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Stickelberger Series and Iwasawa Main Conjecture for Function Fields

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Introduction

The principal objects of study of Number Theory in positive characteristic are the global function fields, which are the natural analogues of the number fields. It is possible to associate to every function field some invariants like, for example, the genus or the class number, that play an important role in the study of the arithmetic properties. Even if these objects have an algebraic nature many analytic tools have been introduced and studied in order to have information on such invariants. For example we can mention the Weil Zeta function, the Artin L-functions and the Goss Zeta function which all have an analytic nature (both in the classical and in the $\mathfrak{p}$-adic setting). The goal of this thesis is to investigate some links between these families of objects and provide theorems that build a bridge between the analytic side and the algebraic side of the theory. In all these theorems an important role will be played by the *Stickelberger Series* $\Theta_S(X)$ which is an algebraic object, but it can be used to generate all sort of L-functions thus providing a kind of "universal series" from which all analytic functions originate. In the second part of this thesis we study the Iwasawa tower generated by the torsion of a Hayes module and use the Stickelberger series to prove a *main conjecture*.

Let F be a global function field defined over a finite field $\mathbb{F}_q$ of characteristic p, fix a prime ∞ of F and let A be the ring of elements regular outside of ∞. We take a finite set of primes S containing ∞ and we denote with F_S the maximal abelian extension of F unramified outside S and with $G_S := Gal(F_S/F)$ its Galois group. The Stickelberger series of S is an element $\Theta_S(X)$ in the power series algebra $\mathbb{Z}[\![G_S]\!][\![X]\!]$.
In Section 1.3 we study the relations between the Stickelberger series and the *Artin L-function* $L(s,\chi)$.

Let F_∞ denote the completion of F at the prime ∞, $\mathbb{C}_\infty$ the completion of a fixed algebraic closure of F_∞ and $\mathbb{S}_\infty$ the topological group $\mathbb{C}_\infty^\times \times \mathbb{Z}_p$. In [Gos2] the author defines a function $\zeta_A(s)$ over $\mathbb{S}_\infty$ that takes values in $\mathbb{C}_\infty$, which represents the analogue in positive characteristic of the Riemann Zeta function, by

$$\zeta_A(s) = \sum \mathfrak{a}^{-s} \quad \text{for } s \in \mathbb{S}_\infty,$$

where the sum is taken over all the non zero ideals $\mathfrak{a}$ of A.
For this function, which is called *Goss Zeta function*, we prove Theorem 1.6.4:

Theorem. *For every $y \in \mathbb{Z}_p$, there exists a continuous ring homomorphism $\Psi_y : \mathbb{Z}[\![G_S]\!][\![X]\!] \to \mathbb{C}_\infty[\![X]\!]$, such that*

$$\Psi_y\left(\Theta_S(X)\right)(x) = \zeta_A(-s) \prod_{\substack{\mathfrak{p}\in S \\ \mathfrak{p}\neq\infty}} (1-\mathfrak{p}^s) \quad \textit{for every } x \in \mathbb{C}_\infty^\times\,,$$

where $s = (x, y) \in \mathbb{S}_\infty$.

There exists also a local version of the Goss Zeta function, which is called the ν-adic Zeta function $\zeta_\nu(s_\nu)$ and that takes value in the complete and algebraically closed field $\mathbb{C}_\nu$ (here ν is a prime of F different from ∞). For this function, Theorem 1.8.4 provides a connection with the Stickelberger series:

Theorem. *Assume that $\nu \in S$. Then for every $y \in \mathbb{Z}_p$ and $j \in \mathbb{Z}/|\mathbb{F}_\nu^\times|$, there exists a continuous ring homomorphism $\Psi_{y,j} : \mathbb{Z}[\![G_S]\!][\![X]\!] \to \mathbb{C}_\nu[\![X]\!]$, such that*

$$\Psi_{y,j}\left(\Theta_S(X)\right)(x) = \zeta_\nu(-s_\nu) \prod_{\substack{\mathfrak{p}\in S \\ \mathfrak{p}\neq \nu,\infty}} (1-\mathfrak{p}^{s_\nu}) \quad \textit{for every } x \in \mathbb{C}_\nu^\times \,,$$

where $s_\nu = (x,y,j) \in \mathbb{S}_\nu := \mathbb{C}_\nu^\times \times \mathbb{Z}_p \times \mathbb{Z}/|\mathbb{F}_\nu^\times|$.

In [ABBL] the authors use a special case of the previous theorem, for the rational function field $F = \mathbb{F}_q[T]$, to prove a *main conjecture* for (the p-parts of) the class groups in the $\mathfrak{p}$-cyclotomic extension generated over $\mathbb{F}_q[T]$ by the $\mathfrak{p}^\infty$-torsion of the Carlitz module ($\mathfrak{p} \neq \infty$ a prime of $\mathbb{F}_q[T]$). In the second chapter of this thesis, for a general function field F, we investigate the extension generated by the torsion of a Hayes module and we are able to prove the main conjecture for the χ part of the Iwasawa module, when χ is a character of type 1 or 2 (Definition 2.3.6). These results were achieved thanks to the work of Greither and Popescu on the Deligne's Picard 1-motive ([GP1] and [GP2]).

We worked under two assumptions: the first one is that the degree of the prime ∞ is 1. This assumption was needed to assure that all the field extensions we are dealing with are geometric, but the reader may observe that this assumption is not really restrictive since we can reduce to this case by extending the costant field of F. The second assumption is that the class number of degree zero divisors $h^0(F)$ is coprime with p.
Let H_A be the Hilbert class field of A and $\Psi : A \to H_A\{\tau\}$ a Hayes module. Fix a prime $\mathfrak{p}$ of F with degree $d_\mathfrak{p}$ and denote with F_n the extension of H_A generated by the $\mathfrak{p}^{n+1}$ torsion of Ψ. The field F_n is an abelian Galois extension of F ramified only at the two primes $\mathfrak{p}$ and ∞. These fields form an Iwasawa tower

$$F \subset H_A \subset F_0 \subset F_1 \subset \cdots \subset F_n \subset \cdots \subset \bigcup_{n\in\mathbb{N}} F_n =: F_\infty \,,$$

indeed if we denote with $\Gamma_n = Gal(F_n/F_0)$ we have that

$$\Gamma_\infty := Gal(F_\infty/F_0) = \varprojlim_n Gal(F_n/F_0) \simeq \mathbb{Z}_p^\infty .$$

We denote with $C_n := Cl^0(F_n)\{p\}$ the p-part of the class group of degree zero divisors of F_n. There is a natural action of $\Gamma_n := Gal(F_n/F_0)$ on C_n, thus this group can be seen as a module over the ring $\mathbb{Z}_p[\Gamma_n]$. These groups form a projective system with respect to the norm maps which allows us to define the limit

$$C_\infty := \varprojlim_n C_n \,,$$

which is a module over the *Iwasawa algebra* $\mathbb{Z}_p[\![\Gamma_\infty]\!]$, but unlike the classical case, in this setting this algebra is not Noetherian, thus we do not have a structure theorem for the finitely generated torsion modules. The main goal of the second chapter is to understand the structure

of the module C_∞ and compute its Fitting ideal. Instead of doing it directly we have to deal with the characters of the group $G_0 := Gal(F_0/F)$ which acts naturally on C_n and C_∞.
Let $\chi \in \mathrm{Hom}(G_0, \mathbb{C}^\times)$ be a complex character of the group G_0 and $W = \mathbb{Z}_p[\zeta]$ the Witt ring generated by a primitive root of unity ζ of order $|G_0|$. For each G_0-module M we define its χ-part as

$$M(\chi) := e_\chi \left(M \otimes_{\mathbb{Z}_p} W\right) ,$$

where $e_\chi \in W[G_0]$ is the *idempotent* of χ.
Theorem 2.4.7 states:

Theorem. *Let χ be a character of type 1 or 2. Then $C_\infty(\chi)$ is a finitely generated torsion module over the Iwasawa algebra $\Lambda := W[\![\Gamma_\infty]\!]$.*

Let $\Theta_\infty(X)$ be the projection of the Stickelberger series $\Theta_S(X)$ to $\mathbb{Z}[\![\Gamma_\infty \times G_0]\!][\![X]\!]$ (here we put $S = \{\mathfrak{p}, \infty\}$) and $\Theta_\infty(X, \chi) = e_\chi \Theta_\infty(X) \in W[\![\Gamma_\infty]\!][\![X]\!]$. Also put

$$\Theta^\sharp_\infty(X,\chi) = \begin{cases} \Theta_\infty(X,\chi) & \text{if } \chi \text{ is of type } 1\,, \\[2ex] \dfrac{\Theta_\infty(X,\chi)}{1-X} & \text{if } \chi \text{ is of type } 2\,, \end{cases}$$

Our main results on the Iwasawa module $C_\infty(\chi)$ is Theorem 2.4.8:

Theorem (Iwasawa main conjecture)**.** *Let χ be a character of type 1 or 2. Then we have*

$$Fitt_\Lambda\left(C_\infty(\chi)\right) = \left(\Theta^\sharp_\infty(1,\chi)\right) .$$

For the characters of type 3 we have no direct information on the structure of $C_n(\chi)$ and $C_\infty(\chi)$, but, working on Pontrjagin duals, we are able to prove a result on the χ-part of the Stickelberger series: for the characters of type 3 which are trivial on the decomposition group of $\mathfrak{p}$, the series $\Theta_n(X,\chi)$ and $\Theta_\infty(X,\chi)$ have a zero of order at least 2 at $X = 1$.

Chapter 1

Stickelberger Series and L-functions

1.1 Setting and notations

In this first chapter we will introduce and study the properties of some analytic and algebraic objects that have been used to study the arithmetic of function fields. The main algebraic object we are dealing with is the Stickelberger series, while the analytic objects we will introduce are the Artin L-functions, the Goss Zeta function and the Goss ν-adic Zeta function. The goal of this chapter is to investigate some links between these families of objects and provide theorems that build a bridge between the analytic side and the algebraic side of the theory.

- F is a global function field of characteristic $p > 0$, i.e., a finite algebraic extension of a field of transcendence degree 1 over a finite field $\mathbb{F}_{p^r} := \mathbb{F}_q$ which we call the *constant field* of F. A more geometric interpretation would be to consider F as the function field of a smooth projective curve X defined over $\mathbb{F}_q$;

- ∞ is a fixed place of F and A is the subring of F of the elements regular outside of ∞;

- for any place ν of F (including ∞), F_ν is the completion of F at ν. Its ring of integers will be denoted by O_ν and $U_1(\nu)$ will be the group of 1-units of F_ν. The residue field $O_\nu/(\nu) := \mathbb{F}_\nu$ is a finite extension of $\mathbb{F}_q$ of degree $d_\nu := [\mathbb{F}_\nu : \mathbb{F}_q]$ (also called the *degree* of ν), its order will be denoted by $\mathbf{N}\nu := q^{d_\nu}$. The degree of a prime ν will be often denoted also by $\deg \nu$;

- $v_\nu : F_\nu \to \mathbb{Z}$ is the (canonical) valuation at ν and π_ν will denote a fixed uniformizer for F_ν, i.e., an element with $v_\nu(\pi_\nu) = 1$;

- the degree of a fractional ideal $\mathfrak{a} = \prod_{\nu \neq \infty} \nu^{n_\nu}$ of A is the quantity $\deg \mathfrak{a} = \sum_{\nu \neq \infty} n_\nu d_\nu$.

1.2 Stickelberger Series

Let S be a finite set of places of F that contains ∞ and we denote with F_S the maximal abelian extension of F unramified outside S and with $G_S := Gal(F_S/F)$ its Galois group. For every place $\nu \notin S$ let ϕ_ν be the Frobenius at ν, i.e., the unique element of G_S that satisfies

$$\phi_\nu(x) \equiv x^{\mathbf{N}\nu} \pmod{\tilde{\nu}}$$

for every $x \in F_S$, where $\tilde{\nu}$ is any place of F_S lying above ν. We observe that the extension F_S/F is unramified at ν, so the decomposition group of ν in G_S is pro-cyclic and topologically generated by ϕ_ν.

Definition 1.2.1. The Stickelberger series of S is defined by the Euler product

$$\Theta_S(X) = \prod_{\nu \notin S} \left(1 - \phi_\nu^{-1} X^{d_\nu}\right)^{-1} .$$

Actually the product above provides a well defined element of $\mathbb{Z}[\![G_S]\!][\![X]\!]$: for every commutative unitary ring R, an element $f(X) \in R[\![X]\!]$ is invertible if and only if $f(0)$ is invertible in R, thus every Euler factor $e_\nu(X) := 1 - \phi_\nu^{-1} X^{d_\nu}$ is invertible in $\mathbb{Z}[\![G_S]\!][\![X]\!]$. Furthermore if we denote with $\mathcal{I}_S$ the set of fractional ideals of A with support outside of S (remember that $\infty \in S$) and with $\phi_\mathfrak{a}$ the Artin symbol associated to $\mathfrak{a} \in \mathcal{I}_S$, i.e.,

$$\mathfrak{a} = \prod \nu^{n_\nu} \Longrightarrow \phi_\mathfrak{a} = \prod \phi_\nu^{n_\nu}$$

(recall that A is a Dedekind domain), then we can write

$$\Theta_S(X) = \sum_{\substack{\mathfrak{a} \in \mathcal{I}_S \\ \mathfrak{a} \geq 0}} \phi_\mathfrak{a}^{-1} X^{\deg \mathfrak{a}} = \sum_{n \geqslant 1} \sum_{\substack{\mathfrak{a} \in \mathcal{I}_S \\ \mathfrak{a} \geq 0 \\ \deg \mathfrak{a} = n}} \phi_\mathfrak{a}^{-1} X^n$$

(where we use the notation $\mathfrak{a} \geq 0$ to denote the integral ideals of A). Since for every positive integer n there exists at most a finite number of primes ν with degree equal to n, the series on the right is clearly an element of $\mathbb{Z}[G_S][\![X]\!]$.

1.3 Artin L-functions

Let K/F be a finite subextension of F_S whose Galois group is G_K and let $S_K \subset S$ bet the set of ramified places together with ∞.

Notation 1.3.1. We remark that we shall always include ∞ in S_K (i.e., if K/F is not ramified at ∞ we put $S_K := \{\nu \text{ prime of } F \text{ ramified in } K/F\} \cup \{\infty\}$).

For every $\nu \notin S_K$ let $\mathrm{Frob}_K \nu \in G_K$ be the Artin symbol. In particular if $\nu \notin S$, $\mathrm{Frob}_K \nu$ is the image of ϕ_ν through the canonical projection $G_S \twoheadrightarrow G_K$.
For every complex character χ of G_K, i.e., an element of $\mathrm{Hom}(G_K, \mathbb{C}^\times)$, we put

$$\chi(\nu) = \begin{cases} \chi(\mathrm{Frob}_K \nu) & \text{if } \nu \notin S_K , \\ 0 & \text{if } \nu \in S_K . \end{cases}$$

Definition 1.3.2. The *Artin L-function* associated to (K, χ) is

$$L_K(s, \chi) = \prod_{\nu \notin S_K} \left(1 - \chi(\nu)(\mathbf{N}\nu)^{-s}\right)^{-1} , \quad \text{for } \mathfrak{Re}(s) > 1$$

(where the condition $\mathfrak{Re}(s) > 1$ guarantees convergence).

Our goal in this section is to provide a link between $L_K(s,\chi)$ and $\Theta_S(X)$.
Let $\Psi : G_S \to \mathbb{C}^\times$ be a continuous character of G_S, i.e., a continuous homomorphism with respect to the natural topologies. We recall that the natural topology on a Galois group is the Krull topology which is generated by the left (or right) cosets of normal subgroups with finite index. With an abuse of notation we denote with Ψ also the ring homomorphism $\mathbb{Z}[\![G_S]\!][\![X]\!] \to \mathbb{C}[\![X]\!]$ induced in a natural way by Ψ.

Theorem 1.3.3. *(a) Let K/F be a finite subextension of F_S with Galois group G_K and let χ be a complex character of G_K. Then there exists a continuous character Ψ of G_S such that*

$$\Psi\left(\Theta_S(X)\right)(q^{-s}) = L_K(s,\chi^{-1}) \prod_{\nu \in S - S_K} \left(1 - \chi^{-1}(\nu) q^{-sd_\nu}\right), \quad \textit{for } \mathfrak{Re}(s) > 1\,. \tag{1.1}$$

(b) Let Ψ be a continuous character of G_S. Then there exists a finite subextension K of F_S with Galois group G_K and a complex character χ of G_K such that equation (1.1) *holds.*

Proof. (a) Let π_K be the canonical projection $G_S \twoheadrightarrow G_K$ and put $\Psi := \chi \circ \pi_K$. Clearly Ψ is a continuous character of G_S. We have that

$$\begin{aligned} \Psi\left(\Theta_S(X)\right) &= \prod_{\nu \notin S} \left(1 - \chi\left(\pi_K(\phi_\nu^{-1})\right) X^{d_\nu}\right)^{-1} \\ &= \prod_{\nu \notin S} \left(1 - \chi^{-1}(\mathrm{Frob}_K \nu) X^{d_\nu}\right)^{-1} \end{aligned}$$

and so

$$\begin{aligned} \Psi\left(\Theta_S(X)\right)(q^{-s}) &= \prod_{\nu \notin S} \left(1 - \chi^{-1}(\mathrm{Frob}_K \nu) q^{-sd_\nu}\right)^{-1} \\ &= \prod_{\nu \notin S_K} \left(1 - \chi^{-1}(\nu) q^{-sd_\nu}\right)^{-1} \prod_{\nu \in S - S_K} \left(1 - \chi^{-1}(\nu) q^{-sd_\nu}\right) \\ &= L_K(s,\chi^{-1}) \prod_{\nu \in S - S_K} \left(1 - \chi^{-1}(\nu) q^{-sd_\nu}\right)\,. \end{aligned}$$

(b) The kernel of Ψ has finite index, indeed Ψ factors through the profinite group $G_S/Ker(\Psi)$, which is topologically isomorphic to $\Psi(G_S)$ due to the first homomorphism theorem. However the only profinite (hence compact) subgroups of $\mathbb{C}^\times$ are the finite subgroups.
We denote with K the fixed field of $Ker(\Psi)$, whose Galois group G_K is isomorphic to the quotient $G_S/Ker(\Psi)$, with χ the character induced by Ψ on G_K and with S_K the set of ramified primes in K/F (recall that it always includes ∞ by our convention recalled in Notation 1.3.1).
Clearly for $\nu \notin S$ the diagram

$$\begin{tikzcd} G_S \arrow[rr, "\Psi"] \arrow[dr, "\pi_K"'] & & \mathbb{C}^\times \\ & G_S/Ker(\Psi) \simeq G_K \arrow[ur, "\chi"'] & \end{tikzcd}$$

shows that $\Psi(\phi_\nu) = \chi(\mathrm{Frob}_K \nu)$ and

$$\begin{aligned}\Psi\left(\Theta_S(X)\right) &= \prod_{\nu \notin S} \left(1 - \Psi(\phi_\nu^{-1}) X^{d_\nu}\right)^{-1} \\ &= \prod_{\nu \notin S} \left(1 - \chi^{-1}(\mathrm{Frob}_K \nu) X^{d_\nu}\right)^{-1} \\ &= \prod_{\nu \notin S_K} \left(1 - \chi^{-1}(\nu) X^{d_\nu}\right)^{-1} \prod_{\nu \in S - S_K} \left(1 - \chi^{-1}(\nu) X^{d_\nu}\right).\end{aligned}$$

From this equation (1.1) follows immediately if we observe that $L_K(s, \chi^{-1})$ is equal to the product $\prod_{\nu \notin S_K} \left(1 - \chi^{-1}(\nu) X^{d_\nu}\right)^{-1}$ evaluated at $X = q^{-s}$.

□

In the final part of this section we will give an application of the previous theorem to prove that the Stickelberger series lies in the Tate algebra.
Let $\mathcal{R}$ be any topological ring. The Tate algebra $\mathcal{R}\langle X\rangle$ is the set of formal power series with coefficients in $\mathcal{R}$, such that the coefficients tend to 0. Note, in particular, that the polynomial ring $\mathcal{R}[X]$ is contained in the Tate algebra. Let W be the ring of integers of a finite extension of $\mathbb{Q}_p$. For the purpose of this thesis and, in particular in Chapter 2, we will be mainly interested in rings of the form $\mathcal{R} = W[\![G]\!]$, where G is the Galois group of an infinite extension of function fields. We recall that the topology on this ring is the weakest such that the projection $\pi : W[\![G]\!] \twoheadrightarrow W[Gal(K/F)]$ is continuous for each finite subextension K/F. A classical result on profinite groups tell that G admits a basis of neighbourhoods of 1_G consisting of open subgroup of finite index which correspond, by Galois Theory, to the finite subextensions. Thus, a sequence of elements a_n of $W[\![G]\!]$ tend to 0 if and only if the sequence of the projections $\pi(a_n)$ is equal to 0 when n is big enough, for each finite subextension K/F.
The coefficients of the Stickelberger series lie in $\mathbb{Z}[\![G_S]\!]$, but to show that $\Theta_S(X)$ is an element of the Tate algebra we have to replace $\mathbb{Z}$ with a non-archimedean complete ring, thus we will use the natural embedding $\mathbb{Z} \hookrightarrow W$ to identify $\Theta_S(X)$ with an element of $W[\![G_S]\!][\![X]\!]$. (Note that we consider $\mathbb{Z}$ with the discreet topology, thus the previous embedding is continuous).

Proposition 1.3.4. *Let W be the ring of integers of a finite extension of $\mathbb{Q}_p$. Then $\Theta_S(X)$ is an element of the Tate algebra $W[\![G_S]\!]\langle X\rangle$.*

Proof. It is enough to show that for each finite subextension K/F the image of the Stickelberger series under the projection $\pi : W[\![G_S]\!][\![X]\!] \twoheadrightarrow W[Gal(K/F)][\![X]\!]$ is a polynomial. We will not work directly with $\Theta_S(X)$ but we will consider $f(X) := (1 - qX)\Theta_S(X)$. Since $(1 - qX)^{-1} = \sum_{n \geq 0} q^n X^n$ is an element of the Tate algebra, i.e. $(1 - qX)$ is a unit in $W[\![G_S]\!]\langle X\rangle$, the thesis will follow immediately if we prove that $f(X)$ is in the Tate algebra. Here we want to underline the necessity of replacing $\mathbb{Z}$ with W: the series $\sum_{n \geq 0} q^n X^n$ is not in the Tate algebra $\mathbb{Z}[\![G_S]\!]\langle X\rangle$.

Let $\Psi : G_S \to \mathbb{C}^\times$ a continuous character. Following part (b) of Theorem 1.3.3 let K be the fixed field of $\ker \Psi$, G_K be the Galois group $Gal(K/F)$, which is finite, and χ be the character induced by Ψ on G_K. We note that χ is the trivial character if and only if Ψ is the trivial character on G_S and, in this case, $K = F$ and $S_K = \{\infty\}$. For $\mathfrak{Re}(s) > 1$ we have

$$\Psi\left(\Theta_S(X)\right)(q^{-s}) = L_K(s, \chi^{-1}) \prod_{\nu \in S - S_K} \left(1 - \chi^{-1}(\nu) q^{-s d_\nu}\right).$$

Now we introduce the full Artin L-function which is defined by the Euler product

$$L(s,\chi^{-1}) = \prod_{\nu} \left(1-\chi^{-1}(\nu)(\mathbf{N}\nu)^{-s}\right)^{-1}, \quad \text{for } \mathfrak{Re}(s) > 1,$$

that differs from $L_K(s,\chi^{-1})$ only for the factors associated to the primes of S_K. Thus we have

$$\begin{aligned}\Psi\left(f(X)\right)(q^{-s}) &= (1-q^{1-s})\Psi\left(\Theta_S(X)\right)(q^{-s}) \\ &= (1-q^{1-s})L(s,\chi^{-1})\prod_{\nu\in S}\left(1-\chi^{-1}(\nu)q^{-sd_\nu}\right).\end{aligned}$$

A theorem of Weil [Wei, VII, Theorem 6] tells that if $\chi \neq \chi_0$ then $L(s,\chi^{-1})$ is a polynomial in q^{-s}, thus $\Psi\left(f(X)\right) \in W[\psi(G_S)][X]$. Another theorem of Weil [Wei, VII, Theorem 4] tells that for $\chi = \chi_0$ we have

$$L(s,\chi_0) = \frac{P(q^{-s})}{(1-q^{-s})(1-q^{1-s})},$$

where $P(X)$ is a polynomial of degree $2g$ (g is the genus of F). Thus also in this case we have proved that $\Psi\left(f(X)\right)$ is a polynomial, because the factor $1-q^{1-s}$ in the denominator of $L(s,\chi_0)$ is killed by the same factor in $\Psi\left(f(X)\right)(q^{-s})$, while the factor $1-q^{-s}$ is killed by one of the terms in the product over the primes of S: here we are using that S is not empty.
We have just proved that $\Psi\left(f(X)\right)$ is a polynomial for each continuous character Ψ, thus for each finite subextension K/F, if we denote by π the projection $G_S \twoheadrightarrow Gal(K/F)$ we have

$$\pi\left(f(X)\right) \in \mathbb{Z}[Gal(K/F)][X]$$

and so $f(X) \in \mathbb{Z}[\![G_S]\!]\langle X\rangle \subset W[\![G_S]\!]\langle X\rangle$, which is our thesis. □

In Chapter 2 we will need to evaluate the Stickelberger series $\Theta_S(X)$ at some point of $W[\![G_S]\!]$, the previous proposition grants us that when we take x in the unit disk $\{x \in W[\![G_S]\!] : |x| \le 1\}$, then the series $\Theta_S(x)$ converges.

1.4 The $\mathbb{S}_\infty$-power of an ideal

Let $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure of F_∞ and put $\mathbb{S}_\infty := \mathbb{C}_\infty^\times \times \mathbb{Z}_p$. The analogue of the Riemann Zeta function for F (i.e., in positive characteristic) has been originally defined for some special values (the integers) by Carlitz in [Car] and later extended by Goss as a $\mathbb{C}_\infty$-valued function whose domain is $\mathbb{S}_\infty$, in [Gos2]. We will see later how the integers embeds in this topological group. This work of Goss may be interpreted a sort of analytic continuation of the function defined by Carlitz.
Before going into the details of the *Goss Zeta function*, we need to define the term I^s for any nonzero fractional ideal I of F and any $s \in \mathbb{S}_\infty$ (see also [Gos1, Chapter 8]).

Definition 1.4.1. A *sign function* on F_∞ is any homomorphism $\mathrm{sgn} : F_\infty^\times \to \mathbb{F}_\infty^\times$ such that its restriction to $\mathbb{F}_\infty^\times$ is the identity. We extend sgn to all F_∞ by defining $\mathrm{sgn}(0) = 0$.

