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Topics in Algebraic Supergeometry over Projective Spaces

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Introduction

"Super mathematics" has quite a long history, starting from the pioneering papers by Berezin [10] and [11], before the discovery of supersymmetry in physics[1]. After its appearance in physics in the 70s, however, super mathematics, and in particular supergeometry, has caught more attention in the mathematical community, and corresponding developments appeared not only in numerous research papers but also in books devoted to the subject, see for example [4], [9], [19], [20], [38], [63] and the recent [16].
In most of the concrete applications of supersymmetry, like in quantum field theory or in supergravity, algebraic properties play a key role, whereas geometry has almost always a marginal role: as noted by Witten at the beginning of [68] "natural physics questions requires only the most basics facts about supermanifolds". This is perhaps the reason why some subtle questions in supergeometry (see [41] and [42] for a deep mathematical approach to supergeometry) have not attracted much attention of physicists and, as a consequence, the necessity of further developments has not been stimulated.
String theory makes exception, though.
The interest of pure mathematics in strings dates back to the early days of the theory, when, in the mid 80s, it was realised that the right mathematical framework needed to provide a rigorous description of bosonic string theory is the one of the algebraic geometry of Riemann surfaces. In particular, in the context of perturbative bosonic string theory, it was realised that the structure of the amplitudes is related to certain invariants on the moduli space of Riemann surfaces [8]. This fact triggered the interest of several mathematicians in string theory, resulting in a first period of fruitful mutual exchanges between the physics of strings and pure mathematics: in this context the work of Beilinson and Manin [7] is a representative one.
Despite some appealing features though, bosonic string theory is plagued by a number of flaws - *e.g.* divergencies and absence of matter in its spectrum - that caused it to be abandoned and relegated to the status of a toy theory. Instead, it was observed that, once properly supplemented with supersymmetry, string theory, or better, *superstring theory*, reveals a number of striking features: it admits gauge groups large enough to include the Standard Model and it has the graviton in its spectrum. From the theoretical physics side, these discoveries instantly made superstring theory into the strongest candidate for a unified theory of matter and interactions and put it in the spotlight of fundamental research. On the other hand, in mathematics, the quest for rigorous foundations of the newborn superstring theory gave rise to new questions, attracting the attention of the mathematical community to the rigorous characterisation of supersymmetry and, in particular, to *supergeometry*, the kind of geometry lying at the very basis of superstring theory. Indeed, perturbative superstring theory is expected to be described in terms of the moduli space of *super Riemann surfaces*, a sort of "supergeometric analog" of ordinary Riemann surfaces, which results to be itself a supermanifold (actually a superstack). The interested reader might want to look, for example, at [29], [47] for references on super Riemann surfaces and their supergeometry. Of particular interest, in the opinion of the author, are the little known papers [53] and [54], that attempted first an algebraic geometric approach to super Riemann surfaces. However, some ambiguities in defining superstring amplitudes at genus higher than one suggested, already in the 80s, that the geometry of such a supermoduli space may not be trivially obtained from the geometry of the bosonic underlying space [3]. More than twenty years of efforts have been necessary in order to unambiguously compute genus two amplitudes: this has been achieved by D'Hoker and

[1] It is fair to say, though, that the introduction of anticommuting variables was proposed yet previously by Schwinger and other physicists, see [35] for a more detailed historical account on the genesis of super-mathematics.

Phong in a series of seven celebrated papers - see in particular the first three [21] [22] [23]. These papers also include some - actually unsuccessful - attempts in defining genus three amplitudes, that renewed the interest of the physical and mathematical community in looking for a solution to the problem of constructing higher genus amplitudes. Through the years, various proposals have been put forward, see *e.g.* [12], [13] and [31], [56].

However, most of such constructions were based on the assumption that the supermoduli space is *projected* (see the first chapter of this thesis for an explanation) so that its supergeometry can be reconstructed by the ordinary geometry of the underlying ordinary moduli space of genus g spin curves. A careful analysis of perturbative superstring theory [66] [67] and of the corresponding role of supergeometry [68] [69] suggested that this could not be the case. Indeed, it has been proved by Donagi and Witten in the groundbreaking paper [25] (see also [26]) that the *supermoduli space is not projected* at least for genus $g \geqslant 5$. Obviously, this result gave rise to new interest in understanding the peculiarities of supergeometry with respect to the usual geometry, in particular from the viewpoint of algebraic geometry: this is the main motivation at the origin of the present thesis.

Before we discuss the structure of the thesis, a brief general consideration. It is the opinion of the author that part of the issues in understanding the geometry of supermanifolds and its distinctive features - such as for example the presence of the so-called non-projected supermanifolds - is due in a certain amount to the absence of explicit constructions and examples in the literature. We have taken a special care in this thesis to keep the exposition as neat and explicit as possible, by providing every step in the proofs and computations and by supplementing every construction and theorem with explicit realisations and examples, in order for every result to be immediately comprehensible. This justifies, in particular, working over (complex) projective spaces - the ordinary manifolds that will appear the most in this thesis -, since these varieties have an obvious open cover that allows for a meaningful and instructive local realisation of the intrinsic constructions that will be discussed.

More in detail, the thesis is structured as follows.

The first chapter is devoted to an introduction to algebraic supergeometry. It has to be stressed, however, that this chapter contains original research as well (see in particular [14]).
In the first section the main definitions are given and the notation that will be used throughout the thesis is laid down. In particular, we introduce the fundamental notions of superspace, local model and, finally, we give the definition of supermanifold. We concentrate on complex supermanifolds, providing the notions of *projected* and *split* complex supermanifolds, that play a central role in the rest of the thesis. A supermanifold comes naturally endowed with a short exact sequence that relates the structure sheaf of a supermanifold with its nilpotent sheaf and the structure sheaf of the reduced manifold $\mathcal{M}_{red}$ underlying the supermanifold, we call it *structural exact sequence*. In the case this short exact sequence splits we say that the associated supermanifold is projected, otherwise we say that the supermanifold is non-projected. A split supermanifold, instead, is a supermanifold that is globally isomorphic to its local model: a non trivial example of this class of supermanifolds is provided by *projective superspaces*, $\mathbb{P}^{n|m}$.
In the second section locally-free sheaves, in particular rank $1|0$ locally-free sheaves - called *even invertible sheaves* - and the related *even Picard group*, are introduced.
The third section is concerned with the tangent and cotangent sheaf of a supermanifold. We first work in full generality showing some short exact sequences these sheaves fit into [14]. Then we specialise to the case of projected supermanifolds, showing that in this case the sequences split and there exists a relationship with the fermionic sheaf $\mathcal{F}_{\mathcal{M}}$ of the supermanifold introduced in the first section [14].
In the third section, we introduce the *Berezinian sheaf* of a supermanifold. It plays a role similar to the canonical sheaf of an ordinary complex manifold. The results obtained in the second section are used to show that the Berezinian of a projected supermanifold can be entirely reconstructed from two sheaves living on the reduced manifold: the canonical sheaf of reduced manifold and the fermionic sheaf [14]. This result, in turn, allows the introduction of a notion of first Chern class for projected supermanifolds, that coincides with the usual first Chern class of the reduced manifold once the odd part of the geometry has been discarded [14]. Finally, we single out a special class

of supermanifolds having trivial Berezinian sheaf and we call them Calabi-Yau supermanifolds, by analogy with the usual definition of Calabi-Yau manifolds in complex algebraic geometry [45], [14]. All of the results, theorems and constructions of this section are supported by the explicit example of projective superspaces. In particular, we compute the Berezinian sheaf of a projective superspace and its first Chern class.

The second chapter of the thesis is entirely dedicated to the study of complex projective superspaces [14]. Projective superspaces are usually considered to be well-understood supermanifolds (they are split supermanifolds, as shown in the first chapter, and various realisations are known). They have also entered several formal constructions in theoretical physics. However, some of their geometric structures and properties have never been investigated in detail, nor established on a rigorous basis.
In particular, in the first section we compute in detail the Čech cohomology of the sheaves of the form $\mathcal{O}_{\mathbb{P}^{n|m}}(\ell)$. These are the pull-back sheaves on $\mathbb{P}^{n|m}$ via the projection map $\pi : \mathbb{P}^{n|m} \to \mathbb{P}^n$ of the ordinary invertible sheaves $\mathcal{O}_{\mathbb{P}^n}(\ell)$ on $\mathbb{P}^n$ [14].
Then, we study the *even Picard group* of projective superspaces, $\mathrm{Pic}_0(\mathbb{P}^{n|m})$, that classifies locally-free sheaves of rank $1|0$ over $\mathbb{P}^{n|m}$. In particular, we show that in the case of the supercurves $\mathbb{P}^{1|m}$ the even Picard group has a continuous part and we give the explicit form of its generators, proving that there exist genuinely supersymmetric invertible sheaves on $\mathbb{P}^{n|m}$ that do *not* come from any ordinary invertible sheaves $\mathcal{O}_{\mathbb{P}^n}(\ell)$ on $\mathbb{P}^n$ [14]. These prove to be non-trivial geometric objects, indeed they have in general non-trivial cohomology, as we show by means of an example. In the third section, using a supersymmetric generalisation of the Euler exact sequence, we study the cohomology of the tangent sheaf of $\mathbb{P}^{n|m}$, which is related to the *infinitesimal automorphisms* and the *first order deformations* of $\mathbb{P}^{n|m}$. In this context, we find that supercurves over $\mathbb{P}^1$ yield again the richest scenario, allowing for many deformations as their odd dimension increases [14].
Finally, with special attention to applications in theoretical physics, the example of the Calabi-Yau supermanifold $\mathbb{P}^{1|2}$ is examined. In particular, we show in full detail how to endow $\mathbb{P}^{1|2}$ with a structure of $\mathcal{N} = 2$ *super Riemann surface*. In this context, we show how to recover from first principles the $\mathcal{N} = 2$ SUSY-preserving automorphisms of $\mathbb{P}^{1|2}$ when structured as a $\mathcal{N} = 2$ super Riemann surface [14]. These SUSY-preserving automorphisms prove to be isomorphic to the Lie superalgebra $\mathfrak{osp}(2|2)$. We give a presentation of $\mathfrak{osp}(2|2)$ relevant for applications of theoretical physics, by exhibiting a particularly meaningful system of generators and displaying their structure equations [14].
Further, following a formal construction based on the path-integral formalism due to Aganagic and Vafa [2], we construct the "mirror supermanifold" (in the sense of Aganagic and Vafa) to $\mathbb{P}^{1|2}$ and we show, by some suitable changes of coordinates, that this supposed mirror supermanifold is again $\mathbb{P}^{1|2}$, that is $\mathbb{P}^{1|2}$ is self-mirror in the sense of Vafa and Aganagic [45]. We stress that this last section has a different flavour compared to the others, as it is based on a formal construction (Aganagic and Vafa) that at present does not have a rigorous mathematical meaning.

In the second chapter of the thesis we studied the geometry of projected, actually split, supermanifolds such as $\mathbb{P}^{n|m}$, whereas the third chapter is dedicated to the study of non-projected supermanifolds instead [15]. We specialise to supermanifolds having odd dimension 2 and we call these $\mathcal{N} = 2$ supermanifolds. Following Manin [41], we provide a detailed construction of the cohomological invariant that obstructs the existence of a projection that splits the structural exact sequence of a supermanifold. In particular, we prove that a supermanifold $\mathcal{M}$ of dimension $n|2$ is described up to isomorphism by the triple $(\mathcal{M}_{red}, \mathcal{F}_{\mathcal{M}}, \omega_{\mathcal{M}})$, where $\mathcal{F}_{\mathcal{M}}$ is the fermionic sheaf of $\mathcal{M}$, actually a locally-free sheaf of $\mathcal{O}_{\mathcal{M}_{red}}$-modules of rank $0|2$, and where $\omega_{\mathcal{M}}$ is a class in $H^1(\mathcal{M}_{red}, \mathcal{T}_{\mathcal{M}_{red}} \otimes Sym^2 \mathcal{F}_{\mathcal{M}})$. The supermanifold is non-projected (and therefore split) if and only if $\omega_{\mathcal{M}}$ is non-zero in $H^1(\mathcal{M}_{red}, \mathcal{T}_{\mathcal{M}_{red}} \otimes Sym^2 \mathcal{F}_{\mathcal{M}})$.
In the second section we specialise to the case that the reduced manifold is a complex projective space $\mathbb{P}^n$ and we prove that there exist non-projected supermanifolds of dimensions $n|2$ over $\mathbb{P}^n$ only for $n = 1$ and $n = 2$.
In the third section, using Grothendieck's splitting theorem for vector bundles over $\mathbb{P}^1$, we give a complete classification of non-projected $\mathcal{N} = 2$ supermanifolds over $\mathbb{P}^1$, here called $\mathbb{P}^1_\omega(m, n)$, by also providing the explicit form of their transition functions. Also, we study the even Picard group

of $\mathbb{P}^1_\omega(m,n)$ - which again has a continuous part - and we give the explicit form of its generators. Finally, using these invertible sheaves, we realise an embedding of $\mathbb{P}^1_\omega(2,2)$ into $\mathbb{P}^{2|2}$, proving that the example given by Witten in [68] of a non-projected supermanifold as a complete intersection into $\mathbb{P}^{2|2}$ is actually nothing but $\mathbb{P}^1_\omega(2,2)$ itself. This is an original section that has not appeared as a paper yet.
In the fourth section we study the non-projected structures for $\mathcal{N}=2$ supermanifolds over $\mathbb{P}^2$, that proves to be the most interesting case. Indeed, a non-projected structure over $\mathbb{P}^2$ exists if and only if the fermionic sheaf is such that $Sym^2\mathcal{F}_\mathcal{M} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$, the canonical sheaf of $\mathbb{P}^2$ [15]. We denote these non-projected supermanifolds with $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$.
Remarkably, we prove that all of these non-projected supermanifolds are Calabi-Yau's and, by studying their even Picard group, we prove that they are *non projective*: they cannot be embedded into any higher-dimensional projective superspace $\mathbb{P}^{n|m}$ [15]. Instead, we show that every non-projected supermanifold $\mathcal{N}=2$ over $\mathbb{P}^2$ can be embedded into a certain super Grassmannian: this is perhaps the main theorem of the chapter. [15]. As explicit examples, we carry out a detailed study of two meaningful cases: when the fermionic sheaf $\mathcal{F}_\mathcal{M}$ is decomposable, given by $\Pi\mathcal{O}_{\mathbb{P}^2}(-1)\oplus\Pi\mathcal{O}_{\mathbb{P}^2}(-2)$ and when it is the cotangent sheaf $\Pi\Omega^1_{\mathbb{P}^2}$ (with reversed parity) over $\mathbb{P}^2$ and we construct the embeddings explicitly.
In the last section of the chapter we study the split loci of the non-projected structures related to the choices $\mathcal{F}_\mathcal{M}=\Pi\mathcal{O}_{\mathbb{P}^2}(-1)\oplus\Pi\mathcal{O}_{\mathbb{P}^2}(-2)$ and $\mathcal{F}_\mathcal{M}=\Pi\Omega^1_{\mathbb{P}^2}$ and we compute their cohomology [15].

The fourth chapter is devoted to the geometry of Π-projective spaces. These particular supermanifolds were introduced by Manin in [42] as the suitable spaces on which one can define the so-called Π-invertible sheaves, the candidates to take over the notion of invertible sheaf in supergeometry.
In this chapter we provide a new construction of Π-projective spaces, in particular we prove that they arise naturally in supergeometry upon considering a non-projected thickening of $\mathbb{P}^n$ related to the cotangent sheaf $\Omega^1_{\mathbb{P}^n}$. More precisely we prove that for $n\geqslant 2$ the Π-projective space $\mathbb{P}^n_\Pi$ can be constructed as the non-projected supermanifold determined by three elements $(\mathbb{P}^n,\Omega^1_{\mathbb{P}^n},\lambda)$, where $\mathbb{P}^n$ is the ordinary complex projective space, $\Omega^1_{\mathbb{P}^n}$ is its cotangent sheaf and λ is a non-zero complex number, that represents the fundamental obstruction class $\omega_\mathcal{M}\in H^1(\mathcal{T}_{\mathbb{P}^n}\otimes\bigwedge^2\Omega^1_{\mathbb{P}^n})\cong\mathbb{C}$ [46]. Likewise, in the case $n=1$ the Π-projective line $\mathbb{P}^1_\Pi$ is the split supermanifold determined by the pair $(\mathbb{P}^1,\Omega^1_{\mathbb{P}^1}\cong\mathcal{O}_{\mathbb{P}^1}(-2))$.
Moreover we show that in any dimension Π-projective spaces are Calabi-Yau supermanifolds [46]. Also, we offer pieces of evidence that, more in general, also Π-Grassmannians can be constructed the same way using the cotangent sheaf of their underlying reduced Grassmannians, provided that also higher, possibly fermionic, obstruction classes can be defined and taken into account [46]. This suggests that this unexpected connection with the cotangent sheaf is characteristic of Π-geometry.
Last we make the connection with the previous chapter, by discussing in more detail the possible embeddings for $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ in relation to Π-projective spaces. In particular, we prove that if we choose the fermionic sheaf to be decomposable, the supermanifold $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ is not only non projective, but also *non Π-projective*: it cannot be embedded into any Π-projective space $\mathbb{P}^n_\Pi$ [15]. However, if we choose the fermionic sheaf of $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ to be the cotangent sheaf $\Pi\Omega^1_{\mathbb{P}^2}$, then $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ is actually the Π-projective plane $\mathbb{P}^2_\Pi$ and as such it has a minimal embedding into the super Grassmannian $G(1|1;\mathbb{C}^{3|3})$ [15].

Chapter 1

Algebraic Supergeometry

This chapter is intended to give a short but self-contained introduction to algebraic supergeometry. In the first section, following in particular the approach of [41], we present some basic material. We give the main definitions and we establish the notations that will be kept throughout this thesis. In particular, attention is paid to lay down the concepts of *split*, *projected* and *non-projected* supermanifold.
In the second section locally-free sheaves on supermanifolds are introduced, with a particular focus on the case of locally-free sheaves of rank $1|0$, we call them *even invertible sheaves.* The classifying space for even invertible sheaves, called *even Picard group* by similarity with the ordinary Picard group, is then introduced and discussed.
The third section is dedicated to two meaningful examples of locally-free sheaves that can be defined on a supermanifold: the tangent and cotangent sheaves. The short exact sequences these sheaves fit into are introduced and studied, in particular in the case of a projected supermanifold.
In the fourth section the *Berezinian sheaf* of a supermanifold is defined and a notion of Calabi-Yau supermanifold is introduced. Again, in the case of a projected supermanifold, using the results of the previous section, the Berezinian sheaf is studied and a supersymmetric version of the first Chern class is laid down.
The constructions are illustrated making use of the example of the complex projective superspace $\mathbb{P}^{n|m}$ throughout the chapter.
The second, third and the fourth sections are modelled on the author's paper [14].

1.1 Main Definitions and Fundamental Constructions

The aim of this section is to introduce the notion of (complex) supermanifolds and to discuss some of the fundamental constructions related to their geometry.
The first step in this direction, lying on the concept of locally-ringed space, is to introduce the basic notion of *superspace.*

Definition 1.1 (Superspace). *A superspace is a pair $(|\mathcal{M}|, \mathcal{O}_{\mathcal{M}})$, where $|\mathcal{M}|$ is a topological space and $\mathcal{O}_{\mathcal{M}}$ is a sheaf of $\mathbb{Z}_2$-graded rings defined over $|\mathcal{M}|$ and such that the stalks $\mathcal{O}_{\mathcal{M},x}$ at every point of $|\mathcal{M}|$ are local rings.*
In other words, a superspace is a locally ringed space having structure sheaf given by a sheaf of $\mathbb{Z}_2$-graded rings.

For the sake of brevity we will call $\mathcal{M}$ the pair $(|\mathcal{M}|, \mathcal{O}_{\mathcal{M}})$ defining the superspace, that is we will define $\mathcal{M} \coloneqq (|\mathcal{M}|, \mathcal{O}_{\mathcal{M}})$.
Once again, that the definition of superspace we have given follows Manin [41] and his algebraic geometrically inclined treatment of supergeometry. It is fair to say, though, that there exists a different, more analitically inclined, approach, which is the one given, for example, in [4], where the structure sheaf gets structured as a sheaf of Fréchet algebras, so a further notion of *semi-norm* should be given.
Before we go further we need to stress two facts. The first one is that in the case of superspace

the requirement about the stalks being local rings reduces to ask that the *even* component of the stalk is a usual commutative local ring, for in superalgebra one has the following lemma.

Lemma 1.1. *Let $A = A_0 \oplus A_1$ a $\mathbb{Z}_2$-graded ring. Then A is local if and only if its even part A_0 is.*

As one can easily realize, this is connected to the fact that the odd elements in $A_1 \subset A$ are nilpotent, and therefore the whole A_1 is contained in every *prime* and *maximal* ideal of the $\mathbb{Z}_2$-graded ring A. This is a very basic but fundamental fact.
Secondly, in the definition above we understood that the restriction morphisms of the $\mathbb{Z}_2$-graded sheaf $\mathcal{O}_{\mathcal{M}}$ are compatible with the grading, that is they never map local odd sections to local even sections and vice-versa. This is actually a general feature of morphisms in supergeometry.
Having defined a superspace as a locally ringed space, one also defines morphisms of superspaces as morphisms of locally ringed spaces.

Definition 1.2 (Morphisms of Superspaces)**.** *Given two superspaces $\mathcal{M}$ and $\mathcal{N}$ a morphism $\varphi : \mathcal{M} \to \mathcal{N}$ is a pair $\varphi := (\phi, \phi^\sharp)$ where*

1. *$\phi : |\mathcal{M}| \to |\mathcal{N}|$ is a continuous map of topological spaces;*
2. *$\phi^\sharp : \mathcal{O}_{\mathcal{N}} \to \phi_* \mathcal{O}_{\mathcal{M}}$ is a morphism of sheaves of $\mathbb{Z}_2$-graded rings, having the property that it preserves the $\mathbb{Z}_2$-grading and that given any point $x \in |\mathcal{M}|$, the homomorphism*
$$\phi^\sharp_x : \mathcal{O}_{\mathcal{N},\phi(x)} \longrightarrow \mathcal{O}_{\mathcal{M},x} \tag{1.1}$$
is local, that is it preserves the (unique) maximal ideal, $\phi^\sharp_x(\mathfrak{m}_{\phi(x)}) \subseteq \mathfrak{m}_x$.

This definition deserves to be commented a little bit further.
First, with an eye to the ordinary theory of schemes in algebraic geometry, we stress that the request that the morphism $\phi^\sharp_x : \mathcal{O}_{\mathcal{N},\phi(x)} \to \mathcal{O}_{\mathcal{M},x}$ preserves the maximal ideal in the second point of the definition above is of particular significance in supergeometry. Indeed it is important to notice that the structure sheaf $\mathcal{O}_{\mathcal{M}}$ of a superspace is in general *not* a sheaf of functions. As long as the structure sheaf $\mathcal{O}_{\mathcal{M}}$ of a certain space or, more in general, of a scheme, is a sheaf of functions, then a section s of $\mathcal{O}_{\mathcal{M}}$ takes values in the field of fractions $k(x) = \mathcal{O}_{\mathcal{M},x}/\mathfrak{m}_x$ that depends on the point $x \in |\mathcal{M}|$, as a function $x \mapsto s(x) \in k(x)$, and the maximal ideal $\mathfrak{m}_x$ contains the germs of functions that vanish at $x \in |\mathcal{M}|$. In the case of superspaces, nilpotent sections - and thus in particular all of the odd sections - would be identically equal to zero as functions on points, and indeed the maximal ideal $\mathfrak{m}_x$ contains the germs of all the nilpotent sections in $\mathcal{O}_{\mathcal{M},x}$. In this context, the request that $\phi^\sharp_x : \mathcal{O}_{\mathcal{N},\phi(x)} \to \mathcal{O}_{\mathcal{M},x}$ is local becomes crucial, while in the case of a genuine sheaf of functions the locality is automatically achieved. In particular, locality implies that a *non* unit element in the stalk $\mathcal{O}_{\mathcal{N},\phi(x)}$, such as a germ of a nilpotent section, can only be mapped to another *non* unit element in $\mathcal{O}_{\mathcal{M},x}$, such as another germ of a nilpotent section. In other words, nilpotent elements cannot be mapped to invertible elements.

Now a fundamental observation is in order. One can always construct a superspace out of two data: a topological space, call it by abuse of notation $|\mathcal{M}|$, and a vector bundle $\mathcal{E}$ over $|\mathcal{M}|$, or, analogously, a locally-free sheaf of $\mathcal{O}_{|\mathcal{M}|}$-modules. Now, we denote $\mathcal{O}_{|\mathcal{M}|}$ the sheaf of continuous functions (with respect to the given topology) on $|\mathcal{M}|$ and we put $\bigwedge^0 \mathcal{E}^* = \mathcal{O}_{|\mathcal{M}|}$. The sheaf of sections of the bundle of exterior algebras $\bigwedge^\bullet \mathcal{E}^*$ has an obvious $\mathbb{Z}_2$-grading (by taking its natural $\mathbb{Z}$-grading mod 2) and therefore in order to realise a superspace it is enough to take the structure sheaf $\mathcal{O}_{\mathcal{M}}$ of the superspace to be the sheaf of sections valued in $\mathcal{O}_{|\mathcal{M}|}$ of the bundle of exterior algebras. This construction is so important to bear its own name.

Definition 1.3 (Local Model $\mathfrak{S}(|\mathcal{M}|, \mathcal{E})$)**.** *Given a pair $(|\mathcal{M}|, \mathcal{E})$, where $|\mathcal{M}|$ is a topological space and $\mathcal{E}$ is a vector bundle over $|\mathcal{M}|$, we call $\mathfrak{S}(|\mathcal{M}|, \mathcal{E})$ the superspace modelled on the pair $(|\mathcal{M}|, \mathcal{E})$, where the structure sheaf is given by the $\mathcal{O}_{|\mathcal{M}|}$-valued sections of the exterior algebra $\bigwedge^\bullet \mathcal{E}^*$.*

Note that we have given a somehow *minimal* definition of local model, indeed we have let $|\mathcal{M}|$ to be no more than a topological space and as such we are only allowed to take $\mathcal{O}_{|\mathcal{M}|}$ to be the sheaf of continuous functions on it. Clearly, we can also work in a richer and more structured category, such as the *differentiable*, *complex analytic* or *algebraic* category. Working in the complex analytic category - the one we will be most concerned with -, for example, we take $|\mathcal{M}|$ to be a complex manifold, $\mathcal{O}_{|\mathcal{M}|}$ to be the sheaf of holomorphic functions on it and $\mathcal{E}$ a holomorphic vector bundle. There are some easy examples of local models that deserve to be mentioned.

Example 1.1 (Affine Superspaces $\mathbb{A}^{p|q}$)**.** *These superspaces are constructed as the local models $\mathfrak{S}(\mathbb{A}^p, \mathcal{O}_{\mathbb{A}^p}^{\oplus q})$, where $\mathbb{A}^p$ is the p-dimensional affine space over the ring (or field) $\mathbb{A}$ and $\mathcal{O}_{\mathbb{A}^p}$ is the sheaf of regular functions over it.*
These are the most common superspaces one encounters in applications of supergeometry to physics. In the differentiable category, modern supersymmetric theories are often formulated in the superspace $\mathbb{R}^{p|q}$, where $\mathcal{O}_{\mathbb{R}^p}$ is the sheaf of $\mathcal{C}^\infty$-functions over $\mathbb{R}^p$: this has the advantage to make supersymmetry manifest, as it becomes a geometric symmetry of the theory.
In what follows we will mostly be concerned with the complex analytic category and with local models of the form $\mathbb{C}^{p|q}$, where $\mathcal{O}_{\mathbb{C}^p}$ is the sheaf of holomorphic functions over $\mathbb{C}^p$.

We are in the position to introduce the notion of supermanifold.

Definition 1.4 (Supermanifold)**.** *A supermanifold is a superspace $\mathcal{M}$ that is locally isomorphic to some local model $\mathfrak{S}(|\mathcal{M}|, \mathcal{E})$.*
In other words, if the topological space $|\mathcal{M}|$ has a (countable) basis $\{U_i\}_{i\in I}$, the structure sheaf $\mathcal{O}_\mathcal{M}$ of the supermanifold $\mathcal{M}$ is described via a collection $\{\psi_{U_i}\}_{i\in I}$ of local isomorphisms of sheaves

$$U_i \longmapsto \psi_{U_i} : \mathcal{O}_\mathcal{M}\lfloor_{U_i} \xrightarrow{\cong} \bigwedge^\bullet \mathcal{E}^*\lfloor_{U_i} \tag{1.2}$$

where we have denoted with $\bigwedge^\bullet \mathcal{E}^*$ the sheaf of sections of the exterior algebra of $\mathcal{E}$.

Let us make some observations before going on, to clarify further the given definition:

1. The definition depends on the category we are working into via the local model $\mathfrak{S}(|\mathcal{M}|, \mathcal{E})$ we choose. In the differentiable and complex analytic category one can restrict the local models to be of the form of $\mathbb{A}^{p|q}$, defined as above (*e.g.* in the case of a differentiable supermanifold, $\mathcal{M}$ is locally isomorphic to $\mathbb{R}^{p|q}$), while in the algebraic category one should allows all local models $\mathfrak{S}(|\mathcal{M}|, \mathcal{E})$ having affine $|\mathcal{M}|$. Generalizing this notion, one is led to the concept of *superscheme.*

2. It is worth stressing out that this point of view, that might appear at first rather abstract, goes along well with the differential geometric intuition behind the concept of manifold: indeed, again, if for example a complex analytic manifold will be a certain object that locally resembles $\mathbb{C}^p$, a complex analytic supermanifold will be an object that locally resemble $\mathbb{C}^{p|q}$ for some p and q: in this case we say that the supermanifold has dimension $p|q$.

3. Finallly, we underline that, in general, the maps in the collection of *local* isomorphisms $\{\psi_{U_i}\}_{i\in I}$ do *not* glue together to give an isomorphism of sheaves! That is, the local isomorphisms do not define in general an isomorphism of sheaves $\psi : \mathcal{O}_\mathcal{M} \to \bigwedge^\bullet \mathcal{E}^*$. If they do, instead, the supermanifold is of a very special kind, as will be explained in the following.

As customary in algebraic geometry, when one is interested in understanding the geometry of a certain geometric object, the reader needs to look at the ring of "functions" that live on it. In our case, a very basic observation to be made is that, given a supermanifold $\mathcal{M}$, because of the $\mathbb{Z}_2$-grading of the structure sheaf $\mathcal{O}_\mathcal{M}$ there will exist a (actually unique) sheaf of ideals $\mathcal{J}_\mathcal{M} \subset \mathcal{O}_\mathcal{M}$ generated by all the nilpotents.

Definition 1.5 (Nilpotent Sheaf $\mathcal{J}_\mathcal{M}$)**.** *Given a supermanifold $\mathcal{M}$ we will call $\mathcal{J}_\mathcal{M}$ the sheaf of ideals generated by all the (nilpotent) odd sections.*

Notice that $\mathcal{J}_\mathcal{M}$ certainly contains the odd part $\mathcal{O}_{\mathcal{M},1}$ of the structure sheaf $\mathcal{O}_\mathcal{M} = \mathcal{O}_{\mathcal{M},0} \oplus \mathcal{O}_{\mathcal{M},1}$, but in the case the supermanifold has more than one odd dimension, it also contains what are called "bosonisations" in physics, which are nilpotent sections but in $\mathcal{O}_{\mathcal{M},0}$. We make this clear by mean of an example. Let us consider a supermanifold having the polynomial superalgebra given

by $\mathbb{C}[x_1, x_2, \theta_1, \theta_2]$ as structure sheaf, we will then have elements of the form $\mathfrak{bos} = f(x_1, x_2)\,\theta_1 \cdot \theta_2$, where $f(x_1, x_2)$ is a polynomial in the even variables x_1 and x_2. These elements are clearly nilpotent and therefore they are contained in $\mathcal{J}_{\mathcal{M}}$, but they are even, $f(x_1, x_2)\,\theta_1 \cdot \theta_2 \in \mathcal{O}_{\mathcal{M},0}$, as the product of two odd generators θ_1 and θ_2 is actually an even element.
If on the one hand the nilpotent sheaf $\mathcal{J}_{\mathcal{M}}$ is expression of the odd geometry of the supermanifold, on the other hand one has also that to every supermanifold $\mathcal{M}$ is attached an ordinary "purely even" manifold, call it $\mathcal{M}_{red}$. This constitutes the spine of the supermanifold, that can be then somehow visualised as an ordinary manifold surrounded by a cloud of nilpotent odd elements. This underlying manifold can be traced back from the supermanifold $\mathcal{M}$ as follows: $\mathcal{J}_{\mathcal{M}}$ defines a sheaf of ideals, therefore it *always* exists a *closed immersion* $\iota : \mathcal{M}_{red} \hookrightarrow \mathcal{M}$, with $\iota := (i, i^\sharp)$ such that

1. $i : |\mathcal{M}| \to |\mathcal{M}|$ is the identity map that maps $|\mathcal{M}|$ to itself.
2. $i^\sharp : \mathcal{O}_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}} \big/ \mathcal{J}_{\mathcal{M}}$ is the quotient morphism at the level of the sheaves and it is sometimes called *augmentation map.*

Notice that looking at the level of the stalks, the morphism $i^\sharp_x : \mathcal{O}_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M},x} \big/ \mathcal{J}_{\mathcal{M},x}$ is clearly surjective for every $x \in \mathcal{O}_{\mathcal{M},x}$, hence it indeed defines a closed immersion. The existence of such a construction allows us to give the following

Definition 1.6 (Reduced Manifold $\mathcal{M}_{red}$). *Given a supermanifold $\mathcal{M} = (|\mathcal{M}|, \mathcal{O}_{\mathcal{M}})$, let $\mathcal{J}_{\mathcal{M}} \subset \mathcal{O}_{\mathcal{M}}$ be its nilpotent sheaf, then we call reduced manifold $\mathcal{M}_{red}$ the ordinary manifold given as a ringed space by the pair $(|\mathcal{M}|, \mathcal{O}_{\mathcal{M}_{red}})$, where $\mathcal{O}_{\mathcal{M}_{red}}$ is defined as $\mathcal{O}_{\mathcal{M}_{red}} := \mathcal{O}_{\mathcal{M}} \big/ \mathcal{J}_{\mathcal{M}}$.*

Incidentally, one might observe that - forgetting the issues relating to the existence of a $\mathbb{Z}_2$-grading - under some circumstances, when Zariski topology is employed, one could look at $\mathcal{M}_{red}$ as a *reduced scheme*, while $\mathcal{M}$ defines a *non reduced scheme*, as its structure sheaf contains nilpotent elements; anyway, without giving any further details we mention that the correct framework to work in would that of *superschemes.*

So far we have thus seen that every supermanifold comes endowed with a surjective morphism of sheaves $i^\sharp : \mathcal{O}_{\mathcal{M}} \twoheadrightarrow \mathcal{O}_{\mathcal{M}_{red}}$, whose kernel is, by construction, given by the nilpotent sheaf: $\mathcal{J}_{\mathcal{M}} = \ker i^\sharp$. Therefore, we have obtained a short exact sequence, actually the most important exact sequence attached to any supermanifold.

Definition 1.7 (Structural Exact Sequence). *Given a supermanifold $\mathcal{M} := (|\mathcal{M}|, \mathcal{O}_{\mathcal{M}})$, let $\mathcal{J}_{\mathcal{M}}$ be its nilpotent sheaf and let $\mathcal{M}_{red} := (|\mathcal{M}|, \mathcal{O}_{\mathcal{M}_{red}})$ be its reduced manifold. Then the structure sheaf $\mathcal{O}_{\mathcal{M}}$, the reduced structure sheaf $\mathcal{O}_{\mathcal{M}_{red}}$ and the nilpotent sheaf $\mathcal{J}_{\mathcal{M}}$ fit together in a short exact sequence of $\mathcal{O}_{\mathcal{M}}$-modules, given by*

$$0 \longrightarrow \mathcal{J}_{\mathcal{M}} \longrightarrow \mathcal{O}_{\mathcal{M}} \longrightarrow \mathcal{O}_{\mathcal{M}_{red}} \longrightarrow 0. \tag{1.3}$$

We call this short exact sequence the structural exact sequence *for the supermanifold $\mathcal{M}$.*

To put things in a different way, the structural exact sequence of a supermanifold says that the structure sheaf $\mathcal{O}_{\mathcal{M}}$ is an *extension* of $\mathcal{O}_{\mathcal{M}_{red}}$ by $\mathcal{J}_{\mathcal{M}}$.

A very natural question that arises looking at the structural exact sequence above is whether it is *split* or not,

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{J}_{\mathcal{M}} \arrow[r] & \mathcal{O}_{\mathcal{M}} \arrow[r, "\iota^\sharp"'] & \mathcal{O}_{\mathcal{M}_{red}} \arrow[l, bend right, dashed, "\pi^\sharp"'] \arrow[r] & 0. \end{tikzcd} \tag{1.4}$$

That is, one might wonder whether there exists a morphism of supermanifolds, we call it $\pi : \mathcal{M} \to \mathcal{M}_{red}$, where $\pi := (\pi, \pi^\sharp)$ are defined as

1. $\pi : |\mathcal{M}| \to |\mathcal{M}|$ is again the identity map that maps the topological space $|\mathcal{M}|$ to itself;
2. $\pi^\sharp : \mathcal{O}_{\mathcal{M}_{red}} \to \mathcal{O}_{\mathcal{M}}$ is a morphism of sheaves of $\mathcal{O}_{\mathcal{M}}$-modules (as we are looking at $\mathcal{O}_{\mathcal{M}_{red}}$ endowed by $\iota^\sharp$ with the structure of $\mathcal{O}_{\mathcal{M}}$-module),

having the property that $\pi \circ \iota = id_{\mathcal{M}_{red}}$. At the level of the structure sheaves, this corresponds to $\pi^\sharp \circ \iota^\sharp = id_{\mathcal{O}_{\mathcal{M}}}$.
In particular, if the morphism $\pi : \mathcal{M} \to \mathcal{M}_{red}$ does exist, then the structure sheaf $\mathcal{O}_{\mathcal{M}}$ is given by the direct sum $\mathcal{O}_{\mathcal{M}} = \mathcal{O}_{\mathcal{M}_{red}} \oplus \mathcal{J}_{\mathcal{M}}$ and the structural exact sequence becomes

$$0 \longrightarrow \mathcal{J}_{\mathcal{M}} \longrightarrow \mathcal{O}_{\mathcal{M}_{red}} \oplus \mathcal{J}_{\mathcal{M}} \longrightarrow \mathcal{O}_{\mathcal{M}_{red}} \longrightarrow 0. \tag{1.5}$$

The supermanifolds that posses such a splitting morphism are given a special name.

Definition 1.8 (Projected Supermanifolds)**.** *Let $\mathcal{M}$ be a supermanifold, if there exists a morphism $\pi : \mathcal{M} \to \mathcal{M}_{red}$, satisfying $\pi \circ \iota = id_{\mathcal{M}_{red}}$, splitting the structural exact sequence of $\mathcal{M}$ as in* (1.4)*, then we say that the supermanifold $\mathcal{M}$ admits a projection on its underlying reduced manifold $\mathcal{M}_{red}$. For short, we say that $\mathcal{M}$ is* projected*.*

The study of obstructions to splitting and the investigation of some examples of non-projected complex supermanifolds will be one the main themes of the present thesis.
Remarkably, the existence of a projection $\pi : \mathcal{M} \to \mathcal{M}_{red}$ has also another very important consequence, namely the structure sheaf $\mathcal{O}_{\mathcal{M}}$ becomes a sheaf of $\mathcal{O}_{\mathcal{M}_{red}}$-modules as well, for, given an open set $U \subset |\mathcal{M}|$, one can define a multiplication by:

$$\begin{aligned} \left(\mathcal{O}_{\mathcal{M}_{red}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}}\right)(U) &\longrightarrow \mathcal{O}_{\mathcal{M}}(U) \\ f \otimes s &\longmapsto \pi^\sharp_U(f) \cdot s. \end{aligned}$$

and this extends to the whole variety $|\mathcal{M}|$ by the properties of sheaves, actually defining a sheaf of $\mathcal{O}_{\mathcal{M}_{red}}$-modules, as claimed.
In general, if the supermanifold is non-projected, the structure sheaf $\mathcal{O}_{\mathcal{M}}$ is *not* a sheaf of $\mathcal{O}_{\mathcal{M}_{red}}$-modules. Likewise, on a non-projected supermanifold, a sheaf of $\mathcal{O}_{\mathcal{M}}$-modules is not in general a sheaf of $\mathcal{O}_{\mathcal{M}_{red}}$-modules: this is a crucial issue in the general theory of supermanifolds.

Now that we know that the presence of a projection $\pi : \mathcal{M} \to \mathcal{M}_{red}$ singles out a class of relatively easier and more tractable supermanifolds, one might further wonder whether there exists any even simpler sub-class of supermanifolds among the projected ones. To answer this question we introduce the following construction: we consider a supermanifold $\mathcal{M}$ having a given odd dimension equal to q, together with its nilpotent sheaf $\mathcal{J}_{\mathcal{M}}$, then it is easily seen that we have a *$\mathcal{J}_{\mathcal{M}}$-adic filtration on* $\mathcal{O}_{\mathcal{M}}$ of length q, that is

$$\mathcal{J}^0_{\mathcal{M}} := \mathcal{O}_{\mathcal{M}} \supset \mathcal{J}_{\mathcal{M}} \supset \mathcal{J}^2_{\mathcal{M}} \supset \mathcal{J}^3_{\mathcal{M}} \supset \ldots \supset \mathcal{J}^q_{\mathcal{M}} \supset \mathcal{J}^{q+1}_{\mathcal{M}} = 0. \tag{1.6}$$

This allows us to give the following definition.

Definition 1.9 ($\mathrm{Gr}\,\mathcal{O}_{\mathcal{M}}$ and $\mathrm{Gr}\,\mathcal{M}$)**.** *Let $\mathcal{M}$ be a supermanifold having odd dimension q together with the $\mathcal{J}_{\mathcal{M}}$-adic filtration of its structure sheaf $\mathcal{O}_{\mathcal{M}}$. Then we call $\mathrm{Gr}^{(i)}\mathcal{O}_{\mathcal{M}} := \mathcal{J}^i_{\mathcal{M}} \big/ \mathcal{J}^{i+1}_{\mathcal{M}}$ the $\mathcal{J}^i_{\mathcal{M}}$-adic component of $\mathcal{O}_{\mathcal{M}}$ and we define the following $\mathbb{Z}_2$-graded sheaf*

$$\mathrm{Gr}\,\mathcal{O}_{\mathcal{M}} := \bigoplus_{i=0}^{q} \mathrm{Gr}^{(i)}\mathcal{O}_{\mathcal{M}} = \mathcal{O}_{\mathcal{M}_{red}} \oplus \mathcal{J}_{\mathcal{M}} \big/ \mathcal{J}^2_{\mathcal{M}} \oplus \ldots \oplus \mathcal{J}^{q-1}_{\mathcal{M}} \big/ \mathcal{J}^q_{\mathcal{M}} \oplus \mathcal{J}^q_{\mathcal{M}}. \tag{1.7}$$

where the $\mathbb{Z}_2$-grading is obtained by taking the obvious $\mathbb{Z}$-grading $\bmod 2$*. We call the superspace $\mathrm{Gr}\,\mathcal{M} := (|\mathcal{M}|, \mathrm{Gr}\,\mathcal{O}_{\mathcal{M}})$ the* split supermanifold associated to *$\mathcal{M}$.*

Clearly, by adding all the successive quotients $\mathcal{J}^i_{\mathcal{M}} \big/ \mathcal{J}^{i+1}_{\mathcal{M}}$ for $i > q$, all yielding zeroes, $\mathrm{Gr}\,\mathcal{O}_{\mathcal{M}}$ can be lifted to a complex $\mathrm{Gr}^{(\bullet)}\mathcal{O}_{\mathcal{M}}$ (of sheaves of $\mathcal{O}_{\mathcal{M}_{red}}$-modules). Once we have this construction, we can make some observations.

- The pair $(|\mathcal{M}|, \mathrm{Gr}^{(0)}\mathcal{O}_{\mathcal{M}})$ defines a supermanifold, or better an ordinary manifold: indeed it is nothing but that the reduced manifold $\mathcal{M}_{red}$ underlying $\mathcal{M}$, for one has that $\mathrm{Gr}^{(0)}\,\mathcal{O}_{\mathcal{M}} := \mathcal{O}_{\mathcal{M}} \big/ \mathcal{J}_{\mathcal{M}}$.

- clearly, the split supermanifold $\mathrm{Gr}\,\mathcal{M}$ associated to $\mathcal{M}$ is projected, as its structure sheaf is given in the form $\mathcal{O}_{\mathcal{M}_{red}} \oplus \mathcal{J}_{\mathcal{M}}$, where in particular the nilpotent sheaf $\mathcal{J}_{\mathcal{M}}$ has a global decomposition in a direct sum of sheaf of $\mathcal{O}_{\mathcal{M}_{red}}$-modules:

$$\mathcal{J}_{\mathcal{M}} \cong \bigoplus_{i=1}^{q} \mathcal{J}^{i}_{\mathcal{M}} \Big/ \mathcal{J}^{i+1}_{\mathcal{M}}. \tag{1.8}$$

  Notice though, that, conversely, a projected supermanifold might still not be isomorphic to any $\mathrm{Gr}\,\mathcal{M}$!

- If we set $\mathcal{O}^{(i)}_{\mathcal{M}} := \mathcal{O}_{\mathcal{M}} / \mathcal{J}^{i+1}$ and we consider the pair $(|\mathcal{M}|, \mathcal{O}^{(i)}_{\mathcal{M}})$, then for

  $i = 0$: we recover again the reduced manifold $\mathcal{M}_{red}$. The construction is the same as $(|\mathcal{M}|, \mathrm{Gr}^{(0)}\mathcal{O}_{\mathcal{M}})$;

  $0 < i < q$: this construction does *not* yield any supermanifold, but just a locally ringed space or maybe a supersymmetric analogue of a non-reduced scheme, since its structure sheaf is not locally isomorphic to any exterior algebras.

  $i > q$: we recover the actual supermanifold $\mathcal{M}$, since $\mathcal{J}^{q+1}_{\mathcal{M}} = 0$.

Having clarified that to any supermanifold $\mathcal{M}$ is associated its split supermanifold $\mathrm{Gr}\,\mathcal{M}$, we now want to make contact between $\mathrm{Gr}\,\mathcal{M}$ and the local model superspace $\mathfrak{S}(|\mathcal{M}|, \mathcal{E})$ based on the pair $(|\mathcal{M}|, \mathcal{E})$. To this end we introduce the following definition.

Definition 1.10 (Fermionic Sheaf $\mathcal{F}_{\mathcal{M}}$)**.** *Let $\mathcal{M}$ be a supermanifold having odd dimension q and let $\mathcal{J}_{\mathcal{M}}$ its nilpotent sheaf. We call the sheaf of locally free $\mathcal{O}_{\mathcal{M}_{red}}$-modules given by the quotient $\mathcal{F}_{\mathcal{M}} := \mathrm{Gr}^{(1)}\mathcal{O}_{\mathcal{M}} = \mathcal{J}_{\mathcal{M}} / \mathcal{J}^2_{\mathcal{M}}$, the* fermionic sheaf *of the supermanifold $\mathcal{M}$.*

The fermionic sheaf is a central object in the theory of supermanifolds, and indeed we will see that it shows up also in the following section when discussing the tangent and cotangent sheaf of a supermanifold. More important, we have that given the topological space $|\mathcal{M}|$ underlying a supermanifold of odd dimension equal to q, it is its fermionic sheaf the object that completes the correspondence with the pair $(|\mathcal{M}|, \mathcal{E})$, making up the local model $\mathfrak{S}(|\mathcal{M}|, \mathcal{E})$. Indeed trough $\mathcal{F}_{\mathcal{M}}$ one recovers the dual of the vector bundle $\mathcal{E}$ over $|\mathcal{M}|$: in other words, given any open set of an open cover of $|\mathcal{M}|$, one has a correspondence

$$\mathcal{F}_{\mathcal{M}}(U) = \big\langle \Theta_1, \ldots, \Theta_q \big\rangle_{\mathcal{O}_{\mathcal{M}_{red}}(U)} \quad \longleftrightarrow \quad \big\langle e^*_1, \ldots e^*_q \big\rangle_{\mathcal{O}_{\mathcal{M}_{red}}(U)} = \mathcal{E}^*(U). \tag{1.9}$$

where we have written $\{\Theta_i\}_{i=1,\ldots,q}$ for a local basis of the fermionic sheaf $\mathcal{F}_{\mathcal{M}}$ - which is a locally-free sheaf, as seen above - and $\{e^*_i\}_{i=1,\ldots,q}$ for a local basis of the dual vector bundle $\mathcal{E}^*$.
Now, as the symmetric powers of the fermionic sheaf $Sym^i \mathcal{F}_{\mathcal{M}}$ corresponds to $\mathrm{Gr}^{(i)}\mathcal{O}_{\mathcal{M}} = \mathcal{J}^i_{\mathcal{M}} \Big/ \mathcal{J}^{i+1}_{\mathcal{M}}$ for $i \geqslant 0$, the correspondence above extends to the whole complexes $\mathrm{Gr}^{(\bullet)}\mathcal{O}_{\mathcal{M}}$ and $\bigwedge^\bullet \mathcal{E}^*$, so that higher *symmetric* powers $Sym^i\,\mathcal{F}_{\mathcal{M}}$ correspond to higher *exterior* powers $\bigwedge^i \mathcal{E}^*$.
This leads to the following lemma.

Lemma 1.2. *Let $\mathcal{M}$ be a supermanifold locally modelled on the pair $(|\mathcal{M}|, \mathcal{E})$, where $|\mathcal{M}|$ is a topological space and $\mathcal{E}$ is a vector bundle, then the associated split supermanifold* $\mathrm{Gr}\,\mathcal{M}$ *to $\mathcal{M}$ is uniquely determined by the pair $(|\mathcal{M}|, \mathcal{E})$ and viceversa.*
In particular, one has the isomorphism $\mathfrak{S}(|\mathcal{M}|, \mathcal{E}) \cong \mathrm{Gr}\,\mathcal{M}$.

Schematically, the relations between the original supermanifolds $\mathcal{M}$, its split associated supermanifold $\mathrm{Gr}\,\mathcal{M}$ and its local model $\mathfrak{S}(|\mathcal{M}|, \mathcal{E})$ goes as follows:

$$\mathcal{M} \rightsquigarrow \mathrm{Gr}\,\mathcal{M} \leftrightsquigarrow \mathfrak{S}(|\mathcal{M}|, \mathcal{E}). \tag{1.10}$$

The previous lemma, or the above diagram, helps us to single out a class of projected supermanifolds, answering the question posed early on in the section: these are the so called split supermanifolds, that constitutes the easiest examples of supermanifolds.

Definition 1.11 (Split Supermanifold)**.** *Let $\mathcal{M}$ be a supermanifold. We say that $\mathcal{M}$ is split if it is globally isomorphic to its local model $\mathfrak{S}(|\mathcal{M}|, \mathcal{E})$ or analogously, if it is isomorphic to its split associated supermanifold* $\operatorname{Gr}\mathcal{O}_{\mathcal{M}}$.

That is, in the case a supermanifold $\mathcal{M}$ is split, the schematic diagram above looks like this

$$\mathcal{M} \leftrightsquigarrow \operatorname{Gr}\mathcal{M} \leftrightsquigarrow \mathfrak{S}(|\mathcal{M}|, \mathcal{E}). \tag{1.11}$$

and one gets to an easy but deep conclusion: *the problem of classifying split supermanifolds having a certain reduced space $\mathcal{M}_{red}$ translates into the classical problem of classifying vector bundles on $\mathcal{M}_{red}$*, thus revealing a connection between the moduli theory of vector bundles and the study of split supermanifolds.

In order to give some concrete realisation of the issues discussed above and to move toward a generalisation of ordinary algebraic geometry to a supersymmetric context, we discuss the most important (non-trivial) examples of complex split supermanifolds: complex projective superspaces.

Example 1.2 (Complex Projective Superspaces)**.** *The complex projective superspace of dimension $n|m$, denoted by $\mathbb{P}^{n|m}$, is the supermanifold defined by the pair $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^{n|m}}) = \mathfrak{S}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(+1)^{\oplus m})$, so that $\mathbb{P}^{n|m}$ has the ordinary complex projective space $\mathbb{P}^n$ as reduced manifold - that completely characterises the topological aspects -, while its structure sheaf $\mathcal{O}_{\mathbb{P}^{n|m}}$ is given by*

$$\mathcal{O}_{\mathbb{P}^{n|m}} = \bigoplus_{k=0}^{\lfloor m/2 \rfloor} \mathcal{O}_{\mathbb{P}^n}(-2k)^{\oplus \binom{m}{2k}} \oplus \Pi \bigoplus_{k=0}^{\lfloor m/2 \rfloor - \delta_{0, m mod 2}} \mathcal{O}_{\mathbb{P}^n}(-2k-1)^{\oplus \binom{m}{2k+1}} \tag{1.12}$$

$$= \bigoplus_{k \text{ even}} \bigwedge^k \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus m} \oplus \bigoplus_{k \text{ odd}} \Pi \bigwedge^k \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus m}, \tag{1.13}$$

where we have inserted the symbol Π as a reminder for the parity reversing.
This expression for the structure sheaf makes completely clear that $\mathbb{P}^{n|m}$ is canonically isomorphic to $\operatorname{Gr}\mathbb{P}^{n|m}$ and the projection $\pi : \mathbb{P}^{n|m} \to \mathbb{P}^n$ embeds, at the level of the structure sheaves, $\mathcal{O}_{\mathbb{P}^n}$ into $\mathcal{O}_{\mathbb{P}^{n|m}}$, as $\mathcal{O}_{\mathbb{P}^n}$ is just a summand in the direct sum above.
The approach we have just presented has the advantage to make apparent the split nature of $\mathbb{P}^{n|m}$, which is what matters here. Though, it is fair to say that in the literature another approach is more common: the one that looks at the complex projective superspace as a quotient by a certain multiplicative group action.
In this perspective one starts considering the usual complex superspace $\mathbb{C}^{n+1|m} := (\mathbb{C}^{n+1}, \mathcal{O}_{\mathbb{C}^{n+1}} \otimes \bigwedge[\xi_1, \ldots, \xi_m])$, whose underlying topological space is given by $\mathbb{C}^{n+1}$ endowed with the complex topology, then one can form the superspace $(\mathbb{C}^{n+1|m})^\times$ simply by considering the obvious restriction of $\mathbb{C}^{n+1|m}$ to the open set $\mathbb{C}^\times := \mathbb{C}^{n+1} \setminus \{0\}$. The complex projective superspace $\mathbb{P}^{n|m}$ is the supermanifold obtained as the quotient of the superspace $(\mathbb{C}^{n+1|m})^\times$ by the action $\mathbb{C}^* \circlearrowright (\mathbb{C}^{n+1|m})^\times$ defined as

$$\lambda \cdot (x_0, \ldots, x_n, \xi_1, \ldots, \xi_m) := (\lambda x_0, \ldots, \lambda x_n, \lambda \xi_1, \ldots, \lambda \xi_m) \tag{1.14}$$

where λ is an element of the multiplicative group $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

We will see in the second chapter that even if projective superspaces are split supermanifolds, they display some interesting unexpected geometric features.

1.2 Locally-Free Sheaves on a Supermanifold and Even Picard Group

In the first section we have defined what is a supermanifold, so now we can start focusing on what can be defined on a supermanifold. One of the most important and useful concept is the one of locally-free sheaves.

Definition 1.12 (Locally-Free Sheaves on a Supermanifold)**.** *Let $\mathcal{M} := (|\mathcal{M}|, \mathcal{O}_{\mathcal{M}})$ be a supermanifold or, more in general, a superscheme. A locally-free sheaf $\mathcal{G}$ of rank $p|q$ on $\mathcal{M}$ is a sheaf of $\mathcal{O}_{\mathcal{M}}$-modules which is locally isomorphic to $\mathcal{O}_{\mathcal{M}}^{\oplus p} \oplus (\Pi\mathcal{O}_{\mathcal{M}})^{\oplus q}$.*

Notice that this will completely replace the idea of "vector superbundle", that will never be mentioned in this thesis. Likewise, we will never talk about "line superbundle": the supersymmetric analog of a line bundle or an invertible sheaf is defined as follows.

