EXPLICIT BOUNDS FOR GENERATORS OF THE CLASS GROUP

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Abstract. Assuming Generalized Riemann's Hypothesis, Bach proved that the class group $\mathcal{C}\ell_{\mathbf{K}}$ of a number field $\mathbf{K}$ may be generated using prime ideals whose norm is bounded by $12\log^2\Delta_{\mathbf{K}}$, and by $(4+o(1))\log^2\Delta_{\mathbf{K}}$ asymptotically, where $\Delta_{\mathbf{K}}$ is the absolute value of the discriminant of $\mathbf{K}$. Under the same assumption, Belabas, Diaz y Diaz and Friedman showed a way to determine a set of prime ideals that generates $\mathcal{C}\ell_{\mathbf{K}}$ and which performs better than Bach's bound in computations, but which is asymptotically worse. In this paper we show that $\mathcal{C}\ell_{\mathbf{K}}$ is generated by prime ideals whose norm is bounded by the minimum of $4.01\log^2\Delta_{\mathbf{K}}$, $4\big(1+(2\pi e^{\gamma})^{-n_{\mathbf{K}}}\big)^2\log^2\Delta_{\mathbf{K}}$ and $4\big(\log\Delta_{\mathbf{K}}+\log\log\Delta_{\mathbf{K}}-(\gamma+\log 2\pi)n_{\mathbf{K}}+1+(n_{\mathbf{K}}+1)\frac{\log(7\log\Delta_{\mathbf{K}})}{\log\Delta_{\mathbf{K}}}\big)^2$. Moreover, we prove explicit upper bounds for the size of the set determined by Belabas, Diaz y Diaz and Friedman's algorithms, confirming that it has size $\asymp(\log\Delta_{\mathbf{K}}\log\log\Delta_{\mathbf{K}})^2$. In addition, we propose a different algorithm which produces a set of generators which satisfies the above mentioned bounds and in explicit computations turns out to be smaller than $\log^2\Delta_{\mathbf{K}}$ except for 7 out of the 31292 fields we tested.

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1. Introduction

Let $\mathbf{K}$ be a number field of degree $n_{\mathbf{K}}\geq 2$, with r_1 (resp. r_2) real (resp. pair of complex) embeddings and denote $\Delta_{\mathbf{K}}$ the absolute value of its discriminant. Throughout the paper $\mathfrak{p}$ will always denote a non-zero prime ideal, ρ a non-trivial zero of $\zeta_{\mathbf{K}}$, γ the imaginary part of such a ρ and, since we are assuming Generalized Riemann's Hypothesis, $\rho=\frac{1}{2}+i\gamma$. The Euler–Mascheroni constant will also be denoted γ, but the context will make it clear.

Buchmann's algorithm is an efficient method to compute both the class group $\mathcal{C}\ell_{\mathbf{K}}$ and a basis for a maximal lattice of the unity group of $\mathbf{K}$. However, it needs as input a set of generators for $\mathcal{C}\ell_{\mathbf{K}}$. Minkowski's bound shows that ideals having a norm (essentially) below $\sqrt{\Delta_{\mathbf{K}}}$ may be used, but it works just for a few fields since the discriminant increases very quickly. Assuming Generalized Riemann's Hypothesis, Eric Bach proved in [Bac90] that ideals with a norm below $12\log^2\Delta_{\mathbf{K}}$ suffice, and that the bound improves up to $(4+o(1))\log^2\Delta_{\mathbf{K}}$ as $\Delta_{\mathbf{K}}$ diverges, where the function in $o(1)$ is not made explicit in that paper, but has order at least $\log^{-2/3}\Delta_{\mathbf{K}}$. This is a remarkable improvement, but for certain applications it is still too large. A different method to find a good bound T for norms of ideal generating $\mathcal{C}\ell_{\mathbf{K}}$ has been proposed by Karim Belabas, Francisco Diaz y Diaz and Eduardo Friedman [BDyDF08]. In all tests their method behaves very well, producing a good bound $T(\mathbf{K})$ (see Section 3 of [BDyDF08]) which is much lower than $4\log^2\Delta_{\mathbf{K}}$. However, the authors prove [BDyDF08, Theorem 4.3] that $T(\mathbf{K})\geq\big(\big(\frac{1}{4n_{\mathbf{K}}}+o(1)\big)\log\Delta_{\mathbf{K}}\log\log\Delta_{\mathbf{K}}\big)^2$, and advance the conjecture that $T(\mathbf{K})\sim\big(\frac{1}{4}\log\Delta_{\mathbf{K}}\log\log\Delta_{\mathbf{K}}\big)^2$. Thus, its impressive performance is the combined effect

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of relatively large/small constants in front of these bounds and of the present computational power, but which will disappear for large $\Delta_{\mathbf{K}}$.

In this paper we first prove in Theorem 3.6 an explicit, easy and better version of Bach's $(4+o(1))\log^2\Delta_{\mathbf{K}}$ bound, in Corollary 3.7 that $4\log^2\Delta_{\mathbf{K}}$ is sufficient for a wide range of fields and in Corollary 3.8 that $4.01\log^2\Delta_{\mathbf{K}}$ is sufficient for all fields. Corollary 3.7 also contains an explicit bound showing that the universal constant 4.01 actually decays exponentially to 4 with the degree of the field.
Secondly, in Theorem 4.4 we prove that $T(\mathbf{K}) \leq \big(\frac{1+o(1)}{4}\log\Delta_{\mathbf{K}}\log\log\Delta_{\mathbf{K}}\big)^2$, and that $T(\mathbf{K}) \leq 3.9(\log\Delta_{\mathbf{K}}\log\log\Delta_{\mathbf{K}})^2$ with only three exceptions which are explicit. In a private communication K. Belabas told us that he also has a proof for the first part of this claim. Together with the lower bound in [BDyDF08] it shows in particular that $T(\mathbf{K}) \asymp (\log\Delta_{\mathbf{K}}\log\log\Delta_{\mathbf{K}})^2$ for fixed $n_{\mathbf{K}}$.

The weight function of [BDyDF08] can be seen as the convolution square of a characteristic function, i.e. of a one step function. Using the convolution square of a two- (resp. three-) steps function, we show in Corollary 5.1 that the bound already improve to $(6.04+o(1))\log^2\Delta_{\mathbf{K}}$ (resp. $(4.81+o(1))\log^2\Delta_{\mathbf{K}}$), where moreover in both cases $o(1) < 0$ for fields of degree $n_{\mathbf{K}} \geq 3$. To further improve the result we propose in Subsections 5.2–5.4 a different algorithm producing a new bound $T_1(\mathbf{K})$, and which is essentially a multistep version of Belabas, Diaz y Diaz and Friedman's algorithm, where the number of steps is not set in advance. By design, it performs better than $T(\mathbf{K})$ and is lower than all the bounds we have proved in the first part of the paper. In Subsection 5.6 we report the conclusions about extensive tests we have conducted on a few thousands of pure and biquadratic fields: in all cases the algorithm produces $T_1(\mathbf{K})$ lower than $\log^2\Delta_{\mathbf{K}}$ except for some biquadratic fields where it is already $\leq 1.004\log^2\Delta_{\mathbf{K}}$.
All 'little-o' terms in these formulas are explicit, simple, of order $\frac{\log\log\Delta_{\mathbf{K}}}{\log\Delta_{\mathbf{K}}}$ and with small coefficients.

We have made two sample implementations of our algorithms. The first one is a script for PARI/GP [PARI15] which can be found at the following address:
`http://users.mat.unimi.it/users/molteni/research/generators/bounds.gp`.
The other is the branch `loic-bnf-optim` of the `git` tree of PARI/GP, available at
`http://pari.math.u-bordeaux.fr/git/pari.git`.

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2. Preliminary

Definition 2.1. *Let $\mathcal{W}$ be the set of functions $F\colon [0,+\infty) \to \mathbf{R}$ such that:*

- *F is continuous;*
- *$\exists\varepsilon > 0$ such that the function $F(x)e^{(\frac{1}{2}+\varepsilon)x}$ is integrable and of bounded variation;*
- *$F(0) > 0$;*
- *$(F(0) - F(x))/x$ is of bounded variation.*

Let then, for $T > 1$, $\mathcal{W}(T)$ be the subset of $\mathcal{W}$ such that:

- *F has support in $[0, \log T]$;*
- *the Fourier cosine transform of F is non-negative.*

Definition 2.2. *For any compactly supported function F on $[0,\infty)$, we set*

$$\mathrm{I}(F) := \int_0^{+\infty} \frac{F(0)-F(x)}{2\sinh(x/2)}\,\mathrm{d}x \quad \text{and} \quad \mathrm{J}(F) := \int_0^{+\infty} \frac{F(x)}{2\cosh(x/2)}\,\mathrm{d}x.$$

Definition 2.3. *Let $T_{\mathcal{C}}(\mathbf{K})$ be the lowest T such that the set $\{\mathfrak{p}\colon \mathrm{N}\mathfrak{p} \le T\}$ generates $\mathcal{C}\ell_{\mathbf{K}}$.*

The main result of [BDyDF08, Th. 2.1] can be reformulated as follows.

Theorem 2.4 (Belabas, Diaz y Diaz, Friedman). *Let $\mathbf{K}$ be a number field satisfying the Riemann Hypothesis for all L-functions attached to non-trivial characters of its ideal class group $\mathcal{C}\ell_{\mathbf{K}}$, and suppose that there exists, for some $T > 1$, an $F \in \mathcal{W}(T)$ such that*

$$2\sum_{\mathfrak{p}} \log \mathrm{N}\mathfrak{p} \sum_{m=1}^{+\infty} \frac{F(m\log \mathrm{N}\mathfrak{p})}{\mathrm{N}\mathfrak{p}^{m/2}} > F(0)(\log \Delta_{\mathbf{K}} - (\gamma + \log 8\pi)n_{\mathbf{K}}) + \mathrm{I}(F)n_{\mathbf{K}} - \mathrm{J}(F)r_1. \tag{2.1}$$

Then $T_{\mathcal{C}}(\mathbf{K}) < T$.

Assuming GRH, Weil's Explicit Formula (see [Lan94, Ch. XVII, Th. 3.1]), as simplified by Poitou in [Poi77], can be written for $F \in \mathcal{W}$ as

$$\begin{aligned} 2\sum_{\gamma} \int_0^{+\infty} F(x)\cos(x\gamma)\,\mathrm{d}x &= 4\int_0^{+\infty} F(x)\cosh\left(\frac{x}{2}\right)\mathrm{d}x \\ &- 2\sum_{\mathfrak{p}} \log \mathrm{N}\mathfrak{p} \sum_{m=1}^{+\infty} \frac{F(m\log \mathrm{N}\mathfrak{p})}{\mathrm{N}\mathfrak{p}^{m/2}} + F(0)(\log \Delta_{\mathbf{K}} - (\gamma + \log 8\pi)n_{\mathbf{K}}) + \mathrm{I}(F)n_{\mathbf{K}} - \mathrm{J}(F)r_1. \end{aligned} \tag{2.2}$$

Hence (2.1) can be stated as

$$2\int_0^{+\infty} F(x)\cosh\left(\frac{x}{2}\right)\mathrm{d}x > \sum_{\gamma}\int_0^{+\infty} F(x)\cos(x\gamma)\,\mathrm{d}x. \tag{2.3}$$

Let Φ be an even, integrable and compactly supported function, and let $F = \Phi * \Phi$. Then

$$\begin{aligned} \int_0^{+\infty} F(x)\cosh\left(\frac{x}{2}\right)\mathrm{d}x &= 2\Big(\int_0^{+\infty} \Phi(x)\cosh\left(\frac{x}{2}\right)\mathrm{d}x\Big)^2, \\ \int_0^{+\infty} F(x)\cos(xt)\,\mathrm{d}x &= 2\Big(\int_0^{+\infty} \Phi(x)\cos(xt)\,\mathrm{d}x\Big)^2, \end{aligned}$$

and F satisfies (2.3) and hence (2.1) if and only if

$$8\Big(\int_0^{+\infty} \Phi(x)\cosh\left(\frac{x}{2}\right)\mathrm{d}x\Big)^2 > \sum_{\gamma} \widehat{\Phi}(\gamma)^2, \tag{2.4}$$

where we have set $\widehat{\Phi}(t) := \int_{\mathbf{R}} \Phi(x)e^{ixt}\,\mathrm{d}x = 2\int_0^{+\infty} \Phi(x)\cos(xt)\,\mathrm{d}x$.

3. Bounds for class group generators

Assume $T > 1$ and let $L := \log T$. Let Φ^+ be a real, non-negative, piecewise continuous function with positively measured support in $[0, L]$, and let

$$\begin{aligned} \Phi^-(x) &:= \Phi^+(-x), \\ \Phi^\circ(x) &:= \Phi^+(L/2 + x), \\ \Phi(x) &:= \Phi^\circ(x) + \Phi^\circ(-x), \\ F &:= \Phi * \Phi. \end{aligned} \tag{3.1}$$

These choices ensure that $F \in \mathcal{W}(T)$.

Proposition 3.1. *Assume* GRH *and let F as in* (3.1). *Then* (2.1) *is satisfied by F if*

$$\begin{aligned} \sqrt{T} \geq{}& \frac{2T\int_0^L \big(\Phi^+(x)\big)^2 \,\mathrm{d}x}{\big(\int_0^L \Phi^+(x)e^{x/2}\,\mathrm{d}x\big)^2}(\log\Delta_{\mathbf{K}} - (\gamma + \log 8\pi)n_{\mathbf{K}}) \\ &- \frac{4T\sum_{\mathfrak{p},m} \log \mathrm{N}\mathfrak{p}\frac{(\Phi^+*\Phi^-)(m\log \mathrm{N}\mathfrak{p})}{\mathrm{N}\mathfrak{p}^{m/2}}}{\big(\int_0^L \Phi^+(x)e^{x/2}\,\mathrm{d}x\big)^2} + \frac{2T\ \mathrm{I}(\Phi^+ * \Phi^-)n_{\mathbf{K}}}{\big(\int_0^L \Phi^+(x)e^{x/2}\,\mathrm{d}x\big)^2} + \frac{2T\int_0^L \Phi^+(x)e^{-x/2}\,\mathrm{d}x}{\int_0^L \Phi^+(x)e^{x/2}\,\mathrm{d}x}. \end{aligned} \tag{3.2}$$

Proof. We have

$$2\int_0^{+\infty} \Phi(x)\cosh\Big(\frac{x}{2}\Big)\,\mathrm{d}x = \frac{1}{T^{1/4}}\int_0^L \Phi^+(x)e^{x/2}\,\mathrm{d}x + T^{1/4}\int_0^L \Phi^+(x)e^{-x/2}\,\mathrm{d}x.$$

Hence

$$\begin{aligned} &8\Big(\int_0^{+\infty}\Phi(x)\cosh\Big(\frac{x}{2}\Big)\,\mathrm{d}x\Big)^2 \\ &\qquad > \frac{2}{\sqrt{T}}\Big(\int_0^L \Phi^+(x)e^{x/2}\,\mathrm{d}x\Big)^2 + 4\int_0^L \Phi^+(x)e^{x/2}\,\mathrm{d}x\int_0^L \Phi^+(x)e^{-x/2}\,\mathrm{d}x. \end{aligned}$$

Moreover

$$\widehat{\Phi}(t) = 2\,\mathrm{Re}\int_{\mathbf{R}} \Phi^\circ(x)e^{ixt}\,\mathrm{d}x = 2\,\mathrm{Re}\big[e^{-i\frac{Lt}{2}}\widehat{\Phi^+}(t)\big]. \tag{3.3}$$

Hence

$$|\widehat{\Phi}(t)|^2 \leq 4\big|e^{-i\frac{Lt}{2}}\widehat{\Phi^+}(t)\big|^2 = 4\big|\widehat{\Phi^+}(t)\big|^2. \tag{3.4}$$