We fix a generator π_∞ of the maximal ideal of F_∞. We will say that the sign function sgn is *normalized* if $\mathrm{sgn}(\pi_\infty) = 1$. Since $U_1(\infty)$ is a pro-p-group and the image of sgn has order prime to p, every sign function is trivial on $U_1(\infty)$. From the decomposition $a = \pi_\infty^{v_\infty(a)}\zeta u$ given by the isomorphism

$$F_\infty^\times \simeq \pi_\infty^{\mathbb{Z}} \times \mathbb{F}_\infty^\times \times U_1(\infty) \tag{1.2}$$

we deduce that for every normalized sign function we have $\operatorname{sgn}(a) = \zeta$.
The 1-unit associated to $a \in F_\infty^\times$ is the element

$$\langle a \rangle_\infty := \frac{a}{\pi_\infty^{v_\infty(a)}\operatorname{sgn}(a)} \in U_1(\infty)\,.$$

From now on we will consider a fixed normalized sign function and the decomposition (1.2) will be written as

$$a = \pi_\infty^{v_\infty(a)} \cdot \operatorname{sgn}(a) \cdot \langle a \rangle_\infty\,.$$

For every $u \in U_1(\infty)$ and $y \in \mathbb{Z}_p$ the series $\displaystyle\sum_{n\geq 0}\binom{y}{n}(u-1)^n$ converges in $U_1(\infty)$, so we put

$$u^y = ((u-1)+1)^y := \sum_{n\geq 0}\binom{y}{n}(u-1)^n\,.$$

We say that an element $a \in F$ is *positive* if $\operatorname{sgn}(a) = 1$ and we denote by A_+ the set of positive integers of F, i.e., the subset positive elements in A.
Let $\mathcal{I}$ be the set of nonzero fractional ideals of F and denote by $\mathcal{P}_+$ the principal fractional ideals with a positive generator. The group $\mathcal{I}/\mathcal{P}_+$ is finite and we put

$$h^+(A) := |\mathcal{I}/\mathcal{P}_+|\,.$$

We also denote with p^t be the maximal power of p that divides $h^+(A)$. We recall that $d_\infty = [\mathbb{F}_\infty : \mathbb{F}_q]$ and, for any $a \in F_\infty^\times$, define the *degree* of a as $\deg(a) = -d_\infty v_\infty(a)$. Note that if $I = (i)$ is principal, then the definition of $\deg(i)$ coincides with the degree of the ideal I, i.e., $\deg(i) = \deg(I) := \log_q |A/I|$.

Remark 1.4.2. We have that $h^+(A) = h^0(F) \cdot d_\infty \cdot (q^{d_\infty} - 1)/(q-1)$, where $h^0(F)$ denotes the cardinality of the class group of degree zero divisors of F: let $\mathcal{P}$ denote the full subgroup of principal ideals of $\mathcal{I}$ and let $h(A)$ be the cardinality of $\mathcal{I}/\mathcal{P}$. Applying ([Ros], Proposition 14.1, part (b)) to our setting we deduce that $h(A) = h^0(F) \cdot d_\infty$. Since $\mathcal{P}_+ \subset \mathcal{P}$, we have a surjective map $\mathcal{I}/\mathcal{P}_+ \to \mathcal{I}/\mathcal{P}$ whose kernel is isomorphic to $\mathcal{P}/\mathcal{P}_+$, thus $h^+(A) = h(A)|\mathcal{P}/\mathcal{P}_+| = h^0(F)d_\infty|\mathcal{P}/\mathcal{P}_+|$.
To compute the cardinality of $\mathcal{P}/\mathcal{P}_+$ consider the following diagram

$$\begin{tikzcd}
F_+ \arrow[r, hook] \arrow[d] & F^\times \arrow[r, two heads, "\operatorname{sgn}"] \arrow[d] & \mathbb{F}_\infty^\times \arrow[d] \\
\mathcal{P}_+ \arrow[r, hook] & \mathcal{P} \arrow[r, two heads] & \mathcal{P}/\mathcal{P}_+
\end{tikzcd}$$

where the vertical map on the right is induced by the central vertical map. Clearly the map $F^\times \to \mathcal{P}$ is surjective and so it is the map $\mathbb{F}_\infty^\times \to \mathcal{P}/\mathcal{P}_+$. Note that the map $F_+ \to \mathcal{P}_+$ is an isomorphism (its kernel is $F_+ \cap \mathbb{F}_q^\times = \{1\}$) thus we deduce, by the snake lemma, that the kernel of the right vertical map is isomorphic to the kernel of the central vertical map, which is $\mathbb{F}_q^\times$. And so we have proved that $|\mathcal{P}/\mathcal{P}_+| = |\mathbb{F}_\infty^\times|/|\mathbb{F}_q^\times| = (q^{d_\infty} - 1)/(q-1)$.

We fix a d_∞-th root of π_∞ and call it π_*: for every integer j we put $s_j = (\pi_*^{-j}, j)$. The map $j \mapsto s_j$ gives us an embedding $\mathbb{Z} \hookrightarrow \mathbb{S}_\infty$.

For every $s = (x, y) \in \mathbb{S}_\infty$ our goal is to define the exponential of a fractional ideal I^s. We start by defining the exponential of a positive element: given $a \in F_\infty^\times$ with $\mathrm{sgn}(a) = 1$ and $s \in \mathbb{S}_\infty$ we put

$$a^s = x^{\deg(a)} \langle a \rangle_\infty^y .$$

The following proposition sums up some fundamental properties which can be found in Section 8.1. of [Gos1]. We report here the proof to make the exposition more clear.

Proposition 1.4.3. *For every $a, b \in F_\infty^\times$ with $sgn(a) = sgn(b) = 1$, one has*

- $a^{s+t} = a^s a^t$ *for every $s, t \in \mathbb{S}_\infty$.*
- $(ab)^s = a^s b^s$ *for every $s \in \mathbb{S}_\infty$.*
- $(a^{s_i})^{s_j} = a^{s_{ij}}$ *for every $i, j \in \mathbb{Z}$.*
- $a^{s_i} = a^i$ *for every $i \in \mathbb{Z}$.*

Proof. This is an easy exercise: let $s = (x_1, y_1)$ and $t = (x_2, y_2)$. Then

- $a^{s+t} = a^{(x_1x_2, y_1+y_2)} = (x_1x_2)^{\deg(a)} \langle a \rangle_\infty^{y_1+y_2} = x_1^{\deg(a)} \langle a \rangle_\infty^{y_1} x_2^{\deg(a)} \langle a \rangle_\infty^{y_2} = a^s a^t$.
- $(ab)^s = x_1^{\deg(ab)} \langle ab \rangle_\infty^{y_1} = x_1^{\deg(a)+\deg(b)} \langle a \rangle_\infty^{y_1} \langle b \rangle_\infty^{y_1} = x_1^{\deg(a)} \langle a \rangle_\infty^{y_1} x_1^{\deg(b)} \langle b \rangle_\infty^{y_1} = a^s b^s$.
- We recall that $a^{s_i} = \pi_*^{-i \deg(a)} \langle a \rangle_\infty^i$ thus we have $\langle a^{s_i} \rangle_\infty = \langle a \rangle_\infty^i$ and

$$\begin{aligned}
\deg(a^{s_i}) &= \deg\left(\pi_*^{-i\deg(a)}\right) \\
&= -d_\infty v_\infty \left(\pi_*^{i d_\infty v_\infty(a)}\right) \\
&= -d_\infty v_\infty \left(\pi_\infty^{i v_\infty(a)}\right) \\
&= -d_\infty i v_\infty(a) \\
&= i \deg(a) .
\end{aligned}$$

  And so $(a^{s_i})^{s_j} = \pi_*^{-j(i\deg(a))} \left(\langle a \rangle_\infty^i\right)^j = \pi_*^{-ij\deg(a)} \langle a \rangle_\infty^{ij} = a^{s_{ij}}$.

- $a^{s_i} = \pi_*^{-i\deg(a)} \langle a \rangle_\infty^i = \pi_*^{i d_\infty v_\infty(a)} \langle a \rangle_\infty^i = \left(\pi_\infty^{v_\infty(a)} \langle a \rangle_\infty\right)^i$ and, since a is positive, this last term is equal to a^i.

□

To simplify notations put

$$e := h^+(A) = |\mathcal{I}/\mathcal{P}_+| .$$

Now we can define the exponential of a fractional ideal: given $I \in \mathcal{I}$, there exists a positive element $\alpha \in F^\times$ such that $I^e = (\alpha)$. We put

$$I^s = x^{\deg I} \langle \alpha \rangle_\infty^{y/e} ,$$

where $\langle\alpha\rangle_\infty^{1/e}$ denotes the unique e-th root of $\langle\alpha\rangle_\infty$ that is a 1-unit. Furthermore we put

$$\langle I\rangle_\infty := \langle\alpha\rangle_\infty^{1/e} .$$

We observe that in general $\langle I\rangle_\infty$ and I^s do not belong to F_∞, but to a suitable extension (see the following proposition).

Proposition 1.4.4. *(a) $\langle I\rangle_\infty$ and I^s are well defined.*

(b) If $I = (\alpha) \in \mathcal{P}_+$ then $I^s = \alpha^s$.

(c) For every $j \in \mathbb{Z}$: I^{s_j} is algebraic over F.

(d) Let $F_{\mathbf{V}}$ be the extension of F obtained by adding every element of the form I^{s_1} with $I \in \mathcal{I}$. Then $F_{\mathbf{V}}/F$ is a finite extension with degree at most e.

(e) Let $F_{\infty,\mathbf{V}}$ be the extension of F_∞ obtained by adding every element of the form $\langle I\rangle_\infty$ with $I \in \mathcal{I}$. Then $F_{\infty,\mathbf{V}}/F$ is a finite extension with degree that divides p^t.

Proof. (a) We suppose $I^e = (\alpha) = (\beta)$ for some positive α, β. Then α and β differ by a unit of A, i.e., by an element $a \in \mathbb{F}_q^\times$ (see, for example [Ros, Proposition 5.1]). Therefore $\alpha = \beta a$ implies $\mathrm{sgn}(\alpha) = \mathrm{sgn}(\beta)\mathrm{sgn}(a)$. However α and β are positive and so $\mathrm{sgn}(a) = 1$. Finally, since $a \in \mathbb{F}_q^\times$, we conclude that $a = \mathrm{sgn}(a) = 1$ and $\alpha = \beta$.

(b) From the equality $I = (\alpha)$ it follows that $\deg(I) = \deg(\alpha)$ and $I^e = (\alpha)^e = (\alpha^e)$. Therefore

$$I^s = x^{\deg(I)}\langle\alpha^e\rangle_\infty^{y/e} = x^{\deg(\alpha)}\langle\alpha\rangle_\infty^y = \alpha^s .$$

(c) Let $I \in \mathcal{I}$ and $\alpha \in F$ be such that $I^e = (\alpha)$. We prove that I^{s_j} is a root of the polynomial $f(X) = X^e - \alpha^j \in F[X]$.
The equality $I^e = (\alpha)$ implies that $e\ \deg(I) = \deg(\alpha) = -d_\infty v_\infty(\alpha)$. Therefore

$$\begin{aligned}
(I^{s_j})^e &= \left(\pi_*^{-j\ \deg(I)}\langle\alpha\rangle_\infty^{j/e}\right)^e \\
&= \pi_*^{-j\ e\ \deg(I)}\langle\alpha\rangle_\infty^j \\
&= \pi_*^{j\ d_\infty\ v_\infty(\alpha)}\langle\alpha\rangle_\infty^j \\
&= \left(\pi_\infty^{v_\infty(\alpha)}\langle\alpha\rangle_\infty\right)^j = \alpha^j .
\end{aligned}$$

(d) We have already proved in the previous step that I^{s_1} is algebraic over F, so it is enough to show that $F_{\mathbf{V}}$ can be generated by a finite number of elements of the form I^{s_1}. Let I and J be representatives of the same equivalence class in $\mathcal{I}/\mathcal{P}_+$ and let α, β and γ be positive elements such that $I^e = (\alpha)$, $J^e = (\beta)$ and $I = \gamma J$. Clearly there exists a constant a such that the equality $\alpha = \gamma^e\beta a$ holds, but the same considerations explained in (a), lead us to conclude that $a = 1$. Furthermore we have that $\deg(I) = \deg(J) + \deg(\gamma)$. Then

$$\begin{aligned}
I^{s_1} &= \pi_*^{-\deg(I)}\langle\alpha\rangle_\infty^{1/e} \\
&= \pi_*^{-\deg(J)-\deg(\gamma)}\langle\gamma\rangle_\infty\langle\beta\rangle_\infty^{1/e} \\
&= J^{s_1}\pi_*^{-\deg(\gamma)}\langle\gamma\rangle_\infty
\end{aligned}$$

$$= \quad J^{s_1}\pi_\infty^{v_\infty(\gamma)}\langle\gamma\rangle_\infty = J^{s_1}\gamma\,.$$

Therefore to generate $F_{\mathbf{V}}$ it is enough to take one fractional ideal I^{s_1} for every equivalence class of $\mathcal{I}/\mathcal{P}_+$.
To prove that the degree of $F_{\mathbf{V}}/F$ is at most e we proceed in the following way:

Step 1: If the equivalence class of J is a power of the class of I in $\mathcal{I}/\mathcal{P}_+$ then $F(J^{s_1}) \subseteq F(I^{s_1})$, indeed if $J = I^r\gamma$ for some positive element γ and some $r \in \mathbb{N}$ we have that $J^{s_1} = I^{rs_1}\gamma$ and so $J^{s_1} \in F(I^{s_1})$.

Step 2: If C is a cyclic component of $\mathcal{I}/\mathcal{P}_+$ and I is a representative for a generator of C, then from the previous step it follows that

$$F(I^{s_1}) \subseteq \bigcup_{[J]\in C} F(J^{s_1}) \subseteq F(I^{s_1})\,.$$

Step 3: We decompose $\mathcal{I}/\mathcal{P}_+$ as the sum of its cyclic components

$$\mathcal{I}/\mathcal{P}_+ = \bigoplus_{i=1}^{r} C_i$$

with $n_i = |C_i|$, and for every cyclic component C_i let I_i be a representative of a generator. We observe that $e = \prod_{i=1}^{r} n_i$.
In the proof of (c) we have proved that I^{s_j} is a root of the polynomial $f(X) = X^e - \alpha^j$, when $I^e = (\alpha)$. However in general $f(X)$ is not the minimal polynomial of I^{s_j}, indeed if there exists $k \in \mathbb{N}$ and a positive β such that $I^k = (\beta)$, the same argument of (c) shows that I^{s_j} is a root of $g(X) = X^k - \beta^j$. Therefore the degree of the extension $F(I^{s_1})/F$ is smaller than or equal to the order of the class of I in $\mathcal{I}/\mathcal{P}_+$. From this it also follows that the minimal polynomial of $I_i^{s_1}$ over F has degree at most n_i.
From the previous step we deduce that $F_{\mathbf{V}} = \bigcup_{i=1}^{r} F(I_i^{s_1})$ and so

$$[F_{\mathbf{V}} : F] \leq \prod_{i=1}^{r}[F(I_i^{s_1}) : F] \leq \prod_{i=1}^{r} n_i = e\,.$$

(e) The proof of this point is similar to the previous one, it is enough to recall that $F_{\infty,\mathbf{V}}$ is obtained by adding to F_∞ the e-th roots of elements in $U_1(\infty)$ and that, since $U_1(\infty)$ is a multiplicative group isomorphic to $\mathbb{Z}_p^\infty$, F_∞ already contains the n-th roots of the elements of $U_1(\infty)$ when n is coprime with p (by Hensel's Lemma).

□

All the objects defined until now depend on the choice of the positive element π_∞ and on the choice of its d_∞-th root π_*. Before concluding this section we want to see what happens when we change these two elements. To do this we will follow the ideas of [Gos1, Section 8.2.]. Let $\pi_{(i)}$, $i = 1, 2$ be generators of the maximal ideal of F_∞, both positive with respect to the sign function and $\pi_{*,(i)}, i = 1, 2$ two fixed d_∞-th roots. We denote with $\langle I\rangle_{\infty,(i)}$ the 1-unit associated

to I, with respect to $\pi_{(i)}$ and with $I^s_{(i)}$ the corresponding exponential.
The elements $\pi_{(1)}$ and $\pi_{(2)}$ differ by a unit in $A_\infty^\times = \mathbb{F}_\infty^\times \times U_1(\infty)$. Let $u = \pi_{(1)}/\pi_{(2)}$, since $\pi_{(1)}$ and $\pi_{(2)}$ are both positive, u is a 1-unit. We denote with u^{1/d_∞} the only d_∞-th root of u which is a 1-unit. We observe that it is uniquely determined by $\pi_{(1)}$ and $\pi_{(2)}$, i.e., it does not depend on $\pi_{*,(1)}$ and $\pi_{*,(2)}$.

Lemma 1.4.5. *For every $I \in \mathcal{I}$ we have*

$$\langle I\rangle_{\infty,(1)} = \left(u^{1/d_\infty}\right)^{\deg(I)} \langle I\rangle_{\infty,(2)}\,.$$

Proof. From the equality

$$\left(u^{1/d_\infty}\right)^{d_\infty} = \frac{\pi_{(1)}}{\pi_{(2)}} = \left(\frac{\pi_{*,(1)}}{\pi_{*,(2)}}\right)^{d_\infty}$$

it follows that there exists a d_∞-th root of unity ζ such that

$$u^{1/d_\infty} = \zeta\frac{\pi_{*,(1)}}{\pi_{*,(2)}}\,.$$

Let α be such that $I^e = (\alpha)$. We have that $e\ \deg(I) = \deg(\alpha) = -d_\infty v_\infty(\alpha)$ and, for $i = 1, 2$

$$\langle I^e\rangle_{\infty,(i)} = \langle \alpha\rangle_{\infty,(i)} = \frac{\alpha}{\pi_{(i)}^{v_\infty(\alpha)}} = \frac{\alpha}{\pi_{*,(i)}^{-e\ \deg(I)}}\,.$$

Therefore recalling that, by Remark 1.4.2, d_∞ divides e, we have

$$\left(\frac{\langle I\rangle_{\infty,(1)}}{\langle I\rangle_{\infty,(2)}}\right)^e = \frac{\pi_{*,(1)}^{e\ \deg(I)}}{\pi_{*,(2)}^{e\ \deg(I)}} = \left(u^{1/d_\infty}\zeta^{-1}\right)^{e\ \deg(I)} = \left(u^{1/d_\infty}\right)^{e\ \deg(I)}\,.$$

Finally, we observe that inside the brackets in the first and in the last term of the equality there are 1-units, so we can extract the e-th root without ambiguity and obtain the thesis. □

Corollary 1.4.6. *There exists a d_∞-th root of unity ζ such that*

$$I_{(1)}^{s_j} = \zeta^{j\deg(I)} I_{(2)}^{s_j}\,.$$

Proof. Let ζ be as in the previous lemma. Then

$$\begin{aligned} I_{(1)}^{s_j} &= \pi_{*,(1)}^{-j\deg(I)}\langle I\rangle_{\infty,(1)}^j \\ &= \left(\pi_{*,(2)}\zeta^{-1}u^{1/d_\infty}\right)^{-j\deg(I)}\left(u^{1/d_\infty}\right)^{j\deg(I)}\langle I\rangle_{\infty,(2)}^j \\ &= \pi_{*,(2)}^{-j\deg(I)}\zeta^{j\deg(I)}\langle I\rangle_{\infty,(2)}^j \\ &= \zeta^{j\deg(I)} I_{(2)}^{s_j}\,. \end{aligned}$$

□

Corollary 1.4.7. *If F contains all the d_∞-th roots of unity then $F_{\mathbf{V}}$ does not depend on the choice of π_∞ and π_*.*

We observe that the hypothesis of the corollary is equivalent saying that d_∞ has the form $p^\ell m$, where m is any divisor of $q-1$.

1.5 The Goss Zeta Function

Recall that by $\mathfrak{a} \geq 0$ we mean the integral ideals of A.

Definition 1.5.1. The Goss Zeta function is defined by the sum

$$\zeta_A(s) = \sum_{\substack{\mathfrak{a}\in\mathcal{I} \\ \mathfrak{a}\geq 0}} \mathfrak{a}^{-s},$$

for $s = (x, y) \in \mathbb{S}_\infty$.

This sum is clearly convergent for $|x|_\infty > 1$ and can also be rewritten as an Euler product just like the classical Riemann Zeta function

$$\zeta_A(s) = \prod_{\nu\neq\infty} \left(1 - \nu^{-s}\right)^{-1}.$$

Since $-s = (x^{-1}, -y)$, this product amounts to

$$\begin{aligned}\zeta_A(s) &= \prod_{\nu\neq\infty} \left(1 - \langle\nu\rangle_\infty^{-y} x^{-\deg(\nu)}\right)^{-1} \\ &= \prod_{\nu\neq\infty} \left(1 - \langle\alpha_\nu\rangle_\infty^{-y/e} x^{-\deg(\nu)}\right)^{-1}\end{aligned}$$

for some α_ν such that $\nu^e = (\alpha_\nu)$.

We recall that for every integer n we have only a finite number of integral ideals $\mathfrak{a}$ with degree equal to n and so the sum

$$a_n(y) = \sum_{\substack{\mathfrak{a}\in\mathcal{I} \\ \mathfrak{a}\geq 0 \\ \deg(\mathfrak{a})=n}} \langle\mathfrak{a}\rangle_\infty^{-y} \in \mathbb{C}_\infty$$

is finite. This allows us to write the Goss Zeta function as a sum over the positive integers.

$$\zeta_A(s) = \sum_{n\geq 0} a_n(y) x^{-n}. \tag{1.3}$$

This last form is very important because we will prove it is convergent for every $(x, y) \in \mathbb{S}_\infty$ and so can be interpreted as an analytic extension to the whole space $\mathbb{S}_\infty$ of the Goss Zeta function.

1.5.1 Convergence of the Goss Zeta Function

In this section we will prove that for every $s = (x, y) \in \mathbb{S}_\infty$ the sum (1.3) is convergent and, in particular, this provides the analytic extension of the Goss Zeta function to the whole space. More precisely we will prove that when n goes to infinity the number $v_\infty(a_n(y))$ diverges faster than a linear polynomial in n, uniformly with respect to y and, since $v_\infty(x^{-n})$ is linear with respect to n, the convergence of the series will readily follow. Indeed we shall prove that, when n is big enough, the number $v_\infty(a_n(y))$ behaves like a polynomial in n of degree 2 (or is larger than that).

We fix $\mathfrak{a}_1, \mathfrak{a}_2, \dots \mathfrak{a}_e \in \mathcal{I}$ representatives of the equivalence classes $C_1, C_2 \dots . C_e$ of $\mathcal{I}/\mathcal{P}_+$. Every non null fractional ideal of C_j can be written uniquely as the product $\mathfrak{a} = \alpha\mathfrak{a}_j$, with $\alpha \in F_+$. Now we have

$$a_n(y) = \sum_{j=1}^{e} \langle \mathfrak{a}_j \rangle_\infty^{-y} \sum_{\substack{\alpha \in F_+ \\ \alpha\mathfrak{a}_j \geq 0 \\ \deg(\alpha\mathfrak{a}_j)=n}} \langle \alpha \rangle_\infty^{-y} . \tag{1.4}$$

Fix an index j, let $n_j = n - \deg(\mathfrak{a}_j)$ and put

$$a_n(C_j, y) = \sum_{\substack{\mathfrak{a} \in C_j \\ \mathfrak{a} \geq 0 \\ \deg(\mathfrak{a})=n}} \langle \mathfrak{a} \rangle_\infty^{-y} \quad \text{and} \quad b_{n,j}(y) = \sum_{\substack{\alpha \in F_+ \\ \alpha\mathfrak{a}_j \geq 0 \\ \deg(\alpha)=n_j}} \langle \alpha \rangle_\infty^{-y},$$

so that $a_n(C_j, y) = \langle \mathfrak{a}_j \rangle_\infty^{-y} \cdot b_{n,j}(y)$. From (1.4) we deduce that our thesis can be obtained just proving that the function $v_\infty(b_{n,j}(y))$ is bounded from below by a polynomial of degree 2 in n for any j, since $|\langle \mathfrak{a}_j \rangle_\infty^{-y}|_\infty = 1$.
We can limit ourselves to integers n such that d_∞ divides n_j, because for the other values of n we have $b_{n,j}(y) = 0$ (there are no α with degree n_j).

Let $\operatorname{div}(x)$ be the divisor associated to x, i.e., the element

$$\operatorname{div}(x) = \sum_\nu \operatorname{ord}_\nu(x)\nu.$$

Without ambiguity we will use this notation for both the elements $\alpha \in F^\times$ and for the fractional ideals $\mathfrak{a} \in \mathcal{I}$. Be careful: the divisors associated to the element α and to the fractional ideal (α) may not coincide since the component at infinity may not be the same. For the same reason the degrees of α and of the divisor associated to α may be different (remember that principal divisors have always degree zero). Contrariwise the degrees of a fractional ideal and of its divisor always coincide. To make this more clear consider the following example.

Example 1.5.2. Let F be the rational function field $\mathbb{F}_q(T)$ and fix as a place at infinity the rational place corresponding to T^{-1}. In this case the ring of ∞-integers is just the polynomial ring $\mathbb{F}_q[T]$. If we denote with $\mathfrak{p}$ the prime corresponding to the element T we have that the divisor associated to the element T is $\mathfrak{p} - \infty$ while the divisor associated to the ideal (T) is just $\mathfrak{p}$.
Furthemore we have that $\deg\ T = 1 = \deg\ (\mathfrak{p})$ and $\deg\ (\mathfrak{p} - \infty) = 0$.

For every $i \in \mathbb{N}$ let $D_i = \operatorname{div}(\mathfrak{a}_j) + (n_j/d_\infty - i)\infty$ and $\mathcal{L}(D_i)$ be its Riemann-Roch space, which is defined by

$$\mathcal{L}(D_i) = \left\{\alpha \in F^\times : \operatorname{div}(\alpha) + D_i \geq 0\right\} .$$

We observe that the divisors D_i are sorted in descending order and so each space $\mathcal{L}(D_i)$ is contained in the previous one, i.e., $\mathcal{L}(D_i) \supseteq \mathcal{L}(D_{i+1})$ for any i. Furthermore the following equality is true

$$\mathcal{L}(D_i) = \left\{\alpha \in \mathcal{L}(D_0) : v_\infty(\alpha) \geq i - \frac{n_j}{d_\infty}\right\} . \tag{1.5}$$

We recall that each $\mathcal{L}(D_i)$ is a vector space over $\mathbb{F}_q$ with finite dimension ℓ_i. Afterwards we will need to know the dimension of some of these spaces and we use the Riemann-Roch theorem to calculate it. Since

$$\deg(D_i) = \deg(\mathfrak{a}_j) + n_j - id_\infty = n - id_\infty\,,$$

for $i < (n - 2g + 2)/d_\infty$, we have

$$\ell_i = \deg(D_i) - g + 1 = n - id_\infty - g + 1\,,$$

where we denote with g the genus of F. We want to study the asymptotic behavior of $b_{n,j}(y)$ for $n \to \infty$, so the number n should be thought as "big" with respect the other numbers. For this reason we can always assume that there exists at least one index i such that the previous equality holds.