Definition 1.13 (Even Invertible Sheaf)**.** *Let $\mathcal{M}$ be a supermanifold or, more in general, a superscheme. An even invertible sheaf $\mathcal{L}$ on $\mathcal{M}$ is a locally-free sheaf of rank $1|0$ on $\mathcal{M}$. That is, $\mathcal{L}$ is locally isomorphic to $\mathcal{O}_{\mathcal{M}}$.*

Also, in the same fashion as in the ordinary theory, we define a supersymmetric analog of the notion of Picard group for invertible sheaves.

Definition 1.14 (Even Picard Group)**.** *Given a supermanifold $\mathcal{M}$, we call the even Picard group $\mathrm{Pic}_0(\mathcal{M})$ of $\mathcal{M}$ the group of isomorphism classes of even invertible sheaves on $\mathcal{M}$,*

$$\mathrm{Pic}_0(\mathcal{M}) := \{\textit{even invertible sheaves on } \mathcal{M}\}/_{\cong} \tag{1.15}$$

where the group operations are given by the tensor product and the dual.

It can be checked that this definition is well-posed.
It is crucial to note that a useful cohomological interpretation, allowing for actual computations, can be given for the even Picard group of a supermanifold. To this end, note that in general, such as in the ordinary context, defining a locally-free sheaf $\mathcal{G}$ of a certain rank on a supermanifold $\mathcal{M}$, amounts to give an open covering of $\mathcal{M}$, call it $\{\mathcal{U}_i\}_{i\in I}$, and the transition functions $\{g_{ij}\}_{i,j\in I}$ between two local frames $e_{\mathcal{U}_i}$ and $e_{\mathcal{U}_j}$ in the intersections $\mathcal{U}_i \cap \mathcal{U}_j$ for $i, j \in I$, so that $e_{\mathcal{U}_i} = g_{ij}e_{\mathcal{U}_j}$. In this fashion, one has the usual correspondence $\mathcal{G} \leftrightarrow (\{\mathcal{U}_i\}_{i\in I}, \{g_{ij}\}_{i,j\in I})$, where we notice that if $\mathcal{G}$ has rank $p|q$ then g_{ij} is a $GL(p|q)$ transformation taking values in $\mathcal{O}_{\mathcal{M}}(\mathcal{U}_i \cap \mathcal{U}_j)$).
It follows that, in the case we are considering an even invertible sheaf, this corresponds to transition functions g_{ij} taking values into $\left(\mathcal{O}^*_{\mathcal{M}}\right)_0 \cong \mathcal{O}^*_{\mathcal{M},0}$ as the transformation needs to be invertible and a parity-preserving one. This has an important consequence, indeed $\mathcal{O}^*_{\mathcal{M},0}$ is a sheaf of *abelian groups*, so that we are allowed to consider its cohomology groups, without confronting the issues related to the definition of non-abelian cohomology (notice that the full sheaf $\mathcal{O}^*_{\mathcal{M}}$ is indeed *not* a sheaf of abelian groups). Clearly, in order to define an even invertible sheaf, the transition functions have to be 1-cocycles valued in the sheaf $\mathcal{O}^*_{\mathcal{M},0}$, so that we have the following easy lemma.

Lemma 1.3 ($\mathrm{Pic}_0(\mathcal{M}) \cong H^1(\mathcal{O}^*_{\mathcal{M},0})$)**.** *Let $\mathcal{M}$ be a supermanifold and let $\mathrm{Pic}_0(\mathcal{M})$ be its even Picard group. Then, the following isomorphism holds:*

$$\mathrm{Pic}_0(\mathcal{M}) \cong H^1(\mathcal{O}^*_{\mathcal{M},0}). \tag{1.16}$$

In what follows, just like Manin in [41] and [42], we will grant ourselves the liberty to call the cohomology group $H^1(\mathcal{O}^*_{\mathcal{M},0})$ the even Picard group of the supermanifold, by implicitly referring to the above isomorphism.
Note that Lemma 1.3 is nothing but the supersymmetric version of the usual isomorphism $\mathrm{Pic}(X) \cong H^1(\mathcal{O}^*_X)$ for ordinary complex manifolds X - and indeed its proof follows exactly the same lines. Likewise, the sheaf $\mathcal{O}^*_{\mathcal{M},0}$ fits into an exact sequence, we call it *even exponential exact sequence*, by the obvious similarity with the ordinary exponential exact sequence:

$$0 \longrightarrow \mathbb{Z}_{\mathcal{M}} \longrightarrow \mathcal{O}_{\mathcal{M},0} \xrightarrow{\exp} \mathcal{O}^*_{\mathcal{M},0} \longrightarrow 0. \tag{1.17}$$

where, beside $\mathcal{O}^*_{\mathcal{M},0}$, we recall that $\mathbb{Z}_{\mathcal{M}}$ is the ordinary sheaf of locally constant functions taking values in $\mathbb{Z}$ and $\mathcal{O}_{\mathcal{M},0}$ is the even part of the structure sheaf. Given an open set of $\mathcal{M}$, the even exponential map above is defined as follows:

$$\begin{aligned} \mathcal{U} \longmapsto \exp_{\mathcal{U}} : \mathcal{O}_{\mathcal{M},0}(\mathcal{U}) &\longrightarrow \mathcal{O}^*_{\mathcal{M},0}(\mathcal{U}) \\ s_0 &\longmapsto \exp_{\mathcal{U}}(s_0) := e^{2\pi i s_0}. \end{aligned}$$

Actually, the only thing that we need to check in order to prove the exactness of the sequence above is the surjectivity of the map $\exp : \mathcal{O}_{\mathcal{M},0} \to \mathcal{O}^*_{\mathcal{M},0}$. This is achieved in the following lemma.

Lemma 1.4 (exp is surjective)**.** *The map* exp *defined above is surjective and* $\ker(\exp) = \mathbb{Z}_{\mathcal{M}}$*.*

Proof. Surjectivity is to be proved locally, on the stalks. Choosing an open set $\mathcal{U} \ni x$, we can take a representative of an element in $\mathcal{O}^*_{\mathcal{M},0,\,x}$ such that the corresponding element in $\mathcal{O}^*_{\mathcal{M},0}(\mathcal{U})$ has the following expansion

$$f_0(x,\theta) = f(x) + N(x,\theta), \qquad f(x) \neq 0. \tag{1.18}$$

Notice that, for the sake of convenience, we have split the contribution on the reduced manifold, $f(z)$ - which is an ordinary non-zero holomorphic function since we are considering an invertible element in $\mathcal{O}^*_{\mathcal{M},0}(\mathcal{U})$ - and we have gathered all the nilpotent contributions in the expansion in the term $N(x,\theta) \in \mathcal{J}_{\mathcal{M}}(\mathcal{U})$, such that $N^m(x,\theta) = 0$ and $N^{m-1}(x,\theta) \neq 0$ for some $m \geqslant 2$, nilpotency index.
Now, since $f(x) \neq 0$, if one wish, it can be collected to give

$$f_0(x,\theta) = f(x)\left(1 + \frac{N(x,\theta)}{f(x)}\right). \tag{1.19}$$

This might be useful in writing the logarithm, defined as to be the (local) inverse of the exponential, that is $\mathcal{U} \mapsto \log_{\mathcal{U}}$ with $\log_{\mathcal{U}}(s_0) = \frac{1}{2\pi i}\log(s_0)$ for $s_0 \in \mathcal{O}^*_{\mathcal{M},0}$. In this way, using the expression above, one finds:

$$\log_{\mathcal{U}}(f_0) = \frac{1}{2\pi i}\log\left(f(x)\right) + \frac{1}{2\pi i}\log\left(1 + \frac{N(x,\theta)}{f(x)}\right) \tag{1.20}$$

$$= \frac{1}{2\pi i}\log\left(f(x)\right) + \frac{1}{2\pi i}\sum_{k=0}^{m-2}\frac{(-1)^k}{k+1}\left(\frac{N(x,\theta)}{f(x)}\right)^{k+1}. \tag{1.21}$$

This is well-defined for $\log\left(f(x)\right)$ is the logarithm of an ordinary holomorphic non-zero function and it is locally single-valued and the remaining part is a finite sum of nilpotents. Therefore over a generic small open set $\mathcal{U} \subset |\mathcal{M}|$ containing x, $f_0 = \exp_{\mathcal{U}}(\log_{\mathcal{U}}(f_0)$, that is exp is surjective. We can now evaluate the exponential of the above quantity to establish the kernel of the map:

$$\exp_{\mathcal{U}}(f_0) = e^{2\pi i(f(x)+N(x,\theta))} = e^{2\pi i f(x)}e^{2\pi i N(x,\theta)} = \tag{1.22}$$

$$= e^{2\pi i f(x)}\left(1 + 2\pi i\sum_{k=1}^{m-1}\frac{N(x,\theta)^k}{k!}\right) = 1_{\mathcal{U}} \tag{1.23}$$

Now the exponential above, $e^{2\pi i f(x)}$, is the usual complex exponential map that has kernel given by the sheaf of locally constant functions taking integral values $\mathbb{Z}$. Let suppose that $f_0 \in \ker(\exp)$, the only way for this to be true is that $\sum_{k=1}^{m-1}\frac{N(x,\theta)^k}{k!} = 0$, which in turn implies that $N(x,\theta) = 0$, indeed, multiplying on the left and on the right side by N^{m-2} one has

$$\left(\sum_{k=1}^{m-1}\frac{N(x,\theta)^k}{k!}\right)\cdot N^{m-2}(x,\theta) = N^{m-1}(x,\theta) \neq 0, \tag{1.24}$$

thus concluding the proof. □

The even exponential sequence first appeared (without a proof) in [42], which has been our main reference. For a different construction the reader might also refer to [4].

1.3 Tangent and Cotangent Sheaf of a Supermanifold

In this section we introduce the tangent and cotangent sheaves of a supermanifold $\mathcal{M}$ and we establish their connection with the fermionic sheaf $\mathcal{F}_{\mathcal{M}}$, that we have discussed in the previous section. We write down the short exact sequences the cotangent sheaf fits into, first by working in a general framework, dealing with a generic, possibly non-projected, supermanifold. Then we specialise to the case of projected supermanifolds. In this context, we study the Berezinian sheaf

- the supergeometric analog of the canonical sheaf - of a projected supermanifold proving that it can be easily reconstructed via the canonical sheaf of the reduced manifold $\mathcal{M}_{red}$ and the fermionic sheaf $\mathcal{F}_{\mathcal{M}}$. Further, following this path, we generalise the notion of *first Chern class* to projected supermanifolds. Throughout the section, we make our constructions explicit using the example provided by the complex projective superspace $\mathbb{P}^{n|m}$.
We start by introducing the following notion in supergeometry, generalising the ordinary one.

Definition 1.15 (Superderivation)**.** *Let A be a superalgebra over a field k, then a superderivation D is a homogeneous a k-linear map $D : A \to A$ of parity $|D|$ that satisfies*

$$D(a \cdot b) = D(a) \cdot b + (-1)^{|D||a|} a \cdot D(b), \tag{1.25}$$

for any $a \in A$ homogeneous of parity $|a|$ and any $b \in A$.

Usually, depending on its parity $|D| \in \{0, 1\}$, a superderivation D is said to be either *even* or *odd*. In particular, on the complex superspace $\mathbb{C}^{p|q}$ having coordinates $(x^1, \dots, x^p|\theta^1, \dots, \theta^q)$, the superderivations of the structure sheaf $\mathcal{O}_{\mathbb{C}^{p|q}}$ are written as $(\partial_{x^1}, \dots, \partial_{x^p}|\partial_{\theta^1}, \dots, \partial_{\theta^q})$, where the $\{\partial_{x^i}\}_{i=1,\dots,p}$ are the *even* superderivations and the $\{\partial_{\theta^j}\}_{j=1,\dots q}$ are the *odd* superderivations.
In order to define how these superderivations act, we let $I := (i_1, \dots, i_q)$ be a multi-index with $i_j = \{0, 1\}$ and $|I| = \sum_{\ell=1}^q i_\ell$ (so that $0 \leqslant |I| \leqslant q$) and we put $\theta^I := (\theta^1)^{i_1} \dots (\theta_j)^{i_j} \dots (\theta^q)^{i_q}$ (where we let $(\theta^j)^0 := 1_{\mathbb{C}}$). Then, an element $f \in \mathcal{O}_{\mathbb{C}^{p|q}}$ can be written uniquely as

$$f = \sum_I f_I(x)\theta^I, \tag{1.26}$$

where $f_I(x) \in \mathcal{O}_{\mathbb{C}^p}$ is an ordinary holomorphic function on $\mathbb{C}^p$ for any multi-index I.
The i-th even derivative of f is then defined as

$$\frac{\partial}{\partial x^i}(f) := \sum_I \left(\frac{\partial f_I}{\partial x^i}(x) \right) \theta^I. \tag{1.27}$$

To define j-th odd derivative of f we want to isolate θ^j in the generic expression of f above. In order to do so, for a multi-index $I := (i_1, \dots, i_q)$, where $i_k \in \{0, 1\}$, we define $I_j := (i_1, \dots, i_{j-1}, 0, i_{j+1}, \dots i_q)$, so we put $i_j = 0$. Then, f can be rewritten as

$$f = \left(\sum_{I_j} f_{I_j}(x)\theta^{I_j} + \sum_{I_j} f_{I_j,j}(x)\theta^j\theta^{I_j} \right), \tag{1.28}$$

where again f_{I_j} and $f_{I_j,j}$ are holomorphic functions on $\mathbb{C}^p$ for any multi-index I_j, having $i_j = 0$.
The j-th odd derivative is then defined as

$$\frac{\partial}{\partial \theta^j}(f) := \sum_{I_j} f_{I_j,j}(x)\theta^{I_j}. \tag{1.29}$$

It is an early result of Leites, see [38], that the $\mathcal{O}_{\mathbb{C}^{p|q}}$-module of the $\mathbb{C}$-linear superderivations is *free* and has dimension $p|q$ with basis given by $\{\partial_{x^1}, \dots \partial_{x^p}|\partial_{\theta^1}, \dots, \partial_{\theta^q}\}$. It follows that, since a (complex) supermanifold $\mathcal{M}$ of dimension $p|q$ is locally isomorphic to $\mathbb{C}^{p|q}$, the $\mathcal{O}_{\mathbb{C}^{p|q}}$-module of superderivations of the structure sheaf $\mathcal{O}_{\mathcal{M}}$ is actually a locally-free sheaf of $\mathcal{O}_{\mathcal{M}}$-modules of rank $p|q$ and we denote if by $\mathcal{SD}er(\mathcal{O}_{\mathcal{M}})$. In the rest of this thesis, though, this sheaf will be referred to as the tangent sheaf of $\mathcal{M}$, as in the following definition.

Definition 1.16 (Tangent Sheaf)**.** *Let $\mathcal{M}$ be a (complex) supermanifold. We denote $\mathcal{T}_{\mathcal{M}}$ the sheaf of superderivations of $\mathcal{O}_{\mathcal{M}}$, $\mathcal{T}_{\mathcal{M}} := \mathcal{SD}er(\mathcal{O}_{\mathcal{M}})$.*

As usual, local sections of $\mathcal{T}_{\mathcal{M}}$ (that is derivations of $\mathcal{O}_{\mathcal{M}}$) will be called *local vector fields.*
The *cotangent sheaf* or *sheaf of* 1*-forms*, is defined as usual starting from the tangent sheaf.

Definition 1.17 (Cotangent Sheaf)**.** *Let $\mathcal{M}$ be a (complex) supermanifold. We denote $\Omega^1_{\mathcal{M}}$ the dual of the tangent sheaf $\mathcal{T}_{\mathcal{M}}$ of $\mathcal{M}$, that is $\Omega^1_{\mathcal{M}} := \mathcal{H}om_{\mathcal{O}_{\mathcal{M}}}(\mathcal{T}_{\mathcal{M}}, \mathcal{O}_{\mathcal{M}})$.*

Before we go on, we notice that, given the tangent sheaf $\mathcal{T}_{\mathcal{M}}$, in [41] Manin distinguishes between two possible choices for the cotangent sheaf. The first one is called $(\Omega^1_{\mathcal{M}})_{ev}$ and it corresponds with what has just been defined to be $\Omega^1_{\mathcal{M}}$ and it is the one that will be used throughout this thesis. The second possibility is to consider *odd* homomorphisms, and defining $(\Omega^1_{\mathcal{M}})_{odd} := \mathcal{H}om_{\mathcal{O}_{\mathcal{M}}}(\mathcal{T}_{\mathcal{M}}, \Pi\mathcal{O}_{\mathcal{M}})$. This is the parity changed version of $\Omega^1_{\mathcal{M}}$ and, as such, if $\Omega^1_{\mathcal{M}}$ has rank $p|q$, then $(\Omega^1_{\mathcal{M}})_{odd}$ has rank $q|p$. This second choice for the cotangent sheaf, $(\Omega^1_{\mathcal{M}})_{odd}$, is often preferred in some applications, in particular when looking at the de Rham theory on supermanifolds [18] [68].
As $\mathcal{T}_{\mathcal{M}}$, the cotangent sheaf $\Omega^1_{\mathcal{M}}$ is a locally-free sheaf of $\mathcal{O}_{\mathcal{M}}$-modules. A local basis for $\Omega^1_{\mathcal{M}}$ is given by $\{dz^1, \ldots, dz^p, d\theta^1, \ldots, d\theta^q\}$, with a duality pairing with the tangent sheaf (locally) given by:

$$\begin{aligned} \langle \cdot, \cdot \rangle_{\mathcal{U}} : (\Omega^1_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{T}_{\mathcal{M}})(\mathcal{U}) &\longrightarrow \mathcal{O}_{\mathcal{M}}(\mathcal{U}) \\ \omega \otimes D &\longmapsto \langle \omega, D \rangle_{\mathcal{U}} := \omega(D) \end{aligned}$$

if D and ω are local sections of $\mathcal{T}_{\mathcal{M}}$ and $\Omega^1_{\mathcal{M}}$ respectively. Given two local sections of the structure sheaf $f, g \in \mathcal{O}_{\mathcal{M}}(\mathcal{U})$, the duality paring reads

$$\langle f\omega, g\, D \rangle_{\mathcal{U}} = (-1)^{|\omega|\cdot|g|} fg \langle \omega, D \rangle_{\mathcal{U}}. \tag{1.30}$$

Along this line, one can define a *differential* $d : \mathcal{O}_{\mathcal{M}} \to \Omega^1_{\mathcal{M}}$, by $f \mapsto df$, putting

$$\langle df, D \rangle_{\mathcal{U}} := D(f). \tag{1.31}$$

This enters the definition of the *de Rham complex* in the theory of supermanifolds, which is a subtle topic we will not discuss any further.
Also, as in the ordinary case, given two supermanifolds $\mathcal{M}$ and $\mathcal{N}$ and morphism $\varphi : \mathcal{M} \to \mathcal{N}$, one can define a morphism of sheaves $d\varphi : \mathcal{T}_{\mathcal{M}} \to \varphi^*\mathcal{T}_{\mathcal{N}}$. A (local) vector superfields X on the supermanifold $\mathcal{M}$ - that is a section of the tangent sheaf, $X \in \mathcal{T}_{\mathcal{M}}(\mathcal{U})$ for a certain open set $\mathcal{U} \subset |\mathcal{M}|$ -, when acting on functions lifted from $\mathcal{N}$, defines a derivation on $\mathcal{N}$ valued in the ring of functions on $\mathcal{M}$. This gives a morphism of $\mathcal{O}_{\mathcal{M}}$-modules, $\varphi^*(\Omega^1_{\mathcal{N}}) \to \mathcal{O}_{\mathcal{M}}$, or equivalently, a section of the sheaf $\varphi^*(\mathcal{T}_{\mathcal{N}})$, which is said to be the image of X with respect to $d\varphi$.
It is important to note that the *restriction* to the reduced manifold $\mathcal{M}_{red}$ splits into complementary even and odd sub-sheaves, actually locally-free sheaves of $\mathcal{O}_{\mathcal{M}_{red}}$-modules:

$$(\mathcal{T}_{\mathcal{M}})_{red} := \mathcal{T}_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}_{red}} = \mathcal{T}_{\mathcal{M},+} \oplus \mathcal{T}_{\mathcal{M},-}, \tag{1.32}$$

$$(\Omega^1_{\mathcal{M}})_{red} := \Omega^1_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}_{red}} = \Omega^1_{\mathcal{M},+} \oplus \Omega^1_{\mathcal{M},-}, \tag{1.33}$$

where, with reference to the first section, we have called $\mathcal{T}_{\mathcal{M},+} := (\mathrm{Gr}^{(0)}\mathcal{T}_{\mathcal{M}})_0$ and $\mathcal{T}_{\mathcal{M},1} := (\mathrm{Gr}^{(0)}\mathcal{T}_{\mathcal{M}})_1$ and likewise for the cotangent sheaf.
Clearly, the even parts $\mathcal{T}_{\mathcal{M},+}$ and $\Omega^1_{\mathcal{M},+}$ in the previous parity splitting are easily identified as $\mathcal{T}_{\mathcal{M}_{red}}$ and $\Omega^1_{\mathcal{M}_{red}}$ respectively, that is the tangent and cotangent sheaf of the reduced manifold $\mathcal{M}_{red}$. In order to identify the odd parts $\mathcal{T}_{\mathcal{M},-}$ and $\Omega^1_{\mathcal{M},-}$, we need to disclose the relationship with the fermionic sheaf we have already mentioned early on in the first section of this chapter. Recalling that the fermionic sheaf has been defined as $\mathcal{F}_{\mathcal{M}} := \mathrm{Gr}^{(1)}\mathcal{O}_{\mathcal{M}}$, we have the following

Corollary 1.1. *Let $\mathcal{M}$ be a (complex) supermanifold, then we have the isomorphism of $\mathcal{O}_{\mathcal{M}_{red}}$-modules*

$$\mathcal{F}_{\mathcal{M}} \cong \left(\mathrm{Gr}^{(0)}\Omega^1_{\mathcal{M}}\right)_1 \qquad\qquad \mathcal{F}^*_{\mathcal{M}} \cong \left(\mathrm{Gr}^{(0)}\mathcal{T}_{\mathcal{M}}\right)_1 \tag{1.34}$$

where the subscript 1 refers to the $\mathbb{Z}_2$-grading.

Proof. We start observing that, locally, a basis of $\mathcal{F}_{\mathcal{M}} = \mathcal{J}_{\mathcal{M}}/\mathcal{J}^2_{\mathcal{M}}$ has the form $\theta^a \bmod \mathcal{J}^2_{\mathcal{M}}$ for $a = 1, \ldots, m$ where m is the odd dimension of $\mathcal{M}$. Moreover we have that

$$\left(\mathrm{Gr}^{(0)}\Omega^1_{\mathcal{M}}\right)_1 := \left(\Omega^1_{\mathcal{M}} \otimes_{\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}}/\mathcal{J}_{\mathcal{M}}\right)_1 \cong \left(\Omega^1_{\mathcal{M}}/\mathcal{J}_{\mathcal{M}}\Omega^1_{\mathcal{M}}\right)_1 \tag{1.35}$$

so that, locally, a basis of $\left(\mathrm{Gr}^{(0)}\Omega^1_{\mathcal{M}}\right)_1$ read $d\theta^j \mathrm{mod} \mathcal{J}_{\mathcal{M}}\Omega^1_{\mathcal{M}}$, again for $j = 1, \ldots, m$ where m is the odd dimension of $\mathcal{M}$. The isomorphism reads

$$\begin{aligned} \mathcal{F}_{\mathcal{M}}\lfloor_{\mathcal{U}} &\longrightarrow \left(\mathrm{Gr}^{(0)}\Omega^1_{\mathcal{M}}\right)_1 \Big\lfloor_{\mathcal{U}} \\ \theta^j \bmod \mathcal{J}^2_{\mathcal{M}} &\longmapsto d\theta^j \bmod \mathcal{J}_{\mathcal{M}}\Omega^1_{\mathcal{M}}. \end{aligned} \tag{1.36}$$

We prove that this is well-defined and independent of the chart: that is, if we let $(y^i|\eta^j)$ be another local coordinates system, we need that the sections $\eta^j \mathrm{mod} \mathcal{J}^2_{\mathcal{M}}$ go to $d\eta^j \mathrm{mod} \mathcal{J}_{\mathcal{M}}\Omega^1_{\mathcal{M}}$. The transformation for the fermionic sheaf $\mathcal{F}_{\mathcal{M}}$ reads $\eta^j \equiv \sum_b f^j_b(x)\theta^b \bmod \mathcal{J}^2_{\mathcal{M}}$, therefore one has

$$\begin{aligned} d\eta^j &= \sum_b \frac{\partial \eta^j}{\partial x^b} dx^b + \sum_b \frac{\partial \eta^j}{\partial \theta^b} d\theta^b \\ &= \sum_b \frac{\partial}{\partial x^b}\left(\sum_c f^j_c(x)\theta^c \bmod \mathcal{J}^2_{\mathcal{M}}\right) dx^b + \sum_b \frac{\partial}{\partial \theta^b}\left(\sum_c f^j_c(x)\theta^c \bmod \mathcal{J}^2_{\mathcal{M}}\right) d\theta^b \\ &= \sum_{b,c} \frac{\partial f^j_c(x)}{\partial x^b}\theta^c \bmod \mathcal{J}^2_{\mathcal{M}}\, dx^b + \sum_b f^j_b(x) \bmod \mathcal{J}^2_{\mathcal{M}}\, d\theta^b \\ &\equiv \sum_b f^j_b(x) d\theta^b \bmod \left(\mathcal{J}_{\mathcal{M}}\Omega^1_{\mathcal{M}}\right), \end{aligned} \tag{1.37}$$

since $\sum_{b,c} \frac{\partial f^j_c(x)}{\partial x^b}\theta^c \bmod \mathcal{J}^2_{\mathcal{M}}\, dx^b \equiv 0 \bmod \mathcal{J}_{\mathcal{M}}\Omega^1_{\mathcal{M}}$, concluding the proof. The other isomorphism, involving the tangent sheaf, is proved in the same fashion. □

Notice that the previous result originally appeared in [41]: here we have just made the maps and computations explicit.
Using Corollary 1.1 we have that the parity splitting for the reduced tangent and cotangent sheaf reads

$$(\mathcal{T}_{\mathcal{M}})_{red} = \mathcal{T}_{\mathcal{M}_{red}} \oplus \mathcal{F}^*_{\mathcal{M}}, \tag{1.38}$$

$$(\Omega^1_{\mathcal{M}})_{red} = \Omega^1_{\mathcal{M}_{red}} \oplus \mathcal{F}_{\mathcal{M}}. \tag{1.39}$$

This is a very useful decomposition that has many applications in the theory of supermanifolds and indeed it will be often used in this thesis.

We now study the cotangent sheaf further, showing which short exact sequences it fits into. It is understood that the same might be done for the tangent sheaf, dualising the short exact sequence. We put ourselves in the most general setting, considering a possibly non-projected supermanifold $\mathcal{M}$: this means that in principle we only have an embedding $\iota : \mathcal{M}_{red} \to \mathcal{M}$, which allows us to have an exact sequence of $\mathcal{O}_{\mathcal{M}}$-modules as follows

$$0 \longrightarrow \mathcal{N}_{\mathcal{O}_{\mathcal{M}}} \longrightarrow \Omega^1_{\mathcal{M}} \xrightarrow{res_{\mathcal{O}_{\mathcal{M}}}} \iota_* \Omega^1_{\mathcal{M}_{red}} \longrightarrow 0 \tag{1.40}$$

where $\mathcal{N}_{\mathcal{O}_{\mathcal{M}}}$ is a suitable sheaf of $\mathcal{O}_{\mathcal{M}}$-module, actually kernel of the map $res_{\mathcal{O}_{\mathcal{M}}} : \Omega^1_{\mathcal{M}} \to \iota_*\Omega^1_{\mathcal{M}_{red}}$, where $\iota_*\Omega^1_{\mathcal{M}_{red}}$ is the push-forward of the sheaf of 1-forms over the reduced variety $\mathcal{M}_{red}$, that is indeed a sheaf of $\mathcal{O}_{\mathcal{M}}$-modules.
Likewise, we can also consider the pull-back of the previous short exact sequence:

$$0 \longrightarrow \mathcal{N}_{\mathcal{O}_{\mathcal{M}_{red}}} \longrightarrow \iota^*\Omega^1_{\mathcal{M}} \xrightarrow{res_{\mathcal{O}_{\mathcal{M}_{red}}}} \Omega^1_{\mathcal{M}_{red}} \longrightarrow 0 \tag{1.41}$$

This gives a short exact sequence of $\mathcal{O}_{\mathcal{M}_{red}}$-modules. Here, similarly as above $\mathcal{N}_{\mathcal{O}_{\mathcal{M}_{red}}}$ is the kernel. Notice that the pull-back by ι makes the short exact sequence well-defined for we have $\iota^*\Omega^1_{\mathcal{M}} = i^{-1}\Omega^1_{\mathcal{M}} \otimes_{i^{-1}\mathcal{O}_{\mathcal{M}}} \mathcal{O}_{\mathcal{M}_{red}}$.
We now wonder if there actually exists a projection $\pi : \mathcal{M} \to \mathcal{M}_{red}$ splitting the exact sequence

above. In presence of the projection, it makes sense to consider the following short exact sequence of $\mathcal{O}_{\mathcal{M}}$-modules:

$$
0 \longrightarrow \pi^*\Omega^1_{\mathcal{M}_{red}} \longrightarrow \Omega^1_{\mathcal{M}} \longrightarrow \mathcal{Q}_{\mathcal{O}_{\mathcal{M}}} \longrightarrow 0 \tag{1.42}
$$

where now $\mathcal{Q}_{\mathcal{O}_{\mathcal{M}}}$ is the quotient $\mathcal{Q}_{\mathcal{O}_{\mathcal{M}}} := \Omega^1_{\mathcal{M}}/\pi^*\Omega^1_{\mathcal{M}_{red}}$ and $\pi^*\Omega^1_{\mathcal{M}_{red}} = \mathcal{O}_{\mathcal{M}} \otimes_{p^{-1}\mathcal{O}_{\mathcal{M}_{red}}} p^{-1}\Omega^1_{\mathcal{M}_{red}}$. This short exact sequence splits,

$$
\begin{tikzcd}
0 \arrow[r] & \pi^*\Omega^1_{\mathcal{M}_{red}} \arrow[r, "imm"'] & \Omega^1_{\mathcal{M}} \arrow[l, bend right, dashed, "proj"'] \arrow[r] & \mathcal{Q}_{\mathcal{O}_{\mathcal{M}}} \arrow[r] & 0.
\end{tikzcd} \tag{1.43}
$$

Notice that being $\mathcal{Q}_{\mathcal{O}_{\mathcal{M}}}$ the quotient $\Omega^1_{\mathcal{M}}/\pi^*\Omega^1_{\mathcal{M}_{red}}$, locally, we have that elements in $\mathcal{Q}_{\mathcal{O}_{\mathcal{M}}}$ are of the form $\mathcal{O}_{\mathcal{M}} \cdot \{dz_1, \dots, dz_p, d\theta_1, \dots, d\theta_q\} \mathrm{mod}\, \mathcal{O}_{\mathcal{M}} \cdot \{dz_1, \dots, dz_p\}$.
Locally, over an open set $\mathcal{U} \subseteq |\mathcal{M}|$ we have:

$$
\begin{array}{ccccc}
\pi^*\Omega^1_{\mathcal{M}_{red}}(\mathcal{U}) & \xrightarrow{imm_{\mathcal{U}}} & \Omega^1_{\mathcal{M}}(\mathcal{U}) & \xrightarrow{proj_{\mathcal{U}}} & \pi^*\Omega^1_{\mathcal{M}_{red}}(\mathcal{U}) \\
\mathcal{O}_{\mathcal{M}} \cdot \{dz_1, \dots, dz_p\} & \longmapsto & \mathcal{O}_{\mathcal{M}} \cdot \{dz_1, \dots, dz_p, 0 \dots, 0\} & \longmapsto & \mathcal{O}_{\mathcal{M}} \cdot \{dz_1, \dots, dz_p\}.
\end{array}
$$

Therefore, when dealing with a projected or a split supermanifold that possess a morphism $\pi : \mathcal{M} \to \mathcal{M}_{red}$, we can consider the sheaf of 1-form $\Omega^1_{\mathcal{M}}$ as given by a direct sum, as follows:

$$
0 \longrightarrow \pi^*\Omega^1_{\mathcal{M}_{red}} \longrightarrow \pi^*\Omega^1_{\mathcal{M}_{red}} \oplus \mathcal{Q}_{\mathcal{O}_{\mathcal{M}}} \longrightarrow \mathcal{Q}_{\mathcal{O}_{\mathcal{M}}} \longrightarrow 0. \tag{1.44}
$$

Now we need the following

Corollary 1.2. *Let $\mathcal{M}$ be a projected supermanifold, with projection given by $\pi : \mathcal{M} \to \mathcal{M}_{red}$. The the following isomorphism holds*

$$
\pi^*\mathcal{F}_{\mathcal{M}} \cong \Omega^1_{\mathcal{M}} \Big/ \pi^*\Omega^1_{\mathcal{M}_{red}}. \tag{1.45}
$$

Proof. Locally elements in $\pi^*\mathcal{F}_{\mathcal{M}}$ can be written as $\theta^a \mathrm{mod} \mathcal{J}^2_{\mathcal{M}}$ for $a = 1, \dots, m$ where m is the odd dimension of $\mathcal{M}$ (notice the abuse of notation with respect to the previous Corollary 1.1) while elements in $\Omega^1_{\mathcal{M}}/\pi^*\Omega^1_{\mathcal{M}_{red}}$ have a local form given by $d\theta^a \,\mathrm{mod}\, \pi^*\Omega^1_{\mathcal{M}_{red}}$, again for $a = 1, \dots, m$ where m odd dimension of $\mathcal{M}$. The isomorphism we are considering reads

$$
\begin{array}{ccc}
\pi^*\mathcal{F}_{\mathcal{M}} & \longrightarrow & \Omega^1_{\mathcal{M}} / \pi^*\Omega^1_{\mathcal{M}} \\
\theta^j \,\mathrm{mod}\, \mathcal{J}^2_{\mathcal{M}} & \longmapsto & d\theta^j \,\mathrm{mod}\, \pi^*\Omega^1_{\mathcal{M}_{red}}.
\end{array} \tag{1.46}
$$

We need this to hold true when passing from chart to chart, that is we need that $\eta^j \,\mathrm{mod}\, \mathcal{J}^2_{\mathcal{M}}$ go to $d\eta^j \,\mathrm{mod}\, \pi^*\Omega^1_{\mathcal{M}_{red}}$, therefore we consider another local chart of $\mathcal{M}$ having local coordinates given by $(y^i|\theta^j)$, and we consider the transformation of dx^i and of $d\theta^j$ for

$$
dy^i = \sum_b \frac{\partial y^i}{\partial x^b} dx^b + \sum_b \frac{\partial y^i}{\partial \theta^b} d\theta^b = \sum_b \frac{\partial y^i}{\partial x^b} dx^b \equiv 0 \,\mathrm{mod}\, \pi^*\Omega^1_{\mathcal{M}_{red}}, \tag{1.47}
$$

as $\partial_{\theta^b} y^i = 0$ since $\mathcal{M}$ is projected and therefore $y = y(x)$. Moreover, remembering that $\eta^j \equiv \sum_b f^j_b(x)\theta^b \,\mathrm{mod}\, \mathcal{J}^2_{\mathcal{M}}$, one has

$$
\begin{aligned}
d\eta^j &= \sum_b \frac{\partial \eta^j}{\partial x^b} dx^b + \sum_b \frac{\partial \eta^j}{\partial \theta^b} d\theta^b \\
&= \sum_b \frac{\partial}{\partial x^b} \left(\sum_c f^j_c(x)\theta^c \,\mathrm{mod}\, \mathcal{J}^2_{\mathcal{M}} \right) dx^b + \sum_b \frac{\partial}{\partial \theta^b} \left(\sum_c f^j_c(x)\theta^c \,\mathrm{mod}\, \mathcal{J}^2_{\mathcal{M}} \right) d\theta^b \\
&= \sum_{b,c} \frac{\partial f^j_c(x)}{\partial x^b} \theta^c \,\mathrm{mod}\, \mathcal{J}^2_{\mathcal{M}}\, dx^b + \sum_b f^j_b(x) \,\mathrm{mod}\, \mathcal{J}^2_{\mathcal{M}}\, d\theta^b \\
&\equiv \sum_b f^j_b(x) d\theta^b \,\mathrm{mod}\, \left(\pi^*\Omega^1_{\mathcal{M}_{red}}\right),
\end{aligned} \tag{1.48}
$$

thus concluding the proof. $\square$

Then, the previous short exact sequence can be re-written in the more useful form

$$0 \longrightarrow \pi^* \Omega^1_{\mathcal{M}_{red}} \longrightarrow \pi^* \Omega^1_{\mathcal{M}_{red}} \oplus \pi^* \mathcal{F}_{\mathcal{M}} \longrightarrow \pi^* \mathcal{F}_{\mathcal{M}} \longrightarrow 0 \quad (1.49)$$

This can be used to express the Berezinian sheaf, that will be introduced in the next section, in a convenient way.

1.4 Berezinian Sheaf, First Chern Class and Calabi-Yau Condition

When passing from the ordinary algebraic geometric setting to an algebraic supergeometric setting, there is one issue in particular that stands out for its peculiarity: the theory of differential forms and integrations (see for example [18] [19] [41] [45] [68]).
The main problem can be sketched as follows: when one tries to generalise the complex of forms to supergeometry using 1-forms $\{d\theta^i\}_{i \in I}$, constructed out of the θ^i, then it comes natural to define wedge products to be *commutative* in the $d\theta$'s, as the θ's are anticommutative elements. This leads to the consequence that forms such as $(d\theta^i)^{\wedge n} := d\theta^i \wedge \ldots \wedge d\theta^i$ do make sense and they are non-zero for an arbitrary value of n, possibly exceeding the odd dimension of the supermanifold.
In other words, this says that in supergeometry the *de Rham complex is not bounded from above*: there is no notion of top-form.
This obviously creates issues in the definition of a coherent notion of integration on supermanifolds. There are actually two possible way out. One is to enlarge the de Rham complex, by supplementing it with the so-called *integral forms*. Without going into details, this makes the de Rham complex into a bi-complex, with a generalised notion of top-form that can be integrated over. The interested reader can refer to the literature cited at the beginning of this section for further information and discussions of integral forms and their properties on supermanifolds.
Another possibility is to look for a supergeometric analog of the canonical sheaf of an ordinary manifold - whose sections are the elements that get integrated over. The key is to observe that the sections of the canonical sheaf transform as *densities* under change of local coordinates, we thus ask for a sheaf defined on the supermanifold $\mathcal{M}$ whose sections transform as densities as well. This calls for finding a supergeometric analog of the notion of determinant (of an automorphism) that enters the transformations of densities such as the sections of the canonical sheaf.
The supergeometric analog of the determinant is known as *Berezianian*. Briefly, given a free $\mathbb{Z}_2$-graded module $A := A^{p|q}$, the Berezianian is a supergroup homomorphisms

$$\mathrm{Ber} : GL(p|q; A) \longrightarrow GL(1|0; A_0) \quad (1.50)$$

that agrees with the determinant when $q = 0$ and it also proves to have similar properties (see [16] [19] [41] [68]). Here $GL(p|q; A)$ are the invertible (even) automorphisms of A. We can thus give the following

Definition 1.18 (Berezinian Sheaf / Berezinian of $\mathcal{M}$). *Let $\mathcal{M}$ be a (complex) supermanifold and let $\mathcal{E}$ be a locally-free sheaf of $\mathcal{O}_{\mathcal{M}}$-modules of rank $p|q$. The Berezianian sheaf of $\mathcal{E}$, we denote it by $\mathcal{B}er(\mathcal{E})$, is the sheaf whose sections transform with the Berezinian of the transition functions of $\mathcal{E}$.*
In particular, we define the Berezinian of the supermanifold $\mathcal{M}$ to be the sheaf $\mathcal{B}er(\Omega^1_{\mathcal{M}})$.

It turns out that the sections of the Berezianian of $\mathcal{M}$ are indeed the objects to call for when one looks for a measure for integration involving nilpotent bits - the so-called *Berezin integral* (see for example [63] and again [68] for details about integration on supermanifolds).
Note, by the way, that if on the one hand the definition of the Berezinian sheaf we have given have the perk of being immediate and suitable for practical computations, on the other hand we might be interested into having this sheaf intrinsically characterized. In this regard, the reader is suggested to refer in particular to the very interesting paper by I.B. Penkov [51]. On the same line,

in the context of differentiable supermanifolds, a nice intrinsic characterization of the Berezinian sheaf a suitable quotient sheaf is provided in [32].

It is in general not obvious how to study the Berezinian of a generic supermanifold, though. The theory we have developed in the previous section, in particular the short exact sequence (1.49), allows for an easy result that simplifies the computation in the case of *projected* supermanifolds. To the best knowledge of the author, this result has never appeared in the literature.

Theorem 1.1 (Berezinian of Projected Supermanifolds)**.** *Let $\mathcal{M}$ be a projected supermanifold, with projection given by $\pi : \mathcal{M} \to \mathcal{M}_{red}$, then one has*

$$\mathrm{Ber}\,(\Omega^1_{\mathcal{M}}) \cong \pi^* \left(\det(\Omega^1_{\mathcal{M}_{red}}) \otimes_{\mathcal{O}_{\mathcal{M}_{red}}} (\det \mathcal{F}_{\mathcal{M}})^{\otimes -1} \right) \tag{1.51}$$

Proof. We have seen that in presence of a projection $\pi : \mathcal{M} \to \mathcal{M}_{red}$, one has that $\Omega^1_{\mathcal{M}} \cong \pi^* \Omega^1_{\mathcal{M}_{red}} \oplus \pi^* \mathcal{F}_{\mathcal{M}}$, then it is enough to take the Berezinian of the both sides of the isomorphism. In particular, the right-hand side reads

$$\begin{aligned} \mathrm{Ber}\,(\pi^*\Omega^1_{\mathcal{M}_{red}} \oplus \pi^* \mathcal{F}_{\mathcal{M}}) &\cong \mathrm{Ber}\,\left(\pi^*\Omega^1_{\mathcal{M}_{red}}\right) \otimes_{\mathcal{O}_{\mathcal{M}}} \mathrm{Ber}\,(\pi^*\mathcal{F}_{\mathcal{M}}) \\ &\cong \pi^* \left(\det(\Omega^1_{\mathcal{M}_{red}}) \otimes_{\mathcal{O}_{\mathcal{M}_{red}}} (\det \mathcal{F}_{\mathcal{M}})^{\otimes -1} \right), \end{aligned} \tag{1.52}$$

thus completing the proof. □

Notice that this result allows us to evaluate the Berezinian of projected supermanifolds by means of completely "classical" elements. Indeed, whenever there is a projection, what one needs is to know the canonical sheaf $\mathcal{K}_{\mathcal{M}_{red}} := \det(\Omega^1_{\mathcal{M}_{red}})$ of the reduced manifold and the determinant sheaf $\det \mathcal{F}_{\mathcal{M}}$ of the fermionic sheaf, that we recall it is a (locally-free) sheaf of $\mathcal{O}_{\mathcal{M}_{red}}$-modules, that is an object living on the reduced manifold.
Using this result, for example, one can easily evaluate the Berezinian sheaf of a projective superspace $\mathbb{P}^{n|m}$. In order to do so, we first define the sheaves $\mathcal{O}_{\mathbb{P}^{n|m}}(\ell)$ as the pull-back sheaves $\pi^*(\mathcal{O}_{\mathbb{P}^n}(\ell))$ via the projection map $\pi : \mathbb{P}^{n|m} \to \mathbb{P}^n$ of $\mathbb{P}^{n|m}$ on its reduced manifold $\mathbb{P}^n$, where we recall that $\pi^*(\mathcal{O}_{\mathbb{P}^n}(\ell)) := \pi^{-1}(\mathcal{O}_{\mathbb{P}^n}(\ell)) \otimes_{\pi^{-1}\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^{n|m}}$. Then, one has the following corollary.

Corollary 1.3 (Berezinian of $\mathbb{P}^{n|m}$ (Version 1))**.** *Let $\mathbb{P}^{n|m}$ be the $n|m$-dimensional projective superspace. Then*

$$\mathrm{Ber}(\Omega^1_{\mathbb{P}^{n|m}}) \cong \mathcal{O}_{\mathbb{P}^{n|m}}(m-n-1). \tag{1.53}$$

Proof. In the case of $\mathbb{P}^{n|m}$ it boils down to consider the following split exact sequence

$$0 \longrightarrow \pi^*\Omega^1_{\mathbb{P}^n} \longrightarrow \Omega^1_{\mathbb{P}^{n|m}} \longrightarrow \pi^*(\Pi \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus m}) \longrightarrow 0. \tag{1.54}$$

Therefore, taking the Berezinian of the short exact sequence, one gets

$$\begin{aligned} \mathrm{Ber}(\Omega^1_{\mathbb{P}^{n|m}}) &\cong \mathrm{Ber}\,\left(\pi^*\Omega^1_{\mathbb{P}^n} \oplus \pi^*\left(\Pi\mathcal{O}^{\oplus m}_{\mathbb{P}^n}\right)\right) \\ &\cong \mathrm{Ber}(\pi^*\Omega^1_{\mathbb{P}^n}) \otimes_{\mathcal{O}_{\mathbb{P}^{n|m}}} \mathrm{Ber}\left(\pi^*\left(\Pi\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus m}\right)\right) \\ &\cong \pi^*\left(\det(\Omega^1_{\mathbb{P}^n}) \otimes_{\mathcal{O}_{\mathbb{P}^n}} \left(\det(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus m})\right)^{\otimes -1}\right) \\ &\cong \pi^*\left(\mathcal{O}_{\mathbb{P}^n}(-n-1) \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(m)\right) \\ &\cong \pi^*\left(\mathcal{O}_{\mathbb{P}^n}(m-n-1)\right) \\ &\cong \mathcal{O}_{\mathbb{P}^{n|m}}(m-n-1), \end{aligned} \tag{1.55}$$

that yields the conclusion. □

In the next chapter we will prove the same result in another way, using an interesting exact sequence that can be defined for projective superspaces $\mathbb{P}^{n|m}$.

Theorem 1.1 above allows also to define a supersymmetric analog for the first Chern class of an ordinary supermanifold, at least in the case we are dealing with a projected supermanifold. Indeed, we can define

Definition 1.19 (First Chern Class of a Projected Supermanifold)**.** *Let $\mathcal{M}$ be a projected supermanifold. Then we define the first Chern class $c_1^s(\Omega^1_{\mathcal{M}}) \in H^2(\mathcal{M}_{red}, \mathbb{Z})$ of the cotangent sheaf $\Omega^1_{\mathcal{M}}$ of $\mathcal{M}$ as*

$$c_1^s(\Omega^1_{\mathcal{M}}) := c_1(\det \Omega^1_{\mathcal{M}_{red}}) - c_1(\det \mathcal{F}_{\mathcal{M}}), \tag{1.56}$$

In particular, we define the first Chern class of $\mathcal{M}$ to be given by

$$c_1^s(\mathcal{M}) := -c_1^s(\Omega^1_{\mathcal{M}}). \tag{1.57}$$

Notice that this reduces to the usual definition of the first Chern class of a variety in the case we set the odd part to zero (recall that $\mathcal{F}_{\mathcal{M}} \subseteq \mathcal{J}_{\mathcal{M}}$), that is we have $c_1^s(\mathcal{M}_{red}) = c_1(\mathcal{M}_{red}) = -c_1(\det \Omega^1_{\mathcal{M}_{red}})$. This construction immediately gives the following corollary.

Corollary 1.4 (First Chern Class of $\mathbb{P}^{n|m}$)**.** *Let $\mathbb{P}^{n|m}$ a projective superspace. Then we have*

$$c_1^s(\mathbb{P}^{n|m}) = n + 1 - m. \tag{1.58}$$

Proof. Since we have that $\mathcal{F}_{\mathbb{P}^{n|m}} = \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus m}$, we have that

$$\begin{aligned} c_1^s(\mathbb{P}^{n|m}) &= -c_1(\det \Omega^1_{\mathbb{P}^n}) + c_1(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus m}) \\ &= -c_1(\mathcal{O}_{\mathbb{P}^n}(-1-n)) + m \cdot c_1(\mathcal{O}_{\mathbb{P}^n}(-1)) \\ &= n + 1 - m, \end{aligned} \tag{1.59}$$

that proves the corollary. □

There is, though, an important remark: when dealing with a *non-projected* supermanifold, no exact sequence comes in our help to study the Berezinian sheaf, moreover it is not obvious how to define a first Chern class. Actually, one then needs to carry out explicit computations, investigating the Berezinian of the change of coordinates of the cotangent sheaf among charts, as we will see later on in this thesis.
There is, however, a distinct class of supermanifolds that can be singled out when looking at the Berezinian sheaf.

Definition 1.20 (Calabi-Yau Supermanifolds)**.** *Let $\mathcal{M}$ be a (complex) supermanifold and let $\mathcal{B}er(\mathcal{M})$ be its Berezinian sheaf. We call $\mathcal{M}$ a Calabi-Yau supermanifold if it has a trivial Berezinian sheaf, that is*

$$\mathcal{B}er(\mathcal{M}) \cong \mathcal{O}_{\mathcal{M}}. \tag{1.60}$$

We will call the triviality condition on the Berezinian sheaf Calabi-Yau condition *henceforth.*

Again, notice that this definition is reasonable in view of the similarity between the canonical sheaf in the context of ordinary algebraic geometry and the Berezianian in the context of algebraic supergeometry: triviality of these two sheaves leads to the notion of Calabi-Yau manifold and supermanifold respectively. There are two facts about Calabi-Yau supermanifolds that is worth stressing out:

1. There is no analog of *Yau's Theorem* for Calabi-Yau supermanifolds and the notion of Ricci-flatness seems problematic in a supergeometric context (see [45] for details). In this regard, it is the opinion of the author that a differential-geometric approach is in general not suitable to tackle the questions of complex supergeometry.
2. the reduced manifold of a Calabi-Yau supermanifold is *not* in general a Calabi-Yau manifold.

We make this second point clear by using as usual complex projective spaces as example.

Example 1.3 ($\mathbb{P}^{n|n+1}$ is a Calabi-Yau Supermanifold)**.** *A well-known fact that can be easily red off the theory developed above is that in the case of a projective superspace $\mathbb{P}^{n|m}$ one satisfies the Calabi-Yau condition given above choosing $m = n+1$: in other words $\mathbb{P}^{n|n+1}$ for any $n > 1$ has trivial Berezinian sheaf and vanishing super first Chern class.*
Notice that the reduced space of $\mathbb{P}^{n|n+1}$ for all n is given by $\mathbb{P}^n$, which is a Fano - not a Calabi-Yau manifold - as $\mathcal{K}_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$.

Actually, Calabi-Yau supermanifolds enter many constructions in theoretical physics (see, in particular [2] [44] [57], [70], and the recent [6]), but they have never really undergone a deep mathematical investigation though.

Chapter 2

Supergeometry of Projective Superspaces

This chapter is entirely dedicated to the study of the geometry of complex projective superspaces, that have been defined in the first chapter.
In the first section the cohomology of the invertible sheaves of the form $\mathcal{O}_{\mathbb{P}^{n|m}}(\ell)$ is computed.
Then, in the second section, the even Picard group of $\mathbb{P}^{n|m}$ is studied and it is established that in the case of supercurves $\mathbb{P}^{1|m}$ it has a continuous part.
In the third section maps and embeddings to projective superspaces are discussed. In particular, the notion of *projective supermanifold* is established.
The fourth section is dedicated to the study of infinitesimal automorphisms and first order deformations of $\mathbb{P}^{n|m}$. In this context, the supercurves $\mathbb{P}^{1|m}$ prove again to be the most interesting case.
In the last sections contact with physics is made. The Calabi-Yau supermanifold $\mathbb{P}^{1|2}$ is given the structure of a $\mathcal{N}=2$ *super Riemann surface* and studied in detail. Last, following a formal construction based on path-integral formalism due to Aganagic and Vafa [2], the "mirror supermanifold" to $\mathbb{P}^{1|2}$ is constructed, showing that it is $\mathbb{P}^{1|2}$ again.

2.1 Cohomology of $\mathcal{O}_{\mathbb{P}^{n|m}}(\ell)$

We have seen in the first chapter that $\mathbb{P}^{n|m} = \mathfrak{S}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(+1)^{\oplus m})$ is a split supermanifold and as such one has that $\mathbb{P}^{n|m} \cong \operatorname{Gr}\mathbb{P}^{n|m}$. In particular, the structure sheaf $\mathcal{O}_{\mathbb{P}^{n|m}}$ is a sheaf of $\mathcal{O}_{\mathbb{P}^n}$-modules given by

$$\mathcal{O}_{\mathbb{P}^{n|m}} = \bigoplus_{k=0}^{\lfloor m/2 \rfloor} \mathcal{O}_{\mathbb{P}^n}(-2k)^{\oplus \binom{m}{2k}} \oplus \Pi \bigoplus_{k=0}^{\lfloor m/2\rfloor - \delta_{0, m \bmod 2}} \mathcal{O}_{\mathbb{P}^n}(-2k-1)^{\oplus \binom{m}{2k+1}}, \tag{2.1}$$

where we have inserted the symbol Π as a reminder for the parity reversing.
In the previous chapter we have defined the ℓ-shifted sheaf $\mathcal{O}_{\mathbb{P}^{n|m}}(\ell)$ as $\mathcal{O}_{\mathbb{P}^{n|m}}(\ell) := \pi^*(\mathcal{O}_{\mathbb{P}^n}(\ell))$ via the projection morphism $\pi : \mathbb{P}^{n|m} \to \mathbb{P}^n$, where we recall that in turn one has $\pi^*(\mathcal{O}_{\mathbb{P}^n}(\ell)) := \pi^{-1}(\mathcal{O}_{\mathbb{P}^n}(\ell)) \otimes_{\pi^{-1}\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^{n|m}}$. These are sheaves of $\mathcal{O}_{\mathbb{P}^n}$-modules of the following form

$$\mathcal{O}_{\mathbb{P}^{n|m}}(\ell) = \bigoplus_{k=0}^{\lfloor m/2 \rfloor} \mathcal{O}_{\mathbb{P}^n}(-2k+\ell)^{\oplus \binom{m}{2k}} \oplus \Pi \bigoplus_{k=0}^{\lfloor m/2\rfloor - \delta_{0, m \bmod 2}} \mathcal{O}_{\mathbb{P}^n}(-2k-1+\ell)^{\oplus \binom{m}{2k+1}}. \tag{2.2}$$

We shall then use the well-known result about the cohomology of the sheaves $\mathcal{O}_{\mathbb{P}^n}(\ell)$ (see for example [49]) in order to compute the cohomology of $\mathcal{O}_{\mathbb{P}^{n|m}}(\ell)$. We recall that

$$h^0(\mathcal{O}_{\mathbb{P}^n}(\ell)) = \binom{\ell+n}{\ell}, \qquad h^n(\mathcal{O}_{\mathbb{P}^n}(\ell)) = \binom{|\ell|-1}{|\ell|-n-1}, \tag{2.3}$$

where $\ell \geqslant 0$ in the first equality and $\ell < 0$ and $|\ell| \geqslant n+1$ in the second equality and all the other cohomology groups are trivial.

It is an easy consequence of the previous decomposition that when dealing with $\mathcal{O}_{\mathbb{P}^{n|m}}(\ell)$ we shall only have non-trivial 0-th and n-th cohomology groups for any m.
In order to make the combinatorics easier when computing the cohomologies, we will consider together the *even* and *odd* dimensions of the cohomology groups, by looking at the sheaves of $\mathcal{O}_{\mathbb{P}^n}$-modules above simply as

$$\mathcal{O}_{\mathbb{P}^{n|m}}(\ell) = \bigoplus_{k=0}^{m} \mathcal{O}_{\mathbb{P}^n}(-k+\ell)^{\oplus\binom{m}{k}}, \tag{2.4}$$

It is fair to say anyway that it would be nice and useful to separate even and odd dimensions of the cohomology group at some point to have a result that clearly takes into account the vector superspace nature of the cohomology groups.