We have $\big|\widehat{\Phi^+}(t)\big|^2 = \widehat{\Phi^+}(t)\overline{\widehat{\Phi^+}(t)} = \widehat{\Phi^+}(t)\widehat{\Phi^-}(t) = \widehat{\Phi^+ * \Phi^-}(t)$. Thus to satisfy (2.4) it is sufficient that

$$\frac{2}{\sqrt{T}}\Big(\int_0^L \Phi^+(x)e^{x/2}\,\mathrm{d}x\Big)^2 + 4\int_0^L \Phi^+(x)e^{x/2}\,\mathrm{d}x\int_0^L \Phi^+(x)e^{-x/2}\,\mathrm{d}x \geq \sum_\gamma \widehat{\digamma}(\gamma), \tag{3.5}$$

where

$$\digamma := 4\Phi^+ * \Phi^-.$$

By Weil's Explicit Formula (2.2),

$$\sum_\gamma \widehat{\digamma}(\gamma) \leq \digamma(0)\big(\log\Delta_{\mathbf{K}} - (\gamma + \log 8\pi)n_{\mathbf{K}}\big) - 2\sum_{\mathfrak{p},m}\log\mathrm{N}\mathfrak{p}\frac{\digamma\big(m\log\mathrm{N}\mathfrak{p}\big)}{\mathrm{N}\mathfrak{p}^{m/2}} \tag{3.6}$$

$$+ \mathrm{I}(F) n_{\mathbf{K}} + 4 \int_0^{+\infty} F(x) \cosh\left(\frac{x}{2}\right) \mathrm{d}x$$

where we cancelled the term $-\mathrm{J}(F) r_1$ because $F \geq 0$. Notice that

$$F(0) = 4 \int_0^L \left(\Phi^+(x)\right)^2 \mathrm{d}x \tag{3.7}$$

and

$$\int_0^{+\infty} F(x) \cosh\left(\frac{x}{2}\right) \mathrm{d}x = \frac{1}{2} \int_{\mathbf{R}} F(x) e^{x/2} \mathrm{d}x$$
$$= 2 \int_0^L \Phi^+(x) e^{x/2} \mathrm{d}x \int_0^L \Phi^+(x) e^{-x/2} \mathrm{d}x. \tag{3.8}$$

The claim follows combining (3.5), (3.6), (3.7) and (3.8). □

3.1. **Upper bound for $T_{\mathbf{e}}(\mathbf{K})$.** The coefficient of $\log \Delta_{\mathbf{K}}$ in Proposition 3.1 is

$$\frac{2T \int_0^L \left(\Phi^+(x)\right)^2 \mathrm{d}x}{\left(\int_0^L \Phi^+(x) e^{x/2} \mathrm{d}x\right)^2}.$$

The Cauchy–Schwarz inequality shows that its minimum value is $\frac{2T}{T-1}$ and it is attained only for $\Phi^+(x) = e^{x/2}$ on $[0, L]$. We are interested into small values for this coefficient, hence this is the best choice we can make. However, this function produces in (3.2) an inequality for T that cannot be solved easily and, moreover, this choice does not give the best possible results for secondary coefficients. To overcome this problem in the next theorem we consider the functions $e^{x/2}\chi_{[L-a,L]}(x)$, where a is a parameter which is fixed in $(0, L]$. This is a suboptimal choice for the coefficient of $\log \Delta_{\mathbf{K}}$ if $a \neq L$, but every value of a independent of T produces an inequality which can be solved easily, still having the correct order for the main term. Furthermore, acting on a we can also minimize the total contribution coming from the other terms in (3.2). Theorem 3.6 is proved using several values of a, and would not be accessible using only the conclusions coming from the choice $a = L$.

Definition 3.2. *Assume $a \in (0, L]$. Let $\Phi_{\mathbf{e}}^+(x) := e^{x/2}\chi_{[L-a,L]}(x)$ and let $F_{\mathbf{e}}$ be the F defined in* (3.1) *when $\Phi^+ = \Phi_{\mathbf{e}}^+$.*

Remark 3.3. We recall that $F_{\mathbf{e}}$ is even with support in $[-L, L]$. Moreover, we find that for every $x \in [0, L]$,

$$F_{\mathbf{e}}(x) = \delta_1(x)(2a - L + x)e^{x/2}\sqrt{T} + \delta_2(x)(L - x)e^{x/2}\sqrt{T}$$
$$+ 2\delta_3(x)\left(e^{-x/2} - e^{x/2-a}\right)T + \delta_4(x)(2a - L - x)e^{-x/2}\sqrt{T}$$

where $\delta_1 := \chi_{[L-2a,L-a)}$, $\delta_2 := \chi_{[L-a,L]}$, $\delta_3 := \chi_{[0,a]}$ and $\delta_4 := \chi_{[0,2a-L]}$.

Definition 3.4. *Let $T_{\mathbf{e}}(\mathbf{K})$ be the minimal T such that the function $F_{\mathbf{e}}$ satisfies* (2.1) *for some a.*

Note that $T_{\mathcal{C}}(\mathbf{K}) \leq T_{\mathbf{e}}(\mathbf{K})$ so that we will state most results about $T_{\mathbf{e}}(\mathbf{K})$ below as results on $T_{\mathcal{C}}(\mathbf{K})$ as well.

Theorem 3.5. *Assume* GRH. *Fix* $T_0 > 1$. *We then have*

$$\sqrt{T_{\mathbf{e}}(\mathbf{K})} \le \max\Big(\sqrt{T_0}, r(\log\Delta_{\mathbf{K}}, n_{\mathbf{K}}, T_0) - \frac{4}{\big(1-T_0^{-1}\big)^2}\sum_{\mathrm{N}\mathfrak{p}^m \le T_0}\Big(\frac{1}{\mathrm{N}\mathfrak{p}^m} - \frac{1}{T_0}\Big)\log\mathrm{N}\mathfrak{p}\Big), \tag{3.9}$$

in particular

$$\sqrt{T_{\mathbf{e}}(\mathbf{K})} \le \max\big(\sqrt{T_0}, r(\log\Delta_{\mathbf{K}}, n_{\mathbf{K}}, T_0)\big), \tag{3.10}$$

where

$$r(\mathcal{L}, n, t) := \frac{2}{1-t^{-1}}\Big(\mathcal{L} + \log t - \Big(\gamma + \log 2\pi - \frac{\log t}{t-1} + \log\big(1-t^{-1}\big)\Big)n\Big).$$

Proof. We are assuming $\Phi^+ = \Phi^+_{\mathbf{e}}$ for some $a \le L$. In this case we have

$$\int_0^L \Phi^+_{\mathbf{e}}(x)e^{x/2}\,\mathrm{d}x = \big(1-e^{-a}\big)T = \int_0^L \big(\Phi^+_{\mathbf{e}}(x)\big)^2\,\mathrm{d}x,$$
$$\int_0^L \Phi^+_{\mathbf{e}}(x)e^{-x/2}\,\mathrm{d}x = a.$$

Moreover, for all $x \in [0, a]$,

$$\Phi^+_{\mathbf{e}} * \Phi^-_{\mathbf{e}}(x) = \big(e^{-x/2} - e^{x/2-a}\big)T.$$

In addition for all $x > a$, $\Phi^+_{\mathbf{e}} * \Phi^-_{\mathbf{e}}(x) = 0$. This means that

$$\begin{aligned}\mathrm{I}(\Phi^+_{\mathbf{e}} * \Phi^-_{\mathbf{e}}) &= \int_0^{+\infty} \frac{\Phi^+_{\mathbf{e}} * \Phi^-_{\mathbf{e}}(0) - \Phi^+_{\mathbf{e}} * \Phi^-_{\mathbf{e}}(x)}{2\sinh(x/2)}\,\mathrm{d}x \\ &= (\log 4)\big(1-e^{-a}\big)T + aT - \big(1-e^{-a}\big)\log\big(e^a - 1\big)T.\end{aligned}$$

We now set $a =: \log T_0$ for some $T_0 > 1$. Since we need to have $L = \log T \ge a$, we get that (3.2) is satisfied for any $T \ge T_0$ such that

$$\sqrt{T} \ge \frac{2}{1-T_0^{-1}}\Big(\log\Delta_{\mathbf{K}} - \frac{2}{1-T_0^{-1}}\sum_{\mathrm{N}\mathfrak{p}^m \le T_0}\Big(\frac{\log\mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^m} - \frac{\log\mathrm{N}\mathfrak{p}}{T_0}\Big) - \Big(\gamma + \log 2\pi - \frac{\log T_0}{T_0 - 1} + \log\big(1-T_0^{-1}\big)\Big)n_{\mathbf{K}} + \log T_0\Big).$$

Since the right-hand side does not depend on T, this proves the first claim. The second is an obvious consequence, because the sum on prime ideals is non-negative. □

3.2. **Upper bounds for class group generators.** Theorem 3.6 below gives an upper bound for $T_{\mathbf{e}}(\mathbf{K})$, and hence for $T_{\mathcal{C}}(\mathbf{K})$. It is essentially the best result we can deduce from Theorem 3.5 (see the remark immediately following the proof). The theorem has Corollaries 3.7 and 3.8 as easy consequences.

Theorem 3.6. *We have*

$$\sqrt{T_{\mathcal{C}}(\mathbf{K})} \le \sqrt{T_{\mathbf{e}}(\mathbf{K})} \le 2\Big(\log\Delta_{\mathbf{K}} + \log\log\Delta_{\mathbf{K}} - (\gamma + \log 2\pi)n_{\mathbf{K}} + 1 + (n_{\mathbf{K}} + 1)\frac{\log(7\log\Delta_{\mathbf{K}})}{\log\Delta_{\mathbf{K}}}\Big).$$

Moreover, if $\log\Delta_{\mathbf{K}} \ge n_{\mathbf{K}}2^{n_{\mathbf{K}}}$, *we have*

$$\sqrt{T_{\mathcal{C}}(\mathbf{K})} \le \sqrt{T_{\mathbf{e}}(\mathbf{K})} \le 2(\log\Delta_{\mathbf{K}} + \log\log\Delta_{\mathbf{K}} - (\gamma + \log 2\pi)n_{\mathbf{K}} + 1).$$

Proof. We use (3.10) with $T_0 = \log\Delta_{\mathbf{K}} + 1$. We have

$$\frac{1}{2}r(\mathcal{L}, n, \mathcal{L}+1) = \mathcal{L} + \log\mathcal{L} - (\gamma + \log 2\pi)n + 1 + (n+1)\frac{\log\mathcal{L}}{\mathcal{L}} - f(\mathcal{L})n + g(\mathcal{L}),$$

where

$$f(\mathcal{L}) := (\gamma + \log 2\pi)\mathcal{L}^{-1} - (1+\mathcal{L}^{-1})^2 - \mathcal{L}^{-2}\log\mathcal{L},$$
$$g(\mathcal{L}) := (1+\mathcal{L}^{-1})\log(1+\mathcal{L}^{-1}).$$

We have $f(\mathcal{L}) \geq 0$ and $g(\mathcal{L}) - 2f(\mathcal{L}) \leq 0$ for any $\mathcal{L} \geq 4$. This proves that

$$(3.11)\quad \frac{1}{2}r(\log\Delta_{\mathbf{K}}, n_{\mathbf{K}}, \log\Delta_{\mathbf{K}}+1) \leq \log\Delta_{\mathbf{K}} + \log\log\Delta_{\mathbf{K}} - (\gamma+\log 2\pi)n_{\mathbf{K}} + 1 + (n_{\mathbf{K}}+1)\frac{\log\log\Delta_{\mathbf{K}}}{\log\Delta_{\mathbf{K}}}$$

for any $\mathbf{K}$ such that $\log\Delta_{\mathbf{K}} \geq 4$. We look under which condition $\frac{1}{2}\sqrt{T_0}$ satisfies the same bound, i.e. when

$$(3.12)\quad \frac{1}{2}\sqrt{\log\Delta_{\mathbf{K}}+1} \leq \log\Delta_{\mathbf{K}} + \log\log\Delta_{\mathbf{K}} - (\gamma+\log 2\pi)n_{\mathbf{K}} + 1 + (n_{\mathbf{K}}+1)\frac{\log\log\Delta_{\mathbf{K}}}{\log\Delta_{\mathbf{K}}}.$$

For $n \geq 2$ and $\mathcal{L} \geq 1$, let

$$h(\mathcal{L}, n) := \mathcal{L} + \log\mathcal{L} - (\gamma+\log 2\pi)n + 1 + (n+1)\frac{\log\mathcal{L}}{\mathcal{L}} - \frac{1}{2}\sqrt{\mathcal{L}+1},$$

so that (3.12) holds true if $h(\log\Delta_{\mathbf{K}}, n_{\mathbf{K}}) \geq 0$. We have $\frac{\partial h}{\partial\mathcal{L}} \geq 0$ if $\mathcal{L} \geq 0.2n - 1$, which is true each time $\mathcal{L} = \log\Delta_{\mathbf{K}}$ and $n = n_{\mathbf{K}}$. This allows to prove that if $\mathcal{L}_0 \geq 0.2n - 1$ satisfies $h(\mathcal{L}_0, n) \geq 0$, then $h(\mathcal{L}, n) \geq 0$ if $\mathcal{L} \geq \mathcal{L}_0$. Case $b = 2.3$ in Table 3 of [Odl76] shows that $\log\Delta_{\mathbf{K}} \geq 2.8n_{\mathbf{K}} - 9.6$: using this inequality we have $h(\log\Delta_{\mathbf{K}}, n_{\mathbf{K}}) \geq 0$ if $n_{\mathbf{K}} \geq 17$. For $2 \leq n_{\mathbf{K}} \leq 16$, we still have $h(\log\Delta_{\mathbf{K}}, n_{\mathbf{K}}) \geq 0$ for $\log\Delta_{\mathbf{K}} \geq \mathcal{L}_0(n_{\mathbf{K}})$ where $\mathcal{L}_0(n_{\mathbf{K}})$ is indicated in the table below. The table also gives the minimum possible $\log\Delta_{\mathbf{K}}$ for the given $n_{\mathbf{K}}$, computed either with "megrez" number field table or with Odlyzko's Table 3.

$n_{\mathbf{K}}$	$\min\log\Delta_{\mathbf{K}}$	$\mathcal{L}_0(n_{\mathbf{K}})$	$n_{\mathbf{K}}$	$\min\log\Delta_{\mathbf{K}}$	$\mathcal{L}_0(n_{\mathbf{K}})$	$n_{\mathbf{K}}$	$\min\log\Delta_{\mathbf{K}}$	$\mathcal{L}_0(n_{\mathbf{K}})$
2	1.098	2.697	7	12.125	13.676	12	24.336	25.675
3	3.135	4.576	8	13.972	16.053	13	27.749	28.096
4	4.762	6.728	9	17.118	18.446	14	29.748	30.520
5	7.383	8.995	10	19.060	20.849	15	33.256	32.948
6	9.184	11.319	11	22.359	23.259	16	35.277	35.378

We note that (3.12) holds also for $n_{\mathbf{K}} = 15$. By (3.10), (3.11) and (3.12), the first claim is proved for $n_{\mathbf{K}} \geq 17$ or $\log\Delta_{\mathbf{K}} \geq \max(4, \mathcal{L}_0(n_{\mathbf{K}}))$ (in an even stronger form, because now we have $\log\log\Delta_{\mathbf{K}}$ instead of $\log(7\log\Delta_{\mathbf{K}})$).