From equality (1.5) it follows that $\alpha \in F^\times$ satisfies both the following conditions

- $\alpha\mathfrak{a}_j \geq 0$
- $\deg(\alpha) = n_j$

if and only if $\alpha \in \mathcal{L}(D_0) - \mathcal{L}(D_1)$ (note that they imply $v_\infty(\alpha) = -\frac{n_j}{d_\infty}$). If we put $X = \{\alpha \in \mathcal{L}(D_0) - \mathcal{L}(D_1) : \mathrm{sgn}(\alpha) = 1\}$, we can rewrite $b_{n,j}(y)$ as

$$b_{n,j}(y) = \sum_{\alpha \in X} \langle \alpha \rangle_\infty^{-y}\,.$$

Lemma 1.5.3. *Let $a \in \mathcal{L}(D_1)$ and $b \in X$. Then $a + b \in X$.*

Proof. Both $\mathcal{L}(D_1)$ and X are subsets of the vector space $\mathcal{L}(D_0)$ and so $a + b$ belongs to $\mathcal{L}(D_0)$. We observe that $v_\infty(a) \geq 1 - n_j/d_\infty$ while $v_\infty(b) = -n_j/d_\infty \neq v_\infty(a)$, therefore

$$v_\infty(a + b) = \min\{v_\infty(a), v_\infty(b)\} = -\frac{n_j}{d_\infty}$$

and $a + b \notin \mathcal{L}(D_1)$.
To complete our proof it remains to show that $a + b$ is positive. Let $m = v_\infty(a) \geq 1 - n_j/d_\infty$ and let $\hat{a} \in O_\infty^\times$ be such that $a = \pi_\infty^m \hat{a}$.
For every $\alpha \in F_\infty^\times$ we have that $\mathrm{sgn}(\alpha) \equiv \alpha\pi_\infty^{-v_\infty(\alpha)} \pmod{\pi_\infty}$ and so we can write $b = \pi_\infty^{-n_j/d_\infty}(1 + \pi_\infty\hat{b})$ with $\hat{b} \in O_\infty$ (note that b is positive). Then

$$a + b = \pi_\infty^{-n_j/d_\infty}(\pi_\infty^{m+n_j/d_\infty}\hat{a} + 1 + \pi_\infty\hat{b}) = \pi_\infty^{v_\infty(a+b)}(1 + \pi_\infty^{m+n_j/d_\infty}\hat{a} + \pi_\infty\hat{b})\,.$$

Hence

$$\begin{aligned} \mathrm{sgn}(a+b) &\equiv (a+b)\pi_\infty^{-v_\infty(a+b)} \pmod{\pi_\infty} \\ &\equiv 1 + \pi_\infty^{m+n_j/d_\infty}\hat{a} + \pi_\infty\hat{b} \pmod{\pi_\infty} \\ &\equiv 1 \pmod{\pi_\infty} \end{aligned} \qquad (1.6)$$

and the thesis is proved. $\square$

The previous lemma tells us that the sum defines an action of $\mathcal{L}(D_1)$ (seen as an abelian group) on the set X by traslation. It is obvious that this action is free, since the equality $g + x = h + x$, with $g, h \in \mathcal{L}(D_1)$ and $x \in X$ can be obtained only for $g = h$. Now we can decompose X as the disjoint union of the orbits of its elements under this action. Let $X_1, X_2, \ldots X_t$ be the orbits of the elements of X (so that with $X_i \cap X_j = \emptyset$ if $i \neq j$ and $X = \bigcup_{l=1}^{t} X_l$) and fix an element $x_l \in X_l$ for each orbit. Then

$$\begin{aligned} b_{n,j}(y) &= \sum_{\alpha \in X} \langle \alpha \rangle_\infty^{-y} \\ &= \sum_{l=1}^{t} \sum_{\alpha \in X_l} \langle \alpha \rangle_\infty^{-y} \\ &= \sum_{l=1}^{t} \sum_{u \in \mathcal{L}(D_1)} \langle u + x_l \rangle_\infty^{-y} . \end{aligned}$$

We put

$$h_n(y) = h_{n,j,l}(y) := \sum_{u \in \mathcal{L}(D_1)} \langle u + x_l \rangle_\infty^{-y} . \tag{1.7}$$

Just like before note that our thesis on the growth of $v_\infty(b_{n,j}(y))$ may be obtained by proving that the growth of the function $v_\infty\left(h_n(y)\right)$ (as a function of n) is greater than a linear polynomial in n.

The following lemma puts together two results of [Tha, Chapter 5]. Here we give a detailed proof.

Lemma 1.5.4. *Let K be a function field with constant field $\mathbb{F}_q$, v any normalized valuation on K and $W \subset K$ an $\mathbb{F}_q$-vector space with finite dimension. Assume that $v(w) > 0$ for every $w \in W$.*

(a) If i is an integer with $0 \leq i < (q-1)\dim_{\mathbb{F}_q} W$, for every $x \in K$ we have

$$\sum_{w \in W} (x + w)^i = 0 .$$[1]

(b) For every $j \in \mathbb{N}^+$ we put $W_j = \{w \in W : v(w) \geq j\}$. Then for every $y \in \mathbb{Z}_p$ we have

$$v\left(\sum_{w \in W} (1 + w)^y\right) \geq (q-1)Q ,$$

where we put $Q = \sum_j \dim_{\mathbb{F}_q} W_j$.

Proof. (a) Let $d = \dim_{\mathbb{F}_q} W$ and we fix a basis $\{e_1, e_2, \ldots, e_d\}$ over $\mathbb{F}_q$. For every $j \in \mathbb{N}$ we have

$$\delta_j = \sum_{c \in \mathbb{F}_q} c^j = \begin{cases} -1 & \text{if } q-1 \text{ divides } j \text{ and } j > 0, \\ 0 & \text{otherwise.} \end{cases}$$

[1] In the sum it may appear a term of the form 0^0. In this case we use the convention that $0^0 := 1$.

Applying the multinomial theorem we have

$$\begin{aligned}\sum_{w\in W}(x+w)^i &= \sum_{c_1,c_2,\dots c_d\in\mathbb{F}_q}(x+c_1e_1+\dots c_de_d)^i\\ &= \sum_{c_1,c_2,\dots c_d\in\mathbb{F}_q}\sum_{\substack{j_1,j_2,\dots j_d\\ j_1+\dots+j_d\le i}}\frac{i!}{j_1!\dots j_d!(i-j_1-\dots-j_d)!}x^{i-j_1-\dots-j_d}(c_1e_1)^{j_1}\dots(c_de_d)^{j_d}\\ &= \sum_{\substack{j_1,j_2,\dots j_d\\ j_1+\dots+j_d\le i}}\frac{i!}{j_1!\dots j_d!(i-j_1-\dots-j_d)!}x^{i-j_1-\dots-j_d}e_1^{j_1}\dots e_d^{j_d}\delta_{j_1}\dots\delta_{j_d}\,.\end{aligned}$$

In this sum each term is equal to zero, indeed we have that $j_1+\dots+j_d\le i<(q-1)d$ and so at least one of the index j is smaller than $q-1$. For that index we have $\delta_j=0$.

(b) The first member of the inequality (seen as a function of y) is continuous and the second member does not depend on y. This implies that it is enough to prove the statement when y is a positive integer because the set of positive integers $\mathbb{N}^+$ is dense in $\mathbb{Z}_p$.

Let J be the greatest positive integer such that $W_J\neq\{0\}$ and so we have

$$\{0\}=W_{J+1}\subsetneq W_J\subseteq W_{J-1}\subseteq\dots\subseteq W_2\subseteq W_1=W.$$

For every $j=1,\dots,J$ we denote with d_j the dimension of W_j and with $d=d_1=\dim_{\mathbb{F}_q}W$ such that

$$0<d_J\le d_{J-1}\le\dots\le d_{j+1}\le d_j\le\dots\le d_2\le d_1=d\,.$$

We fix a basis $\{e_1,e_2,\dots,e_d\}$ of W such that for every j the set $\{e_1,e_2,\dots,e_{d_j}\}$ forms a basis for W_j. Furthermore we put

$$U_j=\mathrm{Span}_{\mathbb{F}_q}\{e_{1+d_{j+1}},\dots,e_{d_j}\}\,.$$

We observe that for every index j we have:

- $W_j=W_{j+1}\oplus U_j$.
- $\dim_{\mathbb{F}_q}U_j=d_j-d_{j+1}$.
- For every $u\in U_j$: $v(u)\ge j$ (in particular we have the equality for every $u\neq0$).

We will prove the following statement: for every index $j\in\{1,2,\dots,J\}$ and for every positive integer $y\in\mathbb{N}^+$ we have the inequality

$$v\left(\sum_{w\in W_j}(1+w)^y\right)\ge(q-1)\left(jd_j+\sum_{l=j+1}^{J}d_l\right)\,. \tag{1.8}$$

Note that for $j=1$ the previous statement is exactly the thesis of (b).
We will prove the inequality by an inductive process: we will first give a proof for $j=J$ and then we will prove it for a generic index j, by assuming it is true for the index $j+1$.

Case $j=J$: by the binomial theorem we have

$$\sum_{w\in W_J}(1+w)^y=\sum_{h=0}^{y}\binom{y}{h}\sum_{w\in W_J}w^h\,.$$

We apply part (a) to the vector space W_J to deduce that the sum $\sum_{w\in W_J} w^h$ is equal to zero when $h < (q-1)d_J$, thus we have

$$\sum_{w\in W_J}(1+w)^y = \sum_{h=(q-1)d_J}^{y}\binom{y}{h}\sum_{w\in W_J} w^h\,.$$

Since v is a non-Archimedean valuation we have

$$v\left(\sum_{h=(q-1)d_J}^{y}\binom{y}{h}\sum_{w\in W_J} w^h\right) \geq \min_{\substack{h\geq (q-1)d_J\\ w\in W_J}}\{v(w^h)\} = \min_{h\geq (q-1)d_J}\{hJ\} = (q-1)Jd_J\,,$$

thus we have proved

$$v\left(\sum_{w\in W_J}(1+w)^y\right) \geq (q-1)Jd_J\,.$$

Now we assume that the inequality (1.8) is true for the index $j+1$ and we prove it for the index j: using the decomposition $W_j = W_{j+1}\oplus U_j$ and the binomial theorem we have

$$\sum_{w\in W_j}(1+w)^y = \sum_{\substack{t\in W_{j+1}\\ u\in U_j}}(1+t+u)^y = \sum_{h=0}^{y}\binom{y}{h}\sum_{t\in W_{j+1}}(1+t)^h\sum_{u\in U_j}u^{y-h}\,.$$

We apply part (a) to the vector space U_j to deduce that the sum $\sum_{u\in U_j} u^{y-h}$ is equal to zero when $y-h < (q-1)\mathrm{dim}_{\mathbb{F}_q}U_j$, i.e., $h > y-(q-1)(d_j-d_{j+1}) =: \hat{y}$, thus we have

$$\sum_{w\in W_j}(1+w)^y = \sum_{h=0}^{\hat{y}}\binom{y}{h}\sum_{u\in U_j}u^{y-h}\sum_{t\in W_{j+1}}(1+t)^h\,.$$

Proceeding like the case $j=J$ and using the assumption for the index $j+1$ we have

$$\begin{aligned}
v\left(\sum_{w\in W_j}(1+w)^y\right) &\geq \min_{\substack{h\leq\hat{y}\\ u\in U_j}}\left\{v\left(u^{y-h}\sum_{t\in W_{j+1}}(1+t)^h\right)\right\}\\
&= \min_{\substack{h\leq\hat{y}\\ u\in U_j}}\left\{(y-h)v(u) + v\left(\sum_{t\in W_{j+1}}(1+t)^h\right)\right\}\\
&\geq \min_{\substack{h\leq\hat{y}\\ u\in U_j}}\left\{(y-h)v(u) + (q-1)\left((j+1)d_{j+1} + \sum_{l=j+2}^{J} d_l\right)\right\}\\
&= (q-1)(d_j-d_{j+1})j + (q-1)\left((j+1)d_{j+1} + \sum_{l=j+2}^{J} d_l\right)\\
&= (q-1)\left(jd_j + \sum_{l=j+1}^{J} d_l\right)\,.
\end{aligned}$$

□

We use this lemma to give an estimate of $h_n(y)$:

Lemma 1.5.5. *Let $h_n(y)$ be the function defined in (1.7). Then we have*

$$v_\infty\left(h_n(y)\right) \geq \frac{(q-1)n^2}{2d_\infty} + O(n)$$

uniformly with respect to y.

Proof. We have

$$\langle u + x_l\rangle_\infty = \pi_\infty^{-v_\infty(u+x_l)}(u+x_l) = \pi_\infty^{n_j/d_\infty} x_l\left(1+\frac{u}{x_l}\right).$$

Recall that, by definition of X, x_l is positive and $v_\infty(x_l) = -n_j/d_\infty$, thus $\pi_\infty^{n_j/d_\infty}x_l$ is a 1-unit and so we can elevate it to the power $-y$, furthermore $v_\infty(u/x_l) = v_\infty(u) - v_\infty(x_l) = v_\infty(u) + n_j/d_\infty > 0$. Therefore from (1.7) we deduce

$$h_n(y) = (\pi_\infty^{n_j/d_\infty}x_l)^{-y}\sum_{u\in\mathcal{L}(D_1)}\left(1+\frac{u}{x_l}\right)^{-y} = (\pi_\infty^{n_j/d_\infty}x_l)^{-y}\sum_{w\in x_l^{-1}\mathcal{L}(D_1)}(1+w)^{-y}.$$

The elements of $x_l^{-1}\mathcal{L}(D_1)$ have positive valuation at ∞ and the vector space $x_l^{-1}\mathcal{L}(D_1)$ satisfies every hypothesis of point (b) of Lemma 1.5.4. Hence we have

$$v_\infty\left(h_n(y)\right) = v_\infty\left(\sum_{w\in x_l^{-1}\mathcal{L}(D_1)}(1+w)^{-y}\right) \geq (q-1)Q,$$

where $Q = \sum_i \dim_{\mathbb{F}_q} W_i$ and $W_i = \{w \in x_l^{-1}\mathcal{L}(D_1) : v_\infty(w) \geq i\}$. Now we observe that the element w is in W_i if and only if $u = x_l w$ is in $\mathcal{L}(D_1)$ and $v_\infty(u) \geq i + v_\infty(x_l) = i - n_j/d_\infty$, i.e, $u \in \mathcal{L}(D_i)$. From this fact we can deduce that the map $W_i \to \mathcal{L}(D_i)$, $w \mapsto x_l w$ is an isomorphism of $\mathbb{F}_q$-vector spaces and so

$$\dim_{\mathbb{F}_q} W_i = \dim_{\mathbb{F}_q}\mathcal{L}(D_i) = \ell_i\,.$$

Finally, with some calculations we obtain that

$$Q = \sum_{i=1}^{\infty}\ell_i \geq \sum_{i=1}^{\lfloor (n-2g+2)/d_\infty\rfloor}\ell_i\,.$$

If d_∞ does not divide $n-2g+2$ the last sum is equal to

$$\sum_{i=1}^{\lfloor (n-2g+2)/d_\infty\rfloor}(n - id_\infty - g + 1)\,,$$

if d_∞ divides $n-2g+2$ the last sum is equal or greater than

$$\sum_{i=1}^{(n-2g+2)/d_\infty - 1}(n - id_\infty - g + 1)\,,$$

thus in both cases we have:

$$\begin{aligned}
Q &\geq \sum_{i=1}^{n/d_\infty+O(1)} (n - id_\infty + O(1)) \\
&= (n+O(1))\left(\frac{n}{d_\infty} + O(1)\right) - d_\infty \sum_{i=1}^{n/d_\infty+O(1)} i \\
&= \frac{n^2}{d_\infty} + O(n) - \frac{d_\infty}{2}\left(\frac{n}{d_\infty} + O(1)\right)\left(1 + \frac{n}{d_\infty} + O(1)\right) \\
&= \frac{n^2}{d_\infty} + O(n) - \frac{n^2}{2d_\infty} + O(n) \\
&= \frac{n^2}{2d_\infty} + O(n)\,.
\end{aligned}$$

□

From this lemma and the previous remarks it follows that

Theorem 1.5.6 (Analytic extension of the Goss Zeta Function)**.** *The series*

$$\sum_{n\geq 0} a_n(y)x^{-n}$$

is absolutely convergent for every $(x, y) \in \mathbb{S}_\infty$ *and is also uniformly convergent on the compact subsets of* $\mathbb{S}_\infty$*.*

1.6 Stickelberger series and Goss Zeta Function

Our goal is to show a link between the Zeta function and the Stickelberger series $\Theta_S(X)$ as we did in Section 1.3 for the Artin L-functions.
Let W_S be the subgroup of G_S generated by all the Artin symbols ϕ_ν with $\nu \notin S$ and let K be the fixed field of the topological closure of W_S. In Section 1.2 we have already observed that the element ϕ_ν is a topological generator of the decomposition group of ν in G_S, therefore the extension K/F is totally split at every prime $\nu \notin S$. From Tchebotarev density theorem we deduce that $K = F$ and so G_S is the topological closure of W_S.

Remark 1.6.1. We note that the group W_S is contained in the projection of the Weil group to G_S.

Lemma 1.6.2. *Let* λ *and* μ *be two distinct primes outside* S*. Then* $\phi_\mu \neq \phi_\lambda$*.*

Proof. Let $\mathbb{I}_F$ be the idéle group of F and H be the subgroup

$$F_\mu^\times \times \prod_{\substack{\nu\neq\mu \\ \nu\notin S}} O_\nu^\times \times \prod_{\nu\in S}\{1\}.$$

Furthermore let K be the class field of $F^\times H$ (as usual $F^\times$ is embedded diagonally in $\mathbb{I}_F$, while $F_\nu^\times$ is embedded via the map which sends the element x to the idéle whose coordinates

are all equal to 1, except for the one corresponding to the prime ν which is equal to x) and $\mathrm{rec} : \mathbb{I}_F \to Gal(K/F)$ be the map induced by the Artin map.
For every prime $\nu \notin S$ we have that $O_\nu^\times$ is contained in the kernel of rec and so the extension K/F is unramified at every prime outside S, in particular we have that K is contained is F_S. Moreover

$$\mathrm{Frob}_K \mu = \mathrm{rec}(\pi_\mu) = 1$$

$$\mathrm{Frob}_K \lambda = \mathrm{rec}(\pi_\lambda) \neq 1$$

since $\pi_\lambda \notin F^\times H$. Therefore the extension K/F is totally split at the prime μ, while λ is inert in K. From this observation it follows that the decomposition groups of μ and λ in G_S do not coincide and so to distinct primes correspond distinct Artin symbols. □

We denote with f the degree of the extension $F_{\infty,\mathbf{V}}/F_\infty$ (recall that f divides p^t by Proposition 1.4.4) and with $N : F_{\infty,\mathbf{V}}^\times \to F_\infty^\times$ the norm map. For any $y \in \mathbb{Z}_p$ and for any $\nu \notin S$ we put

$$\Psi_y(\phi_\nu) = N\left(\langle\nu\rangle_\infty^{-1}\right)^{y/f} .$$

Observe that the norm sends 1-units to 1-units and therefore it is possible to extract the f-th root without ambiguity. From the previous lemma it follows that the map Ψ_y is well defined on Artin symbols.

Lemma 1.6.3. *The map Ψ_y extends to a group homomorphism $\Psi_y : W_S \to \mathbb{C}_\infty^\times$.*

Proof. Since W_S is generated by the Artin symbols every $\sigma \in W_S$ may be written as $\sigma = \prod \phi_\nu^{n_\nu}$ for some integers n_ν. We put $\Psi_y(\sigma) = \prod \Psi_y(\nu)^{n_\nu}$. We have to check that this extension is well defined.
For every $\mathfrak{a} \in \mathcal{I}$ we denote with $\mathbf{i}_\mathfrak{a}$ the idéle whose ν-coordinate is equal to $\pi_\nu^{v_\nu(\mathfrak{a})}$, in particular it is equal to 1 if and only if ν does not belong to the support of $\mathfrak{a}$. We also observe that every idéle of this form is a finite idéle, i.e., it has component at ∞ equal to 1, since fractional ideals do not have ∞ in their support. Clearly the map $\varphi : \mathcal{I} \to \mathbb{I}_F,\ \mathfrak{a} \mapsto \mathbf{i}_\mathfrak{a}$ is an injective homomorphism. Let $\mathrm{rec}_S : \mathbb{I}_F \to G_S$ be the map induced by the Artin homomorphism and $\mathbf{i} \in \mathrm{Im}(\varphi) \cap \ker(\mathrm{rec}_S)$. Since $\ker(\mathrm{rec}_S) = F^\times \cdot \prod_{\nu \notin S} O_\nu^\times$ we have that $\mathbf{i}$ can be written as a product of an element $x \in F^\times$ and an idéle $\mathbf{o} \in \prod_{\nu \notin S} O_\nu^\times$. The component at infinity of $\mathbf{o}$ is equal to 1 since $\infty \in S$, and the same is true for $\mathbf{i}$ because of the definition of the function φ. This implies that x must be equal to 1 and $\mathbf{i} = \mathbf{o}$. Now for every prime $\nu \in S$ different from ∞ the component i_ν of $\mathbf{i}$ must be equal to 1 because it belongs to the kernel of rec_S and for every prime $\nu \notin S$ it must be equal to π_ν^n for some integer n and belongs to $O_\nu^\times$. The only possibility is that $\mathbf{i}$ is the unit.
We have proved that the composition of φ and rec_S is injective. The image of a fractional ideal $\mathfrak{a} = \prod_\nu \nu^{n_\nu}$ under this map is the element $\sigma = \prod_\nu \phi_\nu^{n_\nu}$ and so we have proved that if $\sigma \in W_S$ may be written in two different ways as a product of Artin symbols, then $\Psi_y(\sigma)$ does not depend on the chosen one. □

From the continuity of the norm map and of the root extraction map it follows immediately that $\Psi_y : W_S \to \mathbb{C}_\infty^\times$ is a continuous homomorphism. From what we have previously observed G_S is the topological closure of W_S and, since $\mathbb{C}_\infty$ is a complete topological space, Ψ_y may be extended in a unique way to a continuous homomorphism defined on G_S which we will still call Ψ_y by a little abuse of notation.

Let $\Psi_y : \mathbb{Z}[\![G_S]\!][\![X]\!] \to \mathbb{C}_\infty[\![X]\!]$ be the natural map induced by Ψ_y.

Theorem 1.6.4. *Let $\Theta_S(X)$ be the Stickelberger series and ζ_A the Goss Zeta function. Then, for every $s=(x,y)\in\mathbb{S}_\infty$ we have*

$$\Psi_y\left(\Theta_S(X)\right)(x)=\zeta_A(-s)\prod_{\substack{\nu\in S\\ \nu\neq\infty}}\left(1-\nu^s\right).$$

Proof. Let ν be a prime not in S, n be the order of $[\nu]$ in $\mathcal{I}/\mathcal{P}_+$ and α be a positive element such that $\nu^n=(\alpha)$. Due to what we have already observed $\langle\nu\rangle_\infty$ is the only 1-unit whose n-th power coincides with $\langle\alpha\rangle_\infty$. We write $n=p^h n'$ with $(p,n')=1$ and let u be the only 1-unit whose n'-th power coincides with $\langle\alpha\rangle_\infty$. Since n' is coprime with p, we have that u is in $U_1(\infty)\subset F_\infty$ and that $\langle\nu\rangle_\infty$ is a root of the polynomial $a(X)=X^{p^h}-u\in F_\infty[X]$. Let $b(X)$ be the minimal polynomial of $\langle\nu\rangle_\infty$ over F_∞. Since $\langle\nu\rangle_\infty$ is a root of $a(X)$ and is totally inseparable, it must be $a(X)=b(X)^{p^l}$ and $b(X)=X^{p^k}-v$, where l,k and v satisfy $h=k+l$ and $u=v^{p^l}$. If we denote with K the extension of F_∞ obtained by adding $\langle\nu\rangle_\infty$, we have that K/F_∞ has degree p^k, while $F_{\infty,\mathbf{v}}/K$ has degree f/p^k. Therefore

$$N\left(\langle\nu\rangle_\infty\right)=N_{K,F_\infty}(\langle\nu\rangle_\infty)^{f/p^k}=v^{f/p^k}=\langle\nu\rangle_\infty^f$$

(everything works for $p=2$ as well since in that case $N_{K,F_\infty}(\langle\nu\rangle_\infty)=-v=v$).
From this it follows that $\Psi_y(\phi_\nu)=\langle\nu\rangle_\infty^{-y}$ and that

$$\begin{aligned}\Psi_y\left(\Theta_S(X)\right) &= \prod_{\nu\notin S}\left(1-\Psi_y(\phi_\nu^{-1})X^{d_\nu}\right)^{-1}\\ &= \prod_{\nu\notin S}\left(1-\langle\nu\rangle_\infty^{y}X^{d_\nu}\right)^{-1}\end{aligned}$$

and so

$$\begin{aligned}\Psi_y\left(\Theta_S(X)\right)(x) &= \prod_{\nu\notin S}\left(1-\langle\nu\rangle_\infty^{y}x^{d_\nu}\right)^{-1}\\ &= \prod_{\nu\neq\infty}\left(1-\langle\nu\rangle_\infty^{y}x^{d_\nu}\right)^{-1}\prod_{\substack{\nu\in S\\ \nu\neq\infty}}\left(1-\langle\nu\rangle_\infty^{y}x^{d_\nu}\right)\\ &= \zeta_A(-s)\prod_{\substack{\nu\in S\\ \nu\neq\infty}}\left(1-\nu^s\right).\end{aligned}$$

□

1.7 A special case

In this section we will consider a special case and provide a link between the map Ψ_y of the previous section and the Artin reciprocity map. We assume that the class number of F is equal to 1 and we choose a prime at infinity of degree 1. The particularity of this case is that we can give an explicit description of the idéle class group (Theorem 1.7.1). One example of function field that satisfies these properties is the rational function fields $\mathbb{F}_q(T)$ which have been

studied in [ABBL]. Theorem 1.7.2 at the end of this section will show that Theorem 1.6.4 is a generalization of [ABBL, Theorem 3.8]
Under the assumptions on the class number and the degree of ∞ we have that A is a principal ideal domain and that $h^+(A) = 1$. We also have that the residue field of $\mathbb{F}_\infty$ coincides with the constant field $\mathbb{F}_q$ and so every element of $F^\times$ can be written in a unique way as product of a constant and a positive element of F. Moreover the condition $h^+(A) = 1$ implies that for every prime $\nu \neq \infty$ the corresponding prime ideal of A is principal and can be generated by a positive element $\pi_\nu \in F$. We choose an uniformizer at ∞ in the following way: first we fix a prime $\mathfrak{p} \neq \infty$ of degree 1, let $\pi_\mathfrak{p}$ be its unique positive generator and then put $\pi_\infty := \pi_\mathfrak{p}^{-1}$. Note that this uniformizer is positive and is an element of F [2]. Finally we observe that $F_\mathbf{V} = F$ and $F_{\infty,\mathbf{V}} = F_\infty$.

The Goss Zeta function is defined (as before) by

$$\zeta_A(s) = \sum_{a \in A_+} a^{-s} = \prod_{\nu \neq \infty} \left(1 - \pi_\nu^{-s}\right)^{-1} .$$

The image of the character Ψ_y is contained in $U_1(\infty) \subset F_\infty^\times$ and satisfies $\Psi_y(\phi_\nu) = \langle \pi_\nu \rangle_\infty^{-y}$.

We want to provide a slightly more explicit link between the character Ψ_1 and the Artin map. Let $\mathbb{I}_F$ be the Idéle group of F.

Theorem 1.7.1. *The idéle class group $\mathbb{I}_F/F^\times$ is isomorphic to the product*

$$F_{\infty,+}^\times \times \prod_{\nu \neq \infty} O_\nu^\times =: \mathcal{H},$$

where we have denoted with $F_{\infty,+}^\times$ the kernel of the map $\operatorname{sgn}\colon F_\infty^\times \to \mathbb{F}_q$.