We start considering the 0-th cohomology of $\mathcal{O}_{\mathbb{P}^{n|m}}(\ell)$. We have to deal with two cases: when $m < \ell$, and therefore all the bits in the decomposition are contributing, and when $m \geqslant \ell$ and therefore just the first ℓ contribute.

- $m < \ell$: in this case we sum over all the contributions:

$$\begin{aligned} h^0(\mathcal{O}_{\mathbb{P}^{n|m}}(\ell)) &= \sum_{k=0}^{m} \binom{m}{k}\binom{\ell-k+n}{\ell-k} = \sum_{k=0}^{m} \frac{m!(\ell-k+n)!}{(m-k)!k!(\ell-k)!n!} \\ &= \frac{1}{n!}\sum_{k=0}^{m}\binom{m}{k}\frac{(\ell-k+n)!}{(\ell-k)!} = \frac{1}{n!}\sum_{k=0}^{m}\binom{m}{k}(\ell-k+n)\cdot\ldots\cdot(\ell-k+1) \\ &= \frac{1}{n!}\sum_{k=0}^{m}\binom{m}{k}\left[\frac{d^n}{dx^n}x^{\ell-k+n}\right]_{x=1} = \frac{1}{n!}\frac{d^n}{dx^n}\left[x^{\ell+n}\left(1+\frac{1}{x}\right)^m\right]_{x=1} \\ &= \frac{1}{n!}\frac{d^n}{dx^n}\left[(x+1)^{\ell+n-m}(x+2)^m\right]_{x=0}. \end{aligned} \tag{2.5}$$

- $m \geqslant \ell$: in this case we only sum over the first ℓ contributions:

$$\begin{aligned} h^0(\mathcal{O}_{\mathbb{P}^{n|m}}(\ell)) &= \sum_{k=0}^{\ell} \binom{m}{k}\binom{\ell-k+n}{\ell-k} = \sum_{k=0}^{\ell} \frac{m!(\ell-k+n)!}{(m-k)!k!(\ell-k)!n!} \\ &= \frac{m!}{n!\,\ell!}\sum_{k=0}^{\ell}\binom{\ell}{k}\frac{(\ell-k+n)!}{(m-k)!} = \frac{m!}{n!\,\ell!}\sum_{k=0}^{\ell}\binom{\ell}{k}(\ell-k+n)\cdot\ldots\cdot(m-k+1) \\ &= \frac{m!}{n!\,\ell!}\sum_{k=0}^{\ell}\binom{\ell}{k}\left[\frac{d^{\ell+n-m}}{dx^{\ell+n-m}}x^{\ell-k+n}\right]_{x=1} = \frac{m!}{n!\,\ell!}\frac{d^{\ell+n-m}}{dx^{\ell+n-m}}\left[x^{\ell+n}\left(1+\frac{1}{x}\right)^{\ell}\right]_{x=1} \\ &= \frac{m!}{n!\,\ell!}\frac{d^{\ell+n-m}}{dx^{\ell+n-m}}\left[(x+1)^n(x+2)^{\ell}\right]_{x=0}. \end{aligned} \tag{2.6}$$

We now keep our attention on the contribution given by the n-th cohomology. Again, one needs to distinguish between two cases: namely, when $\ell+n+1 \leqslant 0$ all the summands in the decomposition contribute to the cohomology, while if $\ell+n+1 > 0$ we find that the only bits contributing are the ones having $k \geqslant \ell+n+1$.

- $\ell+n+1 \leqslant 0$: in this case we sum over all the contributions:

$$\begin{aligned} h^n(\mathcal{O}_{\mathbb{P}^{n|m}}(\ell)) &= \sum_{k=0}^{m} \binom{m}{k}\binom{k-\ell-1}{k-\ell-n-1} = \sum_{k=0}^{m} \frac{m!(k-\ell-1)!}{(m-k)!k!n!(k-\ell-n-1)!} \\ &= \frac{1}{n!}\sum_{k=0}^{m}\binom{m}{k}\frac{(k-\ell-1)!}{(k-\ell-n-1)!} = \frac{1}{n!}\sum_{k=0}^{m}\binom{m}{k}(k-\ell-1)\cdot\ldots\cdot(k-\ell-n) \\ &= \frac{1}{n!}\sum_{k=0}^{m}\binom{m}{k}\left[\frac{d^n}{dx^n}x^{k+|\ell|-1}\right]_{x=1} = \frac{1}{n!}\left[\frac{d^n}{dx^n}x^{|\ell|-1}(1+x)^m\right]_{x=1} \\ &= \frac{1}{n!}\left[\frac{d^n}{dx^n}(x+1)^{|\ell|-1}(x+2)^m\right]_{x=0}. \end{aligned} \tag{2.7}$$

Actually, this holds in the case $1 \leqslant |\ell| \leqslant n$. In the special sub-case $\ell = -1$, one finds:

$$
\begin{aligned}
h^n(\mathcal{O}_{\mathbb{P}^{n|m}}(-1)) &= \frac{1}{n!}\left[\frac{d^n}{dx^n}(x+2)^m\right] = \frac{1}{n!} m\cdot(m-1)\cdot\ldots\cdot(m-n+1)\cdot 2^{m-n} \\
&= \binom{m}{n}\cdot 2^{m-n}
\end{aligned} \tag{2.8}
$$

- $\ell + n + 1 > 0$: in this case, the first contribution comes at $k = \ell + n + 1$:

$$
\begin{aligned}
h^n(\mathcal{O}_{\mathbb{P}^{n|m}}(\ell)) &= \sum_{k=\ell+n+1}^{m}\binom{m}{k}\binom{k-\ell-1}{k-\ell-n-1} = \frac{1}{n!}\sum_{k=\ell+1}^{m}\binom{m}{k}\left[\frac{d^n}{dx^n}x^{k-\ell-1}\right]_{x=1} \\
&= \frac{1}{n!}\left[\frac{d^n}{dx^n}\frac{1}{x^{\ell+1}}\left((x+1)^m - \sum_{k=0}^{\ell}\binom{\ell}{k}x^k\right)\right]_{x=1} \\
&= \frac{1}{n!}\left[\frac{d^n}{dx^n}\frac{1}{(x+1)^{\ell+1}}\left((x+2)^m - \sum_{k=0}^{\ell}\binom{\ell}{k}(x+1)^k\right)\right]_{x=0}
\end{aligned} \tag{2.9}
$$

  where we stress that we have changed the sum from $\ell + n + 1$ to $\ell + 1$ since the derivative kills the respective terms, which therefore do not give a contribution. This also holds for $k \geqslant \ell + n + 1 \leqslant m$.

For the sake of notation we introduce the following definitions:

$$
\chi_{m<\ell}(n|m;\ell) := \frac{1}{n!}\frac{d^n}{dx^n}\left[(x+1)^{\ell+n-m}(x+2)^m\right]_{x=0} \tag{2.10}
$$

$$
\chi_{m\geqslant\ell}(n|m;\ell) := \frac{m!}{n!\,\ell!}\frac{d^{\ell+n-m}}{dx^{\ell+n-m}}\left[(x+1)^n(x+2)^\ell\right]_{x=0} \tag{2.11}
$$

$$
\zeta_{\ell+n+1\leqslant 0}(n|m;\ell) := \frac{1}{n!}\left[\frac{d^n}{dx^n}(x+1)^{|\ell|-1}(x+2)^m\right]_{x=0} \tag{2.12}
$$

$$
\zeta_{\ell+n+1>0}(n|m;\ell) := \frac{1}{n!}\left[\frac{d^n}{dx^n}\frac{1}{(x+1)^{\ell+1}}\left((x+2)^m - \sum_{k=0}^{\ell}\binom{\ell}{k}(x+1)^k\right)\right]_{x=0}. \tag{2.13}
$$

In conclusion, we have thus proved the following theorem.

Theorem 2.1. *Let $\mathcal{O}_{\mathbb{P}^{n|m}}(\ell)$ be the sheaf of $\mathcal{O}_{\mathbb{P}^{n|m}}$-modules as above. Then one has the following dimensions in cohomology:*

$$
h^i(\mathcal{O}_{\mathbb{P}^{n|m}}(\ell)) = \begin{cases} \chi_{m<\ell}(n|m;\ell) & i=0,\ m<\ell \\ \chi_{m\geqslant\ell}(n|m;\ell) & i=0,\ m\geqslant\ell \\ \zeta_{\ell+n+1\leqslant 0}(n|m;\ell) & i=n,\ \ell+n+1\leqslant 0 \\ \zeta_{\ell+n+1>0}(n|m;\ell) & i=n,\ \ell+n+1>0. \end{cases} \tag{2.14}
$$

All the other cohomologies are null.

2.2 Invertible Sheaves and Even Picard Group $\mathrm{Pic}_0(\mathbb{P}^{n|m})$

We have explained in section 1.2 of the previous chapter that even invertible sheaves, that is locally-free sheaves of $\mathcal{O}_{\mathcal{M}}$-modules of rank $1|0$ on a supermanifold, are classified (up to isomorphism) by the so-called *even* Picard group, we denote it by $\mathrm{Pic}_0(\mathcal{M})$. This can be proved to be isomorphic to the group $H^1(\mathcal{O}^*_{\mathcal{M},0})$ (notice that $\mathcal{O}^*_{\mathcal{M},0}$ is a sheaf of *abelian* groups), as one might easily get by similarity with the ordinary case [42].
In the following theorem we compute the even Picard group of $\mathbb{P}^{n|m}$.

Theorem 2.2 (Even Picard Group for $\mathbb{P}^{n|m}$)**.** *The even Picard group of the projective superspace $\mathbb{P}^{n|m}$ is given by*

$$\mathrm{Pic}_0(\mathbb{P}^{n|m}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{C}^{\{2^{m-2}(m-2)+1\}} & n = 1,\ m \geqslant 2 \\ \mathbb{Z} & else \end{cases} \tag{2.15}$$

Proof. The main tool to be used in order to compute the even Picard group is the *even* exponential short exact sequence, introduced in section 1.2, that reads

$$0 \longrightarrow \mathbb{Z}_{\mathcal{M}} \longrightarrow \mathcal{O}_{\mathcal{M},0} \longrightarrow \mathcal{O}^*_{\mathcal{M},0} \longrightarrow 0. \tag{2.16}$$

where $\mathcal{O}_{\mathcal{M},0}$ is the even part of structure sheaf of $\mathcal{M}$, and likewise for $\mathcal{O}^*_{\mathcal{M},0}$. We now consider separately the case $n \geqslant 3$, $n = 2$ and $n = 1$.

$n \geqslant 2$ This is the easiest case, as one has $H^i(\mathcal{O}_{\mathbb{P}^{n|m},0}) = 0$ for $i = 1, 2$. So the part of the long exact cohomology sequence we are interested into reduces to

$$0 \longrightarrow \mathrm{Pic}_0(\mathbb{P}^{n|m}) \longrightarrow H^2(\mathbb{Z}_{\mathbb{P}^n}) \cong \mathbb{Z} \longrightarrow 0,$$

so that one has $\mathrm{Pic}_0(\mathbb{P}^{n|m}) \cong \mathbb{Z}$.

$n = 2$ The long exact cohomology sequence reduces to

$$0 \longrightarrow \mathrm{Pic}_0(\mathbb{P}^{2|m}) \longrightarrow H^2(\mathbb{Z}_{\mathbb{P}^2}) \cong \mathbb{Z} \longrightarrow H^2(\mathcal{O}_{\mathbb{P}^{2|m},0}) \longrightarrow H^2(\mathcal{O}^*_{\mathbb{P}^{2|m},0}) \longrightarrow 0,$$

this splits to give $\mathrm{Pic}_0(\mathbb{P}^{2|m}) \cong \mathbb{Z}$ and $H^2(\mathcal{O}_{\mathbb{P}^{2|m},0}) \cong H^2(\mathcal{O}^*_{\mathbb{P}^{2|m},0})$.

$n = 1$ This is the richest case, as one finds

$$0 \longrightarrow H^1(\mathcal{O}_{\mathbb{P}^{1|m},0}) \longrightarrow \mathrm{Pic}_0(\mathbb{P}^{1|m}) \longrightarrow H^2(\mathbb{Z}_{\mathbb{P}^1}) \cong \mathbb{Z} \longrightarrow 0,$$

computing the dimension of $H^1(\mathcal{O}_{\mathbb{P}^{1|m},0})$, one has

$$h^1(\mathcal{O}_{\mathbb{P}^{1|m},0}) = \sum_{k=1}^{\lfloor m/2 \rfloor} \binom{m}{2k}(2k-1) = 2^{m-2}(m-2) + 1. \tag{2.17}$$

Indeed, one can observe that

$$\sum_{k=1}^{\lfloor m/2 \rfloor} \binom{m}{2k}(2k-1) = -\sum_{k=1}^{\lfloor m/2 \rfloor} \binom{m}{2k} + \sum_{k=1}^{\lfloor m/2 \rfloor} 2k\binom{m}{2k}$$

and using that

$$(1+\epsilon x)^m = \sum_{k=0}^{m} \epsilon^k x^k \binom{m}{k} \quad \rightsquigarrow \quad x\frac{d}{dx}(1+\epsilon x)^m = \sum_{k=0}^{m} k\epsilon^k x^k \binom{m}{k}$$

so that for $m \geqslant 2$ one might write

$$x\frac{d}{dx}\left[(1+x)^m + (1-x)^m\right] = 2 \cdot \sum_{k=1}^{\lfloor m/2 \rfloor} 2kx^k \binom{m}{2k}.$$

At $x = 1$, the sum yields

$$m2^{m-2} = \sum_{k=1}^{\lfloor m/2 \rfloor} 2k\binom{m}{2k}.$$

Putting the two bits together one finds

$$-\sum_{k=1}^{\lfloor m/2\rfloor}\binom{m}{2k}+\sum_{k=1}^{\lfloor m/2\rfloor}2k\binom{m}{2k}=(-2^{m-1}+1)+(m2^{m-2})=(m-2)2^{m-2}+1.$$

So that the conclusion follows,

$$\mathrm{Pic}_0(\mathbb{P}^{1|m})\cong\mathbb{Z}\oplus\mathbb{C}^{\{2^{m-2}(m-2)+1\}}. \tag{2.18}$$

These exhaust all the possible cases, concluding the theorem. □

The previous theorem tells us that *all* of the invertible sheaves on $\mathbb{P}^{n|m}$ for $n>1$ are of the form $\mathcal{O}_{\mathbb{P}^{n|m}}(\ell)$. In other words, we can say that all of the invertible sheaves on $\mathbb{P}^{n|m}$ for $n>1$ are the pull-backs via the projection $\pi:\mathbb{P}^{n|m}\to\mathbb{P}^n$ of the invertible sheaves $\mathcal{O}_{\mathbb{P}^n}(\ell)$ on $\mathbb{P}^n$, thus Theorem 2.1 exhausts the cohomology of all invertible sheaves of rank $1|0$ over $\mathbb{P}^{n|m}$ for $n>1$.
This is no longer true in the one-dimensional case: indeed over $\mathbb{P}^{1|m}$, for $m\geqslant 2$, there are invertible sheaves that cannot the obtained by the pull-back of a certain invertible sheaf $\mathcal{O}_{\mathbb{P}^1}(\ell)$ via $\pi:\mathbb{P}^{1|m}\to\mathbb{P}^1$, *i.e.* there are *genuinely supersymmetric invertible sheaves on* $\mathbb{P}^{1|m}$. In view of this, it can be seen that the even supergeometry of projected, actually split supermanifolds, could effectively become a richer geometric setting compared to its ordinary counterpart.
Before we actually give the explicit form of the transition functions of the invertible sheaves on the supercurves $\mathbb{P}^{1|m}$ we fix the notation. We consider $\mathbb{P}^{1|m}$ to be covered by two affine superspaces $(\mathcal{U},\mathbb{C}[z,\theta_1,\ldots,\theta_m])$ and $(\mathcal{V},\mathbb{C}[w,\psi_1\ldots,\psi_m])$, where $\mathcal{U}:=\{[X_0:X_1]\in\mathbb{P}^1 : X_0\neq 0\}$ and $\mathcal{V}:=\{[X_0:X_1]\in\mathbb{P}^1 : X_1\neq 0\}$. The transition functions between the two affine superspaces are the obvious ones and are given by $w=1/z$ for the even part and $\theta_i=\psi_i/z$ for the odd part.
We also set $I=(i_1,\ldots,i_m)$ to be a multi-index with $i_k=\{0,1\}$ such that $|I|=\sum_{k=1}^m i_k\leqslant m$ and we put

$$\psi^I:=\psi_1^{i_1}\ldots\psi_k^{i_k}\ldots\psi_m^{i_m}, \tag{2.19}$$

where, clearly, $\psi_k^0=1_{\mathbb{C}}$. Using this notation, we have the following

Theorem 2.3 (Generators of $H^1(\mathcal{O}^*_{\mathbb{P}^{1|m},0})$)**.** *The cohomology group $H^1(\mathcal{O}^*_{\mathbb{P}^{1|m},0})$ is generated by the following Čech 1-cocycles:*

$$H^1(\mathcal{O}^*_{\mathbb{P}^{1|m},0})\cong\left\langle w^k, 1+\sum_{|I|=1}^{\lfloor m/2\rfloor}\sum_{\ell=1}^{2|I|-1}c_\ell^I\frac{\psi^I}{w^\ell}\right\rangle. \tag{2.20}$$

where $k\in\mathbb{Z}$ and $c_\ell^I\in\mathbb{C}$ for each $|I|=1,\ldots,\lfloor m/2\rfloor$ and $\ell=2|I|-1$.

Proof. One has to explicitly compute the representative of $H^1(\mathcal{O}^*_{\mathbb{P}^{1|m},0})$. In order to achieve this, the usual covering of $\mathbb{P}^1$ given by the two open sets $\{\mathcal{U},\mathcal{V}\}$ can be used, so that one has

$$C^0(\{\mathcal{U},\mathcal{V}\},\mathcal{O}^*_{\mathbb{P}^{1|m},0})=\mathcal{O}^*_{\mathbb{P}^{1|m},0}(\mathcal{U})\times\mathcal{O}^*_{\mathbb{P}^{1|m},0}(\mathcal{V}) \tag{2.21}$$

$$C^1(\{\mathcal{U},\mathcal{V}\},\mathcal{O}^*_{\mathbb{P}^{1|m},0})=\mathcal{O}^*_{\mathbb{P}^{1|m},0}(\mathcal{U}\cap\mathcal{V}). \tag{2.22}$$

The Čech 0-cochains are thus given by pairs of elements of the type $(P(z,\theta_1,\ldots,\theta_m),Q(w,\psi_1,\ldots,\psi_m))$. Making use of the notation we set above, we can write the following expressions for the pair (P,Q):

$$\begin{aligned}P(z,\theta_1,\ldots,\theta_m)&=a+\sum_{k=1}^{\lfloor m/2\rfloor}\sum_{|I|=2k}\tilde{P}_I(z)\theta^I\\&=a+\sum_{i<j}^m\tilde{P}_{ij}(z)\theta^i\theta^j+\sum_{i<j<k<l=1}^m\tilde{P}_{ijkl}(z)\theta^i\theta^j\theta^k\theta^l+\ldots\end{aligned} \tag{2.23}$$

$$Q(w,\psi_1,\ldots,\psi_m) = b + \sum_{k=1}^{\lfloor m/2\rfloor}\sum_{|I|=2k}\tilde{Q}_I(w)\psi^I =$$
$$= b + \sum_{i<j}^{m}\tilde{Q}_{ij}(w)\psi^i\psi^j + \sum_{i<j<k<l=1}^{m}\tilde{Q}_{ijkl}(w)\psi^i\psi^j\psi^k\psi^l + \ldots \tag{2.24}$$

where $a,b\in\mathbb{C}^*$. The boundary map $\delta : C^0(\{\mathcal{U},\mathcal{V}\},\mathcal{O}^*_{\mathbb{P}^{1|m},0}) \to C^1(\{\mathcal{U},\mathcal{V}\},\mathcal{O}^*_{\mathbb{P}^{1|m},0})$ acts as

$$\delta((P,Q)) = Q(w,\psi_1,\ldots,\psi_m)P^{-1}(z,\theta_1,\ldots,\theta_m)\lfloor_{\mathcal{U}\cap\mathcal{V}}. \tag{2.25}$$

Explicitly, one finds

$$\delta((P,Q)) = \frac{b}{a} + \sum_{i<j=1}^{m}\left(\frac{\tilde{Q}_{ij}(w)}{a} + \frac{b}{a^2}\frac{\tilde{P}_{ij}(1/w)}{w^2}\right)\psi_i\psi_j +$$
$$+ \sum_{i<j<k<l=1}^{m}\left(\frac{\tilde{Q}_{ijkl}(w)}{a} - \frac{b}{a^3}\frac{\tilde{P}_{ijkl}(1/w)}{w^4} - \frac{1}{a^2}\frac{\tilde{Q}_{ij}(w)\tilde{P}_{kl}(1/w)}{w^2}\right)\psi_i\psi_j\psi_k\psi_l + \ldots \tag{2.26}$$

Clearly, one immediately sees that $H^0(\mathcal{O}^*_{\mathbb{P}^{1|m},0}) \cong \mathbb{C}^*$, as the group is represented by the constant cocycles (a,a) with $a\neq 0$.
On the other hand, the elements in $\mathcal{O}^*_{\mathbb{P}^{1|m},0}(\mathcal{U}\cap\mathcal{V})$ are given by expressions having the following form

$$W(w,1/w,\psi_1,\ldots,\psi_m) = cw^k + \sum_{k=1}^{\lfloor m/2\rfloor}\sum_{|I|=2k}\tilde{W}_I(w,1/w)\psi^I$$
$$= cw^k + \sum_{i<j}^{m}\tilde{W}_{ij}(w,1/w)\psi^i\psi^j + \sum_{i<j<k<l=1}^{m}\tilde{W}_{ijkl}(w,1/w)\psi^i\psi^j\psi^k\psi^l + \ldots \tag{2.27}$$

where again, clearly $c\in\mathbb{C}^*$, $k\in\mathbb{Z}$ and $\tilde{W}_I\in\mathbb{C}[w,1/w]$ for all the multi-indices I. Confronting the expressions in (2.26) and (2.27) one sees that b/a can be used to set the coefficient c of w^k to 1. Also, for every power in the θ's, the polynomials $\tilde{Q}_I(w)$ kill the regular part of the corresponding $\tilde{W}_I$ and the mixed terms, such as for example $\tilde{Q}_{ij}(w)\tilde{P}_{kl}(1/w)/w^2$, in (2.26), do not contribute anyway, as they enter in lower-order powers in the theta's, so that they are completely fixed.
We thus see that the non-exact 1-cocycles are given by transition functions having the following form

$$H^1(\mathcal{O}^*_{\mathbb{P}^{1|m},0}) \cong \left\langle w^k, 1 + \sum_{|I|=1}^{\lfloor m/2\rfloor}\sum_{\ell=1}^{2|I|-1} c^I_\ell\frac{\psi^I}{w^\ell}\right\rangle \tag{2.28}$$

where $k\in\mathbb{Z}$ and each of the $(m-2)2^{m-2}+1$ coefficients c^I_ℓ is a complex number. □

Recalling that an even invertible sheaf on a supermanifold can be defined exactly as an ordinary invertible sheaf in algebraic geometry, that is by giving an open covering and the transition functions between the open sets of the covering, if we choose the open cover of $\mathcal{M}_{red} = \mathbb{P}^1$ to be given by $\{\mathcal{U},\mathcal{V}\}$ as above, we can adopt the following notation for the even invertible sheaves of $\mathbb{P}^{1|m}$:

$$\mathcal{O}_{\mathbb{P}^{1|m}}(k) \longleftrightarrow \{w^k\}, \tag{2.29}$$
$$\mathcal{L}_{\mathbb{P}^{1|m}}(c_1,\ldots,c_{f(m)}) \longleftrightarrow \left\{1 + \sum_{|I|=1}^{\lfloor m/2\rfloor}\sum_{\ell=1}^{2|I|-1} c^I_\ell\frac{\psi^I}{w^\ell}\right\}, \tag{2.30}$$

for $k\in\mathbb{Z}$, $c^I_\ell\in\mathbb{C}$ and $f(m) = (m-2)2^{m-2}+1$. Note that $\mathcal{O}_{\mathbb{P}^{1|m}}(k) := \pi^*\mathcal{O}_{\mathbb{P}^1}(k)$, where $\pi : \mathcal{M}\to\mathcal{M}_{red}$ is the projection map. Having set these conventions, we get the following theorem.

Theorem 2.4 (Even Picard Group $\mathrm{Pic}_0(\mathbb{P}^{1|m})$)**.** *The even Picard group of $\mathbb{P}^{1|m}$ is generated by the following even invertible sheaves*

$$\mathrm{Pic}_0(\mathbb{P}^{1|m}) \cong \left\langle \mathcal{O}_{\mathbb{P}^{1|m}}(k), \mathcal{L}_{\mathbb{P}^{1|m}}(c_1, 0, \ldots, 0), \ldots, \mathcal{L}_{\mathbb{P}^{1|m}}(0, \ldots, c_{f(m)}) \right\rangle, \tag{2.31}$$

for $k = \pm 1$, $c_1, \ldots c_{f(m)} \in \mathbb{C}$ and where $f(m) = (m-2)2^{m-2} + 1$.

Proof. Taking into account the notation adopted above, this is a consequence of the previous theorem and of the isomorphism $\mathrm{Pic}_0(\mathbb{P}^{1|m}) \cong H^1(\mathcal{O}^*_{\mathbb{P}^{1|m},0})$. $\square$

Before we go on, we stress that one can check that $\mathrm{Pic}_0(\mathbb{P}^{1|m})$, as seen via the isomorphism with $\mathbb{Z} \oplus \mathbb{C}^{(m-2)2^{m-2}+1}$, has the structure of an abelian group with addition, that is

$$\begin{aligned} \mathbb{Z} \oplus (\mathbb{C} \oplus \ldots \oplus \mathbb{C}) \times \mathbb{Z} \oplus (\mathbb{C} \oplus \ldots \oplus \mathbb{C}) &\longrightarrow \mathbb{Z} \oplus (\mathbb{C} \oplus \ldots \oplus \mathbb{C}) \\ ((k, c_1, \ldots, c_{f(m)}), (\tilde{k}, \tilde{c}_1, \ldots, \tilde{c}_{f(m)})) &\longmapsto (k + \tilde{k}, c_1 + \tilde{c}_1, \ldots c_{f(m)} + \tilde{c}_{f(m)}). \end{aligned} \tag{2.32}$$

where $f(m) = (m-2)2^{m-2} + 1$.

At this point, it is fair to say that whereas we have been able to compute the cohomology of the invertible sheaves of the kind $\mathcal{O}_{\mathbb{P}^{n|m}}(\ell)$, it is instead not certainly a trivial task to deduce a general formula for the cohomology of the most general invertible supersymmetric sheaf on $\mathbb{P}^{1|n}$ for $n \geqslant 2$, originating by tensor product of the generators shown above.

At this stage, it would be easy to provide a general formula for the genuinely supersymmetric generators of the even Picard group above, but this would not help to solve the general question. We thus limit ourselves to provide the reader with an example, as to show that these invertible sheaves have an interesting non-trivial cohomology.

Example 2.1 (The Cohomology of a Supersymmetric Invertible Sheaf)**.** *We consider the following supersymmetric invertible sheaf on $\mathbb{P}^{1|3}$:*

$$\mathcal{L}_{\mathbb{P}^{1|3}} := \left\{ \{\mathcal{U}, \mathcal{V}\}, \quad e_{\mathcal{U}} = \left(1 + \sum_{i<j;i,j=1}^{3} \frac{\psi_i \psi_j}{w}\right) e_{\mathcal{V}} \right\} \tag{2.33}$$

for $e_{\mathcal{U}}$ and $e_{\mathcal{V}}$ two local frames on the open sets $\mathcal{U}$ and $\mathcal{V}$ respectively. Notice this is a generator of the even Picard group for $\mathbb{P}^{1|3}$. It is easy to actually compute Čech cohomology. We have that

$$\begin{aligned} C^0(\{\mathcal{U}, \mathcal{V}\}, \mathcal{L}_{\mathbb{P}^{1|3}}) &:= \mathcal{L}_{\mathbb{P}^{1|3}}(\mathcal{U}) \times \mathcal{L}_{\mathbb{P}^{1|3}}(\mathcal{V}) \ni (P(z, \theta_1, \theta_2) e_{\mathcal{U}},\ Q(w, \psi_1, \psi_2) e_{\mathcal{V}}) \\ C^1(\{\mathcal{U}, \mathcal{V}\}, \mathcal{L}_{\mathbb{P}^{1|3}}) &:= \mathcal{L}_{\mathbb{P}^{1|3}}(\mathcal{U} \cap \mathcal{V}) \ni W(w, 1/w, \psi_1, \psi_2) e_{\mathcal{V}} \end{aligned} \tag{2.34}$$

where $P \in \mathbb{C}[z, \theta_1, \theta_2]$, $Q \in \mathbb{C}[w, \psi_1 \psi_2]$ and $W \in \mathbb{C}[w, 1/w, \psi_1, \psi_2]$.
By following the usual strategy, we change coordinates as to get

$$\begin{aligned} P(z, \theta_1, \theta_2) e_{\mathcal{U}} &= \left(A^{(0)}(z) + \sum_{i=1}^{3} A_i^{(1)}(z) \theta_i + \sum_{i<j;i,j=1}^{3} A_{ij}^{(2)}(z) \theta_i \theta_j + A^{(3)}(z) \theta_1 \theta_2 \theta_3 \right) e_{\mathcal{U}} \\ &= \left(A^{(0)}(1/w) + \sum_{i=1}^{3} A_i^{(1)}(1/w) \frac{\psi_i}{w} + \sum_{i<j;i,j=1}^{3} \left(\frac{A_{ij}^{(2)}(1/w)}{w^2} + \frac{A^{(0)}(1/w)}{w} \right) \psi_i \psi_j + \right. \\ &\quad \left. + \sum_{i<j;i,j=1}^{3} \left(\sum_{i=1}^{3} (-1)^{i-1} \frac{A_i^{(1)}(1/w)}{w^2} + \frac{A^{(3)}(1/w)}{w^3} \right) \psi_1 \psi_2 \psi_3 \right) e_{\mathcal{V}}. \end{aligned} \tag{2.35}$$

One can clearly see that there is no way to get a globally defined holomorphic section, that is to extend $P(z, \theta_1, \theta_2) e_{\mathcal{U}}$ to the whole $\mathbb{P}^{1|3}$ without hitting a singularity, and this tells that $h^0(\mathcal{L}_{\mathbb{P}^{1|3}}) = 0|0$.
Instead, considering $(Q - P)\lfloor_{\mathcal{U} \cap \mathcal{V}}$, upon using the expression above for P in the chart $\mathcal{V}$, one finds that $h^1(\mathcal{L}_{\mathbb{P}^{1|3}}) = 3|2$, and in particular, it is generated by the following elements:

$$H^1(\mathcal{L}_{\mathbb{P}^{1|3}}) = \left\langle \frac{\psi_1 \psi_2}{w}, \frac{\psi_1 \psi_3}{w}, \frac{\psi_2 \psi_3}{w} \,\middle|\, \frac{\psi_1 \psi_2 \psi_3}{w}, \frac{\psi_1 \psi_2 \psi_3}{w^2} \right\rangle_{\mathbb{C}} \tag{2.36}$$

where we have written the representative in the chart $\mathcal{V}$: notice that all of these elements are nilpotent, they live in $\mathcal{J}_{\mathbb{P}^{1|3}}(\mathcal{U} \cap \mathcal{V})$. The cohomology of $\mathcal{L}_{\mathbb{P}^{1|3}}$ is thus given by

$$h^i(\mathcal{L}_{\mathbb{P}^{1|3}}) = \begin{cases} 0|0 & i = 0 \\ 3|2 & i = 1. \end{cases} \tag{2.37}$$

Similar computations can be easily done for any invertible sheaves of this kind: in general, one would again find a vanishing zeroth cohomology group, while a non-vanishing - and possibly very rich as the fermionic dimension of $\mathbb{P}^{1|m}$ increases - first cohomology group.

2.3 Maps and Embeddings into Projective Superspaces

For future application in this thesis we now discuss how to set up a map, or better an embedding, into a projective superspace. Indeed, in the next chapter, beside realising examples of non-projected supermanifolds, we will be interested into understanding whether it is possible to realise an embedding of these non-projected supermanifolds into some supermanifold *with a universal property*, such as, for example, a projective superspace $\mathbb{P}^{n|m}$.

We first review the general framework, referring mainly to [16] and [38] for further details. As in the ordinary theory, a *sub-supermanifold* is defined in general as a pair $(\mathcal{N}, \iota)$, were $\mathcal{N}$ is a supermanifold and $\iota := (\iota, \iota^\sharp) : (\mathcal{N}, \mathcal{O}_{\mathcal{N}}) \to (\mathcal{M}, \mathcal{O}_{\mathcal{M}})$ is an *injective* morphism with some regularity property. In particular, depending on these regularity properties, we can distinguish between two kind of sub-supermanifolds. We start from the milder notion.

Definition 2.1 (Immersed Supermanifold)**.** *Let $\iota := (i, i^\sharp) : (|\mathcal{N}|, \mathcal{O}_{\mathcal{N}}) \to (|\mathcal{M}|, \mathcal{O}_{\mathcal{M}})$ be a morphism of supermanifolds. We say that $(\mathcal{N}, \iota)$ is an immersed supermanifold if $i : |\mathcal{N}| \to |\mathcal{M}|$ is injective and the differential $(d\iota)_x : \mathcal{T}_{\mathcal{N}}(x) \to \mathcal{T}_{\mathcal{M}}(i(x))$ is injective for all $x \in |\mathcal{N}|$.*

Making stronger requests, we can give instead the following definition.

Definition 2.2 (Embedded Supermanifold)**.** *Let $\iota := (i, i^\sharp) : (|\mathcal{N}|, \mathcal{O}_{\mathcal{N}}) \to (|\mathcal{M}|, \mathcal{O}_{\mathcal{M}})$ be a morphism of supermanifolds. We say that $(\mathcal{N}, \iota)$ is an embedded supermanifold if it is an immersed submanifold and $i : |\mathcal{M}| \to |\mathcal{N}|$ is an homeomorphism onto its image.*
In particular, if $\iota(|\mathcal{N}|) \subset |\mathcal{M}|$ is a closed subset of $|\mathcal{M}|$ we will say that $(\mathcal{N}, \iota)$ is a closed embedded supermanifold.

In this thesis, we will mostly deal with closed embedded supermanifolds. Remarkably, it is possible to show that a morphism $\iota : \mathcal{N} \to \mathcal{M}$ is an embedding *if and only if* the corresponding morphism $\iota^\sharp : \mathcal{O}_{\mathcal{M}} \to \mathcal{O}_{\mathcal{N}}$ is a surjective morphism of sheaves (see [16]). Notice that, for example, given a supermanifold $\mathcal{M}$, one always has a natural closed embedding: the map $\iota : \mathcal{M}_{red} \to \mathcal{M}$, that embeds the reduced manifold underlying the supermanifold into the supermanifold itself.
It is anyway fair to say that these definitions apply only to honest *non-singular* supermanifolds, and it is somehow tricky to generalise them, for example, to superschemes, as hinted in [16] and [38].

We now specialise to embedding into projective superspaces $\mathbb{P}^{n|m}$. In a similar way as in ordinary algebraic geometry, after Grothendieck, setting up such an embedding, calls for a search for *very ample* locally-free sheaves of $\mathcal{O}_{\mathcal{M}}$-modules of rank $1|0$.
The first step into this direction is to bring to a supergeometric context the ordinary invertible sheaves of $\mathcal{O}_{\mathbb{P}^n}$-modules $\mathcal{O}_{\mathbb{P}^n}(\ell)$ classified by the Picard group $\mathrm{Pic}\,(\mathbb{P}^n) \cong H^1(\mathcal{O}^*_{\mathbb{P}^n}) \cong \mathbb{Z}$. As we have seen in the first and in the second section of this chapter, this can be achieved using the fact that projective superspaces are *split* supermanifold, that is we have a projection, we write it $\pi : \mathbb{P}^{n|m} \to \mathbb{P}^n$ as usual. We can thus pull-back the invertible sheaves $\mathcal{O}_{\mathbb{P}^n}(\ell)$ to $\mathbb{P}^{n|m}$ by the projection map and we define the sheaves $\mathcal{O}_{\mathbb{P}^{n|m}}(\ell)$ as the pull-back sheaves $\pi^*(\mathcal{O}_{\mathbb{P}^n}(\ell))$ as above, where we recall that $\pi^*(\mathcal{O}_{\mathbb{P}^n}(\ell)) := \pi^{-1}(\mathcal{O}_{\mathbb{P}^n}(\ell)) \otimes_{\pi^{-1}\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^{n|m}}$. Again, the most obvious way to deal with $\mathcal{O}_{\mathbb{P}^{n|m}}(\ell)$ is clearly to look at them as locally-free sheaves of $\mathcal{O}_{\mathbb{P}^n}$-modules: as such, they amount to twist the structure sheaf $\mathcal{O}_{\mathbb{P}^{n|m}}$ seen as a sheaf of $\mathcal{O}_{\mathbb{P}^n}$-modules (as displayed in (2.1)) by $\mathcal{O}_{\mathbb{P}^n}(\ell)$.
In particular it is important to focus on $\mathcal{O}_{\mathbb{P}^{n|m}}(1)$. Clearly, specifying a result obtained above by

discriminating even and odd dimension, its zeroth-cohomology is given, as a sheaf of $\mathcal{O}_{\mathbb{P}^n}$-modules, by

$$H^0(\mathcal{O}_{\mathbb{P}^{n|m}}(1)) = H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \oplus \Pi \bigoplus_{i=1}^{m} H^0(\mathcal{O}_{\mathbb{P}^n}) \cong \mathbb{C}^{n+1|m}, \tag{2.38}$$

and, with obvious notation, $\mathcal{O}_{\mathbb{P}^{n|m}}(1)$ is globally-generated by the sections $\{X_0, \ldots, X_n | \Theta_1, \ldots, \Theta_n\}$, that is we have a surjection $H^0(\mathcal{O}_{\mathbb{P}^{n|m}}(1)) \otimes \mathcal{O}_{\mathbb{P}^{n|m}} \to \mathcal{O}_{\mathbb{P}^{n|m}}(1)$.
The sheaf $\mathcal{O}_{\mathbb{P}^{n|m}}(1)$ plays an important role as one is to set up an embedding of a certain complex supermanifold $\mathcal{M}$ into a projective superspace $\mathbb{P}^{n|m}$. Indeed, as in ordinary algebraic geometry, one has that if $\mathcal{E}$ is a certain globally-generated sheaf of $\mathcal{O}_{\mathcal{M}}$-modules of rank $1|0$, having $n+1|m$ global sections $\{s_0, \ldots, s_n | \xi_1, \ldots, \xi_m\}$, then there exists a morphism $\phi_{\mathcal{E}} : \mathcal{M} \to \mathbb{P}^{n|m}$ such that $\mathcal{E} = \phi_{\mathcal{E}}^*(\mathcal{O}_{\mathbb{P}^{n|m}}(1))$ and such that $s_i = \phi_{\mathcal{E}}^*(X_i)$ and $\xi_j = \phi_{\mathcal{E}}^*(\Theta_j)$ for $i = 0, \ldots, n$ and $j = 1, \ldots, m$. Notice that also the converse is true, that is given a morphism $\phi : \mathcal{M} \to \mathbb{P}^{n|m}$, then there exists a globally generated sheaf of $\mathcal{O}_{\mathcal{M}}$-modules $\mathcal{E}_\phi$ such that it is generated by the global sections $\phi^*(X_i)$ and $\phi^*(\Theta_j)$ for $i = 0, \ldots, n$ and $j = 1, \ldots, m$. Relying on this result, we can give the following definition.

Definition 2.3 (Projective Supermanifold)**.** *We say that a complex supermanifold $\mathcal{M}$ is projective if there exists a morphism $\phi : \mathcal{M} \to \mathbb{P}^{n|m}$ such that ϕ is injective on $\mathcal{M}_{red}$ and its differential $d\phi$ is injective everywhere on $\mathcal{T}_{\mathcal{M}}$.*

We have thus that setting up a morphism from a supermanifolds $\mathcal{M}$ to $\mathbb{P}^{n|m}$ calls for a search for a suitable (very ample) locally-free sheaf of $\mathcal{O}_{\mathcal{M}}$-modules of rank $1|0$: this, in turn, leads to consider the *even Picard group* of $\mathcal{M}$, that classifies such locally-free sheaves.
Before we go on we stress that an empty even Picard group $\mathrm{Pic}_0(\mathcal{M})$ it is enough to guarantee the non-existence of the embedding into projective super space $\phi : \mathcal{M} \to \mathbb{P}^{n|m}$ as ϕ, as morphism of supermanifolds, is a parity-preserving one. We anticipate that this observation will be crucial when studying the existence of embeddings of non-projected $\mathcal{N} = 2$ supermanifold over $\mathbb{P}^2$ later on in this thesis.

2.4 Infinitesimal Automorphisms and First Order Deformations

We are now interested into studying the infinitesimal automorphisms and first order deformations for $\mathbb{P}^{n|m}$ by computing the cohomology of the tangent sheaf $\mathcal{T}_{\mathbb{P}^{n|m}}$ of $\mathbb{P}^{n|m}$.
The main tool that we will exploit is a generalisation to a supergeometric setting of the ordinary Euler exact sequence [41] [42], that reads

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^{n|m}} \longrightarrow \mathcal{O}_{\mathbb{P}^{n|m}}(1) \otimes \mathbb{C}^{n+1|m} \longrightarrow \mathcal{T}_{\mathbb{P}^{n|m}} \longrightarrow 0. \tag{2.39}$$

In the following we will write $\mathcal{O}_{\mathbb{P}^{n|m}}(1)^{\oplus n+1|m} = \mathcal{O}_{\mathbb{P}^{n|m}}(1) \otimes \mathbb{C}^{n+1|m}$.
In passing, we notice that this short exact sequence give another way to compute the Berezinian of the projective superspaces, we have already computed in the first chapter. Indeed the following theorem holds true.

Theorem 2.5 (Berezinian of $\mathbb{P}^{n|m}$ (Version 2))**.** *Let $\mathbb{P}^{n|m}$ be the $n|m$-dimensional projective superspace. Then*

$$\mathrm{Ber}(\Omega^1_{\mathbb{P}^{n|m}}) \cong \mathcal{O}_{\mathbb{P}^{n|m}}(m - n - 1). \tag{2.40}$$

Proof. We consider the dual of the supergeometric version of the Euler exact sequence in (2.39), that is

$$0 \longrightarrow \Omega^1_{\mathbb{P}^{n|m}} \longrightarrow \mathcal{O}_{\mathbb{P}^{n|m}}(-1)^{\oplus n+1|m} \longrightarrow \mathcal{O}_{\mathbb{P}^{n|m}} \longrightarrow 0. \tag{2.41}$$

Since $\mathrm{Ber}(\mathcal{O}_{\mathbb{P}^{n|m}})$ is trivial, taking into account the multiplicative behaviour of the Berezinian with respect to exact sequence, one finds

$$
\begin{aligned}
\mathrm{Ber}(\Omega^1_{\mathbb{P}^{n|m}}) \cong \mathrm{Ber}\left(\mathcal{O}_{\mathbb{P}^{n|m}}(-1)^{\oplus n+1|m}\right) \cong \mathrm{Ber}\left(\mathcal{O}_{\mathbb{P}^{n|m}}(-1)^{\oplus n+1} \oplus \Pi\mathcal{O}_{\mathbb{P}^{n|m}}(-1)^{\oplus m}\right)\\
\mathcal{O}_{\mathbb{P}^{n|m}}(-n-1) \otimes_{\mathcal{O}_{\mathbb{P}^{n|m}}} \mathcal{O}_{\mathbb{P}^{n|m}}(m) \cong \mathcal{O}_{\mathbb{P}^{n|m}}(m-n-1),
\end{aligned}
\tag{2.42}
$$

which concludes the proof. □

We now look at the cohomology exact sequence associated to the Euler exact sequence:

$$
0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^{n|m}}) \xrightarrow{\tilde{e}_0} H^0(\mathcal{O}_{\mathbb{P}^{n|m}}(1)^{\oplus n+1|m}) \longrightarrow H^0(\mathcal{T}_{\mathbb{P}^{n|m}}) \longrightarrow H^1(\mathcal{O}_{\mathbb{P}^{n|m}}) \longrightarrow \dots
$$

$$
\dots \longrightarrow H^{n-1}(\mathcal{T}_{\mathbb{P}^{n|m}}) \longrightarrow H^n(\mathcal{O}_{\mathbb{P}^{n|m}}) \xrightarrow{\tilde{e}_n} H^n(\mathcal{O}_{\mathbb{P}^{n|m}}(1)^{\oplus n+1|m}) \longrightarrow H^n(\mathcal{T}_{\mathbb{P}^{n|m}}) \longrightarrow 0.
$$

These are the only relevant parts of the long exact sequence in cohomology associated to the Euler sequence, since, considering the $\mathcal{O}_{\mathbb{P}^n}$-module structure of the sheaf of algebras $\mathcal{O}_{\mathbb{P}^{n|m}}$ obtained by the projection map $\pi : \mathbb{P}^{n|m} \to \mathbb{P}^n$, one has the factorisation in a direct sum as in (2.1) and Theorem 2.1 holds true.
Actually, the map $\tilde{e}_n : H^n(\mathcal{O}_{\mathbb{P}^{n|m}}) \longrightarrow H^n(\mathcal{O}_{\mathbb{P}^{n|m}}(1) \otimes \mathbb{C}^{n+1|m})$ in cohomology deserves some special attention. One has the following theorem.

Theorem 2.6. *The map*

$$
\tilde{e}_n : H^n(\mathcal{O}_{\mathbb{P}^{n|m}}) \longrightarrow H^n(\mathcal{O}_{\mathbb{P}^{n|m}}(1) \otimes \mathbb{C}^{n+1|m}). \tag{2.43}
$$

has maximal rank. In particular it is injective if $m \neq n+1$.

Proof. We use the Serre duality on a supermanifold (see [65], Proposition 3, for a thorough discussion). The dualising sheaf of $\mathbb{P}^{n|m}$ is given by $\mathrm{Ber}(\Omega^1_{\mathbb{P}^{n|m}})$, that is the so called Berezinian sheaf of $\mathbb{P}^{n|m}$, which has been shown to be isomorphic to $\mathcal{O}_{\mathbb{P}^{n|m}}(m-n-1)$. Given a sheaf $\mathcal{E}_{\mathbb{P}^{n|m}}$ of $\mathcal{O}_{\mathbb{P}^{n|m}}$-module, Serre duality then reads

$$
H^i(\mathcal{E}_{\mathbb{P}^{n|m}}) \cong H^{n-i}(\mathcal{E}^*_{\mathbb{P}^{n|m}} \otimes \mathcal{O}_{\mathbb{P}^{n|m}}(m-n-1))^*. \tag{2.44}
$$

By functoriality of Serre duality, we see therefore that the map ((2.43)) can be written as

$$
\tilde{e}_n : H^0(\mathcal{O}_{\mathbb{P}^{n|m}}(m-n-1))^* \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^{n|m}}(m-n-2) \otimes \mathbb{C}^{n+1|m})^*, \tag{2.45}
$$

which is the dual to the map $H^0(\mathcal{O}_{\mathbb{P}^{n|m}}(m-n-2) \otimes \mathbb{C}^{n+1|m}) \overset{(X_0,\dots,\Theta_m)}{\longrightarrow} H^0(\mathcal{O}_{\mathbb{P}^{n|m}}(m-n-1))$ defined by multiplication of matrices of global sections.
Setting X_i^* and Θ_j^* to be the *dual* base to $\langle X_0, \dots, X_n, \Theta_1, \dots \Theta_m\rangle$, that generates the vector superspace $H^0(\mathcal{O}_{\mathbb{P}^{n|m}}(1))$, we can consider the superspace $\mathcal{U}_{n+1|m}$, spanned by $\langle X_0^*, \dots, X_n^*, \Theta_1^*, \dots, \Theta_m^*\rangle$, and we may write

$$
\begin{aligned}
H^0(\mathcal{O}_{\mathbb{P}^{n|m}}(m-n-1))^* &= Sym^{m-n-1}\left(\mathcal{U}_{n+1|m}\right)\\
H^0(\mathcal{O}_{\mathbb{P}^{n|m}}(m-n-2))^* &= Sym^{m-n-2}\left(\mathcal{U}_{n+1|m}\right),
\end{aligned}
\tag{2.46}
$$

where Sym denotes the symmetric power functor in the supercommutative setting. In other words, this actually means that we are writing these spaces as the superspace of the homogeneous forms in X_i^*, Θ_j^* of global degrees $m-n-1$ and $m-n-2$, respectively. As usual, the dual operation to the multiplication by a variable X_i^* or Θ_j^*, is the derivation $\partial_{X_i^*}$ or $\partial_{\Theta_j^*}$, respectively. Therefore the map (2.45) can be written as the *super gradient map*

$$
\tilde{e}_n : Sym^{m-n-1}(\mathcal{U}_{n+1|m}) \xrightarrow{\tilde{\nabla}_{(X_i^*, \Theta_j^*)}} Sym^{m-n-2}(\mathcal{U}_{n+1|m} \otimes \mathbb{C}^{n+1|m}). \tag{2.47}
$$

where the super gradient map is given by

$$\tilde{\nabla}_{(X_i^*,\Theta_j^*)} := \begin{pmatrix} \partial_{X_0^*} \\ \vdots \\ \partial_{X_n^*} \\ -\partial_{\Theta_1^*} \\ \vdots \\ -\partial_{\Theta_m^*} \end{pmatrix} \tag{2.48}$$

where the minus signs in front of the odd derivatives are due to the super transposition.
Now it is obvious by inspection that this map has non-zero kernel if and only if $m = n+1$, in which case the first space consists in the constant homogeneous forms, and the second space is zero. $\square$

The previous theorem together with the cohomology of the sheaves $\mathcal{O}_{\mathbb{P}^{n|m}}(\ell)$, allows us to compute the cohomology of the tangent space of projective super spaces $\mathbb{P}^{n|m}$. Notice that, surprisingly, some attention must be paid in the case the projective superspace is Calabi-Yau in the sense explained above (*i.e.* trivial Berezinian sheaf), corresponding to $m = n + 1$.
In the case $n > 1$ one finds:

Automorphisms: taking into account the even and odd dimensions, we have that of $h^0(\mathcal{T}_{\mathbb{P}^{n|m}})$ matches the dimension of $\mathfrak{sl}(n+1|m)$, the Lie superalgebra of the Lie supergroup $PGL(n+1|m)$, as somewhat expected by similarity with the ordinary case on $\mathbb{P}^n$. In particular, we have

$$h^0(\mathcal{T}_{\mathbb{P}^{n|m}}) = n^2 + m^2 + 2n|2nm + 2m \qquad n > 1, \;\; \forall m, \tag{2.49}$$

that indeed equals $\dim \mathfrak{sl}(n+1|m)$.

Deformations: dimensional reasons assure that, in the case $n > 2$, the supermanifold $\mathbb{P}^{n|m}$ is rigid for all m. Moreover, in the case $n = 2$, Theorem 2.6 guarantees that when $m \neq 3$, we have $h^1(\mathcal{T}_{\mathbb{P}^{n|m}}) = 0$, since $\tilde{e}_2 : H^2(\mathcal{O}_{\mathbb{P}^{2|m}}) \to H^2(\mathcal{O}_{\mathbb{P}^{2|m}}^{\oplus 3|m}(1))$ is injective and therefore $\mathbb{P}^{2|m}$ is rigid also whenever $m \neq 3$.
The only case that actually needs to be treated carefully is that of the Calabi-Yau supermanifold $\mathbb{P}^{2|3}$: indeed, in this case Theorem 2.6 is not helping us, and further, since we are working over the projective plane $\mathbb{P}^2$ the second cohomology groups could, in principle, be non-zero. We have, thus, the following exact sequence:

$$0 \longrightarrow H^1(\mathcal{T}_{\mathbb{P}^{2|3}}) \longrightarrow H^2(\mathcal{O}_{\mathbb{P}^{2|3}}) \longrightarrow H^2(\mathcal{O}_{\mathbb{P}^{2|3}}(+1)^{\oplus 3|3}) \longrightarrow H^2(\mathcal{T}_{\mathbb{P}^{2|3}}) \longrightarrow 0. \tag{2.50}$$

A direct computation, or the use of the previous formulas, shows that $H^2(\mathcal{O}_{\mathbb{P}^{2|3}}) \cong \mathbb{C}^{0|1}$ and $H^2(\mathcal{O}_{\mathbb{P}^{2|3}}(+1)^{\oplus 3|3}) = 0$, so one has that $h^1(\mathcal{T}_{\mathbb{P}^{2|3}}) = 0|1$ and therefore $\mathbb{P}^{2|3}$ possess a single *odd* deformation. This is the only projective superspace having a first order deformation whenever $n \geqslant 2$. We will see that the situation is much different over $\mathbb{P}^1$.

We summarise these results in the following

Theorem 2.7 (Infinitesimal Automorphisms and First-Order Deformations for $\mathbb{P}^{n|m}$)**.** *Let $\mathbb{P}^{n|m}$ be a projective superspace such that $n > 1$. Then one has*

$$\begin{aligned} &h^0(\mathcal{T}_{\mathbb{P}^{n|m}}) = \dim \mathfrak{sl}(n+1|m) = n^2 + m^2 + 2n|2nm + 2m \\ &h^1(\mathcal{T}_{\mathbb{P}^{n|m}}) = 0|0, \end{aligned} \tag{2.51}$$

the only exception being the Calabi-Yau supermanifold $\mathbb{P}^{2|3}$ which is such that $h^1(\mathcal{T}_{\mathbb{P}^{2|3}}) = 0|1$.

In the next subsection we will focus our attention on the case of supercurves over $\mathbb{P}^1$.

2.4.1 Supercurves over $\mathbb{P}^1$ and the Calabi-Yau Supermanifold $\mathbb{P}^{1|2}$

We now repeat what has been done in the previous section in the case of supercurves.
We start considering supercurves of the kind $\mathbb{P}^{1|m}$, where $m \neq 2$. In this case, as seen above, the map $\tilde{e}_1 : H^1(\mathcal{O}_{\mathbb{P}^{1|m}}) \to H^1(\mathcal{O}_{\mathbb{P}^{1|m}}(1)^{\oplus 2|m})$ is injective and the long exact sequence in cohomology splits in two short exact sequences.
Using some tricks similar to those employed in the proof of Theorem 2.2 to solve the combinatorics, it is possible to give the even and odd dimensions of the cohomology groups involved in long exact sequence related to the Euler exact sequence. We get the following

$$\begin{cases} h^0(\mathcal{O}_{\mathbb{P}^{1|m}}) = 1|0 \\ h^1(\mathcal{O}_{\mathbb{P}^{1|m}}) = (m-2)2^{m-2}+1\,|\,(m-2)2^{m-2} \\ h^0(\mathcal{O}_{\mathbb{P}^{1|m}}(+1)^{\oplus 2|m}) = m^2+4\,|\,2m \\ h^1(\mathcal{O}_{\mathbb{P}^{1|m}}(+1)^{\oplus 2|m}) = (m^2-2m-8)2^{m-2}+m^2+4\,|\,(m^2-2m-8)2^{m-2}+4m. \end{cases} \tag{2.52}$$

We can thus conclude that

$$h^0(\mathcal{T}_{\mathbb{P}^{1|m}}) = m^2+3\,|\,4m \tag{2.53}$$

This is, again, what we expected, since this number corresponds to the dimension of the super Lie algebra $\mathfrak{sl}(2|m)$, connected to the super group $PGL(2|m)$, the "superisation" of $PGL(2,\mathbb{C})$, the group of automorphisms of $\mathbb{P}^1$.
As for the first-order deformations, we finds that

$$h^1(\mathcal{T}_{\mathbb{P}^{1|m}}) = (m^2-3m-6)2^{m-2}+m^2+3\,|\,(m^2-3m-6)2^{m-2}+4m. \tag{2.54}$$

We observe that we have no (first-order) deformations in the case of $\mathbb{P}^{1|1}$ and for $\mathbb{P}^{1|3}$. We anticipate that we also have no deformations in the Calabi-Yau case $\mathbb{P}^{1|2}$, that will be discussed in the next paragraph. We start having deformations from $\mathbb{P}^{1|4}$, where we find for example $h^1(\mathcal{T}_{\mathbb{P}^{1|4}}) = 11|8$.
Before we go on we notice that, of course, $H^2(\mathcal{T}_{\mathbb{P}^{1|m}}) = 0$, therefore following the supersymmetric generalisation of a well-known result by Kodaira and Spencer ([58], page 21) due to A. Yu. Vaintrob [61], we have that for any $m \geqslant 4$, the complex supermanifold $\mathbb{P}^{1|m}$ has no *obstruction classes* and there exists a Kuranishi family whose base space is a complex *supermanifold* having indeed dimension equal to $h^1(\mathcal{T}_{\mathbb{P}^{1|m}})$. It would be certainly interesting to study this family in detail to get acquainted with the - still rather mysterious - *odd deformations* appearing in the theory of supermanifolds.