To complete the proof and to extend the claim to $2 \leq n_{\mathbf{K}} \leq 16$ and $\log\Delta_{\mathbf{K}} \leq \max(4, \mathcal{L}_0(n_{\mathbf{K}}))$ we use a different strategy. Let

$$\ell(n,t) := \frac{1}{2}(t^{1/2} - t^{-1/2}) - \log t + \Big(\gamma + \log 2\pi - \frac{\log t}{t-1} + \log\big(1 - t^{-1}\big)\Big)n,$$

for $n > 0$ and $t > 1$. It is the function such that

$$r(\ell(n,t), n, t) = \sqrt{t},$$

hence, if $\log\Delta_{\mathbf{K}} = \ell(n_{\mathbf{K}},T_0)$, then by (3.10) T_0 is an upper bound for $T_{\mathbf{e}}(\mathbf{K})$; note that this corresponds to the case $a = L = \log T$ in Theorem 3.5. Observe that ℓ is increasing as a function of t, and that it diverges to $-\infty$ and to $+\infty$ for $t \to 1^-$ and $t \to +\infty$, respectively, for every fixed n. As a consequence, for given $n_{\mathbf{K}}$ and $\log\Delta_{\mathbf{K}}$ there is a unique T_0 such that $\ell(n_{\mathbf{K}},T_0) = \log\Delta_{\mathbf{K}}$, and this T_0 is also an upper-bound for $T_{\mathbf{e}}(\mathbf{K})$.
Thus, for $2 \leq n_{\mathbf{K}} \leq 16$ (only a finite set of cases) and $\log\Delta_{\mathbf{K}} \leq \max(4,\mathcal{L}_0(n_{\mathbf{K}}))$ (a bounded range for $\log\Delta_{\mathbf{K}}$) we set T_0 such that $\log\Delta_{\mathbf{K}} = \ell(n_{\mathbf{K}},T_0)$ and we directly check that

$$\frac{1}{2\sqrt{T_0}} + \Big(\frac{\log T_0}{T_0-1} - \log(1-T_0^{-1})\Big)n_{\mathbf{K}} \leq 1 + \log\Big(\frac{\ell(n_{\mathbf{K}},T_0)}{T_0}\Big) + (n_{\mathbf{K}}+1)\frac{\log(7\ell(n_{\mathbf{K}},T_0))}{\ell(n_{\mathbf{K}},T_0)}$$

which is equivalent to

$$\frac{1}{2}\sqrt{T_0} \leq \log\Delta_{\mathbf{K}} + \log\log\Delta_{\mathbf{K}} - (\gamma+\log 2\pi)n_{\mathbf{K}} + 1 + (n_{\mathbf{K}}+1)\frac{\log(7\log\Delta_{\mathbf{K}})}{\log\Delta_{\mathbf{K}}}.$$

(Note that now $\log(7\log\Delta_{\mathbf{K}})$ appears, as in the claim.)

For the second claim of the theorem, we use (3.9) still with $T_0 = \log\Delta_{\mathbf{K}}+1$. We compute a lower bound for the sum of prime ideals choosing two prime ideals $\mathfrak{p}_0$ and $\mathfrak{p}_1$ above respectively 2 and 3. We get

$$\sum_{\mathrm{N}\mathfrak{p}^m \leq T_0}\Big(\frac{1}{\mathrm{N}\mathfrak{p}^m} - \frac{1}{T_0}\Big)\log\mathrm{N}\mathfrak{p} \geq \Big(\frac{1}{\mathrm{N}\mathfrak{p}_0} - \frac{1}{T_0}\Big)\log\mathrm{N}\mathfrak{p}_0 + \delta_{n_{\mathbf{K}},2}\Big(\frac{1}{\mathrm{N}\mathfrak{p}_1} - \frac{1}{T_0}\Big)\log\mathrm{N}\mathfrak{p}_1,$$

where $\delta_{n_{\mathbf{K}},2}$ is 1 if $n_{\mathbf{K}} = 2$ and 0 otherwise. Note that this holds in any case because if $\mathfrak{p}_0$ or $\mathfrak{p}_1$ does not appear in the original sum, then the chosen lower bound is negative. In its turn this is

$$\geq n_{\mathbf{K}}(\log 2)\Big(\frac{1}{2^{n_{\mathbf{K}}}} - \frac{1}{T_0}\Big) + n_{\mathbf{K}}(\log 3)\delta_{n_{\mathbf{K}},2}\Big(\frac{1}{3^{n_{\mathbf{K}}}} - \frac{1}{T_0}\Big), \tag{3.13}$$

because the inert case gives the least contribution. Since $\max(4,\mathcal{L}_0(n_{\mathbf{K}})) \leq n_{\mathbf{K}}2^{n_{\mathbf{K}}}$ for all $n_{\mathbf{K}} \leq 16$, (3.11) holds if $\log\Delta_{\mathbf{K}} \geq n_{\mathbf{K}}2^{n_{\mathbf{K}}}$. Hence to prove the second claim it is sufficient to prove that if $\log\Delta_{\mathbf{K}} \geq n_{\mathbf{K}}2^{n_{\mathbf{K}}}$, then

$$\begin{aligned}(n_{\mathbf{K}}+1)&\frac{\log\log\Delta_{\mathbf{K}}}{\log\Delta_{\mathbf{K}}} \\ &\leq 2n_{\mathbf{K}}(\log 2)\Big(\frac{1}{2^{n_{\mathbf{K}}}} - \frac{1}{\log\Delta_{\mathbf{K}}+1}\Big) + n_{\mathbf{K}}(\log 3)\delta_{n_{\mathbf{K}},2}\Big(\frac{1}{3^{n_{\mathbf{K}}}} - \frac{1}{\log\Delta_{\mathbf{K}}+1}\Big)\end{aligned} \tag{3.14}$$

and

$$\sqrt{\log\Delta_{\mathbf{K}}+1} \leq 2(\log\Delta_{\mathbf{K}} + \log\log\Delta_{\mathbf{K}} - (\gamma+\log 2\pi)n_{\mathbf{K}} + 1).$$

The second statement is elementary and is true for any $n_{\mathbf{K}} \geq 2$. For (3.14), we observe that the left-hand side is decreasing in $\log\Delta_{\mathbf{K}}$ while the right-hand side is increasing, hence it is sufficient to verify it with $\log\Delta_{\mathbf{K}}$ substituted by $n_{\mathbf{K}}2^{n_{\mathbf{K}}}$. One can see that it is true for $n_{\mathbf{K}} \geq 7$ and for $2 \leq n_{\mathbf{K}} \leq 6$ and $\log\Delta_{\mathbf{K}} \geq \mathcal{L}_1(n_{\mathbf{K}})$ as indicated in table below

$n_{\mathbf{K}}$	$n_{\mathbf{K}}2^{n_{\mathbf{K}}}$	$\mathcal{L}_1(n_{\mathbf{K}})$
2	8	15.670
3	24	35.173
4	64	78.801
5	160	174.859
6	384	384.395

To fill the gap, we use (3.9), (3.13) and $T_0 = \log \Delta_{\mathbf{K}} + 7$. □

Remark. The first claim is somehow the best we can hope from (3.10). Indeed the optimal T_0 for (3.10) is such that

$$\log \Delta_{\mathbf{K}} = T_0 - (n_{\mathbf{K}} + 1) \log T_0 + \Big(\gamma + \log 2\pi - 2\frac{\log T_0}{T_0 - 1} + \log(1 - T_0^{-1})\Big) n_{\mathbf{K}} - 1$$

for all but a finite number of fields of degree $n_{\mathbf{K}} \leq 22$. Using this formula, one checks that the best bound we can get from (3.10) is

$$2(\log \Delta_{\mathbf{K}} + \log\log \Delta_{\mathbf{K}} - (\gamma + \log 2\pi) n_{\mathbf{K}} + 1 + \epsilon(\log \Delta_{\mathbf{K}}))$$

where $\epsilon(\log \Delta_{\mathbf{K}}) \sim (n_{\mathbf{K}} + 1) \log\log \Delta_{\mathbf{K}} / \log \Delta_{\mathbf{K}}$ for $\log \Delta_{\mathbf{K}} \to \infty$ and fixed $n_{\mathbf{K}}$.

Remark. Using the full strength of (3.9), one can prove that for quadratic fields the second claim of Theorem 3.6 is true for $T_{\mathcal{C}}(\mathbf{K})$ also for $\log \Delta_{\mathbf{K}} \leq n_{\mathbf{K}} 2^{n_{\mathbf{K}}} = 8$ with only the four exceptions $\mathbf{Q}[\sqrt{-15}]$, $\mathbf{Q}[\sqrt{-5}]$, $\mathbf{Q}[\sqrt{-23}]$ and $\mathbf{Q}[\sqrt{-6}]$ (for which $T_{\mathcal{C}}(\mathbf{K}) = 2$) and the ten fields of discriminant in $[-11, 13] \cup \{-19\}$ (for which the class group is trivial).

We now prove that, for fixed $n_{\mathbf{K}}$, the absolute upper bound for $T_{\mathcal{C}}(\mathbf{K}) / \log^2 \Delta_{\mathbf{K}}$ is near 4 and that the asymptotic limit $4 \log^2 \Delta_{\mathbf{K}}$ is true for a very large set of fields.

Corollary 3.7. *We have*

$$T_{\mathcal{C}}(\mathbf{K}) \leq T_{\mathbf{e}}(\mathbf{K}) \leq 4\big(1 + \big(2\pi e^{\gamma}\big)^{-n_{\mathbf{K}}}\big)^2 \log^2 \Delta_{\mathbf{K}}.$$

Moreover,

$$\text{if} \quad \log \Delta_{\mathbf{K}} \leq \frac{1}{e}\big(2\pi e^{\gamma}\big)^{n_{\mathbf{K}}} \quad \text{then} \quad T_{\mathcal{C}}(\mathbf{K}) \leq T_{\mathbf{e}}(\mathbf{K}) \leq 4 \log^2 \Delta_{\mathbf{K}}.$$

Notice that $2\pi e^{\gamma} > 11.19$.

Proof. It is sufficient to prove that $T_{\mathbf{e}}(\mathbf{K}) \leq 4 \log^2 \Delta_{\mathbf{K}}$ if $\log \Delta_{\mathbf{K}} \leq n_{\mathbf{K}} 2^{n_{\mathbf{K}}}$, because the second statement of Theorem 3.6 already proves both statements in the remaining ranges.

The right-hand side of the first statement of Theorem 3.6 is $2 \log \Delta_{\mathbf{K}} + 2f(\log \Delta_{\mathbf{K}}, n_{\mathbf{K}})$ with

$$f(\mathcal{L}, n) := \log \mathcal{L} + 1 - (\gamma + \log 2\pi) n + (n+1) \frac{\log(7\mathcal{L})}{\mathcal{L}}.$$

We just need to check that $f(\log \Delta_{\mathbf{K}}, n_{\mathbf{K}}) \leq 0$ if $\log \Delta_{\mathbf{K}} \leq n_{\mathbf{K}} 2^{n_{\mathbf{K}}}$. As a function of $\mathcal{L} \geq 1$, for fixed $n \geq 2$, $\frac{\partial f}{\partial \mathcal{L}} = \frac{n+1}{\mathcal{L}}\big(\frac{1}{n+1} - \frac{\log(7\mathcal{L}) - 1}{\mathcal{L}}\big)$ is negative then positive, hence to check that f is negative, it is sufficient to check its value for the minimum and the maximum $\mathcal{L}$ we are interested in. We have $f(n2^n, n) < 0$ for any $n \geq 2$ and $f(\log 23, n) < 0$ for any $n \geq 3$. Thus $f(\log \Delta_{\mathbf{K}}, n_{\mathbf{K}}) < 0$ for any field of degree $n_{\mathbf{K}} \geq 3$. For quadratic fields, we come back to (3.10) with $T_0 = 2 \log \Delta_{\mathbf{K}}$ to directly check that $T_{\mathbf{e}}(\mathbf{K}) \leq 4 \log^2 \Delta_{\mathbf{K}}$ if $\log \Delta_{\mathbf{K}} \leq n_{\mathbf{K}} 2^{n_{\mathbf{K}}} = 8$. □

Remark. The upper bound for the quotient $T_{\mathbf{e}}(\mathbf{K}) / \log^2 \Delta_{\mathbf{K}}$ tends obviously very fast to 4. For instance, for $n_{\mathbf{K}} \geq 10$, we have $T_{\mathcal{C}}(\mathbf{K}) \leq T_{\mathbf{e}}(\mathbf{K}) \leq (4 + 2.6 \cdot 10^{-10}) \log^2 \Delta_{\mathbf{K}}$, but notice that the second claim in Corollary 3.7 shows that $T_{\mathcal{C}}(\mathbf{K}) \leq T_{\mathbf{e}}(\mathbf{K}) \leq 4 \log^2 \Delta_{\mathbf{K}}$ if $\Delta_{\mathbf{K}} \leq \exp(10^{10})$.

For a much tighter range of discriminants one can prove bounds of the form $T_{\mathbf{e}}(\mathbf{K}) \leq c\log^2\Delta_{\mathbf{K}}$ with $c<4$. For instance, we have the psychologically important bound $T_{\mathcal{C}}(\mathbf{K}) \leq T_{\mathbf{e}}(\mathbf{K}) \leq \log^2\Delta_{\mathbf{K}}$ as soon as

$$\log\Delta_{\mathbf{K}} + 2\log\log\Delta_{\mathbf{K}} + 2 + 2(n_{\mathbf{K}}+1)\frac{\log(7\log\Delta_{\mathbf{K}})}{\log\Delta_{\mathbf{K}}} \leq 2(\gamma+\log 2\pi)n_{\mathbf{K}}.$$

For a given degree $n_{\mathbf{K}} \geq 4$, this happens for $\Delta_{\mathbf{K}}$ lower than a certain limit. As $n_{\mathbf{K}}$ goes to infinity, the limit corresponds to a root-discriminant tending to $(2\pi e^{\gamma})^2 = 125.23\ldots$. There are infinitely many fields satisfying this condition. Indeed, consider the field $F = \mathbf{Q}[\cos(2\pi/11), \sqrt{2}, \sqrt{-23}]$. Martinet [Mar78] proved that the Hilbert class field tower of F is infinite because F satisfies Golod–Shafarevich's condition. Since $n_F = 20$ and $\log\Delta_F \leq 90.6$, this shows that there is an infinite number of fields $\mathbf{K}$ such that $\log\Delta_{\mathbf{K}} \leq 4.53n_{\mathbf{K}}$. For one of those fields, we have $T_{\mathcal{C}}(\mathbf{K}) \leq T_{\mathbf{e}}(\mathbf{K}) \leq \log^2\Delta_{\mathbf{K}}$ if $n_{\mathbf{K}} \geq 47$ and the quotient improves when the degree increases, with $\limsup\{T_{\mathbf{e}}(\mathbf{K})/\log^2\Delta_{\mathbf{K}}\colon \log\Delta_{\mathbf{K}} \leq 4.53n_{\mathbf{K}}\} \leq 0.88$.

As a second example, consider $F = \mathbf{Q}[x]/(f)$, where $f = x^{10} + 223x^8 + 18336x^6 + 10907521x^4 + 930369979x^2 + 18559139599$. Hajir and Maire [HM01a] proved that the Hilbert class field tower of F is infinite because F satisfies Golod–Shafarevich's condition. Since $\log\Delta_F \leq 44.4$, this shows that there is an infinite number of fields $\mathbf{K}$ such that $\log\Delta_{\mathbf{K}} \leq 4.44n_{\mathbf{K}}$. For one of those fields, we have $T_{\mathcal{C}}(\mathbf{K}) \leq T_{\mathbf{e}}(\mathbf{K}) \leq \log^2\Delta_{\mathbf{K}}$ if $n_{\mathbf{K}} \geq 34$ with $\limsup\{T_{\mathbf{e}}(\mathbf{K})/\log^2\Delta_{\mathbf{K}}\colon \log\Delta_{\mathbf{K}} \leq 4.44n_{\mathbf{K}}\} \leq 0.84$.