Proof. First we prove that every idéle of $\mathcal{H}$ identifies a different equivalence class in $\mathbb{I}_F/F^\times$. Let $\mathbf{i} = (i_\infty, i_{\nu_1}, i_{\nu_2}, \dots)$ and $\mathbf{j} = (j_\infty, j_{\nu_1}, j_{\nu_2}, \dots)$ be two idéles in $\mathcal{H}$ and assume that they belong to the same equivalence class. Let $x \in F^\times$ be such that $\mathbf{i} = x\mathbf{j}$. For every $\nu \neq \infty$ we have that $i_\nu = xj_\nu$, but since both i_ν and j_ν are units in F_ν, it follows that $v_\nu(x) = 0$. Moreover from the product formula we have $v_\infty(x) = -\sum_{\nu \neq \infty} d_\nu v_\nu(x) = 0$ and so x is a constant. Finally since $i_\infty = xj_\infty$ and both i_∞ and j_∞ are positive, we deduce that $x = 1$.
To complete the proof (and provide an explicit isomorphism) we have to show that every equivalence class of $\mathbb{I}_F/F^\times$ contains an idéle of $\mathcal{H}$. Let $\mathbf{i} = (i_\infty, i_{\nu_1}, i_{\nu_2}, \dots)$ be any idéle and consider the element

$$x_\mathbf{i} = \operatorname{sgn}(i_\infty)^{-1} \prod_{\nu \neq \infty} \pi_\nu^{-v_\nu(i_\nu)} ,$$

which is in $F^\times$ because there are only finitely many ν with $v_\nu(i_\nu) \neq 0$. It is easy to check that the idéle $x\mathbf{i}$ is in $\mathcal{H}$ and that the map

$$\mathbb{I}_F/F^\times \longrightarrow \mathcal{H}$$

$$[\mathbf{i}] \mapsto x_\mathbf{i}\mathbf{i}$$

is an isomorphism. ∎

[2] If F is the rational function field $\mathbb{F}_q(T)$, here we are simply taking as uniformizer the element $\pi_\infty = 1/(T-\alpha)$, where α is any element of $\mathbb{F}_q$ and $\mathfrak{p} = (T - \alpha)$.

Let $C_F := \mathbb{I}_F/F^\times$ be the idéle class group of F and consider the composition of the Artin map with the projection $G_F^{\mathrm{ab}} := Gal(\overline{F}/F)^{\mathrm{ab}} \twoheadrightarrow G_S$. The map we obtain in this way has the group $O_S = \prod_{\nu \notin S} O_\nu^\times$ as its kernel and, therefore, the Artin map induces a continuous embedding

$$\mathrm{rec}_S : C_F/O_S \hookrightarrow G_S \,.$$

This embedding is not surjective since the group G_S is profinite, while the quotient C_F/O_S is not. If we denote with $\widehat{C_F/O_S}$ the profinite completion [3] of C_F/O_S, the map rec_S extends in a unique way to an isomorphism of topological groups

$$\widehat{\mathrm{rec}}_S : \widehat{C_F/O_S} \xrightarrow{\sim} G_S \,.$$

Using the isomorphism in Theorem 1.7.1, one has that the quotient C_F/O_S is isomorphic to

$$\pi_\infty^{\mathbb{Z}} \times U_1(\infty) \times \prod_{\substack{\nu \in S \\ \nu \neq \infty}} O_\nu^\times \,.$$

Its profinite completion is the group

$$\widehat{\langle \pi_\infty \rangle} \times U_1(\infty) \times \prod_{\substack{\nu \in S \\ \nu \neq \infty}} O_\nu^\times$$

where $\widehat{\langle \pi_\infty \rangle} \simeq \widehat{\mathbb{Z}}$.

Theorem 1.7.2. *We denote with π_S the canonical projection $\widehat{C_F/O_S} \twoheadrightarrow U_1(\infty)$. Then the following diagram is commutative*

$$\begin{tikzcd} G_S \arrow[rr, "\Psi_1"] \arrow[dr, "\widehat{\mathrm{rec}}_S^{-1}"'] & & U_1(\infty) \\ & \widehat{C_F/O_S} \arrow[ur, two heads, "\pi_S"'] & \end{tikzcd} .$$

In general for every $y \in \mathbb{Z}_p$ we have the following commutative diagram

$$\begin{tikzcd} G_S \arrow[r, "\Psi_y"] \arrow[d, "\widehat{\mathrm{rec}}_S^{-1}"'] & U_1(\infty) \\ \widehat{C_F/O_S} \arrow[r, two heads, "\pi_S"'] & U_1(\infty) \arrow[u, "y"'] \end{tikzcd}$$

where y denotes the raise-to-the-power y map.

[3] For a topological group G, the profinite completion $\widehat{G}$ is the inverse limit of its finite quotients with respect to the natural projection maps.

Proof. We have already noted that G_S is the topological closure of the group W_S, generated by the Artin symbols. Since all the maps in the theorem are continuous, it is enough to show that $\Psi_1(\phi_\nu) = \pi \circ \widehat{rec}_S^{-1}(\phi_\nu)$ for every $\nu \notin S$.
We recall that the global Artin map is the product on each idéle component of the local Artin maps and that the local Artin map rec_ν sends π_ν to ϕ_ν, since the extension F_S/F is unramified at ν. Let $\mathbf{i}_\nu$ be the idéle whose ν-coordinate is equal to π_ν and whose μ-coordinates for $\mu \neq \nu$ are all equal to 1. The image of the equivalence class $[\mathbf{i}_\nu] \in \mathbb{I}_F/F^\times$ under the map rec_S is equal to ϕ_ν. As noted above, the hypothesis $h^+(A) = 1$ implies that we have a positive generator $\pi_\nu \in F$ for ν, hence

$$\phi_\nu = \mathrm{rec}_S\left([\mathbf{i}_\nu]\right) = \mathrm{rec}_S\left([\pi_\nu^{-1}\mathbf{i}_\nu]\right)$$

and, recalling that the idéle $\pi_\nu^{-1}\mathbf{i}_\nu$ belongs to $\mathcal{H}$, we obtain that

$$\pi_S \circ \widehat{rec}_S^{-1}(\phi_\nu) = \langle \pi_\nu^{-1} \rangle_\infty = \Psi_1(\phi_\nu)\,.$$

For a general $y \in \mathbb{Z}_p$ the thesis is obtained simply by observing that for every Artin symbol ϕ_ν we have $\Psi_y(\phi_\nu) = \Psi_1(\phi_\nu)^y$ by definition. □

1.8 ν-adic Zeta Function

In this section we will define a ν-adic analogue of the Goss Zeta function.
Fix a place ν different from ∞ and let $F_\nu, \mathbb{C}_\nu, \mathbb{F}_\nu$ and π_ν be the ν-adic versions of the objects defined for the place ∞. Fix an algebraic closure $\overline{F}$ of F and let $\sigma : \overline{F} \hookrightarrow \mathbb{C}_\nu$ be an F-embedding. All the objects that we shall define later on depend on σ, but we will omit this dependency to simplify the notations.
Let $F_{\nu,\mathbf{V}} := \sigma(F_\mathbf{V})F_\nu$, which will play the role of the field $F_{\infty,\mathbf{V}}$, and let $\mathbb{S}_\nu = \mathbb{C}_\nu^\times \times \mathbb{Z}_p \times \mathbb{Z}/|\mathbb{F}_\nu^\times|$, which is a subgroup of the group of $\mathbb{C}_\nu^\times$-valued characters on $F_\nu^\times$. As done in Proposition 1.4.4, we can prove that $F_{\nu,\mathbf{V}}$ is a totally inseparable extension of F_ν with degree less than or equal to p^t (the maximal power of p that divides $h^+(A)$). Since this extension is totally inseparable the residue field of $F_{\nu,\mathbf{V}}$ is still $\mathbb{F}_\nu$ and so the cyclic group $\mathbb{Z}/|\mathbb{F}_\nu^\times|$ acts on the multiplicative group of this residue field by raise to the power. Now we take an element $s_\nu = (x, y, j) \in \mathbb{S}_\nu$ and we define the exponential $I^{s_\nu} \in \mathbb{C}_\nu^\times$ for every fractional ideal $I \in \mathcal{I}$ coprime with ν. We recall that the element $I^{s_1} \in F_\mathbf{V}$ is a root of the polynomial $X^e - \alpha$ where α is the unique positive generator of I^e (here s_1 is the element $(\pi_*^{-1}, 1) \in \mathbb{S}_\infty$) and so the valuation at ν of I^{s_1} is equal to zero. This implies that the element $\sigma(I^{s_1})$ is a unit in $F_{\nu,\mathbf{V}}$ and so can be written uniquely as a product

$$\sigma(I^{s_1}) = \omega(I)\langle I \rangle_\nu$$

for some $\omega(I) \in \mathbb{F}_\nu^\times$ and $\langle I \rangle_\nu$ a 1-unit of $F_{\nu,\mathbf{V}}$. With the notation above it is easy to check that the map which sends I to $\omega(I)$ is a group homomorphism $\omega : \mathcal{I}_\nu \to \mathbb{F}_\nu^\times$ defined on the group $\mathcal{I}_\nu$ of the fractional ideals coprime with ν.
Finally if we take $s_\nu = (x, y, j) \in \mathbb{S}_\nu$ and I coprime with ν we can define

$$I^{s_\nu} = x^{\deg(I)}\omega(I)^j \langle I \rangle_\nu^y.$$

We can embed $\mathbb{Z}$ in $\mathbb{S}_\nu$ via the map $j \in \mathbb{Z} \mapsto s_{\nu,j} = (1, j, j) \in \mathbb{S}_\nu$. One can show that this ν-adic exponential satisfies the following properties (analogous to the ones of the exponential defined is Section 1.4)

Proposition 1.8.1. *For every $I, J \in \mathcal{I}$ coprime with ν, one has*

- $I^{s_\nu + t_\nu} = I^{s_\nu} I^{t_\nu}$ *for every $s_\nu, t_\nu \in \mathbb{S}_\nu$.*
- $(IJ)^{s_\nu} = I^{s_\nu} J^{s_\nu}$ *for every $s_\nu \in \mathbb{S}_\nu$.*
- $(I^{s_{\nu,i}})^{s_{\nu,j}} = I^{s_{\nu,ij}}$ *for every $i, j \in \mathbb{Z}$.*
- $I^{s_{\nu,j}} = \sigma(I^{s_j})$ *for every $j \in \mathbb{Z}$. In particular $I^{s_{\nu,j}}$ is algebraic over F.*

Proof. The proofs of the first 3 properties are similar to the analogous properties of the exponential defined in Section 1.4 and so are left to the reader. For the last one we have

$$I^{s_{\nu,j}} = \omega(I)^j \langle I \rangle_\nu^j = \sigma\left(I^{s_1}\right)^j = \sigma(I^{s_j}).$$

□

Definition 1.8.2. The *ν-adic Goss Zeta function* is defined by

$$\zeta_\nu(s_\nu) = \sum_{\substack{\mathfrak{a} \in \mathcal{I} \\ \mathfrak{a} \geq 0 \\ \nu \nmid \mathfrak{a}}} \mathfrak{a}^{-s_\nu} = \prod_{\mathfrak{p} \neq \nu, \infty} \left(1 - \mathfrak{p}^{-s_\nu}\right)^{-1}.$$

Now we want to obtain the function $\zeta_\nu(s_\nu)$ as the image of the Sickelberger series under an appropriate map. This will be done for all the primes $\nu \neq \infty$ which are in S, so for the rest of this section we will assume that $\nu \in S$. Let f_ν be the degree of the extension $F_{\nu,\mathbf{V}}/F_\nu$ and denote by $N_\nu : F_{\nu,\mathbf{V}}^\times \to F_\nu^\times$ the norm map. We fix $(y, j) \in \mathbb{Z}_p \times \mathbb{Z}/|\mathbb{F}_\nu^\times|$ and for every $\mathfrak{p} \notin S$ we put

$$\Psi_{y,j}(\phi_\mathfrak{p}) = N_\nu(\langle \mathfrak{p} \rangle_\nu^{-1})^{y/f_\nu} \omega(\mathfrak{p})^{-j}.$$

In Lemma 1.6.2 we have proved that different primes correspond to different Artin symbols and so $\Psi_{y,j}(\phi_\mathfrak{p})$ is well defined.

Lemma 1.8.3. *For every $(y, j) \in \mathbb{Z}_p \times \mathbb{Z}/|\mathbb{F}_\nu^\times|$ the map $\Psi_{y,j}$ extends to a continuous ring homomorphism $\mathbb{Z}[\![G_S]\!][\![X]\!] \to \mathbb{C}_\nu[\![X]\!]$.*

Proof. Let $\tau = \prod \phi_\mathfrak{p}^{n_\mathfrak{p}} \in W_S$. By the proof of Lemma 1.6.3 we have that the map that sends $\prod \mathfrak{p}^{n_\mathfrak{p}}$ to $\prod \phi_\mathfrak{p}^{n_\mathfrak{p}}$ is injective on the set of fractional ideals with support outside of S. Then we can put $\Psi_{y,j}(\tau) := \prod \Psi_{y,j}(\phi_\mathfrak{p})^{n_\mathfrak{p}}$ without ambiguity. Now since W_S is dense in G_S we have that $\Psi_{y,j}$ extends in a unique way as a continuous map defined over G_S because $\mathbb{C}_\nu$ is complete. □

We have the following

Theorem 1.8.4. *Let $\Theta_S(X)$ be the Stickelberger series and ζ_ν the ν-adic Goss Zeta function. If we assume $\nu \in S$ then for every $s_\nu = (x, y, j) \in \mathbb{S}_\nu$ we have*

$$\Psi_{y,j}\left(\Theta_S(X)\right)(x) = \zeta_\nu(-s_\nu) \prod_{\substack{\mathfrak{p} \in S \\ \mathfrak{p} \neq \nu, \infty}} (1 - \mathfrak{p}^{s_\nu}).$$

Proof. Le $\mathfrak{p}$ be a prime not in S, n the exact order of $[\mathfrak{p}]$ in $\mathcal{I}/\mathcal{P}_+$ and α a positive element such that $\mathfrak{p}^n = (\alpha)$. We have that $(\mathfrak{p}^n)^{s_1} = (\alpha)^{s_1} = \alpha$ and so

$$\omega(\mathfrak{p})^n \langle \mathfrak{p} \rangle_\nu^n = \sigma(\mathfrak{p}^{s_1})^n = \sigma(\alpha)^{s_1} = \alpha\,,$$

which implies that $\langle \mathfrak{p} \rangle_\nu^n = \langle \alpha \rangle_\nu$.
From now on with proceed like in Theorem 1.6.4. Write $n = p^h n'$ with $(p, n') = 1$ and let u be the unique 1-unit whose n' power coincide with $\langle \alpha \rangle_\nu$. Since n' is coprime with p, we have that u is in F_ν and that $\langle \mathfrak{p} \rangle_\nu$ is a root of the polynomial $a(X) = X^{p^h} - u \in F_\nu[X]$. Let $b(X)$ be the minimal polynomial of $\langle \mathfrak{p} \rangle_\nu$ over F_ν. Since $\langle \mathfrak{p} \rangle_\nu$ is a root of $a(X)$ and it is a totally inseparable element, it must be $a(X) = b(X)^{p^l}$ and $b(X) = X^{p^k} - v$, where l, k and v satisfy $h = k + l$ and $u = v^{p^l}$. If we denote with K the extension of F_ν obtained by adding $\langle \mathfrak{p} \rangle_\nu$, we have that K/F_ν is an extension of degree p^k and $F_{\nu,\mathbf{V}}/K$ has degree f_ν/p^k. Therefore

$$N_\nu\left(\langle \mathfrak{p} \rangle_\nu\right) = N_{K,F_\infty}(\langle \mathfrak{p} \rangle_\nu)^{f_\nu/p^k} = v^{f_\nu/p^k} = \langle \mathfrak{p} \rangle_\nu^{f_\nu}\,.$$

From this we obtain that $\Psi_{y,j}(\phi_\mathfrak{p}) = \langle \mathfrak{p} \rangle_\nu^{-y} \omega(\mathfrak{p})^{-j}$ and

$$\begin{aligned}\Psi_{y,j}\left(\Theta_S(X)\right) &= \prod_{\mathfrak{p} \notin S} \left(1 - \Psi_{y,j}(\phi_\mathfrak{p}^{-1}) X^{d_\mathfrak{p}}\right)^{-1} \\ &= \prod_{\mathfrak{p} \notin S} \left(1 - \langle \mathfrak{p} \rangle_\nu^y \omega(\mathfrak{p})^j X^{d_\mathfrak{p}}\right)^{-1}\end{aligned}$$

and so

$$\begin{aligned}\Psi_{y,j}\left(\Theta_S(X)\right)(x) &= \prod_{\mathfrak{p} \notin S} \left(1 - \langle \mathfrak{p} \rangle_\nu^y \omega(\mathfrak{p})^j x^{d_\mathfrak{p}}\right)^{-1} \\ &= \prod_{\mathfrak{p} \neq \nu,\infty} \left(1 - \langle \mathfrak{p} \rangle_\nu^y \omega(\mathfrak{p})^j x^{d_\mathfrak{p}}\right)^{-1} \prod_{\substack{\mathfrak{p} \in S \\ \mathfrak{p} \neq \nu,\infty}} \left(1 - \langle \mathfrak{p} \rangle_\nu^y \omega(\mathfrak{p})^j x^{d_\mathfrak{p}}\right) \\ &= \zeta_\nu(-s_\nu) \prod_{\substack{\mathfrak{p} \in S \\ \mathfrak{p} \neq \nu,\infty}} \left(1 - \mathfrak{p}^{s_\nu}\right)\,.\end{aligned}$$

□

1.9 Properties of ν-adic Zeta Function

In this section we will investigate some properties of the ν-adic Zeta function and in particular its values at integers.
For each pair of non negative integers j and n, we define

$$S_n(j) = \sum_{\substack{\mathfrak{a} \geq 0 \\ \deg \mathfrak{a} = n}} \mathfrak{a}^{s_j}$$

and the power series

$$Z(X, j) = \sum_{n \geq 0} S_n(j) X^n$$

whose coefficients lie in $F_\mathbf{V}$.

Lemma 1.9.1. *The series $Z(X,j)$ is actually a polynomial of degree less than or equal to $d_\infty + g - 1 + \lfloor \frac{j}{q-1} \rfloor$.*

Proof. Fix a non negative integer j. We prove that when n is bigger than $d_\infty + g - 1 + \lfloor \frac{j}{q-1} \rfloor$, $S_n(j)$ is equal to 0. To prove this we shall apply the same tools used to prove the convergence of the Goss Zeta function.
Let C_h for $h = 1, \ldots, e$ be the classes of $\mathcal{I}/\mathcal{P}_+$ and, for each h, fix a representative $\mathfrak{a}_h \in C_h$. Now define

$$S_n(C_h, j) = \sum_{\substack{\mathfrak{a}\geq 0 \\ \deg(\mathfrak{a})=n \\ \mathfrak{a}\in C_h}} \mathfrak{a}^{s_j} = \mathfrak{a}_h^{s_j} \cdot \sum_{\substack{\alpha\in\mathcal{P}_+ \\ \alpha\mathfrak{a}_h\geq 0 \\ \deg(\alpha)=n-\deg(\mathfrak{a}_h)}} \alpha^j \, .$$

Clearly $S_n(j)$ is the sum of the $S_n(C_h, j)$ (as h varies) so it is enough to show that each of them is equal to 0. Fix an index h and let $n_h = n - \deg(\mathfrak{a}_h)$. We will consider only indices h such that d_∞ divides n_h, otherwise the thesis is trivial. Just like in Section 1.5.1 we denote with X the set of positive elements α such that $\alpha\mathfrak{a}_h$ is an integral ideal and whose degree is equal to n_h, and with $\mathcal{L}(D_1)$ the Riemann-Roch space associated to the divisor $D_1 = \mathrm{div}(\mathfrak{a}_h) + (n_h/d_\infty - 1)\infty$. The additive group of $\mathcal{L}(D_1)$ acts by traslation on X and the action is free, so we can decompose X as the union of its orbits under this action: $X = \bigcup_{l=1}^{t} X_l$. Fix a representative $x_l \in X_l$ for each orbit. Then we have

$$S_n(C_h, j) = \mathfrak{a}_h^{s_j} \sum_{l=1}^{t} \sum_{u\in\mathcal{L}(D_1)} (u + x_l)^j = \mathfrak{a}_h^{s_j} \sum_{l=1}^{t} x_l^j \sum_{w\in x_l^{-1}\mathcal{L}(D_1)} (w+1)^j \, .$$

Observe that $v_\infty(x_l) = -n_h/d_\infty$ and $v_\infty(u) \geq 1 - n_h/d_\infty$. This implies that $v_\infty(w)$ is positive for every $w \in x_l^{-1}\mathcal{L}(D_1)$ and that the vector space $x_l^{-1}\mathcal{L}(D_1)$ satisfies the hypothesis of Lemma 1.5.4 part (a). Therefore the inner sum is zero when $j < (q-1)\ell_1 = (q-1)(n-g+1)$. $\square$

The polynomials $Z(X,j)$ are strictly related to the special values of the Goss Zeta function since we have that $Z(1,j) = \zeta_A(-s_j)$ for any $j \in \mathbb{N}$.

We shall also need some ν-adic version of this polynomials, which will be used to interpolate the special values of the ν-adic Goss Zeta function.

Definition 1.9.2. The *ν-adic L-series* is defined by

$$L_\nu(X, y, \omega^i) = \sum_{n\geq 0} \left(\sum_{\substack{\mathfrak{a}\geq 0\, ,\ \nu\nmid\mathfrak{a} \\ \deg(\mathfrak{a})=n}} \omega(\mathfrak{a})^i \langle\mathfrak{a}\rangle_\nu^y \right) X^n \, ,$$

for any $y \in \mathbb{Z}_p$ and $i \in \mathbb{Z}/|\mathbb{F}_\nu^*|$.

From the definition one immediately has

Proposition 1.9.3. *For every $s_\nu = (x, y, i) \in \mathbb{S}_\nu$ we have*

$$L_\nu(x, y, \omega^i) = \zeta_\nu(-s_\nu). \tag{1.9}$$

Proof. We recall that for a fractional ideal $\mathfrak{a}$ coprime with ν we have $\mathfrak{a}^{s_\nu} = \omega(\mathfrak{a})^i \langle \mathfrak{a} \rangle_\nu^y x^{\deg \mathfrak{a}}$ and so

$$\begin{aligned} L_\nu(x, y, \omega^i) &= \sum_{n \geq 0} \left(\sum_{\substack{\mathfrak{a} \geq 0\,,\ \nu \nmid \mathfrak{a} \\ \deg(\mathfrak{a}) = n}} \omega(\mathfrak{a})^i \langle \mathfrak{a} \rangle_\nu^y x^n \right) \\ &= \sum_{n \geq 0} \left(\sum_{\substack{\mathfrak{a} \geq 0\,,\ \nu \nmid \mathfrak{a} \\ \deg(\mathfrak{a}) = n}} \mathfrak{a}^{s_\nu} \right) \\ &= \sum_{\substack{\mathfrak{a} \in \mathcal{I} \\ \mathfrak{a} \geq 0\,,\ \nu \nmid \mathfrak{a}}} \mathfrak{a}^{s_\nu} \\ &= \zeta_\nu(-s_\nu). \end{aligned}$$

□

The following theorem provides a link between this power series and the polynomial $Z(X, j)$ for some particular values of i and j. It also shows that for these values of i and j the series $L_\nu(X, y, \omega^i)$ is actually a polynomial.

Theorem 1.9.4. *Assume that $\nu \in S$.*

(a) Let i and j be two non negative integers, such that $i \equiv j \pmod{q^{d_\nu} - 1}$. Then

$$L_\nu(X, j, \omega^i) = Z(X, j)(1 - \nu^{s_j} X^{d_\nu})\,.$$

In particular $L_\nu(X, j, \omega^i)$ in a polynomial.

(b) For every $y \in \mathbb{Z}_p$ we have that

$$L_\nu(X, y, \omega^i) \equiv Z(X, i) \pmod{\overline{\nu}}\,,$$

where $\overline{\nu}$ denotes any prime of $F_{\mathbf{V}}$ which lies above ν.