We are left with the Calabi-Yau supermanifold $\mathbb{P}^{1|2}$: in this case, the map $\tilde{e}_1 : H^1(\mathcal{O}_{\mathbb{P}^{1|2}}) \to H^1(\mathcal{O}_{\mathbb{P}^{1|2}}(1)^{\oplus 2|2})$ is not injective, the long exact sequence does not split into two short exact sequence as for $\mathbb{P}^{1|m}$, for $m \neq 2$, and something interesting happens.
The key is to observe that in the case $m = 2$ we get $h^1(\mathcal{O}_{\mathbb{P}^{1|2}}(+1)^{\oplus 2|2}) = 0|0$, so we immediately have that $h^1(\mathcal{T}_{\mathbb{P}^{1|2}}) = 0|0$, which tells us that $\mathbb{P}^{1|2}$ is rigid, as anticipated. We are left with the following sequence:

$$0 \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^{1|2}}) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^{1|2}}(+1)^{\oplus 2|2}) \longrightarrow H^0(\mathcal{T}_{\mathbb{P}^{1|2}}) \longrightarrow H^1(\mathcal{O}_{\mathbb{P}^{1|2}}) \longrightarrow 0. \tag{2.55}$$

Computing the dimensions we finds:

$$\begin{cases} h^0(\mathcal{O}_{\mathbb{P}^{1|2}}) = 1|0 \\ h^1(\mathcal{O}_{\mathbb{P}^{1|2}}) = 1|0 \\ h^0(\mathcal{O}_{\mathbb{P}^{1|2}}(+1)^{\oplus 2|2}) = 8|8, \end{cases} \tag{2.56}$$

therefore (2.55) reads

$$0 \to \mathbb{C}^{1|0} \longrightarrow \mathbb{C}^{8|8} \longrightarrow H^0(\mathcal{T}_{\mathbb{P}^{1|2}}) \longrightarrow \mathbb{C}^{1|0} \to 0, \tag{2.57}$$

so as for the dimensions we have

$$h^0(\mathcal{T}_{\mathbb{P}^{1|2}}) = 8|8. \tag{2.58}$$

This is somehow surprising for this dimension does *not* correspond to the dimension of the super Lie algebra $\mathfrak{sl}(2|2)$, connected to $PGL(2|2)$: we would indeed find $\dim \mathfrak{sl}(2|2) = 7|8 \neq 8|8$!
The Calabi-Yau supermanifold $\mathbb{P}^{1|2}$ stands out as the *unique* exception among projective super spaces having $h^0(\mathcal{T}_{\mathbb{P}^{n|m}}) \neq \dim \mathfrak{sl}(n|m)$ (see [45]). There is indeed one more "infinitesimal automorphism" to be taken into account beside those coming from $\mathfrak{sl}(2|2)$: it is given by the field $\theta_1\theta_2\partial_z \in H^0(\mathcal{T}_{\mathbb{P}^{1|2}})$ (here represented in one of the two chart covering $\mathbb{P}^{1|2}$), which is defined everywhere. Physically, we would say that this is the *only* existing *bosonisation* of the even (local) coordinate z.
Notice that one might think that following the same line - that is considering bosonisations of the even coordinates - one might discover many more everywhere-defined vector fields enlarging the symmetry transformations of $\mathbb{P}^{n|m}$: this is *not* the case as the previous results show. Indeed, such supposedly everywhere defined vector fields are not allowed by the transformation properties of the local coordinates $\mathbb{P}^{n|m}$: the correct compensations that makes them into global vector fields happen only in the case of one even and two odd coordinates, corresponding to $\mathbb{P}^{1|2}$. The reader might convince himself by considering the $\theta\theta$-bosonisation in the case of $\mathbb{P}^{1|3}$ or $\mathbb{P}^{2|2}$. Going up in the order of bosonisation only makes the situation worse. We summarise the results for the supercurves $\mathbb{P}^{1|m}$ in the following

Theorem 2.8 (Infinitesimal Automorphisms and First Order Deformations for $\mathbb{P}^{1|m}$)**.** *Let $\mathbb{P}^{1|m}$ be a supercurve over $\mathbb{P}^1$. Then one has*

$$h^0(\mathcal{T}_{\mathbb{P}^{1|m}}) = \dim \mathfrak{sl}(2|m) = m^2 + 3 \,|\, 4m, \tag{2.59}$$

the only exception being $\mathbb{P}^{1|2}$, that is such that $h^0(\mathcal{T}_{\mathbb{P}^{1|2}}) = 8|8$.
Moreover, if $m \geqslant 4$ one finds

$$h^1(\mathcal{T}_{\mathbb{P}^{1|m}}) = (m^2 - 3m - 6)2^{m-2} + m^2 + 3 \,|\, (m^2 - 3m - 6)2^{m-2} + 4m. \tag{2.60}$$

If $m < 3$ the supermanifold $\mathbb{P}^{1|m}$ is rigid.

Getting back to the Calabi-Yau case $\mathbb{P}^{1|2}$, for future use we can be even more explicit and find a basis of global sections generating $H^0(\mathcal{T}_{\mathbb{P}^{1|2}})$.
The most generic section, (in the local chart having coordinates $z|\theta_1, \theta_2$), has the form

$$\begin{aligned} s(z, \theta_1, \theta_2) &= (a(z) + b_1(z)\theta_1 + b_2(z)\theta_2 + c(z)\theta_1\theta_2)\, \partial_z \\ &\quad + \sum_{i=1}^{2} (A^{(i)}(z) + B_1^{(i)}(z)\theta_1 + B_2^{(i)}(z)\theta_2 + C^{(i)}(z)\theta_1\theta_2)\partial_{\theta_i}. \end{aligned} \tag{2.61}$$

By passing to the chart $w|\phi_1, \phi_2$ one has the transformation

$$z = \frac{1}{w}, \qquad \theta_i = \frac{\phi_i}{w}, \qquad i = 1, 2, \tag{2.62}$$

so that the local generators $\{\partial_z, \partial_{\theta_i}\}$ for $i = 1, 2$ of $\mathcal{T}_{\mathbb{P}^{1|2}}$, transform as

$$\partial_z = -(w^2\partial_w + w\phi_1\partial_{\phi_1} + w\phi_2\partial_{\phi_2}), \qquad \partial_{\theta_i} = w\partial_{\phi_i}, \qquad i = 1, 2. \tag{2.63}$$

Imposing the absence of singularities when changing local charts, from $(z|\theta_1, \theta_2)$ to $(w|\phi_1, \phi_2)$ - that is computing explicitly $H^0(\mathcal{T}_{\mathbb{P}^{1|2}})$ - we get the following

Theorem 2.9 (Global Sections of $\mathcal{T}_{\mathbb{P}^{1|2}}$)**.** *A basis of the vector superspace $H^0(\mathcal{T}_{\mathbb{P}^{1|2}})$ is given by the sections*

$$\begin{aligned} &V_1 = \partial_z, \qquad V_2 = z\partial_z, \qquad V_3 = z^2\partial_z + z\theta_1\partial_{\theta_1} + z\theta_2\partial_{\theta_2}, \qquad V_4 = \theta_1\theta_2\partial_z, \\ &V_5 = \theta_1\partial_{\theta_1}, \qquad V_6 = \theta_2\partial_{\theta_1}, \qquad V_7 = \theta_1\partial_{\theta_2}, \qquad V_8 = \theta_2\partial_{\theta_2}, \end{aligned} \tag{2.64}$$

$$\begin{aligned} &\Xi_1 = \theta_1\partial_z, \qquad \Xi_2 = z\theta_1\partial_z + \theta_1\theta_2\partial_{\theta_2}, \qquad \Xi_3 = \theta_2\partial_z, \qquad \Xi_4 = z\theta_2\partial_z - \theta_1\theta_2\partial_{\theta_1}, \\ &\Xi_5 = \partial_{\theta_1}, \qquad \Xi_6 = z\partial_{\theta_1}, \qquad \Xi_7 = \partial_{\theta_2}, \qquad \Xi_8 = z\partial_{\theta_2}. \end{aligned} \tag{2.65}$$

Notice that $h^0(\mathcal{T}_{\mathbb{P}^{1|2}}) = 8|8$, as expected upon using homological methods. Actually, the explicit form of the sections could be found using the fact that $\mathbb{P}^{1|2}$ is split, and as such $\mathcal{O}_{\mathbb{P}^{1|2}}$ is a locally-free sheaf of $\mathcal{O}_{\mathbb{P}^1}$-modules. In particular, also the tangent sheaf can be looked at as a locally-free sheaf of $\mathcal{O}_{\mathbb{P}^1}$-modules and, by using one of the exact sequences introduced in section 1.3, one finds:

$$
\begin{aligned}
\mathcal{T}_{\mathbb{P}^{1|2}} &\cong \left(\mathcal{O}_{\mathbb{P}^1}(2) \otimes \left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \Pi\left(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}\right)\right)\right) \oplus \\
&\quad \oplus \Pi\left(\mathcal{O}_{\mathbb{P}^1}(+1)^{\oplus 2} \otimes \left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \Pi\left(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}\right)\right)\right) && (2.66)\\
&\cong \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 5} \oplus \Pi\left(\mathcal{O}_{\mathbb{P}^1}(+1)^{\oplus 4} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}\right). && (2.67)
\end{aligned}
$$

It is then easy to identify the global sections by the usual identifications.
In the next section we will start from the global sections (2.64) and (2.65) to study the $\mathcal{N} = 2$ super Riemann surface structure we can endow $\mathbb{P}^{1|2}$ with.

2.5 $\mathbb{P}^{1|2}$ as $\mathcal{N} = 2$ Super Riemann Surface

We now make explicit the $\mathcal{N} = 2$ super Riemann surface structure of $\mathbb{P}^{1|2}$. Relying on [42] and [69], we can give the following definition.

Definition 2.4 ($\mathcal{N} = 2$ Super Riemann Surface). *A $\mathcal{N} = 2$ super Riemann surface is a triple $(\mathcal{M}, \mathcal{D}_1, \mathcal{D}_2)$ where $\mathcal{M}$ is a complex supermanifold of dimension $1|2$ and $\mathcal{D}_1, \mathcal{D}_2$ are two rank $0|1$ locally-free sub-sheaves of the tangent sheaf $\mathcal{T}_{\mathcal{M}}$, whose sum is direct in $\mathcal{T}_{\mathcal{M}}$, which satisfy the following conditions:*

1. *$\mathcal{D}_1, \mathcal{D}_2$ are* integrable. *That is, if we let D_i, for $i = 1, 2$, be the local generators of $\mathcal{D}_i$, then*

$$
D_i^2 := \frac{1}{2}\{D_i, D_i\} = f D_i, \tag{2.68}
$$

   for some odd local section $f \in (\mathcal{O}_{\mathcal{M}})_1$.

2. *The* Frobenius form

$$
\begin{aligned}
F : \mathcal{D}_1 \otimes \mathcal{D}_2 &\longrightarrow (\mathcal{T}_{\mathcal{M}})_0 \cong \mathcal{T}_{\mathcal{M}}/\mathcal{D}_1 \oplus \mathcal{D}_2 \\
D_1 \otimes D_2 &\longmapsto \{D_1, D_2\} \,\mathrm{mod}\, (\mathcal{D}_1 \oplus \mathcal{D}_2)
\end{aligned} \tag{2.69}
$$

   is an isomorphism.

We call the sub-sheaves $\mathcal{D}_1, \mathcal{D}_2 \subset \mathcal{T}_{\mathcal{M}}$ satisfying these conditions the structure distributions *of the $\mathcal{N} = 2$ super Riemann surface.*

Note that, looking at integrability, the second condition in the definition above is equivalent to say that the sheaf $\mathcal{D}_1 \oplus \mathcal{D}_2$ is *non* integrable, the obstruction to integrability being that the anticommutator $\{D_1, D_2\}$ is linearly independent of D_1 and D_2. In this way $\{D_1, D_2, \{D_1, D_2\}\}$ gives a local basis for the tangent sheaf $\mathcal{T}_{\mathcal{M}}$ at any point, which implies the existence of the following exact sequence

$$
0 \longrightarrow \mathcal{D}_1 \oplus \mathcal{D}_2 \longrightarrow \mathcal{T}_{\mathcal{M}} \longrightarrow \mathcal{D}_1 \otimes \mathcal{D}_2 \longrightarrow 0. \tag{2.70}
$$

We discuss these feature in the following easy example.

Example 2.2 ($\mathbb{C}^{1|2}$ as $\mathcal{N} = 2$ super Riemann surface). *In order to endow the complex superspace $\mathcal{M} = \mathbb{C}^{1|2}$ with a $\mathcal{N} = 2$ super Riemann surface structure one takes the sub-bundles generated, for example, by the global sections*

$$
D_{0,1} = \partial_{\theta_1} + \theta_2 \partial_z, \qquad D_{0,2} = \partial_{\theta_2} + \theta_1 \partial_z, \tag{2.71}
$$

which are integrable (indeed $\{D_{0,i}, D_{0,i}\} = 0$ for $i = 1, 2$) and have anticommutator given by

$$
\{D_{0,1}, D_{0,2}\} = 2\partial_z, \tag{2.72}
$$

so that $D_{0,1}, D_{0,2}, \{D_{0,1}, D_{0,2}\}$ generate the whole $\mathcal{T}_{\mathbb{C}^{1|2}}$, that is

$$
\mathcal{T}_{\mathbb{C}^{1|2}} \cong \mathrm{Span}_{\mathcal{O}_{\mathbb{C}^{1|2}}}\{D_{0,1},\, D_{0,2},\, \{D_{0,1}, D_{0,2}\}\}. \tag{2.73}
$$

This is an example of non-compact $\mathcal{N} = 2$ super Riemann surface.

We note that in the previous example the sections $D_{0,1}, D_{0,2}, \{D_{0,1}, D_{0,2}\}$ defines a *global* $\mathcal{N} = 2$ super Riemann surfaces structure on $\mathbb{C}^{1|2}$. The situation is different in the case of $\mathbb{P}^{1|2}$. Indeed, the "defining sections" $\{D_{0,1},\, D_{0,2}, \{D_{0,1}, D_{0,2}\}\}$ for the $\mathcal{N} = 2$ super Riemann surface structure of $\mathbb{C}^{1|2}$, remain global sections of tangent sheaf even for the supermanifold $\mathbb{P}^{1|2}$, since, looking at the previous Theorem 2.9 one has

$$D_{0,1} = \partial_{\theta_1} + \theta_2 \partial_z = \Xi_3 + \Xi_5, \qquad D_{0,2} = \partial_{\theta_2} + \theta_1 \partial_z = \Xi_1 + \Xi_7, \tag{2.74}$$

$$\{D_{0,1}, D_{0,2}\} = 2\partial_z = 2V_1. \tag{2.75}$$

The big difference resides in that such sections are not sufficient to generate the whole $\mathcal{T}_{\mathbb{P}^{1|2}}$, since ∂_z has a double zero in $w = 0$. We say that these sections define a *local* $\mathcal{N} = 2$ super Riemann surface structure on $\mathbb{P}^{1|2}$.
We can study this particular local $\mathcal{N} = 2$ super Riemann surface structure in some detail. We choose the usual open cover of $\mathcal{M}_{red} = \mathbb{P}^1$ as above, given by two open sets $\mathcal{U}$ and $\mathcal{V}$ having local coordinates given by $z|\theta_1, \theta_2$ and $w|\phi_1, \phi_2$ respectively and related by the transformations in (2.62). We are interested into identifying the structure distributions singled out by the derivations (2.74). To this end, we put

$$D_{\mathcal{U},1} := \partial_{\theta_1} + \theta_2 \partial_z, \qquad D_{\mathcal{U},2} := \partial_{\theta_2} + \theta_1 \partial_z. \tag{2.76}$$

and we study their transformations on the intersection $\mathcal{U} \cap \mathcal{V}$.
In the intersection $\mathcal{U} \cap \mathcal{V}$, one has

$$\begin{aligned} D_{\mathcal{U},1} &= w\partial_{\phi_1} + \frac{\phi_2}{w}\left(-w^2 \partial_w - w\phi_1 \partial_{\phi_1} - w\phi_2 \partial_{\phi_2}\right) \\ &= w\partial_{\phi_1} - w\phi_2 \partial_w + \phi_1 \phi_2 \partial_{\phi_1} \\ &= (w + \phi_1 \phi_2)\left(\partial_{\phi_1} - \phi_2 \partial_2\right) \\ &= (w + \phi_1 \phi_2)\, D_{\mathcal{V},1}, \end{aligned} \tag{2.77}$$

where we have put $D_{\mathcal{V},1} := \partial_{\phi_1} - \phi_2 \partial_2$. Similar transformation applies to $D_{\mathcal{U},2}$.
We now recall that, in the notation of section 2.2, the even Picard group of $\mathbb{P}^{1|2}$ is given by $\mathrm{Pic}_0(\mathbb{P}^{1|2}) \cong \mathbb{Z} \oplus \mathbb{C}$, and generated by

$$\mathrm{Pic}_0(\mathbb{P}^{1|2}) \cong \langle \mathcal{O}_{\mathbb{P}^{1|2}}(\pm 1), \mathcal{L}_{\mathbb{P}^{1|2}}(c) \rangle \tag{2.78}$$

for $c \in \mathbb{C}$, where $\mathcal{L}_{\mathbb{P}^{1|2}}(c) \leftrightarrow \{\{\mathcal{U}, \mathcal{V}\}, (1 + c\phi_1\phi_2/w)\}$. We thus observe that taking $\mathcal{O}_{\mathbb{P}^{1|2}}(1)$ and choosing $c = 1$ in $\mathcal{L}_{\mathbb{P}^{1|2}}(c)$, we have that their tensor product yields

$$\mathcal{O}_{\mathbb{P}^{1|2}}(1) \otimes \mathcal{L}_{\mathbb{P}^{1|2}}(1) \longleftrightarrow \{\{\mathcal{U}, \mathcal{V}\}, (w + \phi_1 \phi_2)\}, \tag{2.79}$$

which is exactly the invertible sheaf identified by the transformations of the derivations $D_{\mathcal{U},1}, D_{\mathcal{U},2}$, up to a parity change.
We can thus conclude that

$$\mathcal{D}_1 \cong \Pi\left(\mathcal{O}_{\mathbb{P}^{1|2}}(1) \otimes \mathcal{L}_{\mathbb{P}^{1|2}}(1)\right), \qquad \mathcal{D}_2 \cong \Pi\left(\mathcal{O}_{\mathbb{P}^{1|2}}(1) \otimes \mathcal{L}_{\mathbb{P}^{1|2}}(1)\right). \tag{2.80}$$

This is interesting, as it shows that the most natural $\mathcal{N} = 2$ super Riemann surfaces structure that $\mathbb{P}^{1|2}$ can be endowed with is related to *genuinely supersymmetric* invertible sheaves - such as $\mathcal{L}_{\mathbb{P}^{1|2}}(c)$ -, that do *not* come from any pull-backs of invertible sheaves over $\mathbb{P}^1$ by the projection $\pi : \mathbb{P}^{1|2} \to \mathbb{P}^1$.

We have seen that $D_{\mathcal{U},1}$ and $D_{\mathcal{U},2}$ as above defined a *local* $\mathcal{N} = 2$ super Riemann surfaces structure, as their commutator yields a vector field proportional to ∂_z, that has a zero of order 2 at $w = 0$. Since $\mathbb{P}^{1|2}$ has a large vector superspace of global sections, $h^0(\mathcal{T}_{\mathbb{P}^{1|2}}) = 8|8$, one can actually look for more general global odd sections satisfying the integrability condition.
The most general form that a global odd section can take is

$$D_{odd} = \sum_{i=1}^{8} \alpha_i \Xi_i, \tag{2.81}$$

where Ξ's that appeared in (2.9) are such that $\mathrm{Span}_{\mathbb{C}}\{\Xi_1, \ldots, \Xi_8\} \cong \big(H^0(\mathcal{T}_{\mathbb{P}^{1|2}})\big)_1$ and where α_i for $i = 1, \ldots, 8$ are complex numbers. We then impose the integrability condition in the form $D^2_{odd} = 0$. This leads to the following conditions

$$\begin{cases} \alpha_1\alpha_5 + \alpha_7\alpha_3 = 0, \\ \alpha_2\alpha_6 + \alpha_8\alpha_4 = 0, \\ \alpha_1\alpha_6 + \alpha_2\alpha_5 + \alpha_3\alpha_8 + \alpha_4\alpha_7 = 0. \end{cases} \tag{2.82}$$

Solving, we find, for example, the sections

$$D_1 = \alpha_1(\Xi_3 + \Xi_5) + \alpha_2(\Xi_4 + \Xi_6), \tag{2.83}$$

$$D_2 = \beta_1(\Xi_1 + \Xi_7) + \beta_2(\Xi_2 + \Xi_8), \tag{2.84}$$

again for $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$. The anticommutator reads

$$\{D_1, D_2\} = 2[\alpha_1\beta_1 V_1 + \alpha_2\beta_1(V_5 + V_2) + \alpha_1\beta_2(V_8 + V_2) + \alpha_2\beta_2 V_3],$$

or more explicitly

$$\{D_1, D_2\} = 2[\alpha_1\beta_1\partial_z + \alpha_2\beta_1(\theta_1\partial_{\theta_1} + z\partial_z) + \alpha_1\beta_2(\theta_2\partial_{\theta_2} + z\partial_z) + \alpha_2\beta_2(z^2\partial_z + z\theta_1\partial_{\theta_1} + z\theta_2\partial_{\theta_2})].$$

Where we note that $\{D_1, D_2\}$ is non-zero at $w = 0$.
Also, notice that the sections D_1 and D_2 of equations (2.83), (2.84) can be re-written in the more meaningful form

$$D_1 = [\alpha_1 + \alpha_2(z - \theta_1\theta_2)](\Xi_3 + \Xi_5), \tag{2.85}$$

$$D_2 = [\beta_1 + \beta_2(z + \theta_1\theta_2)](\Xi_1 + \Xi_7). \tag{2.86}$$

It can be checked that the map $\mathcal{O}_{\mathbb{P}^{1|2}} \to \mathcal{T}_{\mathbb{P}^{1|2}}$ defined by $f \mapsto fD_i$ is injective and therefore one has that D_1 and D_2 generate two invertible sheaves of rank $0|1$, we call them $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively as above. Also, as explained above, we see that now $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_1 \otimes \mathcal{D}_2$ generate the whole $\mathcal{T}_{\mathbb{P}^{1|2}}$, since the triple $\{D_1, D_2, \{D_1, D_2\}\}$ does.

The defining superderivations in the form D_1 and D_2 prove very useful when it comes to investigate the *automorphisms* of the $\mathcal{N} = 2$ super Riemann surface structure. Indeed, the automorphisms of $\mathbb{P}^{1|2}$ are generated by the vector superspace of all global sections of $\mathcal{T}_{\mathbb{P}^{1|2}}$ determined above. We have to select the sub-algebra of global sections *acting internally* on the invertible sheaves $\mathcal{D}_1$, $\mathcal{D}_2$, *i.e.* the sub-algebra of the global sections whose commutators or anticommutators with the D_i is proportional to the D_i. By a direct inspection we see that the automorphisms of the $\mathcal{N} = 2$ super structure are generated by a $4|4$-dimensional linear superspace with basis given by $\{U_1, \ldots U_4, \Sigma_1, \ldots, \Sigma_4\}$, where

$$\begin{gathered} U_1 := V_1, \qquad U_2 := V_2 + V_5, \qquad U_3 := V_3, \qquad U_4 := V_2 + V_8, \\ \Sigma_1 := \Xi_1 + \Xi_7, \qquad \Sigma_2 := \Xi_2 + \Xi_8, \qquad \Sigma_3 := \Xi_3 + \Xi_5, \qquad \Sigma_4 := \Xi_4 + \Xi_6. \end{gathered} \tag{2.87}$$

These generators satisfy the super commutation relations

$$\begin{aligned} &[U_1, U_2] = U_1, \quad [U_1, U_3] = U_2 + U_4, \quad [U_1, U_4] = U_1, \\ &[U_2, U_3] = U_3, \quad [U_2, U_4] = 0, \quad [U_3, U_4] = -U_3; \end{aligned}$$

$$\begin{aligned} &\{\Sigma_1, \Sigma_2\} = 0, \quad \{\Sigma_1, \Sigma_3\} = 2U_1, \quad \{\Sigma_1, \Sigma_4\} = 2U_2, \\ &\{\Sigma_2, \Sigma_3\} = 2U_4, \quad \{\Sigma_2, \Sigma_4\} = 2U_3, \quad \{\Sigma_3, \Sigma_4\} = 0; \end{aligned}$$

$$\begin{aligned} &[U_1, \Sigma_1] = 0, \quad [U_1, \Sigma_2] = \Sigma_1, \quad [U_1, \Sigma_3] = 0, \quad [U_1, \Sigma_4] = \Sigma_3, \\ &[U_2, \Sigma_1] = 0, \quad [U_2, \Sigma_2] = \Sigma_2, \quad [U_2, \Sigma_3] = -\Sigma_3, \quad [U_2, \Sigma_4] = 0, \\ &[U_3, \Sigma_1] = -\Sigma_2, \quad [U_3, \Sigma_2] = 0, \quad [U_3, \Sigma_3] = -\Sigma_4, \quad [U_3, \Sigma_4] = 0, \\ &[U_4, \Sigma_1] = -\Sigma_1, \quad [U_4, \Sigma_2] = 0, \quad [U_4, \Sigma_3] = 0, \quad [U_4, \Sigma_4] = \Sigma_4. \end{aligned}$$

Something better can be done in order to write the resulting superalgebra in a more meaningful and, in particular, physically relevant form. We define

$$
\begin{aligned}
&H := U_1, \qquad K := U_3, \qquad D := \frac{1}{2}\left(U_2 + U_4\right), \qquad Y := \frac{1}{2}\left(U_2 - U_4\right), \\
&Q_1 := \frac{1}{\sqrt{2}}\left(\Sigma_1 - i\Sigma_3\right), \quad Q_2 := \frac{1}{\sqrt{2}}\left(\Sigma_3 - i\Sigma_1\right), \quad S_1 = -\frac{1}{\sqrt{2}}\left(\Sigma_2 - i\Sigma_4\right), \quad S_2 := -\frac{1}{\sqrt{2}}\left(\Sigma_4 - i\Sigma_2\right).
\end{aligned}
\tag{2.88}
$$

For completeness, we write these elements in terms of the (local) basis of the tangent space:

$$
\begin{aligned}
&\textbf{Bosonic generators:} \qquad \begin{cases} H := \partial_z, \\ K := z^2\partial_z + z\theta_1\partial_{\theta_1}, \\ D := z\partial_z + \frac{1}{2}(\theta_1\partial_{\theta_1} + \theta_2\partial_{\theta_2}), \\ Y := \frac{1}{2}\left(\theta_1\partial_{\theta_1} - \theta_2\partial_{\theta_2}\right); \end{cases} \\
&\textbf{Fermionic generators:} \qquad \begin{cases} Q_1 := \frac{1}{2}\left(\theta_1\partial_z + \partial_{\theta_2} - i(\partial_{\theta_1} + \theta_2\partial_z)\right), \\ Q_2 := \frac{1}{2}\left(\partial_{\theta_1} + \theta_2\partial_z - i(\theta_1\partial_z + \partial_{\theta_2})\right), \\ S_1 := \frac{1}{2}\left((-z\theta_1 + iz\theta_2)\partial_z + i(z - \theta_1\theta_2)\partial_{\theta_1} + (-\theta_1\theta_2 - z)\partial_{\theta_2}\right), \\ S_2 := \frac{1}{2}\left((-z\theta_2 + iz\theta_1)\partial_z + (-z + \theta_1\theta_2)\partial_{\theta_1} + i(z + \theta_1\theta_2)\partial_{\theta_2}\right). \end{cases}
\end{aligned}
\tag{2.89}
$$

These definitions allow us to prove, by simply computing the supercommutators, the following

Theorem 2.10 ($\mathcal{N} = 2$ SUSY Algebra)**.** *Let $(\mathbb{P}^{1|2}, \mathcal{D}_1, \mathcal{D}_2)$ be the $\mathcal{N} = 2$ super Riemann surfaces constructed from $\mathbb{P}^{1|2}$. Then the algebra of the $\mathcal{N} = 2$ SUSY-preserving infinitesimal automorphisms is generated by $\left\{H, K, D, Y \,\middle|\, Q_1, Q_2, S_1, S_2\right\}$ and it corresponds to the Lie superalgebra $\mathfrak{osp}(2|2)$ of the orthosymplectic Lie supergroup $OSp(2|2)$, as it satisfies the following structure equations:*

$$
\begin{aligned}
&\{Q_i, Q_j\} = -2i\delta_{ij}H, \qquad \{S_i, S_j\} = -2i\delta_{ij}K, \qquad \{Q_i, S_j\} = +2i\delta_{ij}D - 2\epsilon_{ij}Y, \\
&[H, Q_i] = 0, \qquad [H, S_i] = -Q_i, \qquad [H, Q_i] = S_i, \qquad [H, S_i] = 0, \\
&[D, Q_i] = -\frac{1}{2}Q_i, \qquad [D, S_i] = \frac{1}{2}S_i, \qquad [Y, Q_i] = \frac{1}{2}\epsilon_{ij}Q_j, \qquad [Y, S_i] = \frac{1}{2}\epsilon_{ij}S_j, \\
&[Y, H] = 0, \qquad [Y, D] = 0, \qquad [Y, K] = 0,
\end{aligned}
\tag{2.90}
$$

together with the structure equations of the closed (bosonic) sub-algebra $\mathfrak{o}(2,1)$:

$$
[H, D] = H, \qquad [H, K] = 2D, \qquad [D, K] = K. \tag{2.91}
$$

We stress that, as the reader with some expertise in supersymmetric QFT's might have easily noticed, the above form has the merit to make manifest all the physically relevant elements of the superalgebra, such as the translations, rotations, supersymmetries, dilatations and so on. This shows a direct connection with physical theories, which is sometimes left hidden in the more mathematical oriented literature.

It is anyway fair to stress that some attention needs to to be paid here. Indeed, even if $\mathfrak{osp}(2|2)$ is actually the Lie superalgebra of automorphisms of $\mathbb{P}^{1|2}$ as $\mathcal{N} = 2$ super Riemann surface, the related supergroup $OSp(2|2)$, defined as

$$
OSp(2|2) := \left\{A \in GL(2|2) : \; A^{st} I_{2|2} A = I_{2|2}\right\} \quad \text{where} \quad I_{2|2} := \left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right), \tag{2.92}
$$

is *not* the supergroup of automorphisms of $\mathbb{P}^{1|2}$ as $\mathcal{N} = 2$ super Riemann surface. Instead, it turns out (see [42]) that the supergroup of automorphisms of $\mathbb{P}^{1|2}$ as a $\mathcal{N} = 2$ super Riemann surfaces - call it $\mathbb{P}^{1|2}_{\mathcal{N}=2}$ is obtained as a suitable quotient of $OSp(2|2)$, indeed we have

$$
1 \longrightarrow \mathbb{Z}_2 \longrightarrow OSp(2|2) \longrightarrow Aut\,(\mathbb{P}^{1|2}_{\mathcal{N}=2}) \longrightarrow 1. \tag{2.93}
$$

where $\mathbb{Z}_2 = \{\pm 1\}$. One can see that $Aut(\mathbb{P}^{1|2}_{\mathcal{N}=2})$ has two connected components as $OSp(2|2)$: an automorphism that does not belong to the identity component interchanges the two structure distributions, $\mathcal{D}_1 \leftrightarrow \mathcal{D}_2$.

Before we conclude this section, though, it is fair to say that at, the present state, the study of $\mathcal{N} = 2$ super Riemann surfaces structures is still at its beginning and, as such, it is yet to be properly developed. For example, very little is known about the supermoduli space of (compact) $\mathcal{N} = 2$ super Riemann surfaces. In particular, in genus 0 such a supermoduli space is expected to be a point [27]. With explicit reference to the present section, this result would then call for a deeper study even in the case of the relatively easy example of $\mathbb{P}^{1|2}$ we have been concerned here. For instance, among other issues, it would be interesting to clarify the relationship between the two $\mathcal{N} = 2$ super Riemann surfaces structures we have constructed.

2.6 Aganagic-Vafa's Mirror Supermanifold for $\mathbb{P}^{1|2}$

In [2], Aganagic and Vafa gave some prescriptions based on path-integral formalism to construct the "mirror" of a certain supermanifold. More precisely, they developed a path-integral computation that relates a certain *Landau-Ginzburg model* associated to a complex projective superspace $\mathbb{P}^{n|m}$, to a *σ-model* on a supermanifold $\widetilde{\mathcal{M}}$, embedded into a certain product of projective superspaces (we refer directly to [2] for further details). In what follows, we employ the prescription of Aganagic and Vafa and we construct the dual of the Landau-Ginzburg model associated to the Calabi-Yau supermanifold $\mathbb{P}^{1|2}$. This turns out to be given by a σ-model on a Calabi-Yau supermanifold in $\mathbb{P}^{1|1} \times \mathbb{P}^{1|1}$, which - to a more careful analysis - is again $\mathbb{P}^{1|2}$. In other words, the construction of Aganagic and Vafa, maps $\mathbb{P}^{1|2}$ to itself!
Before we go on to the actual computation, we stress that this section has a completely different flavour compared to the others, as it is based on the formal construction of [2] that cannot be given a rigorous mathematical meaning, mainly because of the issues related to the definition of path-integrals and their measures.

In order to construct the Landau-Ginzburg model attached to $\mathbb{P}^{1|2}$, we focus on the holomorphic part of the superpotential, where X_I, Y_I for $I = 0, 1$ are *bosonic/even* super fields and η_I, χ_I for $I = 0, 1$ are *fermionic/odd* super fields (*i.e.* the lowest component of their expansion is a bosonic field and a fermionic field, respectively), while t is the so called *Kähler parameter*. This superpontential is given by

$$\mathcal{W}_{\mathbb{P}^{1|2}}(X, Y, \eta, \xi) = \int \prod_{I=0}^{1} \mathcal{D}Y_I \mathcal{D}X_I \mathcal{D}\eta_I \mathcal{D}\chi_I \delta \left(\sum_{I=0}^{1} (Y_I - X_I) - t \right) \cdot \exp \left\{ \sum_{I=0}^{1} e^{-Y_I} + e^{-X_I} + e^{-X_I} \eta_I \chi_I \right\}.$$

By a field redefinition,

$$X_1 = \hat{X}_1 + Y_0, \qquad Y_1 = \hat{Y}_1 + Y_0, \tag{2.94}$$

the path-integral above can be recast in the form:

$$\int \mathcal{D}Y_0 \mathcal{D}X_0 \mathcal{D}\hat{Y}_1 \mathcal{D}\hat{X}_1 \prod_{I=0}^{1} \mathcal{D}\eta_I \mathcal{D}\chi_I \delta \left(Y_0 - X_0 + Y_1 - X_1 - t \right) \cdot \exp \left\{ e^{-Y_0} + e^{-X_0} + e^{-\hat{Y}_1 - Y_0} + e^{-\hat{X}_1 - Y_0} + e^{-X_0} \eta_0 \chi_0 + \eta_1 \chi_1 e^{-\hat{X}_1 - Y_0} \right\}.$$

Integrating in X_0, the delta imposes the following constraint on the bosonic fields:

$$X_0 = Y_0 + (Y_1 - X_1) - t. \tag{2.95}$$

Plugging this inside the previous path integral one gets

$$\int \mathcal{D}Y_0\mathcal{D}\hat{Y}_1\mathcal{D}\hat{X}_1 \prod_{I=0}^{1} \mathcal{D}\eta_I\mathcal{D}\chi_I \exp\left\{e^{-Y_0} + e^{-Y_0-(Y_1-X_1)+t} + e^{-\hat{Y}_1-Y_0} + e^{-\hat{X}_1-Y_0}\right\}$$
$$\cdot \exp\left\{e^{-Y_0-(Y_1-X_1)+t}\eta_0\chi_0 + \eta_1\chi_1 e^{-\hat{X}_1-Y_0}\right\}.$$

The fermionic $\mathcal{D}\eta_0\mathcal{D}\chi_0$ integration reads

$$\begin{aligned}&\int \mathcal{D}\eta_0\mathcal{D}\chi_0 \exp\left\{e^{-Y_0-(Y_1-X_1)+t}\eta_0\chi_0\right\} = \\ &= \int \mathcal{D}\eta_0\mathcal{D}\chi_0 e^{-Y_0-(Y_1-X_1)+t}\left(1+\eta_0\chi_0\right) = -e^{-Y_0-(Y_1-X_1)+t},\end{aligned} \tag{2.96}$$

and therefore one obtains that

$$-\int \mathcal{D}Y_0\mathcal{D}\hat{Y}_1\mathcal{D}\hat{X}_1\mathcal{D}\eta_1\mathcal{D}\chi_1 e^{-Y_0-(Y_1-X_1)+t}$$
$$\cdot \exp\left\{e^{-Y_0}\left(1+e^{-(Y_1-X_1)+t} + e^{-\hat{Y}_1} + e^{-\hat{X}_1} + \eta_1\chi_1 e^{-\hat{X}_1}\right)\right\}.$$

Here, e^{-Y_0} might be interpreted as a Lagrange multiplier and we perform the coordinate charge

$$e^{-Y_0} = \Lambda, \qquad \mathcal{D}Y_0 = -\Lambda^{-1}\mathcal{D}\Lambda, \tag{2.97}$$

such that the integral reads

$$\int \Lambda^{-1}\mathcal{D}\Lambda\mathcal{D}\hat{Y}_1\mathcal{D}\hat{X}_1\mathcal{D}\eta_1\mathcal{D}\chi_1\Lambda e^{-(Y_1-X_1)+t}$$
$$\cdot \exp\left\{\Lambda\left(1+e^{-(Y_1-X_1)+t} + e^{-\hat{Y}_1} + e^{-\hat{X}_1} + \eta_1\chi_1 e^{-\hat{X}_1}\right)\right\}.$$

Finally, by another field redefinition, namely

$$e^{-\hat{X}_1} = x_1, \qquad \mathcal{D}\hat{X}_1 = -\frac{\mathcal{D}x_1}{x_1}, \tag{2.98}$$

$$e^{-\hat{Y}_1} = x_1 y_1, \qquad \mathcal{D}\hat{Y}_1 = -\frac{\mathcal{D}y_1}{y_1}, \tag{2.99}$$

$$\eta_1 = \frac{\tilde{\eta}_1}{x_1}, \qquad \mathcal{D}\eta = x_1\mathcal{D}\tilde{\eta}, \tag{2.100}$$

we notice that the Berezinian enters the transformation of the measure. In fact, the path-integral acquires the following form:

$$\begin{aligned}\mathcal{W}_{\mathbb{P}^{1|2}} &= \int \mathcal{D}\Lambda\frac{\mathcal{D}y_1}{y_1}\frac{\mathcal{D}x_1}{x_1}(x_1\mathcal{D}\tilde{\eta}_1)\mathcal{D}\chi_1\left(y_1 e^t\right)\exp\left\{\Lambda\left(1+e^t y_1 + x_1 + x_1 y_1 + \tilde{\eta}_1\chi_1\right)\right\} \\ &= \int \mathcal{D}\Lambda\mathcal{D}y_1\mathcal{D}x_1\mathcal{D}\tilde{\eta}_1\mathcal{D}\chi_1 e^t \exp\left\{\Lambda\left(1+e^t y_1 + x_1 + x_1 y_1 + \tilde{\eta}_1\chi_1\right)\right\}.\end{aligned} \tag{2.101}$$

By noticing that the factor e^t is not integrated over, and performing the integration over the Lagrange multiplier Λ, one obtains that the theory is constrained on the super hypersurface

$$1 + x_1 + x_1 y_1 + \tilde{\eta}\chi + e^t y_1 = 0. \tag{2.102}$$

By redefining the field $\tilde{y}_1 = 1 + y_1$, a more symmetric form can be achieved:

$$1 + x_1\tilde{y}_1 + \tilde{\eta}\chi + e^t(\tilde{y}_1 - 1) = 0. \tag{2.103}$$

Casting the equation in homogeneous form, we have

$$\mathbb{P}^{1|1}\times\mathbb{P}^{1|1} \supset X_0\tilde{Y}_0 + X_1\tilde{Y}_1 + \tilde{\eta}\chi + e^t(X_0\tilde{Y}_1 - X_0\tilde{Y}_0) = 0. \tag{2.104}$$

This is a *superquadric*, call it $\mathcal{Q}$, in $\mathbb{P}^{1|1} \times \mathbb{P}^{1|1}$, with homogeneous coordinates $[X_0 : X_1 : \tilde{\eta}]$ and $[\tilde{Y}_0 : \tilde{Y}_1 : \chi]$ respectively, and it is a Calabi-Yau supermanifold. In the following treatment, we will drop the tildes and we will just call the homogenous coordinates of the super projective spaces $[X_0 : X_1 : \eta] \equiv [X_0 : X_1 : \tilde{\eta}]$ and $[Y_0 : Y_1 : \eta] \equiv [\tilde{Y}_0 : \tilde{Y}_1 : \chi]$. We now re-write the equation for $\mathcal{Q}$ in the following form:

$$X_0((1-e^t)Y_0 + e^t Y_1) + X_1 Y_1 + \eta\chi = 0. \tag{2.105}$$

Setting

$$\ell(Y_0, Y_1) := (1-e^t)Y_0 + e^t Y_1, \tag{2.106}$$

it is not hard to see that the reduced part $\mathcal{Q}_{red}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ is obtained just by setting the odd coordinates to zero, as

$$\mathbb{P}^1 \times \mathbb{P}^1 \supset X_0\, \ell(Y_0, Y_1) + X_1 Y_1 = 0, \tag{2.107}$$

and one can realize that $\mathcal{Q}_{red} \cong \mathbb{P}^1$.
We are interested into fully identifying $\mathcal{Q}$ as a known variety; to this end, we observe that, as embedded into $\mathbb{P}^{1|1} \times \mathbb{P}^{1|1}$, it is covered by the Cartesian product of the usual four open sets:

$$\begin{aligned}
U_0 \times V_0 &= \{[X_0 : X_1 : \eta] : X_0 \neq 0\} \times \{[Y_0 : Y_1 : \chi] : Y_0 \neq 0\},\\
U_0 \times V_1 &= \{[X_0 : X_1 : \eta] : X_0 \neq 0\} \times \{[Y_0 : Y_1 : \chi] : Y_1 \neq 0\},\\
U_1 \times V_0 &= \{[X_0 : X_1 : \eta] : X_1 \neq 0\} \times \{[Y_0 : Y_1 : \chi] : Y_0 \neq 0\},\\
U_1 \times V_1 &= \{[X_0 : X_1 : \eta] : X_1 \neq 0\} \times \{[Y_0 : Y_1 : \chi] : Y_1 \neq 0\}.
\end{aligned} \tag{2.108}$$

Moreover, one needs all the above four open sets to cover $\mathcal{Q}$, because

$$\begin{aligned}
\mathcal{Q}_{red} \cap \{X_0 = 0\} &= [0 : 1] \times [1 : 0] \in U_1 \times V_0,\\
\mathcal{Q}_{red} \cap \{X_1 = 0\} &= [1 : 0] \times [1 : 1 - e^{-t}] \in U_0 \times V_0,\\
\mathcal{Q}_{red} \cap \{Y_0 = 0\} &= [1 : -e^t] \times [0 : 1] \in U_0 \times V_1,\\
\mathcal{Q}_{red} \cap \{X_0 = X_1 = 1\} &= [1 : 1] \times [e^t + 1 : e^t - 1] \in U_1 \times V_1.
\end{aligned} \tag{2.109}$$

Therefore, we would like to find a suitable change of coordinates allowing us to use fewer open sets. It turns out that one can reduce to use only two open sets. Indeed, by switching coordinates to

$$Y_0' := \ell(Y_0, Y_1), \qquad Y_1' := Y_1, \tag{2.110}$$
$$X_0' := X_0, \qquad X_1' := X_1, \tag{2.111}$$
$$\eta' := \eta, \qquad \chi' := \chi, \tag{2.112}$$

the equation for $\mathcal{Q}$ becomes

$$X_0' Y_0' + X_1' Y_1' + \eta'\chi' = 0. \tag{2.113}$$

Then, by exchanging Y_0' with Y_1' and dropping the primes for convenience, one obtains the following equation for $\mathcal{Q}$:

$$X_0 Y_1 + X_1 Y_0 + \eta\chi = 0. \tag{2.114}$$

Since

$$\mathcal{Q}_{red} \cap \{X_0 = 0\} = \mathcal{Q}_{red} \cap \{Y_0 = 0\} = [0 : 1] \times [0 : 1] \in U_1 \times V_1, \tag{2.115}$$
$$\mathcal{Q}_{red} \cap \{X_1 = 0\} = \mathcal{Q}_{red} \cap \{Y_1 = 0\} = [1 : 0] \times [1 : 0] \in U_0 \times V_0, \tag{2.116}$$

this change of coordinates allows us to cover $\mathcal{Q}$ by just two open sets, namely by :

$$U_{\mathcal{Q}} := \mathcal{Q} \cap (U_0 \times V_0), \tag{2.117}$$
$$V_{\mathcal{Q}} := \mathcal{Q} \cap (U_1 \times V_1). \tag{2.118}$$

Therefore, by choosing the (affine) coordinates

$$U_{\mathcal{Q}}: \ z := \frac{X_1}{X_0}, \quad u := \frac{Y_1}{Y_0}, \quad \theta_0 := \frac{\eta}{X_0}, \quad \theta_1 := \frac{\chi}{Y_0}, \tag{2.119}$$

$$V_{\mathcal{Q}}: \ w := \frac{X_0}{X_1}, \quad v := \frac{Y_0}{Y_1}, \quad \phi_0 := -\frac{\eta}{X_1}, \quad \phi_1 := \frac{\chi}{Y_1}, \tag{2.120}$$

we obtain the following affine equations for $\mathcal{Q}$ on $U_{\mathcal{Q}}$ and $V_{\mathcal{Q}}$:

$$U_{\mathcal{Q}}: \ z + u + \theta_0\theta_1 = 0, \tag{2.121}$$

$$V_{\mathcal{Q}}: \ w + v - \phi_0\phi_1 = 0, \tag{2.122}$$

describing lines in $\mathbb{C}^{2|2}$. We notice that these two equations are glued together using the relations

$$w = \frac{1}{z}, \qquad v = \frac{1}{u}, \tag{2.123}$$

$$\phi_0 = -w\theta_0, \qquad \phi_1 = v\theta_1. \tag{2.124}$$

Finally, we would like to characterise the variety $\mathcal{Q}$ by its transition functions, in order to identify it with a known one. By the previous equation, we may take as *proper* bosonic coordinates u and v, as

$$z = -u - \theta_0\theta_1, \tag{2.125}$$

$$w = -v + \phi_0\phi_1. \tag{2.126}$$

We already know that $v = \frac{1}{u}$ and $\phi_1 = \frac{\theta_1}{u}$, so we still have to deal with ϕ_0 :

$$\phi_0 = -\frac{\theta_0}{z} = \frac{\theta_0}{u + \theta_0\theta_1} = \frac{\theta_0(u - \theta_0\theta_1)}{(u + \theta_0\theta_1)(u - \theta_0\theta_1)} = \frac{\theta_0 u}{u^2} = \frac{\theta_0}{u}, \tag{2.127}$$

implying that the variety $\mathcal{Q} \subset \mathbb{P}^1 \times \mathbb{P}^1$ is actually nothing but $\mathbb{P}^{1|2}$.

Chapter 3

$\mathcal{N} = 2$ Non-Projected Supermanifolds over $\mathbb{P}^n$

This chapter is dedicated to the study of the geometry of non-projected supermanifolds having odd dimension equal to 2 - we call them $\mathcal{N} = 2$ supermanifolds - over projective spaces.
In the first section we provide the construction of the cohomological invariant that obstructs the existence of a projection that splits the structural exact sequence of a supermanifold.
Then, in the second section we specialise to the case the reduced manifold is a projective space $\mathbb{P}^n$ and we prove that $\mathcal{N} = 2$ non-projected supermanifolds exist only over the projective line $\mathbb{P}^1$ and the projective plane $\mathbb{P}^2$.
The third section section is dedicated to a classification of non-projected supermanifolds over $\mathbb{P}^1$ - we call them $\mathbb{P}^1_\omega(m, n)$. Also, we study the even invertible sheaves that can be defined over these non-projected supermanifolds, by computing their even Picard group. We then use these even invertible sheaves to explicitly realise an embedding of a particular non-projected supermanifold, namely $\mathbb{P}^2_\omega(2, 2)$, into $\mathbb{P}^{2|2}$, making contact with an example of non-projected supermanifold discussed by Witten in [68].
The fourth section is dedicated to the study of the geometry of non-projected supermanifolds over $\mathbb{P}^2$ - we call them $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$. In particular, it is proved that all of these supermanifolds are Calabi-Yau's and they are all non-projective, *i.e.* they cannot be embedded into any projective superspace $\mathbb{P}^{n|m}$. Instead, we prove that all of these non-projected supermanifolds can always be embedded into a certain super Grassmannian. Last, we realise explicitly these embeddings in two meaningful cases and we study the cohomology at their *split locus*.

3.1 Obstruction to the Splitting of a $\mathcal{N} = 2$ Supermanifold

In this first section we start studying the event in which a (complex) supermanifold does not admit a projection on its reduced part. In particular, we will single out a cohomological invariant that detects an obstruction to split the structural exact sequence (1.3) attached to a particular supermanifold.

First of all, it is important to notice that in the case a supermanifold has odd dimension equal to one there cannot be any obstruction, as the following obvious theorem establishes.

Theorem 3.1 (Supermanifolds of dimension $n|1$). *Let $\mathcal{M} := (|\mathcal{M}|, \mathcal{O}_\mathcal{M})$ a (complex) supermanifold of odd dimension 1. Then $\mathcal{M}$ is defined up to isomorphism by the pair $(\mathcal{M}_{red}, \mathcal{F}_\mathcal{M})$ and in fact, $\mathcal{M} = \mathfrak{S}(\mathcal{M}, \mathcal{F}^*_\mathcal{M})$*

Proof. If the parity splitting reads $\mathcal{O}_\mathcal{M} = \mathcal{O}_{\mathcal{M},0} \oplus \mathcal{O}_{\mathcal{M},1}$ and the odd dimension of the supermanifold is 1, then $\mathcal{J}^2_\mathcal{M} = 0$ and one naturally has that $\mathcal{O}_{\mathcal{M},1} \cong \mathcal{J}_\mathcal{M} \cong \mathcal{F}_\mathcal{M}$, a (locally free) sheaf of $\mathcal{O}_{\mathcal{M}_{red}}$-modules of rank 1, having odd parity, and $\mathcal{O}_{\mathcal{M},0} \cong \mathcal{O}_{\mathcal{M}_{red}}$, so there can't be any bosonisation extending $\mathcal{O}_{\mathcal{M}_{red}}$. $\square$

Obstruction to projectedness might appear in the case the odd dimension of the supermanifold $\mathcal{M}$ is at least 2. Indeed, on the one hand, by looking again at the parity splitting of the structure sheaf $\mathcal{O}_{\mathcal{M}} = \mathcal{O}_{\mathcal{M},0} \oplus \mathcal{O}_{\mathcal{M},1}$, we still have that the odd part $\mathcal{O}_{\mathcal{M},1}$ coincides with the fermionic sheaf $\mathcal{F}_{\mathcal{M}} := \mathcal{J}_{\mathcal{M}} \big/ \mathcal{J}^2_{\mathcal{M}}$: this also tells us that $\mathcal{O}_{\mathcal{M},1} (\cong \mathcal{F}_{\mathcal{M}})$ is a sheaf of $\mathcal{O}_{\mathcal{M}_{red}}$-modules and not only a sheaf of $\mathcal{O}_{\mathcal{M},0}$-modules (notice that this is in general no longer true in the case the fermionic dimension of $\mathcal{M}$ is greater than 2). On the other hand, the even part of the structure sheaf $\mathcal{O}_{\mathcal{M},0}$, which is a sheaf of rings, is an *extension* of $\mathcal{O}_{\mathcal{M}_{red}}$ by $Sym^2 \mathcal{F}_{\mathcal{M}}$:

$$0 \longrightarrow Sym^2 \mathcal{F}_{\mathcal{M}} \longrightarrow \mathcal{O}_{\mathcal{M},0} \xrightarrow{\iota_0^\sharp} \mathcal{O}_{\mathcal{M}_{red}} \longrightarrow 0, \tag{3.1}$$

where $Sym^2 \mathcal{F}_{\mathcal{M}} = \mathrm{Gr}^{(2)} \mathcal{O}_{\mathcal{M}}$ is a sheaf of $\mathcal{O}_{\mathcal{M},0}$-ideals with square zero, that actually corresponds to $\mathcal{J}^2_{\mathcal{M}}$ as $\mathrm{Gr}^{(2)} \mathcal{O}_{\mathcal{M}} := \mathcal{J}^2_{\mathcal{M}} \big/ \mathcal{J}^3_{\mathcal{M}}$ and $\mathcal{J}^3_{\mathcal{M}} = 0$ as $\mathcal{M}$ has fermionic dimension 2. The map $\iota_0^\sharp : \mathcal{O}_{\mathcal{M},0} \to \mathcal{O}_{\mathcal{M}_{red}}$ is a homomorphism of sheaves of rings, which is induced by the inclusion morphism $\iota^\sharp : \mathcal{O}_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}_{red}}$, well defined in a general setting, not restricted to odd dimension 2.
We recall that, in general, there is *no* homomorphism of sheaves of rings $\pi_0^\sharp : \mathcal{O}_{\mathcal{M}_{red}} \to \mathcal{O}_{\mathcal{M},0} \subset \mathcal{O}_{\mathcal{M}}$ splitting $\iota_0^\sharp$ and $\mathcal{O}_{\mathcal{M},0}$ is *not* a sheaf of $\mathcal{O}_{\mathcal{M}_{red}}$-modules: that is, a supermanifold $\mathcal{M}$ having odd dimension equal to 2 does not in general admit a projection $\pi : \mathcal{M} \to \mathcal{M}_{red}$.
Obstruction theory for complex supermanifolds has been first discussed in the seminal work of Green [30]. Here we will show where the obstruction to splitting lies following [41]: due to the difficulty of the cited reference, we will provide the reader with a complete proof. We start by proving the following fundamental lemma.