As a third example, consider the field $F = \mathbf{Q}[x]/(f)$, where $f = x^{12} + 339x^{10} - 19752x^8 - 2188735x^6 + 284236829x^4 + 4401349506x^2 + 15622982921$. In [HM01b], the authors proved that F admits an infinite tower of extensions ramified at most above a single prime ideal of F of norm 9. Since $\log(9\Delta_F)/12 \leq 4.41$, there is an infinite number of fields $\mathbf{K}$ such that $\log\Delta_{\mathbf{K}} \leq 4.41n_{\mathbf{K}}$. For one of those fields, we have $T_{\mathcal{C}}(\mathbf{K}) \leq T_{\mathbf{e}}(\mathbf{K}) \leq \log^2\Delta_{\mathbf{K}}$ if $n_{\mathbf{K}} \geq 32$ with $\limsup\{T_{\mathbf{e}}(\mathbf{K})/\log^2\Delta_{\mathbf{K}}\colon \log\Delta_{\mathbf{K}} \leq 4.41n_{\mathbf{K}}\} \leq 0.82$.
Assuming GRH, Serre [Ser75] proved that there are only finitely many fields such that $\log\Delta_{\mathbf{K}} \leq cn_{\mathbf{K}}$ for every $c < \gamma + \log 8\pi$. Suppose that $\log\Delta_{\mathbf{K}} \leq (\gamma+\log 8\pi)n_{\mathbf{K}}$, then $T_{\mathcal{C}}(\mathbf{K}) \leq T_{\mathbf{e}}(\mathbf{K}) \leq \log^2\Delta_{\mathbf{K}}$ if $n_{\mathbf{K}} \geq 11$. Serre's result does not rule out the possibility that there are infinitely many such fields; in this case $\limsup\{T_{\mathbf{e}}(\mathbf{K})/\log^2\Delta_{\mathbf{K}}\colon \log\Delta_{\mathbf{K}} \leq (\gamma+\log 8\pi)n_{\mathbf{K}}\} \leq \big(\frac{4\log 2}{\gamma+\log 8\pi}\big)^2 \leq 0.54$.

Corollary 3.8. *Assume* GRH. *Then*

$$T_{\mathcal{C}}(\mathbf{K}) \leq T_{\mathbf{e}}(\mathbf{K}) \leq 4.01\log^2\Delta_{\mathbf{K}}.$$

Proof. For $n_{\mathbf{K}} \geq 3$ or $n_{\mathbf{K}} = 2$ and $\log\Delta_{\mathbf{K}} \leq (2\pi e^{\gamma})^{n_{\mathbf{K}}}/e$, the claim follows from Corollary 3.7. For $n_{\mathbf{K}} = 2$ and $\log\Delta_{\mathbf{K}} \geq (2\pi e^{\gamma})^{n_{\mathbf{K}}}/e$ we apply a different argument. Let

$$f_{\mathbf{K}}(n,t) = \log t - \Big(\gamma + \log 2\pi - \frac{\log t}{t-1} + \log\big(1-t^{-1}\big)\Big)n - \frac{2}{1-t^{-1}}\sum_{\mathrm{N}\mathfrak{p}^m \leq t}\Big(\frac{1}{\mathrm{N}\mathfrak{p}^m} - \frac{1}{t}\Big)\log\mathrm{N}\mathfrak{p},$$

so that (3.9) can be written as

$$\sqrt{T_{\mathbf{e}}(\mathbf{K})} \leq \max\Big(\sqrt{T_0}, \frac{2}{1-T_0^{-1}}(\log\Delta_{\mathbf{K}} + f_{\mathbf{K}}(n_{\mathbf{K}}, T_0))\Big).$$

Suppose we have a T_0 such that $f_{\mathbf{K}}(n_{\mathbf{K}}, T_0) \leq 0$, then we have

$$\sqrt{T_{\mathbf{e}}(\mathbf{K})} \leq \max\Big(\sqrt{T_0}, \frac{2}{1-T_0^{-1}}\log\Delta_{\mathbf{K}}\Big),$$

and hence

$$T_{\mathbf{e}}(\mathbf{K}) \leq \frac{4}{(1-T_0^{-1})^2}\log^2\Delta_{\mathbf{K}}$$

if $\log\Delta_{\mathbf{K}} \geq \frac{1}{2}(T_0^{1/2} - T_0^{-1/2})$. Recalling that $n_{\mathbf{K}} = 2$, we choose $T_0 = 935$: in this case

$$f_{\mathbf{K}}(2, 935) \leq 2 - \frac{935}{467}\sum_{\mathrm{N}\mathfrak{p}^m \leq 935}\Big(\frac{1}{\mathrm{N}\mathfrak{p}^m} - \frac{1}{935}\Big)\log\mathrm{N}\mathfrak{p}.$$

The value of the sum on prime ideals depends on $\mathbf{K}$, but it is always larger than what we get assuming that all primes are inert. This gives

$$f_{\mathbf{K}}(2, 935) \leq 2 - \frac{935}{467}\sum_{p^{2m} \leq 935}\Big(\frac{1}{p^{2m}} - \frac{1}{935}\Big)\log(p^2) \leq -0.02$$

which therefore produces $T_{\mathbf{e}}(\mathbf{K}) \leq 4.0086\log^2\Delta_{\mathbf{K}}$ for $\log\Delta_{\mathbf{K}} \geq 15.3$. The proof is complete because $(2\pi e^{\gamma})^2/e \geq 46$. □

3.3. Lower bound for $T_{\mathbf{e}}(\mathbf{K})$.

Proposition 3.9. *Assume* GRH. *Then*

$$\sqrt{T_{\mathbf{e}}(\mathbf{K})} \geq (1 + o(1))\frac{\log\Delta_{\mathbf{K}}}{n_{\mathbf{K}}}.$$

Proof. Let $S(T)$ denote the Dirichlet series appearing on the left-hand side of (2.1). Then, introducing the generalized von Mangoldt function $\tilde{\Lambda}_{\mathbf{K}}(n) := \sum_{\mathrm{N}\mathfrak{p}^m = n}\log\mathrm{N}\mathfrak{p}$ we get

$$S(T) = \sum_n \frac{2F_{\mathbf{e}}(\log n)}{\sqrt{n}}\tilde{\Lambda}_{\mathbf{K}}(n). \tag{3.15}$$

Using $\tilde{\Lambda}_{\mathbf{K}}(n) \leq n_{\mathbf{K}}\Lambda(n)$ in (3.15) and introducing Stieltjes' integral notation, we have

$$\frac{S(T)}{n_{\mathbf{K}}} \leq \int_{2^-}^{+\infty}\frac{2F_{\mathbf{e}}(\log x)}{\sqrt{x}}\,\mathrm{d}\psi(x).$$

Let $g(x) = \frac{2F_{\mathbf{e}}(\log x)}{\sqrt{x}}$ and notice that it is a continuous function which is derivable except at most in $T^{\pm 1}$ and $(Te^{-2a})^{\pm 1}$ with a derivative which is continuous where it exists and bounded. Thus, with a partial integration we get:

$$\begin{aligned}\frac{S(T)}{n_{\mathbf{K}}} &\leq -\int_2^{+\infty} g'(x)\psi(x)\,\mathrm{d}x \leq -\int_2^{+\infty} g'(x)x\,\mathrm{d}x + \int_2^{+\infty}|g'(x)||\psi(x) - x|\,\mathrm{d}x\\ &= \int_2^{+\infty} g(x)\,\mathrm{d}x + \int_2^{+\infty}|g'(x)||\psi(x) - x|\,\mathrm{d}x + 2g(2).\end{aligned}$$

Since under RH $|\psi(x)-x|\le 2\sqrt{x}\log^2 x$ for every $x\ge 2$ (Schoenfeld proved that RH implies $|\psi(x)-x|\le\frac{1}{8\pi}\sqrt{x}\log^2 x$ as soon as $x\ge 74$, a direct computation shows that inequality for the intermediate range $x\in[2,74]$) we get

$$
\begin{aligned}
&\le \int_2^{+\infty} g(x)\,\mathrm{d}x + 2\int_2^{+\infty}|g'(x)|\sqrt{x}\,\log^2 x\,\mathrm{d}x + 3F_{\mathbf{e}}(\log 2)\\
&= 2\int_2^{+\infty}\frac{F_{\mathbf{e}}(\log x)}{\sqrt{x}}\,\mathrm{d}x + 2\int_2^{+\infty}\Big|\Big(\frac{F_{\mathbf{e}}(\log x)}{\sqrt{x}}\Big)'\Big|\sqrt{x}\,\log^2 x\,\mathrm{d}x + 3F_{\mathbf{e}}(\log 2)\\
&= 2\int_{\log 2}^{+\infty} F_{\mathbf{e}}(x)e^{x/2}\,\mathrm{d}x + \int_{\log 2}^{+\infty}|2F'_{\mathbf{e}}(x)-F_{\mathbf{e}}(x)|x^2\,\mathrm{d}x + 3F_{\mathbf{e}}(\log 2).
\end{aligned}
$$

We extend the range of the integrals, getting

$$
\frac{S(T)}{n_{\mathbf{K}}}\le 2\int_0^{+\infty}F_{\mathbf{e}}(x)e^{x/2}\,\mathrm{d}x + \int_0^{+\infty}|2F'_{\mathbf{e}}(x)-F_{\mathbf{e}}(x)|x^2\,\mathrm{d}x + 3F_{\mathbf{e}}(\log 2). \tag{3.16}
$$

We notice that

$$
\int_0^{+\infty}F_{\mathbf{e}}(x)e^{x/2}\,\mathrm{d}x \le \int_{\mathbf{R}}F_{\mathbf{e}}(x)e^{x/2}\,\mathrm{d}x = \Big(\int_{\mathbf{R}}\Phi_{\mathbf{e}}(x)e^{x/4}\,\mathrm{d}x\Big)^2. \tag{3.17}
$$

We observe that, since $\Phi_{\mathbf{e}}\ge 0$, $\max F_{\mathbf{e}} = F_{\mathbf{e}}(0)$ and that the non-negative part of the support of $F_{\mathbf{e}}$ is included in $[0,a]\cup[L-2a,L]$ (the intervals may overlap) hence

$$
\begin{aligned}
\int_0^{+\infty}|F_{\mathbf{e}}(x)|x^2\,\mathrm{d}x &= \int_0^L F_{\mathbf{e}}(x)x^2\,\mathrm{d}x \le F_{\mathbf{e}}(0)\Big(\int_0^a x^2\,\mathrm{d}x + \int_{L-2a}^L x^2\,\mathrm{d}x\Big)\\
&= F_{\mathbf{e}}(0)\big(2aL^2-4a^2L+3a^3\big)\le 2aF_{\mathbf{e}}(0)L^2.
\end{aligned} \tag{3.18}
$$

Moreover, from Remark 3.3, we see that the function is piecewise of the form $(ax+b)e^{x/2}+(cx+d)e^{-x/2}$, with a, b, c and d constants, with at most four pieces. Deriving the expression we find that it can have at most three variations in each piece. The total variation of $F_{\mathbf{e}}$ on $[0,L]$ is thus at most $12\max F_{\mathbf{e}} = 12F_{\mathbf{e}}(0)$. It follows that

$$
\int_0^{+\infty}|F'_{\mathbf{e}}(x)|x^2\,\mathrm{d}x\le 12F_{\mathbf{e}}(0)L^2. \tag{3.19}
$$

Plugging (3.17), (3.18) and (3.19) into (3.16) we get

$$
\begin{aligned}
\frac{S(T)}{n_{\mathbf{K}}} &\le 2\Big(\int_{\mathbf{R}}\Phi_{\mathbf{e}}(x)e^{x/2}\,\mathrm{d}x\Big)^2 + 2(a+12)F_{\mathbf{e}}(0)L^2 + 3F_{\mathbf{e}}(0)\\
&= 2\Big(e^{-a}T^{3/4}\int_0^a e^x\,\mathrm{d}x + T^{-1/4}\int_0^a\mathrm{d}x\Big)^2 + 2(a+12)F_{\mathbf{e}}(0)L^2 + 3F_{\mathbf{e}}(0)\\
&= 2\Big(\big(1-e^{-a}\big)T^{3/4}+aT^{-1/4}\Big)^2 + 2(a+12)F_{\mathbf{e}}(0)L^2 + 3F_{\mathbf{e}}(0)\\
&= 2\big(1-e^{-a}\big)^2T^{3/2} + 4a\big(1-e^{-a}\big)T^{1/2} + a^2T^{-1/2} + 2(a+12)F_{\mathbf{e}}(0)L^2 + 3F_{\mathbf{e}}(0).
\end{aligned}
$$

We have $\mathrm{J}(F_{\mathbf{e}})\le\frac{\pi}{2}F_{\mathbf{e}}(0)$ hence in order to satisfy (2.1) we must have

$$
\begin{aligned}
2\big(1-e^{-a}\big)^2T^{3/2} &+ 4a\big(1-e^{-a}\big)T^{1/2} + a^2T^{-1/2} + 2(a+12)F_{\mathbf{e}}(0)L^2 + 3F_{\mathbf{e}}(0)\\
&\ge\frac{S(T)}{n_{\mathbf{K}}}\ge F_{\mathbf{e}}(0)\Big(\frac{\log\Delta_{\mathbf{K}}}{n_{\mathbf{K}}}-\Big(\gamma+\log 8\pi+\frac{\pi}{2}\Big)\Big)
\end{aligned}
$$

which we can simplify to

$$(1+o(1))\sqrt{T} \geq \frac{F_{\mathbf{e}}(0)}{2\big(1-e^{-a}\big)^2 T}\Big(\frac{\log \Delta_{\mathbf{K}}}{n_{\mathbf{K}}} - 2(a+12)L^2 + O(1)\Big).$$

Since $F_{\mathbf{e}}(0) = \int_{\mathbf{R}} (\Phi_{\mathbf{e}}(y))^2 \,\mathrm{d}y \geq 2\int_{L/2-a}^{L/2} e^{L/2+y}\,\mathrm{d}y = 2\big(1-e^{-a}\big)T$, this requires

$$\big(1+o(1)\big)^2\sqrt{T} \geq \frac{1}{1-e^{-a}}\Big(\frac{\log \Delta_{\mathbf{K}}}{n_{\mathbf{K}}} - 2(a+12)L^2 + O(1)\Big).$$

In the given range for a, we can assume that the right-hand side is positive otherwise the claim is evident. In that case the minimum for the main term is obviously $a = \log T$, hence we can assume that

$$(1+o(1))\sqrt{T} \geq \frac{1}{1-T^{-1}}\Big(\frac{\log \Delta_{\mathbf{K}}}{n_{\mathbf{K}}} + O(\log^3 \log \Delta_{\mathbf{K}})\Big).$$

The claim follows. □

4. Upper bound for $T(\mathbf{K})$

Belabas, Diaz y Diaz and Friedman [BDyDF08, Section 3] applied Theorem 2.4 with $F(x) = F_L(x) := (L-x)\chi_{[-L,L]}(x) = (\Phi * \Phi)(x)$, where Φ is the characteristic function of $[-L/2, L/2]$, with $L = \log T$, $T > 1$. (Actually they chose $F = \frac{1}{L}\Phi * \Phi$, but the difference does not matter since (2.1) is homogeneous). For this weight function, (2.1) reads

$$2\sum_{\substack{\mathfrak{p},m\\ \mathrm{N}\mathfrak{p}^m < T}} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{m/2}}\Big(1 - \frac{\log \mathrm{N}\mathfrak{p}^m}{L}\Big) > \log \Delta_{\mathbf{K}} - (\gamma + \log 8\pi)n_{\mathbf{K}} + \frac{\mathrm{I}(F_L)}{L}n_{\mathbf{K}} - \frac{\mathrm{J}(F_L)}{L}r_1 \tag{4.1}$$

with

$$\mathrm{I}(F_L) = \frac{\pi^2}{2} - 4\,\mathrm{dilog}\left(\frac{1}{\sqrt{T}}\right) + \mathrm{dilog}\left(\frac{1}{T}\right) \leq \frac{\pi^2}{2}$$

and

$$\mathrm{J}(F_L) = \frac{\pi L}{2} - 4C + 4\,\mathrm{Im}\,\mathrm{dilog}\left(\frac{i}{\sqrt{T}}\right) \geq \frac{\pi L}{2} - 4C,$$

where $\mathrm{dilog}\, x = -\int_0^x \frac{\log(1-u)}{u}\,\mathrm{d}u$ and $C = \sum_{k\geq 0}(-1)^k(2k+1)^{-2} = 0.9159\ldots$ is Catalan's constant. Note that Belabas, Diaz y Diaz and Friedman use the estimated values for $\mathrm{I}(F_L)$ and $\mathrm{J}(F_L)$ instead of their exact values: this is a legitimate simplification which affects the conclusions only by very small quantities. In this way they produce a quick algorithm giving a bound $T(\mathbf{K})$ for $T_{\mathcal{C}}(\mathbf{K})$ which, in explicit computations, is very small. Unfortunately they also prove that it is $\geq ((\frac{1}{4n_{\mathbf{K}}} + o(1))\log \Delta_{\mathbf{K}} \log\log \Delta_{\mathbf{K}})^2$, and that therefore it is asymptotically worse than Bach's bound. In the same paper they advance the conjecture that $(\log \Delta_{\mathbf{K}} \log\log \Delta_{\mathbf{K}})^2$ is the correct size of $T(\mathbf{K})$, guessing that $T(\mathbf{K}) = ((\frac{1}{4} + o(1))\log \Delta_{\mathbf{K}} \log\log \Delta_{\mathbf{K}})^2$. In this section we prove for $T(\mathbf{K})$ the corresponding upper bound and some explicit bounds.