Proof. (a) Fix a prime $\mathfrak{p}$ different from ν and ∞ and consider $f_\mathfrak{p}(X) = 1 - \omega(\mathfrak{p})^i \langle \mathfrak{p} \rangle_\nu^j X^{d_\mathfrak{p}}$ as an element of $F_{\nu,\mathbf{V}}[\![X]\!]$. Clearly $f_\mathfrak{p}(X)$ is invertible since $f_\mathfrak{p}(0) = 1$ and its inverse is given by

$$f_\mathfrak{p}(X)^{-1} = \sum_{n \geq 0} \left(\omega(\mathfrak{p})^i \langle \mathfrak{p} \rangle_\nu^j X^{d_\mathfrak{p}} \right)^n \,.$$

Let

$$f_n^{-1}(X) := \prod_{\deg(\mathfrak{p}) = n} f_\mathfrak{p}(X)^{-1}$$

then the product

$$\prod_{n \geq 0} f_n^{-1}(X)$$

is convergent with respect to the X-adic topology.
The map that sends every ideal I coprime with ν to the 1-unit $\langle I\rangle_\nu$ is multiplicative and the same is true for the map ω defined on this set of ideals. This fact, together with the unique factorization of ideals in A, allows us to conclude that the limit of the product above is equal to $L_\nu(X,j,\omega^i)$, i.e.,

$$L_\nu(X,j,\omega^i) = \prod_{\mathfrak{p}\neq\nu,\infty} \left(1-\omega(\mathfrak{p})^i\langle\mathfrak{p}\rangle_\nu^j X^{d_\mathfrak{p}}\right)^{-1} . \tag{1.10}$$

In the proof of Theorem 1.8.4 we have seen that this product is equal to

$$\Psi_{j,i}\left(\Theta_S(X)\right)\cdot \prod_{\substack{\mathfrak{p}\in S\\ \mathfrak{p}\neq\infty,\nu}} \left(1-\omega(\mathfrak{p})^i\langle\mathfrak{p}\rangle_\nu^j X^{d_\mathfrak{p}}\right)^{-1} .$$

Since $i\equiv j \pmod{q^{d_\nu}-1}$, we have that $\omega(\mathfrak{p})^i=\omega(\mathfrak{p})^j$ and so

$$\Psi_{j,i}(\phi_\mathfrak{p}^{-1}) = \omega(\mathfrak{p})^i\langle\mathfrak{p}\rangle_\nu^j = \omega(\mathfrak{p})^j\langle\mathfrak{p}\rangle_\nu^j = \sigma\left(\mathfrak{p}^{s_1}\right)^j .$$

We recall that $\mathfrak{p}^{s_1}$ lies in $F_{\mathbf{V}}$ and that σ is the identity on this field, so

$$\Psi_{j,i}(\phi_\mathfrak{p}^{-1}) = \mathfrak{p}^{s_j} = \pi_*^{-jd_\mathfrak{p}}\langle\mathfrak{p}\rangle_\infty^j = \Psi_j(\phi_\mathfrak{p}^{-1})\pi_*^{-jd_\mathfrak{p}} .$$

From this equality if follows that

$$\begin{aligned}\Psi_{j,i}\left(\Theta_S(X)\right) &= \prod_{\mathfrak{p}\notin S}\left(1-\Psi_{j,i}(\phi_\mathfrak{p}^{-1})X^{d_\mathfrak{p}}\right)^{-1}\\ &= \prod_{\mathfrak{p}\notin S}\left(1-\Psi_j(\phi_\mathfrak{p}^{-1})\pi_*^{-jd_\mathfrak{p}}X^{d_\mathfrak{p}}\right)^{-1} \\ &= \Psi_j\left(\Theta_S(\pi_*^{-j}X)\right) .\end{aligned} \tag{1.11}$$

We have proved that

$$L_\nu(X,j,\omega^i) = \Psi_j\left(\Theta_S(\pi_*^{-j}X)\right)\cdot \prod_{\substack{\mathfrak{p}\in S\\ \mathfrak{p}\neq\infty,\nu}} \left(1-\mathfrak{p}^{s_j}X^{d_\mathfrak{p}}\right)^{-1} . \tag{1.12}$$

Since

$$Z(X,j) = \sum_{\substack{\mathfrak{a}\in\mathcal{I}\\ \mathfrak{a}\geq 0}} \mathfrak{a}^{s_j}X^{\deg\mathfrak{a}} = \sum_{n\geq 0}\left(\sum_{\substack{\mathfrak{a}\geq 0\\ \deg\mathfrak{a}=n}}\langle\mathfrak{a}\rangle_\infty^j\right)(\pi_*^{-j}X)^n ,$$

the same arguments used to obtain (1.10), applied now to $Z(X,j)$, allow us to write

$$Z(X,j) = \prod_{\mathfrak{p}\neq\infty}\left(1-\langle\mathfrak{p}\rangle_\infty^j(\pi_*^{-j}X)^{d_\mathfrak{p}}\right)^{-1} . \tag{1.13}$$

The product (1.13) can be written as

$$\Psi_j\left(\Theta_S(\pi_*^{-j}X)\right)\cdot\prod_{\substack{\mathfrak{p}\in S\\ \mathfrak{p}\neq\infty}}\left(1-\mathfrak{p}^{s_j}X^{d_{\mathfrak{p}}}\right)^{-1}.$$

Thus, joining together (1.12) and (1.13), we obtain that

$$L_\nu(X,j,\omega^i)=Z(X,j)(1-\nu^{s_j}X^{d_\nu}).$$

(b) For every ideal $\mathfrak{a}$ we have that

$$\langle\mathfrak{a}\rangle_\nu^y\equiv 1\equiv\langle\mathfrak{a}\rangle_\nu^i \pmod{\overline{\nu}}.$$

Hence

$$\begin{aligned}
L_\nu(X,y,\omega^i) &= \sum_{n\geq 0}\left(\sum_{\substack{\mathfrak{a}\geq 0,\ \nu\nmid\mathfrak{a}\\ \deg\mathfrak{a}=n}}\omega(\mathfrak{a})^i\langle\mathfrak{a}\rangle_\nu^y\right)X^n\\
&\equiv \sum_{n\geq 0}\left(\sum_{\substack{\mathfrak{a}\geq 0,\ \nu\nmid\mathfrak{a}\\ \deg\mathfrak{a}=n}}\omega(\mathfrak{a})^i\langle\mathfrak{a}\rangle_\nu^i\right)X^n \pmod{\overline{\nu}}\\
&= L_\nu(X,i,\omega^i) \pmod{\overline{\nu}}\\
&= Z(X,i)(1-\nu^{s_i}X^{d_\nu}) \pmod{\overline{\nu}}\\
&\equiv Z(X,i) \pmod{\overline{\nu}}.
\end{aligned}$$

□

Chapter 2

Stickelberger series and class groups

2.1 Introduction

In this chapter we will apply the results of Chapter 1 to the study of class group of degree zero divisors. We will first introduce the Hayes modules and use them to build an Iwasawa tower. Then we will study the behaviour of the p-part of the class groups of degree zero divisors of the fields in the tower and of their inverse limit taken with respect to the norm maps. The study of these objects is one of principal part of Iwasawa Theory. The Iwasawa Main Conjecture relates the characteristic ideals of Iwasawa modules to the ν-adic L-functions, thus providing a link between the algebraic and the analytic side of the theory. A version of the Main Conjecture for $\mathbb{Z}_p^d$-extensions of global function fields was proved by R. Crew in [Cre] using mainly geometric tools and, later, with a different approach by Burns in [Bur], with the contribute of Kueh, Lai and Tan ([KLT]). The Iwasawa extensions we will consider in this chapter are $\mathbb{Z}_p^\infty$-extensions, thus the inverse limit of the p-part of the class groups turns out to be a module over a non-Noetherian Iwasawa algebra and, unlike the classical case, we do not have a structure theorem for finitely generated torsion modules like [Was, Theorem 13.12] and so we are forced to study its Fitting ideal instead of the characteristic ideal which cannot be defined in this setting. For an alternative approach to this problem the reader may look, for example, at [BBL], where the authors study $\mathbb{Z}_p^\infty$-extensions of global function fields using $\mathbb{Z}_p^d$-filtrations.

Through this chapter we will assume that $d_\infty = 1$ and that p does not divide $h^0(F)$. Under these assumptions we have the following simplifications:

- The residue field $\mathbb{F}_\infty$ coincides with the field of costants $\mathbb{F}_q$.
- Every principal ideal admits a positive generator.
- The class number of the ring of integers A is equal to $h^0(F)$.
- The field $F_{\infty,\mathbf{V}}$ coincides with the field F_∞.
- For every $a \in F$: $\deg(a) = -v_\infty(a)$.
- $\pi_* = \pi_\infty$.

2.2 Hayes extensions

Let H_A be the Hilbert class field of A, i.e., the maximal abelian extension of F which is unramified at every prime and totally split at ∞. Obviously we have that H_A is a subfield of F_S for any choice of the set S. Since the prime ∞ has degree 1 we have that the constant field of H_A is $\mathbb{F}_q$. It is a well known fact from class field theory that $Pic(A) \simeq Gal(H_A/F)$ and the isomorphism is provided by the Artin reciprocity map. In particular the class of a fractional ideal $\mathfrak{a}$ is sent to to its Frobenius in $Gal(H_A/F)$ and, in case the support of $\mathfrak{a}$ is disjoint from S, this is simply the restriction of its Artin symbol $\phi_{\mathfrak{a}} \in G_S$.

Definition 2.2.1. We denote with $H_A\{\tau\}$ the ring of skew-polynomials in the variable τ with coefficient in the field H_A. A *Hayes module* is an homomorphism of $\mathbb{F}_q$-algebras $\Phi : A \to H_A\{\tau\}$, such that:

(a) the image of A is not contained in H_A;

(b) for every $a \in A$ the coefficient of degree 0 of $\Phi_a := \Phi(a)$ is equal to a;

(c) for every $a \in A$ the degree of Φ_a, seen as a polynomial in τ, is equal to $\deg(a)$ (i.e., Φ *has rank 1*);

(d) for every $a \in A$, the leading coefficient of Φ_a is $\operatorname{sgn}(a)$ (i.e., Φ is *sgn-normalized*).

For details on the Hayes module the reader may refer to ([Gos1] Chapter 7, [Hay] and [Shu]).

Remark 2.2.2. In the general theory a Hayes module is defined over the narrow class field H_A^+, i.e., the abelian extension of F that is naturally isomorphic to $Pic^+(A) := \mathcal{I}/\mathcal{P}_+$ by class field theory. In our context we have a simplification because the fields H_A and H_A^+ coincide since the degree of the prime at infinity is equal to 1.

We use a Hayes module to define an action of A on the algebraic closure of F. For every $a \in A$ and $x \in \overline{F}$ we put $a \cdot x := \Phi_a(x)$. This action defines a structure of A-module on $\overline{F}$. If we take an integral ideal $\mathfrak{a}$ of A we can consider the left ideal of $H_A\{\tau\}$ generated by all the elements Φ_a with $a \in \mathfrak{a}$. Since $H_A\{\tau\}$ is right-euclidean, we have that every left ideal is principal. We denote by $\Phi_{\mathfrak{a}}$ the unique monic generator of this ideal.

Definition 2.2.3. An element $x \in \overline{F}$ is said to be *of* $\mathfrak{a}$*-torsion* if $\Phi_{\mathfrak{a}}(x) = 0$. Since $\Phi_{\mathfrak{a}}$ is a non zero polynomial (for $\mathfrak{a} \neq 0$) we have that the set $\Phi[\mathfrak{a}]$ of all elements of $\mathfrak{a}$-torsion is finite.

Proposition 2.2.4. *Let $\mathfrak{a} \neq 0$ be an integral ideal and let $\Phi[\mathfrak{a}]$ be the $\mathfrak{a}$-torsion of Φ. Then*

(a) $\Phi[\mathfrak{a}]$ is an $A/\mathfrak{a}$-module;

(b) $\Phi[\mathfrak{a}] \simeq A/\mathfrak{a}$;

(c) if $\lambda_{\mathfrak{a}}$ is a generator of $\Phi[\mathfrak{a}]$ as $A/\mathfrak{a}$-module, then every other generator is of the form $\Phi_b(\lambda_{\mathfrak{a}})$ for some $b \in (A/\mathfrak{a})^\times$.

We use the $\mathfrak{a}$-torsion of the Hayes module Φ to define extensions of the field F analogous to the cyclotomic extension of $\mathbb{Q}$. We denote by $F(\mathfrak{a})$ the extension of H_A generated by $\Phi[\mathfrak{a}]$, i.e., $F(\mathfrak{a}) := H_A(\Phi[\mathfrak{a}])$. Equivalently this extension can be obtained simply by adding an $A/\mathfrak{a}$-generator of $\Phi[\mathfrak{a}]$ to H_A.
The following theorem summarizes the properties of the Hayes extension contained in [Gos1] Proposition 7.5.4, Corollary 7.5.6, Proposition 7.5.8 and Proposition 7.5.18.

Theorem 2.2.5. *The following hold*

(a) $F(\mathfrak{a})$ *is a geometric, abelian Galois extension of* F*;*

(b) $Gal\,(F(\mathfrak{a})/H_A) \simeq (A/\mathfrak{a})^\times$ *via an isomorphism which sends* $a \in A$ *to the element* $\sigma_a \in Gal\,(F(\mathfrak{a})/H_A)$ *which verifies* $\sigma_a(\lambda) = \Phi_a(\lambda)$ *for every* $\lambda \in \Phi[\mathfrak{a}]$*;*

(c) the only ramified primes in $F(\mathfrak{a})/H_A$ *are the primes of* H_A *dividing* $\mathfrak{a}$ *and* ∞*;*

(d) the inertia group of ∞ *coincides with its decomposition group and is isomorphic to to* $\mathbb{F}_q^\times$ *via the isomorphism in (b);*

(e) if $\mathfrak{p}^n$ *is the exact power of* $\mathfrak{p}$ *dividing* $\mathfrak{a}$*, then the inertia group of* $\mathfrak{p}$ *is isomorphic to* $(A/\mathfrak{p}^n)^\times$ *via the isomorphism in (b);*

(f) if I *is an ideal of* A *coprime with* $\mathfrak{a}$ *and* $\sigma_I \in Gal\,(F(\mathfrak{a})/F)$ *is its Artin symbol, then we have that* $\sigma_I(\lambda) = \Phi_I(\lambda)$ *for every* $\lambda \in \Phi[\mathfrak{a}]$*.*

Now we fix a prime $\mathfrak{p}$ for the rest of the section and we set $S = \{\mathfrak{p}, \infty\}$. For each non negative integer n we put $F_n := F(\mathfrak{p}^{n+1})$ and $G_n = Gal(F_n/F)$. From point (b) of Theorem 2.2.5 we deduce that F_n/F_0 is a p-extension and that F_0/F is an extension of order $h^0(F)(q^{d_\mathfrak{p}} - 1)$. Since we have assumed $h^0(F)$ coprime with p we can decompose G_n as the product $G_0 \times \Gamma_n$ where $\Gamma_n = Gal(F_n/F_0)$ is a p-group and G_0 is the part of order coprime with p.
The fields F_n form an Iwasawa tower: if we denote with F_∞ the union of all the fields F_n, with

$$G_\infty := Gal(F_\infty/F) = \varprojlim_n G_n$$

and with

$$\Gamma_\infty := Gal(F_\infty/F_0) = \varprojlim_n \Gamma_n$$

we have that $\Gamma_\infty \simeq \mathbb{Z}_p^\infty$. Observe that the only primes ramified in F_∞/F are $\mathfrak{p}$ and ∞ and so F_∞ is a subextension of F_S. The following diagram gives a recap of the fields and Galois groups introduced above.

$$\begin{tikzcd}
F \arrow[r, dash, "Pic(A)"] \arrow[rr, dash, bend left, "G_0"] \arrow[rrr, dash, bend right, "G_n"'] \arrow[rrrr, dash, bend right, "G_\infty"'] \arrow[rrrrr, dash, bend right, "G_S"'] & H_A \arrow[r, dash] & F_0 \arrow[r, dash, "\Gamma_n"] \arrow[rr, dash, bend left, "\Gamma_\infty"] & F_n \arrow[r, dash] & F_\infty \arrow[r, dash] & F_S
\end{tikzcd}$$

2.2.1 Primes in Hayes extensions

Now we focus on the behaviour of the primes in these extensions: as we have already observed any prime different from $\mathfrak{p}$ and ∞ is unramified in all the extensions.
The prime $\mathfrak{p}$ is unramified in H_A/F and totally ramified in F_∞/H_A, in the sense that each prime

of H_A lying above $\mathfrak{p}$ is totally ramified. Since F_∞/F is an abelian extension we have that all the primes of H_A lying above $\mathfrak{p}$ have the same inertia group (isomorphic to $Gal(F_\infty/H_A)$) in G_∞. The prime ∞ is totally split in H_A/F, then it ramifies (not totally in general) in F_0/H_A, with inertia group isomorphic to $\mathbb{F}_q^\times$, and then it is again totally split in the remaining extensions.

2.3 Fitting ideals for Tate modules

Fix an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$ and denote with γ the arithmetic Frobenius, which is a topological generator of the pro-cyclic Galois group $G_{\mathbb{F}} := Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. For every field L, we denote by L^{ar} the compositum of L with $\overline{\mathbb{F}}_q$. In the case of the field F_n (resp. F_∞) we have that F_n^{ar} (resp. F_∞^{ar}) is a Galois extension of F, whose Galois group is isomorphic to $G_n \times G_{\mathbb{F}}$ (resp. $G_\infty \times G_{\mathbb{F}}$) since the constant field of F_n (resp. F_∞) is $\mathbb{F}_q$.

2.3.1 The modules $H_n(\nu)$

For every prime ν of F there exists only a finite number of primes of F_n^{ar} that lie above ν, because the extension F_n/F is finite and F_n^{ar}/F_n is arithmetic (we recall that in an arithmetic extension every prime splits in finitely many places). The following proposition tells us the exact number of primes lying above ν.

Proposition 2.3.1. *The number of primes of F_n^{ar} lying above ν in equal to*

- $d_\nu \cdot [F_n : F]$ *if* $\nu \neq \mathfrak{p}, \infty$*;*
- $d_{\mathfrak{p}} \cdot h^0(F)$ *if* ν *is equal to* $\mathfrak{p}$*;*
- $h^0(F) \cdot q^{nd_{\mathfrak{p}}} \cdot (q^{d_{\mathfrak{p}}} - 1)/(q-1)$ *if* ν *is equal to* ∞*.*

Proof. For every field K and every positive integer t we denote with $K(t)$ the compositum of K with $\mathbb{F}_{q^t}$. One can see that $F_n(t)$ is the compositum of the two fields F_n and $F(t)$ which are disjoint over F, since F_n/F is a geometric extension, $F(t)/F$ is arithmetic and $\mathbb{F}_{q^t}$ is the costant field of $F_n(t)$. We also oberve that F_n^{ar} is the union of all the fields $F_n(t)$ when t varies over all positive integers.
Fix a prime $\overline{\nu}$ of $F_n(t)$ which lies above ν and let e be the ramification index of $\overline{\nu}$ over ν and f be its inertia degree. Clearly these two numbers do not depend on the choice of $\overline{\nu}$ since $F_n(t)/F$ is an abelian extension. We have also that e does not depend on t since $F_n(t)/F_n$ is arithmetic. Now we have

$$f = [\mathbb{F}_{\overline{\nu}} : \mathbb{F}_\nu] = \frac{[\mathbb{F}_{\overline{\nu}} : \mathbb{F}_{q^t}][\mathbb{F}_{q^t} : \mathbb{F}_q]}{[\mathbb{F}_\nu : \mathbb{F}_q]} = t \cdot \frac{\deg(\overline{\nu})}{d_\nu}.$$

The degree of the extension $F_n(t)/F$ is equal to $[F_n : F] \cdot t$ so, if we denote with r the number of primes of $F_n(t)$ lying above ν, from the Kummer formula we have

$$[F_n : F] \cdot t = r \cdot e \cdot f = r \cdot e \cdot t \cdot \frac{\deg(\overline{\nu})}{d_\nu},$$

which can be re-arranged to obtain

$$r = \frac{d_\nu \cdot [F_n : F]}{e \cdot \deg(\overline{\nu})}.$$

When the integer t is big enough we have that the degree of $\overline{\nu}$ is equal to one. So to obtain the thesis it is enough to recall that every prime different from $\mathfrak{p}$ and ∞ is unramified if F_n/F, that for the prime $\mathfrak{p}$ the ramification index in equal to $[F_n : H_A] = [F_n : F]/h^0(F)$ and that for the prime ∞ the ramification index is equal to $|\mathbb{F}_q^\times| = q - 1$. $\square$

Before going on we recall here the definition of Fitting ideal of a finitely generated module. For an in-depth discussion the reader may refer to [Nor, Chapter 3] or the appendix of [MW]. Let R be any commutative and unitary ring. For our purpose R will be one of the rings $W[\Gamma_\bullet]$ or $W[\Gamma_\bullet][\![G_\mathbb{F}]\!]$ where W is an appropriate local ring, but we give here the general definition. Let M be a finitely generated module over M and fix a set of generators $e_1, \ldots, e_r$. A *relation vector* between the generators is an element $\underline{a} = (a_1, \ldots, a_r) \in R^r$ such that $\sum a_i e_i = 0$. A *matrix of relations* is any $q \times r$ matrix, with $q \geq r$, whose entries are in R and whose rows are relation vectors.

Definition 2.3.2. The Fitting ideal of the finitely generated R-module M is the ideal generated by the determinants of the $r \times r$ minors of all the matrices of relations of M. It will be denoted by $\mathrm{Fitt}_R\, M$.

We want to point out that here we have called Fitting ideals what the author of [Nor] refers to as 0-Fitting invariant of M.

Remark 2.3.3.
- It may appear that the definition above depends on the choice of the set of generators $\{e_i\}$, but it can be proved that the Fitting ideal does not change when we take a different set of generators.
- It is easy to check that we can limit ourselves to determinants of the $r \times r$ matrices of relations.
- One can prove that the Fitting ideal is contained if the annihilator of the module. Thus the definition appears interesting only for torsion modules.

We denote with $H_n(\nu)$ the $\mathbb{Z}_p$-free module generated by the set of primes of F_n^{ar} lying above ν. Let $I_n(\nu) \subset G_n$ be the inertia group of ν: we have that $H_n(\nu)$ is also a free $\mathbb{Z}_p[G_n/I_n(\nu)]$-module of rank d_ν. Moreover there is a natural action of the group $G_\mathbb{F}$ on $H_n(\nu)$ and we are interested in studying the structure of $H_n(\nu)$ as a $\mathbb{Z}_p[G_n][\![G_\mathbb{F}]\!]$-module.

For $\nu \in \{\mathfrak{p}, \infty\}$ we will denote by Fr_ν any lift to G_n of the Frobenius map that belongs to $G_n/I_n(\nu)$. For the prime ∞ the inertia group $I_n(\infty)$ is contained in G_0 and does not depend on n, so we shall simply denote it with I_∞. Since the decomposition and inertia groups of ∞ coincide, we can choose $\mathrm{Fr}_\infty = 1$. The same choice can be done for the prime $\mathfrak{p}$ if and only if the prime $\mathfrak{p}$ is totally split in H_A. For all the other primes we simply denote with Fr_ν the Frobenius map in G_n.

Definition 2.3.4. The *Euler factor* at ν is

$$e_\nu(X) := 1 - \mathrm{Fr}_\nu^{-1} X^{d_\nu} \in \mathbb{Z}_p[G_n][\![X]\!]\,.$$

Since we will also need to specialize the variable X at γ^{-1}, we put

$$e_\nu := e_\nu(\gamma^{-1}) = 1 - \mathrm{Fr}_\nu^{-1} \gamma^{-d_\nu} \in \mathbb{Z}_p[G_n][\![G_\mathbb{F}]\!]\,.$$

The statements of [GP2, Lemmas 2.1 and 2.2] adapted to our setting translate into the following

Lemma 2.3.5. *For $\nu \in \{\mathfrak{p}, \infty\}$ we denote with $Aug_{\nu,n}$ the augumentation ideal of $\mathbb{Z}_p[G_n][\![G_{\mathbb{F}}]\!]$ associated to the inertia group $I_n(\nu)$, i.e., the ideal generated by the elements of the form $\tau - 1$, with $\tau \in I_n(\nu)$. Then*

(a) if $\nu \neq \mathfrak{p}, \infty$, then $Fitt_{\mathbb{Z}_p[G_n][\![G_{\mathbb{F}}]\!]}(H_n(\nu)) = (e_\nu)$;

(b) $Fitt_{\mathbb{Z}_p[G_n][\![G_{\mathbb{F}}]\!]}(H_n(\infty)) = (e_\infty, Aug_{\infty,n})$;

(c) $Fitt_{\mathbb{Z}_p[G_n][\![G_{\mathbb{F}}]\!]}(H_n(\mathfrak{p})) = (e_{\mathfrak{p}}, Aug_{\mathfrak{p},n})$.

From this lemma and the fact that the $H_n(\nu)$ are free $\mathbb{Z}_p[G_n/I_n(\nu)]$-modules, we have the isomorphisms

- if $\nu \neq \mathfrak{p}, \infty$
$$H_n(\nu) \simeq \mathbb{Z}_p[G_n][\![G_{\mathbb{F}}]\!]/(e_\nu)\,;$$
- if $\nu = \infty$
$$H_n(\infty) \simeq \mathbb{Z}_p[G_0/\mathbb{F}_q^\times \times \Gamma_n][\![G_{\mathbb{F}}]\!]/(e_\infty)\,;$$
- if $\nu = \mathfrak{p}$
$$H_n(\mathfrak{p}) \simeq \mathbb{Z}_p[Pic(A)][\![G_{\mathbb{F}}]\!]/(e_{\mathfrak{p}})\,.$$

2.3.2 Complex characters

Let $\chi \in \mathrm{Hom}(G_0, \mathbb{C}^\times)$ be a complex character for G_0. The character χ takes values in the set of roots of unity of order $|G_0| = h^0(F)(q^{d_{\mathfrak{p}}} - 1)$ and so we need to consider modules over the Witt ring $W = \mathbb{Z}_p[\zeta]$, where ζ denotes any primitive root of unity of order $|G_0|$. Recall that we are assuming $(|G_0|, p) = 1$, we put

$$e_\chi := \frac{1}{|G_0|} \sum_{g \in G_0} \chi(g^{-1}) g \in W[G_0]$$

for the idempotent associated to χ. For any $W[G_0]$-module M, we denote its χ-part by $M(\chi) := e_\chi M$. If M is a $\mathbb{Z}_p[G_0]$-module we first turn it into a $W[G_0]$-module by tensoring with W over $\mathbb{Z}_p$ and then we consider the χ-part of this tensor product (all this will often be tacitly assumed and forgotten in the notations). We also recall that W is a flat $\mathbb{Z}_p$-module and so the functor $W \otimes_{\mathbb{Z}_p} -$ is exact in the category of $\mathbb{Z}_p$-modules. Finally note that if the action of G_0 is trivial on M, then

$$M(\chi) = \begin{cases} W \otimes_{\mathbb{Z}_p} M & \text{if } \chi = \chi_0\,, \\ 0 & \text{if } \chi \neq \chi_0\,. \end{cases}$$

Definition 2.3.6. Let χ be a character of G_0. We will distinguish 3 types of characters:

- χ is said to be of *type 1* if $\chi(I_\infty) \neq 1$;
- χ is said to be of *type 2* if $\chi(I_\infty) = 1$ and $\chi(Gal(F_0/H_A)) \neq 1$;
- χ is said to be of *type 3* if $\chi(Gal(F_0/H_A)) = 1$.

Among the characters of type 3 there is the trivial one which will be denoted by χ_0.

2.3.3 The theorem of Greither and Popescu

Now we can start our computations: since $d_\infty = 1$ and we have chosen $\mathrm{Fr}_\infty = 1$ we have that $e_\infty = 1 - \gamma^{-1}$. From Lemma 2.3.5 we have that $H_n(\infty) \simeq \mathbb{Z}_p[G_0/I_\infty \times \Gamma_n]$ and so

$$H_n(\infty)(\chi) \simeq \begin{cases} 0 & \text{if } \chi \text{ is of type } 1\,, \\ W[\Gamma_n] & \text{otherwise}\,. \end{cases} \tag{2.1}$$

For the prime $\mathfrak{p}$ we have $H_n(\mathfrak{p}) \simeq \mathbb{Z}_p[Pic(A)][\![G_\mathbb{F}]\!]/(1 - \mathrm{Fr}_\mathfrak{p}^{-1}\gamma^{-d_\mathfrak{p}})$ and so we have the following exact sequence

$$(1 - \mathrm{Fr}_\mathfrak{p}^{-1}\gamma^{-d_\mathfrak{p}})\mathbb{Z}_p[Pic(A)][\![G_\mathbb{F}]\!] \hookrightarrow \mathbb{Z}_p[Pic(A)][\![G_\mathbb{F}]\!] \twoheadrightarrow H_n(\mathfrak{p})\,. \tag{2.2}$$

We can tensor the previous exact sequence with W and multiply by e_χ to obtain

$$e_\chi(1 - \mathrm{Fr}_\mathfrak{p}^{-1}\gamma^{-d_\mathfrak{p}})W[Pic(A)][\![G_\mathbb{F}]\!] \hookrightarrow e_\chi W[Pic(A)][\![G_\mathbb{F}]\!] \twoheadrightarrow H_n(\mathfrak{p})(\chi)\,. \tag{2.3}$$

If we observe that $e_\chi(1 - \mathrm{Fr}_\mathfrak{p}^{-1}\gamma^{-d_\mathfrak{p}}) = \left(1 - \chi(\mathrm{Fr}_\mathfrak{p}^{-1})\gamma^{-d_\mathfrak{p}}\right) e_\chi$ and $e_\chi W[Pic(A)][\![G_\mathbb{F}]\!] = W[\![G_\mathbb{F}]\!]$, we can conclude that

$$H_n(\mathfrak{p})(\chi) \simeq \begin{cases} 0 & \text{if } \chi \text{ is of type 1 or 2}\,, \\ W[\![G_\mathbb{F}]\!]/\left(1 - \chi(\mathrm{Fr}_\mathfrak{p}^{-1})\gamma^{-d_\mathfrak{p}}\right) & \text{otherwise}\,. \end{cases} \tag{2.4}$$

Now consider the group of divisors $H_n(\infty) \oplus H_n(\mathfrak{p})$ which is the free $\mathbb{Z}_p$-module of divisors of F_n^{ar} whose support is contained in S. We denote with D_n its subgroup of divisors of degree zero, i.e., the kernel of the map

$$\deg : H_n(\infty) \oplus H_n(\mathfrak{p}) \to \mathbb{Z}_p\,,$$

and with $D_n(\chi)$ its χ-part. Since there is no action of G_0 on $\mathbb{Z}_p$ we have that when χ is not the trivial character, $D_n(\chi)$ is simply the sum $H_n(\infty)(\chi) \oplus H_n(\mathfrak{p})(\chi)$, i.e.,

$$D_n(\chi) \simeq \begin{cases} 0 & \text{if } \chi \text{ is of type } 1\,, \\ W[\Gamma_n] & \text{if } \chi \text{ is of type } 2\,, \\ W[\Gamma_n] \oplus W[\![G_\mathbb{F}]\!]/\left(1 - \chi(\mathrm{Fr}_\mathfrak{p}^{-1})\gamma^{-d_\mathfrak{p}}\right) & \text{if } \chi \text{ is of type 3 and } \chi \neq \chi_0\,. \end{cases} \tag{2.5}$$

Let X_n be the projective curve defined over $\mathbb{F}_q$ and associated with F_n and $Jac(X_n)(\overline{\mathbb{F}}_q)$ the set of $\overline{\mathbb{F}}_q$-rational points of its Jacobian. We recall that for each abelian group M, the multiplication by p defines a surjective map $M[p^{n+1}] \to M[p^n]$ from the p^{n+1}-torsion subgroup of M to the p^n-torsion subgroup. One can use this maps to define a projective limit which is called the p-adic Tate module of M and we will denote it with $T_p(M)$ (note that the Tate module depends only on the p-part of M). For more details the reader may refer to the classical work of Tate [Tat] or [Mum, Chapter IV].
We will denote $T_p(F_n) := T_p\left(Jac(X_n)(\overline{\mathbb{F}}_q)\right)$ the p-adic Tate module of the Jacobian of X_n, defined over the algebraically closed field $\overline{\mathbb{F}}_q$. Our task is to study the structure of $T_p(F_n)$ as a

Galois module over $\mathbb{Z}_p[\Gamma_n][\![G_{\mathbb{F}}]\!]$. More precisely we will study its χ-part when χ is of type 1 or 2.

