
^N225	FTSEMIB.MI	0,990	3,7924	4,2733	4,4442	-0,0384	**0,3744**	-0,9974	**0,6386**
^N225	^FCHI	0,950	2,4664	3,4896	3,3217	0,0505	**0,4695**	-0,9912	**0,5125**
^N225	^FCHI	0,975	3,1575	3,7596	3,8466	-0,0226	**0,5005**	-0,9947	**0,5355**
^N225	^FCHI	0,990	3,7982	4,0289	4,4512	-0,0949	**0,3764**	-0,9976	**0,6276**
^N225	^HSI	0,950	2,5277	3,6804	3,3680	0,0928	**0,4825**	-0,9908	**0,4935**
^N225	^HSI	0,975	3,2068	3,6804	3,8788	-0,0512	**0,5275**	-0,9922	**0,5115**
^N225	^HSI	0,990	3,8186	3,8710	4,4776	-0,1355	**0,3534**	-0,9977	**0,6547**
^N225	SXDP.Z	0,950	2,3817	2,9504	3,2435	-0,0904	**0,4695**	-0,9745	**0,4835**
^N225	SXDP.Z	0,975	3,0666	4,3077	3,7887	0,1370	**0,3413**	-0,9938	**0,5065**
^N225	SXDP.Z	0,990	3,7606	4,3077	4,4080	-0,0228	**0,4675**	-0,9947	**0,5375**
^N225	KO	0,950	2,5635	3,5989	3,3926	0,0608	**0,5035**	-0,9792	**0,4765**
^N225	KO	0,975	3,2319	4,0578	3,8957	0,0416	**0,4855**	-0,9886	**0,5105**
^N225	KO	0,990	3,8290	4,2954	4,4921	-0,0438	**0,4785**	-0,9923	**0,4885**

For each couple of indexes (first two columns) at each level α (third column) there are displayed the theoretical VaR for GC distribution (fourth column), the empirical ES evaluated on the first sample (fifth column), the theoretical ES for GC distribution (sixth column), the statistic tests Z_1 and Z_2 (seventh and ninth columns) and the associated p-values for the GC distribution (eight and tenth columns). The significance level is fixed at 1%.

as it fits well the tails of the empirical distributions of financial returns and as such it can be conveniently used to compute risk measures like the Value at Risk and the expected shortfall. An application to a portfolio of a set of financial asset indexes provides evidence of the effectiveness of the GCS densities as it results from their in and out of sample performance in both VaR and expected shortfall estimation.

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