4.1. **Estimation of the number of zeros.** We first prove an estimation of the number of zeros of Dedekind's zeta function that we will use to prove the main result of this section.

Definition 4.1. *Let, for* $t \in \mathbf{R}$*,*

$$N_{\mathbf{K}}(t) := \#\{\rho \colon |\gamma| \le t\},$$

where the number is intended including the multiplicity.

Trudgian [Tru15] gives an estimation of $N_{\mathbf{K}}(t)$ for $t \ge 1$. The bound depends on a parameter η which we take equal to 0.05. In that case the formula is:

$$\forall t \ge 1, \quad N_{\mathbf{K}}(t) = \frac{t}{\pi} \log\Big(\Delta_{\mathbf{K}}\Big(\frac{t}{2\pi e}\Big)^{n_{\mathbf{K}}}\Big) + R_{\mathbf{K}}(t),$$

with

$$|R_{\mathbf{K}}(t)| \le 0.247(\log \Delta_{\mathbf{K}} + n_{\mathbf{K}} \log t) + 8.851 n_{\mathbf{K}} + 3.024.$$

For $t \in (0, 1]$, we use a different strategy.

Proposition 4.2. *Assume* GRH*. We have*

$$\forall t \in (0,1], \quad N_{\mathbf{K}}(t) \le 0.637 t \Big(\log \Delta_{\mathbf{K}} - 2.45 n_{\mathbf{K}} + S\Big(\frac{3.03}{t}\Big)\Big)$$

where

$$S(U) := 960 \frac{\big((U-4)e^{\frac{U}{4}} + (U+4)e^{-\frac{U}{4}}\big)^2}{U^5}.$$

Proof. We use an analog of [Oma00, Proposition 1], but with a different weight: the function $F := 30\phi * \phi$ with $\phi(x) := \big(\frac{1}{4} - x^2\big)\chi_{[-\frac{1}{2},\frac{1}{2}]}(x)$. We then have

$$F(x) = \begin{cases} -|x|^5 + 5|x|^3 - 5x^2 + 1 & \text{if } |x| \le 1, \\ 0 & \text{if } 1 < |x|, \end{cases}$$

and

$$\widehat{F}(t) = 30\big(\widehat{\phi}(t)\big)^2 = 120 \frac{\big(2\sin\big(\frac{t}{2}\big) - t\cos\big(\frac{t}{2}\big)\big)^2}{t^6}.$$

Observe that $\widehat{F}$ is decreasing on $[0, 8.98]$. Consider, for $U > 0$,

$$F_U(x) := F\Big(\frac{x}{U}\Big).$$

We then have

$$\widehat{F_U}(t) = U\widehat{F}(Ut) = 120U \frac{\big(2\sin\big(\frac{Ut}{2}\big) - Ut\cos\big(\frac{Ut}{2}\big)\big)^2}{(Ut)^6}.$$

Using Weil's Explicit Formula (2.2) we have

$$U \sum_{\gamma} \widehat{F}(U\gamma) = 4\int_0^{+\infty} F_U(x)\cosh\Big(\frac{x}{2}\Big)\,\mathrm{d}x - 2\sum_{\mathfrak{p},m} \frac{\log \mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}^{\frac{m}{2}}} F_U(m\log \mathrm{N}\mathfrak{p}) + \log\Delta_{\mathbf{K}} - (\gamma + \log 8\pi)n_{\mathbf{K}} + \mathrm{I}(F_U)n_{\mathbf{K}} - \mathrm{J}(F_U)r_1. \tag{4.2}$$

We have

$$\begin{aligned} 4\int_0^{+\infty} F_U(x)\cosh\Big(\frac{x}{2}\Big)\,\mathrm{d}x &= 4\int_0^{+\infty} F\Big(\frac{x}{U}\Big)\cosh\Big(\frac{x}{2}\Big)\,\mathrm{d}x \\ &= 4U\int_0^1 (1-5x^2+5x^3-x^5)\cosh\Big(\frac{Ux}{2}\Big)\,\mathrm{d}x \\ &= 960\frac{\big((U-4)e^{\frac{U}{4}}+(U+4)e^{-\frac{U}{4}}\big)^2}{U^5} = S(U). \end{aligned}$$

Moreover,

$$\mathrm{I}(F_U) = \int_0^{+\infty} \frac{1-F_U(x)}{2\sinh(x/2)}\,\mathrm{d}x = U\int_0^{+\infty}(1-F(x))\frac{e^{-Ux/2}}{1-e^{-Ux}}\,\mathrm{d}x.$$

Integrating by parts, which is possible because F is C^2, it becomes

$$= 5\int_0^1 (2x-3x^2+x^4)\log\Big(\frac{1+e^{-Ux/2}}{1-e^{-Ux/2}}\Big)\,\mathrm{d}x$$

from which we readily see that $\mathrm{I}(F_U)$ is decreasing. Removing the positive terms $\sum_{\mathfrak{p},m}$ and $\mathrm{J}(F_U)r_1$ from (4.2), we get

$$\forall U>0,\quad U\sum_\gamma \widehat{F}(U\gamma) \le \log\Delta_{\mathbf{K}} - (\gamma+\log 8\pi - \mathrm{I}(F_U))n_{\mathbf{K}} + S(U).$$

Let $t\in(0,1]$ and c such that $0<c\le 8.98$, then setting $U=\frac{c}{t}$ and using $\mathrm{I}(F_{c/t})\le \mathrm{I}(F_c)$ we have

$$\sum_\gamma \widehat{F}\Big(\frac{c\gamma}{t}\Big) \le \frac{t}{c}\Big(\log\Delta_{\mathbf{K}} - (\gamma+\log 8\pi-\mathrm{I}(F_c))n_{\mathbf{K}} + S\Big(\frac{c}{t}\Big)\Big)$$

and

$$\widehat{F}(c)N_{\mathbf{K}}(t) \le \sum_{|\gamma|\le t}\widehat{F}\Big(\frac{c\gamma}{t}\Big) \le \sum_\gamma \widehat{F}\Big(\frac{c\gamma}{t}\Big)$$

so that for all $t\in(0,1]$ and all $c\in(0,8.98]$ we have

$$N_{\mathbf{K}}(t) \le \frac{c^5 t}{120(2\sin(c/2)-c\cos(c/2))^2}\Big(\log\Delta_{\mathbf{K}} - \big(\gamma+\log 8\pi - \mathrm{I}(F_c)\big)n_{\mathbf{K}} + S\Big(\frac{c}{t}\Big)\Big).$$

The value of c minimizing the coefficient of $t\log\Delta_{\mathbf{K}}$ is $3.051\ldots$. The claim follows setting $c=3.03$. ☐

Definition 4.3. *Let $M_{\mathbf{K}}(t)$ be the function*

$$M_{\mathbf{K}}(t) := \begin{cases} 0.637t\big(\log\Delta_{\mathbf{K}} - 2.45n_{\mathbf{K}} + S\big(\frac{3.03}{t}\big)\big) & \text{if } 0<t<1 \\ \frac{t}{\pi}\log\big(\Delta_{\mathbf{K}}\big(\frac{t}{2\pi e}\big)^{n_{\mathbf{K}}}\big) + 0.247(\log\Delta_{\mathbf{K}} + n_{\mathbf{K}}\log t) + 8.851n_{\mathbf{K}} + 3.024 & \text{if } 1\le t. \end{cases}$$

Recalling the previous proposition we thus have

$$\forall t>0,\quad N_{\mathbf{K}}(t)\le M_{\mathbf{K}}(t).$$

4.2. **The bound.** Now we are in position to prove the announced upper bounds for $T(\mathbf{K})$.

Theorem 4.4. *Assume* GRH. *We have for any fixed* $n_{\mathbf{K}}$*,*

$$\limsup_{\Delta_{\mathbf{K}}\to\infty} \frac{T(\mathbf{K})}{\big(\log\Delta_{\mathbf{K}}\log\log\Delta_{\mathbf{K}}\big)^2} \leq \frac{1}{16}.$$

Moreover, for any field $\mathbf{K} \notin \big\{\mathbf{Q}[\sqrt{-1}], \mathbf{Q}[\sqrt{-3}], \mathbf{Q}[\sqrt{5}]\big\}$ *we have*

$$T(\mathbf{K}) \leq 3.9\big(\log\Delta_{\mathbf{K}}\log\log\Delta_{\mathbf{K}}\big)^2.$$

Remark. Computing $T(\mathbf{K})$ for the whole "megrez" number field table [megrez08], we find that the quotient $T(\mathbf{K})/(\log\Delta_{\mathbf{K}}\log\log\Delta_{\mathbf{K}})^2$ is mostly ≤ 0.27 for them. In fact, apart the fields appearing as exceptions in the theorem and for which the quotient is ≥ 10, there are only six more fields for which it is ≥ 1: the quadratic fields of discriminant in $\{-11,-8,-7,8,12,13\}$.

Proof. Assume $T > e$, $L = \log T$ and $F := F_L = \Phi * \Phi$ with $\Phi := \chi_{[-L/2,L/2]}$. Then (2.4) becomes

$$4\Big(\sqrt{T} - 2 + \frac{1}{\sqrt{T}}\Big) \geq \sum_{\gamma} \frac{1-\cos(L\gamma)}{\gamma^2},$$

but to estimate $T(\mathbf{K})$ we have to add the function

$$G(T) = \Big(4\operatorname{dilog}\Big(\frac{1}{\sqrt{T}}\Big) - \operatorname{dilog}\Big(\frac{1}{T}\Big)\Big)n_{\mathbf{K}} + 4\operatorname{Im}\operatorname{dilog}\Big(\frac{i}{\sqrt{T}}\Big)r_1$$

to the right hand side, as a consequence of the approximations used in [BDyDF08] for $\mathrm{I}(F_L)$ and $\mathrm{J}(F_L)$. Thus, introducing $f(t) := \frac{1-\cos t}{t^2}$, the condition becomes

$$4\Big(\sqrt{T} - 2 + \frac{1}{\sqrt{T}}\Big) \geq L^2\sum_{\gamma} f(L\gamma) + G(T), \tag{4.3}$$

and we need an upper bound for the sum appearing on the right hand side. The function $G(T)$ may be easily estimated, for $T \geq e$, as

$$G(T) \leq \frac{8n_{\mathbf{K}}}{\sqrt{T}}\sum_{k=0}^{\infty}\frac{T^{-2k}}{(4k+1)^2} \leq \frac{8.05n_{\mathbf{K}}}{\sqrt{T}}. \tag{4.4}$$

The main contribution to the sum on zeros comes from those which are close to 0, the remaining ones being easily and quite well estimated via the partial summation formula. Thus, we consider first the range $|t| \leq 1$. The best absolute bound for $f(Lt)$ in this range is $\frac{1}{2}L^2$, and if we bound the sum $\sum_{|\gamma|\leq} f(L\gamma)$ simply as $\sup_{t\in[0,1]}|f(Lt)|N_{\mathbf{K}}(1)$ then we get a term of size $\frac{1}{2}L^2\log\Delta_{\mathbf{K}}$. With this bound (4.3) would become

$$(4+o(1))\sqrt{T} > \Big(\frac{1}{2}+o(1)\Big)L^2\log\Delta_{\mathbf{K}}$$

forcing T to $\geq (\frac{1}{4}+o(1))(\log\Delta_{\mathbf{K}})^2(\log\log\Delta_{\mathbf{K}})^4$, which is much larger than what we want to prove.

We overcome this problem using the conclusion of Proposition 4.2 to bound $N_{\mathbf{K}}(t)$ for $t \leq 1$, but in itself this is not yet sufficient since that bound diverges as t goes to 0. Hence, for very small γ we apply a more involved argument. In a way similar to the proof of Proposition 4.2 but using $F := \Phi * \Phi$ with $\Phi := \chi_{[-1/2,1/2]}$, we have,

$$\forall U > 0, \qquad 2U\sum_\gamma f(U\gamma) \leq \log\Delta_{\mathbf{K}} - (\gamma + \log 8\pi - \mathcal{I}(U))n_{\mathbf{K}} + \frac{8}{U}\big(e^{U/4} - e^{-U/4}\big)^2,$$

where $\mathcal{I}(U) := \mathrm{I}(F_U) = \frac{1}{U}\big(\frac{\pi^2}{2} + 4\,\mathrm{dilog}(e^{-U/2}) - \mathrm{dilog}(e^{-U})\big)$. Hence

$$\forall U \geq 3.545, \quad 2U\sum_\gamma f(U\gamma) \leq \log\Delta_{\mathbf{K}} - 2.6016\, n_{\mathbf{K}} + \frac{8}{U}e^{\frac{U}{2}}. \tag{4.5}$$

Setting $U = L$, this bound gives immediately a bound for $L^2\sum_\gamma f(L\gamma)$ of the right order $\frac{1}{2}L\log\Delta_{\mathbf{K}}$ for the part depending on the discriminant. Unfortunately, it also contains the term $4e^{\frac{L}{2}} = 4\sqrt{T}$ which makes the bound completely useless when inserted in (4.3). As a consequence we have to modify a bit this approach, and we use (4.5) with $U := L - 2\log L$. In fact, we notice that $\forall t \geq 0$, $f'(t) \leq 0.014$. This means that

$$\forall t \geq 0, \quad f(Lt) \leq f((L - 2\log L)t) + 0.028t\log L. \tag{4.6}$$

Below $\gimel := \big(0.014L^2\log L\big)^{-1/3}$ we bound $f(Lt)$ using (4.6) otherwise we use the trivial bound $f(Lt) \leq \frac{2}{(Lt)^2}$. We thus define

$$\forall t \geq 0, \quad g(t) := \begin{cases} 0.028tL^2\log L & \text{if } 0 \leq t \leq \gimel, \\ \frac{2}{t^2} & \text{if } \gimel < t. \end{cases}$$

Note that we have chosen $\gimel$ in such a way that g is continuous. With this definition of g, we thus have $L^2 f(Lt) \leq L^2 f((L - 2\log L)t)\chi_{[0,\gimel]}(t) + g(t)$.
Since we use (4.5) with $U = L - 2\log L$, we need that $L - 2\log L \geq 3.545$ so that we suppose $T \geq 2000$. This, in turn, means that $\gimel \leq 1$. In this way we get

$$L^2\sum_\gamma f(L\gamma) \leq L^2\sum_{|\gamma|\leq\gimel} f((L - 2\log L)\gamma) + \sum_\gamma g(\gamma). \tag{4.7}$$

By (4.5), for the first part we have

$$L^2\sum_{|\gamma|\leq\gimel} f((L - 2\log L)\gamma) \leq \frac{L^2}{2(L - 2\log L)}\Big(\log\Delta_{\mathbf{K}} - 2.6016n_{\mathbf{K}} + \frac{8\sqrt{T}}{L(L - 2\log L)}\Big). \tag{4.8}$$