Following the definitions and the properties given in [GP1, Section 2] we consider the Deligne's Picard 1-motive $\mathcal{M}_{S,\emptyset}$ associated to the field F_n^{ar} and to the set of primes of F_n^{ar} which lie above the primes of S. We can take the empty set as auxiliary set of primes in the definition of $\mathcal{M}_{S,\emptyset}$ because we are only interested in the study of the p-adic Tate module of the Jacobian (see [GP1, Remark 2.7]). In what follows we will simply denote the Deligne's Picard 1-motive with $\mathcal{M}_n$.
The multiplication by p induces a surjective map on the p^m-torsion of $\mathcal{M}_n$ for every positive integer m and so we can define the p-adic Tate module of $\mathcal{M}_n$ as

$$T_p(\mathcal{M}_n) = \varprojlim_{m} \mathcal{M}_n[p^m]\,.$$

We denote with $\Theta_n(X)$ (resp. $\Theta_\infty(X)$) the projection of the Stickelberger series $\Theta_S(X)$ to $\mathbb{Z}[G_n][\![X]\!]$ (resp. $\mathbb{Z}[\![G_\infty]\!][\![X]\!]$), which is easily seen to be the Stickelberger series associated to the extension F_n/F (resp. F_∞/F) since the set of ramified primes in this subextension is exactly S.

In [GP1, Theorem 4.3] the authors prove the following

Theorem 2.3.7. *One has*

$$Fitt_{\mathbb{Z}_p[G_n][\![G_{\mathbb{F}}]\!]}\left(T_p(\mathcal{M}_n)\right) = \left(\Theta_n(\gamma^{-1})\right)\,.$$

Note that evaluating the Stickelberger series $\Theta_n(X)$ at $X = \gamma^{-1}$ makes sense because of Proposition 1.3.4.

2.3.4 Fitting ideals for Tate modules: finite level

In [GP1, after Definition 2.6] the authors provide the following exact sequence

$$0 \longrightarrow T_p(F_n) \longrightarrow T_p(\mathcal{M}_n) \longrightarrow D_n \longrightarrow 0\,. \tag{2.6}$$

For every character χ we denote by $\Theta_n(X,\chi)$ the only element of $W[\Gamma_n][\![X]\!]$ that satisfies $\Theta_n(X,\chi)e_\chi = e_\chi\Theta_n(X)$.

Theorem 2.3.8. *Let $\chi \in \widehat{G_0}$ be a character not of type 3. Then we have*

$$\mathrm{Fitt}_{W[\Gamma_n][\![G_{\mathbb{F}}]\!]}\left(T_p(F_n)(\chi)\right) = \left(\Theta_n^\sharp(\gamma^{-1},\chi)\right)\,,$$

where we put

$$\Theta_n^\sharp(\gamma^{-1},\chi) = \begin{cases} \Theta_n(\gamma^{-1},\chi) & \text{if } \chi \text{ is of type } 1\,,\\[2ex] \dfrac{\Theta_n(\gamma^{-1},\chi)}{1-\gamma^{-1}} & \text{if } \chi \text{ is of type } 2\,.\end{cases}$$

Proof. From Theorem 2.3.7 we have that the Fitting ideal of $T_p(\mathcal{M}_n)$ over the ring $\mathbb{Z}_p[G_n][\![G_{\mathbb{F}}]\!]$ is the ideal generated by $\Theta_n(\gamma^{-1})$. We want to use this information to determine the Fitting ideal of $T_p(\mathcal{M}_n)(\chi)$ over the ring $W[\Gamma_n][\![G_{\mathbb{F}}]\!]$.
Since $\mathbb{Z}_p[G_n][\![G_{\mathbb{F}}]\!]$ is a Noetherian ring and $T_p(\mathcal{M}_n)$ is finitely generated we have a presentation

$$\mathbb{Z}_p[G_n][\![G_{\mathbb{F}}]\!]^{\oplus r} \xrightarrow{\ \alpha\ } \mathbb{Z}_p[G_n][\![G_{\mathbb{F}}]\!]^{\oplus s} \twoheadrightarrow T_p(\mathcal{M}_n),$$

with $r \geq s$ because $T_p(\mathcal{M}_n)$ is a torsion module. Note that we cannot assume, in general, that α is injective. The map α can be identified with a $r \times s$ matrix whose entries are in $\mathbb{Z}_p[G_n][\![G_{\mathbb{F}}]\!]$ and the ideal generated by the determinants of the $s \times s$ minors of α is $\big(\Theta_n(\gamma^{-1})\big)$. Now we take the χ-part in the previous presentation to obtain

$$W[\Gamma_n][\![G_{\mathbb{F}}]\!]^{\oplus r} \xrightarrow{\alpha_\chi} W[\Gamma_n][\![G_{\mathbb{F}}]\!]^{\oplus s} \twoheadrightarrow T_p(\mathcal{M}_n)(\chi),$$

i.e., a presentation for $T_p(\mathcal{M}_n)(\chi)$ as a module over $W[\Gamma_n][\![G_{\mathbb{F}}]\!]$. Clearly the ideal generated by the determinants of the $s \times s$ minors of α_χ is generated by $e_\chi \Theta_n(\gamma^{-1}) = \Theta_n(\gamma^{-1}, \chi)$ and it is the Fitting ideal of $T_p(\mathcal{M}_n)(\chi)$ over $W[\Gamma_n][\![G_{\mathbb{F}}]\!]$.
We have already observed that tensor product $W \otimes_{\mathbb{Z}_p} -$ is an exact functor and so it preserves exact sequences. The same happens when we take the χ part and so, from (2.6), we obtain

$$0 \longrightarrow T_p(F_n)(\chi) \longrightarrow T_p(\mathcal{M}_n)(\chi) \longrightarrow D_n(\chi) \longrightarrow 0\,. \tag{2.7}$$

- If χ is of type 1, from (2.5) we have that $D_n(\chi) = 0$, thus

$$T_p(F_n)(\chi) \simeq T_p(\mathcal{M}_n)(\chi)$$

  and so the thesis is proved because in this case $\Theta_n^\sharp(\gamma^{-1}, \chi) = \Theta_n(\gamma^{-1}, \chi)$.

- If χ is of type 2, from (2.5) we have that $D_n(\chi) = W[\Gamma_n] \simeq W[\Gamma_n][\![G_{\mathbb{F}}]\!]/(1-\gamma^{-1})$ which is a cyclic $W[\Gamma_n][\![G_{\mathbb{F}}]\!]$-module. This allows us to apply [CG, Lemma 3] to the previous exact sequence and to obtain

$$(1-\gamma^{-1})\mathrm{Fitt}_{W[\Gamma_n][\![G_{\mathbb{F}}]\!]}\left(T_p(F_n)(\chi)\right) = \mathrm{Fitt}_{W[\Gamma_n][\![G_{\mathbb{F}}]\!]}\left(T_p(\mathcal{M}_n)(\chi)\right) = \left(\Theta_n(\gamma^{-1}, \chi)\right)\,.$$

□

When χ is a character of type 3 things get more complicated, since $D_n(\chi)$ is not cyclic and so we cannot apply [CG, Lemma 3] as we did in the previous theorem. We shall consider those characters in Section 2.5.

2.3.5 Fitting ideals for Tate modules: infinite level

Until now we have studied the Galois module $T_p(F_n)$ for a fixed integer n. Now we will consider two indices $n > m \geq 0$ and study the relation between $T_p(F_n)$ and $T_p(F_m)$.
We put $\Gamma_m^n = Gal(F_n/F_m)$ and we recall that this extension of fields is totally ramified at every prime that lies above $\mathfrak{p}$ and unramified at every other prime (moreover the number of primes in F_m above $\mathfrak{p}$ is the same for any m and coincides with the number of primes of H_A lying above $\mathfrak{p}$). We denote with $\overline{C}_n$ the p-part of the class group of degree zero divisors of F_n^{ar} and we recall that $T_p(F_n) = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \overline{C}_n)$. Thus the norm map from $\overline{C}_n$ to $\overline{C}_m$ induces a map $N_m^n : T_p(F_n) \to T_p(F_m)$ and likewise the inclusion map induces a map $i_n^m : T_p(F_m) \to T_p(F_n)$. We define

$$T_p(F_\infty)(\chi) = \varprojlim_n T_p(F_n)(\chi)\,,$$

where the limit is taken with respect to the norm maps. The limit $T_p(F_\infty)(\chi)$ is a module over the profinite (non noetherian) algebra $\Lambda_{\mathbb{F}} := W[\![\Gamma_\infty]\!][\![G_{\mathbb{F}}]\!]$. Our next goals are to prove that, for characters of type 1 and 2, this module is finitely generated and torsion over $\Lambda_{\mathbb{F}}$ and to compute its Fitting ideal.

There is a natural Galois action of Γ_m^n on $T_p(\mathcal{M}_n)$ and an inclusion map $i_n^m : T_p(\mathcal{M}_m) \to T_p(\mathcal{M}_n)$ that satisfies $T_p(\mathcal{M}_n)^{\Gamma_m^n} = i_n^m(T_p(\mathcal{M}_m))$ (see [GP1, Theorem 3.1]). There is also a norm map $N_m^n : T_p(\mathcal{M}_n) \to T_p(\mathcal{M}_m)$ such that the compositum $N_m^n \circ i_n^m$ is the multiplication by $[F_n : F_m]$.
We have one last norm map defined from D_n to D_m and inclusion map from D_m to D_n whose compositum in again the multiplication by $[F_n : F_m]$.
All this norm and inclusion maps defined on $T_p(F_\bullet), T_p(\mathcal{M}_\bullet)$ and $D_\bullet$ are compatible with the exact sequence (2.6) in the sense that the following diagram is commutative for every index m and n.

$$\begin{tikzcd}
0 \arrow[r] & T_p(F_n) \arrow[r] \arrow[d, shift right, "N_m^n"'] & T_p(\mathcal{M}_n) \arrow[r] \arrow[d, shift right, "N_m^n"'] & D_n \arrow[r] \arrow[d, shift right, "N_m^n"'] & 0 \\
0 \arrow[r] & T_p(F_m) \arrow[r] \arrow[u, bend right, "i_m^n"'] & T_p(\mathcal{M}_m) \arrow[r] \arrow[u, bend right, "i_m^n"'] & D_n \arrow[r] \arrow[u, bend right, "i_m^n"'] & 0
\end{tikzcd}$$

This diagram suggests to investigate the behaviour of the norm map on $T_p(\mathcal{M}_n)$ first and then move to $T_p(F_n)$. We denote with $I_{\Gamma_m^n}$ the augumentation ideal of $W[\Gamma_n]$ associated to the subgroup Γ_m^n. The next lemma follows immediately from Theorem 3.9 of [GP1].

Lemma 2.3.9. *The norm map $N_m^n : T_p(\mathcal{M}_n) \to T_p(\mathcal{M}_m)$ is surjective and its kernel is $I_{\Gamma_m^n} T_p(\mathcal{M}_n)$.*

Proof. From [GP1, Theorem 3.9 part (2)] we have that $T_p(\mathcal{M}_n)$ is free over $\mathbb{Z}_p[\Gamma_m^n]$ (because Γ_m^n is a p-group). Thus

$$\hat{H}^i\left(\Gamma_m^n, T_p(\mathcal{M}_n)\right) = 0 \quad \text{for every integer } i\,,$$

where we denoted by $\hat{H}^i(\Gamma_m^n, \bullet)$ the i-th group of Tate cohomology.
Specializing the previous equality at $i = 0$ we obtain

$$N_m^n\left(T_p(\mathcal{M}_n)\right) = T_p(\mathcal{M}_n)^{\Gamma_m^n} = i_n^m\left(T_p(\mathcal{M}_m)\right).$$

In a similar way we obtain the second part of the lemma by specializing at $i = -1$. □

Now we can study the norm map on $T_p(F_n)$.

Proposition 2.3.10. *Let χ be a character of type 1 or 2. Then the norm map $N_m^n : T_p(F_n)(\chi) \to T_p(F_m)(\chi)$ is surjective and its kernel is $I_{\Gamma_m^n} T_p(F_n)(\chi)$.*

Proof. Consider the exact sequence (2.7). If χ is a character of type 1 we have that $T_p(F_n)(\chi) \simeq T_p(\mathcal{M}_n)(\chi)$ and so the thesis is simply a restatment of the previous lemma (recalling that taking the χ-part of a module is an exact functor).
Now assume χ is of type 2 so that $D_n(\chi) \simeq W[\Gamma_n]$ and note that this is a Γ_m^n-cohomologically trivial module. Since $T_p(\mathcal{M}_n)(\chi)$ is also cohomologically trivial by (the proof of) [GP1, Theorem 3.9], we have that

$$\hat{H}^i\left(\Gamma_m^n, T_p(F_n)(\chi)\right) = 0 \quad \text{for every } i \in \mathbb{Z}\,. \tag{2.8}$$

Specializing at $i=-1$ we obtain

$$\ker\{N^n_m : T_p(F_n)(\chi) \to T_p(F_m)(\chi)\} = I_{\Gamma^n_m} T_p(F_n)(\chi)\,.$$

To prove surjectivity of the norm map we take Γ^n_m-invariants in (2.7) to obtain

$$0 \to T_p(F_n)(\chi)^{\Gamma^n_m} \longrightarrow T_p(\mathcal{M}_m)(\chi) \longrightarrow D_n(\chi)^{\Gamma^n_m} \longrightarrow \hat{H}^1\left(\Gamma^n_m, T_p(F_n)(\chi)\right) = 0\,.$$

Note that $D_n(\chi)^{\Gamma^n_m} \simeq W[\Gamma_n]^{\Gamma^n_m} \simeq W[\Gamma_m]$ and compare the previous exact sequence with (2.7) with m in place of n. We readily get that $T_p(F_n)(\chi)^{\Gamma^n_m} \simeq T_p(F_m)(\chi)$. Now specializing (2.8) at $i=0$ we obtain

$$N^n_m\left(T_p(F_n)(\chi)\right) = T_p(F_n)(\chi)^{\Gamma^n_m} \simeq T_p(F_m)(\chi)\,.$$

□

We are now ready to state our first main theorem.

Theorem 2.3.11. *Let χ be a character of type 1 or 2. Then $T_p(F_\infty)(\chi)$ is a finitely generated torsion $\Lambda_{\mathbb{F}}$-module.*

Proof. Fix an index m, and denote with $\mathfrak{I}_m$ the augumentation ideal of $W[\![\Gamma_\infty]\!]$ associated to the subgroup $Gal(F_\infty/F_m)$. In particular we have

$$\mathfrak{I}_m = \varprojlim_n I_{\Gamma^n_m}\,.$$

We also put

$$\widetilde{\mathfrak{I}}_m = \Lambda_{\mathbb{F}} \otimes_{W[\![\Gamma_\infty]\!]} \mathfrak{I}_m$$

for the corresponding ideal of $\Lambda_{\mathbb{F}}$. Now Proposition 2.3.10 yields $T_p(F_m)(\chi) = N^n_m\left(T_p(F_n)(\chi)\right)$. Applying the first homomorphism theorem and again Proposition 2.3.10 we also have that

$$N^n_m\left(T_p(F_n)(\chi)\right) \simeq T_p(F_n)(\chi)/\ker N^n_m = T_p(F_n)(\chi)/I_{\Gamma^n_m} T_p(F_n)(\chi)\,,$$

thus

$$T_p(F_m)(\chi) \simeq T_p(F_n)(\chi)/I_{\Gamma^n_m} T_p(F_n)(\chi)\,.$$

The previous equality holds for every $n > m$ and so we have

$$T_p(F_m)(\chi) \simeq T_p(F_\infty)(\chi)/\mathfrak{I}_m T_p(F_\infty)(\chi) = T_p(F_\infty)(\chi)/\widetilde{\mathfrak{I}}_m T_p(F_\infty)(\chi)\,.$$

The module on the left is finitely generated over

$$W[\Gamma_m][\![G_{\mathbb{F}}]\!] = \Lambda_{\mathbb{F}}/\widetilde{\mathfrak{I}}_m$$

and, since the ideals $\widetilde{\mathfrak{I}}_m$ form an open filtration of the profinite algebra $\Lambda_{\mathbb{F}}$, we can apply the generalized Nakayama Lemma (see [BH]) and obtain that $T_p(F_\infty)(\chi)$ is a finitely generated module over $\Lambda_{\mathbb{F}}$.
Now we define the element $\Theta^\sharp_\infty(\gamma^{-1},\chi) \in \Lambda_{\mathbb{F}}$ as

$$\Theta^\sharp_\infty(\gamma^{-1},\chi) = \begin{cases} \Theta_\infty(\gamma^{-1},\chi) & \text{if } \chi \text{ is of type } 1\,, \\ \dfrac{\Theta_\infty(\gamma^{-1},\chi)}{1-\gamma^{-1}} & \text{if } \chi \text{ is of type } 2\,, \end{cases}$$

which is simply the inverse limit of the generators of the Fitting ideals of $T_p(F_m)(\chi)$ over $W[\Gamma_m][\![G_{\mathbb{F}}]\!]$ (computed in Theorem 2.3.8). At the finite level we have $\Theta_n^\sharp(\gamma^{-1},\chi)T_p(F_n)(\chi)=0$ because the Fitting ideal of a module is contained in the annihilator. Taking the inverse limit of the previous equalities we obtain $\Theta_\infty^\sharp(\gamma^{-1},\chi)T_p(F_\infty)(\chi)=0$ and so $T_p(F_\infty)(\chi)$ is torsion. $\square$

Now that we have proved that the module $T_p(F_\infty)(\chi)$ is a finitely generated torsion module over $\Lambda_{\mathbb{F}}$, we know that its Fitting ideal is well defined. With our second main theorem we compute a generator of that ideal via an inverse limit process.

Theorem 2.3.12. *Let χ be a character of type* 1 *or* 2*. Then we have*

$$\mathit{Fitt}_{\Lambda_{\mathbb{F}}}\left(T_p(F_\infty)(\chi)\right)=\left(\Theta_\infty^\sharp(\gamma^{-1},\chi)\right).$$

Proof. From the equality

$$\left(\Theta_\infty^\sharp(\gamma^{-1},\chi)\right)=\varprojlim_n\left(\Theta_n^\sharp(\gamma^{-1},\chi)\right)=\varprojlim_n \mathrm{Fitt}_{W[\Gamma_n][\![G_{\mathbb{F}}]\!]}\left(T_p(F_n)(\chi)\right)$$

we reduce the proof to showing the equality

$$\mathrm{Fitt}_{\Lambda_{\mathbb{F}}}\left(T_p(F_\infty)(\chi)\right)=\varprojlim_n \mathrm{Fitt}_{W[\Gamma_n][\![G_{\mathbb{F}}]\!]}\left(T_p(F_n)(\chi)\right)$$

We denote with N_m^∞ the projection $T_p(F_\infty)(\chi)\twoheadrightarrow T_p(F_m)(\chi)$. These maps are obviously compatible with the norm maps, in the sense that $N_m^\infty=N_m^n\circ N_n^\infty$. Let $t_1,\dots,t_r$ be generators of $T_p(F_\infty)(\chi)$ over $\Lambda_{\mathbb{F}}$, then we have that $N_m^\infty(t_1),\dots,N_m^\infty(t_r)$ generate $T_p(F_m)(\chi)$ over $W[\Gamma_m][\![G_{\mathbb{F}}]\!]$, since $T_p(F_m)(\chi)=T_p(F_\infty)(\chi)/\widetilde{\mathfrak{I}}_m T_p(F_\infty)(\chi)$.
Then for every integer n we have the following exact sequence

$$0\to K_n\longrightarrow W[\Gamma_n][\![G_{\mathbb{F}}]\!]^{\oplus r}\longrightarrow T_p(F_n)(\chi)\to 0$$

where the map on the right is given by $(w_1,\dots,w_r)\mapsto\sum_i w_iN_n^\infty(t_i)$. We also have the exact sequence at the infinite level

$$0\to K_\infty\longrightarrow\Lambda_{\mathbb{F}}^{\oplus r}\longrightarrow T_p(F_\infty)(\chi)\to 0\,.$$

The previous exact sequences fits into the diagram

$$\begin{tikzcd}
0 \arrow[r] & K_n \arrow[r]\arrow[d, "k_m^n"] & W[\Gamma_n][\![G_{\mathbb{F}}]\!]^{\oplus r} \arrow[r]\arrow[d, "\pi_m^n"] & T_p(F_n)(\chi) \arrow[r]\arrow[d, "N_m^n"] & 0\\
0 \arrow[r] & K_m \arrow[r] & W[\Gamma_m][\![G_{\mathbb{F}}]\!]^{\oplus r} \arrow[r] & T_p(F_m)(\chi) \arrow[r] & 0
\end{tikzcd}$$

where k_m^n denotes the restriction of the projection π_m^n to the kernel K_n. The kernel of π_m^n is $(I_{\Gamma_m^n}W[\Gamma_n][\![G_{\mathbb{F}}]\!])^{\oplus r}$, while the kernel of N_m^n is $I_{\Gamma_m^n}T_p(F_n)(\chi)$ due to Proposition 2.3.10 and so the map between these two kernels is surjective. The map π_m^n is clearly surjective thus, by the snake lemma, we have that k_m^n is also surjective and so the previous diagram satisfies the

Mittag-Leffler condition which allows us to take the inverse limit. Comparing this limit with the exact sequence at level infinity we obtain that

$$K_\infty = \varprojlim_n K_n.$$

We denote with $\mathcal{M}_{r\times r}(K_m)$ the set of $r \times r$ matrices whose entries are in $W[\Gamma_m][\![G_{\mathbb{F}}]\!]$ and such that each row, seen as a vector in $W[\Gamma_m][\![G_{\mathbb{F}}]\!]^{\oplus r}$, is in K_m and with k_m^n the natural extension of the map $K_n \to K_m$ to $\mathcal{M}_{r\times r}(K_n) \to \mathcal{M}_{r\times r}(K_m)$. From the surjectivity of the first map we clearly have that also the extension is surjective. From the definition we have that $\mathrm{Fitt}_{W[\Gamma_m][\![G_{\mathbb{F}}]\!]}(T_p(F_m)(\chi))$ is the ideal generated by all $\det M_m$, with $M_m \in \mathcal{M}_{r\times r}(K_m)$. Note that, since the diagram above is commutative, we have that for each M_m

$$\pi_m^n(\det M_n) = \det(k_m^n(M_n)).$$

In what follows we will use the same notation also for the infinite level.
Now if we take $M_\infty \in \mathcal{M}_{r\times r}(K_\infty)$ we have that $\pi_m^\infty(\det M_\infty) \in \mathrm{Fitt}_{W[\Gamma_m][\![G_{\mathbb{F}}]\!]}(T_p(F_m)(\chi))$ and so $\pi_m^\infty(\mathrm{Fitt}_{\Lambda_{\mathbb{F}}}(T_p(F_\infty)(\chi))) \subseteq \mathrm{Fitt}_{W[\Gamma_m][\![G_{\mathbb{F}}]\!]}(T_p(F_m)(\chi))$. Since this is true for each index m we have that

$$\mathrm{Fitt}_{\Lambda_{\mathbb{F}}}(T_p(F_\infty)(\chi)) \subseteq \varprojlim_m \mathrm{Fitt}_{W[\Gamma_m][\![G_{\mathbb{F}}]\!]}(T_p(F_m)(\chi)).$$

For the other inclusion we need a little bit more work and, in particular, we have to deal with some topological properties of the rings $W[\Gamma_m][\![G_{\mathbb{F}}]\!]$. Essentially we will follow the arguments of [GK, Theorem 2.1].
Each element of $\mathrm{Fitt}_{W[\Gamma_m][\![G_{\mathbb{F}}]\!]}(T_p(F_m)(\chi))$ may be written as a linear combination

$$x_m = \sum_{i=1}^{s} \lambda_i \det M_m^{(i)} \tag{2.9}$$

with $\lambda_i \in W[\Gamma_m][\![G_{\mathbb{F}}]\!]$ and $M_m^{(i)} \in \mathcal{M}_{r\times r}(K_m)$. If we denote with $\overline{M}_m^{(i)}$ the matrix whose first row is equal to the first row of $M_m^{(i)}$ multiplied by λ_i and whose other rows are all equal to the corresponding row of $M_m^{(i)}$ we have that $\overline{M}_m^{(i)}$ is in $\mathcal{M}_{r\times r}(K_m)$ and that $\det \overline{M}_m^{(i)} = \lambda_i \det M_m^{(i)}$ and so it is not restrictive to assume that all the coefficients λ_i in the linear combination (2.9) are equal to 1. One can also show that the number s of the terms in that sum may be chosen independently from m: this follows easly from the fact that also the number of elements needed to generate $T_p(F_m)(\chi)$ (and $T_p(F_\infty)(\chi)$) can be chosen independently from m.
Now we put $\mathcal{B}_m := \mathcal{M}_{r\times r}(K_m)^{\oplus s}$ and define the non-linear operator $\phi_m : \mathcal{B}_m \to W[\Gamma_m][\![G_{\mathbb{F}}]\!]$, by $\phi_m\left(M_m^{(1)}, \ldots, M_m^{(s)}\right) = \sum_i \det M_m^{(i)}$ (as usual we define also an operator ϕ_∞ for the infinite level). This operator is clearly continuous because of the continuity of the determinant (on $\mathcal{B}_m$ we put the natural topology induced from $W[\Gamma_m][\![G_{\mathbb{F}}]\!]$). From what we have observed we have that the image of ϕ_m is exactly the Fitting ideal of $T_p(F_m)(\chi)$. The map k_m^n which we have previously extended to $\mathcal{M}_{r\times r}(K_n)$ can also be extended to $\mathcal{B}_n$ and this extension fits into the diagram

$$\begin{tikzcd}
\mathcal{B}_n \arrow[r, "\phi_n"] \arrow[d, "k_m^n"'] & W[\Gamma_n][\![G_{\mathbb{F}}]\!] \arrow[d, "\pi_m^n"] \\
\mathcal{B}_m \arrow[r, "\phi_m"'] & W[\Gamma_m][\![G_{\mathbb{F}}]\!].
\end{tikzcd}$$