Lemma 3.1. *Let $\mathcal{M} := (|\mathcal{M}|, \mathcal{O}_{\mathcal{M}})$ be a (complex) supermanifold having odd dimension 2. Then the even part of the structure sheaf $\mathcal{O}_{\mathcal{M},0}$ uniquely defines a class $\omega_{\mathcal{M}} \in H^1(\mathcal{M}_{red}, \mathcal{T}_{\mathcal{M}_{red}} \otimes Sym^2 \mathcal{F}_{\mathcal{M}})$.*

Proof. Let us consider an open cover $\mathcal{U} = \{\mathcal{U}_i\}_{i \in I}$ of $|\mathcal{M}|$, such that for every open set $\mathcal{U}_i$ we do have homomorphisms of sheaves or rings

$$\pi^\sharp_{\mathcal{U}_i} : \mathcal{O}_{\mathcal{U}_i, red} \longrightarrow \mathcal{O}_{\mathcal{U}_i, 0} \tag{3.2}$$

such that $\iota^\sharp_{\mathcal{U}_i,0} \circ \pi^\sharp_{\mathcal{U}_i} = id_{\mathcal{O}_{\mathcal{U}_i,red}}$. This can be done if the open sets $\mathcal{U}_i$ are chosen so that on each $\mathcal{U}_i$ some coordinate system $\underline{z}|\underline{\theta}$ is defined. We denote π_i the map $\pi^\sharp_{\mathcal{U}_i}$ for the sake of brevity, and we refer to the collection $\{\pi_i\}_{i\in I}$ as the *local splittings* for the short exact sequence (3.1). Let us now define the following morphism

$$\omega_{ij} := (\pi_i - \pi_j) \Big\lfloor_{\mathcal{U}_i \cap \mathcal{U}_j} : \mathcal{O}_{\mathcal{U}_i \cap \mathcal{U}_j, red} \longrightarrow \ker \left(\iota^\sharp_{\mathcal{U}_i \cap \mathcal{U}_j, 0} \right) \cong Sym^2 \mathcal{F}_{\mathcal{U}_i \cap \mathcal{U}_j}. \tag{3.3}$$

This is well-defined, indeed the difference above takes vales in the kernel of $\iota^\sharp_{\mathcal{U}_i \cap \mathcal{U}_j, 0}$, as

$$\begin{aligned} \iota^\sharp_{\mathcal{U}_i \cap \mathcal{U}_i, 0} \circ (\pi_i - \pi_j) \Big\lfloor_{\mathcal{U}_i \cap \mathcal{U}_j} &= \iota^\sharp_{\mathcal{U}_i \cap \mathcal{U}_j, 0} \circ (\pi_i) \lfloor_{\mathcal{U}_i \cap \mathcal{U}_j} - \iota^\sharp_{\mathcal{U}_i \cap \mathcal{U}_j, 0} \circ (\pi_j) \lfloor_{\mathcal{U}_i \cap \mathcal{U}_j} \\ &= id_{\mathcal{O}_{\mathcal{U}_i \cap \mathcal{U}_j, red}} - id_{\mathcal{O}_{\mathcal{U}_i \cap \mathcal{U}_j, red}} = 0 \end{aligned} \tag{3.4}$$

and by the short exact sequence (3.1) we see that this kernel is given by $Sym^2 \mathcal{F}_{\mathcal{U}_i \cap \mathcal{U}_j}$.
Now let f, g be local sections of $\mathcal{O}_{\mathcal{U}_i \cap \mathcal{U}_j, red}$, then remembering that we write $\pi_i(f)$ instead of $\pi_i \lfloor_{\mathcal{U}_i \cap \mathcal{U}_j} (f)$ for the sake of brevity, we have

$$\begin{aligned} \omega_{ij}(f \cdot g) &= \pi_i(f \cdot g) - \pi_j(f \cdot g) \\ &= \pi_i(f)\, \pi_i(g) - \pi_j(f)\, \pi_j(g) \\ &= \pi_i(f)\, \pi_i(g) \underbrace{- \pi_i(f)\, \pi_j(g) + \pi_i(f)\, \pi_j(g)}_{=0} - \pi_j(f)\, \pi_j(g) \\ &= \pi_i(f) \underbrace{(\pi_i(g) - \pi_j(g))}_{\omega_{ij}(g)} + \underbrace{(\pi_i(f) - \pi_j(f))}_{\omega_{ij}(f)} \pi_j(g) \\ &= \pi_i(f)\, \omega_{ij}(g) + \omega_{ij}(f)\, \pi_j(g), \end{aligned} \tag{3.5}$$

where we have used the definition of ω_{ij} and the fact that the π_i are homomorphisms of sheaves of rings. Recall that the $\mathcal{O}_{\mathcal{U}_i,0}$-module structure of $Sym^2\mathcal{F}_{\mathcal{U}_i}$ is the same as its $\mathcal{O}_{\mathcal{U}_i,red}$-module structure, as $Sym^2\mathcal{F}_{\mathcal{U}_i} \cong \mathcal{J}^2_{\mathcal{M}}/\mathcal{J}^3_{\mathcal{M}}$. By looking at the last map, this implies that

$$\omega_{ij}(f\cdot g) = \pi_i(f)\,\omega_{ij}(g) + \omega_{ij}(f)\,\pi_j(g) = f\cdot\omega_{ij}(g) + \omega_{ij}(f)\cdot g, \tag{3.6}$$

so that we have that the map $\omega_{ij} : \mathcal{O}_{\mathcal{U}_i\cap\mathcal{U}_j,red} \to Sym^2\mathcal{F}_{\mathcal{U}_i\cap\mathcal{U}_j}$ satisfies the Leibniz rule and therefore it is a derivation on $\mathcal{O}_{\mathcal{U}_i\cap\mathcal{U}_j,red}$ valued in the sheaf $Sym^2\mathcal{F}_{\mathcal{U}_i\cap\mathcal{U}_j}$.
So far we have proved that $\omega_{ij} \in (\mathcal{T}_{\mathcal{M}_{red}}\otimes Sym^2\mathcal{F}_{\mathcal{M}})(\mathcal{U}_i\cap\mathcal{U}_j)$, actually the choice of the local splittings $\{\pi_i\}_{i\in I}$ define a Čech 1-cocycle, $(\omega_{ij})_{i,j\in I} \in Z^1(\mathcal{U},\mathcal{T}_{\mathcal{M}_{red}}\otimes Sym^2\mathcal{F}_{\mathcal{M}})$, since by definition

$$(\omega_{ij}+\omega_{jk}+\omega_{ki})\big\lfloor_{\mathcal{U}_i\cap\mathcal{U}_j\cap\mathcal{U}_k} = (\pi_i-\pi_j+\pi_j-\pi_k+\pi_k-\pi_i)\big\lfloor_{\mathcal{U}_i\cap\mathcal{U}_j\cap\mathcal{U}_k} = 0. \tag{3.7}$$

This is clear once it is observed that $(\omega_{ij})_{i,j\in I}$ is defined as the application of the Čech coboundary operator δ on the 0-cochain given by the collection of the local splitting $\{\pi_i\}_{i\in I}$.
Now we want to prove that, by changing the local splitting, than the $(\omega_{ij})_{ij\in I}$ only changes by a coboundary term. To this end we let $\{\pi'_i\}_{i\in I}$ be another choice of local splittings: if we require $\iota^\sharp_{\mathcal{U}_i}\circ\pi'_i = id_{\mathcal{O}_{\mathcal{U}_i,red}}$ to hold, then the only possibility is that

$$\pi'_i = \pi_i + \psi_i \quad \text{where} \quad \psi_i : \mathcal{O}_{\mathcal{U}_i,red} \longrightarrow Sym^2\mathcal{F}_{\mathcal{U}_i}(\subseteq \mathcal{O}_{\mathcal{U}_i,0}), \tag{3.8}$$

otherwise the composition with $\iota^\sharp_{\mathcal{U}_i}$ does not yield a local identity. Moreover, as π'_i is a homomorphism of sheaves of rings, one has that $\pi'_i(f\cdot g) = \pi'_i(f)\,\pi'_i(g)$. The left hand side reads

$$\pi'_i(f\cdot g) = (\pi_i+\psi_i)(f\cdot g) = \pi_i(f\cdot g) + \psi_i(f\cdot g),$$

while the right hand side reads

$$\pi'_i(f)\,\pi'_i(g) = (\pi_i+\psi_i)(f)(\pi_i+\psi_i)(g) = \pi_i(f)\,\pi_i(g) + \pi_i(f)\,\psi_i(g) + \psi_i(f)\,\pi_i(g)$$

remembering that $\psi_i(f)\,\psi_i(g) = 0$ as ψ_i is valued in $Sym^2\mathcal{F}_{\mathcal{M}}$. Then one has that

$$\psi_i(f\cdot g) = \pi_i(f)\,\psi_i(g) + \psi_i(f)\,\pi_i(g). \tag{3.9}$$

This proves that ψ_i is a derivation valued in $Sym^2\mathcal{F}_{\mathcal{M}}$, $\psi_i \in (\mathcal{T}_{\mathcal{M}_{red}}\otimes Sym^2\mathcal{F}_{\mathcal{M}})(\mathcal{U}_i)$, and therefore $\{\pi'_i\}_{i\in I}$ defines the Čech 1-cocycle $\omega'_{ij} := \omega_{ij}+\psi_i-\psi_j$, so that the 1-cocycle $(\omega_{ij})_{i,j\in I}$ only changes by a coboundary. Thus $\mathcal{O}_{\mathcal{M},0}$ (and therefore $\mathcal{O}_{\mathcal{M}}$) uniquely defines a cohomology class $\omega_{\mathcal{M}} \in H^1(\mathcal{M},\mathcal{T}_{\mathcal{M}_{red}}\otimes Sym^2\mathcal{F}_{\mathcal{M}})$. This completes the proof of the lemma. □

With this at hand, we are now in the position to prove the main theorem, which is actually a simple consequence of the above lemma.

Theorem 3.2 (Obstruction to Splitting). *Let $\mathcal{M}$ be a (complex) supermanifold of odd dimension 2. Then $\mathcal{M}$ is projected (and hence split) if and only if the obstruction class $\omega_{\mathcal{M}}$ is zero in $H^1(\mathcal{M}_{red},\mathcal{T}_{\mathcal{M}_{red}}\otimes Sym^2\mathcal{F}_{\mathcal{M}})$.*

Proof. Using the same notation as in the above lemma, assume that $\omega_{\mathcal{M}}$ is trivial. Then, there exist local splittings $\{\pi_i\}_{i\in I}$ such that $\omega_{ij} := (\pi_i-\pi_j)\big\lfloor_{\mathcal{U}_i\cap\mathcal{U}_j}$ is a coboundary, that is $\omega_{ij} = (\psi_i-\psi_j)\big\lfloor_{\mathcal{U}_i\cap\mathcal{U}_j}$, for some $\{\psi_i\}_{i\in I}$ such that $\psi_i : \mathcal{O}_{\mathcal{U}_i,red} \to \mathcal{O}_{\mathcal{U}_i,0}$. Then we can define $\pi'_i = \pi_i - \psi_i$ so that

$$\omega' = (\pi'_i - \pi'_j)\big\lfloor_{\mathcal{U}_i\cap\mathcal{U}_j} = (\psi_i-\psi_i-\psi_j+\psi_j)\big\lfloor_{\mathcal{U}_i\cap\mathcal{U}_j} = 0.$$

This implies that $\pi'_i = \pi'_j$ on the intersections $\mathcal{U}_i\cap\mathcal{U}_j$, therefore (restoring the original notation) we have a *global* homomorphism of sheaves of rings $\pi^\sharp_0 : \mathcal{O}_{\mathcal{M}_{red}} \to \mathcal{O}_{\mathcal{M},0}$ such that $\pi^\sharp_0\lfloor_{\mathcal{U}_i} = \pi_i$ and such that $\iota^\sharp_0\circ\pi^\sharp_0 = id_{\mathcal{M}_{red}}$.
Conversely, let $\mathcal{M}$ be projected, that is let $\pi^\sharp_0 : \mathcal{O}_{\mathcal{M}_{red}} \to \mathcal{O}_{\mathcal{M},0}$ be a global homomorphism splitting (3.1), then it is enough to put $\pi_i := (\pi^\sharp_0)\big\lfloor_{\mathcal{U}_i}$ to get a collection of local splittings defining a trivial cocycle. □

The theorem above offers a simple way to detect when a complex supermanifold having odd dimension equal 2 fails to be projected by means of a cohomological invariant that can be computed by ordinary algebraic geometric methods. The knowledge of $\omega_{\mathcal{M}}$ for a supermanifold $\mathcal{M}$ of dimension $n|2$ is a fundamental ingredient in the characterisation of the given supermanifold.

Theorem 3.3 (Supermanifolds of dimension $n|2$)**.** *Let $\mathcal{M} := (|\mathcal{M}|, \mathcal{O}_{\mathcal{M}})$ be a (complex) supermanifold of dimension $n|2$. Then $\mathcal{M}$ is defined up to isomorphism by the triple $(\mathcal{M}_{red}, \mathcal{F}_{\mathcal{M}}, \omega_{\mathcal{M}})$ where $\mathcal{F}_{\mathcal{M}}$ is a rank $0|2$ sheaf of locally-free $\mathcal{O}_{\mathcal{M}_{red}}$-modules, the so called fermionic sheaf, and $\omega_{\mathcal{M}} \in H^1(\mathcal{M}_{red}, \mathcal{T}_{\mathcal{M}_{red}} \otimes Sym^2 \mathcal{F}_{\mathcal{M}})$.*

Proof. Clearly, the pair $(\mathcal{M}_{red}, \mathcal{F}_{\mathcal{M}})$ is enough to completely characterise the underlying complex manifold and the odd part $\mathcal{O}_{\mathcal{M},1}$ of the structure sheaf $\mathcal{O}_{\mathcal{M}}$ of the supermanifold. The even part of the structure sheaf is determined as an extension of $\mathcal{O}_{\mathcal{M}_{red}}$ by $Sym^2\mathcal{F}_{\mathcal{M}}$. Given $\omega_{\mathcal{M}} \in H^1(\mathcal{M}_{red}, \mathcal{T}_{\mathcal{M}_{red}} \otimes Sym^2\mathcal{F}_{\mathcal{M}})$, this can be realised as follows. Let $\mathcal{U} = \{\mathcal{U}_i\}_{i\in I}$ be an open covering of $|\mathcal{M}|$ such that $\omega_{\mathcal{M}}\lfloor_{\mathcal{U}_i}$ is trivial and such that on the intersections $\mathcal{U}_i \cap \mathcal{U}_j$ for $i \neq j$ one has that $\omega_{\mathcal{M}}$ is represented by a cocycle $(\omega_{ij})_{i,j\in I}$. Then, we construct the sheaves $\mathcal{O}_{\mathcal{U}_i,0} := (\mathcal{O}_{\mathcal{M}_{red}} \oplus Sym^2\mathcal{F}_{\mathcal{M}})\lfloor_{\mathcal{U}_i}$ for $\mathcal{U}_i \in \mathcal{U}$ and we glue them on the intersections $\mathcal{U}_i \cap \mathcal{U}_j$ using $(\omega_{ij})_{ij\in I}$:

$$\begin{aligned} \left(\left(\mathcal{O}_{\mathcal{M}_{red}} \oplus Sym^2\mathcal{F}_{\mathcal{M}}\right)\lfloor_{\mathcal{U}_i}\right)\big\lfloor_{\mathcal{U}_j} &\longrightarrow \left(\left(\mathcal{O}_{\mathcal{M}_{red}} \oplus Sym^2\mathcal{F}_{\mathcal{M}}\right)\lfloor_{\mathcal{U}_j}\right)\big\lfloor_{\mathcal{U}_i} \\ (f_{red},\ g_{\theta\theta}) &\longmapsto (f_{red},\ g_{\theta\theta} + \omega_{ij}(f_{red})). \end{aligned}$$

This procedure gives the extension of $\mathcal{O}_{\mathcal{M}_{red}}$ by $Sym^2\mathcal{F}_{\mathcal{M}}$, that is the even part of the structure sheaf $\mathcal{O}_{\mathcal{M},0}$ (see [33], chapter III, section 6), thus concluding the proof. □

Before going on, the following important observations are in order:

- The last part of the previous theorem can be spelled-out by saying that in presence of a non-trivial extension, *i.e.* when we are dealing with a non-projected supermanifold, the transition functions coming from the underlying manifold $\mathcal{M}_{red}$ get a correction coming from $\omega_{\mathcal{M}}$ as they are lifted to $\mathcal{M}$. More precisely, if $\{\mathcal{U}_i\}_{i\in I}$ is an open covering of $|\mathcal{M}|$ such that in a certain intersection $\mathcal{U}_i \cap \mathcal{U}_j$ the transition functions of $\mathcal{M}_{red}$ are given by certain functions $z_{\ell i} = z_{\ell i}(\underline{z}_j)$ for $\ell = 1, \ldots, n$, then the *even* transition functions of a non-projected $n|2$-dimensional supermanifold will be given explicitly by

$$z_{\ell i}(\underline{z}_j, \underline{\theta}_j) = z_{\ell i}(\underline{z}_j) + \omega_{ij}(\underline{z}_j, \underline{\theta}_j)(z_{\ell i}) \qquad \ell = 1, \ldots, n, \tag{3.10}$$

  where the zetas and the thetas are respectively even and odd local coordinates for $\mathcal{M}$ and we recall that ω_{ij} is actually a derivation acting on $z_{\ell i}$. Notice also that the two thetas can only appear in ω_{ij} through their product (thus respecting parity!), indeed ω_{ij} is a derivation taking values in $Sym^2\mathcal{F}_{\mathcal{M}}$;

- choosing $\omega'_{\mathcal{M}} = \lambda\,\omega_{\mathcal{M}}$ for $\lambda \in \mathbb{C}^*$ defines an isomorphic extension $\mathcal{O}'_{\mathcal{M},0}$ of $\mathcal{O}_{\mathcal{M}_{red}}$ by $Sym^2\mathcal{F}_{\mathcal{M}}$, however the isomorphism with $\mathcal{O}_{\mathcal{M},0}$ is not the identity on $\mathcal{O}_{\mathcal{M}_{red}}$ and $Sym^2\mathcal{F}_{\mathcal{M}}$.

The crucial issue of finding a set of invariants that completely characterises complex supermanifolds having odd dimension greater than 2 (up to isomorphisms) and given reduced complex manifold remains - to the best knowledge of the author - still completely open. This is because there are sharp limitations to the definition of *higher obstruction classes* beyond $\omega_{\mathcal{M}} \in H^1(\mathcal{M}_{red}, \mathcal{T}_{\mathcal{M}_{red}} \otimes Sym^2\mathcal{F}_{\mathcal{M}})$, as discussed for example in the fourth chapter of [9], and recently in [25]. Remarkably, none of the issues related to higher obstruction classes affects the first obstruction class $\omega_{\mathcal{M}}$ we have introduced, which can therefore be legitimately called *fundamental obstruction class*: it is really an invariant (this is not true for higher obstructions) for the supermanifold $\mathcal{M}$ and it obstructs the existence of a projection.
In the next section we will start providing concrete examples of constructions of non-projected supermanifolds having projective spaces as reduced manifolds.

3.2 $\mathcal{N} = 2$ Non-Projected Supermanifolds over Projective Spaces

In this section we apply Theorem 3.3 of the previous section to the case the underlying manifolds are ordinary projective spaces $\mathbb{P}^n$, our aim being to identify the obstructions to the existence of a projection and therefore to single out all the non-projected supermanifolds of odd dimension 2 having $\mathbb{P}^n$ as reduced space.

Since we are working over $\mathbb{P}^n$, if we take the fermionic sheaf $\mathcal{F}_{\mathcal{M}}$ to be a locally-free sheaf of $\mathcal{O}_{\mathbb{P}^n}$-module having dimension $0|2$, then it follows that there exists an isomorphism $Sym^2\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^n}(k)$ for some k, since all of the invertible sheaves over $\mathbb{P}^n$ are of the form $\mathcal{O}_{\mathbb{P}^n}(k)$ for some k (indeed $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$).
The basic tool to be exploited here in association with Theorem 3.3 is the (twisted) Euler sequence for the tangent space over $\mathbb{P}^n$, that reads

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(k) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(k+1)^{\oplus n+1} \longrightarrow \mathcal{T}_{\mathbb{P}^n}(k) \longrightarrow 0. \tag{3.11}$$

We now examine the $\mathcal{N} = 2$ supersymmetric extensions of projective space $\mathbb{P}^n$ for every $n = 1, 2, \ldots$.

n = 1 : In the case of $\mathbb{P}^1$, one has to study whenever $H^1(\mathcal{T}_{\mathbb{P}^1} \otimes Sym^2\mathcal{F}_{\mathcal{M}}) = H^1(\mathcal{T}_{\mathbb{P}^1}(k))$ is non-zero. This is easily achieved, since recalling that over $\mathbb{P}^1$ one has $\mathcal{T}_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2)$, it amounts to find a k such that $H^1(\mathcal{O}_{\mathbb{P}^1}(2+k)) \neq 0$. This is realised in the cases $k = -\ell \leqslant -4$, and one finds $H^1(\mathcal{O}_{\mathbb{P}^1}(2-\ell)) \cong \mathbb{C}^{\ell-3}$. These cohomology groups have a well-known description: they are the $\mathbb{C}$-vector spaces with bases given by $\left\{\frac{1}{(X_0)^j(X_1)^{\ell-2-j}}\right\}_{j=1}^{\ell-3}$, where X_0 and X_1 are the homogeneous coordinates of $\mathbb{P}^1$, see for example the proof of Theorem 5.1 Chapter III of [33]. As a result, the non-projected supermanifolds over $\mathbb{P}^1$ are those such that $Sym^2\mathcal{F}_{\mathcal{M}} \cong \mathcal{O}_{\mathbb{P}^1}(-\ell)$ with $\ell \geqslant 4$.

n = 2 : The case over $\mathbb{P}^2$ is by far the most interesting, and - surprisingly - it has been forgotten by Manin, as he studies fermionic super-extensions over projective spaces in [41]. Since over $\mathbb{P}^2$ one has $H^1(\mathcal{O}_{\mathbb{P}^2}(k)) = H^1(\mathcal{O}_{\mathbb{P}^2}(k+1)) = 0$, the long exact sequence in cohomology induced by the Euler short exact sequence splits in two exact sequences. The one we are concerned with reads

$$0 \longrightarrow H^1(\mathcal{T}_{\mathbb{P}^2}(k)) \longrightarrow H^2(\mathcal{O}_{\mathbb{P}^2}(k)) \longrightarrow H^2(\mathcal{O}_{\mathbb{P}^2}(k+1))^{\oplus 3} \longrightarrow H^2(\mathcal{T}_{\mathbb{P}^2}(k)) \longrightarrow 0.$$

Now it is convenient to distinguish between three different sub-cases.

$k > -3$: This is the easiest one, since $H^2(\mathcal{O}_{\mathbb{P}^2}(k)) = 0$, which implies that $H^1(\mathcal{T}_{\mathbb{P}^2}(k))$ is zero.

$k = -3$: In this case we have that $H^2(\mathcal{O}_{\mathbb{P}^2}(-2))^{\oplus 3} = 0$, so we get an isomorphism

$$H^1(\mathcal{T}_{\mathbb{P}^2}(-3)) \cong H^2(\mathcal{O}_{\mathbb{P}^2}(-3)) \cong \mathbb{C}, \tag{3.12}$$

and, again, this cohomology group is generated by the cohomology class $[\frac{1}{X_0X_1X_2}]$ induced by the 2-cocycle defined by $\frac{1}{X_0X_1X_2} \in \Gamma(\mathcal{U}_0 \cap \mathcal{U}_1 \cap \mathcal{U}_2, \mathcal{O}_{\mathbb{P}^2}(-3))$.

$k < -3$: In this case both $H^2(\mathcal{O}_{\mathbb{P}^2}(k))$ and $H^2(\mathcal{O}_{\mathbb{P}^2}(k+1))$ are non-zero. Therefore, this makes a little bit harder to explicitly evaluate $H^1(\mathcal{T}_{\mathbb{P}^2}(k))$ directly. Though, this can be achieved upon using *Bott formulas* (see for example [49]) that give the dimension of cohomology groups of the (twisted) cotangent bundles of projective spaces. First of all, we observe that using Serre duality one gets $H^1(\mathcal{T}_{\mathbb{P}^2}(k)) \cong H^1(\mathcal{T}^*_{\mathbb{P}^2}(-k-3))^*$. In general, Bott formulas guarantee that $H^q(\bigwedge^p T^*_{\mathbb{P}^n}(k)) = 0$ if $q \neq n$ and $q, k \neq 0$. In our specific case we have $q = 1, n = 2, p = 1$ and $-k-3 < -6$, therefore $H^1(\mathcal{T}_{\mathbb{P}^2}(k)) = 0$.

The above computation yields that the only non-projected supermanifold having underlying manifold isomorphic to $\mathbb{P}^2$ will have a fermionic sheaf $\mathcal{F}_{\mathcal{M}}$ such that $Sym^2\mathcal{F}_{\mathcal{M}} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$.

n > 2 : In this case it is easy to conclude that $H^1(\mathcal{T}_{\mathbb{P}^n}(k)) = 0$ since in the long exact sequence in cohomology this group sits between $H^1(\mathcal{O}_{\mathbb{P}^n}(k+1))^{\oplus n+1}$ and $H^2(\mathcal{O}_{\mathbb{P}^n}(k))$ and both of these groups are zero for every k if $n > 2$.

The above results allow us to classify the non-projected supermanifolds having $\mathbb{P}^1$ or $\mathbb{P}^2$ as reduced space. In the next section we take on the case over $\mathbb{P}^1$ case.

3.3 Non-Projected Supermanifolds over $\mathbb{P}^1$

In this section we classify all of the non-projected supermanifolds having $\mathbb{P}^1$ as reduced manifold. This classification is actually rather straightforward and it directly relies on the fact that vector bundles over $\mathbb{P}^1$ have no continuous moduli. Indeed, one has the following

Theorem 3.4 (Non-projected $\mathcal{N} = 2$ Supermanifolds over $\mathbb{P}^1$)**.** *Every non-projected $\mathcal{N} = 2$ supermanifold over $\mathbb{P}^1$ is characterised up to isomorphism by a triple $(\mathbb{P}^1, \mathcal{F}_{\mathcal{M}}, \omega)$ where $\mathcal{F}_{\mathcal{M}}$ is a rank $0|2$ sheaf of $\mathcal{O}_{\mathbb{P}^1}$-modules such that $\mathcal{F}_{\mathcal{M}} \cong \Pi\mathcal{O}_{\mathbb{P}^1}(m) \oplus \Pi\mathcal{O}_{\mathbb{P}^1}(n)$ with $m + n = -\ell$, $\ell \geqslant 4$ and ω is a non-zero cohomology class $\omega \in H^1(\mathcal{O}_{\mathbb{P}^1}(2-\ell))$.*

Proof. We know that the obstruction to splitting is obtained for a fermionic sheaf of rank $0|2$ such that $Sym^2\mathcal{F}_{\mathcal{M}} \cong \mathcal{O}_{\mathbb{P}^1}(-\ell)$, $\ell \geqslant 4$: in this case one gets non-zero obstruction classes in $H^1(\mathcal{T}_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}(-\ell)) \cong H^1(\mathcal{O}_{\mathbb{P}^1}(2-\ell)) \cong \mathbb{C}^{\ell-3}$.
It remains to show that the only sheaves that yield such an isomorphism are of the given form. This is a consequence of the *Grothendieck splitting theorem* (see [33] or [49]), that states that every locally-free sheaf of $\mathcal{O}_{\mathbb{P}^1}$-modules of arbitrary rank n is isomorphic to a direct sum of invertible sheaves, that in turn are all of the form $\mathcal{O}_{\mathbb{P}^1}(k)$ for some $k \in \mathbb{Z}$. In other words, if we let $\mathcal{E}$ be a locally-free sheaf of rank n over $\mathbb{P}^1$, then we have

$$\mathcal{E} \cong \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(k_i) \tag{3.13}$$

uniquely up to permutation of the terms at the right hand side. Since we have that $\mathcal{F}_{\mathcal{M}}$ is a locally-free sheaf of rank $0|2$ over $\mathbb{P}^1$, it will be of the form $\mathcal{F}_{\mathcal{M}} \cong \Pi\mathcal{O}_{\mathbb{P}^1}(m) \oplus \Pi\mathcal{O}_{\mathbb{P}^1}(n)$ for some $n, m \in \mathbb{Z}$ as a consequence of Grothendieck theorem. The only way one can have $Sym^2\mathcal{F}_{\mathcal{M}} \cong \mathcal{O}_{\mathbb{P}^1}(-\ell)$ is to choose m and n such that $m + n = -\ell$, and this concludes the proof. □

Adapting the notation, the previous theorem justifies the following definition.

Definition 3.1 (The Supermanifolds $\mathbb{P}^1_\omega(m,n)$)**.** *We denote $\mathbb{P}^1_\omega(m,n)$, with $m \geqslant n$, an $\mathcal{N} = 2$ supermanifold arising from a triple $(\mathbb{P}^1, \mathcal{F}_{\mathcal{M}}, \omega)$, where the fermionic sheaf $\mathcal{F}_{\mathcal{M}} = \Pi\mathcal{O}_{\mathbb{P}^1}(-m) \oplus \Pi\mathcal{O}_{\mathbb{P}^1}(-n)$ is such that $m + n = \ell$, $\ell \geqslant 4$ and ω is a (possibly zero) class in $H^1(\mathcal{O}_{\mathbb{P}^1}(2-\ell))$.*

In view of Theorem 3.4, clearly, $\mathbb{P}^1_\omega(m,n)$ is a non-projected supermanifold if and only if ω is a non-zero class in $H^1(\mathcal{O}_{\mathbb{P}^1}(2-\ell))$.

We now look for the explicit form the of the transition functions of a non-projected supermanifold $\mathbb{P}^1_\omega(m,n)$, as to be able to construct the supermanifold also by the patching technique for future simply physical applications.
Working over $\mathbb{P}^1$ leads to consider a set of homogeneous coordinates $[X_0 : X_1] \in \mathbb{P}^1$ and a set of affine coordinates and their algebras over the two open sets of the covering $\mathcal{U} := \{\mathcal{U}, \mathcal{V}\}$ of $\mathbb{P}^1$. In particular, working modulo $\mathcal{J}_{\mathcal{M}}$ we have:

$$\mathcal{U} := \{X_0 \neq 0\} \quad \rightsquigarrow \quad z \bmod \mathcal{J}^2_{\mathcal{M}} := \frac{X_1}{X_0}, \tag{3.14}$$

$$\mathcal{V} := \{X_1 \neq 0\} \quad \rightsquigarrow \quad w \bmod \mathcal{J}^2_{\mathcal{M}} := \frac{X_0}{X_1}. \tag{3.15}$$

The transition functions are given by:

$$\mathcal{U} \cap \mathcal{V} : \qquad z \bmod \mathcal{J}^2_{\mathcal{M}} = \frac{1}{w} \bmod \mathcal{J}^2_{\mathcal{M}}, \tag{3.16}$$

Remember that these expressions are given $\bmod\, \mathcal{J}^2_{\mathcal{M}}$ instead of $\bmod\, \mathcal{J}_{\mathcal{M}}$ since in $\mathcal{N} = 2$ one has that $(\mathcal{J}_{\mathcal{M}})_0 = \mathcal{J}^2_{\mathcal{M}}$.

Passing to the fermionic sheaf $\mathcal{F}_{\mathcal{M}} = \Pi\mathcal{O}_{\mathbb{P}^1}(m) \oplus \Pi\mathcal{O}_{\mathbb{P}^1}(n)$, we denote (θ_1, θ_2) a local basis of $\mathcal{F}_{\mathcal{M}}$ on $\mathcal{U}$ and (ψ_1, ψ_2) a local basis of $\mathcal{F}_{\mathcal{M}}$ on $\mathcal{V}$ respectively, so that one can write

$$\mathcal{U} := \{X_0 \neq 0\} \quad \rightsquigarrow \quad \theta_1 := \Pi\left(\frac{1}{X_0^{-m}}\right), \quad \theta_2 := \Pi\left(\frac{1}{X_0^{-n}}\right), \tag{3.17}$$

$$\mathcal{V} := \{X_1 \neq 0\} \quad \rightsquigarrow \quad \psi_1 := \Pi\left(\frac{1}{X_1^{-m}}\right), \quad \psi_2 := \Pi\left(\frac{1}{X_1^{-n}}\right), \tag{3.18}$$

where the Π's are there to remember the odd parity. The transition functions therefore are given by

$$\mathcal{U} \cap \mathcal{V} : \qquad \theta_1 = \frac{\psi_1}{w^{-m}}, \qquad \theta_2 = \frac{\psi_1}{w^{-n}}. \tag{3.19}$$

Before we go on, a comment on the notation might be helpful. Usually the θ's and the ψ's are looked at as sections of the odd part of structure sheaf, $(\mathcal{O}_{\mathbb{P}^1_\omega(m,n)})_1$. The identification above makes sense recalling that when $\mathcal{N} = 2$ one has in general $(\mathcal{O}_{\mathcal{M}})_1 \cong \mathcal{F}_{\mathcal{M}}$, so that for $\mathbb{P}^1_\omega(m,n)$ the odd sections θ's and ψ's can indeed be identified with local sections in $\Pi\mathcal{O}_{\mathbb{P}^1}(m) \oplus \Pi\mathcal{O}_{\mathbb{P}^1}(n)$, once the parity is taken into account. This identification is pretty useful, as the symmetric product in $Sym^2\mathcal{F}_{\mathcal{M}}$ can be represented as product of odd elements in $(\mathcal{O}_{\mathcal{M}})_1$. In our case in particular one gets

$$\theta_1\theta_2 = \frac{1}{X_0^{-m-n}} = \frac{1}{X_0^{\ell}} \in \mathcal{O}_{\mathbb{P}^1}(-\ell)(\mathcal{U}_0), \qquad\qquad \psi_1\psi_2 = \frac{1}{X_1^{-m-n}} = \frac{1}{X_1^{\ell}} \in \mathcal{O}_{\mathbb{P}^1}(-\ell)(\mathcal{U}_1). \tag{3.20}$$

Before we go into to derivation of the explicit form of the transition functions for a supermanifold of the family $\mathbb{P}^1_\omega(m,n)$, we recall that under our assumptions

$$\omega \ni H^1(\mathcal{T}_{\mathbb{P}^1} \otimes Sym^2\mathcal{F}_{\mathcal{M}}) \cong H^1(\mathcal{O}_{\mathbb{P}^1}(2-\ell)) \cong \left\langle \left\{ \left[\frac{1}{(X_0)^j (X_1)^{\ell-2-j}} \right] \right\}_{j=1}^{\ell-3} \right\rangle_{\mathbb{C}}. \tag{3.21}$$

Therefore, as an element of the vector space $H^1(\mathcal{T}_{\mathbb{P}^1} \otimes Sym^2\mathcal{F}_{\mathcal{M}})$, the most general obstruction class ω can be represented expanded over its base as

$$\omega = \lambda_1 \left[\frac{1}{X_0 X_1^{\ell-3}}\right] + \ldots + \lambda_{\ell-3}\left[\frac{1}{X_0^{\ell-3}X_1}\right], \qquad \lambda_i \in \mathbb{C}, \quad i = 1, \ldots, \ell-3. \tag{3.22}$$

This gives explicitly the isomorphism $H^1(\mathcal{T}_{\mathbb{P}^1} \otimes Sym^2\mathcal{F}_{\mathcal{M}}) \cong \mathbb{C}^{\ell-3}$ that is understood when we say that a class ω is represented by a choice of $\{\lambda_1, \ldots, \lambda_{\ell-3}\}$. In particular, we will call the choice $\lambda_i = 0$ for all $i = 1, \ldots, \ell-3$ the *split locus* of the family, as it leads to a split supermanifold.
We are now ready to give the explicit form of the transition functions of $\mathbb{P}^1_\omega(m,n)$.

Theorem 3.5 (Transition Functions of $\mathbb{P}^1_\omega(m,n)$)**.** *The transition functions of an element of the family $\mathbb{P}^1_\omega(m,n)$ are given in $\mathcal{U} \cap \mathcal{V}$ by*

$$z = \frac{1}{w} + \sum_{j=1}^{\ell-3} \lambda_j \frac{\psi_1\psi_2}{w^{2+j}}, \tag{3.23}$$

$$\theta_1 = \frac{\psi_1}{w^{-m}}, \tag{3.24}$$

$$\theta_2 = \frac{\psi_1}{w^{-n}}, \tag{3.25}$$

where $\lambda_i \in \mathbb{C}$ for $i = 1, \ldots, \ell-3$.

Proof. The odd transition functions for θ_i $i = 1, 2$ have already been found in (3.19). We are thus left to find the even transition function. We first recall that a section of $\mathcal{T}_{\mathbb{P}^1}$ defined by ∂_z satisfies the transformation law $\partial_z = -w^2\partial_w$, hence it has a double zero on $[0:1] \in \mathbb{P}^1$ and it can be identified with the section $X_0^2 \in H^0(\mathcal{O}_{\mathbb{P}^1}(2))$, up to a non-zero scalar factor. Moreover, since

$\omega \in H^1(\mathcal{O}_{\mathbb{P}^1}(2-\ell))$ one chooses, as above, $\omega = \left\langle \left\{ \left[\lambda_j \middle/ (X_0)^j (X_1)^{\ell-2-j} \right] \right\}_{j=1}^{\ell-3} \right\rangle$. Therefore one gets the following identifications:

$$\left[\left\{ \frac{\lambda_j}{(X_0)^j (X_1)^{\ell-2-j}} \right\}_{j=1}^{\ell-3} \right] = \left[\left\{ \frac{\lambda_j X_0^2}{(X_0)^{j+2}(X_1)^{\ell-2-j}} \right\}_{j=1}^{\ell-3} \right] = \left[\left\{ \frac{\lambda_j}{(X_0)^{j+2}(X_1)^{\ell-j-2}} \right\}_{j=1}^{\ell-3} \partial_z \right].$$

Now, using (3.20) above, one sees in particular that $\psi_1\psi_2 = 1 \big/ X_1^{-m-n} = 1 \big/ X_1^{\ell}$. Under these identifications one gets

$$\omega = \left[\left\{ \frac{\lambda_j}{(X_0)^{j+2}(X_1)^{\ell-j-2}} \right\}_{j=1}^{\ell-3} \partial_z \right] = \left[\left\{ \lambda_j \left(\frac{X_1}{X_0} \right)^{j+2} \right\}_{j=1}^{\ell-3} \frac{1}{X_1^{\ell}} \partial_z \right] = \left[\left\{ \frac{\lambda_j}{w^{j+2}} \right\}_{j=1}^{\ell-3} \psi_1\psi_2\partial_z \right],$$

Remembering that $z \operatorname{mod} \mathcal{J}_{\mathcal{M}}^2 = (1/w) \operatorname{mod} \mathcal{J}_{\mathcal{M}}^2$, and plugging the previous result into (3.10), one has the conclusion:

$$z(w, \psi_1, \psi_2) = \frac{1}{w} + \sum_{j=1}^{\ell-3} \lambda_j \frac{\psi_1\psi_2}{w^{2+j}} \partial_z z = \frac{1}{w} + \sum_{j=1}^{\ell-3} \lambda_j \frac{\psi_1\psi_2}{w^{2+j}}. \tag{3.26}$$

□

The previous theorem exhaust *all* of the possible non-projected (non-singular) $\mathcal{N} = 2$ supermanifolds over the projective line $\mathbb{P}^1$: they are abstractly classified by the Theorem 3.4 at the beginning of this section and they can be constructed explicitly by patching charts via the transition functions we have just given.

3.3.1 Even Picard Group of $\mathbb{P}^1_\omega(m,n)$

In this section we start studying the even Picard group of an element of the family $\mathbb{P}^1_\omega(m,n)$ to see what sort of *even* invertible sheaves can be defined on a supermanifold belonging to the family.
We first recall that in general the even part of the structure sheaf of a $\mathcal{N} = 2$ supermanifold fits into the short exact sequence (3.1), that can be exponentiated to give

$$0 \longrightarrow Sym^2 \mathcal{F}_{\mathcal{M}} \xrightarrow{\ i\ } \mathcal{O}^*_{\mathcal{M},0} \xrightarrow{\ j\ } \mathcal{O}^*_{\mathcal{M}_{red}} \longrightarrow 0, \tag{3.27}$$

Where $i : Sym^2\mathcal{F}_{\mathcal{M}} \to \mathcal{O}^*_{\mathcal{M},0}$ sends a section $s \in Sym^2\mathcal{F}_{\mathcal{M}}$ to $1 + s \in \mathcal{O}^*_{\mathcal{M},0}$ and $j : \mathcal{O}^*_{\mathcal{M},0} \to \mathcal{O}_{\mathcal{M}_{red}}$ is simply the restriction of invertible elements under the inclusion map (viewed as a morphism of super ringed spaces) $\iota : \mathcal{M}_{red} \to \mathcal{M}$, of the underlying reduced manifold in the supermanifold lying above him (we recall that this map always exists).
The short exact sequence (3.78) enters the proof of the following theorem.

Theorem 3.6 (Even Picard Group of $\mathbb{P}^1_\omega(m,n)$)**.** *The even Picard group of $\mathbb{P}^1_\omega(m,n)$ is given by*

$$\operatorname{Pic}_0(\mathbb{P}^1_\omega(m,n)) \cong \mathbb{Z} \oplus \mathbb{C}^{\ell-1}. \tag{3.28}$$

Proof. We first recall that for the family $\mathbb{P}^1_\omega(m,n)$ we have that $m + n = -\ell$ and $\ell \geqslant 4$ and that $Sym^2\mathcal{F}_{\mathcal{M}} \cong \mathcal{O}_{\mathbb{P}^1}(-\ell)$, so that (3.78) looks like

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-\ell) \longrightarrow \mathcal{O}^*_{\mathbb{P}^1_\omega(m,n),0} \longrightarrow \mathcal{O}^*_{\mathbb{P}^1} \longrightarrow 0. \tag{3.29}$$

Taking the long exact sequence in cohomology one then finds the following:

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathcal{O}_{\mathbb{P}^1}(-\ell)) \arrow[r] & H^0(\mathcal{O}^*_{\mathbb{P}^1_\omega(m,n),0}) \arrow[r] & H^0(\mathcal{O}^*_{\mathbb{P}^1}) \arrow[dll, out=-10, in=170] \\ & H^1(\mathcal{O}_{\mathbb{P}^1}(-\ell)) \arrow[r] & H^1(\mathcal{O}^*_{\mathbb{P}^1_\omega(m,n),0}) \arrow[r] & H^1(\mathcal{O}^*_{\mathbb{P}^1}) \arrow[r] & 0. \end{tikzcd}$$

Clearly, we have that $H^0(\mathcal{O}_{\mathbb{P}^1}(-\ell)) \cong 0$ and $H^1(\mathcal{O}_{\mathbb{P}^1}(-\ell)) \cong \mathbb{C}^{\ell-1}$, while as for the sheaf $\mathcal{O}^*_{\mathbb{P}^1}$ we have $H^0(\mathcal{O}_{\mathbb{P}^1}) \cong \mathbb{C}^*$ and $\operatorname{Pic}(\mathbb{P}^1) = H^1(\mathcal{O}^*_{\mathbb{P}^1}) \cong \mathbb{Z}$. Also, considering the boundary map, $\delta : H^0(\mathcal{O}^*_{\mathbb{P}^1}) \to H^1(\mathcal{O}^1_{\mathbb{P}}(-\ell))$, we see that this can only be the zero map, as there are no ways one can map a non-nilpotent non-zero element in $H^0(\mathcal{O}^*_{\mathbb{P}^1}) \cong \mathbb{C}^*$ to a nilpotent 1-cocycle. This implies that the exact sequence splits into two exact sequence yielding the isomorphism $H^0(\mathcal{O}^*_{\mathbb{P}^1_\omega(m,n),0}) \cong \mathbb{C}^*$ and the short exact sequence

$$0 \longrightarrow \mathbb{C}^{\ell-1} \longrightarrow \operatorname{Pic}_0(\mathbb{P}^1_\omega(m,n)) \longrightarrow \mathbb{Z} \longrightarrow 0, \tag{3.30}$$

that gives the result. $\square$

As we are working over $\mathbb{P}^1$, that is covered by just two sets $\{\mathcal{U}, \mathcal{V}\}$, it is actually possible to do more and provide explicitly the free generators of the Picard group $\operatorname{Pic}_0(\mathbb{P}^{1|2}_\omega)$, as we did for supercurves over $\mathbb{P}^1$ in the first chapter.

Theorem 3.7 (Generators of $H^1(\mathcal{O}^*_{\mathbb{P}^1_\omega(m,n)})$). *The cohomology group $H^1(\mathcal{O}^*_{\mathbb{P}^1_\omega(m,n)})$ is generated by the following Čech 1-cocycles*

$$H^1(\mathcal{O}^*_{\mathbb{P}^1_\omega(m,n)}) \cong \left\langle w^k, 1 + c_1 \frac{\psi_1\psi_2}{w^1}, \ldots, 1 + c_{\ell-1}\frac{\psi_1\psi_2}{w^{\ell-1}} \right\rangle, \tag{3.31}$$

where $k \in \mathbb{Z}$ and $c_1, \ldots, c_{\ell-1}$ are non-zero complex numbers.

Proof. We aim to compute the Čech cohomology valued in the sheaf $\mathcal{O}^*_{\mathbb{P}^1_\omega(m,n),0}$ explicitly. Given the usual covering $\{\mathcal{U}, \mathcal{V}\}$ of $\mathbb{P}^1$, we have

$$C^0\left(\{\mathcal{U},\mathcal{V}\}, \mathcal{O}^*_{\mathbb{P}^1_\omega(m,n),0}\right) = \mathcal{O}^*_{\mathbb{P}^1_\omega(m,n),0}(\mathcal{U}) \times \mathcal{O}^*_{\mathbb{P}^1_\omega(m,n),0}(\mathcal{V}) \tag{3.32}$$

$$C^1\left(\{\mathcal{U},\mathcal{V}\}, \mathcal{O}^*_{\mathbb{P}^1_\omega(m,n),0}\right) = \mathcal{O}^*_{\mathbb{P}^1_\omega(m,n),0}(\mathcal{U}\cap\mathcal{V}) \tag{3.33}$$

where the elements of Čech 0-cochain $\mathcal{O}^*_{\mathbb{P}^1_\omega(m,n),0}(\mathcal{U}) \times \mathcal{O}^*_{\mathbb{P}^1_\omega(m,n),0}(\mathcal{V})$ are given by pairs of elements of the kind $(P(z,\theta_1\theta_2), Q(w,\psi_1\psi_2))$ such that

$$P(z,\theta_1\theta_2) = a + \tilde{P}(z)\theta_1\theta_2 \qquad a \in \mathbb{C}^*,\ \tilde{P} \in \mathbb{C}[z] \tag{3.34}$$

$$Q(w,\theta_1\theta_2) = b + \tilde{Q}(w)\psi_1\psi_2 \qquad b \in \mathbb{C}^*,\ \tilde{Q} \in \mathbb{C}[w]. \tag{3.35}$$

Clearly, the boundary map $\delta : C^0\left(\{\mathcal{U},\mathcal{V}\}, \mathcal{O}^*_{\mathbb{P}^1_\omega(m,n),0}\right) \to C^1\left(\{\mathcal{U},\mathcal{V}\}, \mathcal{O}^*_{\mathbb{P}^1_\omega(m,n),0}\right)$ acts as follows

$$\delta\big((P(z,\theta_1\theta_2), Q(w,\psi_1\psi_2))\big) = Q(w,\psi_1\psi_2)P^{-1}(z,\theta_1\theta_2)\big\lfloor_{\mathcal{U}\cap\mathcal{V}} \tag{3.36}$$

so that, in full generality, the image of δ is given by

$$\begin{aligned}\delta\big((P,Q)\big) &= (b + \tilde{Q}(w)\psi_1\psi_2)\left(\frac{1}{a} - \tilde{P}(1/w)\frac{\psi_1\psi_2}{a^2 w^\ell}\right)\\ &= \frac{b}{a} + \left(\frac{\tilde{Q}(w)}{a} - \frac{b}{a^2}\frac{\tilde{P}(1/w)}{w^\ell}\right)\psi_1\psi_2\end{aligned} \tag{3.37}$$

Out of this expression, incidentally, one immediately see that $H^0\left(\mathcal{O}_{\mathbb{P}^1_\omega(m,n)}\right) \cong \mathbb{C}^*$ as it is given by the constant elements $(a,a) \in Z^1\left(\mathcal{O}_{\mathbb{P}^1_\omega(m,n)}\right)$, where $a \neq 0$.
On the other hand, one has that the elements in $\mathcal{O}^*_{\mathbb{P}^1_\omega(m,n),0}(\mathcal{U}\cap\mathcal{V})$ are given by expression of the form

$$W(w, 1/w, \psi_1\psi_2) = c\, w^k + \tilde{W}(w,1/w)\,\psi_1\psi_2, \tag{3.38}$$

where $c \in \mathbb{C}^*$, $k \in \mathbb{Z}$, $\tilde{W} \in \mathbb{C}[w,1/w]$. Now, confronting the previous expression with the image of the map δ in the (3.37), one sees that b/a can be used to set the coefficient of c of w^k to one and thus the non-exact 1-cocycles are indeed given by transition functions of the form

$$\left\{w^k, 1 + c_1\frac{\psi_1\psi_2}{w^j}, \ldots, 1 + c_{\ell-1}\frac{\psi_1\psi_2}{w^{\ell-1}}\right\}, \tag{3.39}$$

concluding the theorem. $\square$

As already done in the previous chapter for the supercurves $\mathbb{P}^{1|m}$, we call the even invertible sheaves on $\mathbb{P}^1_\omega(m,n)$ characterised by transition functions having the above form as follows:

$$\mathcal{O}_{\mathbb{P}^1_\omega(m,n)}(k) \longleftrightarrow \{w^k\}, \tag{3.40}$$

$$\mathcal{L}_{\mathbb{P}^1_\omega(m,n)}(c_1,\ldots,c_{\ell-1}) \longleftrightarrow \left\{1+\sum_{j=1}^{\ell-1} c_j \frac{\psi_1\psi_2}{w^j}\right\}, \tag{3.41}$$

for $k \in \mathbb{Z}$, $m+n=-\ell$, $\ell \geqslant 4$ and $c_1,\ldots,c_{\ell-1} \in \mathbb{C}$. We have the following theorem.

Theorem 3.8 (Even Picard Group $\mathrm{Pic}_0(\mathbb{P}^1_\omega(m,n))$)**.** *The even Picard group of $\mathbb{P}^1_\omega(m,n)$ is generated by the following even invertible sheaves*

$$\mathrm{Pic}_0(\mathbb{P}^1_\omega(m,n)) = \left\langle \mathcal{O}_{\mathbb{P}^1_\omega(m,n)}(\pm k), \mathcal{L}_{\mathbb{P}^1_\omega(m,n)}(c_1,0,\ldots,0),\ldots,\mathcal{L}_{\mathbb{P}^1_\omega(m,n)}(0,\ldots,0,c_{\ell-1})\right\rangle. \tag{3.42}$$

for $k=\pm 1$, $c_1,\ldots,c_{\ell-1} \in \mathbb{C}$ and $m+n=-\ell$, $\ell \geqslant 4$.

Proof. Taking into account the notation adopted above, this theorem is just a consequence of the previous one by the isomorphism $\mathrm{Pic}_0(\mathcal{M}) \cong H^1(\mathcal{O}^*_{\mathcal{M},0})$. □

3.3.2 Embedding of $\mathbb{P}^1_\omega(2,2)$: an Example by Witten

We now focus on a specific non-projected supermanifold in the family $\mathbb{P}^1_\omega(m,n)$ and we construct the embedding of it into an ordinary projective superspace.
We choose to deal with probably the easiest example of a non-projected supermanifold belonging to the family, which is given by $\mathbb{P}^1_\omega(2,2)$: this corresponds to the choice of a fermionic sheaf of the form $\mathcal{F}_\mathcal{M} = \Pi\mathcal{O}_{\mathbb{P}^1}(-2)\oplus\Pi\mathcal{O}_{\mathbb{P}^1}(-2)$ so that $Sym^2\mathcal{F}_\mathcal{M} \cong \mathcal{O}_{\mathbb{P}^1}(-4)$. This leads to an obstruction class $\omega \in H^1(\mathcal{O}(-2)) \cong \mathbb{C}$: since the obstruction cohomology group is one-dimensional, we represent ω by a complex number λ (see (3.23)), which we choose to fix to be one, so that the even transition function of our supermanifold $\mathbb{P}^1_\omega(2,2)$ is given by

$$z = \frac{1}{w} + \frac{\psi_1\psi_2}{w^3}. \tag{3.43}$$

Notice that, choosing a different non-zero value for λ, we would have gotten an (non-canonically) isomorphic non-projected supermanifold: indeed choosing ω or $\lambda\omega$ for $\lambda \in \mathbb{C}^*$ leads to isomorphic supermanifolds, as stressed in the second observation following Theorem 3.3.

In the case of $\mathbb{P}^1_\omega(2,2)$, the even Picard group is given by $\mathrm{Pic}_0(\mathbb{P}^1_\omega(2,2)) \cong \mathbb{Z}\oplus\mathbb{C}^{\oplus 3}$, having, in the isomorphisms $\mathrm{Pic}_0(\mathbb{P}^1_\omega(2,2)) \cong H^1(\mathcal{O}^*_{\mathbb{P}^1_\omega(2,2)})$ generators given by

$$H^1(\mathcal{O}^*_{\mathbb{P}^1_\omega(2,2)}) \cong \left\langle w^k, 1+c_1\frac{\psi_1\psi_2}{w}, 1+c_2\frac{\psi_1\psi_2}{w^2}, 1+c_3\frac{\psi_1\psi_2}{w^3}\right\rangle. \tag{3.44}$$

with $k \in \mathbb{Z}$ and $c_1,c_2,c_3 \in \mathbb{C}$.
Now, in order to explicitly find an embedding we let $e_\mathcal{U}$ and $e_\mathcal{V}$ be two local frames in the open sets of the covering of $\mathbb{P}^1$ and we let $\mathcal{O}_{\mathbb{P}^1_{\omega(2,2)}}(2) \leftrightarrow \{(\mathcal{U},\mathcal{V}), e_\mathcal{U} = w^2 e_\mathcal{V}\}$ and $\mathcal{L}_{\mathbb{P}^1_\omega(2,2)}(0,-1,0) \leftrightarrow \{(\mathcal{U},\mathcal{V}), e_\mathcal{U} = (1-\psi_1\psi_2/w^2)e_\mathcal{V}\}$ and we consider the invertible sheaf given by their tensor product:

$$\widehat{\mathcal{L}}_{\mathbb{P}^1_\omega(2,2)} := \mathcal{O}_{\mathbb{P}^1_\omega(2,2)}(2)\otimes\mathcal{L}_{\mathbb{P}^1_\omega(2,2)}(0,-1,0) \leftrightarrow \{(\mathcal{U},\mathcal{V}), e_\mathcal{U} = (w^2-\psi_1\psi_2)e_\mathcal{V}\}. \tag{3.45}$$

Then the following lemma holds true.