Notice that in this way the term containing $\sqrt{T}$ is actually $O(\sqrt{T}/\log T)$ and does not interfere any more.
We now estimate the second part of (4.7). We have

$$\sum_\gamma g(\gamma) = \sum_{|\gamma|\leq\gimel} g(\gamma) + \sum_{\gimel<|\gamma|} g(\gamma) \leq g(\gimel)N_{\mathbf{K}}(\gimel) + \int_{\gimel^+}^{+\infty} g(t)\,\mathrm{d}N_{\mathbf{K}}(t) = -\int_{\gimel}^{+\infty} g'(t)N_{\mathbf{K}}(t)\,\mathrm{d}t.$$

Since g' is non-positive on $[\gimel,\infty)$ we can estimate $N_{\mathbf{K}}$ by $M_{\mathbf{K}}$, getting

$$\leq -\int_{\gimel}^{+\infty} g'(t)M_{\mathbf{K}}(t)\,\mathrm{d}t = -\int_{\gimel}^{1} g'(t)M_{\mathbf{K}}(t)\,\mathrm{d}t - \int_1^{+\infty} g'(t)M_{\mathbf{K}}(t)\,\mathrm{d}t.$$

The first integral can be estimated by noticing that S is increasing so that $M_{\mathbf{K}}(t) \leq \frac{t}{\beth} M_{\mathbf{K}}(\beth)$ in $[\beth, 1]$. The second integral can be computed. In this way we have

$$\begin{aligned}
\sum_\gamma g(\gamma) &\leq 2.55\Big(\log\Delta_{\mathbf{K}} - 2.45 n_{\mathbf{K}} + S\Big(\frac{3.03}{\beth}\Big)\Big)\int_\beth^1 \frac{\mathrm{d}t}{t^2} + g(1)M_{\mathbf{K}}(1) + \int_1^{+\infty} g(t)F'_{\mathbf{K}}(t)\,\mathrm{d}t \\
&= 2.55\Big(\log\Delta_{\mathbf{K}} - 2.45 n_{\mathbf{K}} + S\Big(\frac{3.03}{\beth}\Big)\Big)\Big(\frac{1}{\beth} - 1\Big) \\
&\quad + \frac{2}{\pi}(\log\Delta_{\mathbf{K}} - n_{\mathbf{K}}\log(2\pi e)) + 2\cdot 0.247\log\Delta_{\mathbf{K}} + 2(8.851 n_{\mathbf{K}} + 3.024) \\
&\quad + 2\int_1^{+\infty}\Big(\frac{1}{\pi}\Big(\log\Delta_{\mathbf{K}} + n_{\mathbf{K}}\log\big(\frac{t}{2\pi}\big)\Big) + \frac{0.247 n_{\mathbf{K}}}{t}\Big)\frac{\mathrm{d}t}{t^2} \\
&= 2.55\Big(\log\Delta_{\mathbf{K}} - 2.45 n_{\mathbf{K}} + S\Big(\frac{3.03}{\beth}\Big)\Big)\Big(\frac{1}{\beth} - 1\Big) \\
&\quad + \frac{4}{\pi}(\log\Delta_{\mathbf{K}} - n_{\mathbf{K}}\log 2\pi) + 0.247(2\log\Delta_{\mathbf{K}} + n_{\mathbf{K}}) + 2(8.851 n_{\mathbf{K}} + 3.024).
\end{aligned} \tag{4.9}$$

Inserting (4.8) and (4.9) in (4.7) we finally obtain

$$\begin{aligned}
L^2\sum_\gamma f(L\gamma) &\leq \frac{L^2}{2(L-2\log L)}\Big(\log\Delta_{\mathbf{K}} - 2.6016 n_{\mathbf{K}} + \frac{8\sqrt{T}}{L(L-2\log L)}\Big) \\
&\quad + 2.55\Big[\log\Delta_{\mathbf{K}} - 2.45 n_{\mathbf{K}} + S\Big(3.03\big(0.014L^2\log L\big)^{1/3}\Big)\Big]\Big[\big(0.014L^2\log L\big)^{1/3} - 1\Big] \\
&\quad + \frac{4}{\pi}(\log\Delta_{\mathbf{K}} - n_{\mathbf{K}}\log 2\pi) + 0.247(2\log\Delta_{\mathbf{K}} + n_{\mathbf{K}}) + 2(8.851 n_{\mathbf{K}} + 3.024) \\
&\leq \Big(\frac{L^2}{2(L-2\log L)} + 0.62\big(L^2\log L\big)^{1/3} - 0.78\Big)\log\Delta_{\mathbf{K}} + \frac{4L}{(L-2\log L)^2}\sqrt{T} \\
&\quad + \Big(-\frac{1.3008L^2}{L-2\log L} - 1.5057\big(L^2\log L\big)^{1/3} + 21.857\Big)n_{\mathbf{K}} \\
&\quad + 2.55\,S\Big(3.03\big(0.014L^2\log L\big)^{1/3}\Big)\Big(\big(0.014L^2\log L\big)^{1/3} - 1\Big) + 6.05.
\end{aligned}$$

We are assuming $T \geq 2000$, thus the coefficient of $n_{\mathbf{K}}$ is smaller than $3.17 - 1.3L$ and we get

$$\begin{aligned}
L^2\sum_\gamma f(L\gamma) &\leq \Big(\frac{L^2}{2(L-2\log L)} + 0.62\big(L^2\log L\big)^{\frac{1}{3}} - 0.78\Big)\log\Delta_{\mathbf{K}} + \frac{4L}{(L-2\log L)^2}\sqrt{T} \\
&\quad + 2.55\,S\Big(3.03\big(0.014L^2\log L\big)^{\frac{1}{3}}\Big)\Big(\big(0.014L^2\log L\big)^{1/3} - 1\Big) + (3.17 - 1.3L)n_{\mathbf{K}} + 6.05.
\end{aligned} \tag{4.10}$$

We can now deal with the first part of the proposition, that is

$$\limsup_{\Delta_{\mathbf{K}}\to\infty} \frac{T(\mathbf{K})}{\big(\log\Delta_{\mathbf{K}}\log\log\Delta_{\mathbf{K}}\big)^2} \leq \frac{1}{16}$$

inserting (4.10) in (4.3), with the bound (4.4). Obviously, T diverges as $\Delta_{\mathbf{K}}$ goes to infinity; in particular, the restriction $T \geq 2000$ does not matter. Moreover, we observe that the second term in the first line of (4.10) is $o(\log T)\log\Delta_{\mathbf{K}}$ and that the second line is $O(\sqrt{T}/\log T)$, hence the first claim is proved.

We now study the second part of the proposition, which means the claimed inequality

$$T(\mathbf{K}) \leq 3.9\big(\log\Delta_{\mathbf{K}}\log\log\Delta_{\mathbf{K}}\big)^2.$$

We note that, to have $T < 2000$ in (4.3) with (4.4) and (4.10), we need $\log\Delta_{\mathbf{K}} < 5.15+0.63n_{\mathbf{K}}$. According to Table 3 in [Odl76] (entry $b = 1.3$), this may happen only if $n_{\mathbf{K}} \le 5$ and also in this case, only when $\Delta_{\mathbf{K}} \le 607$ (resp. 1141, 2143, 4023) for fields of degree 2 (resp. 3, 4, 5).

For fields of degree $n_{\mathbf{K}} \ge 6$ and $\Delta_{\mathbf{K}} \ge 1.7{\cdot}10^5$, by elementary arguments one sees from (4.3), (4.4) and (4.10) that

$$T(\mathbf{K}) \le 3.6\big(\log\Delta_{\mathbf{K}}\log\log\Delta_{\mathbf{K}}\big)^2.$$

According to Table 3 in [Odl76], this covers in particular all fields with degree $n_{\mathbf{K}} \ge 7$.

For fields of degree $n_{\mathbf{K}} \le 5$, we see that

$$T(\mathbf{K}) \le 3.9\big(\log\Delta_{\mathbf{K}}\log\log\Delta_{\mathbf{K}}\big)^2 \tag{4.11}$$

as soon as $\Delta_{\mathbf{K}} \ge 3\cdot 10^6$ for quadratic fields, or $\Delta_{\mathbf{K}} \ge 10^6$ for $3 \le n_{\mathbf{K}} \le 5$. There remains a finite number of fields of degree $2 \le n_{\mathbf{K}} \le 6$. All those with $n_{\mathbf{K}} \ge 3$ appear in "megrez" number field table [megrez08] and for all fields, including the quadratic ones, we use the algorithm indicated in [BDyDF08] as implemented in [PARI15]. For $\mathbf{K} \in \big\{\mathbf{Q}[\sqrt{-3}], \mathbf{Q}[\sqrt{-1}]\big\}$ we find $T(\mathbf{K}) = 5$ and for $\mathbf{K} = \mathbf{Q}[\sqrt{5}]$, $T(\mathbf{K}) = 7$. All other fields satisfy (4.11). □

5. Multi-step

5.1. **Bounds for two and three-steps.** The original choice $F = \Phi * \Phi$ with Φ the characteristic function of $[-L/2, L/2]$ is of the type considered in (3.1) with $\Phi^+(x) = \chi_{[0,L]}(x)$, i.e. a function assuming only one value in $[0, L]$. We call this choice the *one-step* case. The following corollary shows that the performance of the algorithm significantly improves already when $\Phi^+(x)$ is allowed to assume two or three values in the $[0, L]$ interval, as long as it is zero when x is close to $L/2$ (so that $\Phi(x) = 0$ if x is close to 0). In particular, the extra factor $\log\log\Delta_{\mathbf{K}}$ disappears. We call these choices *two-* and *three-steps.*

Corollary 5.1. *Let* $\Phi^+(x) = b\chi_{[L-2a,L]}(x) + \chi_{[L-a,L]}(x)$ *and define* F *as in* (3.1). *Denote respectively* $T_{c2}(\mathbf{K})$ *('two steps') and* $T_{c3}(\mathbf{K})$ *('three steps') the lowest* T *such that* (2.1) *is satisfied by such an* F*, with respectively* $b = 0$ *and* $b \ne 0$*. We have*

$$\sqrt{T_{c2}(\mathbf{K})} \le \max\big(2.456\log\Delta_{\mathbf{K}} - 5.623n_{\mathbf{K}} + 14, \sqrt{13}\big),$$

where the result is obtained with $a = 2.5$*, and*

$$\sqrt{T_{c3}(\mathbf{K})} \le \max\big(2.193\log\Delta_{\mathbf{K}} - 6.19n_{\mathbf{K}} + 16, \sqrt{32}\big),$$

which is obtained with $a = 1.722$ *and* $b = \frac{e^{-a/2}}{1-e^{-a/2}}$*.*

Proof. The claim is proved directly applying Proposition 3.1 with $\Phi^+(x) = b\chi_{[L-2a,L]}(x) + \chi_{[L-a,L]}(x)$. Conditions $\sqrt{T_{c2}} \ge \sqrt{13}$ and $\sqrt{T_{c3}} \ge \sqrt{32}$ come from the need to ensure that the support of Φ^+ be in $[0, L]$, so that we need to assume $L > a$ for the two-steps and $L > 2a$ for the three-steps, respectively. □

5.2. **The algorithm.** Our aim is to find a good T for the number field $\mathbf{K}$ as fast as possible exploiting the bilinearity of the convolution product. We introduce some definitions to make the discussion easier.

Definition 5.2. *Let $\mathcal{S}$ be the real vector space of even and compactly supported step functions and, for $T > 1$, let $\mathcal{S}(T)$ be the subspace of $\mathcal{S}$ of functions supported in $[-L/2, L/2]$, with $L = \log T$.*

Definition 5.3. *For any integer $N \geq 1$ and positive real δ we define the subspace $\mathcal{S}_d(N,\delta)$ of $\mathcal{S}(e^{2N\delta})$ made of functions which are constant for all $k \in \mathbf{N}$ on $[k\delta, (k+1)\delta)$.*

The elements of $\mathcal{S}_d(N,\delta)$ are thus step functions with fixed step width δ. If $N \geq 1$, $\delta > 0$ and $T = e^{2N\delta}$ we have

$$\mathcal{S}_d(N,\delta) \subset \mathcal{S}(T) \subset \mathcal{S}, \tag{5.1a}$$

$$\forall \Phi \in \mathcal{S}(T), \qquad \Phi * \Phi \in \mathcal{W}(T), \tag{5.1b}$$

$$\mathcal{S}_d(N,\delta) \subset \mathcal{S}_d(N+1,\delta), \tag{5.1c}$$

$$\forall k \geq 1, \quad \mathcal{S}_d\Big(kN, \frac{\delta}{k}\Big) \subseteq \mathcal{S}_d(N,\delta). \tag{5.1d}$$

If, for some $T > 1$, $\Phi \in \mathcal{S}(T)$ and $F = \Phi * \Phi$ satisfies (2.1) then, according to Theorem 2.4, $T_{\mathcal{C}}(\mathbf{K}) < T$. This leads us to define the linear form $\ell_{\mathbf{K}}$ on $\bigcup_{T>1} \mathcal{W}(T)$ by

$$\ell_{\mathbf{K}}(F) = -2\sum_{\mathfrak{p}} \log \mathrm{N}\mathfrak{p} \sum_{m=1}^{+\infty} \frac{F(m \log \mathrm{N}\mathfrak{p})}{\mathrm{N}\mathfrak{p}^{m/2}} + F(0)(\log \Delta_{\mathbf{K}} - (\gamma + \log 8\pi) n_{\mathbf{K}}) + \mathrm{I}(F) n_{\mathbf{K}} - \mathrm{J}(F) r_1$$

and the quadratic form $q_{\mathbf{K}}$ on $\mathcal{S}$ by $q_{\mathbf{K}}(\Phi) = \ell_{\mathbf{K}}(\Phi * \Phi)$. From Theorem 2.4 we deduce the following consequence.

Corollary 5.4. *Let $\mathbf{K}$ be a number field satisfying* GRH *and $T > 1$. If the restriction of $q_{\mathbf{K}}$ to $\mathcal{S}(T)$ has a negative eigenvalue then $T_{\mathcal{C}}(\mathbf{K}) < T$.*

Definition 5.5. *A* bound *for $\mathbf{K}$ is an $L = \log T$ with T as in Theorem 2.4.*

Note that $q_{\mathbf{K}}$ is a continuous function as a function from $(\mathcal{S}(T), \|.\|_1)$ to $\mathbf{R}$. Therefore if L is a bound for $\mathbf{K}$ then there exists an $L' < L$ such that L' is a bound for $\mathbf{K}$. Note also that, in terms of T, only the norms of prime ideals are relevant, which means that we do not need the smallest possible T to get the best result.

Remark 5.6. If $T > 1$ and $\Phi \in \mathcal{S}(T)$, then for any $\varepsilon > 0$ there exists $N \geq 1$, $\delta > 0$ and $\Phi_\delta \in \mathcal{S}_d(N,\delta)$ such that $\|\Phi * \Phi - \Phi_\delta * \Phi_\delta\|_\infty \leq \varepsilon$ and $e^{2N\delta} \leq T$. Hence we do not loose anything in terms of bounds for $\mathbf{K}$ if we consider only the subspaces of the form $\mathcal{S}_d(N,\delta)$.

As we will see later, we can compute $q_{\mathbf{K}}(\Phi)$ for a generic $\Phi \in \mathcal{S}(T)$ combining its values for $\Phi = \chi_{[-L/2,L/2]}$ at different L's. Thus, let $\texttt{GRHcheck}(\mathbf{K}, L)$ be the function that returns the right hand side of (4.1) minus its left hand side (without the approximations for $\mathrm{I}(F_L)$ and $\mathrm{J}(F_L)$), and $\texttt{BDyDF}(\mathbf{K})$ be the function which implements the algorithm of [BDyDF08, Section 3]. The computation of $\texttt{BDyDF}(\mathbf{K})$ is very fast because the only arithmetic information we need on $\mathbf{K} \simeq \mathbf{Q}[x]/(P)$ is the splitting information for primes $p < T$ and is determined easily for nearly all p. Indeed if p does not divide the index of $\mathbf{Z}[x]/(P)$ in $\mathcal{O}_{\mathbf{K}}$, then the splitting of p in $\mathbf{K}$ is determined by the factorization of $P \mod p$. We can also store such splitting information for all p that we consider and do not recompute it each time we test whether a given L is a bound for $\mathbf{K}$.