Now take a sequence $(x_m)_{m\in\mathbb{N}} \in \varprojlim \mathrm{Fitt}_{W[\Gamma_m][\![G_{\mathbb{F}}]\!]}(T_p(F_m)(\chi))$, to prove the thesis we have to show that there exists an element $b_\infty \in \mathcal{B}_\infty$ such that $\phi_\infty(b_\infty) = (x_m)_{m\in\mathbb{N}}$.
For each integer m we put $\Omega_m = \phi_m^{-1}(x_m)$. The topological ring $W[\Gamma_m][\![G_{\mathbb{F}}]\!]$ is a Hausdorff space, thus finite sets are closed and, from the continuity of ϕ_m, we have that also Ω_m is closed. Furthermore, since $\mathcal{B}_m$ is compact (because $W[\Gamma_m][\![G_{\mathbb{F}}]\!]$ is compact) and Ω_m is contained in the direct sum of $s \cdot r^2$ copies of $W[\Gamma_m][\![G_{\mathbb{F}}]\!]$, we get that also Ω_m is compact. For each $\omega_n \in \Omega_n$ we have that

$$\phi_m\left(k_m^n(\omega_n)\right) = \pi_m^n\left(\phi_n(\omega_n)\right) = \pi_m^n(x_n) = x_m$$

thus the image of Ω_n under k_m^n is contained in Ω_m. This allows us to define a new set

$$\overline{\Omega}_m = \bigcap_{n>m} k_m^n(\Omega_n) \subset \Omega_m.$$

From the compactness of Ω_n and the continuity of k_m^n we deduce that the set $\overline{\Omega}_m$ is not empty and it is also compact. Clearly the image of $\overline{\Omega}_n$ under the map k_m^n is contained in $\overline{\Omega}_m$, but we will now prove that it is exactly $\overline{\Omega}_m$. Take an element $\overline{\omega}_m \in \overline{\Omega}_m$. From the definition we have that there exists, for each $n > m$, an element $\omega_n \in \Omega_n$ such that $k_m^n(\omega_n) = \overline{\omega}_m$. Now fix a positive index h and for $n > m+h$ consider $k_{m+h}^n(\omega_n) \in \Omega_{m+h}$ as a sequence in n. Since Ω_{m+h} is compact we have that it admits a convergent subsequence. We denote with $\overline{\omega}_{m+h}$ the limit of this subsequence, thus for each integer $n > m$ we have defined an element $\overline{\omega}_n \in \Omega_n$. Then we have

$$\begin{aligned} k_{m+h}^n(\overline{\omega}_n) &= k_{m+h}^n\left(\lim_{\substack{t\to\infty \\ t \text{ in the subsequence}}} k_n^t(\omega_t)\right) \\ &= \lim_{\substack{t\to\infty \\ t \text{ in the subsequence}}} \left(k_{m+h}^n \circ k_n^t(\omega_t)\right) \\ &= \lim_{\substack{t\to\infty \\ t \text{ in the subsequence}}} \left(k_{m+h}^t(\omega_t)\right) \\ &= \overline{\omega}_{m+h}\,. \end{aligned}$$

The previous equality shows us that $\overline{\omega}_{m+h}$ is in $\overline{\Omega}_{m+h}$ and, since $\overline{\omega}_m = k_m^{m+h}(\overline{\omega}_{m+h})$, we have proved the surjectivity of the map k_m^{m+h}. But we have also proved that the sequence $\overline{\omega}_m$ is coherent and, since $\phi_m(\overline{\omega}_m) = x_m$ for each integer m we have that $\overline{\omega}_\infty := \varprojlim \overline{\omega}_m$ is in $\mathcal{B}_\infty$ and $\phi_\infty(\overline{\omega}_\infty) = (x_m)_{m\in\mathbb{N}}$. □

2.4 Fitting ideals for the class groups

In this section we will investigate the p-part of the class groups of degree zero divisors of the fields F_n in the Iwasawa tower and their inverse limit. As in the previous section we shall see the inverse limit as a module over the non-Noetherian Iwasawa algebra $\Lambda := W[\![\Gamma]\!]$ (which is the quotient of the algebra $\Lambda_{\mathbb{F}}$ with respect to the augumentation ideal of $G_{\mathbb{F}}$), we will prove that it is finitely generated and torsion and we will compute its Fitting ideal as a limit of the Fitting ideals at the finite levels.

2.4.1 Fitting ideals for the class groups: finite level

Let $C_n := Cl^0(F_n)\{p\}$ be the p-torsion of the class groups of degree zero divisors of F_n. In particular the natural action of Γ_n makes it a module over the commutative ring $\mathbb{Z}_p[\Gamma_n]$. Since C_n is finite its Fitting ideal over $\mathbb{Z}_p[\Gamma_n]$ is well defined. Using the computation done for $T_p(F_n)(\chi)$ we will be able to compute also the χ-part of the Fitting ideal of $W \otimes_{\mathbb{Z}_p} C_n$ over $W[\Gamma_n]$.

Denote with $T_p(F_n)_{G_{\mathbb{F}}}$ the quotient $T_p(F_n)/(1-\gamma^{-1})T_p(F_n)$. Note that the ideal $(1-\gamma^{-1})$ of $\mathbb{Z}_p[\Gamma_n][\![G_{\mathbb{F}}]\!]$ is the augumentation ideal of $G_{\mathbb{F}}$ (recall that $G_{\mathbb{F}}$ is pro-cyclic and γ is the arithmetic Frobenius, which is a topological generator) and so the action of $G_{\mathbb{F}}$ on $T_p(F_n)_{G_{\mathbb{F}}}$ is trivial.

Lemma 2.4.1. *The module $T_p(F_n)_{G_{\mathbb{F}}}$ is isomorphic to C_n as module over $\mathbb{Z}_p[G_n]$.*

Proof. The proof of this lemma will be achieved by applying certain functors to some well known exact sequences.
We start with the classical sequence

$$0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

and apply the contravariant funtor $\mathrm{Hom}(*, C_n)$ to obtain

$$\to \mathrm{Hom}(\mathbb{Q}_p, C_n) \to \mathrm{Hom}(\mathbb{Z}_p, C_n) \to \mathrm{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, C_n) \to \mathrm{Ext}^1(\mathbb{Q}_p, C_n) \to .$$

Since $\mathbb{Q}_p$ is a field and C_n is finite we have that $\mathrm{Hom}(\mathbb{Q}_p, C_n) = \mathrm{Ext}^1(\mathbb{Q}_p, C_n) = 0$ and so

$$C_n \simeq \mathrm{Hom}(\mathbb{Z}_p, C_n) \simeq \mathrm{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, C_n). \tag{2.10}$$

We recall that $\overline{C}_n$ is the p-part of the class group of degree zero divisors of F_n^{ar}, which satisfies the equality $T_p(F_n) = \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \overline{C}_n)$. If we consider the multiplication by $1-\gamma^{-1}$ on $\overline{C}_n$ we have the following exact sequence (where surjectivity on the right is due to Lang's Theorem, see, for example, [Ser, Chapter VI, §4]):

$$0 \longrightarrow C_n \longrightarrow \overline{C}_n \xrightarrow{1-\gamma^{-1}} \overline{C}_n \longrightarrow 0.$$

If we apply the functor $\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, *)$ to the previus sequence we get

$$\begin{tikzcd} {} \arrow[r] & \mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, C_n) \arrow[r] & T_p(F_n) \arrow[r, "1-\gamma^{-1}"] & T_p(F_n) \arrow[r] & \mathrm{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, C_n) \arrow[d] \\ & & & & \mathrm{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, \overline{C}_n). \end{tikzcd}$$

The group C_n is a finite p-group thus $\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, C_n) = 0$ and since the costant field of F_n^{ar} is algebrically closed we have that $\overline{C}_n$ is divisible hence $\mathrm{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, \overline{C}_n) = 0$. From the last sequence we get

$$\mathrm{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, C_n) \simeq T_p(F_n)/(1-\gamma^{-1})T_p(F_n)$$

and combining this with (2.10) we obtain the thesis. □

As we have previously anticipated we want to use our computation of the Fitting ideal of $T_p(F_n)(\chi)$, which is a module over $W[\Gamma_n][\![G_{\mathbb{F}}]\!]$, to compute the Fitting ideal of $C_n(\chi)$, which is a module over $W[\Gamma_n]$. To do this we need the following general result on Fitting ideals.

Lemma 2.4.2. *Let R be any unitary ring and M a finitely generated torsion module over R. Let I be any non trivial ideal of R and denote with π the canonical projection $R \twoheadrightarrow R/I$. Then we have*

$$Fitt_{R/I}(M/IM) = \pi\left(Fitt_R(M)\right).$$

Proof. Let $m_1, m_2, \ldots, m_r$ be a set of generators of M. Clearly the image $\overline{m}_h$ of the elements m_h under the canonical projection $M \twoheadrightarrow M/IM$ is a set of generators of M/IM as a R/I-module and so any relation between the generators of M determines a relation between the generators of M/IM. Thus

$$\pi\left(\mathrm{Fitt}_R(M)\right) \subseteq \mathrm{Fitt}_{R/I}(M/IM).$$

To prove the other inclusion we fix a relation between the generators of M/IM and prove that it comes from a relation between the generators of M, i.e., let $a_1, a_2, \ldots, a_r \in R$ be such that

$$\sum_{h=1}^{r} \pi(a_h)\overline{m}_h = 0 \quad (\text{in } M/IM),$$

then $\sum_h a_h m_h \in IM$ and so there exist $i_1, i_2, \ldots, i_r \in I$ such that

$$\sum_{h=1}^{r} a_h m_h = \sum_{h=1}^{r} i_h m_h.$$

From the previous equality we deduce that the coefficients $b_h := a_h - i_h$ forms a relation between the generators m_h of M and we also have that this relation induces the original one between the generators of M/IM since $\pi(b_h) = \pi(a_h)$. Thus we have proved the inclusion

$$\mathrm{Fitt}_{R/I}(M/IM) \subseteq \pi\left(\mathrm{Fitt}_R(M)\right).$$

□

Now we are ready to compute the Fitting ideal of $C_n(\chi)$. Let $\pi : W[\Gamma_n][\![G_{\mathbb{F}}]\!] \to W[\Gamma_n]$ be the map that sends $\gamma \mapsto 1$, since $W[\Gamma_n] = W[\Gamma_n][\![G_{\mathbb{F}}]\!]/I_{G_{\mathbb{F}}}$ we have that this map is the canonical projection. Thus, combining Lemma 2.4.1, Lemma 2.4.2 and the computations of Theorem 2.3.8 we obtain

Theorem 2.4.3. *Let $\chi \in \widehat{G_0}$ be a character not of type 3. Then we have*

$$Fitt_{W[\Gamma_n]}\left(C_n(\chi)\right) = \left(\Theta_n^{\sharp}(1,\chi)\right),$$

where

$$\Theta_n^{\sharp}(1,\chi) = \begin{cases} \Theta_n(1,\chi) & \text{if } \chi \text{ is of type } 1, \\ \dfrac{\Theta_n(\gamma^{-1},\chi)}{1-\gamma^{-1}}\Big|_{\gamma=1} & \text{if } \chi \text{ is of type } 2. \end{cases}$$

2.4.2 Fitting ideals for the class groups: infinite level

Now we consider the fields F_n for different indices n. Let

$$C_\infty := \varprojlim_n C_n$$

where the limit is made with respect to the norm maps $N_m^n : C_n \to C_m$ with $n > m \geq 0$. Since every group C_n has an action of the Galois group Γ_n it turns out that C_∞ has a structure of module over

$$\mathbb{Z}_p[[\Gamma]] := \varprojlim_n \mathbb{Z}_p[\Gamma_n].$$

We will also need to consider the maps $i_n^m : C_n \to C_m$ which are the maps induced by the immersions $i_n^m : \mathrm{Div}(F_m) \hookrightarrow \mathrm{Div}(F_n)$ with $n > m \geq 0$. We recall that for a divisor $D = \sum_\nu n_\nu \nu$ of F_m we have $i_n^m(D) := \sum_\nu \sum_{w|\nu} e(w|\nu)w$, where $e(w|\nu)$ is the ramification index of w over ν. In particular one can see from the Kummer formula that $\deg(i_n^m(D)) = [F_n : F_m] \cdot \deg(D)$, thus the image of a degree zero divisor by the map i_n^m still has degree zero. We also observe that the image of $\mathrm{Div}(F_m)$ by i_n^m is contained in the subset of Γ_m^n-invariant elements of $\mathrm{Div}(F_n)$.
The following proposition gives us some information about the injectivity and surjectivity of the maps N and i on the class groups.

Proposition 2.4.4. *Let $F_0 \subseteq K \subset E \subset F_\infty$ with E/F_0 a finite extension.*

(a) The norm map $N_K^E : Cl^0(E) \to Cl^0(K)$ is surjective.

(b) The map $i_K^E : Cl^0(K) \to Cl^0(E)$ is injective.

Proof. (a) Fix a prime ν of K which lies above ∞ and let B be the ring of elements regular outside ν. As we have observed in Section 2.2.1 the prime at infinity does not have inertia in F_∞/F and so the degree of ν is equal to 1. This implies that $Cl(B) \simeq Cl^0(K)$.
Let C be the integral closure of B in E: there is a natural map $Cl^0(E) \to Cl(C)$ which is surjective because the extension E/K is totally split at ν and so, again, every prime of E which lies above ν has degree equal to one. So we are in the following setting:

$$Cl^0(E) \twoheadrightarrow Cl(C) \to Cl(B) \simeq Cl^0(K),$$

where the map $Cl(C) \to Cl(B)$ is the natural norm map. To complete the proof we will have to show that this map is surjective.
Let $H(B)$ (resp. $H(C)$) be the Hilbert class field, i.e, the maximal abelian extension of K (resp. E) which is unramified everywhere and totally split at ν (resp. at the prime of E which lies above ν). By class field theory we have the isomorphisms $Gal(H(B)/K) \simeq Cl(B)$ and $Gal(H(C)/E) \simeq Cl(C)$ induced by the Artin map. Now we observe that the two fields E and $H(B)$ are disjoint over K because E/K is totally ramified at every prime of K which lies above $\mathfrak{p}$ and $H(B)/K$ is unramified at every prime, thus we have the canonical isomorphism $Gal(EH(B)/E) \simeq Gal(H(B)/K)$.
The restriction map

$$Res : Gal\,(H(C)/E) \twoheadrightarrow Gal\,(EH(B)/E) \simeq Gal\,(H(B)/K)$$

fits into the following commutative diagram (from class field theory)

$$\begin{tikzcd} Cl(C) \arrow[r, "\simeq"] \arrow[d, "N"'] & Gal\,(H(C)/E) \arrow[d, "Res"] \\ Cl(B) \arrow[r, "\simeq"] & Gal\,(H(B)/K) \end{tikzcd}$$

and so the surjectivity is proved.

(b) Let $G := Gal(E/K)$ and for every field L we denote with $\mathcal{P}_L$ the set of principal divisors of L. Taking the G-cohomology in the exact sequence

$$0 \longrightarrow \mathbb{F}_q^\times \longrightarrow E^\times \longrightarrow \mathcal{P}_E \longrightarrow 0 \tag{2.11}$$

we obtain

$$0 \longrightarrow \mathbb{F}_q^\times \longrightarrow K^\times \longrightarrow \mathcal{P}_E^G \longrightarrow H^1(G,\mathbb{F}_q^\times) \longrightarrow H^1(G,E^\times) \longrightarrow$$
$$\longrightarrow H^1(G,\mathcal{P}_E) \longrightarrow H^2(G,\mathbb{F}_q^\times).$$

From Hilbert 90 we have that $H^1(G,E^\times)=0$ and, since G is a p-group we also have that $H^i(G,\mathbb{F}_q^\times)$ for every $i\geq 1$, thus $H^1(G,\mathcal{P}_E)=0$ and the following exact sequence holds

$$0 \longrightarrow \mathbb{F}_q^\times \longrightarrow K^\times \longrightarrow \mathcal{P}_E^G \longrightarrow 0\ .$$

If we compare this last sequence with the analogue of (2.11) for the field K we have that $\mathcal{P}_E^G = \mathcal{P}_K$.
Taking the G-cohomology in

$$0 \longrightarrow \mathcal{P}_E \longrightarrow \mathrm{Div}^0(E) \longrightarrow Cl^0(E) \longrightarrow 0\ .$$

we obtain

$$0 \longrightarrow \mathcal{P}_E^G = \mathcal{P}_K \longrightarrow \mathrm{Div}^0(E)^G \longrightarrow Cl^0(E)^G \longrightarrow H^1(G,\mathcal{P}_E) = 0\ ,$$

which fits into the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{P}_K \arrow[r] \arrow[d, equal] & \mathrm{Div}^0(K) \arrow[r] \arrow[d, hook] & Cl^0(K) \arrow[r] \arrow[d, "i_K^E"] & 0 \\ 0 \arrow[r] & \mathcal{P}_K \arrow[r] & \mathrm{Div}^0(E)^G \arrow[r] & Cl^0(E)^G \arrow[r] & 0. \end{tikzcd} \tag{2.12}$$

Applying the snake lemma we obtain the thesis.

□

From diagram (2.12) we also deduce that

$$Cl^0(E)^G/i_K^E\left(Cl^0(K)\right) \simeq \mathrm{Div}^0(E)^G/i_K^E\left(\mathrm{Div}^0(K)\right) \tag{2.13}$$

Now we take a character χ of G_0. Before computing the limit of the class groups we have to study the kernel of the norm map $N_K^E(\chi) : Cl^0(E)(\chi) \to Cl^0(K)(\chi)$ for the intermediate finite extensions E/K.

Lemma 2.4.5. *Let $F_0 \subseteq K \subset E \subset F_\infty$ with E/F_0 a finite extension and let $G := Gal(E/K)$. Assume that $|G| = p$. Then the group $\Delta := Gal(F_0/H_A)$ acts trivially on $Cl^0(E)^G/i_K^E\left(Cl^0(K)\right)$.*

Proof. As we have already observed in (2.13) we have the isomorphism

$$Cl^0(E)^G/i_K^E\left(Cl^0(K)\right) \simeq \mathrm{Div}^0(E)^G/i_K^E\left(\mathrm{Div}^0(K)\right) .$$

If we consider the two maps

$$\mathrm{Div}^0(E)^G \hookrightarrow \mathrm{Div}(E)^G \twoheadrightarrow \mathrm{Div}(E)^G/i_K^E\left(\mathrm{Div}(K)\right)$$

we have that the kernel of the composition of this two maps is $\mathrm{Div}^0(E)^G \cap i_K^E\left(\mathrm{Div}(K)\right) = i_K^E\left(\mathrm{Div}^0(K)\right)$ and so there is an injection

$$\mathrm{Div}^0(E)^G/i_K^E\left(\mathrm{Div}^0(K)\right) \hookrightarrow \mathrm{Div}(E)^G/i_K^E\left(\mathrm{Div}(K)\right).$$

Thus it is enough to show that Δ acts trivially on $\mathrm{Div}(E)^G/i_K^E\left(\mathrm{Div}(K)\right)$.
Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the set of primes of K which lie above $\mathfrak{p}$ and $\mathfrak{P}_1, \dots, \mathfrak{P}_s$ the set of primes of E which lie above $\mathfrak{p}$. We recall that the extension E/K is totally ramified at $\mathfrak{p}$ and so we can assume for every $j = 1, \dots, s$ that $\mathfrak{P}_j$ is the unique prime of E which lies over $\mathfrak{p}_j$, moreover the only extension where the prime $\mathfrak{p}$ (may) split is H_A/F and so s divides $h^0(F)$, i.e., is coprime with p. In particular we have that

$$i_K^E(\mathfrak{p}_j) = p\mathfrak{P}_j .$$

Now we prove that the group $\mathrm{Div}(E)^G/i_K^E\left(\mathrm{Div}(K)\right)$ is isomorphic to the sum of s copies of $\mathbb{Z}/p$ and that a set of generators is $\left\{\mathfrak{P}_j + i_K^E\left(\mathrm{Div}(K)\right)\right\}_{j=1,\dots,s}$. We can write $\mathrm{Div}(K) = \bigoplus_\nu \mathbb{Z}\nu$ (where ν runs through all the primes of K) and $\mathrm{Div}(E) = \bigoplus_\nu H_\nu$, with $H_\nu = \bigoplus_{w|\nu} \mathbb{Z}w$ (where w runs through the set of primes of E which lie above ν). Now for the ramified primes we have $H_{\mathfrak{p}_j} = \mathbb{Z}\mathfrak{P}_j = H_{\mathfrak{p}_j}^G$ and for the unramified primes, if we denote with G_ν the decomposition group of ν in G, we have that $H_\nu = \mathbb{Z}[G/G_\nu]w$ and so $H_\nu^G = i_K^E\left(\mathbb{Z}\nu\right)$. We have proved that

$$\mathrm{Div}(E)^G = \bigoplus_{j=1}^{s} \mathbb{Z}\mathfrak{P}_j \oplus \bigoplus_{\nu \neq \mathfrak{p}_j} i_K^E\left(\mathbb{Z}\nu\right)$$

and

$$i_K^E\left(\mathrm{Div}(K)\right) = \bigoplus_{j=1}^{s} p\mathbb{Z}\mathfrak{P}_j \oplus \bigoplus_{\nu \neq \mathfrak{p}_j} i_K^E\left(\mathbb{Z}\nu\right)$$

thus

$$\mathrm{Div}(E)^G/i_K^E\left(\mathrm{Div}(K)\right) = \bigoplus_{j=1}^{s} \left(\mathbb{Z}/p\right)\mathfrak{P}_j .$$

Now we observe that if we take a set of integers $\alpha_1, \dots, \alpha_s$ each of them coprime with p, we have that also the set $\left\{\alpha_j\mathfrak{P}_j + i_K^E\left(\mathrm{Div}(K)\right)\right\}_{j=1,\dots,s}$ generates $\mathrm{Div}(E)^G/i_K^E\left(\mathrm{Div}(K)\right)$: indeed for each index j, by the Bezout identity, we have that there exist two integers m and n such that $n\alpha_j = 1 + mp$ and so

$$\begin{aligned} n\left(\alpha_j\mathfrak{P}_j + i_K^E\left(\mathrm{Div}(K)\right)\right) &= \mathfrak{P}_j + mp\mathfrak{P}_j + i_K^E\left(\mathrm{Div}(K)\right) \\ &= \mathfrak{P}_j + mi_K^E(\mathfrak{p}_j) + i_K^E\left(\mathrm{Div}(K)\right) \end{aligned}$$

$$= \mathfrak{P}_j + i_K^E\left(\mathrm{Div}(K)\right).$$

Now we consider the field E^{Δ} (resp. K^{Δ}) which is the subfield of E (resp. K) fixed by the elements of Δ. Since Δ is a subgroup of G_0 (with cardinality coprime with p) there is a canonical isomorphism $G^{\Delta} := Gal(E^{\Delta}/K^{\Delta}) \cong G$ and since the extension F_0/H_A is totally ramified at the primes above $\mathfrak{p}$ we have that there are exactly s primes in E^{Δ} (resp. K^{Δ}) above $\mathfrak{p}$.
We denote with $\mathfrak{P}_j^{\Delta}$ (resp. $\mathfrak{p}_j^{\Delta}$) the unique prime of E^{Δ} (resp. K^{Δ}) which lies below $\mathfrak{P}_j$ (resp. $\mathfrak{p}_j$).
As above we can prove that the group $\mathrm{Div}(E^{\Delta})^{G^{\Delta}}/i_{K^{\Delta}}^{E^{\Delta}}\left(\mathrm{Div}(K^{\Delta})\right)$ is isomorphic to the sum of s copies of $\mathbb{Z}/p$ and that a set of generators is $\left\{\mathfrak{P}_j^{\Delta} + i_{K^{\Delta}}^{E^{\Delta}}\left(\mathrm{Div}(K^{\Delta})\right)\right\}_{j=1,\dots,s}$. To conclude we observe that $i_{E^{\Delta}}^{E}(\mathfrak{P}_j^{\Delta}) = |\Delta|\mathfrak{P}_j$ and, since $|\Delta|$ is coprime with p, the image of these classes under the map $i_{E^{\Delta}}^{E}$ is a set of generators for $\mathrm{Div}(E)^G/i_K^E\left(\mathrm{Div}(K)\right)$ and clearly the action of Δ on this classes is trivial. □

We use the previuos lemma to prove:

Proposition 2.4.6. *Let $F_0 \subseteq K \subset E \subset F_\infty$ with E/F_0 a finite extension and let $G := Gal(E/K)$. Then for χ of type 1 or 2 we have*

$$ker\left(N_K^E\right)(\chi) = I_G Cl^0(E)(\chi)$$

where I_G denotes the augumentation of G.