Lemma 3.2. *Let $\widehat{\mathcal{L}}_{\mathbb{P}^1_\omega(2,2)}$ be the even invertible sheaf over $\mathbb{P}^1_\omega(2,2)$ defined above. Then $\widehat{\mathcal{L}}_{\mathbb{P}^1_\omega(2,2)}$ admits the following global sections*

1. $Y_0 := \{e_\mathcal{U},\ (w^2-\psi_1\psi_2)e_\mathcal{V}\}$
2. $Y_1 := \{ze_\mathcal{U},\ we_\mathcal{V}\}$
3. $Y_2 := \{(z^2-\theta_1\theta_2)e_\mathcal{U},\ e_\mathcal{V}\}$

4. $\Xi_1 := \{\theta_1 e_{\mathcal{U}},\ \psi_1 e_{\mathcal{V}}\}$

5. $\Xi_2 := \{\theta_2 e_{\mathcal{U}},\ \psi_2 e_{\mathcal{V}}\}$,

it is ample and it allows for an embedding $\phi_2 : \mathbb{P}^1_\omega(2,2) \hookrightarrow \mathbb{P}^{2|2}$ *whose imagine is given by the equation*

$$\Xi_1\Xi_2 - Y_1^2 + Y_0Y_2 = 0. \tag{3.46}$$

Proof. The proof that the five sections defined locally above are indeed global sections for $\widehat{\mathcal{L}}_{\mathbb{P}^1_\omega(2,2)}$ amounts to the direct check that the given local definitions agree on $\mathcal{U} \cap \mathcal{V}$. Let us check for example that Y_2 is a global section:

$$\begin{aligned} Y_2 &= \left(z^2 - \theta_1\theta_2\right) e_{\mathcal{U}} \\ &= \left(z^2 - \theta_1\theta_2\right)\left(w^2 - \psi_1\psi_2\right) e_{\mathcal{V}} \\ &= \left(\left[\frac{1}{w} + \frac{\psi_1\psi_2}{w^3}\right]^2 + \frac{\psi_1\psi_2}{w^4}\right)\left(w^2 - \psi_1\psi_2\right) e_{\mathcal{V}} \\ &= \left(\frac{1}{w^2} + 2\frac{\psi_1\psi_2}{w^4} + \frac{\psi_1\psi_2}{w^4}\right)\left(w^2 - \psi_1\psi_2\right) e_{\mathcal{V}} \\ &= \left(\frac{1}{w^2} + \frac{\psi_1\psi_2}{w^4}\right)\left(w^2 - \psi_1\psi_2\right) e_{\mathcal{V}} \\ &= e_{\mathcal{V}} \end{aligned} \tag{3.47}$$

It is immediate to get the equation satisfied by the global sections, by looking at their local definition, for example on the open set $\mathcal{U}$:

$$\left[\Xi_1\Xi_2 - Y_1^2 - Y_0Y_2\right]\lfloor_{\mathcal{U}} = \theta_1\theta_2 - z^2 + z^2 - \theta_1\theta_2 = 0. \tag{3.48}$$

therefore one indeed gets $\Xi_1\Xi_2 - Y_1^2 + Y_0Y_2 = 0$.
To ensure that the corresponding map $\phi_2 : \mathbb{P}^1_\omega(2,2) \to \mathbb{P}^{2|2}$ is actually an embedding we need to check that it is injective at the level of geometric points (i.e. between the reduced manifolds) and that its super differential is injective on the super tangent space $\mathcal{T}_{\mathbb{P}^1_\omega(2,2)}$. To achieve this, one can define the map $\phi_2 : \mathbb{P}^1_\omega(2,2) \to \mathbb{P}^{2|2}$ locally: on the open set $\mathcal{U}$ one has

$$\phi_{\mathcal{U},2} : (z, \theta_1, \theta_2) \longmapsto [1 : z : z^2 - \theta_1\theta_2 : \theta_1 : \theta_2], \tag{3.49}$$

while on $\mathcal{V}$ one gets

$$\phi_{\mathcal{U},2} : (z, \theta_1, \theta_2) \longmapsto [w^2 - \psi_1\psi_2 : w : 1 : \psi_1 : \psi_2] \tag{3.50}$$

This map is seen to be injective over $\mathbb{P}^1$. The super differential reads

$$d\phi_{\mathcal{U},2} = \begin{pmatrix} 0 & 1 & 2z\theta_1\theta_2 & 0 & 0 \\ 0 & 0 & -\theta_2 & 1 & 0 \\ 0 & 0 & \theta_1 & 0 & 1 \end{pmatrix} \qquad d\phi_{\mathcal{V},m} = \begin{pmatrix} 2w & 1 & 0 & 0 & 0 \\ -\psi_2 & 0 & 0 & 1 & 0 \\ +\psi_1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

so that it $d\phi_2$ is injective and this completes the proof. □

Interestingly, the embedding we have realised is related with the example of non-projected supermanifold given by Witten in [68], page 8, as explained in the following remark.

Remark 3.1 (An Example by Witten)**.** *In order to make the concept of non-projected supermanifold more accessible, Witten proposes the following supercurve in* $\mathbb{P}^{2|2}$

$$Y_0^2 + Y_1^2 + Y_2^2 + \Xi_1\Xi_2 = 0. \tag{3.51}$$

as an example of non-projected supermanifold. Actually, even though he offers an explanation for the non-projectedness of this supermanifold, that does not appear as completely straightforward to the author of the present thesis.
In the opinion of the author, a framework that make clear that this supermanifold is actually non-projected is the one developed above. Indeed, the example put forward by Witten is nothing but the embedding of $\mathbb{P}^1_\omega(2,2)$ *in* $\mathbb{P}^{2|2}$ *we have constructed above.*
We have shown in the previous theorem that the supermanifold $Y_0Y_2 - Y_1^2 + \Xi_1\Xi_2 = 0$ *in* $\mathbb{P}^{2|2}$ *is the image of* $\mathbb{P}^1_\omega(2,2)$ *through the embedding* ϕ_2*, one can then make use of a certain transformation in* $PGL(3|2)$ *- that is an automorphism supergroup of* $\mathbb{P}^{2|2}$ *- to bring the equation in the form displayed by Witten. In particular, the transformation*

$$PGL(3|2) \ni [T] = \left(\begin{array}{ccc|cc} 1 & 0 & i & 0 & 0 \\ 0 & i & 0 & 0 & 0 \\ 1 & 0 & -i & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right) \tag{3.52}$$

does the job. This shows that the non-projective supermanifold displayed by Witten is indeed isomorphic to the non-projected supermanifold we have called $\mathbb{P}^1_\omega(2,2)$*.*

Incidentally, we stress the limits of the previous Lemma 3.2: indeed, it holds only for a single element of the family, the supermanifold $\mathbb{P}^1_\omega(2,2)$.
Fixing the obstruction class to be one-dimensional - that amounts to fix $Sym^2\mathcal{F}_\mathcal{M} \cong \mathcal{O}_{\mathbb{P}^1}(-4)$ or in the notation adopted $m + n = 4$ - and considering a non-isomorphic element of the family, which can be therefore represented as $\mathbb{P}^1_\omega(m, 4-m)$ for $m \neq 2$, one needs to choose a different invertible sheaf, which is likely of the form $\mathcal{O}_{\mathbb{P}^1_\omega(m,4-m)}(m) \leftrightarrow \{\{\mathcal{U},\mathcal{V}\}, e_\mathcal{U} = w^m e_\mathcal{V}\}$ tensored by another one represented by a unipotent element of the even Picard group, in order to realise the embedding. We will not go further this direction, but pass instead to the case of non-projected $\mathcal{N} = 2$ supermanifolds of higher dimension and their embeddings, which proves to be much more interesting.

3.4 Non-Projected Supermanifolds over $\mathbb{P}^2$

In this section we repeat what we have previously done over $\mathbb{P}^1$, now studying the non-projected $\mathcal{N} = 2$ structures over $\mathbb{P}^2$. We recall that this exhausts the possibilities for non-projected $\mathcal{N} = 2$ structures over projective spaces, as there are no non-projected $\mathcal{N} = 2$ supermanifolds over $\mathbb{P}^n$ whenever $n > 2$.
In the same spirit as Theorem 3.4, we prove the following

Theorem 3.9 (Non-Projected $\mathcal{N} = 2$ Supermanifolds over $\mathbb{P}^2$)**.** *Every non-projected* $\mathcal{N} = 2$ *supermanifold over* $\mathbb{P}^2$ *is characterised up to isomorphism by a triple* $(\mathbb{P}^2, \mathcal{F}_\mathcal{M}, \omega)$ *where* $\mathcal{F}_\mathcal{M}$ *is a rank* $0|2$ *sheaf of* $\mathcal{O}_{\mathbb{P}^2}$*-modules such that* $Sym^2\mathcal{F}_\mathcal{M} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$ *and* ω *is a non-zero cohomology class* $\omega \in H^1(\mathcal{T}_{\mathbb{P}^2}(-3))$*.*

Proof. It is enough to use the result of Theorem 3.3 together with the the computations of section 3.2, in particular equation (3.12), that tells that the obstruction to splitting can be obtained only for sheaf $\mathcal{F}_\mathcal{M}$ such that their second symmetric power - in the supersymmetric sense - is isomorphic to the canonical sheaf of $\mathbb{P}^2$. □

Notice that the situation is rather more complicated over $\mathbb{P}^2$ compared to the case over $\mathbb{P}^1$ we have discussed above. Indeed even if over $\mathbb{P}^2$ the obstruction can only be one-dimensional in contrast with the case over $\mathbb{P}^1$, we have that locally-free sheaves of $\mathcal{O}_{\mathbb{P}^2}$-modules do not in general split as direct sums of invertible sheaves, and they might have a moduli space. The condition for a supermanifold over $\mathbb{P}^2$ to be non-projected fixes the first Chern class of the fermionic sheaf $\mathcal{F}_\mathcal{M}$, but this is not enough to uniquely fix a moduli space for these sheaves, as one would need to fix their second Chern class as well. From this point of view, the previous theorem is not really a *classification result* and in particular it is much less exhaustive compared with its analog over $\mathbb{P}^1$, that exploits Grothendieck's splitting theorem for vector bundles over $\mathbb{P}^1$ to provide a specific form

for the fermonic sheaf of the supermanifold.
By the way, the previous theorem justifies the following definition.

Definition 3.2 (The Supermanifolds $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$)**.** *We denote $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ a supermanifold arising from a triple $(\mathbb{P}^2, \mathcal{F}_\mathcal{M}, \omega)$ where the fermionic sheaf $\mathcal{F}_\mathcal{M}$ is a locally-free sheaf of $\mathcal{O}_{\mathbb{P}^2}$-modules of rank $0|2$ such that $Sym^2\mathcal{F}_\mathcal{M} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$ and such that $\omega_\mathcal{M}$ is a (possibly zero) cohomology class in $H^1(\mathcal{T}_{\mathbb{P}^2}(-3)) \cong \mathbb{C}$.*

Clearly, a supermanifold of the kind $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ is not projected if and only if ω is a non-zero class in $H^1(\mathcal{T}_{\mathbb{P}^2}(-3))$.

We now look for the explicit form of the transition functions for a supermanifold in the family $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$.
Working over $\mathbb{P}^2$ leads to consider a set of homogeneous coordinates $[X_0 : X_1 : X_2]$ on $\mathbb{P}^2$ and in turn the set of the affine coordinates and their algebras over the three open sets of the covering $\mathcal{U} := \{\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2\}$ of $\mathbb{P}^2$. In particular, working modulo $\mathcal{J}_\mathcal{M}$, we will have the following

$$\begin{aligned}
&\mathcal{U}_0 := \{X_0 \neq 0\} && \rightsquigarrow && z_{10} \operatorname{mod} \mathcal{J}^2_\mathcal{M} := \frac{X_1}{X_0}, && z_{20} \operatorname{mod} \mathcal{J}^2_\mathcal{M} := \frac{X_2}{X_0};\\
&\mathcal{U}_1 := \{X_1 \neq 0\} && \rightsquigarrow && z_{11} \operatorname{mod} \mathcal{J}^2_\mathcal{M} := \frac{X_0}{X_1}, && z_{21} \operatorname{mod} \mathcal{J}^2_\mathcal{M} := \frac{X_2}{X_1};\\
&\mathcal{U}_2 := \{X_2 \neq 0\} && \rightsquigarrow && z_{12} \operatorname{mod} \mathcal{J}^2_\mathcal{M} := \frac{X_0}{X_2}, && z_{22} \operatorname{mod} \mathcal{J}^2_\mathcal{M} := \frac{X_1}{X_2}.
\end{aligned} \tag{3.53}$$

The transition functions between these charts therefore look like

$$\begin{aligned}
&\mathcal{U}_0 \cap \mathcal{U}_1: && z_{10} \operatorname{mod} \mathcal{J}^2_\mathcal{M} = \frac{1}{z_{11}} \operatorname{mod} \mathcal{J}^2_\mathcal{M}, && z_{20} \operatorname{mod} \mathcal{J}^2_\mathcal{M} = \frac{z_{21}}{z_{11}} \operatorname{mod} \mathcal{J}^2_\mathcal{M};\\
&\mathcal{U}_0 \cap \mathcal{U}_2: && z_{10} \operatorname{mod} \mathcal{J}^2_\mathcal{M} = \frac{z_{22}}{z_{12}} \operatorname{mod} \mathcal{J}^2_\mathcal{M}, && z_{20} \operatorname{mod} \mathcal{J}^2_\mathcal{M} = \frac{1}{z_{12}} \operatorname{mod} \mathcal{J}^2_\mathcal{M};\\
&\mathcal{U}_1 \cap \mathcal{U}_2: && z_{11} \operatorname{mod} \mathcal{J}^2_\mathcal{M} = \frac{z_{12}}{z_{22}} \operatorname{mod} \mathcal{J}^2_\mathcal{M}, && z_{21} \operatorname{mod} \mathcal{J}^2_\mathcal{M} = \frac{1}{z_{22}} \operatorname{mod} \mathcal{J}^2_\mathcal{M}.
\end{aligned} \tag{3.54}$$

Again, the reason why we give expressions for the local bosonic coordinates z_{ij} and their transformation functions mod $\mathcal{J}^2_\mathcal{M}$ instead of mod $\mathcal{J}_\mathcal{M}$ is that, as $\mathcal{N} = 2$, one has $(\mathcal{J}_\mathcal{M})_0 = \mathcal{J}^2_\mathcal{M}$.

Moreover we will denote θ_{1i}, θ_{2i} a basis of the rank $0|2$ sheaf $\mathcal{F}_\mathcal{M}$ on any of the open sets $\mathcal{U}_i$, for $i = 0, 1, 2$, and, since $\mathcal{J}^3_\mathcal{M} = 0$, the transition functions among these bases will have the form

$$\mathcal{U}_i \cap \mathcal{U}_j: \qquad \begin{pmatrix} \theta_{1i} \\ \theta_{2i} \end{pmatrix} = M_{ij} \cdot \begin{pmatrix} \theta_{1j} \\ \theta_{2j} \end{pmatrix}, \tag{3.55}$$

with M_{ij} a 2×2 matrix with coefficients in $\mathcal{O}_{\mathbb{P}^2}(\mathcal{U}_i \cap \mathcal{U}_j)$. Note that in the transformation (3.55) one can write M_{ij} as a matrix with coefficients given by some even rational functions of z_{1j}, z_{2j}, because of the definitions (3.53) and the facts that $\theta_{hj} \in \mathcal{J}_\mathcal{M}$ and $\mathcal{J}^3_\mathcal{M} = 0$.

Finally we note the transformation law for the products $\theta_{1i}\theta_{2i}$, which is given by

$$\theta_{1i}\theta_{2i} = (\det M_{ij})\theta_{1j}\theta_{2j}. \tag{3.56}$$

Since $\det M$ is a transition function for the invertible sheaf $Sym^2\mathcal{F}_\mathcal{M} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$ over $\mathcal{U}_i \cap \mathcal{U}_j$, this can be written, up to constant changes of bases in $\mathcal{F}\lfloor_{\mathcal{U}_i}$ and $\mathcal{F}\lfloor_{\mathcal{U}_j}$, in the more precise form

$$\theta_{1i}\theta_{2i} = \left(\frac{X_j}{X_i}\right)^3 \theta_{1j}\theta_{2j}. \tag{3.57}$$

This also means that we can identify the base $\theta_{1i}\theta_{2i}$ of $Sym^2\mathcal{F}_\mathcal{M}\lfloor_{\mathcal{U}_i}$ with the standard base $\frac{1}{X_i^3}$ of $\mathcal{O}_{\mathbb{P}^2}(-3)$ over $\mathcal{U}_i$.

The relations and transition functions given above are those that *all* the supermanifolds of the kind $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ share, regardless the specific form of its fermionic sheaf $\mathcal{F}_\mathcal{M}$. In the following theorem, in particular, we give the explicit form of the even transition functions.

Theorem 3.10 (Transition Functions for $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$)**.** *The transition functions for an element of the family $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ from coordinates on $\mathcal{U}_0$ to coordinates on $\mathcal{U}_1$ are given by*

$$\begin{pmatrix} z_{10} \\ z_{20} \\ \theta_{10} \\ \theta_{20} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{z_{11}} \\ \dfrac{z_{21}}{z_{11}} + \lambda \dfrac{\theta_{11}\theta_{21}}{(z_{11})^2} \\ M \begin{pmatrix} \theta_{11} \\ \theta_{21} \end{pmatrix} \end{pmatrix} \tag{3.58}$$

where $\lambda \in \mathbb{C}$ is a representative of the class $\omega \in H^1(\mathcal{T}_{\mathbb{P}^2}(-3)) \cong \mathbb{C}$ and M is a 2×2 matrix with coefficients in $\mathbb{C}[z_{11}, z_{11}^{-1}, z_{21}]$ such that $\det M = 1/z_{11}^3$. *Similar transformations hold between the other pairs of open sets.*

Proof. The part of the transformation law (3.58) that relates the fermionic coordinates θ_{10}, θ_{20} and θ_{11}, θ_{21} has already been discussed above. We are therefore left to explain the part of the transformation (3.58) that relates the bosonic coordinates z_{10}, z_{20} and z_{11}, z_{21}. Writing the general transformation (3.10) in this particular case, yields the following

$$\begin{aligned} z_{10} &= \frac{1}{z_{01}} + \omega(z_{10}) \\ z_{20} &= \frac{z_{21}}{z_{01}} + \omega(z_{20}), \end{aligned}$$

with ω a derivation of $\mathcal{O}_{\mathbb{P}^2}$ with values in $Sym^2\mathcal{F}_\mathcal{M}$, which identifies an element $\omega_\mathcal{M} \in H^1(\mathcal{T}_{\mathbb{P}^2} \otimes Sym^2\mathcal{F}_\mathcal{M})$. Recall that by Theorem 3.3 it is only the cohomology class ω that matters in defining the structure of the supermanifold $\mathcal{M}$. In particular $\mathcal{M}$ is non-projected if and only if $\omega \in H^1(\mathcal{T}_{\mathbb{P}^2} \otimes Sym^2\mathcal{F}_{\mathcal{M}_{\mathbb{P}^2}})$ is non-zero. As we have seen, the only possibility for this space to be non-zero is $Sym^2\mathcal{F}_\mathcal{M} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$, so that ω lies in $H^1(\mathcal{T}_{\mathbb{P}^2}(-3))$. Indeed this space is non-null as can be seen by the (twisted) Euler exact sequence for the tangent space, which reads

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-3) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3} \longrightarrow \mathcal{T}_{\mathbb{P}^2}(-3) \longrightarrow 0. \tag{3.59}$$

The long exact sequence in cohomology yields the following isomorphism:

$$\delta : H^1(\mathcal{T}_{\mathbb{P}^2} \otimes \mathcal{O}_{\mathbb{P}^2}(-3)) \xrightarrow{\cong} H^2(\mathcal{O}_{\mathbb{P}^2}(-3)) \cong \mathbb{C} \tag{3.60}$$

where δ is the connecting homomorphism. We now will make this isomorphism more explicit. Recall that the untwisted Euler sequence is

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2} \xrightarrow{e} \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3} \xrightarrow{\pi_*} \mathcal{T}_{\mathbb{P}^2} \longrightarrow 0 \tag{3.61}$$

where, if we write formally $\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3} = \mathcal{O}_{\mathbb{P}^2}(1)\partial_{X_0} \oplus \mathcal{O}_{\mathbb{P}^2}(1)\partial_{X_1} \oplus \mathcal{O}_{\mathbb{P}^2}(1)\partial_{X_2}$, we have

$$e(f) = f(X_0\partial_{X_0} + X_1\partial_{X_1} + X_2\partial_{X_2}) \tag{3.62}$$

$$\pi_*(X_i\partial_{X_j}) = \partial_{(X_j/X_i)}. \tag{3.63}$$

The last relation takes place over the open set $\mathcal{U}_i$, with affine coordinates X_j/X_i, for $j \neq i$. This holds because, fibrewise, the Euler sequence is provided by the differentials $\pi_* : \mathcal{T}_{(\mathbb{C}^3)^*,v} \to \mathcal{T}_{\mathbb{P}^2,[v]}$ of the canonical projection $\pi : (\mathbb{C}^3)^* \to \mathbb{P}^2$. In particular, over $\mathcal{U}_0$ we have the local splitting of $\mathcal{O}_\mathcal{M}$ given by identifying $z_{10} = X_1/X_0$, $z_{20} = X_2/X_0$ and fermionic coordinates given by the chosen local base θ_{10}, θ_{20} of $\mathcal{F}_\mathcal{M}$, and we get $\partial_{z_{20}} = \pi_*(X_0\partial_{X_2})$. By similar reasons we can write $\partial_{z_{11}} = \pi_*(X_1\partial_{X_0})$ over $\mathcal{U}_1$ and $\partial_{z_{22}} = \pi_*(X_2\partial_{X_1})$ over $\mathcal{U}_2$. Now consider the local section $\frac{1}{X_0X_1X_2} \in \mathcal{O}_{\mathbb{P}^2}(-3)(\mathcal{U}_0 \cap \mathcal{U}_1 \cap \mathcal{U}_2)$, whose class $[\frac{1}{X_0X_1X_2}]$ is a basis of $H^2(\mathcal{O}_{\mathbb{P}^2}(-3))$. We make the following calculation on local sections over $\mathcal{U}_0 \cap \mathcal{U}_1 \cap \mathcal{U}_2$ of the sequence (3.59)

$$\begin{aligned} e\left(\frac{1}{X_0X_1X_2}\right) &= \frac{X_0\partial_{X_0} + X_1\partial_{X_1} + X_2\partial_{X_2}}{X_0X_1X_2} \\ &= \frac{1}{X_0^3}\left(\frac{X_0}{X_1}\right)X_0\partial_{X_2} + \frac{1}{X_1^3}\left(\frac{X_1}{X_2}\right)X_1\partial_{X_0} + \frac{1}{X_2^3}\left(\frac{X_2}{X_0}\right)X_2\partial_{X_1} \\ &= \theta_{10}\theta_{20}\left(\frac{X_0}{X_1}\right)X_0\partial_{X_2} + \theta_{11}\theta_{21}\left(\frac{X_1}{X_2}\right)X_1\partial_{X_0} + \theta_{12}\theta_{22}\left(\frac{X_2}{X_0}\right)X_2\partial_{X_1}. \end{aligned} \tag{3.64}$$

By applying π_* to both the first and the last expression above we obtain

$$
\begin{aligned}
0 &= \frac{\theta_{10}\theta_{20}}{z_{10}}\partial_{z_{20}} + \frac{\theta_{11}\theta_{21}}{z_{21}}\partial_{z_{11}} + \frac{\theta_{12}\theta_{22}}{z_{12}}\partial_{z_{22}} \\
&= \frac{\theta_{11}\theta_{21}}{z_{11}^2}\partial_{z_{20}} + \frac{\theta_{12}\theta_{22}}{z_{22}^2}\partial_{z_{11}} + \frac{\theta_{10}\theta_{20}}{z_{20}^2}\partial_{z_{22}}
\end{aligned}
\tag{3.65}
$$

where, for the last equality, we have used the transformations (3.54) and (3.57). The final result is that the assignments of local sections of $\mathcal{T}_{\mathbb{P}^2} \otimes Sym^2\mathcal{F}$

$$
\begin{aligned}
\omega_{01} &= \frac{\theta_{11}\theta_{21}}{z_{11}^2}\partial_{z_{20}} \text{ on } \mathcal{U}_0 \cap \mathcal{U}_1, \\
\omega_{12} &= \frac{\theta_{12}\theta_{22}}{z_{22}^2}\partial_{z_{11}} \text{ on } \mathcal{U}_1 \cap \mathcal{U}_2, \\
\omega_{20} &= \frac{\theta_{10}\theta_{20}}{z_{20}^2}\partial_{z_{22}} \text{ on } \mathcal{U}_0 \cap \mathcal{U}_2
\end{aligned}
\tag{3.66}
$$

satisfy the cocycle condition

$$
\omega_{01}\lfloor_{\mathcal{U}_0\cap\mathcal{U}_1\cap\mathcal{U}_2} + \omega_{12}\lfloor_{\mathcal{U}_0\cap\mathcal{U}_1\cap\mathcal{U}_2} + \omega_{20}\lfloor_{\mathcal{U}_0\cap\mathcal{U}_1\cap\mathcal{U}_2} = 0 \tag{3.67}
$$

and therefore define a cohomology class $[\omega] \in H^1(\mathcal{T}_{\mathbb{P}^2} \otimes Sym^2\mathcal{F}_{\mathcal{M}})$. Moreover, by definition of the connecting homomorphism δ, one has

$$
\delta([\omega]) = \left[\frac{1}{X_0X_1X_2}\right] \in H^2(\mathcal{O}_{\mathbb{P}^2}(-3)). \tag{3.68}
$$

In particular, from the class $[\lambda\omega] \in H^1(\mathcal{T}_{\mathbb{P}^2} \otimes Sym^2\mathcal{F}_{\mathcal{M}})$, one obtains the claimed transformation

$$
z_{10} = \frac{1}{z_{01}} + \lambda\omega_{01}(z_{10}) = \frac{1}{z_{01}}, \tag{3.69}
$$

$$
z_{20} = \frac{z_{21}}{z_{01}} + \lambda\omega_{01}(z_{20}) = \frac{z_{21}}{z_{01}} + \lambda\frac{\theta_{11}\theta_{21}}{z_{11}^2}. \tag{3.70}
$$

□

This theorem provides the general form of the even transition functions for supermanifolds in the family $\mathbb{P}^2_\omega(\mathcal{F}_{\mathcal{M}})$: in the following subsections we will use this result to study some more geometry of these particular non-projected supermanfolds.

3.4.1 $\mathbb{P}^2_\omega(\mathcal{F}_{\mathcal{M}})$ is a Calabi-Yau Supermanifold

In the present subsection we prove that all *all* of the supermanifolds over $\mathbb{P}^2$ of the form $\mathbb{P}^2_\omega(\mathcal{F}_{\mathcal{M}})$, are Calabi-Yau supermanifolds in the sense of the Definition 1.20, regardless the choice made for the fermionic sheaf $\mathcal{F}_{\mathcal{M}}$.

Theorem 3.11 ($\mathbb{P}^2_\omega(\mathcal{F}_{\mathcal{M}})$ is Calabi-Yau)**.** *All of the supermanifolds of the form $\mathbb{P}^2_\omega(\mathcal{F}_{\mathcal{M}})$ are Calabi-Yau supermanifolds. That is,*

$$
\mathcal{B}er\left(\mathbb{P}^2_\omega(\mathcal{F}_{\mathcal{M}})\right) \cong \mathcal{O}_{\mathbb{P}^2_\omega(\mathcal{F}_{\mathcal{M}})}. \tag{3.71}
$$

Proof. We can work locally, considering transformations between $\mathcal{U}_0$ and $\mathcal{U}_1$. Then, using the results of the previous section, we can write the transition functions for an element of the family $\mathbb{P}^2_\omega(\mathcal{F}_{\mathcal{M}})$ as in (3.58) where $\lambda \in \mathbb{C}$ is a representative of the class $\omega \in H^1(\mathcal{T}_{\mathbb{P}^2}(-3)) \cong \mathbb{C}$. We can now compute the (super) Jacobian of this transformation, obtaining:

$$
\mathrm{Jac}\,(\Phi) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \tag{3.72}
$$

where one has

$$A = \begin{pmatrix} -\frac{1}{(z_{11})^2} & 0 \\ -\frac{z_{21}}{(z_{11})^2} - 2\lambda \frac{\theta_{11}\theta_{21}}{(z_{11})^3} & \frac{1}{z_{11}} \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ \lambda \frac{\theta_{21}}{(z_{11})^2} & -\lambda \frac{\theta_{11}}{(z_{11})^2} \end{pmatrix}$$

$$C = \begin{pmatrix} \partial_{z_{11}} M \begin{pmatrix} \theta_{11} \\ \theta_{21} \end{pmatrix} & \partial_{z_{21}} M \begin{pmatrix} \theta_{11} \\ \theta_{21} \end{pmatrix} \end{pmatrix} \qquad D = M. \tag{3.73}$$

Then, we can compute the Berezinian of this Jacobian matrix using the well-known formula $\mathrm{Ber}\,(\mathrm{Jac}\,(\Phi)) = \det(A - BD^{-1}C) \det D^{-1}$. We have that

$$A - BD^{-1}C = \begin{pmatrix} -\frac{1}{z_{11}} & 0 \\ * & \frac{1}{z_{11}} - H \end{pmatrix} \quad \text{where} \quad H = \frac{\lambda}{(z_{11})^2} (\theta_{21}, -\theta_{11})\, M^{-1} \partial_{z_{21}} M \begin{pmatrix} \theta_{11} \\ \theta_{21} \end{pmatrix}. \tag{3.74}$$

Now, on the one hand we have that $\det M = \frac{1}{(z_{11})^3}$ and therefore $\partial_{z_{21}} \det M = 0$. On the other hand, denoting $\delta := \partial_{z_{21}}$, we find by explicit computation that

$$H = \frac{\lambda}{(z_{11})^2} (\theta_{21}, -\theta_{11}) \frac{1}{\det M} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \delta a & \delta b \\ \delta c & \delta d \end{pmatrix} \begin{pmatrix} \theta_{11} \\ \theta_{21} \end{pmatrix} = \lambda z_{11} \,\mathrm{Tr}(M^{-1}\delta M)\, \theta_{11}\theta_{21} \tag{3.75}$$

and upon noticing that $0 = \delta(\det M) = \mathrm{Tr}(M^{-1}\delta M)$, we find that $H = 0$. Therefore one has

$$\mathrm{Ber}\,(\mathrm{Jac}\,(\Phi)) = \det \begin{pmatrix} -\frac{1}{(z_{11})^2} & 0 \\ * & \frac{1}{z_{11}} \end{pmatrix} \det M^{-1} = -1 \tag{3.76}$$

which concludes the proof. □

Before we pass to another peculiar property of the family of non-projected supermanifolds $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$, we stress that in the previous theorem we have been forced to use a "brute force" computation using the transition functions of the supermanifolds of the family. Indeed, to the best knowledge of the author, there are no results such as Theorem 1.1 - that hold for *projected* supermanifolds - in the case of non-projected supermanifolds and no exact sequences come in our help: this forces us to carry out explicit computations and a direct evaluation of the Berezinian sheaf.

3.4.2 $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ is Non Projective

We continue studying the properties of the family $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ and we consider the locally-free sheaves of rank $1|0$, i.e. the even invertible sheaves, that can be defined over these non-projected supermanifolds, by studying the even Picard group $\mathrm{Pic}_0(\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M}))$. We have seen that in a previous section that the even Picard group of the non-projected $\mathcal{N} = 2$ supermanifolds over $\mathbb{P}^1$ is actually pretty rich, and in particular - even if we have explicitly considered just a single example - all of these non-projected supermanifolds can be embedded into some projective superspace $\mathbb{P}^{n|m}$. We will see that this is *not* the case when dealing with non-projected $\mathcal{N} = 2$ supermanifolds over $\mathbb{P}^2$.

Theorem 3.12 ($\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ is Non Projective)**.** *The even Picard group of the non-projected supermanifolds of the family $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ is trivial, regardless how one chooses the fermionic sheaf $\mathcal{F}_\mathcal{M}$:*

$$\mathrm{Pic}_0(\mathbb{P}^{2|2}_\omega) = 0. \tag{3.77}$$

In particular, the non-projected supermanifolds of the family $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ are non-projective, regardless how one chooses the fermionic sheaf $\mathcal{F}_\mathcal{M}$.

Proof. For the sake on notation we put $\mathcal{M} := \mathbb{P}^{2|2}_\omega(\mathcal{F}_\mathcal{M})$ and, remembering that $Sym^2\mathcal{F}_\mathcal{M} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$, we consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-3) \xrightarrow{\ \exp\ } \mathcal{O}^*_{\mathcal{M},0} \longrightarrow \mathcal{O}^*_{\mathbb{P}^2} \longrightarrow 1. \tag{3.78}$$

As already seen above, this is the multiplicative version of the structural exact sequence, with the first map defined by $\exp(h) = 1 + h$, as $h \in \mathcal{O}_{\mathbb{P}^2}(-3) = (\mathcal{J}_\mathcal{M})_0$ and $(\mathcal{J}_\mathcal{M})_0^2 = 0$. The exact sequence above gives the following piece of long exact cohomology sequence

$$0 \longrightarrow H^1(\mathcal{O}^*_{\mathcal{M},0}) \longrightarrow H^1(\mathcal{O}^*_{\mathbb{P}^2}) \xrightarrow{\ \delta\ } H^2(\mathcal{O}_{\mathbb{P}^2}(-3)) \longrightarrow \cdots . \tag{3.79}$$

Now, one has $\mathrm{Pic}(\mathbb{P}^2) = H^1(\mathcal{O}^*_{\mathbb{P}^2}) \cong \mathbb{Z}$ and $H^2(\mathcal{O}_{\mathbb{P}^2}(-3)) \cong \mathbb{C}$, so everything reduces to decide whether the connecting homomorphism $\delta : \mathrm{Pic}(\mathbb{P}^2) \to H^2(\mathcal{O}_{\mathbb{P}^2}(-3))$ is the zero map or it is an injective map $\mathbb{Z} \to \mathbb{C}$. This can be checked directly, by a diagram-chasing computation, by looking at the following diagram of cochain complexes,

$$\begin{tikzcd} C^2(\mathcal{O}_{\mathbb{P}^2}(-3)) \arrow[r, "i"] & C^2(\mathcal{O}^*_{\mathcal{M},0}) & \\ & C^1(\mathcal{O}^*_{\mathcal{M},0}) \arrow[u] \arrow[r, two heads, "j"] & C^1(\mathcal{O}^*_{\mathbb{P}^2}), \end{tikzcd} \tag{3.80}$$

that is obtained by considering the short exact sequence (3.78) and the Čech cochain complexes of the sheaves involved in the sequence.
One then picks the generating line bundle $\langle \mathcal{O}_{\mathbb{P}^2}(1)\rangle_{\mathcal{O}_{\mathbb{P}^2}} \cong \mathrm{Pic}(\mathbb{P}^2)$ and, given the usual covering $\mathcal{U} := \{\mathcal{U}_i\}_{i=0}^2$ of $\mathbb{P}^2$ as above, $\mathcal{O}_{\mathbb{P}^2}(1)$ can be represented by the cocycle $g_{ij} \in Z^1(\mathcal{U}, \mathcal{O}^*_{\mathbb{P}^2})$ given by the transition functions of the line bundle itself. Explicitly, in homogeneous coordinates, these cocycles are given by

$$\mathcal{O}_{\mathbb{P}^2}(1) \longleftrightarrow \left\{ g_{01} = \frac{X_0}{X_1},\ g_{12} = \frac{X_1}{X_2},\ g_{20} = \frac{X_2}{X_0} \right\}. \tag{3.81}$$

Since the map $j : C^1(\mathcal{O}^*_{\mathcal{M},0}) \to C^1(\mathcal{O}^*_{\mathbb{P}^2})$ is surjective, these cocycles are, in particular, image of elements in $C^1(\mathcal{O}^*_{\mathcal{M},0})$. More precisely we have

$$j(z_{11}) = g_{01}, \quad j(z_{22}) = g_{12}, \quad j(z_{20}) = g_{20},$$

hence we can consider the lifting $\sigma = \{z_{11}, z_{22}, z_{20}\}$ of $\{g_{01}, g_{12}, g_{20}\}$ to $C^1(\mathcal{O}^*_{\mathcal{M},0})$. We stress that this is *not* a cocycle in $C^1(\mathcal{O}^*_{\mathcal{M},0})$. Now, by going up in the diagram to $C^2(\mathcal{O}^*_{\mathcal{M},0})$ by means of the Čech boundary map $\delta : C^1(\mathcal{O}^*_{\mathcal{M},0}) \to C^2(\mathcal{O}^*_{\mathcal{M},0})$, and using the bosonic transformation laws induced by the derivations (3.66), one finds the following element:

$$\begin{aligned} \delta(\sigma) &= z_{11} \cdot z_{22} \cdot z_{20}\lfloor_{\mathcal{U}_0 \cap \mathcal{U}_1 \cap \mathcal{U}_2} \\ &= \left(\frac{z_{12}}{z_{22}} + \lambda \frac{\theta_{12}\theta_{22}}{z_{22}^2} \right) z_{22} z_{20} \\ &= 1 + \lambda \frac{\theta_{12}\theta_{22}}{z_{22}} z_{20} = 1 + \lambda \theta_{12}\theta_{22} \left(\frac{X_2}{X_1} \right) \left(\frac{X_2}{X_0} \right) \qquad \text{by (3.53)} \\ &= 1 + \lambda \left(\frac{1}{X_2^3} \right) \left(\frac{X_2}{X_1} \right) \left(\frac{X_2}{X_0} \right) = 1 + \frac{\lambda}{X_0 X_1 X_2}. \end{aligned}$$

We have that the element $1 + \frac{\lambda}{X_0X_1X_2}$ is the image of $\frac{\lambda}{X_0X_1X_2}$ through the map i. Hence we find that $\delta : \mathrm{Pic}(\mathbb{P}^2) \longrightarrow H^2(\mathcal{O}_{\mathbb{P}^2}(-3))$ maps $[\mathcal{O}_{\mathbb{P}^2}(1)] \longmapsto [\frac{\lambda}{X_0X_1X_2}]$, which is non-zero for $\lambda \neq 0$, i.e. for $\mathcal{M}$ non-projected. This leads to the conclusion that $\mathrm{Pic}_0(\mathbb{P}^{2|2}_\omega(\mathcal{F}_\mathcal{M})) = H^1(\mathcal{O}^*_{\mathbb{P}^{2|2}_\omega(\mathcal{F}_\mathcal{M}),0}) = 0$, i.e. the only locally-free sheaf of rank $1|0$ on $\mathcal{M}$ is $\mathcal{O}_\mathcal{M}$.

In particular there are no locally-free sheaves of rank $1|0$ to realise an embedding in a projective superspace, that is the non-projected supermanifold $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ is non-projective. □

The previous theorem leads to an interesting conclusion. Indeed it makes clear the substantial difference between complex algebraic supergeometry and the usual complex algebraic geometry, where projective spaces are the prominent ambient spaces. This fact was already known by Manin (see, for example [41], [42]), who produced many examples of non-projective supermanifolds. However, we anticipate that later on in this chapter we will show that any supermanifold of the form $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ can always be embedded in some super Grassmannian. In order to achieve this, in the next section we will review the geometry of super Grassmannians.

3.5 The Geometry of Super Grassmannians

In this section we introduce some elements of the geometry of super Grassmannians. All the results we expose in this section are due to Y. Manin and his school [41], [42], [52]. For a thorough and deep treatment of super Grassmannian and super flag varieties we suggest the interested reader to refer to [41] and [42]. Here we intend to give a self contained and elementary exposition of Super Grassmannians with emphasis on their general non-projectdness and their non-projectivity, *i.e.* non-embeddability into projective superspaces.

Super Grassmannians are the supersymmetric generalisation of the ordinary Grassmannians. $G(a|b; V^{n|m})$ is a universal parameter space for $a|b$-dimensional linear subspaces of a given $n|m$-dimensional space $V^{n|m}$. For the sake of clarity we will only work in the analytic category, always choosing to deal with the simplest possible situation, that is choosing the $n|m$-dimensional space $V^{n|m}$ to be a super vector space of the kind $\mathbb{C}^{n|m}$. In what follows we briefly review how to construct a super Grassmannian by patching together the "charts" covering it. This is a rather straightforward generalisation of the usual construction of ordinary Grassmannians via their *big cells.*

1. Let $\mathbb{C}^{n|m}$ be such that $n|m = c_0|c_1 + d_0|d_1$. We look at $\mathbb{C}^{n|m}$ as given by $\mathbb{C}^{c_0+d_0} \oplus (\Pi\mathbb{C})^{c_1+d_1}$. This is obviously freely-generated, and we will write its elements as row vectors with respect to a certain basis, $\mathbb{C}^{n|m} = \mathrm{Span}\{e^0_1, \ldots, e^0_n | e^1_1, \ldots, e^1_m\}$, where the upper indices refer to the $\mathbb{Z}_2$-parity.

2. Consider a collection of indices $I = I_0 \cup I_1$ such that I_0 is a collection of d_0 out of the n indices of $\mathbb{C}^n$ and I_1 is a collection of d_1 indices out of m indices of $\Pi\mathbb{C}^m$. If $\mathcal{I}$ is the set of such collections of indices I we have that
$$\mathrm{card}(\mathcal{I}) = \mathrm{card}(\mathcal{I}_0 \times \mathcal{I}_1) = \binom{n}{d_0} \cdot \binom{m}{d_1}. \tag{3.82}$$
This will be the number of *super big cells* covering the super Grassmannian.

3. Choosing a certain element $I \in \mathcal{I}$ we associate to it a set of even and odd (complex) variables $\{x^{\alpha\beta}_I \,|\, \xi^{\alpha\beta}_I\}$. These are arranged as to fill in the places of a $d_0|d_1 \times n|m = a|b \times (c_0+d_0)|(c_1+d_1)$ super matrix in a way such that the columns having indices in $I \in \mathcal{I}_I$ forms a $(d_0 + d_1) \times (d_0 + d_1)$ unit matrix if brought together. For example a certain choice of $I \in \mathcal{I}$ yields the following
$$\mathcal{Z}_I := \left(\begin{array}{c|c||c|c} x_I & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & 0 & \xi_I \\ \hline\hline \xi_I & 0 & \begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix} & x_I \end{array}\right), \tag{3.83}$$
where we picked that particular $I \in \mathcal{I}$ that underlines the presence of the $(d_0+d_1) \times (d_0+d_1)$ unit matrix.

4. We can now define the superspace $\mathcal{U}_I \to \mathrm{Spec}\,\mathbb{C} \cong \{pt\}$ to be the analytic superspace $\{pt\} \times \mathbb{C}^{d_0\cdot c_0 + d_1 \cdot c_1 | d_0 \cdot c_1 + d_1 \cdot c_0} \cong \mathbb{C}^{d_0\cdot c_0 + d_1 \cdot c_1 | d_0 \cdot c_1 + d_1 \cdot c_0}$, where $\{x^{\alpha\beta}_I \,|\, \xi^{\alpha\beta}_I\}$ are the complex coordinates over the point. When represented as above, the superspace $\mathcal{U}_I$ is called a *super big cell* of the Grassmannian.

5. We now aim to patch together two superspaces $\mathcal{U}_I$ and $\mathcal{U}_J$ for two different $I, J \in \mathcal{I}$. Given $\mathcal{Z}_I$ the super big cell attached to $\mathcal{U}_I$ we consider the super submatrix $\mathcal{B}_{IJ}$ formed by the columns having indices in J. Let $\mathcal{U}_{IJ} = \mathcal{U}_I \cap \mathcal{U}_J$ be the (maximal) sub superspace of $\mathcal{U}_I$ such that on $\mathcal{U}_{IJ}$ we have that $\mathcal{B}_{IJ}$ is invertible. Clearly the odd coordinates do not affect the invertibility, so that it is enough that the two determinants of the even parts of the matrix $\mathcal{B}_{IJ}$ that are respectively a $d_0 \times d_0$ and a $d_1 \times d_1$ matrix are different from zero. If this is the case, on the superspace $\mathcal{U}_{IJ}$ we have common coordinates $\{x^{\alpha\beta}_I \,|\, \xi^{\alpha\beta}_I\}$ and $\{x^{\alpha\beta}_J \,|\, \xi^{\alpha\beta}_J\}$,

and the rule to pass from one system of coordinates to the other one is provided on $\mathcal{U}_{IJ}$ by $\mathcal{Z}_J = \mathcal{B}^{-1}_{IJ}\mathcal{Z}_I$.
Let us make it clearer by means of the following explicit example. Consider the following two super big cells:

$$\mathcal{Z}_I := \left(\begin{array}{ccc||cc} 1 & 0 & x_1 & 0 & \xi_1 \\ 0 & 1 & x_2 & 0 & \xi_2 \\ \hline\hline 0 & 0 & \eta & 1 & y \end{array}\right), \qquad \mathcal{Z}_J := \left(\begin{array}{ccc||cc} 1 & \tilde{x}_1 & 0 & 0 & \tilde{\xi}_1 \\ 0 & \tilde{x}_2 & 1 & 0 & \tilde{\xi}_2 \\ \hline\hline 0 & \tilde{\eta} & 0 & 1 & \tilde{y} \end{array}\right). \tag{3.84}$$

Looking at $\mathcal{Z}_I$, we see that the columns belonging to J are the first, the third and the fourth, so that

$$\mathcal{B}_{IJ} = \left(\begin{array}{cc|c} 1 & x_1 & 0 \\ 0 & x_2 & 0 \\ \hline 0 & \eta & 1 \end{array}\right). \tag{3.85}$$

Computing the determinant of the upper-right 2×2 matrix, we have invertibility of $\mathcal{B}_{IJ}$ corresponds to $x_2 \neq 0$ (as seen from the point of view of $\mathcal{U}_I$. Likewise we would have found $\tilde{x}_2 \neq 0$ by looking at $\mathcal{Z}_J$ and $\mathcal{U}_J$). The inverse of $\mathcal{B}^{-1}_{IJ}$ reads

$$\mathcal{B}^{-1}_{IJ} = \left(\begin{array}{cc|c} 1 & -x_1/x_2 & 0 \\ 0 & 1/x_1 & 0 \\ \hline 0 & \eta/x_2 & 1 \end{array}\right) \tag{3.86}$$

so that we can compute the coordinates of $\mathcal{U}_J$ as functions of the ones of $\mathcal{U}_I$ via $\mathcal{Z}_J = \mathcal{B}^{-1}_{IJ}\mathcal{Z}_I$:

$$\left(\begin{array}{ccc||cc} 1 & \tilde{x}_1 & 0 & 0 & \tilde{\xi}_1 \\ 0 & \tilde{x}_2 & 1 & 0 & \tilde{\xi}_2 \\ \hline\hline 0 & \tilde{\eta} & 0 & 1 & \tilde{y} \end{array}\right) = \left(\begin{array}{ccc||cc} 1 & -x_1/x_2 & 0 & 0 & \xi_1 - \xi_2 x_1/x_2 \\ 0 & 1/x_2 & 1 & 0 & \xi_2/x_2 \\ \hline\hline 0 & -\eta/x_2 & 0 & 1 & y_1 - \eta\xi_2/x_2 \end{array}\right). \tag{3.87}$$

The change of coordinates can be read out of this. Observe that the denominator x_2 is indeed invertible on $\mathcal{U}_{IJ}$.

6. Patching together the superspaces $\mathcal{U}_I$ we obtain the Grassmannian supermanifold $G(d_0|d_1;\mathbb{C}^{n|m})$ as the quotient

$$G(d_0|d_1;\mathbb{C}^{n|m}) := \bigcup_{I\in\mathcal{I}}/\mathcal{R}, \tag{3.88}$$

where we have written $\mathcal{R}$ for the equivalence relations generated by the change of coordinates that have been described above. As a (complex) supermanifold a super Grassmannian has dimension

$$\dim_{\mathbb{C}} G(d_0|d_1;\mathbb{C}^{n|m}) = d_0(n-d_0) + d_1(m-d_1)|d_0(m-d_1) + d_1(n-d_0). \tag{3.89}$$

We stress that the maps $\psi_{\mathcal{U}_I} : \mathcal{U}_I \to G(d_0|d_1;\mathbb{C}^{n|m})$ are isomorphisms onto (open) sub superspaces of the super Grassmannian, so that the various super big cells offer a *local* description of it, in the same way a usual (complex) supermanifold is locally isomorphic to a superspace of the kind $\mathbb{C}^{n|m}$.

The easiest possible example of super Grassmannians are projective superspaces, that are realised as $\mathbb{P}^{n|m} = G(1|0;\mathbb{C}^{n+1|m})$, exactly as in the ordinary case. These are *split* supermanifolds, a feature that they do not in general share with a generic Grassmannian $G(d_0|d_1;\mathbb{C}^{n|m})$, as we shall see in a moment.
There are still some elements to introduce before we go on, though. For convenience, in what follows we will call G a super Grassmannian of the kind $G(d_0|d_1;\mathbb{C}^{n|m})$ and we give the following definition, see [41].

Definition 3.3 (Tautological Sheaf)**.** *Let G be a super Grassmannian and let it be covered by the super big cells $\{\mathcal{U}_I\}_{I\in\mathcal{I}}$. We call tautological sheaf $\mathcal{S}_G$ of the super Grassmannian G the sheaf of locally-free $\mathcal{O}_G$-modules of rank $d_0|d_1$ defined as*

$$\mathcal{U} \cap \mathcal{U}_I \longmapsto \mathcal{S}_G(\mathcal{U}\cap\mathcal{U}_I) := \big\langle \text{rows of the matrix } \mathcal{Z}_I \big\rangle_{\mathcal{O}_G(\mathcal{U}\cap\mathcal{U}_I)}. \tag{3.90}$$

The reader can convince himself that this is a well-posed definition, since one has that $\mathcal{S}_G(\mathcal{U}_I)\lfloor_{\mathcal{U}_{IJ}}$ and $\mathcal{S}_G(\mathcal{U}_J)\lfloor_{\mathcal{U}_{JI}}$ get identified by means of the transition functions $\mathcal{B}_{IJ}$.
We now aim to have some insight about the geometry of a super Grassmannian by looking at its reduced space - which encloses all the topological information -, at the filtration of its trivial sheaf $\mathcal{O}_G$ and its tautological sheaf $\mathcal{S}_G$. Even if they do miss some crucial information, they offer an easy and useful approximation of the structure of these sheaves.
We start observing that given a super Grassmannian G, we automatically have two *even* sub Grassmannians.

Definition 3.4 (G_0 and G_1)**.** *Let $G = G(d_0|d_1;\mathbb{C}^{n|m})$ be a super Grassmannian. Then we call G_0 and G_1 the two purely even sub Grassmannians defined as*

$$G_0 := G(d_0|0;\mathbb{C}^{n|0}), \qquad\qquad G_1 := G(0|d_1;\mathbb{C}^{0|m}). \tag{3.91}$$

Given a super big cell $\mathcal{U}_I$, these can be visualised as the upper-left and the lower-right part respectively and they come endowed with their tautological sheaves, we call them $\mathcal{S}_0$ and $\mathcal{S}_1$. Notice, though, that $\mathcal{S}_1$ defines a sheaf of locally-free $\mathcal{O}_{G_1}$-modules and, as such, it has rank $0|d_1$.
Given an ordinary complex Grassmannian G of the kind $G(d;\mathbb{C}^n)$ and its tautological sheaf $\mathcal{S}_G$, we can also define the *sheaf orthogonal to the tautological sheaf*, we call it $\widetilde{\mathcal{S}}$, whose dual fits into the short exact sequence

$$0 \longrightarrow \mathcal{S}_G \longrightarrow \mathcal{O}_G^{\oplus n} \longrightarrow \widetilde{\mathcal{S}}^*_G \longrightarrow 0. \tag{3.92}$$

Notice that in the case the Grassmannian corresponds to a certain projective space $G(1|0;\mathbb{C}^{n+1}) = \mathbb{P}^n$, the sheaf orthogonal to the tautological sheaf can be red off the Euler exact sequence twisted by the tautological sheaf itself $\mathcal{S}_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-1)$, and, indeed, we have that $\widetilde{\mathcal{S}}^*_G \cong \mathcal{T}_{\mathbb{P}^n}(-1)$, so that $\widetilde{\mathcal{S}}_G \cong \Omega^1_{\mathbb{P}^n}(+1)$.
Finally notice that in the case of a super Grassmannian $G(d_0|d_1;n|m)$ the sequence (3.92) generalises to the canonical sequence

$$0 \longrightarrow \mathcal{S}_G \longrightarrow \mathcal{O}_G^{\oplus n|m} \longrightarrow \widetilde{\mathcal{S}}^*_G \longrightarrow 0. \tag{3.93}$$

We now have all the ingredients to state the following theorem

Theorem 3.13. *Let $G = G(d_0|d_1;\mathbb{C}^{n|m})$ be a super Grassmannian and let G_0 and G_1 their even sub Grassmannians together with the sheaves $\mathcal{S}_0, \mathcal{S}_1$ and $\widetilde{\mathcal{S}}_0, \widetilde{\mathcal{S}}_1$. Then the following (canonical) isomorphisms hold true*

1) $G_{red} \cong G_0 \times G_1$;

2) $\mathrm{Gr}\,\mathcal{O}_G \cong Sym\,(\mathcal{S}_0 \otimes \widetilde{\mathcal{S}}_1 \oplus \widetilde{\mathcal{S}}_0 \otimes \mathcal{S}_1)$,

where by Sym we mean the super-symmetric algebra over $\mathcal{O}_{G_0\times G_1}$.

Proof. See [41]. □

As a fundamental and easiest example yet having all the features characterising the geometry of super Grassmannians, we discuss the super Grassmannian $G(1|1,\mathbb{C}^{2|2})$. This super Grassmannian is such that $\dim_{\mathbb{C}} G(1|1;\mathbb{C}^{2|2}) = 2|2$, therefore the methods we have developed early on to study supermanifolds having fermionic dimension 2 apply.

The Geometry of $G(1|1;\mathbb{C}^{2|2})$: in studying the geometry of $G(1|1;\mathbb{C}^{2|2})$, we call it G for short, we first wish to address to its reduced manifold. This is easily identified using the previous theorem.

Lemma 3.3 ($G(1|1;\mathbb{C}^{2|2})_{red} \cong \mathbb{P}^1_0 \times \mathbb{P}^1_1$)**.** *Let G be the super Grassmannian as above, then*

$$G(1|1;\mathbb{C}^{2|2})_{red} \cong \mathbb{P}^1_0 \times \mathbb{P}^1_1. \tag{3.94}$$

Proof. Keeping the same notation as above, we have that $G_0 = G(1|0;\mathbb{C}^{2|0})$ and $G_1 = G(0|1;\mathbb{C}^{0|2})$, and therefore, topologically, we have that $G_0 \cong \mathbb{P}^1_0$ and $G_1 \cong \mathbb{P}^1_1$, where the subscripts are there to distinguish the two copies of projective lines. The conclusion follows by the first point of previous theorem. □

Notice that we would have easily gotten to the same conclusion by looking at the big cells of this super Grassmannian, after having set the nilpotents to zero.
We have therefore the following situation

$$
\begin{tikzcd}
& \mathbb{P}^1_0 \times \mathbb{P}^1_1 \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\
\mathbb{P}^1_0 & & \mathbb{P}^1_1
\end{tikzcd}
\tag{3.95}
$$

that helps us to recover the geometric data of G_{red} and G out of those of the two copies of projective lines.
Along this line, remembering that $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(\ell_1,\ell_2)$ is the external tensor product $\mathcal{O}_{\mathbb{P}^1_0}(\ell_1)\boxtimes\mathcal{O}_{\mathbb{P}^1_1}(\ell_2) := \pi_0^*\mathcal{O}_{\mathbb{P}^1_0}(\ell_1)\otimes_{\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}}\pi_1^*\mathcal{O}_{\mathbb{P}^1_1}(\ell_2)$, and since the tautological sheaf on $\mathbb{P}^1$ is $\mathcal{O}_{\mathbb{P}^1}(-1)$, we have that

$$
\mathcal{S}_0 = \mathcal{O}_{\mathbb{P}^1_0}(-1)\boxtimes\mathcal{O}_{\mathbb{P}^1_1} = \mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(-1,0), \tag{3.96}
$$

$$
\mathcal{S}_1 = \Pi\mathcal{O}_{\mathbb{P}^1_0}\boxtimes\mathcal{O}_{\mathbb{P}^1_1}(-1) = \Pi\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(0,-1). \tag{3.97}
$$

Similarly, observing that the sheaf dual to the tautological sheaf on $\mathbb{P}^1$ is given again by the sheaf $\mathcal{O}_{\mathbb{P}^1}(+1)$, as the (twisted) Euler sequence reads

$$
0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \longrightarrow \mathcal{T}_{\mathbb{P}^1}(-1) \longrightarrow 0\ , \tag{3.98}
$$

and therefore $\widetilde{\mathcal{S}}_{\mathbb{P}^1} \cong (\mathcal{T}_{\mathbb{P}^1}(-1))^* \cong \Omega^1_{\mathbb{P}^1}(+1) \cong \mathcal{O}_{\mathbb{P}^1}(-1)$, one has the following results:

$$
\widetilde{\mathcal{S}}_0 = \mathcal{O}_{\mathbb{P}^1_0}(-1)\boxtimes\mathcal{O}_{\mathbb{P}^1_1} = \mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(-1,0), \tag{3.99}
$$

$$
\widetilde{\mathcal{S}}_1 = \Pi\mathcal{O}_{\mathbb{P}^1_0}\boxtimes\mathcal{O}_{\mathbb{P}^1_1}(-1) = \Pi\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(0,-1). \tag{3.100}
$$

This is enough to single out the fermionic sheaf of G, as $\mathcal{F}_G = \mathrm{Gr}^{(1)}\,\mathcal{O}_G$ and therefore by virtue of the second point of the previous theorem, we have that $\mathcal{F}_G \cong \mathcal{S}_0\otimes\widetilde{\mathcal{S}}_1 \oplus \widetilde{\mathcal{S}}_0\otimes\mathcal{S}_1$, thus

$$
\mathcal{F}_G \cong \Pi\left(\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(-1,-1)\oplus\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(-1,-1)\right). \tag{3.101}
$$

This in turns shows that

$$
Sym^2\mathcal{F}_G = \mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(-2,-2). \tag{3.102}
$$

We can now prove the following theorem.