We denote $q_{\mathbf{K},N,\delta}$ the restriction of $q_{\mathbf{K}}$ to $\mathcal{S}_d(N,\delta)$. According to Corollary 5.4, if $q_{\mathbf{K},N,\delta}$ has a negative eigenvalue then $2N\delta$ is a bound for $\mathbf{K}$. This justifies the following definition.

Definition 5.7. *The pair (N,δ) is $\mathbf{K}$-good when $q_{\mathbf{K},N,\delta}$ has a negative eigenvalue.*

We can reinterpret Functions `GRHcheck` and `BDyDF` saying that if `GRHcheck`$(\mathbf{K},2\delta)$ is negative then $(1,\delta)$ is $\mathbf{K}$-good and that $\left(1,\frac{1}{2}\log \texttt{BDyDF}(\mathbf{K})\right)$ is $\mathbf{K}$-good.

The fundamental step for our algorithm is the following: given $\delta > 0$ we look for the smallest N such that (N,δ) is $\mathbf{K}$-good. Looking for such an N can be done fairly easily with this setup. For any $i \geq 1$, let Φ_i be the characteristic function of $(-i\delta, i\delta)$. Then $(\Phi_i)_{1\leq i\leq N}$ is a basis of $\mathcal{S}_d(N,\delta)$. We have $\Phi_i * \Phi_i = F_{2i\delta} = (2i\delta - |x|)\chi_{[-2i\delta,2i\delta]}(x)$. We observe that

$$\Phi_i * \Phi_j = F_{(i+j)\delta} - F_{|i-j|\delta} \ .$$

This means that the matrix A_N of $q_{\mathbf{K},N,\delta}$ can be computed by computing only the values of $\ell_{\mathbf{K}}(F_{i\delta})$ for $1 \leq i \leq 2N$ and subtracting those values.

We then stop when the determinant of A_N is negative or when $2N\delta \geq$ `BDyDF`$(\mathbf{K})$. This does not guarantee that we stop as soon as there is a negative eigenvalue. Indeed, consider the following sequence of signatures:

$$(0,p,0) \to (1,p,0) \to (1,p,1) \to (0,p+1,2) \to \cdots$$

here a signature is (z,p,m) where z is the dimension of the kernel and p (resp. m) the dimension of a maximal subspace where $q_{\mathbf{K}}$ is positive (resp. negative) definite. We should have stopped when the signature was $(1,p,1)$ however the determinant was zero there. Our algorithm will stop as soon as there is an odd number of negative eigenvalues (and no zero) or we go above `BDyDF`$(\mathbf{K})$. Such unfavorable sequence of signatures is however very unlikely and does not happen in practice.

The corresponding algorithm is presented in Function `NDelta`. We have added a limit $N_{\max}$ for N which is not needed right now but will be used later. Note that (Φ_i) is a basis adapted to the inclusion (5.1c) so that we only need to compute the edges of the matrix A_N at each step. The test $\det A < 0$ in line 13 can be efficiently implemented using Cholesky LDL^* decomposition because it is incremental; moreover, if the last coefficient of D is negative, the last line of L^{-1} is a vector v such that $vA^tv < 0$ so that we can check the result.

One way to use this function is to compute $T =$ `BDyDF`$(\mathbf{K})$ and for some $N_{\max} \geq 2$, let $\delta = \frac{L}{2N_{\max}}$ and $N =$ `NDelta`$(\mathbf{K},\delta,N_{\max})$. Using the inclusion (5.1d), we see that (N,δ) is $\mathbf{K}$-good and that $N \leq N_{\max}$, so that we have improved the bound.

5.3. **Adaptive steps.** Unfortunately Function `NDelta` is not very efficient mostly for two reasons. To explain them and to improve the function we introduce some extra notations.

For any $\delta > 0$, let N_δ be the minimal N such that (N,δ) is $\mathbf{K}$-good. Observe that Function `NDelta` computes N_δ, as long as $N_\delta \leq N_{\max}$ and no zero eigenvalue prevents success. Obviously, using (5.1c), we see that for any $N \geq N_\delta$, (N,δ) is $\mathbf{K}$-good. We have observed numerically that the sequence $N\delta_N$ is roughly decreasing, i.e. for most values of N we have $N\delta_N \geq (N+1)\delta_{N+1}$.

For any $N \geq 1$, let δ_N be the infimum of the δ's such that (N,δ) is $\mathbf{K}$-good. It is not necessarily true that if $\delta \geq \delta_N$ then (N,δ) is $\mathbf{K}$-good, however we have never found a counterexample. The function $\delta \mapsto \delta N_\delta$ is piecewise linear with discontinuities at points where

N_δ changes; the function is increasing in the linear pieces and decreasing at the discontinuities. This means that if we take $0 < \delta_2 < \delta_1$ but we have $N_{\delta_2} > N_{\delta_1}$ then we may have $\delta_2 N_{\delta_2} > \delta_1 N_{\delta_1}$ so the bound we get for δ_2 is not necessarily as good as the one for δ_1.
The resolution of Function `NDelta` is not very good: going from $N-1$ to N the bound for the norm of the prime ideals is multiplied by $e^{2\delta}$. This is the first reason reducing the efficiency of the function. The second one is that if $N_{\max}$ is above 20 or so, the number $\delta = \frac{\log \texttt{BDyDF}(\mathbf{K})}{2N_{\max}}$ has no specific reason to be near δ_{N_δ}; as discussed above, this means that we can get a better bound for $\mathbf{K}$ by choosing δ to be just above either δ_{N_δ} or δ_{1+N_δ}. Both reasons derive from the same facts and give a bound for $\mathbf{K}$ that can be overestimated by at most 2δ for the considered $N = \texttt{NDelta}(\mathbf{K}, \delta, N_{\max})$.

To improve the result, we can use once again inclusion (5.1d) and determine a good approximation of δ_N for $N = 2^n$. We determine first by dichotomy a δ_0 such that (N_0, δ_0) is $\mathbf{K}$-good for some $N_0 \geq 1$. For any $k \geq 0$, we take $N_{k+1} = 2N_k$ and determine by dichotomy a δ_{k+1} such that (N_{k+1}, δ_{k+1}) is $\mathbf{K}$-good; we already know that $\frac{\delta_k}{2}$ is an upper bound for δ_{k+1} and we can either use 0 as a lower bound or try to find a lower bound not too far from the upper bound because the upper bound is probably not too bad. The algorithm is described in Function `Bound`. It uses a subfunction $\texttt{OptimalT}(\mathbf{K}, N, T_\ell, T_h)$ which returns the smallest integer $T \in [T_\ell, T_h]$ such that $\texttt{NDelta}(\mathbf{K}, L/(2N), N) > 0$. The algorithm does not return a bound below those proved in Theorem 3.6 and Corollary 3.8.

5.4. **Further refinements.** To improve the speed of the algorithm, we decided to make the dichotomy in $\texttt{OptimalT}(\mathbf{K}, N, T_\ell, T_h)$, not on all value of T but only on the norms of the prime ideals in $[T_\ell, T_h]$.

To reduce the time used to compute the determinants, we tried to use steps of width 4δ in $[-L/2, L/2]$ and of width 2δ in the rest of $[-3L/4, 3L/4]$, to halve the dimension of $\mathcal{S}_d(N, \delta)$. It worked in the sense that we found substantially the same T faster. However we decided that the total time of the algorithm is not high enough to justify the increase in code complexity.

Input: a number field $\mathbf{K}$
Input: a positive real δ
Input: a positive integer $N_{\max}$
Output: an $N \leqslant N_{\max}$ such that (N, δ) is $\mathbf{K}$-good or 0

$tab \leftarrow (2N_{\max}+1)$-dimensional array;
$tab[0] \leftarrow 0$;
$A \leftarrow N_{\max} \times N_{\max}$ identity matrix;
$N \leftarrow 0$;
while $N < N_{\max}$ **do**
  $N \leftarrow N + 1$;
  $tab[2N-1] \leftarrow (2N-1)\texttt{GRHcheck}(\mathbf{K}, (2N-1)\delta)$;
  $tab[2N] \leftarrow 2N\texttt{GRHcheck}(\mathbf{K}, 2N\delta)$;
  for $i \leftarrow 1$ **to** N **do**
    $A[N, i] \leftarrow tab[N+i] - tab[N-i]$;
    $A[i, N] \leftarrow A[N, i]$;
  end
  if $\det A < 0$ **then**
    return N;
  end
end
return 0;

Function NDelta($\mathbf{K}$,δ,$N_{\max}$)

Input: a number field $\mathbf{K}$
Output: a bound for the norm of a system of generators of $\mathcal{C}\ell_{\mathbf{K}}$

if $\log \Delta_{\mathbf{K}} < n_{\mathbf{K}} 2^{n_{\mathbf{K}}}$ **then**
  $T_0 \leftarrow 4\Big(\log \Delta_{\mathbf{K}} + \log\log \Delta_{\mathbf{K}} - (\gamma + \log 2\pi) n_{\mathbf{K}} + 1 + (n_{\mathbf{K}} + 1)\frac{\log(7\log \Delta_{\mathbf{K}})}{\log \Delta_{\mathbf{K}}}\Big)^2$;
else
  $T_0 \leftarrow 4(\log \Delta_{\mathbf{K}} + \log\log \Delta_{\mathbf{K}} - (\gamma + \log 2\pi) n_{\mathbf{K}} + 1)^2$;
end
$T_0 \leftarrow \min\left(T_0, 4.01 \log^2 \Delta_{\mathbf{K}}\right)$;
$N \leftarrow 8$; $\delta \leftarrow 0.0625$;
while $\texttt{NDelta}(\mathbf{K}, \delta, N) = 0$ **do**
  $\delta \leftarrow \delta + 0.0625$;
end
$T_h \leftarrow \texttt{OptimalT}(\mathbf{K}, N, e^{2N(\delta - 0.0625)}, e^{2N\delta})$;
$T \leftarrow T_h + 1$;
while $T_h < T \,||\, T > T_0$ **do**
  $T \leftarrow T_h$; $N \leftarrow 2N$;
  $T_h \leftarrow \texttt{OptimalT}(\mathbf{K}, N, 1, T_h)$;
end
return T;

Function Bound($\mathbf{K}$)

5.5. **Theoretical performance.** We denote $T_1(\mathbf{K})$ the result of Function `Bound`. The algorithm reaches bounds of the same quality as those of $T_{\mathbf{e}}(\mathbf{K})$.

Corollary 5.8. *Assume* GRH. *Then Function* `Bound` *terminates. Moreover we have*

$$\begin{aligned}
\sqrt{T_1(\mathbf{K})} &\leq 2\log\Delta_{\mathbf{K}} + 2\log\log\Delta_{\mathbf{K}} + 2 - 2(\gamma+\log 2\pi)n_{\mathbf{K}} + 2(n_{\mathbf{K}}+1)\frac{\log(7\log\Delta_{\mathbf{K}})}{\log\Delta_{\mathbf{K}}},\\
\sqrt{T_1(\mathbf{K})} &\leq 2\log\Delta_{\mathbf{K}} + 2\log\log\Delta_{\mathbf{K}} + 2 - 2(\gamma+\log 2\pi)n_{\mathbf{K}} \qquad \textit{if } \log\Delta_{\mathbf{K}} \geq n_{\mathbf{K}}2^{n_{\mathbf{K}}},\\
T_1(\mathbf{K}) &\leq 4\big(1+\big(2\pi e^{\gamma}\big)^{-n_{\mathbf{K}}}\big)^2\log^2\Delta_{\mathbf{K}},\\
T_1(\mathbf{K}) &\leq 4\log^2\Delta_{\mathbf{K}} \qquad \textit{if } \log\Delta_{\mathbf{K}} \leq \frac{1}{e}\big(2\pi e^{\gamma}\big)^{n_{\mathbf{K}}},\\
T_1(\mathbf{K}) &\leq 4.01\log^2\Delta_{\mathbf{K}}.
\end{aligned}$$

Proof. Consider one of the bounds of Theorem 3.6 or of Corollary 3.8. It is associated to a certain $F_{\mathbf{e}} = \Phi_{\mathbf{e}} * \Phi_{\mathbf{e}}$ with a certain $a = \log T_0$ and T given by the bound. As in Remark 5.6, For any $\varepsilon > 0$, there exists a step function Φ with support in $[-L/2, L/2]$ and 2^N steps, for N large enough, such that $\|F - \Phi * \Phi\|_\infty < \varepsilon$. Since the inequality in (2.1) is strict, we can take ε small enough so that $\Phi * \Phi$ satisfies (2.1). Hence, for N large enough, the algorithm will find a negative eigenvalue in $\mathcal{S}_d(2^N, 2^{-N-1}L)$ and hence it will terminate. The bound T that it gives obviously satisfies the first two and last inequality of the statement of the corollary. Since the intermediate inequalities are consequences of the first two, T will also satisfy the intermediates inequalities. □

5.6. **Effective performance.**

5.6.1. *Various fields.* We tested the algorithm on several fields. Let first $\mathbf{K} = \mathbf{Q}[x]/(P)$ where

$$P = x^3 + 559752270111028720x + 55137512477462689.$$

The polynomial P has been chosen so that for all primes $2 \leq p \leq 53$ there are two prime ideals of norms p and p^2. This ensures that there are lots of small norms of prime ideals. We have $T(\mathbf{K}) = 19162$. There are 2148 non-zero prime ideals with norms up to $T(\mathbf{K})$. We found that $T_1(\mathbf{K}) = 11071$ and that there are 1343 non-zero prime ideals of norms up to $T_1(\mathbf{K})$.

We tested also the algorithm on the set of 4686 fields of degree 2 to 27 and small discriminant coming from a benchmark of [PARI15]. The mean value of $T_1(\mathbf{K})/T(\mathbf{K})$ for those fields is lower than 1/2.

We have tested the cyclotomic fields $\mathbf{K} = \mathbf{Q}[\zeta_n]$ for $n \leq 250$. For them we have found that the quotient $T_1(\mathbf{K})/T(\mathbf{K})$ becomes smaller and smaller as the degree increases, reaching the value 1/2 for the higher order cyclotomic fields. However, we have observed that the fraction is generally higher than what we get for the generic fields with comparable degree and discriminant. Certainly cyclotomic fields are not typical fields: for instance for them the class number grows more than exponentially as a function of the order [Wash97, Th. 4.20]. The weight function that we observe in the tests for generic fields contains several parameters and therefore could have generic profiles but actually always shows two bumps, one centered at the origin and one near the end of the support. On the contrary, the weight producing $T(\mathbf{K})$ has a unique bump in the origin, by design. The fact that the original algorithm already produces a good bound for cyclotomic of small order in some sense means that the second bump is not necessary, and this is probably due to the existence of a lots of ideals of small norms. However, we admit we do not have any convincing explanation of this phenomenon.

5.6.2. *Pure fields, small discriminants.* We computed $T(\mathbf{K})$ and $T_1(\mathbf{K})$ for fields of the form $\mathbf{Q}[x]/(P)$ with $P = x^n \pm p$ and p is the first prime after 2^a for a certain family of integers n and a such that $\log \Delta_{\mathbf{K}} \leq 250$. We limited the discriminant because, while at the time of writing the record for which the Buchmann algorithm has been successfully completed has $\log \Delta_{\mathbf{K}} \geq 646$, this has been done for only very few fields with $\log \Delta_{\mathbf{K}} \geq 100 \log 10 \simeq 230$. We computed the family of $\frac{T_1(\mathbf{K})}{T(\mathbf{K})}$ for each fixed degree. The results are presented in Figure 1. We can see that in the right-half of the graph, the fields adopt the asymptotical behavior where $\frac{T_1(\mathbf{K})}{T(\mathbf{K})} \asymp (\log\log \Delta_{\mathbf{K}})^{-2}$.