Proof. We will proceed by induction on $|G|$. We need to consider two basic cases: for $|G| = 1$ both members of the equality are zero and so there is nothing to prove.
For $|G| = p$ we have that χ may be seen as a non trivial character of $\Delta = Gal(F_0/H_A)$ because it is of type 1 or 2 and, by the previous lemma, we have that Δ acts trivially on $Cl^0(E)^G/i_K^E\left(Cl^0(K)\right)$, thus we have

$$\left(Cl^0(E)^G/i_K^E\left(Cl^0(K)\right)\right)(\chi) = 0,$$

i.e., $Cl^0(E)^G(\chi) = i_K^E\left(Cl^0(K)\right)(\chi)$. The group G is cyclic and we denote with g a fixed generator, thus $I_G Cl^0(E)(\chi) = (g-1)Cl^0(E)(\chi)$. We also recall the well known isomorphism $I_G Cl^0(E)(\chi) \simeq Cl^0(E)(\chi)/Cl^0(E)^G(\chi)$. Then we have the following two sequences:

$$0 \longrightarrow \ker\left(N_K^E(\chi)\right) \longrightarrow Cl^0(E)(\chi) \xrightarrow{N_K^E} Cl^0(K)(\chi) \longrightarrow 0,$$

$$0 \longrightarrow Cl^0(K)(\chi) \xrightarrow{i_K^E} Cl^0(E)(\chi) \xrightarrow{1-g} I_G Cl^0(E)(\chi) \longrightarrow 0.$$

The first one is exact because of the surjectivity of the norm (Proposition 2.4.4, part (a)), the second one is exact because of what we have previously observed. Comparing the cardinalities of the groups in the two sequences we deduce that $|\ker\left(N_K^E\right)(\chi)| = |I_G Cl^0(E)(\chi)|$, but since $I_G Cl^0(E)(\chi) \subseteq \ker\left(N_K^E\right)(\chi)$, we have the equality between them.
For the inductive step we can now assume $|G| = p^l > p$, thus we can take an intermediate field $K \subsetneqq E' \subsetneqq E$ and let $G_1 = Gal(E/E')$ and $G_2 = Gal(E'/K)$ with both G_1 and G_2 with cardinality strictly smaller than G. Then by the inductive hypothesis we have

$$\ker\left(N_{E'}^E\right)(\chi) = I_{G_1} Cl^0(E)(\chi)$$

and

$$\ker\left(N_K^{E'}\right)(\chi) = I_{G_2}Cl^0(E')(\chi).$$

From Proposition 2.4.4, part (a), we have that $N_{E'}^E : Cl^0(E)(\chi) \to Cl^0(E')(\chi)$ is surjective and so

$$N_{E'}^E\left(I_G Cl^0(E)(\chi)\right) = I_{G_2}Cl^0(E')(\chi).$$

If we take $x \in \ker\left(N_K^E\right)(\chi)$ since $N_K^E = N_K^{E'} \circ N_{E'}^E$ we have that $N_{E'}^E(x) \in \ker\left(N_K^{E'}\right)(\chi) = I_{G_2}Cl^0(E')(\chi)$, thus there exists $g \in I_G$ and $y \in Cl^0(E)(\chi)$ such that $N_{E'}^E(x) = N_{E'}^E(gy)$. This is equivalent to $x - gy \in \ker\left(N_{E'}^E\right)(\chi)$ and so

$$x \in \ker\left(N_{E'}^E\right)(\chi) + I_G Cl^0(E)(\chi) = I_G Cl^0(E)(\chi),$$

since $\ker\left(N_{E'}^E\right)(\chi) = I_{G_1}Cl^0(E)(\chi) \subseteq I_G Cl^0(E)(\chi)$. We have proved that

$$\ker\left(N_K^E\right)(\chi) \subseteq I_G Cl^0(E)(\chi).$$

The other inclusion is trivial. □

Now following the proof of Theorem 2.3.11 we can finally prove:

Theorem 2.4.7. *Let χ be a character of type 1 or 2. Then $C_\infty(\chi)$ is a finitely generated torsion Λ-module.*

Proof. We recall that for every index m we have denoted with $\mathfrak{I}_m$ the augumentation ideal of $W[\![\Gamma_\infty]\!]$ associated to the subgroup $Gal(F_\infty/F_m)$ which is the inverse limit of the augumentation ideals $I_{\Gamma_m^n}$.
From Proposition 2.4.4 part (a) we have $C_m(\chi) = N_m^n\left(C_n(\chi)\right)$ and from Proposition 2.4.6 we deduce

$$N_m^n\left(C_n(\chi)\right) \simeq C_n(\chi)/\mathrm{ker}N_m^n = C_n(\chi)/I_{\Gamma_m^n}C_n(\chi)\,,$$

thus

$$C_m(\chi) \simeq C_n(\chi)/I_{\Gamma_m^n}C_n(\chi)\,.$$

The previous equality holds for every $n > m$ and so we have

$$C_m(\chi) \simeq C_\infty(\chi)/\mathfrak{I}_m C_\infty(\chi)\,.$$

The module on the left is finite and, in particular, finitely generated over

$$W[\Gamma_m] = \Lambda/\mathfrak{I}_m.$$

The ideals $\mathfrak{I}_m$ form an open filtration of the Iwasawa algebra Λ, thus we can deduce that $C_\infty(\chi)$ is a finitely generated module over Λ because of the generalized Nakayama Lemma ([BH]).
To show that $C_\infty(\chi)$ is also torsion we define the element $\Theta_\infty^\sharp(1,\chi) \in \Lambda$ as

$$\Theta_\infty^\sharp(1,\chi) = \begin{cases} \Theta_\infty(1,\chi) & \text{if } \chi \text{ is of type } 1\,, \\[2ex] \dfrac{\Theta_\infty(\gamma^{-1},\chi)}{1-\gamma^{-1}}\Big|_{\gamma=1} & \text{if } \chi \text{ is of type } 2\,, \end{cases}$$

which is simply the inverse limit of the elements $\Theta_n^\sharp(1,\chi)$ which generate the Fitting ideals of the various modules $C_m(\chi)$ over $W[\Gamma_m]$ (Theorem 2.4.3).
Now it is easy to see that $\Theta_\infty^\sharp(1,\chi)C_\infty(\chi) = 0$, since $\Theta_n^\sharp(1,\chi)C_n(\chi) = 0$ for every n, and so $C_\infty(\chi)$ is a torsion Λ-module. □

Now we can prove the main conjecture. The reader may see that the arguments of the proof are similar to the ones of Theorem 2.3.12, indeed this is a consequence of the surjectivity of the norm maps and of the computations on the kernels.

Theorem 2.4.8 (*Main Conjecture*)**.** *Let χ be a character of type 1 or 2. Then we have*

$$\mathit{Fitt}_\Lambda\left(C_\infty(\chi)\right) = \left(\Theta_\infty^\sharp(1,\chi)\right).$$

Proof. The equality

$$\left(\Theta_\infty^\sharp(1,\chi)\right) = \varprojlim_n \left(\Theta_n^\sharp(1,\chi)\right) = \varprojlim_n \mathrm{Fitt}_{W[\Gamma_n]}\left(C_n(\chi)\right)$$

reduce the proof to proving that

$$\mathrm{Fitt}_\Lambda\left(C_\infty(\chi)\right) = \varprojlim_n \mathrm{Fitt}_{W[\Gamma_n]}\left(C_n(\chi)\right).$$

We denote with N_m^∞ the projection $C_\infty(\chi) \twoheadrightarrow C_m(\chi)$ and with $t_1,\dots,t_r$ a fixed set of generators of $C_\infty(\chi)$ over Λ. From the equality $C_m(\chi) = C_\infty(\chi)/\mathfrak{I}_m C_\infty(\chi)$ we deduce that $N_m^\infty(t_1),\dots,N_m^\infty(t_r)$ generate $C_m(\chi)$ over $W[\Gamma_m]$.
For every integer n we have the following exact sequence

$$0 \to K_n \longrightarrow W[\Gamma_n]^{\oplus r} \longrightarrow C_n(\chi) \to 0$$

where the map on the right is given by $(w_1,\dots,w_r) \mapsto \sum_i w_i N_n^\infty(t_i)$, which holds also at the infinite level

$$0 \to K_\infty \longrightarrow \Lambda^{\oplus r} \longrightarrow C_\infty(\chi) \to 0$$

and which fits into the diagram

$$\begin{tikzcd}
0 \arrow[r] & K_n \arrow[r] \arrow[d, "k_m^n"] & W[\Gamma_n]^{\oplus r} \arrow[r] \arrow[d, "\pi_m^n"] & C_n(\chi) \arrow[r] \arrow[d, "N_m^n"] & 0 \\
0 \arrow[r] & K_m \arrow[r] & W[\Gamma_m]^{\oplus r} \arrow[r] & C_m(\chi) \arrow[r] & 0\,.
\end{tikzcd}$$

As usual k_m^n denotes the restriction of the projection π_m^n to the kernel K_n. The kernel of π_m^n is $(I_{\Gamma_m^n}W[\Gamma_n])^{\oplus r}$, while the kernel of N_m^n is $I_{\Gamma_m^n}C_n(\chi)$ due to Proposition 2.4.6 and so the map between these two kernels is surjective. The map π_m^n is surjective thus, by the snake lemma, we have that k_m^n is also surjective and so the previous diagram satisfies the Mittag-Leffler condition which allows us to take the inverse limit. Comparing this limit with the exact sequence at level infinity we obtain that

$$K_\infty = \varprojlim_n K_n.$$

Now, to conclude the proof, the reader may follow the same technical arguments of the second part of the proof of Theorem 2.3.12. □

Now using this Main Conjecture and the theorems of Chapter 1 on the Goss Zeta function and on the ν-adic Zeta function we expect to find some correlation between the Fitting ideal of C_∞ and the special values of the Zeta functions.

2.5 The characters of type 3

In this section we give a look at the χ-part of the modules C_n and $T_p(F_n)$ when χ is a character of type 3. Note that the trivial character χ_0 is one of these characters. Unlike the case of characters of type 1 and 2 we will not be able to obtain results directly on these modules, but we have to deal with their duals ($\mathbb{Z}_p$-duals and Pontrjagin duals). At the end of the section we will also focus on the problems that occur when we try to define the projective limits of these modules.

Throughout this section we will denote with χ a character of type 3, which may be identified canonically with a character of the Picard group $\Delta = Gal(H_A/F) \simeq Pic(A)$. For every G_0-module M we denote with $M^* := \mathrm{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p)$ the $\mathbb{Z}_p$-dual of M. Note that when we consider the scalar extension we have $M^* \otimes_{\mathbb{Z}_p} W = \mathrm{Hom}_W(M \otimes_{\mathbb{Z}_p} W, W)$.
From our computations (2.1) and (2.4) we have that for type 3 characters:

$$H_n(\infty)(\chi) = W[\Gamma_n] \quad \text{and} \quad H_n(\mathfrak{p})(\chi) = W[\![G_{\mathbb{F}}]\!]/\left(1 - \chi(\mathrm{Fr}_{\mathfrak{p}}^{-1})\gamma^{-d_{\mathfrak{p}}}\right) \tag{2.14}$$

where $\mathrm{Fr}_{\mathfrak{p}} \in \Delta$ is the Artin symbol of the prime $\mathfrak{p}$.
We denote, as usual, with D_n the kernel of the degree map

$$\deg : H_n(\infty) \oplus H_n(\mathfrak{p}) \to \mathbb{Z}_p.$$

Then we have that if χ is different from the trivial character χ_0

$$D_n(\chi) \simeq W[\Gamma_n] \oplus W[\![G_{\mathbb{F}}]\!]/\left(1 - \chi(\mathrm{Fr}_{\mathfrak{p}}^{-1})\gamma^{-d_{\mathfrak{p}}}\right) , \tag{2.15}$$

since there is no action of Δ on $\mathbb{Z}_p$ and so $\mathbb{Z}_p(\chi) = 0$.

For the trivial character $\chi = \chi_0$ we first need to consider the degree map on the submodule $H_n(\mathfrak{p})(\chi_0)$. We denote with $D_{n,\mathfrak{p}}(\chi_0)$ the kernel of this map and define

$$\begin{aligned} D_{n,\mathfrak{p}}(\chi_0) \oplus H_n(\infty)(\chi_0) &\to H_n(\mathfrak{p})(\chi_0) \oplus H_n(\infty)(\chi_0) \\ (x, y) &\mapsto (x - \deg y \cdot \mathbf{1}\, , y) \ , \end{aligned}$$

where $\mathbf{1}$ is the element which corresponds to the unity under the isomorphism $H_n(\mathfrak{p})(\chi_0) \simeq W[\![G_{\mathbb{F}}]\!]/\left(1 - \gamma^{-d_{\mathfrak{p}}}\right)$. Clearly this map is injective and its image is contained in $D_n(\chi_0)$, but one can easily see that this map is also onto $D_n(\chi_0)$ with inverse the map

$$\begin{aligned} D_n(\chi_0) &\to D_{n,\mathfrak{p}}(\chi_0) \oplus H_n(\infty)(\chi_0) \\ (x, y) &\mapsto (x + \deg y \cdot \mathbf{1}\, , y) \ . \end{aligned}$$

We have shown that

$$D_n(\chi_0) \simeq D_{n,\mathfrak{p}}(\chi_0) \oplus H_n(\infty)(\chi_0) \, .$$

Now we compute $D_{n,\mathfrak{p}}(\chi_0)$: since the primes of F_n^{ar} which lie above $\mathfrak{p}$ have all degree 1 and the action of γ (and also γ^{-1}) simply permutes these primes, we have that γ does not change the

degree of an element, i.e., $\deg(\gamma^{-1}x) = \deg x$ for each element $x \in H_n(\mathfrak{p})(\chi_0)$. This implies that $(1-\gamma^{-1})H_n(\mathfrak{p})(\chi_0)$ is contained in $D_{n,\mathfrak{p}}(\chi_0)$, but from the isomorphism $H_n(\mathfrak{p})(\chi_0) \simeq W[\![G_{\mathbb{F}}]\!]/\left(1-\gamma^{-d_\mathfrak{p}}\right)$ we deduce that $(1-\gamma^{-1})H_n(\mathfrak{p})(\chi_0)$ is exactly $D_{n,\mathfrak{p}}(\chi_0)$.
Thus we have proved that

$$D_n(\chi) \simeq \begin{cases} W[\Gamma_n] \oplus W[\![G_{\mathbb{F}}]\!]/\left(1-\chi(\mathrm{Fr}_\mathfrak{p}^{-1})\gamma^{-d_\mathfrak{p}}\right) & \text{if } \chi \neq \chi_0\,, \\ \\ W[\Gamma_n] \oplus W[\![G_{\mathbb{F}}]\!]/\left(\frac{1-\gamma^{-d_\mathfrak{p}}}{1-\gamma^{-1}}\right) & \text{if } \chi = \chi_0\,. \end{cases} \tag{2.16}$$

Now we want to give a resolution of $D_n(\chi)$ as a module over $W[\Gamma_n][\![G_{\mathbb{F}}]\!]$. First we start considering the case $\chi \neq \chi_0$. We denote with $n(\Gamma_n) \in W[\Gamma_n][\![G_{\mathbb{F}}]\!]$ the element

$$n(\Gamma_n) = \sum_{\sigma\in\Gamma_n} \sigma\,.$$

It is easy to see that for each $\sigma \in \Gamma_n$ we have $(\sigma-1)n(\Gamma_n) = 0$. Now we denote with I_{Γ_n} the augumentation ideal of $W[\Gamma_n][\![G_{\mathbb{F}}]\!]$ associated to Γ_n, i.e., the ideal generated by the elements $\sigma - 1$, with $\sigma \in \Gamma_n$. From what we have just observed we have that the multiplication by $n(\Gamma_n)$ on the quotient $W[\Gamma_n][\![G_{\mathbb{F}}]\!]/\left(1-\chi(\mathrm{Fr}_\mathfrak{p}^{-1})\gamma^{-d_\mathfrak{p}}\right)$ induces a map

$$H_n(\mathfrak{p})(\chi) = W[\Gamma_n][\![G_{\mathbb{F}}]\!]/\left(I_{\Gamma_n}\,, 1-\chi(\mathrm{Fr}_\mathfrak{p}^{-1})\gamma^{-d_\mathfrak{p}}\right) \xrightarrow{n(\Gamma_n)} W[\Gamma_n][\![G_{\mathbb{F}}]\!]/\left(1-\chi(\mathrm{Fr}_\mathfrak{p}^{-1})\gamma^{-d_\mathfrak{p}}\right)$$

whose cokernel is $W[\Gamma_n][\![G_{\mathbb{F}}]\!]/\left(n(\Gamma_n)\,, 1-\chi(\mathrm{Fr}_\mathfrak{p}^{-1})\gamma^{-d_\mathfrak{p}}\right)$. So we are in the following situation:

$$H_n(\mathfrak{p})(\chi) \xrightarrow{n(\Gamma_n)} W[\Gamma_n][\![G_{\mathbb{F}}]\!]/\left(1-\chi(\mathrm{Fr}_\mathfrak{p}^{-1})\gamma^{-d_\mathfrak{p}}\right) \twoheadrightarrow W[\Gamma_n][\![G_{\mathbb{F}}]\!]/\left(n(\Gamma_n)\,, 1-\chi(\mathrm{Fr}_\mathfrak{p}^{-1})\gamma^{-d_\mathfrak{p}}\right). \tag{2.17}$$

We will show now that this sequence is exact, i.e., that the map of the left is injective. To do this we observe that all the modules in the sequence are free over W and we count the ranks. From (2.14) we have that the module on the left has rank $d_\mathfrak{p}$. For the central module we have

$$W[\Gamma_n][\![G_{\mathbb{F}}]\!]/\left(1-\chi(\mathrm{Fr}_\mathfrak{p}^{-1})\gamma^{-d_\mathfrak{p}}\right) \simeq \left(W[\![G_{\mathbb{F}}]\!]/\left(1-\chi(\mathrm{Fr}_\mathfrak{p}^{-1})\gamma^{-d_\mathfrak{p}}\right)\right)[\Gamma_n]$$

and so the rank is $d_\mathfrak{p}|\Gamma_n|$. For the module on the right we have

$$W[\Gamma_n][\![G_{\mathbb{F}}]\!]/\left(n(\Gamma_n)\,, 1-\chi(\mathrm{Fr}_\mathfrak{p}^{-1})\gamma^{-d_\mathfrak{p}}\right) \simeq \left(\left(W[\Gamma_n]/n(\Gamma_n)\right)[\![G_{\mathbb{F}}]\!]\right)/\left(1-\chi(\mathrm{Fr}_\mathfrak{p}^{-1})\gamma^{-d_\mathfrak{p}}\right)$$

whose rank is

$$d_\mathfrak{p}(|\Gamma_n| - |\Gamma_n/\Gamma_n|) = d_\mathfrak{p}|\Gamma_n| - d_\mathfrak{p}.$$

Thus we have proved that (2.17) is exact.
Now simply recalling that

$$H_n(\infty)(\chi) = W[\Gamma_n] \simeq W[\Gamma_n][\![G_{\mathbb{F}}]\!]/(1-\gamma^{-1})$$

we obtain the following resolution for $D_n(\chi)$

$$0 \longrightarrow D_n(\chi) \longrightarrow W[\Gamma_n][\![G_{\mathbb{F}}]\!]/(1-\gamma^{-1}) \oplus W[\Gamma_n][\![G_{\mathbb{F}}]\!]/\left(1-\chi(\mathrm{Fr}_\mathfrak{p}^{-1})\gamma^{-d_\mathfrak{p}}\right) \longrightarrow$$
$$\longrightarrow W[\Gamma_n][\![G_{\mathbb{F}}]\!]/\left(n(\Gamma_n)\,, 1-\chi(\mathrm{Fr}_\mathfrak{p}^{-1})\gamma^{-d_\mathfrak{p}}\right) \longrightarrow 0. \tag{2.18}$$

Proceeding in a similar way for the trivial character, and using the computation done for $D_n(\chi_0)$ one obtains the resolution

$$D_n(\chi_0) \hookrightarrow W[\Gamma_n][\![G_{\mathbb{F}}]\!]/(1-\gamma^{-1}) \oplus W[\Gamma_n][\![G_{\mathbb{F}}]\!]/\,(v_{\mathfrak{p}}) \twoheadrightarrow W[\Gamma_n][\![G_{\mathbb{F}}]\!]/\,(n(\Gamma_n)\,,v_{\mathfrak{p}})\,, \tag{2.19}$$

where we have put $v_{\mathfrak{p}} := (1-\gamma^{-d_{\mathfrak{p}}})/(1-\gamma^{-1})$.

Taking the χ-part in the exact sequence [GP1, after Definition 2.6] one obtains

$$T_p(F_n)(\chi) \hookrightarrow T_p(\mathcal{M}_n)(\chi) \twoheadrightarrow D_n(\chi)$$

which can be joined with (2.18), to obtain

$$\begin{tikzcd} T_p(F_n)(\chi) \arrow[r, hook] & T_p(\mathcal{M}_n)(\chi) \arrow[r] & W[\Gamma_n][\![G_{\mathbb{F}}]\!]/(1-\gamma^{-1}) \oplus W[\Gamma_n][\![G_{\mathbb{F}}]\!]/\,\big(1-\chi(\mathrm{Fr}_{\mathfrak{p}}^{-1})\gamma^{-d_{\mathfrak{p}}}\big) \arrow[d, two heads] \\ & & W[\Gamma_n][\![G_{\mathbb{F}}]\!]/\,\big(n(\Gamma_n)\,,1-\chi(\mathrm{Fr}_{\mathfrak{p}}^{-1})\gamma^{-d_{\mathfrak{p}}}\big) \end{tikzcd}$$

for $\chi \neq \chi_0$. All the four modules in the previous sequence are free and finitely generated over W (and so also over $\mathbb{Z}_p$). The module $T_p(\mathcal{M}_n)(\chi)$ has projective dimension 1 over $W[\Gamma_n][\![G_{\mathbb{F}}]\!]$ because of [GP1, Theorem 3.9] and the same is trivially true also for the third module. This allows us to apply [GP2, Lemma 2.4] and obtain

$$(1-\gamma^{-1})\left(1-\chi(\mathrm{Fr}_{\mathfrak{p}}^{-1})\gamma^{-d_{\mathfrak{p}}}\right)\mathrm{Fitt}_{W[\Gamma_n][\![G_{\mathbb{F}}]\!]}T_p(F_n)(\chi)^* = \Theta_n(\gamma^{-1},\chi)\cdot\left(n(\Gamma_n)\,,1-\chi(\mathrm{Fr}_{\mathfrak{p}}^{-1})\gamma^{-d_{\mathfrak{p}}}\right). \tag{2.20}$$

Proceeding in a similar way for the trivial character and using the resolution of $D_n(\chi_0)$ above, we obtain

$$(1-\gamma^{-d_{\mathfrak{p}}})\mathrm{Fitt}_{W[\Gamma_n][\![G_{\mathbb{F}}]\!]}T_p(F_n)(\chi_0)^* = \Theta_n(\gamma^{-1},\chi_0)\cdot(n(\Gamma_n)\,,v_{\mathfrak{p}})\,. \tag{2.21}$$

Now we focus a little bit more on equation (2.20): consider the natural projection $\pi : W[\Gamma_n][\![G_{\mathbb{F}}]\!] \to W[\Gamma_n]$ which maps γ to 1. If $\mathrm{Fr}_{\mathfrak{p}}$ lies in the kernel of the character χ, i.e., the character is trivial on the decomposition group of $\mathfrak{p}$, the left hand side of the equality has a zero of order (at least) 2 when evaluated at $\gamma = 1$ and the ideal $\big(n(\Gamma_n)\,,1-\chi(\mathrm{Fr}_{\mathfrak{p}}^{-1})\gamma^{-d_{\mathfrak{p}}}\big)$ becomes principal and generated by the non zero element $n(\Gamma_n)$, thus the Stickelberger series $\Theta_n(X,\chi)$ has a zero of order at least 2 in $X = 1$. If $\mathrm{Fr}_{\mathfrak{p}}$ does not belong to the kernel of χ, then the ideal $\big(n(\Gamma_n)\,,1-\chi(\mathrm{Fr}_{\mathfrak{p}}^{-1})\gamma^{-d_{\mathfrak{p}}}\big)$ does not become principal (in general) when projected to $W[\Gamma_n]$. Similar consideration allows us to conclude that the Stickelberger series has a zero at $X = 1$, but in this case we cannot say that the order is greater than one.

The same consideration applied to equation (2.21) lead us to conclude that also the Stickelberger series $\Theta_n(X,\chi_0)$ has a zero at $X = 1$, of order at least 1. Thus if we put

$$\Theta_n^\sharp(\gamma^{-1},\chi) = \begin{cases} \frac{\Theta_n(\gamma^{-1},\chi)}{(1-\gamma^{-1})^2} & \text{if } \mathrm{Fr}_{\mathfrak{p}} \in \ker\chi \text{ and } \chi \neq \chi_0\,, \\[2ex] \dfrac{\Theta_n(\gamma^{-1},\chi)}{1-\gamma^{-1}} & \text{otherwise}\,, \end{cases}$$

we have

Theorem 2.5.1. *Let $\chi \in \widehat{G}_0$ be a character of type 3. Then we have*

$$Fitt_{W_n[\![G_{\mathbb{F}}]\!]} T_p(F_n)(\chi)^* = \begin{cases} \Theta_n^\sharp(\gamma^{-1},\chi)\cdot\left(\frac{n(\Gamma_n)}{1-\chi(Fr_{\mathfrak{p}}^{-1})\gamma^{-d_{\mathfrak{p}}}}\,,1\right) & \text{if } Fr_{\mathfrak{p}}\notin ker\chi\,, \\ \Theta_n^\sharp(\gamma^{-1},\chi)\cdot\left(\frac{n(\Gamma_n)}{1+\gamma^{-1}+\cdots+\gamma^{-d_{\mathfrak{p}}+1}}\,,1-\gamma^{-1}\right) & \text{if } Fr_{\mathfrak{p}}\in ker\chi \text{ and } \chi\neq\chi_0\,, \\ \Theta_n^\sharp(\gamma^{-1},\chi_0)\cdot\left(\frac{n(\Gamma_n)}{1+\gamma^{-1}+\cdots+\gamma^{-d_{\mathfrak{p}}+1}}\,,1\right) & \text{if } \chi=\chi_0\,. \end{cases}$$

Now we move to the study of the class group of F_n as a module over $W[\Gamma_n]$. In the beginning of [GP2, Section 3] it is shown that

$$C_n^\vee \simeq T_p(F_n)^*/(1-\gamma^{-1})T_p(F_n)^*,$$

where $C_n^\vee = \mathrm{Hom}(C_n, \mathbb{Q}_p/\mathbb{Z}_p)$ is the Pontrjagin dual of C_n. Thus we can now apply Lemma 2.4.2 to obtain

$$\mathrm{Fitt}_{W[\Gamma_n]} C_n(\chi)^\vee = \pi\left(\mathrm{Fitt}_{W[\Gamma_n][\![G_{\mathbb{F}}]\!]} T_p(F_n)(\chi)^*\right),$$

where $\pi : W[\Gamma_n][\![G_{\mathbb{F}}]\!] \to W[\Gamma_n]$ is the map which sends γ to 1. Then we have

Theorem 2.5.2. *Let $\chi \in G_0$ be a character of type 3. Then we have*

$$Fitt_{W[\Gamma_n]} C_n(\chi)^\vee = \begin{cases} \Theta_n^\sharp(1,\chi)\cdot\left(\frac{n(\Gamma_n)}{1-\chi(Fr_{\mathfrak{p}}^{-1})}\,,1\right) & \text{if } Fr_{\mathfrak{p}}\notin ker\chi\,, \\ \Theta_n^\sharp(1,\chi)\cdot\left(\frac{n(\Gamma_n)}{d_{\mathfrak{p}}}\right) & \text{if } Fr_{\mathfrak{p}}\in ker\chi \text{ and } \chi\neq\chi_0\,, \\ \Theta_n^\sharp(1,\chi_0)\cdot\left(\frac{n(\Gamma_n)}{d_{\mathfrak{p}}}\,,1\right) & \text{if } \chi=\chi_0\,. \end{cases}$$

Remark 2.5.3. We would like to point out some particular cases: if the decomposition group of the prime $\mathfrak{p}$ is contained is the kernel of the non trivial character χ, then the Fitting ideal of $C_n(\chi)^\vee$ is principal as we see from the previous theorem.
Another interesting case is when the degree of the prime $\mathfrak{p}$ is coprime with p. In this case the degree $d_{\mathfrak{p}}$ is invertible in $\mathbb{Z}_p$ and so the fractional ideal $\left(\frac{n(\Gamma_n)}{d_{\mathfrak{p}}}\,,1\right)$ actually is an integral ideal, which has 1 as one of its generators. Thus in this case the Fitting ideal of $C_n(\chi_0)^\vee$ is principal and it is generated by $\Theta_n^\sharp(1,\chi_0)$.

Now it would be interesting to proceed like for characters of type 1 and 2 and make some kind of projective limits of the modules $T_p(F_n)(\chi)^*$ and $C_n(\chi)^\vee$ and study the Fitting ideal of this limit. However at this point it is not clear which are the maps we have to consider to make this limit: note that the natural norm maps N_n^{n+m} from $T_p(F_{n+m})(\chi)$ to $T_p(F_n)(\chi)$ and from $C_{n+m}(\chi)$ to $C_n(\chi)$ induce maps on the dual with the opposite direction, so it is not possible to use them to make a projective limit. Another problem is that when considering the natural projection $\pi_n^{n+m} : W[\Gamma_{n+m}] \to W[\Gamma_n]$ we have that $\pi_n^{n+m}(n(\Gamma_{n+m})) = |A/\mathfrak{p}|^m \cdot n(\Gamma_n)$, thus the generators of the Fitting ideals of $C_n(\chi)^\vee$ are not compatible with respect of this projection map, and cannot be used to define an element in the algebra $W[\![\Gamma_\infty]\!]$.

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