Theorem 3.14 ($G(1|1;\mathbb{C}^{2|2})$ is Non-Projected)**.** *The supermanifold $G = G(1|1;\mathbb{C}^{2|2})$ is in general non-projected. In particular, one finds that $H^1(\mathcal{T}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}\otimes Sym^2\mathcal{F}_G)\cong\mathbb{C}\oplus\mathbb{C}$.*

Proof. In order to compute the cohomology group $H^1(\mathcal{T}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}\otimes Sym^2\mathcal{F}_G)$, we observe that in general, on the product of two varieties, we have $\mathcal{T}_{X\times Y}\cong p_1^*\mathcal{T}_X\oplus p_2^*\mathcal{T}_Y$, where the p_i are the projections on the factors, so that, in particular, we find

$$
\mathcal{T}_{\mathbb{P}^1_0\times\mathbb{P}^1_1} \cong \pi_0^*\mathcal{T}_{\mathbb{P}^1_0}\oplus\pi_1^*\mathcal{T}_{\mathbb{P}^1_1} \cong \pi_0^*\mathcal{O}_{\mathbb{P}^1_0}(2)\oplus\pi_1^*\mathcal{O}_{\mathbb{P}^1_1}(2) = \mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(2,0)\oplus\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(0,2).
$$

Taking the tensor product with $Sym^2\mathcal{F}_G$ as found above, we have

$$
\begin{aligned}
\mathcal{T}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}\otimes Sym^2\mathcal{F}_G &\cong \left(\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(2,0)\oplus\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(0,2)\right)\otimes\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(-2,-2)\\
&\cong \mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(0,-2)\oplus\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(-2,0).
\end{aligned}
\tag{3.103}
$$

Now, the Künneth formula reduces the computation of the cohomology sheaves on product varieties to the cohomology of their constituent,

$$
H^n\left(X\times Y, p_1^*\mathcal{F}_X\otimes_{\mathcal{O}_{X\times Y}}p_2^*\mathcal{G}_Y\right)\cong\bigoplus_{i+j=n}H^i\left(X,\mathcal{F}_X\right)\otimes H^j(Y,\mathcal{F}_Y), \tag{3.104}
$$

so that we find

$$\begin{aligned} H^1(\mathcal{T}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}\otimes Sym^2\mathcal{F}_G) &\cong H^1(\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(0,-2)\oplus\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(-2,0)) \\ &\cong H^1(\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(0,-2))\oplus H^1(\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(-2,0)) \\ &\cong H^0(\mathcal{O}_{\mathbb{P}^1_0})\otimes H^1(\mathcal{O}_{\mathbb{P}^1_1})(-2)\oplus H^1(\mathcal{O}_{\mathbb{P}^1_0})(-2)\otimes H^0(\mathcal{O}_{\mathbb{P}^1_1}) \\ &\cong \mathbb{C}\oplus\mathbb{C}, \end{aligned} \tag{3.105}$$

thus concluding the proof. □

There are various ways to find the representatives in the obstruction cohomology group for this non-projected supermanifold. We will now use the super big cells of $G(1|1;\mathbb{C}^{2|2})$ to identifies these representatives and to establish that in the isomorphisms $H^1(\mathcal{T}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}\otimes Sym^2\mathcal{F}_G)\cong\mathbb{C}\oplus\mathbb{C}$ the cohomology class corresponds to the choice $\omega_G=(1,1)$.
We start by observing that, since the reduced manifold underlying $G(1|1;\mathbb{C}^{2|2})$ has the topology of $\mathbb{P}^1_0\times\mathbb{P}^1_1$, it is covered by four standard open sets. If we call $\mathcal{U}^{(0)}=\{\mathcal{U}^{(0)}_\ell\}_{\ell=0,1}$ the usual open sets covering $\mathbb{P}^1_0$ and likewise $\mathcal{U}^{(1)}=\{\mathcal{U}^{(1)}_\ell\}_{\ell=0,1}$ the open sets covering $\mathbb{P}^1_1$, we then have a system of open sets covering their product $\mathbb{P}^1_0\times\mathbb{P}^1_1$ given by

$$\begin{aligned} \mathcal{U}_1 &:= \mathcal{U}^{(0)}_0\times\mathcal{U}^{(1)}_0 = \left\{([X_0:X_1],[Y_0:Y_1])\in\mathbb{P}^1_0\times\mathbb{P}^1_1 : X_0\neq 0,\ Y_0\neq 0\right\}, \\ \mathcal{U}_2 &:= \mathcal{U}^{(0)}_1\times\mathcal{U}^{(1)}_0 = \left\{([X_0:X_1],[Y_0:Y_1])\in\mathbb{P}^1_0\times\mathbb{P}^1_1 : X_1\neq 0,\ Y_0\neq 0\right\}, \\ \mathcal{U}_3 &:= \mathcal{U}^{(0)}_0\times\mathcal{U}^{(1)}_1 = \left\{([X_0:X_1],[Y_0:Y_1])\in\mathbb{P}^1_0\times\mathbb{P}^1_1 : X_0\neq 0,\ Y_1\neq 0\right\}, \\ \mathcal{U}_3 &:= \mathcal{U}^{(0)}_1\times\mathcal{U}^{(1)}_1 = \left\{([X_0:X_1],[Y_0:Y_1])\in\mathbb{P}^1_0\times\mathbb{P}^1_1 : X_1\neq 0,\ Y_1\neq 0\right\}. \end{aligned} \tag{3.106}$$

These correspond to the following matrices $\mathcal{Z}_{\mathcal{U}_i}$, that easily allows us to read the coordinates on the big cells:

$$\mathcal{Z}_{\mathcal{U}_1} := \left(\begin{array}{cc||cc} 1 & x_1 & 0 & \xi_1 \\ \hline\hline 0 & \eta_1 & 1 & y_1 \end{array}\right), \qquad \mathcal{Z}_{\mathcal{U}_2} := \left(\begin{array}{cc||cc} x_2 & 1 & 0 & \xi_2 \\ \hline\hline \eta_2 & 0 & 1 & y_2 \end{array}\right), \tag{3.107}$$

$$\mathcal{Z}_{\mathcal{U}_3} := \left(\begin{array}{cc||cc} 1 & x_3 & \xi_3 & 0 \\ \hline\hline 0 & \eta_3 & y_3 & 1 \end{array}\right), \qquad \mathcal{Z}_{\mathcal{U}_4} := \left(\begin{array}{cc||cc} x_4 & 1 & \xi_4 & 0 \\ \hline\hline \eta_4 & 0 & y_4 & 1 \end{array}\right). \tag{3.108}$$

By following the procedure illustrated above or by (allowed!) rows and columns operations on the $\mathcal{Z}_{\mathcal{U}_i}$ we can find the transition rules of the coordinates between the various charts,

$$\mathcal{U}_1\cap\mathcal{U}_2 \rightsquigarrow \begin{cases} x_1 = x_2^{-1} \\ \xi_1 = \xi_2 x_2^{-1} \\ \eta_1 = -\eta_2 x_2^{-1} \\ y_1 = y_2 + \xi_2\eta_2 x_2^{-1} \end{cases} \qquad \mathcal{U}_1\cap\mathcal{U}_3 \rightsquigarrow \begin{cases} x_1 = x_3 - \xi_3\eta_3 y_3^{-1} \\ \xi_1 = -\xi_3 y_3^{-1} \\ \eta_1 = \eta_3 y_3^{-1} \\ y_1 = y_3^{-1} \end{cases}$$

$$\mathcal{U}_1\cap\mathcal{U}_4 \rightsquigarrow \begin{cases} x_1 = x_4^{-1} + \xi_4\eta_4 x_4^{-2} y_4^{-1} \\ \xi_1 = -\xi_4 x_4^{-1} y_4^{-1} \\ \eta_1 = -\eta_4 x_4^{-1} y_4^{-1} \\ y_1 = y_4^{-1} - \xi_4\eta_4 x_4^{-1} y_4^{-2} \end{cases} \qquad \mathcal{U}_2\cap\mathcal{U}_3 \rightsquigarrow \begin{cases} x_2 = x_3^{-1} + \xi_3\eta_3 x_3^{-2} y_3^{-1} \\ \xi_2 = -\xi_3 x_3^{-1} y_3^{-1} \\ \eta_2 = -\eta_3 x_3^{-1} y_3^{-1} \\ y_2 = y_3^{-1} - \xi_3\eta_3 x_3^{-1} y_3^{-2} \end{cases}$$

$$\mathcal{U}_2\cap\mathcal{U}_4 \rightsquigarrow \begin{cases} x_2 = x_4 - \xi_4\eta_4 y_4^{-1} \\ \xi_2 = -\xi_4 y_4^{-1} \\ \eta_2 = \eta_4 y_4^{-1} \\ y_2 = y_4^{-1} \end{cases} \qquad \mathcal{U}_3\cap\mathcal{U}_4 \rightsquigarrow \begin{cases} x_3 = x_4^{-1} \\ \xi_3 = \xi_4 x_4^{-1} \\ \eta_3 = -\eta_4 x_4^{-1} \\ y_3 = y_4 + \xi_4\eta_4 x_4^{-1} \end{cases} \tag{3.109}$$

together with their inverses. By looking at these transformation rules, we therefore have that in the isomorphism above the class is represented by $(1,1)\in\mathbb{C}\oplus\mathbb{C}$ and the cocycles representing ω

are thus given by $\omega = (\omega_{12}, \omega_{13}, \omega_{14}, \omega_{23}, \omega_{24}, \omega_{34})$, where the ω_{ij} are (in tensor notation)

$$
\begin{aligned}
\omega_{12} &= \frac{\xi_2 \eta_2}{x_2} \otimes \partial_{y_1}, & \omega_{13} &= -\frac{\xi_3 \eta_3}{y_3} \otimes \partial_{x_1}, \\
\omega_{14} &= +\frac{\xi_4 \eta_4}{x_4^2 y_4} \otimes \partial_{x_1} - \frac{\xi_4 \eta_4}{x_4 y_4^2} \otimes \partial_{y_1}, & \omega_{23} &= +\frac{\xi_3 \eta_3}{x_3^2 y_3} \otimes \partial_{x_2} - \frac{\xi_3 \eta_3}{x_3 y_3^2} \otimes \partial_{y_2}, \\
\omega_{24} &= -\frac{\xi_4 \eta_4}{y_4} \otimes \partial_{x_2}, & \omega_{34} &= +\frac{\xi_1 \eta_4}{x_4} \otimes \partial_{y_3}. \qquad (3.110)
\end{aligned}
$$

One can get to the same result also by another kind of computation. First we observe that $H^1(\mathcal{O}_{\mathbb{P}^1_0 \times \mathbb{P}^1_1}(-2,0)) \oplus H^1(\mathcal{O}_{\mathbb{P}^1_0 \times \mathbb{P}^1_1}(0,-2))$ is generated by two elements

$$
H^1(\mathcal{O}_{\mathbb{P}^1_0 \times \mathbb{P}^1_1}(-2,0)) \oplus H^1(\mathcal{O}_{\mathbb{P}^1_0 \times \mathbb{P}^1_1}(0,-2)) \cong \left\langle \frac{1}{X_0 X_1} \boxtimes 1,\ 1 \boxtimes \frac{1}{Y_0 Y_1} \right\rangle_{\mathcal{O}_{\mathbb{P}^1_0 \otimes \mathbb{P}^1_1}}, \qquad (3.111)
$$

thus corresponding, in general, to a pair $(\ell_1, \ell_2) \in \mathbb{C} \oplus \mathbb{C}$. We can look at these generators in the intersections, keeping in mind that $\mathcal{F}_G \cong \Pi \mathcal{O}_{\mathbb{P}^1_0 \times \mathbb{P}^1_1}(-1,-1) \oplus \Pi \mathcal{O}_{\mathbb{P}^1_0 \times \mathbb{P}^1_1}(-1,-1)$, in order to identify the cocycles that enter in the transition functions:

$\mathcal{U}_1 \cap \mathcal{U}_2$: The following identifications can be made

$$
\begin{aligned}
\xi_1 &= \Pi \left(\frac{1}{X_0} \boxtimes \frac{1}{Y_0}, 0 \right), & \eta_1 &= \Pi \left(0, \frac{1}{X_0} \boxtimes \frac{1}{Y_0} \right), \\
\xi_2 &= \Pi \left(\frac{1}{X_1} \boxtimes \frac{1}{Y_0}, 0 \right), & \eta_2 &= \Pi \left(0, \frac{1}{X_1} \boxtimes \frac{1}{Y_0} \right). \qquad (3.112)
\end{aligned}
$$

Notice these gives the transition functions above between ξ_1 and ξ_2 and between η_1 and η_2. Now, in the intersection $\mathcal{U}_1 \cap \mathcal{U}_2$ only the bit $H^1(\mathcal{O}_{\mathbb{P}^1_0 \times \mathbb{P}^1_1}(-2,0))$ gives contributions and we have therefore

$$
\begin{aligned}
\omega_{12} &= \pm \ell_1 \left(\frac{1}{X_0 X_1} \boxtimes 1 \right) = \pm \ell_1 \left(\frac{1}{X_0 X_1} \boxtimes \frac{Y_0^2}{Y_0^2} \right) = \pm \ell_1 \left(\frac{1}{X_0 X_1} \boxtimes \frac{1}{Y_0^2} \right) \otimes \partial_{y_1} \\
&= \pm \ell_1 \left(\frac{X_1}{X_0} \right) \left(\Pi \left(\frac{1}{X_1} \boxtimes \frac{1}{Y_0}, 0 \right) \odot \Pi \left(0, \frac{1}{X_1} \boxtimes \frac{1}{Y_0} \right) \right) \otimes \partial_{y_1} \\
&= \pm \ell_1 \frac{\xi_2 \eta_2}{x_2} \otimes \partial_{y_1} \qquad (3.113)
\end{aligned}
$$

where we have denoted by $\odot$ the (super) symmetric product of the two local sections on $\mathcal{F}_G$, as represented above.

$\mathcal{U}_1 \cap \mathcal{U}_3$: This time we have a contribution from $H^1(\mathcal{O}_{\mathbb{P}^1_0 \times \mathbb{P}^1_1}(0,-2))$ and, therefore, we have to deal with $\omega_{13} = \ell_2 \left(1 \boxtimes 1/Y_0 Y_1\right)$. By a completely analogous treatment to the one above, we find that

$$
\omega_{13} = \pm \ell_2 \left(1 \boxtimes \frac{1}{Y_0 Y_1} \right) = \pm \ell_2 \frac{\xi_3 \eta_3}{y_3} \otimes \partial_{x_1}. \qquad (3.114)
$$

$\mathcal{U}_1 \cap \mathcal{U}_4$: In this case we have both the contributions, therefore

$$
\omega_{14} = \pm \ell_1 \left(\frac{1}{X_0 X_1} \boxtimes 1 \right) \pm \ell_2 \left(1 \boxtimes \frac{1}{Y_0 Y_1} \right), \qquad (3.115)
$$

so that by analogous manipulations as the one above we find

$$
\omega_{14} = \pm \ell_1 \frac{\xi_4 \eta_4}{x_4 y_4^2} \otimes \partial_{y_1} \pm \ell_2 \frac{\xi_4 \eta_4}{x_4^2 y_4} \otimes \partial_{x_1}. \qquad (3.116)
$$

All the other ω_{ij} are identified in the same way and enter one of these categories.
To conclude, imposing the cocycle conditions we fix the various signs of the ℓ_1 and ℓ_2 above, that agree with the one we found above by looking at the coordinates of the big cells: choosing $(\ell_1 = 1, \ell_2 = 1)$ - this can always be done up to a change of coordinates -, we regain the same even transition functions as above.
In other words, using the theorem classifying the complex supermanifold of dimension $n|2$, we have that $G(1|1;\mathbb{C}^{2|2})$ can be defined up to isomorphism as follows

Definition 3.5 ($G(1|1;\mathbb{C}^{2|2})$ as a Non-Projected Supermanifold)**.** *The super Grassmnannian $G(1|1;\mathbb{C}^{2|2})$ can be defined up to isomorphism as the $2|2$ dimensional supermanifold characterised by the triple $(\mathbb{P}^1_0 \times \mathbb{P}^1_1, \mathcal{F}_G, \omega_G)$ where $\mathcal{F}_G = \Pi\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(-1,-1) \oplus \Pi\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(-1,-1)$ and where $\omega_G = (\ell_1, \ell_2)$, with $\ell_1 \neq 0$ and $\ell_2 \neq 0$, in the isomorphism $\omega_G \in H^1(\mathcal{T}_{\mathbb{P}^1_0\times\mathbb{P}^1_1} \otimes Sym^2\mathcal{F}_G) \cong \mathbb{C}\oplus\mathbb{C}$.*

Actually, apart from projective superspaces, super Grassmannians are in general non-projected: the case of $G(1|1;\mathbb{C}^{2|2})$ we treated, being of odd dimension 2, allows a detailed study through its obstruction class and it is the first non-trivial example of non-projected super Grassmannian.
Now, we add a bit of knowledge about the geometry of $G(1|1;\mathbb{C}^{2|2})$, showing also that it is *not* a projective supermanifold.

Theorem 3.15 ($G(1|1;\mathbb{C}^{2|2})$ is Non-Projective)**.** *Let $G(1|1;\mathbb{C}^{2|2})$ be super Grassmannian defined as above. Then $G(1|1;\mathbb{C}^{2|2})$ is non-projective.*

Proof. In order to prove the non-projectivity of $G := G(1|1;\mathbb{C}^{2|2})$ we consider the following short exact sequence, modelled out of (3.78)

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(-2,-2) \longrightarrow \mathcal{O}^*_{G,0} \longrightarrow \mathcal{O}^*_{\mathbb{P}^1_0\times\mathbb{P}^1_1} \longrightarrow 0. \tag{3.117}$$

Taking into account ordinary results in algebraic geometry, one has $H^0(\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(-2,-2)) = 0 = H^1(\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(-2,-2))$, whereas $H^2(\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(-2,-2)) \cong \mathbb{C}$. Likewise, one finds $H^0(\mathcal{O}^*_{\mathbb{P}^1_0\times\mathbb{P}^1_1}) \cong \mathbb{C}^*$ and $\mathrm{Pic}(\mathbb{P}^1_0 \times \mathbb{P}^1_1) = H^1(\mathcal{O}^*_{\mathbb{P}^1_0\times\mathbb{P}^1_1}) \cong \mathbb{Z}\oplus\mathbb{Z}$, using the ordinary exponential exact sequence. This is enough to realise that the long exact sequence in cohomology induced by the sequence above splits in two exact sequences. The first one gives an isormorphism $H^0(\mathcal{O}_{G,0}) \cong \mathbb{C}^*$. The second one instead reads

$$0 \longrightarrow H^1(\mathcal{O}^*_{G,0}) \longrightarrow \mathrm{Pic}(\mathbb{P}^1_0 \times \mathbb{P}^1_1) \cong \mathbb{Z}\oplus\mathbb{Z} \longrightarrow H^2(\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(-2,-2)) \cong \mathbb{C} \longrightarrow \cdots \tag{3.118}$$

This says that in order to establish the fate of the cohomology group $H^1(\mathcal{O}^*_{G,0})$ we have to look at the boundary map in cohomology $\delta : \mathrm{Pic}(\mathbb{P}^1_0 \times \mathbb{P}^1_1) \to H^2(\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(-2,-2))$. Let us consider the following diagram of cochain complexes

$$\begin{tikzcd} C^2(\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(-2,-2)) \arrow[r, tail] & C^2(\mathcal{O}^*_{G,0}) \\ & C^1(\mathcal{O}^*_{G,0}) \arrow[u] \arrow[r, two heads] & C^1(\mathcal{O}^*_{\mathbb{P}^1_0\times\mathbb{P}^1_1}), \end{tikzcd} \tag{3.119}$$

obtained by combining (3.78) with the Čech cochain complexes of the sheaves that appear.
Now, since $\langle \mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(1,0), \mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(0,1)\rangle_{\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}} \cong \mathrm{Pic}(\mathbb{P}^1_0 \times \mathbb{P}^1_1)$, given the usual cover of $\mathbb{P}^1_0 \times \mathbb{P}^1_1$ by the open sets $\mathcal{U}_i$ as above, $\mathcal{O}^*_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(1,0)$ can be represented by six cocycles $g_{ij} \in Z^1(\mathcal{U}_i \cap \mathcal{U}_j, \mathcal{O}^*_{\mathbb{P}^1_0\times\mathbb{P}^1_1})$. Explicitly, these cocycles are the transition functions of the line bundle

$$\mathcal{O}^*_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(1,0) \longleftrightarrow \left\{ g_{12} = \frac{X_1}{X_0}, g_{13} = 1, g_{14} = \frac{X_1}{X_0}, g_{23} = \frac{X_0}{X_1}, g_{24} = 1, g_{34} = \frac{X_1}{X_0} \right\},$$

where, with an abuse of notation, we forget about the second bit of the external tensor product, which is the identity. Since the map $j : C^1(\mathcal{O}^*_{G,0}) \to C^1(\mathcal{O}^*_{\mathbb{P}^1_0\times\mathbb{P}^1_1})$ is surjective, these cocycles are images of elements in $C^1(\mathcal{O}^*_{G,0})$, but since j is induced by the inclusion of the reduced variety $\mathbb{P}^1_0\times\mathbb{P}^1_1$

into G, then the cochains in $C^1(\mathcal{O}^*_{G,0})$ are exactly the $\{g_{ij}\}_{ij\in I}$ we have written above. Notice these are no longer cocycles in $\mathcal{O}^*_{G,0}$. Now, using the Čech coboundary map $\delta(j^*\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(1,0))$ over G, one finds, for example, the following relation:

$$g_{12}\cdot g_{23}\cdot g_{31}\lfloor_{\mathcal{U}_1\cap\mathcal{U}_2\cap\mathcal{U}_3} = 1\boxtimes 1 + \frac{1}{X_0X_1}\boxtimes\frac{1}{Y_0Y_1}. \tag{3.120}$$

Indeed, by looking at the affine coordinates in the big cells, these reads $x_2x_3 = 1+\frac{\xi_2\eta_2}{x_2y_2}$ and setting as above

$$\xi_2 = \Pi\left(\frac{1}{X_1}\boxtimes\frac{1}{Y_0},0\right), \qquad \eta_2 = \Pi\left(0,\frac{1}{X_1}\boxtimes\frac{1}{Y_0}\right), \tag{3.121}$$

and taking their (super) symmetric product we find that $\frac{\xi_2\eta_2}{x_2y_2} = \frac{1}{X_0X_1}\boxtimes\frac{1}{Y_0Y_1}$. Now, by commutativity, this element is in the kernel of the map $j : C^2(\mathcal{O}^*_{G,0}) \to C^2(\mathcal{O}^*_{\mathbb{P}^1_0\times\mathbb{P}^1_1})$, that equals the image of the map $i : C^2(\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(-2,-2)) \to C^2(\mathcal{O}^*_{G,0})$, therefore there exists an element $N\in C^2(\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(-2,-2))$ such that $i(N) = 1\boxtimes 1+\frac{1}{X_0X_1}\boxtimes\frac{1}{Y_0Y_1}$ and it is a cocycle. Considering that the map i is induced by the map $\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(-2,-2)\ni a\boxtimes b\mapsto 1\boxtimes 1+a\boxtimes b\in\mathcal{O}^*_{G,0}$, we have that the element $1\boxtimes+\frac{1}{X_0X_1}\boxtimes\frac{1}{Y_0Y_1}$ is the image of $1\frac{1}{X_0X_1}\boxtimes\frac{1}{Y_0Y_1}$ through the map i. By symmetry, the same applies to the second generator of $\mathrm{Pic}(\mathbb{P}^1_0\times\mathbb{P}^1_1)$, given by $\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(0,1)$, so that the map $\delta : \mathrm{Pic}(\mathbb{P}^1_0\times\mathbb{P}^1_1)\cong\mathbb{Z}\oplus\mathbb{Z}\to H^2(\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(-2,-2))\cong\mathbb{C}$ reads $\mathbb{Z}\oplus\mathbb{Z}\ni(a,b)\longmapsto a+b\in\mathbb{C}$. By exactness, it is then clear that the only invertible sheaves on $\mathbb{P}^1_0\times\mathbb{P}^1_1$ that lift to the whole G are those of the kind $\mathcal{O}_{\mathbb{P}^1_0\times\mathbb{P}^1_1}(a,-a)$, as the composition of the maps gives $(a,-a)\mapsto(a,-a)\mapsto a-a=0$ as it should. As these invertible sheaves have no cohomology, they cannot give any embedding in projective superspaces and this completes the proof. □

Notice that the theorem above says that we have $\mathrm{Pic}(\mathbb{P}^1_0\times\mathbb{P}^1_1)\neq 0$ (actually $\mathrm{Pic}(\mathbb{P}^1_0\times\mathbb{P}^1_1)\cong\mathbb{Z}$), but still there are no *ample* invertible sheaves that allow for an embedding of $G(1|1;\mathbb{C}^{2|2})$ into some projective superspaces.
This has a fundamental consequence, as non-projectivity is not confined to this particular super Grassmannian only.

Theorem 3.16 (Super Grassmannians are Non-Projective)**.** *The super Grassmannian space $G(a|b;\mathbb{C}^{m|n})$ for $0<a<n$ and $0<b<m$ is non-projective.*

Proof. Following [41] it is enough to observe that the inclusion $\mathbb{C}^{2|2}\subset\mathbb{C}^{a+1|b+1}$ induces an inclusion $G(1|1;\mathbb{C}^{2|2})\hookrightarrow G(1|1;\mathbb{C}^{a+1|b+1})$. This last super Grassmannian is isomorphic, as for the usual Grassmannians, to $G(a|b;(\mathbb{C}^{a+1|b+1})^*)$, that in turn embeds into $G(a|b;\mathbb{C}^{n|m})$. This leads to $G(1|1;\mathbb{C}^{2|2})\hookrightarrow G(a|b;\mathbb{C}^{n|m})$: as $G(1|1;\mathbb{C}^{2|2})$ is non-projective, so it is $G(a|b;\mathbb{C}^{n|m})$. □

This shows that, in the context of algebraic supergeometry, it is no longer true that projective superspaces represent a privileged ambient and it is therefore a substantial departure from usual complex algebraic geometry, as we have already mentioned above.

3.6 Embedding of $\mathbb{P}^2_\omega(\mathcal{F}_{\mathcal{M}})$ into Super Grassmannians

3.6.1 The universal property of super Grassmannians

The super Grassmannians have the following universal property.

Universal Property: for any superscheme $\mathcal{M}$ and any locally-free sheaf of $\mathcal{O}_{\mathcal{M}}$-modules $\mathcal{E}$ of rank $a|b$ on $\mathcal{M}$ and any vector sub superspace $V\subset H^0(\mathcal{E})$ with $V\cong\mathbb{C}^{n|m}$ such that the evaluation map $V\otimes\mathcal{O}_{\mathcal{M}}\to\mathcal{E}$ is surjective, then there exists a *unique* map $\Phi:\mathcal{M}\to G(a|b,V)$ such that the inclusion $\mathcal{E}^*\to V^*\otimes\mathcal{O}_{\mathcal{M}}$ is the pull-back of the inclusion $\mathcal{S}_G\to\mathcal{O}_G^{\oplus n|m}$ from the sequence (3.93).

In this case, once a local basis $\{e_1,\dots,e_r|f_1,\dots,f_s\}$ is fixed for $\mathcal{E}$ over some open set $\mathcal{U}$, then, over $\mathcal{U}$, the evaluation map $V\otimes\mathcal{O}_{\mathcal{M}}\to\mathcal{E}$ is defined by a $(r|s)\times(n|m)$ matrix $M_{\mathcal{U}}$ with coefficients in $\mathcal{O}_{\mathcal{M}}(\mathcal{U})$, and any reduction of $M_{\mathcal{U}}$ into a standard form of type (3.83) by means of elementary row operations, is a local representation of the map Φ.

3.6.2 The Embedding Theorem

In this subsection we will prove the following result.

Theorem 3.17. *Let $\mathcal{M} = \mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ and $\mathcal{T}_\mathcal{M}$ its tangent sheaf. Let $V = H^0(Sym^k\mathcal{T}_\mathcal{M})$. Then, for any $k \gg 0$ the evaluation map $V \otimes \mathcal{O}_\mathcal{M} \to Sym^k\mathcal{T}_\mathcal{M}$ induces an embedding $\Phi_k : \mathcal{M} \longrightarrow G(2k|2k, V)$.*

This is the main result of this chapter, showing that any non-projected non-projective $\mathcal{N} = 2$ supermanifold of the family $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ embeds into some super Grassmannian.
For the sake of notation we put $\mathcal{M} = \mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ in what follows and likewise we will refer to the structure sheaf of $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ simply as $\mathcal{O}_\mathcal{M}$.
With reference to what we have explained in the first chapter, section 1.1, we first introduce the following definition.

Definition 3.6 (The Superscheme $\mathcal{M}^{(2)}$)**.** *We call $\mathcal{M}^{(2)}$ the sub superscheme of $\mathcal{M}$ given by the pair $(\mathbb{P}^2, \mathcal{O}_{\mathcal{M}^{(2)}})$, where we have posed $\mathcal{O}_{\mathcal{M}^{(2)}} := \mathcal{O}_\mathcal{M} \big/ \mathcal{J}^2_\mathcal{M}$.*

We stress that this is *not* actually a supermanifold: indeed it fails to be locally isomorphic to any local model of the kind $\mathbb{C}^{p|q}$, and, more generally, it is locally isomorphic to an affine superscheme for some super ring. Anyway, incidentally, the reader can observe that the superscheme $\mathcal{M}^{(2)}$ behaves as an honest *commutative* scheme, since having modded out $\mathcal{J}^2_\mathcal{M}$ there is no anticommutativity left.
We characterise the geometry of $\mathcal{M}^{(2)}$ in the following lemma.

Lemma 3.4 (The Geometry of $\mathcal{M}^{(2)}$)**.** *Let $\mathcal{M}^{(2)}$ be the superscheme characterised by the pair $(\mathbb{P}^2, \mathcal{O}_{\mathcal{M}^{(2)}})$, where $\mathcal{O}_{\mathcal{M}^{(2)}} = \mathcal{O}_\mathcal{M} \big/ \mathcal{J}^2_\mathcal{M}$. Then $\mathcal{M}^{(2)}$ is projected and its structure sheaf, as a sheaf of $\mathcal{O}_{\mathbb{P}^2}$-algebras, is $\mathcal{O}_{\mathcal{M}^{(2)}} \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{F}_\mathcal{M}$.*

Proof. It is enough to observe that the parity splitting of the structure sheaf reads $\mathcal{O}_{\mathcal{M}^{(2)}} = \mathcal{O}_{\mathcal{M},0} \big/ \mathcal{J}^2_\mathcal{M} \oplus \mathcal{O}_{\mathcal{M},1} \big/ \mathcal{J}^2_\mathcal{M}$, hence the defining short exact sequence for the even part reduces to an isomorphism $\mathcal{O}^{(2)}_{\mathcal{M},0} \cong \mathcal{O}_{\mathbb{P}^2}$. As the variety has odd dimension 2, we therefore must have that the structure sheaf gets endowed with a structure of $\mathcal{O}_{\mathbb{P}^2}$-module given by $\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{F}_\mathcal{M}$, that actually coincides with the parity splitting. We observe that in the $\mathcal{O}_{\mathbb{P}^2}$-algebra $\mathcal{O}_{\mathcal{M}^{(2)}} \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{F}_\mathcal{M}$ the product $\mathcal{F}_\mathcal{M} \otimes_{\mathcal{O}_{\mathbb{P}^2}} \mathcal{F}_\mathcal{M} \to \mathcal{O}_{\mathbb{P}^2}$ is null. □

We are now ready to move towards a proof of Theorem 3.17. The first requirement is that the morphism Φ_k is well defined. This is a consequence of the following Lemma.

Lemma 3.5. *Let $\mathcal{M}$ and $\mathcal{M}^{(2)}$ be as above. The following facts hold.*

1. *The restriction maps $V \longrightarrow H^0(Sym^k\mathcal{T}_\mathcal{M}\lfloor_{\mathcal{M}^{(2)}})$ and $V \longrightarrow H^0(Sym^k\mathcal{T}_\mathcal{M}\lfloor_{\mathbb{P}^2})$ are surjective for $k \gg 0$.*
2. *The locally-free sheaf of $\mathcal{O}_\mathcal{M}$-modules $Sym^k\mathcal{T}_\mathcal{M}$ is generated by global sections, i.e. the evaluation map $V \otimes \mathcal{O}_\mathcal{M} \to Sym^k\mathcal{T}_\mathcal{M}$ is surjective, for $k \gg 0$.*

Proof. Let us consider the composition of linear maps

$$V \longrightarrow H^0(Sym^k\mathcal{T}_\mathcal{M}\lfloor_{\mathcal{M}^{(2)}}) \longrightarrow H^0(Sym^k\mathcal{T}_\mathcal{M}\lfloor_{\mathbb{P}^2}) \longrightarrow Sym^k\mathcal{T}_\mathcal{M}(x), \tag{3.122}$$

with $Sym^k\mathcal{T}_\mathcal{M}(x)$ the fibre at x. By the supercommutative version of the Nakayama Lemma - see for example lemma 4.7.1 in [63] - to prove fact 2 one has to show that for any $x \in \mathbb{P}^2$ the linear map $V \to Sym^k\mathcal{T}_\mathcal{M}(x)$ is surjective. Therefore we can reduce ourselves to show the surjectivity of all the linear maps in the composition, which will also include a proof of fact 1. For simplicity of notation we set $\mathcal{E}_k := Sym^k\mathcal{T}_\mathcal{M}$. To set the surjectivity of the last map, we observe that, since $\mathcal{T}_\mathcal{M}\lfloor_{\mathbb{P}^2} \cong \mathcal{T}_{\mathbb{P}^2} \oplus \mathcal{F}^*_\mathcal{M}$, one has

$$\begin{aligned} \mathcal{E}_k\lfloor_{\mathbb{P}^2} &\cong Sym^k(\mathcal{T}_{\mathbb{P}^2} \oplus \mathcal{F}^*_\mathcal{M}) \\ &= (Sym^k\mathcal{T}_{\mathbb{P}^2}) \oplus (Sym^{k-1}\mathcal{T}_{\mathbb{P}^2} \otimes \mathcal{F}^*_\mathcal{M}) \oplus (Sym^{k-2}\mathcal{T}_{\mathbb{P}^2} \otimes Sym^2\mathcal{F}^*_\mathcal{M}), \end{aligned} \tag{3.123}$$

all the other summands being 0, since $\mathcal{F}_\mathcal{M} = \Pi E$ for some vector bundle E of rank 2 and $Sym^i\mathcal{F}_\mathcal{M} = \Pi^i \bigwedge^i E$.

Now one can use the well-known ampleness of the vector bundle $\mathcal{T}_{\mathbb{P}^2}$ (see [34] for the definition of an ample vector bundle in algebraic geometry) to conclude that all the higher cohomology groups $H^i(Sym^k\mathcal{T}_{\mathbb{P}^2}(-i))$, $H^i(Sym^{k-1}\mathcal{T}_{\mathbb{P}^2}\otimes\mathcal{F}^*_{\mathcal{M}}(-i))$, $H^i(Sym^{k-2}\mathcal{T}_{\mathbb{P}^2}\otimes Sym^2\mathcal{F}^*_{\mathcal{M}}(-i))$ vanish for $k\gg 0$, and hence all these vector bundles are generated by global sections, since they are 0-regular.

Alternatively, one can use the exact sequences

$$0\longrightarrow(Sym^{m-1}\mathcal{O}^{\oplus 3}_{\mathbb{P}^2})(m-1)\longrightarrow(Sym^m\mathcal{O}^{\oplus 3}_{\mathbb{P}^2})(m)\longrightarrow Sym^m\mathcal{T}_{\mathbb{P}^2}\longrightarrow 0,\qquad(3.124)$$

deduced from the Euler sequence, tensor them with $Sym^j\mathcal{F}_{\mathcal{M}}$ for $j=0,1,2$ and use the fact that $H^i(\mathcal{F}_{\mathcal{M}}(m))=0$ for any $i>0$ and that $\mathcal{F}_{\mathcal{M}}(m)$ is generated by global sections, for any $m\gg 0$, to deduce the same conclusions for $Sym^m\mathcal{T}_{\mathbb{P}^2}\otimes Sym^j\mathcal{F}_{\mathcal{M}}$.

Recall the exact sequence

$$0\longrightarrow\mathcal{E}_k\otimes\mathcal{J}_{\mathcal{M}}\longrightarrow\mathcal{E}_k\longrightarrow\mathcal{E}_k\lfloor_{\mathbb{P}^2}\longrightarrow 0,\qquad(3.125)$$

and observe that, as $\mathcal{J}^3_{\mathcal{M}}=0$, one has that $\mathcal{J}_{\mathcal{M}}$ is a $\mathcal{O}_{\mathcal{M}}/\mathcal{J}^2_{\mathcal{M}}$-module, i.e. a $\mathcal{O}_{\mathcal{M}^{(2)}}$-module. As such, by Lemma 3.4 one also knows that $\mathcal{J}_{\mathcal{M}}$, and hence also $\mathcal{E}_k\otimes\mathcal{J}_{\mathcal{M}}$, has a structure of a $\mathcal{O}_{\mathbb{P}^2}$-module, given as

$$\mathcal{E}_k\otimes\mathcal{J}_{\mathcal{M}}\cong(\mathcal{E}_k\lfloor_{\mathbb{P}^2}\otimes Sym^2\mathcal{F})\oplus(\mathcal{E}_k\lfloor_{\mathbb{P}^2}\otimes\mathcal{F})\cong(\mathcal{E}_k\lfloor_{\mathbb{P}^2}(-3))\oplus(\mathcal{E}_k\lfloor_{\mathbb{P}^2}\otimes\mathcal{F}).\qquad(3.126)$$

Similarly, let us consider the exact sequence

$$0\longrightarrow\mathcal{E}_k\otimes\mathcal{J}^2_{\mathcal{M}}\longrightarrow\mathcal{E}_k\longrightarrow\mathcal{E}_k\lfloor_{\mathcal{M}^{(2)}}\longrightarrow 0,\qquad(3.127)$$

where $\mathcal{E}_k\otimes\mathcal{J}^2_{\mathcal{M}}\cong\mathcal{E}_k\lfloor_{\mathbb{P}^2}(-3)$ is a $\mathcal{O}_{\mathbb{P}^2}$-module. Similarly as above, one can show that $H^1(\mathcal{E}_k\lfloor_{\mathbb{P}^2}\otimes\mathcal{F})=0$ and $H^1(\mathcal{E}_k\lfloor_{\mathbb{P}^2}(-3))=0$ for $k\gg 0$, hence one has that $H^0(\mathcal{E}_k)\to H^0(\mathcal{E}_k\lfloor_{\mathcal{M}^{(2)}})$ and $H^0(\mathcal{E}_k)\to H^0(\mathcal{E}\lfloor_{\mathbb{P}^2})$ are surjective for $k\gg 0$. □

This Lemma allows us to prove our main theorem 3.17.

Proof of Theorem 3.17. The preliminary results above, show that for $k\gg 0$ the morphism $\Phi_k:\mathcal{M}\to G(n|m,V)$ is globally defined, with $n|m$ the rank of the sheaf $Sym^k\mathcal{T}_{\mathcal{M}}$.
Note that $n|m$ can be computed from the formula (3.123) for the restriction of $Sym^k\mathcal{T}_{\mathcal{M}}$ to $\mathbb{P}^2$, where its even and odd summands are, respectively, $(Sym^k\mathcal{T}_{\mathcal{M}})_0=Sym^k\mathcal{T}_{\mathbb{P}^2}\oplus(Sym^{k-2}\mathcal{T}_{\mathbb{P}^2}\otimes Sym^2\mathcal{F}^*_{\mathcal{M}})$ and $(Sym^k\mathcal{T}_{\mathcal{M}})_1=Sym^{k-1}\mathcal{T}_{\mathbb{P}^2}\otimes\mathcal{F}^*_{\mathcal{M}}$, from which we get $n|m=2k|2k$.

At the level of the reduced manifolds, Φ_k defines a morphism $\Phi_k\lfloor_{\mathbb{P}^2}=(\phi_0,\phi_1)\colon\mathbb{P}^2\longrightarrow G_0\times G_1$ which is associated to the surjections

$$\begin{aligned}V_0\otimes\mathcal{O}_{\mathbb{P}^2}&\twoheadrightarrow H^0((\mathcal{E}_k)_0\lfloor_{\mathbb{P}^2})\otimes\mathcal{O}_{\mathbb{P}^2}\twoheadrightarrow(\mathcal{E}_k)_0\lfloor_{\mathbb{P}^2}\\ V_1\otimes\mathcal{O}_{\mathbb{P}^2}&\twoheadrightarrow H^0((\mathcal{E}_k)_0\lfloor_{\mathbb{P}^2})\otimes\mathcal{O}_{\mathbb{P}^2}\twoheadrightarrow(\mathcal{E}_k)_1\lfloor_{\mathbb{P}^2}.\end{aligned}$$

They define embeddings of $\mathbb{P}^2$ into the ordinary Grassmannians $G_0=G(2k;V_0)$ and $G_1=G(2k;V_1)$ for $k\gg 0$ by well-known vanishing theorems in projective algebraic geometry.
Indeed, for any vector bundle E on $\mathbb{P}^2$ the evaluation map $H^0(E)\to E$ defines an embedding into a Grassmannian if, when composed with the restriction $E\to E/\mathfrak{m}^2_xE$, for any $x\in\mathbb{P}^2$ and $\mathfrak{m}_x$ the maximal ideal in the stalk $\mathcal{O}_{\mathbb{P}^2,x}$, one gets a surjection $H^0(E)\to H^0(E/\mathfrak{m}^2_xE)$. This map is part of the exact sequence of cohomology associated to

$$0\longrightarrow\mathfrak{m}^2_xE\longrightarrow E\longrightarrow E\big/\mathfrak{m}^2_xE\longrightarrow 0,$$

so the surjection above follows if $H^1(\mathfrak{m}^2_xE)=0$. In our case E is either $E=\mathcal{E}_0\lfloor_{\mathbb{P}^2}=Sym^k\mathcal{T}_{\mathbb{P}^2}\oplus Sym^{k-2}\mathcal{T}_{\mathbb{P}^2}(-3)$ or $E=\mathcal{E}_1\lfloor_{\mathbb{P}^2}=Sym^{k-1}\mathcal{T}_{\mathbb{P}^2}\otimes\mathcal{F}_{\mathcal{M}}$, and the vanishing of $H^1(\mathfrak{m}^2_xE)=0$ can be shown in both cases by means of the Euler sequence, by the same arguments as above.
In conclusion, we have shown that $\Phi_k:\mathcal{M}\to G(2k|2k,V)$ is injective at the level of geometrical points.

A similar criterion as in the ordinary algebraic geometry case applies to show the injectivity of the tangent map $d\Phi_k(x) : \mathcal{T}_{\mathcal{M}}(x) \to \mathcal{T}_{G(2k|2k,V)}(x)$ at any geometrical point $x \in \mathbb{P}^2$. The maximal ideal of x in $\mathcal{O}_{\mathcal{M},x}$ is $\mathfrak{M}_x := \mathfrak{m}_x + \mathcal{J}_{\mathcal{M},x}$ where $\mathfrak{m}_x$ is the ordinary reduced part and $\mathcal{J}_{\mathcal{M},x}$ is the nilpotent part (notice that $\mathfrak{M}_x = \ker(\mathcal{O}_{\mathcal{M},x} \to \mathbb{C})$), and one can define the sub superscheme $\mathcal{V}_x$ of $\mathcal{M}$ with reduced manifold $\{x\}$ and structure sheaf $\mathcal{O}_{\mathcal{M},x}/\mathfrak{M}_x^2$. Note that $(\mathfrak{M}_x^2)_0 = \mathfrak{m}_x^2 + \mathcal{J}_{\mathcal{M},x}^2$ and $(\mathfrak{M}_x)_1 = \mathfrak{m}_x \mathcal{J}_{\mathcal{M},x}$, from which it follows

$$\mathcal{O}_{\mathcal{M},x} \big/ \mathfrak{M}_x^2 \cong \mathcal{O}_{\mathbb{P}^2} \big/ \mathfrak{m}_x^2 \oplus \left(\mathcal{J}_{\mathcal{M},x} \big/ \mathfrak{m}_x \mathcal{J}_{\mathcal{M},x} \right) = \mathcal{O}_{\mathbb{P}^2} \big/ m_x^2 \oplus \mathcal{F}_{\mathcal{M}}(x) . \tag{3.128}$$

Note also that the tangent space of the superscheme $\mathcal{V}_x = (x, \mathcal{O}_{\mathcal{M},x}/\mathfrak{M}_x^2)$ is the same as the tangent space $\mathcal{T}_{\mathcal{M}}(x) = (\mathfrak{M}_x/\mathfrak{M}_x^2)^*$. From these observations one gets the analogous result as in the classical case that the surjectivity of the restriction map $H^0(\mathcal{E}_k) \to H^0(\mathcal{E}_k \otimes \mathcal{O}_{\mathcal{M},x}/\mathfrak{M}_x^2) = H^0(\mathcal{E}_k/\mathfrak{M}_x^2\mathcal{E}_k)$ ensures the injectivity of the tangent map $d\Phi_k$. Moreover observe that the superscheme embedding $\mathcal{V}_x \to \mathcal{M}$ factorises through $\mathcal{M}^{(2)}$, as $\mathcal{O}_{\mathcal{M},x}/\mathfrak{M}_x^2$ is also an $\mathcal{O}_{\mathcal{M}}/\mathcal{J}_{\mathcal{M}}^2$-module. Then the restriction map factorises as follows

$$H^0(\mathcal{E}_k) \twoheadrightarrow H^0(\mathcal{E}_k\lfloor_{\mathcal{M}^{(2)}}) \longrightarrow H^0(\mathcal{E}_k/\mathfrak{M}_x^2\mathcal{E}_k),$$

and we will show that the second map is surjective as well, using the fact that $\mathcal{E}_k\lfloor_{\mathcal{M}^{(2)}}$ is a $\mathcal{O}_{\mathbb{P}^2}$-module and by applying similar arguments as above, based on the vanishing of the higher cohomology of $H^i(\mathbb{P}^2, \mathcal{G}(k))$, with $\mathcal{G}$ any fixed coherent sheaf, for $k \gg 0$. Indeed in our case we have $\mathcal{E}_k\lfloor_{\mathcal{M}^{(2)}} \cong \mathcal{E}_k\lfloor_{\mathbb{P}^2} \oplus (\mathcal{E}_k\lfloor_{\mathbb{P}^2} \otimes \mathcal{F}_{\mathcal{M}})$ as a $\mathcal{O}_{\mathbb{P}^2}$- module, so the decomposition (3.123) still applies to give the structure of $\mathcal{E}_k\lfloor_{\mathcal{M}^{(2)}}$ as a $\mathcal{O}_{\mathbb{P}^2}$- module. Setting $\overline{\mathfrak{M}_x^2}$ the ideal sheaf of V_x in $\mathcal{M}^{(2)}$, one has the exact sequence

$$0 \longrightarrow \overline{\mathfrak{M}_x^2}\mathcal{E}_k\lfloor_{\mathcal{M}^{(2)}} \longrightarrow \mathcal{E}_k|_{\mathcal{M}^{(2)}} \longrightarrow \mathcal{E}_k/\mathfrak{M}_x^2\mathcal{E}_k \longrightarrow 0,$$

so we are left to prove $H^1(\overline{\mathfrak{M}_x^2}\mathcal{E}_k\lfloor_{\mathcal{M}^{(2)}}) = 0$ for $k \gg 0$. Now $\overline{\mathfrak{M}_x^2}\mathcal{E}_k\lfloor_{\mathcal{M}^{(2)}} = \overline{\mathfrak{M}_x^2}\mathcal{E}_k\lfloor_{\mathbb{P}^2} \oplus (\overline{\mathfrak{M}_x^2}\mathcal{E}_k\lfloor_{\mathbb{P}^2} \otimes \mathcal{F}_{\mathcal{M}})$ as a $\mathcal{O}_{\mathbb{P}^2}$-module, therefore the decomposition (3.123) and the Euler sequences (3.124) apply to our case, showing that $H^1(\overline{\mathfrak{M}_x^2}\mathcal{E}_k|_{\mathcal{M}^{(2)}}) = 0$ holds because of the vanishing of the higher cohomology groups $H^i(\mathbb{P}^2, \mathcal{G}(k))$ for any $\mathcal{G}$ coherent sheaf and $k \gg 0$, as a consequence of Serre's theorem (see [33], Theorem 5.2, page 228). □

In the following remarks we stress some limitations and possible generalisations of the previous theorem. Here, with abuse of notation, we write $\mathcal{F}_{\mathcal{M}}$ for the parity changed version of the fermionic sheaf $\Pi\mathcal{F}_{\mathcal{M}}$, which therefore becomes a locally-free sheaf of $\mathcal{O}_{\mathcal{M}_{red}}$-modules of rank $2|0$.

Remark 3.1. Theorem 3.17 is not effective, since it does not give any estimate on k and on the super dimension of $V = H^0(\mathcal{E}_k)$ and hence it does not identify the target super Grassmannian of the embedding Φ_k. In fact k depends heavily on the choice of $\mathcal{F}_{\mathcal{M}}$. However, it seems possible to calculate a uniform k and $\dim V$ under some boundedness conditions on $\mathcal{F}_{\mathcal{M}}$, such as $\mathcal{F}_{\mathcal{M}}^*$ globally-generated, or $\mathcal{F}_{\mathcal{M}}$ semistable.

Remark 3.2. If one wants to generalise the result of Theorem 3.17 to other non-projected supermanifolds, with reduced manifold $\mathcal{M}_{red}$ with $\dim \mathcal{M}_{red} \geqslant 2$, then the tangent sheaf $\mathcal{T}_{\mathcal{M}_{red}}$ will not in general be ample (this happens only for $\mathcal{M}_{red}$ a projective space, by a celebrated theorem of S. Mori, [43]) and therefore one faces the problem of finding a suitable ample locally-free sheaf of $\mathcal{O}_{\mathcal{M}_{red}}$-modules E on $\mathcal{M}_{red}$ that *can be extended* to a locally-free sheaf $\mathcal{E}$ on $\mathcal{M}$. This is a delicate problem that is certainly worth to study.

Before we go on to the next section, in view of the first remark above, we propose the following

Problem. *Find a* fixed *super Grassmannian* $G = G(2k|2k, V)$, i.e. *a uniform* k *and* $\dim V$, *so that* $\mathcal{M} = \mathbb{P}_\omega^{2|2}(\mathcal{F}_{\mathcal{M}})$ *can be embedded in* G, *in the case when* $\mathcal{F}_{\mathcal{M}}^*$ *is ample, or in the case when it is stable, with given* $c_1(\mathcal{F}_{\mathcal{M}}) = -3$ *and* $c_2(\mathcal{F}_{\mathcal{M}}) = n$.

In the next section we will show that $\mathcal{T}_{\mathcal{M}}$ (i.e. $k = 1$) already provides an explicit embedding into a super Grassmannian, for two significant special choices of the fermionic sheaf $\mathcal{F}_{\mathcal{M}}$.

3.7 Two Homogenous $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ and Their Embeddings

With a slight abuse of notation we will give the following definition.

Definition 3.7 (Homogeneous $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$)**.** *We say that a supermanifold of the family $\mathcal{M} = \mathbb{P}^{2|2}_\omega(\mathcal{F}_\mathcal{M})$ is* homogeneous *if its fermionic sheaf $\mathcal{F}_\mathcal{M}$ is homogeneous, i.e. $\phi^*\mathcal{F}_\mathcal{M} = \mathcal{F}_\mathcal{M}$ for any $\phi \in PGL(3)$.*

By a theorem of Van de Ven, the only homogeneous, rank 2 sheaves of $\mathcal{O}_{\mathbb{P}^2}$-modules on $\mathbb{P}^2$ are those of type $\mathcal{O}_{\mathbb{P}^2}(a) \oplus \mathcal{O}_{\mathbb{P}^2}(b)$ or $\Omega^1_{\mathbb{P}^2}(c)$, with $a, b, c \in \mathbb{Z}$ (see [62] or Theorem 2.2.2 in [49]). Since we have to impose $Sym^2\mathcal{F}_\mathcal{M} \cong \mathcal{O}_{\mathbb{P}^2}(-3)$ in order for the supermanifold to be of the kind $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$, we will put $a + b = -3$ and $c = 0$. We will also impose $\mathcal{F}^*_\mathcal{M}$ ample, in view of the Problem concluding the previous section.
We will thus consider the following choices for a *homogeneous fermionic sheaf $\mathcal{F}_\mathcal{M}$*:

- **decomposable**: $\mathcal{F}_\mathcal{M} := \Pi\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \Pi\mathcal{O}_{\mathbb{P}^2}(-2)$.
- **non-decomposable**: $\mathcal{F}_\mathcal{M} := \Pi\Omega^1_{\mathbb{P}^2}$.

Notice that taking into consideration the possibility of a non-decomposable locally-free sheaf does represent a substantial novelty. Indeed, even if in the mathematical literature this possibility is in principle taken into account, there are no actual realisations so far to the best knowledge of the author. From the physical point of view, instead, dealing with supermanifolds boiled down for long time to take care of a certain anti-commutating behaviour of some variables and the only non-trivial supermanifold that theoretical physicists have been concerned with - ordinary split projective superspaces - have indeed decomposable fermionic sheaves.

In the following subsections we deal in full detail with the two scenarios sketched above, namely the one of a decomposable sheaf and the one of a non-decomposable sheaf.

3.7.1 Decomposable Sheaf: $\mathcal{F}_\mathcal{M} = \Pi\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \Pi\mathcal{O}_{\mathbb{P}^2}(-2)$

We have the following theorem.

Theorem 3.18 (Transition functions (1))**.** *Let $\mathbb{P}^{2|2}_\omega(\mathcal{F}_\mathcal{M})$ be the non-projected supermanifold with $\mathcal{F}_\mathcal{M} = \Pi\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \Pi\mathcal{O}_{\mathbb{P}^2}(-2)$. Then, its transition functions take the following form:*

$$
\begin{array}{lllll}
\mathcal{U}_0 \cap \mathcal{U}_1 : & z_{10} = \dfrac{1}{z_{11}}, & z_{20} = \dfrac{z_{21}}{z_{11}} + \lambda\dfrac{\theta_{11}\theta_{21}}{(z_{11})^2}; & \theta_{10} = \dfrac{\theta_{11}}{z_{11}}, & \theta_{20} = \dfrac{\theta_{21}}{(z_{11})^2}; \\
\mathcal{U}_1 \cap \mathcal{U}_2 : & z_{11} = \dfrac{z_{12}}{z_{22}} + \lambda\dfrac{\theta_{12}\theta_{22}}{(z_{22})^2}, & z_{21} = \dfrac{1}{z_{22}}; & \theta_{11} = \dfrac{\theta_{12}}{z_{22}}, & \theta_{21} = \dfrac{\theta_{22}}{(z_{22})^2}; \\
\mathcal{U}_2 \cap \mathcal{U}_0 : & z_{12} = \dfrac{1}{z_{20}}, & z_{22} = \dfrac{z_{10}}{z_{20}} + \lambda\dfrac{\theta_{10}\theta_{20}}{(z_{20})^2}; & \theta_{12} = \dfrac{\theta_{10}}{z_{10}}, & \theta_{22} = \dfrac{\theta_{20}}{(z_{10})^2}.
\end{array}
\tag{3.129}
$$

Proof. It follows immediately from Theorem 3.10, taking into account the transition matrix for the given $\mathcal{F}_\mathcal{M}$, that have the form $M = \begin{pmatrix} \frac{1}{z_{01}} & 0 \\ 0 & \frac{1}{z_{01}^2} \end{pmatrix}$ on $\mathcal{U}_0 \cap \mathcal{U}_1$ and similar form on the other two intersections of the fundamental open sets. □

We now consider the tangent sheaf $\mathcal{T}_\mathcal{M}$ and we aim to calculate its global sections, representing them in a given chart, and we set up the embedding into the related super Grassmannian.