Let $t(\mathbf{K})$ denote the time needed to compute $T(\mathbf{K})$ and $t_1(\mathbf{K})$ the additional time needed to compute $T_1(\mathbf{K})$. In Figure 2, we have drawn the points $\frac{t_1(\mathbf{K})}{t(\mathbf{K})}$ for the four families of fields we have tested. We have removed three points with $\frac{t_1(\mathbf{K})}{t(\mathbf{K})} \geq 25$ (in details: 30.17, 30.31 and 47.19) whose $\log \Delta_{\mathbf{K}}$ is respectively 160.81, 162.20 and 167.74. In spite of the relatively large value for the quotient, the value of $t_1(\mathbf{K})$ in all cases has been lower than 2s (and actually larger than 0.35s in only 22 out of the 8308 fields, including the three fields for which $t_1(\mathbf{K})/t(\mathbf{K}) \geq 25$, all having $\log \Delta_{\mathbf{K}} \geq 153$). This shows that the time needed to compute $T_1(\mathbf{K})$ is already limited with respect to the time needed for the full Buchmann algorithm.

5.6.3. *Pure fields.* We once again computed $T(\mathbf{K})$ and $T_1(\mathbf{K})$ for fields of the form $\mathbf{Q}[x]/(P)$ with $P = x^n \pm p$ and p is the first prime after 10^a for a certain family of integers n and a. The graph of $\frac{T_1(\mathbf{K})}{T(\mathbf{K})}$ looks like a continuation of the right-half of Figure 1 so that we do not draw it once again. The graph of $\frac{T_1(\mathbf{K})}{T(\mathbf{K})}(\log\log \Delta_{\mathbf{K}})^2$ is much more regular and looks to have a non-zero limit, see Figure 3 below. We plotted the graph of $\frac{T_1(\mathbf{K})}{\log^2 \Delta_{\mathbf{K}}}$ for the same fields in Figure 4 as well. We computed the mean of $\frac{T_1(\mathbf{K})}{T(\mathbf{K})}(\log\log \Delta_{\mathbf{K}})^2$ and the maximum of $\frac{T_1(\mathbf{K})}{\log^2 \Delta_{\mathbf{K}}}$ for each fixed degree. The results are summarized below:

P	$a \leq$	$\log \Delta_{\mathbf{K}} \leq$	mean of $\frac{T_1(\mathbf{K})}{T(\mathbf{K})}(\log\log \Delta_{\mathbf{K}})^2$	$1 - \max\left(\frac{T_1(\mathbf{K})}{\log^2 \Delta_{\mathbf{K}}}\right) \geq$
$x^2 - p$	3999	9212	13.19	$2 \cdot 10^{-5}$
$x^6 + p$	1199	13818	13.38	$9 \cdot 10^{-6}$
$x^{21} - p$	328	15169	13.68	$4 \cdot 10^{-5}$

The small discriminants are (obviously) much less sensitive to the new algorithm. We reduced the range for each series to have $\log \Delta_{\mathbf{K}} \leq 500$. The results are as follows:

P	$a \leq$	mean of $\frac{T_1(\mathbf{K})}{T(\mathbf{K})}(\log\log \Delta_{\mathbf{K}})^2$	$1 - \max\left(\frac{T_1(\mathbf{K})}{\log^2 \Delta_{\mathbf{K}}}\right) \geq$
$x^2 - p$	218	12.35	0.018
$x^6 + p$	43	13.66	0.073
$x^{21} - p$	10	17.19	0.279

5.6.4. *Biquadratic fields.* We repeated the computations above also for biquadratic fields $\mathbf{Q}[\sqrt{p_1}, \sqrt{p_2}]$ where each p_i is the first prime after 2^{a_i} (respectively 10^{a_i}) for certain families of integers a_i and included them in Figures 1–4.

In the case where p_i is the first prime after 10^{a_i}, we found that the mean of $\frac{T_1(\mathbf{K})}{T(\mathbf{K})}(\log\log \Delta_{\mathbf{K}})^2$ is 13.63 for the 7119 fields computed, and 13.88 if we restrict the family to the 1537 ones with $\log \Delta_{\mathbf{K}} \leq 500$, while the maximum of $\frac{T_1(\mathbf{K})}{\log^2 \Delta_{\mathbf{K}}}$ is lower than 1.0038 for all fields and 0.957 for the fields with $\log \Delta_{\mathbf{K}} \leq 500$.

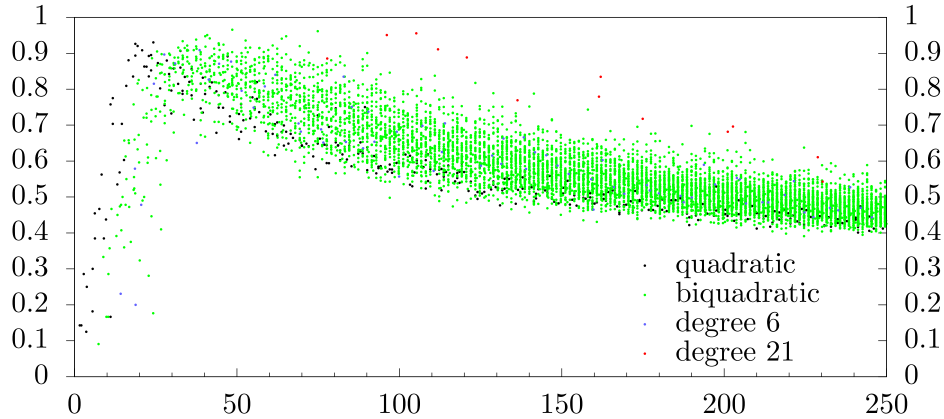

FIGURE 1: $\frac{T_1(\mathbf{K})}{T(\mathbf{K})}$ for some fields of small discriminant; in abscissa $\log \Delta_{\mathbf{K}}$.

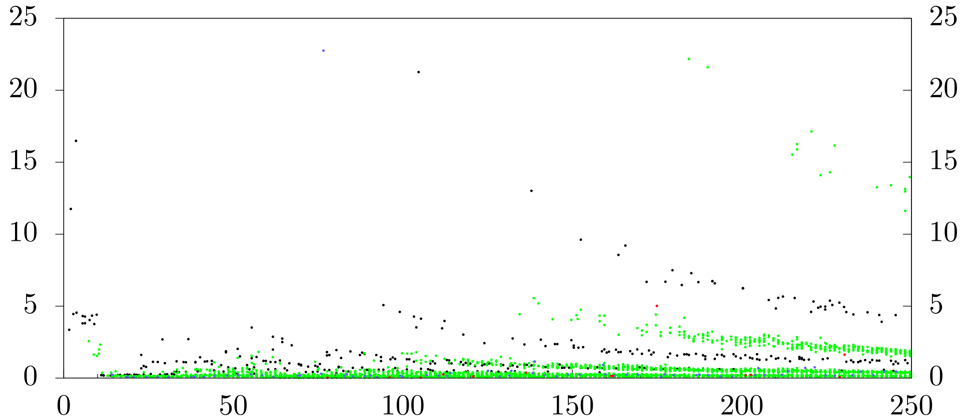

FIGURE 2: $\frac{t_1(\mathbf{K})}{t(\mathbf{K})}$ for some fields of small discriminant; in abscissa $\log \Delta_{\mathbf{K}}$.

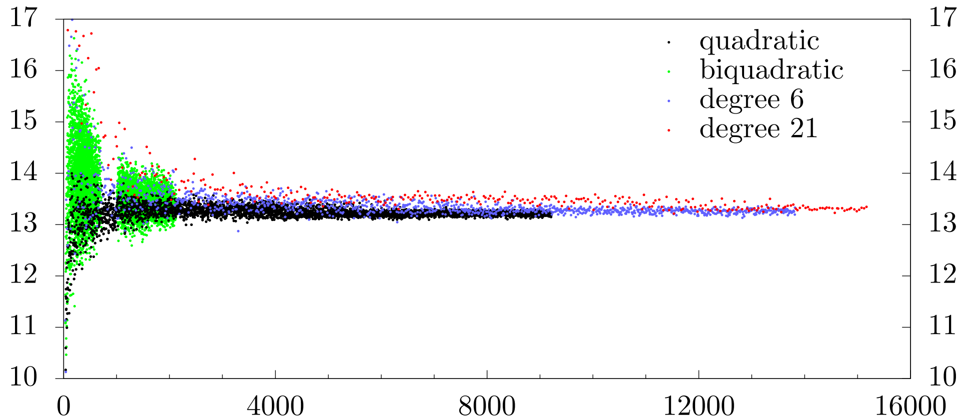

FIGURE 3: $\frac{T_1(\mathbf{K})}{T(\mathbf{K})}(\log\log \Delta_{\mathbf{K}})^2$ for some fields; in abscissa $\log \Delta_{\mathbf{K}}$.

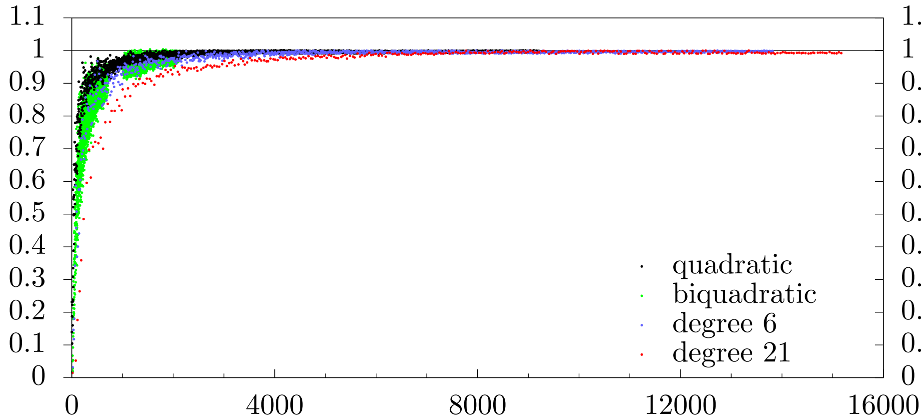

FIGURE 4: $\frac{T_1(\mathbf{K})}{\log^2 \Delta_{\mathbf{K}}}$ for some fields; in abscissa $\log \Delta_{\mathbf{K}}$.

5.7. **A simplified algorithm.** Since the expression of $F_{\mathbf{e}}$ as given in Remark 3.3 is simpler for $a = L/2$, we can implement a variant of the algorithm of Belabas, Diaz y Diaz and Friedman with that weight. We have for $x \geq 0$,

$$F_{\mathbf{e}}(x) = \Big(\big(x - 2 + 2e^{-x}\sqrt{T}\big)\chi_{[0,L/2)}(x) + (L-x)\chi_{[L/2,L]}(x)\Big)e^{x/2}\sqrt{T}$$

so that (2.1) becomes

$$\sum_{\mathrm{N}\mathfrak{p}^m < \sqrt{T}} \Big(m \log \mathrm{N}\mathfrak{p} - 2 + 2\frac{\sqrt{T}}{\mathrm{N}\mathfrak{p}^m}\Big) \log \mathrm{N}\mathfrak{p} + \sum_{\sqrt{T} \leq \mathrm{N}\mathfrak{p}^m < T} (L - m \log \mathrm{N}\mathfrak{p}) \log \mathrm{N}\mathfrak{p}$$
$$> \big(\sqrt{T} - 1\big)(\log \Delta_{\mathbf{K}} - (\gamma + \log 8\pi) n_{\mathbf{K}}) + \frac{\mathrm{I}(F_{\mathbf{e}})}{2\sqrt{T}} n_{\mathbf{K}} - \frac{\mathrm{J}(F_{\mathbf{e}})}{2\sqrt{T}} r_1$$

where

$$\frac{\mathrm{I}(F_{\mathbf{e}})}{2\sqrt{T}} = \big(\sqrt{T} - 1\big) \log\Big(\frac{4}{1 - T^{-1/2}}\Big) - \frac{L^2}{8} + \frac{L}{2} - \frac{\pi^2}{12} - \operatorname{dilog}(-T^{-1/2})$$
$$\frac{\mathrm{J}(F_{\mathbf{e}})}{2\sqrt{T}} = \big(\sqrt{T} + 1\big) \log\Big(\frac{2}{1 + T^{-1/2}}\Big) + \frac{L^2}{8} - \frac{L}{2} - \frac{\pi^2}{24} - \operatorname{dilog}(-T^{-1/2}) + \frac{1}{2} \operatorname{dilog}(-T^{-1}).$$

Applying Theorem 3.5 with $T_0 = \sqrt{T}$, we can see that the result $T_2(\mathbf{K})$ of this algorithm satisfies

$$\sqrt{T_2(\mathbf{K})} \leq 2 \log \Delta_{\mathbf{K}} + 2 \log\log \Delta_{\mathbf{K}} - (\gamma + \log 2\pi) n_{\mathbf{K}} + 1 + 2 \log 2 + c n_{\mathbf{K}} \frac{\log\log \Delta_{\mathbf{K}}}{\log \Delta_{\mathbf{K}}}$$

for some absolute constant c. This means that the asymptotical expansion is nearly optimal: the first term that changes with respect to Theorem 3.6 is the constant term which increases from 2 to $1 + 2\log 2 \simeq 2.38$. For small discriminants this algorithm is sometimes worse than `BDyDF` and always significantly worse than `Bound`, given in Subsections 5.2–5.4. For larger discriminants, it gives only sightly bigger results than `Bound` but is always faster. Numerically, in our experiments with the above mentioned fields, $T_2(\mathbf{K}) \geq T(\mathbf{K})$ if $\log \Delta_{\mathbf{K}} \leq 48$ (resp. 142, 83, 162) for $n_{\mathbf{K}} = 2$ (resp. 4, 6, 21) for a total of 401 fields out the 12648 tested.

6. A final comment

We have proved that

$$(1+o(1))\frac{\log\Delta_{\mathbf{K}}}{n_{\mathbf{K}}}\leq\sqrt{T_{\mathbf{e}}(\mathbf{K})}\leq(2+o(1))\log\Delta_{\mathbf{K}}.$$

We have three reasons to believe that the "true" behavior is

$$\sqrt{T_{\mathbf{e}}(\mathbf{K})}\sim\log\Delta_{\mathbf{K}}$$

as $\Delta_{\mathbf{K}}\to\infty$, for fixed $n_{\mathbf{K}}$.

The first one is computational. We have observed that $\sqrt{T_1(\mathbf{K})}/\log\Delta_{\mathbf{K}}$ seems to tend to 1 (from below, see Figure 4) for several series of pure fields and one series of biquadratic fields. We also tested some restricted cases with $\Phi^+_{\mathbf{e}}$ and $a=L$ and the Φ^+ of the form indicated in Corollary 5.1 with $a=\log 4$ and $b=0$: in all cases we observed the same phenomenon, which is that the experimental result seems to be half the one we can prove.

The second one is related to the upper bound. The function $\widehat{F}(t)=4(\mathrm{Re}[e^{-iLt/2}\widehat{\Phi^+}(t)])^2$ in (3.3) is estimated with $\widehat{F}(t)=4|\widehat{\Phi^+}(t)|^2$ in (3.4). This step removes the quick oscillations of $e^{-iLt/2}$ and allows the conclusion of the argument, but it overestimates the contribution of this object, which would be of this size only in the case where the γ's were placed very close to the maxima of $\cos^2(Lt)$ and which would be smaller by a factor $1/2$, in mean, for uniformly spaced zeros. Unfortunately, the actual information we have for the vertical distribution of zeros is not strong enough to distinguish between these two behaviors.

The third one is related to the lower bound in Proposition 3.9. For its computation we have considered each prime integer as totally split. This allows an explicit bound, but it should be corrected by a factor $1/n_{\mathbf{K}}$, because this is the density of the totally split primes, and the contribution of the other primes should be negligible.

All three reasons indicate that $1\cdot\log\Delta_{\mathbf{K}}$ should be the correct asymptotic. We are unable to prove it, though.

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