The transition functions of the tangent bundle $\mathcal{T}_\mathcal{M}$ can be determined by S_3-symmetry once one is known. Indeed if we denote by $\tilde{\mathcal{U}}_i$ for $i = 0, 1, 2$ the three affine supermanifolds given by the pairs $\tilde{\mathcal{U}}_i := (\mathcal{U}_i, \mathbb{C}[z_{1i}, z_{2i}, \theta_{1i}, \theta_{2i}])$, then, for example we have

$$
\mathrm{Jac}_{10} : \mathcal{T}_{\tilde{\mathcal{U}}_1}(\mathcal{U}_0 \cap \mathcal{U}_1) \longrightarrow \mathcal{T}_{\tilde{\mathcal{U}}_0}(\mathcal{U}_0 \cap \mathcal{U}_1) \tag{3.130}
$$

and the morphism Jac_{10} can be represented as a matrix with respect to the local basis given by the derivations $\{\partial_{z^{10}}, \partial_{z_{20}}, \partial_{\theta_{10}}, \partial_{\theta_{20}}\}$. Using the chain rule, from the transition functions of $\mathcal{T}_\mathcal{M}$ we get

$$\begin{aligned}
\partial_{z_{10}} &= -(z_{11})^2\partial_{z_{11}} + [-z_{11}z_{21} + \theta_{11}\theta_{21}]\partial_{z_{21}} - \theta_{11}z_{11}\partial_{\theta_{11}} - 2\theta_{21}z_{11}\partial_{\theta_{21}}, \\
\partial_{z_{20}} &= z_{11}\partial_{z_{21}}, \\
\partial_{\theta_{10}} &= -\theta_{21}\partial_{z_{21}} + z_{11}\partial_{\theta_{11}}, \\
\partial_{\theta_{20}} &= z_{11}\theta_{11}\partial_{z_{21}} + (z_{11})^2\partial_{\theta_{21}},
\end{aligned} \tag{3.131}$$

therefore the map Jac_{10} has the following matrix representation

$$[\mathrm{Jac}_{10}] = \left(\begin{array}{cc|cc} -(z_{11})^2 & -z_{11}z_{21} + \theta_{11}\theta_{21} & -\theta_{11}z_{11} & -2\theta_{21}z_{11} \\ 0 & z_{11} & 0 & 0 \\ \hline 0 & -\theta_{21} & z_{11} & 0 \\ 0 & z_{11}\theta_{11} & 0 & (z_{11})^2 \end{array}\right) \tag{3.132}$$

that acts on the basis represented as a column vector. Now we look for the explicit form of the global sections generating $\mathcal{T}_{\mathcal{M}}$, as to explicitly set up the embedding in a super Grassmannian. We stress that in order to keep the discussion the most general possibile (and for the sake of a future use) we will keep explicit a *parameter* $\lambda \in \mathbb{C}$ representing the cohomology class $\omega_{\mathcal{M}} \in H^1(\mathcal{T}_{\mathbb{P}^2}(-3)) \cong \mathbb{C}$, which we recall to be the same λ appearing in the transition functions provided by (3.129).

Theorem 3.19 (Generators of $H^0(\mathcal{T}_{\mathcal{M}})$)**.** *The tangent sheaf $\mathcal{T}_{\mathcal{M}}$ of $\mathcal{M}$ has $12|12$ global sections. In particular, in the local chart $\mathcal{U}_0$, a basis for $H^0(\mathcal{T}_{\mathcal{M}})$ is given by* $\mathrm{Span}_{\mathbb{C}}\{\mathcal{V}_1, \ldots, \mathcal{V}_{12}| \,\Xi_1, \ldots, \Xi_{12}\}$, *where*

$$\begin{aligned}
&\mathcal{V}_1 = \partial_{z_{10}}, \qquad \mathcal{V}_2 = \partial_{z_{20}}, \qquad \mathcal{V}_3 = z_{20}\partial_{z_{10}}, \qquad \mathcal{V}_4 = z_{10}\partial_{z_{20}}, \qquad \mathcal{V}_5 = z_{10}\partial_{z_{10}} - z_{20}\partial_{z_{20}}, \\
&\mathcal{V}_6 = \theta_{10}\partial_{\theta_{20}}, \qquad \mathcal{V}_7 = z_{10}\theta_{10}\partial_{\theta_{20}}, \qquad \mathcal{V}_8 = z_{20}\theta_{10}\partial_{\theta_{20}}, \\
&\mathcal{V}_9 = \theta_{10}\partial_{\theta_{10}} + z_{20}\partial_{z_{20}}, \qquad \mathcal{V}_{10} = \theta_{20}\partial_{\theta_{20}} + z_{20}\partial_{z_{20}}, \\
&\mathcal{V}_{11} = (z_{10})^2\partial_{z_{10}} + (z_{10}z_{20} + \lambda\theta_{10}\theta_{20})\partial_{z_{20}} + z_{10}\theta_{10}\partial_{\theta_{10}} + 2z_{10}\theta_{20}\partial_{\theta_{20}}, \\
&\mathcal{V}_{12} = (z_{10}z_{20} - \lambda\theta_{10}\theta_{20})\partial_{z_{10}} + (z_{20})^2\partial_{z_{20}} + z_{20}\theta_{10}\partial_{\theta_{10}} + 2z_{20}\theta_{20}\partial_{\theta_{20}}, \\
\\
&\Xi_1 = \partial_{\theta_{10}}, \qquad \Xi_2 = \partial_{\theta_{20}}, \qquad \Xi_3 = \theta_{10}\partial_{z_{10}}, \qquad \Xi_4 = \theta_{10}\partial_{z_{20}}, \qquad \Xi_5 = z_{10}\partial_{\theta_{20}}, \qquad \Xi_6 = z_{20}\partial_{\theta_{20}}, \\
&\Xi_7 = (z_{10})^2\partial_{\theta_{20}} - \lambda z_{10}\theta_{10}\partial_{z_{20}}, \qquad \Xi_8 = (z_{20})^2\partial_{\theta_{20}} + \lambda z_{20}\theta_{10}\partial_{z_{10}}, \\
&\Xi_9 = z_{10}\partial_{\theta_{10}} + \lambda\theta_{20}\partial_{z_{20}}, \qquad \Xi_{10} = -z_{20}\partial_{\theta_{10}} + \lambda\theta_{20}\partial_{z_{10}}, \\
&\Xi_{11} = z_{10}\theta_{10}\partial_{z_{10}} + z_{20}\theta_{10}\partial_{z_{20}} + 2\theta_{10}\theta_{20}\partial_{\theta_{20}}, \\
&\Xi_{12} = (z_{10}z_{20} - \lambda\theta_{10}\theta_{20})\partial_{\theta_{20}} - \lambda z_{20}\theta_{10}\partial_{z_{20}},
\end{aligned} \tag{3.133}$$

where $\lambda \in \mathbb{C}$ is a complex number representing the cohomology class $H^1(\mathcal{T}^2_{\mathbb{P}}(-3)) \cong \mathbb{C}$.

Proof. The theorem is proved by evaluating the zero-th Čech cohomology group of $\mathcal{T}_{\mathcal{M}}$, by means of a lengthy computation in charts. $\square$

The actual embedding is realised by the following construction.

Construction 3.1 (Embedding)**.** *The tangent sheaf $\mathcal{T}_{\mathcal{M}}$ of $\mathcal{M}$ allows for an embedding $i : \mathbb{P}^{2|2}_{\omega} \to G(2|2, \mathbb{C}^{12|12})$, constructed by means of the global sections of the tangent sheaf in the following way:*

1. *we represent the $12|12$ global sections of $\mathcal{T}_{\mathcal{M}}$ in a local chart, say $\mathcal{U}_0$, as above, expanding them onto a local basis of generators of $\mathcal{T}_{\mathcal{M}}$, that has dimension $2|2$;*

2. *the coefficients of the expansion are mapped into $12|12$ columns, so that the resulting matrix is a super Grassmannian of the kind $G(2|2, \mathbb{C}^{12|12})$ represented in a certain super big-cell.*

We notice that the functorial properties of the Grassmannian makes the map well-defined in case it is in a super big-cell.

In particular, the global sections we have found above lead to the following image into $G(2|2,\mathbb{C}^{12|12})$:

$$
i(\mathcal{M}) = \left(\begin{array}{cc|c|cc|c} 1 & 0 & A_{1\times 10} & 0 & 0 & B_{1\times 10} \\ 0 & 1 & A_{2\times 10} & 0 & 0 & B_{2\times 10} \\ \hline 0 & 0 & C_{1\times 10} & 1 & 0 & D_{1\times 10} \\ 0 & 0 & C_{2\times 10} & 0 & 1 & D_{2\times 10} \end{array}\right) \tag{3.134}
$$

where we have highlighted the super big-cell singled out by the four global sections $\{\mathcal{V}_1 = \partial_{z_1}, \mathcal{V}_2 = \partial_{z_2}, \Xi_1 = \partial_{\theta_1}, \Xi_2 = \partial_{\theta_2}\}$ in the chart $\mathcal{U}_0$ and the $A_{i\times 10}, B_{i\times 10}, C_{i\times 10}, D_{i\times 10}$ for $i = 1, 2$, make up four 2×10 matrices, as follows:

$$
A := \begin{pmatrix} A_{1\times 10} \\ A_{2\times 10} \end{pmatrix} = \begin{pmatrix} z_2 & 0 & z_1 & 0 & 0 & 0 & 0 & 0 & z_1^2 & z_1z_2 - \lambda\theta_1\theta_2 \\ 0 & z_1 & -z_2 & 0 & 0 & 0 & z_2 & z_2 & z_1z_2 + \lambda\theta_1\theta_2 & z_2^2 \end{pmatrix},
$$

$$
B := \begin{pmatrix} B_{1\times 10} \\ B_{2\times 10} \end{pmatrix} = \begin{pmatrix} \theta_1 & 0 & 0 & 0 & 0 & \lambda z_2\theta_1 & 0 & \lambda\theta_2 & z_1\theta_1 & 0 \\ 0 & \theta_1 & 0 & 0 & -\lambda z_1\theta_1 & 0 & \lambda\theta_2 & 0 & z_2\theta_1 & -\lambda z_2 \end{pmatrix},
$$

$$
C := \begin{pmatrix} C_{1\times 10} \\ C_{2\times 10} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \theta_1 & 0 & z_1\theta_1 & z_2\theta_1 \\ 0 & 0 & 0 & \theta_1 & z_1\theta_1 & z_2\theta_1 & 0 & \theta_2 & 2z_1\theta_2 & 2z_2\theta_2 \end{pmatrix},
$$

$$
D := \begin{pmatrix} D_{1\times 10} \\ D_{2\times 10} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & z_1 & -z_2 & 0 & 0 \\ 0 & 0 & z_1 & z_2 & z_1^2 & z_2^2 & 0 & 0 & 2\theta_1\theta_2 & z_1z_2 - \lambda\theta_1\theta_2 \end{pmatrix}, \tag{3.135}
$$

where the subscript referring to the chart $\mathcal{U}_0$ of $\mathcal{M}$ has been suppressed for readability purpose. The following theorem confirms that the map i is indeed an embedding.

Theorem 3.20. *Let $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ be the non-projected supermanifold endowed with a fermionic sheaf $\mathcal{F}_\mathcal{M} := \Pi\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \Pi\mathcal{O}_{\mathbb{P}^2}(-2)$. Then the map $i : \mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M}) \to G(2|2,\mathbb{C}^{12|12})$ is an embedding of supermanifolds.*

Proof. One can check from the expressions above that the map is injective on the geometric points, that is on $\mathbb{P}^2$, and moreover its super differential is injective. This can be checked, for example, by representing the super differential as a 4×80 matrix, where the four 1×80 rows are given by the derivatives of a row vector $(A_{i\times 10}, B_{i\times 10}, C_{i\times 10}, D_{i\times 10})$ with respect to $\partial_{z_1}, \partial_{z_2}, \partial_{\theta_1}, \partial_{\theta_2}$. The resulting matrix has rank 4. □

Actually, to simplify the proof and avoid cumbersome computation, one can consider just a subset of the global sections found above in order to prove global generation and injectivity of the differential. Namely, let us consider the global sections given by

$$
S := \{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_5, \mathcal{V}_9 - \mathcal{V}_{10}, \Xi_1, \Xi_2\} \subset H^0(\mathcal{T}_\mathcal{M}). \tag{3.136}
$$

These define a sub-matrix of the $12|12 \times 4|4$ matrix given, having columns given by coordinates of the global sections with respect to the basis $\partial_{z_1}, \partial_{z_2}, \partial_{\theta_1}, \partial_{\theta_2}$ in the chart $\mathcal{U}_0$. It has the following form, writing the columns in a suitable order

$$
i(S) = \left(\begin{array}{c|cccc|cc} & \mathcal{V}_9 - \mathcal{V}_{10} & \mathcal{V}_5 & \mathcal{V}_1 & \mathcal{V}_2 & \Xi_1 & \Xi_2 \\ \hline \partial_{z_1} & 0 & z_1 & 1 & 0 & 0 & 0 \\ \partial_{z_2} & 0 & -z_2 & 0 & 1 & 0 & 0 \\ \hline \partial_{\theta_1} & \theta_1 & 0 & 0 & 0 & 1 & 0 \\ \partial_{\theta_2} & -\theta_2 & 0 & 0 & 0 & 0 & 1 \end{array}\right). \tag{3.137}
$$

This gives a *linear* embedding of $\mathcal{U}_0$ into a super big-cell of the super Grassmannian, which proves both global generation and injectivity of the differential over $\mathcal{U}_0$. By symmetry, or if one prefers, by the homogeneity of $\mathcal{M}$ and $\mathcal{T}_\mathcal{M}$ with respect to the action of $PGL(3)$, the same result holds true over $\mathcal{U}_1$ and $\mathcal{U}_2$ as well.

3.7.2 Non-Decomposable Sheaf: $\mathcal{F}_{\mathcal{M}} = \Pi\Omega^1_{\mathbb{P}^2}$

If we take $\Pi\Omega^1_{\mathbb{P}^2}$ to be the fermionic sheaf of the supermanifold $\mathbb{P}^2_\omega(\mathcal{F}_{\mathcal{M}})$, then we let θ_{10}, θ_{20} transform as dz_{10} and dz_{20} respectively, obtaining the transformations

$$\begin{aligned}
&\mathcal{U}_0 \cap \mathcal{U}_1: && \theta_{10} = -\frac{\theta_{11}}{(z_{11})^2}, && \theta_{20} = -\frac{z_{21}}{(z_{11})^2}\theta_{11} + \frac{\theta_{21}}{z_{11}};\\
&\mathcal{U}_2 \cap \mathcal{U}_0: && \theta_{12} = -\frac{\theta_{20}}{(z_{20})^2}, && \theta_{22} = \frac{\theta_{10}}{z_{10}} - \frac{z_{10}}{(z_{20})^2}\theta_{20};\\
&\mathcal{U}_1 \cap \mathcal{U}_2: && \theta_{11} = -\frac{z_{12}}{(z_{22})^2}\theta_{22} + \frac{\theta_{12}}{z_{22}}, && \theta_{21} = -\frac{\theta_{22}}{(z_{22})^2}.
\end{aligned} \tag{3.138}$$

Just like above, we now look for the complete form of the transition functions. By Theorem 3.10, we have the following result.

Theorem 3.21 (Transition functions (2))**.** *Let $\mathbb{P}^2_\omega(\mathcal{F}_{\mathcal{M}})$ be the non-projected supermanifold with $\mathcal{F}_{\mathcal{M}} = \Pi\Omega^1_{\mathbb{P}^2}$. Then, its transition functions take the following form:*

$$\begin{aligned}
&\mathcal{U}_0 \cap \mathcal{U}_1: && z_{10} = \frac{1}{z_{11}}, \quad z_{20} = \frac{z_{21}}{z_{11}} + \lambda\frac{\theta_{11}\theta_{21}}{(z_{11})^2}; && \theta_{10} = -\frac{\theta_{11}}{(z_{11})^2}, \quad \theta_{20} = -\frac{z_{21}}{(z_{11})^2}\theta_{11} + \frac{\theta_{21}}{z_{11}};\\
&\mathcal{U}_1 \cap \mathcal{U}_2: && z_{11} = \frac{z_{12}}{z_{22}} - \lambda\frac{\theta_{12}\theta_{22}}{(z_{22})^2}, \quad z_{21} = \frac{1}{z_{22}}; && \theta_{12} = -\frac{\theta_{20}}{(z_{20})^2}, \quad \theta_{22} = \frac{\theta_{10}}{z_{10}} - \frac{z_{10}}{(z_{20})^2}\theta_{20};\\
&\mathcal{U}_2 \cap \mathcal{U}_0: && z_{12} = \frac{1}{z_{20}}, \quad z_{22} = \frac{z_{10}}{z_{20}} - \lambda\frac{\theta_{10}\theta_{20}}{(z_{20})^2}; && \theta_{11} = -\frac{z_{12}}{(z_{22})^2}\theta_{22} + \frac{\theta_{12}}{z_{22}}, \quad \theta_{21} = -\frac{\theta_{22}}{(z_{22})^2}.
\end{aligned} \tag{3.139}$$

Proof. Again, it follows immediately from Theorem 3.10, taking into account the transition matrix for the given $\mathcal{F}_{\mathcal{M}}$, that can be red from above. □

We now repeat what done above in the case we choose $\mathcal{F}_{\mathcal{M}} = \Pi\Omega^1_{\mathbb{P}^2}$. We again consider the tangent sheaf $\mathcal{T}_{\mathcal{M}}$ first. By looking at the transition functions we have found above in 3.139, by means of the chain rule, we can find the full transition functions of the tangent sheaf between the charts covering $\mathbb{P}^2$. For example, on $\mathcal{U}_0 \cap \mathcal{U}_1$, we find:

$$\begin{aligned}
\partial_{z_{10}} &= -(z_{11})^2\partial_{z_{11}} + (-z_{11}z_{21} + \lambda\theta_{11}\theta_{21})\,\partial_{z_{21}} - 2z_{11}\theta_{11}\partial_{\theta_{11}} + (-z_{21}\theta_{11} - z_{11}\theta_{21})\,\partial_{\theta_{21}},\\
\partial_{z_{20}} &= z_{11}\partial_{z_{21}} + \theta_{11}\partial_{\theta_{21}},\\
\partial_{\theta_{10}} &= -\lambda\,(z_{21}\theta_{11} - z_{11}\theta_{21})\,\partial_{z_{21}} - (z_{11})^2\partial_{\theta_{11}} + (-z_{11}z_{21} - \lambda\theta_{11}\theta_{21})\,\partial_{\theta_{21}},\\
\partial_{\theta_{20}} &= +\lambda\theta_{11}\partial_{z_{21}} + z_{11}\partial_{\theta_{21}},
\end{aligned} \tag{3.140}$$

so that the map Jac_{10} has the following matrix representation

$$[\mathrm{Jac}_{10}] = \left(\begin{array}{cc|cc}
-(z_{11})^2 & -z_{11}z_{21} + \lambda\theta_{11}\theta_{21} & -2z_{11}\theta_{11} & -z_{21}\theta_{11} - z_{11}\theta_{21}\\
0 & z_{11} & 0 & \theta_{11}\\
\hline
0 & -\lambda\,(z_{21}\theta_{11} - z_{11}\theta_{21}) & -(z_{11})^2 & (-z_{11}z_{21} - \lambda\theta_{11}\theta_{21})\\
0 & \lambda\theta_{11} & 0 & z_{11}
\end{array}\right)$$

As above, we now look for the explicit form of the global sections of the tangent sheaf to set up the embedding into a certain Grassmannian. Just like above, we keep the parameter λ explicit.

Theorem 3.22 (Generators of $H^0(\mathcal{T}_{\mathcal{M}})$)**.** *The tangent sheaf $\mathcal{T}_{\mathcal{M}}$ of $\mathcal{M}$ has $8|9$ global sections. In particular, in the local chart $\mathcal{U}_0$, one has that $H^0(\mathcal{T}_{\mathcal{M}}) \cong \mathrm{Span}_{\mathbb{C}}\{\mathcal{V}_1, \ldots, \mathcal{V}_8 | \, \Xi_1, \ldots, \Xi_9\}$, where*

$$\begin{aligned}
&\mathcal{V}_1 = \partial_{z_1}, \qquad \mathcal{V}_2 = \partial_{z_2}, \qquad \mathcal{V}_3 = z_1\partial_{z_1} + \theta_1\partial_{\theta_1},\\
&\mathcal{V}_4 = z_2\partial_{z_1} + \theta_2\partial_{\theta_1}, \qquad \mathcal{V}_5 = z_2\partial_{z_2} + \theta_2\partial_{\theta_2}, \qquad \mathcal{V}_6 = z_1\partial_{z_2} + \theta_1\partial_{\theta_2},\\
&\mathcal{V}_7 = (z_1)^2\partial_{z_1} + (z_1z_2 - \lambda\theta_1\theta_2)\partial_{z_2} + 2z_1\theta_1\partial_{\theta_1} + (z_2\theta_1 + z_1\theta_2)\partial_{\theta_2},\\
&\mathcal{V}_8 = (z_1z_2 + \lambda\theta_1\theta_2)\partial_{z_1} + (z_2)^2\partial_{z_2} + (z_2\theta_1 + z_1\theta_2)\partial_{\theta_2},
\end{aligned} \tag{3.141}$$

$$
\begin{aligned}
&\Xi_1 = \partial_{\theta_1}, \qquad\qquad \Xi_2 = \partial_{\theta_2}, \qquad\qquad \Xi_3 = \theta_1\partial_{z_1} + \theta_2\partial_{z_2},\\
&\Xi_4 = z_1\partial_{\theta_1} - \lambda\theta_2\partial_{z_2}, \qquad\qquad \Xi_5 = z_2\partial_{\theta_1} + \lambda\theta_2\partial_{z_1},\\
&\Xi_6 = z_2\partial_{\theta_2} + \lambda\theta_2\partial_{z_2}, \qquad\qquad \Xi_7 = z_1\partial_{\theta_2} + \lambda\theta_1\partial_{z_2},\\
&\Xi_8 = \lambda(z_2\theta_1 - z_1\theta_2)\partial_{z_2} + (z_1)^2\partial_{\theta_1} + (z_1z_2 + \lambda\theta_1\theta_2)\partial_{\theta_2},\\
&\Xi_9 = \lambda(z_1\theta_2 - z_2\theta_1)\partial_{z_1} + (z_1z_2 - \lambda\theta_1\theta_2)\partial_{\theta_1} + (z_1)^2\partial_{\theta_2},
\end{aligned} \tag{3.142}
$$

where $\lambda \in \mathbb{C}$ is a complex number representing the cohomology class $\omega \in H^1(\mathcal{T}^2_{\mathbb{P}}(-3)) \cong \mathbb{C}$.

Proof. As above, the theorem is proved evaluating the zero-th Čech cohomology group of the tangent bundle $\mathcal{T}_{\mathcal{M}}$, by means of a lengthy computation in charts. □

We now follow the same construction as above in order to construct the explicit embedding into a super Grassmannian. In particular, we get the following image into $G(2|2, \mathbb{C}^{8|9})$:

$$
i(\mathcal{M}) = \left(\begin{array}{cc|c|cc|c}
1 & 0 & A_{1\times 6} & 0 & 0 & B_{1\times 7}\\
0 & 1 & A_{2\times 6} & 0 & 0 & B_{2\times 7}\\
\hline
0 & 0 & C_{1\times 6} & 1 & 0 & D_{1\times 7}\\
0 & 0 & C_{2\times 6} & 0 & 1 & D_{2\times 7}
\end{array}\right) \tag{3.143}
$$

where we have employed the same representation as above, by highlighting the super big-cell singled out by the four global sections $\{\mathcal{V}_1 = \partial_{z_1}, \mathcal{V}_2 = \partial_{z_2}, \Xi_1 = \partial_{\theta_1}, \Xi_2 = \partial_{\theta_2}\}$ in the chart $\mathcal{U}_0$ and where the $A_{i\times 6}$ and $C_{i\times 6}$ for $i = 1, 2$ are made up by two 2×6 matrices, while $B_{i\times 7}$ and $D_{i\times 10}$ for $i = 1, 2$ are made up by two 2×7 matrices, as follows

$$
A := \begin{pmatrix} A_{1\times 6}\\ A_{2\times 6}\end{pmatrix} = \begin{pmatrix} z_1 & z_2 & 0 & 0 & (z_1)^2 & z_1z_2 - \lambda\theta_1\theta_2\\ 0 & 0 & z_2 & z_1 & z_1z_2 - \lambda\theta_1\theta_2 & (z_2)^2 \end{pmatrix},
$$

$$
B := \begin{pmatrix} B_{1\times 7}\\ B_{2\times 7}\end{pmatrix} = \begin{pmatrix} \theta_1 & 0 & \lambda\theta_2 & 0 & 0 & 0 & \lambda z_1\theta_2 - z_2\theta_1\\ \theta_2 & -\lambda\theta_2 & 0 & \lambda\theta_2 & \lambda\theta_1 & \lambda z_1\theta_2 - z_2\theta_1 & 0 \end{pmatrix},
$$

$$
C := \begin{pmatrix} C_{1\times 6}\\ C_{2\times 6}\end{pmatrix} = \begin{pmatrix} \theta_1 & \theta_2 & 0 & 0 & 2z_1\theta_1 & z_2\theta_1 + z_1\theta_2\\ 0 & 0 & \theta_2 & \theta_1 & z_2\theta_1 + z_1\theta_2 & 2z_2\theta_2 \end{pmatrix},
$$

$$
D := \begin{pmatrix} D_{1\times 7}\\ D_{2\times 7}\end{pmatrix} = \begin{pmatrix} 0 & z_1 & z_2 & 0 & 0 & (z_1)^2 & z_1z_2 - \lambda\theta_1\theta_2\\ 0 & 0 & 0 & z_2 & z_1 & z_1z_2 + \lambda\theta_1\theta_2 & (z_1)^2 \end{pmatrix}, \tag{3.144}
$$

where the superscript referring to the chart $\mathcal{U}_0$ of $\mathbb{P}^{2|2}_{\omega}$ has been suppressed. The following theorem confirms that the map i is indeed an embedding.

Theorem 3.23. *Let $\mathbb{P}^2_{\omega}(\mathcal{F}_{\mathcal{M}})$ be the non-projected supermanifold endowed with the fermionic sheaf $\mathcal{F}_{\mathcal{M}} := \Pi\Omega^1_{\mathbb{P}^2}$. Then the map $i : \mathbb{P}^2_{\omega}(\mathcal{F}_{\mathcal{M}}) \to G(2|2, \mathbb{C}^{8|9})$ is an embedding of supermanifolds.*

Proof. One can check from the expressions above that the map is injective on the geometric points and its super differential is injective. The explicit check can be carried out as explained above in the decomposable case, yielding a rank 4 matrix. □

As before, one can avoid cumbersome computation by considering only a certain subset of the global sections to prove global generation and injectivity of the differential. Over $\mathcal{U}_0$, we consider the following subset

$$
S := \{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_5, \Xi_1, \Xi_2\}, \tag{3.145}
$$

whose representative matrix is given by

$$
i(S) = \left(\begin{array}{c|cccc|cc}
 & \mathcal{V}_5 & \mathcal{V}_3 & \mathcal{V}_1 & \mathcal{V}_2 & \Xi_1 & \Xi_2\\
\hline
\partial_{z_1} & 0 & z_1 & 1 & 0 & 0 & 0\\
\partial_{z_2} & z_2 & 0 & 0 & 1 & 0 & 0\\
\hline
\partial_{\theta_1} & 0 & \theta_1 & 0 & 0 & 1 & 0\\
\partial_{\theta_2} & \theta_2 & 0 & 0 & 0 & 0 & 1
\end{array}\right). \tag{3.146}
$$

This, again, provides a *linear* embedding of $\mathcal{U}_0$ into a super big-cell of the super Grassmannian $G(2|2, \mathbb{C}^{8|9})$. By symmetry or homogeneity this results extends to the $\mathcal{U}_1$ and $\mathcal{U}_2$ as well.

3.7.3 The Split Locus and its Cohomology

Having let λ explicit in the transition functions of the homogeneous supermanifolds $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ in (3.18) and (3.21), as to stress the dependence from the (obstruction) cohomology class $H^1(\mathcal{T}_{\mathbb{P}^2} \otimes Sym^2\mathcal{F}_\mathcal{M}) \cong \mathbb{C}$, bears an advantage: indeed it can be seen that, keeping the fermionic sheaf $\mathcal{F}_\mathcal{M}$ fixed, one can set up a (flat) *family of compact complex supermanifolds* $\mathcal{X}$ by letting λ vary. In other words, we have a (flat) morphism

$$\begin{tikzcd} \mathcal{X} \arrow[d, "\varphi"] \\ \operatorname{Spec}\mathbb{C}[\lambda] \end{tikzcd} \tag{3.147}$$

such that its fibre $\mathcal{X}_{\tilde{\lambda}} := \varphi^{-1}(\tilde{\lambda})$ above a certain $\{\tilde{\lambda}\} \in \operatorname{Spec}\mathbb{C}[\lambda]$ corresponds to the compact complex supermanifold $\mathbb{P}^{2|2}_\omega(\mathcal{F}_\mathcal{M})$ having the obstruction class $\omega_\mathcal{M} \in H^1(\mathcal{T}_{\mathbb{P}^2}(-3))$ represented by $\tilde{\lambda} \in \mathbb{C} \cong \operatorname{Spec}\mathbb{C}[\lambda]$.
Clearly, looking at the *central fibre*, above $\{0\} \in \operatorname{Spec}\mathbb{C}\,[\lambda]$, corresponds to set $\lambda = 0$ in the transition functions of $\mathcal{M}$, that is, $\mathcal{X}_0 = \varphi^{-1}(0)$ corresponds to a split supermanifold: we call it *split locus* of the family $\varphi : \mathcal{X} \to \operatorname{Spec}\mathbb{C}[\lambda]$.

Let us now take on our explicit examples. We start from the decomposable case, choosing $\mathcal{F}_\mathcal{M} = \Pi\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)$. The split locus of the family corresponds to the *weighted projective superspace*, we call it $\mathbb{P}^{2|2}(-1,-2)$, having a structure sheaf given by

$$\mathcal{O}_{\mathbb{P}^{2|2}(-1,-2)} := \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3) \oplus \Pi\left[\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)\right]. \tag{3.148}$$

This split structure sheaf leads to the obvious transition functions of the split type. For $\mathbb{P}^{2|2}(-1,-2)$ we get

$$z_{10} = \frac{1}{z_{11}}, \qquad z_{20} = \frac{z_{21}}{z_{11}}, \qquad \theta_{10} = \frac{\theta_{11}}{z_{11}}, \qquad \theta_{20} = \frac{\theta_{21}}{(z_{11})^2}. \tag{3.149}$$

Upon using these transition functions, one gets the following result.

Corollary 3.1 (Global Sections at the Split Locus)**.** *The tangent sheaf $\mathcal{T}_\mathcal{M}$ of $\mathcal{M} = \mathbb{P}^{2|2}(-1,-2)$, defined as above, has $13|12$ global sections. A basis for $H^0(\mathcal{T}_\mathcal{M})$ in the local chart $\mathcal{U}_0$ is given by*

$$\begin{aligned}
&\tilde{\mathcal{V}}_1 = \partial_{z_{10}} \quad \tilde{\mathcal{V}}_2 = z_{20}\partial_{z_{10}} \quad \tilde{\mathcal{V}}_3 = \partial_{z_{20}} \quad \tilde{\mathcal{V}}_4 = z_{10}\partial_{z_{20}} \quad \tilde{\mathcal{V}}_5 = z_{10}\partial_{z_{10}} \quad \tilde{\mathcal{V}}_6 = z_{20}\partial_{z_{20}} \\
&\tilde{\mathcal{V}}_7 = \theta_{10}\partial_{\theta_{20}} \quad \tilde{\mathcal{V}}_8 = z_{10}\theta_{10}\partial_{\theta_{20}} \quad \tilde{\mathcal{V}}_9 = z_{20}\theta_{10}\partial_{\theta_{20}} \quad \tilde{\mathcal{V}}_{10} = \theta_{10}\partial_{\theta_{10}} \quad \tilde{\mathcal{V}}_{11} = \theta_{20}\partial_{\theta_{20}} \\
&\tilde{\mathcal{V}}_{12} = (z_{10})^2\partial_{z_{10}} + z_{10}z_{20}\partial_{z_{20}} + z_{10}\theta_{10}\partial_{\theta_{10}} + 2z_{10}\theta_{20}\partial_{\theta_{20}} \\
&\tilde{\mathcal{V}}_{13} = z_{10}z_{20}\partial_{z_{10}} + (z_{20})^2\partial_{z_{20}} + z_{20}\theta_{10}\partial_{\theta_{10}} + 2z_{20}\theta_{20}\partial_{\theta_{20}} \\
\\
&\tilde{\Xi}_1 = \theta_{10}\partial_{z_{10}} \quad \tilde{\Xi}_2 = \theta_{10}\partial_{z_{20}} \quad \tilde{\Xi}_3 = \partial_{\theta_{10}} \quad \tilde{\Xi}_4 = \partial_{\theta_{20}} \quad \tilde{\Xi}_5 = z_{10}\partial_{\theta_{20}} \quad \tilde{\Xi}_6 = z_{20}\partial_{\theta_{20}} \\
&\tilde{\Xi}_7 = (z_{10})^2\partial_{\theta_{20}} \quad \tilde{\Xi}_8 = (z_{20})^2\partial_{\theta_{20}} \quad \tilde{\Xi}_9 = z_{10}\partial_{\theta_{10}} \quad \tilde{\Xi}_{10} = z_{20}\partial_{\theta_{10}} \\
&\tilde{\Xi}_{11} = z_{10}\theta_{10}\partial_{z_{10}} + z_{20}\theta_{10}\partial_{z_{20}} + 2\theta_{10}\theta_{20}\partial_{\theta_{20}} \quad \tilde{\Xi}_{12} = z_{10}z_{20}\partial_{\theta_{20}}.
\end{aligned} \tag{3.150}$$

There are some interesting facts to notice when looking at the family $\varphi : \mathcal{X} \to \operatorname{Spec}\mathbb{C}[\lambda]$.

1. All the fibres corresponding to non-split supermanifolds are isomorphic. Only the split-locus corresponding to $\mathbb{P}^{2|2}(-1,-2)$ makes exception: it is the only non-isomorphic supermanifold

in the family and indeed the cohomology *jumps* to different values above it. In other words we find

$$R^0\varphi_* \mathcal{T}_{\mathcal{X}/\mathrm{Spec}\,\mathbb{C}[\lambda]}(\tilde{\lambda}) \cong \begin{cases} \mathbb{C}^{13|12} & \text{above the split-locus, } \tilde{\lambda} = 0 \\ \mathbb{C}^{12|12} & \text{elsewhere, } \tilde{\lambda} \neq 0 \end{cases} \tag{3.151}$$

$$R^1\varphi_* \mathcal{T}_{\mathcal{X}/\mathrm{Spec}\,\mathbb{C}[\lambda]}(\tilde{\lambda}) \cong \begin{cases} \mathbb{C}^{1|0} & \text{above the split-locus, } \tilde{\lambda} = 0 \\ 0 & \text{elsewhere, } \tilde{\lambda} \neq 0 \end{cases} \tag{3.152}$$

where $\mathcal{T}_{\mathcal{X}/\mathrm{Spec}\,\mathbb{C}[\lambda]}$ is the *relative tangent sheaf* to $\varphi : \mathcal{X} \to \mathrm{Spec}\,\mathbb{C}[\lambda]$ and R^i is the *right-derived cohomology functor*.

2. At the level of the global sections, the difference between the split case and the non-split case (after modding out the $\theta\theta$-terms) resides in that the two global sections $\tilde{\mathcal{V}}_5$ and $\tilde{\mathcal{V}}_6$ of $\mathcal{T}_{\mathcal{M}}$ fail to be global in the non-split case treated above. Instead, they need to be arranged together to form a global section, which is $\mathcal{V}_5 = \tilde{\mathcal{V}}_5 - \tilde{\mathcal{V}}_6$.

3. The previous corollary and the first observation above have another consequence yet. Indeed, one finds that, in contrast with the non-split case $\tilde{\lambda} \neq 0$ which is *rigid* - it has no *deformations*, accounted in the first cohomology group of the tangent sheaf -, one finds $H^1(\mathcal{T}_{\mathcal{M}}) \cong \mathbb{C}^{1|0}$ above the split-locus. By the way, deformation theory for supermanifolds and, in particular, for non-projected supermanifolds, has yet to be properly addressed and developed in the literature, thus we refrain to comment further the results we have found above.

Now, consider instead the non-decomposable sheaf $\mathcal{F}_{\mathcal{M}} = \Pi\Omega^1_{\mathbb{P}^2}$. The supermanifold above the *split locus* $\lambda = 0$, we call it $\mathbb{P}^{2|2}(\Pi\Omega^1_{\mathbb{P}^2})$, has structure sheaf given by

$$\mathcal{O}_{\mathbb{P}^{2|2}(\Pi\Omega^1_{\mathbb{P}^2})} := \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3) \oplus \Pi\Omega^1_{\mathbb{P}^2}. \tag{3.153}$$

This structure sheaf has the obvious transition functions of the split type. Clearly, while the odd transition functions remain the same, one needs to set $\lambda = 0$ in (3.139) for the even transition functions, that become the usual transition functions of $\mathbb{P}^2$. These lead to the following corollary.

Corollary 3.2 (Global Sections at the Split Locus)**.** *The tangent sheaf $\mathcal{T}_{\mathcal{M}}$ of $\mathcal{M} = \mathbb{P}^{2|2}(\Pi\Omega^1_{\mathbb{P}^2})$, defined as above, has $9|9$ global sections. A basis for $H^0(\mathcal{T}_{\mathcal{M}})$ in the local chart $\mathcal{U}_0$ is given by*

$$\begin{aligned} &\tilde{\mathcal{V}}_1 = \partial_{z_1} \qquad \tilde{\mathcal{V}}_2 = \partial_{z_2} \qquad \tilde{\mathcal{V}}_3 = \theta_1\partial_{\theta_1} + \theta_2\partial_{\theta_2}, \qquad \tilde{\mathcal{V}}_4 = z_1\partial_{z_1} - \theta_2\partial_{\theta_2} \\ &\tilde{\mathcal{V}}_5 = z_2\partial_{z_1} + \theta_2\partial_{\theta_1} \qquad \tilde{\mathcal{V}}_6 = z_2\partial_{z_2} + \theta_2\partial_{\theta_2} \qquad \tilde{\mathcal{V}}_7 = z_1\partial_{z_2} + \theta_1\partial_{\theta_2} \\ &\tilde{\mathcal{V}}_8 = (z_1)^2\partial_{z_1} + z_1z_2\partial_{z_2} + 2z_1\theta_1\partial_{\theta_1} + (z_2\theta_1 + z_1\theta_2)\partial_{\theta_2} \\ &\tilde{\mathcal{V}}_9 = z_1z_2\partial_{z_1} + (z_2)^2\partial_{z_2} + (z_2\theta_1 + z_1\theta_2)\partial_{\theta_2} + 2z_2\theta_2\partial_{\theta_2} \end{aligned} \tag{3.154}$$

$$\begin{aligned} &\tilde{\Xi}_1 = \partial_{\theta_1} \qquad \tilde{\Xi}_2 = \partial_{\theta_2} \qquad \tilde{\Xi}_3 = \theta_1\partial_{z_1} + \theta_2\partial_{z_2}, \qquad \tilde{\Xi}_4 = z_1\partial_{\theta_1} \qquad \tilde{\Xi}_5 = z_2\partial_{\theta_1} \\ &\tilde{\Xi}_6 = z_2\partial_{\theta_2} \qquad \tilde{\Xi}_7 = z_1\partial_{\theta_2} \quad \tilde{\Xi}_8 = (z_1)^2\partial_{\theta_1} + z_1z_2\partial_{\theta_2}, \qquad \tilde{\Xi}_9 = z_1z_2\partial_{\theta_1} + (z_1)^2\partial_{\theta_2}. \end{aligned} \tag{3.155}$$

We stress the following facts.

1. Again, all of the fibres corresponding to non-split supermanifolds are isomorphic, while the split-locus corresponding to $\mathbb{P}^{2|2}(\Pi\Omega^1_{\mathbb{P}^2})$ makes exception and its cohomology *jumps* to different values. We find

$$R^0\varphi_* \mathcal{T}_{\mathcal{X}/\mathrm{Spec}\,\mathbb{C}[\lambda]}(\tilde{\lambda}) \cong \begin{cases} \mathbb{C}^{9|9} & \text{above the split-locus, } \tilde{\lambda} = 0 \\ \mathbb{C}^{8|9} & \text{elsewhere, } \tilde{\lambda} \neq 0 \end{cases} \tag{3.156}$$

$$R^1\varphi_* \mathcal{T}_{\mathcal{X}/\mathrm{Spec}\,\mathbb{C}[\lambda]}(\tilde{\lambda}) \cong \begin{cases} \mathbb{C}^{1|1} & \text{above the split-locus, } \tilde{\lambda} = 0 \\ \mathbb{C}^{0|1} & \text{elsewhere, } \tilde{\lambda} \neq 0 \end{cases} \tag{3.157}$$

where we have used the same notation as above for the relative tangent sheaf and the right derived cohomology functor.

2. If we mod out the $\theta\theta$-terms, and we compare all the global sections of a generic non-projected supermanifolds having $\tilde{\lambda} \neq 0$ with the global sections above the split-locus we find that these differ by the sections $\tilde{\mathcal{V}}_3$ and $\tilde{\mathcal{V}}_4$ and in particular one has $\mathcal{V}_3 = \tilde{\mathcal{V}}_4 + \tilde{\mathcal{V}}_3$.

3. In contrast with the decomposable case treated above, we have that also the generic fibre, over $\tilde{\lambda} \neq 0$, does have an infinitesimal odd deformation, but it does not have any even deformation, while above the split-locus we find both even and odd deformations.

Chapter 4

Supergeometry of Π-Projective Spaces

4.1 Introduction and Motivation

We have seen in the previous chapter that a remarkable difference between ordinary algebraic geometry and super algebraic geometry is concerned with the role of *projective superspaces* as a natural set-up and ambient space. Indeed, there are important examples of supermanifolds that fail to be *projective*, *i.e.* they do not posses any (ample) invertible sheaf that allows for an embedding into projective superspaces [52] and this is the case of the family of non-projected supermanifolds $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ we have introduced in the previous chapter. Also, as discussed early on, there is no natural generalisation of the Plücker map, so that super Grassmannians cannot in general be embedded into projective superspaces.

This led Manin to suggest that in a supergeometric setting, invertible sheaves might not play the same fundamental role they play in ordinary algebraic geometry. Instead, together with Skornyakov, he proposed as a suitable substitute of invertible sheaves in algebraic supergeometry, the notion of Π-*invertible sheaves.* These are locally-free sheaves of rank $1|1$ endowed with a specific *odd* symmetry, locally exchanging the even and odd components, called Π-*symmetry.* The spaces allowing for such sheaves to be defined were first constructed by Manin, see [41]: these are called Π-projective spaces $\mathbb{P}^n_\Pi$ and more in general Π-Grassmannians. The relevance of these geometric objects became apparent along with the generalisation to a supersymmetric context of the theory of elliptic curves and theta functions due to Levin. In particular, it was realised in [39] and [40] that the correct supergeometric generalisation of theta functions, called *supertheta functions*, should not be sections of certain invertible sheaves, but instead sections of Π-invertible sheaves and every supersymmetric elliptic curve can be naturally embedded into a certain product of Π-projective spaces $\mathbb{P}^n_\Pi$ by means of supertheta functions. Recently, following an observation due to Deligne, Kwok has provided in [36] a different description of Π-projective spaces $\mathbb{P}^n_\Pi$ by constructing them as suitable quotients by the algebraic supergroup $\mathbb{G}_m^{1|1} = \mathbb{D}^*$, the multiplicative version of the *super skew field* $\mathbb{D}$, which is a non-commutative associative superalgebra, thus making apparent a connection between Π-projective geometry and the broader universe of non-commutative geometry.

In this chapter we will provide a new construction of Π-projective spaces $\mathbb{P}^n_\Pi$, showing how they arise naturally as non-projected supermanifolds over $\mathbb{P}^n$, upon choosing the fermionic sheaf of the supermanifold to be the cotangent sheaf $\Omega^1_{\mathbb{P}^n}$. More precisely, we will show that for $n > 1$ any Π-projective space can be defined by three ordinary objects, a projective space $\mathbb{P}^n$, the sheaf of 1-forms $\Omega^1_{\mathbb{P}^n}$ defined on it and a certain cohomology class, actually the *fundamental obstruction class*, $\omega \in H^1(\mathbb{P}^n, \mathcal{T}_{\mathbb{P}^n} \otimes \bigwedge^2 \Omega^1_{\mathbb{P}^n})$, where $\mathcal{T}_{\mathbb{P}^n}$ is the tangent sheaf of $\mathbb{P}^n$. In the case $n = 1$ one does not need any cohomology class and the data coming from the projective line $\mathbb{P}^1$ and the cotangent sheaf $\Omega^1_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$ are enough to describe the Π-projective line $\mathbb{P}^1_\Pi$. Moreover we show that Π-projective spaces are *all* Calabi-Yau supermanifolds, that is, they have trivial Berezinian sheaf, a feature that makes them particularly interesting for physical applications.

Moreover, we provide some pieces of evidence that the relation with the cotangent sheaf of the

underlying manifold is actually a characterising one in Π-geometry. Indeed, not only Π-projective spaces, but also more in general Π-Grassmannians can be constructed as certain non-projected supermanifolds starting from the cotangent sheaf of the underlying reduced Grassmannias. We show by means of an example that in this context the non-projected structure of the supermanifold becomes in general more complicated and, in addition to the fundamental one, also higher obstruction classes enter the description.
Later on we make contact with the previous chapter, by reconsidering the embedding of the two non-projected homogeneous supermanifolds $\mathbb{P}^2_\omega(\mathcal{F}_{\mathcal{M}})$ that arise upon choosing a decomposable fermionic sheaf of the kind $\mathcal{F}_{\mathcal{M}} = \Pi\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \Pi\mathcal{O}_{\mathbb{P}^2}(-2)$ or a non-decomposable fermionic sheaf of the kind $\mathcal{F}_{\mathcal{M}} = \Pi\Omega^1_{\mathbb{P}^2}$. In particular we will show that, choosing a decomposable fermionic sheaf, leads to a *non* Π-projective supermanifold: that is, the supermanifold cannot be embedded into any Π-projective space $\mathbb{P}^n_\Pi$. On the other hand, choosing the fermionic sheaf to be the non-decomposable sheaf $\mathcal{F}_{\mathcal{M}} = \Pi\Omega^1_{\mathbb{P}^2}$, we find that this supermanifold is actually the Π-projective plane $\mathbb{P}^2_\Pi$, and as such it has a minimal embedding into the super Grassmannian $G(1|1, \mathbb{C}^{3|3})$.

4.2 Π-Projective Geometry and Π-Grassmannians

In this section we will give a short introduction to Π-projective geometry and Π-Grassmannians, for more details and thorough treatment we invite the reader to refer to [41] and [42] as usual.

As far as the author is concerned the most straightforward and immediate way to introduce Π-projective spaces and their geometry is via super Grassmannians. This approach has also the merit to make clear that Π-projective spaces are in general embedded in super Grassmannians, showing that super Grassmannians are once again good embedding spaces in a supergeometric context, as discussed in the previous chapter. The notation for super Grassmannians is based on section 3.5 of the previous chapter.

Our starting point is the definition of Π-symmetry. We will give a general definition on sheaves.

Definition 4.1 (Π-Symmetry)**.** *Let $\mathcal{G}$ be a locally-free sheaf of $\mathcal{O}_{\mathcal{M}}$-modules of rank $n|n$ on a supermanifold $\mathcal{M}$, a Π-symmetry is an isomorphism such that $p_\Pi : \mathcal{G} \longrightarrow \Pi\mathcal{G}$ and such that $p_\Pi^2 = id$.*

We now work locally and use simply the supercommutative free $\mathbb{C}$-module $\mathbb{C}^{n|n} = \mathbb{C}^n \oplus \Pi\mathbb{C}^n$, instead of a generic sheaf: we can therefore choose a certain basis of even elements such that $\mathbb{C}^n = \mathrm{Span}\{e_1, \dots, e_n\}$ and we generate a basis for the whole $\mathbb{C}^{n|n}$ as follows

$$\mathbb{C}^{n|n} = \mathrm{Span}\{e_1, \dots, e_n \,|\, p_\Pi e_1, \dots, p_\Pi e_n\}. \tag{4.1}$$

Clearly, the action of p_Π exchanges the generators of $\mathbb{C}^n$ and $\Pi\mathbb{C}^n$.
We observe that somehow the presence of a Π-symmetry should remind us of a "physical supersymmetry", as it transform even elements in odd elements and viceversa. Also, as supersymmetry requires a Hilbert space allowing for the same amount of bosonic and fermionic states, similarly Π-symmetry imposes an equal number of even and odd dimensions for a certain "ambient space", as it might be the supercommutative free module $\mathbb{C}^{n|n}$ above.
Along this line, one can give the following

Definition 4.2 (Π-Symmetric Submodule)**.** *Let M be a supercommutative free A-module such that $M = A^n \oplus \Pi A^n$. Then we say that a super submodule $S \subset M$ is Π-symmetric if it is stable under the action of p_Π.*

This has as a consequence the following obvious lemma:

Lemma 4.1. *Let M be a supercommutative free A-module such that $M = A^n \oplus \Pi A^n$ together with a basis given by $\{e_1, \dots, e_n \,|\, p_\Pi e_1, \dots, p_\Pi e_n\}$. Then a super submodule of M is Π-symmetric if and only if for every element $v = \sum_{i=1}^n x^i e_i + \xi^i p_\Pi e_i$ it also contains $v_\Pi = \sum_{i=1}^n (-\xi^i e_i + x^i p_\Pi e_i)$*

The proof is clear, as v_Π is nothing but the Π-transformed partner of v. Notice, though, the presence of a minus sign due to parity reasons.
Such Π-symmetric submodules allow us to define Π-projective superspaces, we call them $\mathbb{P}^n_\Pi$, and,

more in general, Π-symmetric super Grassmannians. The construction follows closely the one of super Grassmannians of the kind $G(1|1; n+1|n+1)$, but we only take into account Π-symmetric free submodules characterised as by the Lemma 4.1. These, in turn, allow us to write down the $n+1$ affine super cells covering $\mathbb{P}^n_\Pi$, each of these related to an affine supermanifold of the kind $\tilde{\mathcal{U}}_i := (\mathcal{U}_i, \mathbb{C}[z_{1i}, \ldots, z_{ni}, \theta_{1i}, \ldots, \theta_{ni}]) \cong \mathbb{C}^{n|n}$ and where $\mathcal{U}_i$ are the usual open sets covering $\mathbb{P}^n$. We try to make these considerations explicit by considering the case of the the Π-projective line, we call it $\mathbb{P}^1_\Pi$.

Example 4.1 (Π-Projective Line $\mathbb{P}^1_\Pi$)**.** *This is the classifying space of the* Π*-symmetric* $1|1$ *dimensional super subspaces of* $\mathbb{C}^{2|2}$*, corresponding to the super Grassmannian* $G_\Pi(1|1; 2|2)$*, where subscript refers to the presence of the* Π*-symmetry with respect to the ordinary case treated previously. This is covered by two affine superspaces, each isomorphic to* $\mathbb{C}^{1|1}$*, having coordinates in the super big-cells notation given by*

$$\mathcal{Z}_{\mathcal{U}_0} := \left(\begin{array}{cc||cc} 1 & x_0 & 0 & \xi_0 \\ \hline 0 & -\xi_0 & 1 & x_0 \end{array} \right) \qquad \mathcal{Z}_{\mathcal{U}_1} := \left(\begin{array}{cc||cc} x_1 & 1 & \xi_1 & 0 \\ \hline -\xi_1 & 0 & x_1 & 1 \end{array} \right). \tag{4.2}$$

It is then not hard to find the transition functions in the intersections of the charts either by means of allowed rows and column operation or by the method explained above. By rows and columns operations, for example, one finds:

$$\begin{aligned}
&\left(\begin{array}{cc||cc} 1 & x_0 & 0 & \xi_0 \\ \hline 0 & -\xi_0 & 1 & x_0 \end{array} \right) \xrightarrow{R_0/x_0, R_1/x_0} \left(\begin{array}{cc||cc} 1/x_0 & 1 & 0 & \xi_0/x_0 \\ \hline 0 & -\xi_0/x_0 & 1/x_0 & 1 \end{array} \right) \\
&\left(\begin{array}{cc||cc} 1/x_0 & 1 & 0 & \xi_0/x_0 \\ \hline 0 & -\xi_0/x_0 & 1/x_0 & 1 \end{array} \right) \xrightarrow{R_0 - \xi_0/x_0 R_1} \left(\begin{array}{cc||cc} 1/x_0 & 1 & -\xi_0/x_0^2 & 0 \\ \hline 0 & -\xi_0/x_0 & 1/x_0 & 1 \end{array} \right) \\
&\left(\begin{array}{cc||cc} 1/x_0 & 1 & -\xi_0/x_0^2 & 0 \\ \hline 0 & -\xi_0/x_0 & 1/x_0 & 1 \end{array} \right) \xrightarrow{R_1 + \xi_0/x_0 R_0} \left(\begin{array}{cc||cc} 1/x_0 & 1 & -\xi_0/x_0^2 & 0 \\ \hline \xi_0/x_0^2 & 0 & 1/x_0 & 1 \end{array} \right).
\end{aligned}$$

One can then read the transition functions in the intersection of the affine charts, characterising the structure sheaf $\mathcal{O}_{\mathbb{P}^1_\Pi}$ *of the* Π*-projective line:*

$$x_1 = \frac{1}{x_0}, \qquad \xi_1 = -\frac{\xi_1}{x_0^2}. \tag{4.3}$$

This leads to the conclusion that the Π*-projective line* $\mathbb{P}^{1|1}_\Pi$ *is the* $1|1$*-dimensional supermanifold that is completely characterised by the pair* $(\mathbb{P}^1, \Pi\mathcal{O}_{\mathbb{P}^1}(-2))$*. We emphasise that* $\mathcal{O}_{\mathbb{P}^1}(-2) \cong \Omega^1_{\mathbb{P}^1}$ *over* $\mathbb{P}^1$*: we will see in what follows that this is not by accident.*

Before we go on, some easy remarks are in order. First, one can immediately observe that the Π-projective line is substantially different compared with the projective superline $\mathbb{P}^{1|1}$ (this is also remarked in [28]). Indeed, as explained in Theorem 3.1, being $\mathbb{P}^{1|1}$ a $1|1$ dimensional supermanifold is completely characterised by the pair $(|\mathcal{M}| = \mathbb{P}^1, \mathcal{F}_{\mathcal{M}} = \Pi\mathcal{O}_{\mathbb{P}^1}(-1))$.
Also, without going into details, we note that while $\mathbb{P}^{1|1}$ can be structured as *super Riemann surface* (see [41] or [69]), the Π-projective line $\mathbb{P}^1_\Pi$ cannot. Indeed, the fermionic bundle $\mathcal{F}_{\mathbb{P}^{1|1}_\Pi} = (\mathcal{O}_{\mathbb{P}^1_\Pi})_1$ is given by $\mathcal{O}_{\mathbb{P}^1}(-2)$ and this does not define a *theta characteristic* on $\mathbb{P}^1$. Indeed, there is just one such, and it is given by $\mathcal{O}_{\mathbb{P}^1}(-1)$, therefore the only genus zero super Riemann surface is given by the ordinary $\mathbb{P}^{1|1} = (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1))$. This has a certain importance in the mathematical formulation of superstring perturbation theory [69].

Before we go on, we recall that in case a supermanifold $\mathcal{M}$ has odd dimension greater than 1 it is no longer true in general that the supermanifold is completely determined by the pair $(\mathcal{M}_{red}, \mathcal{F}_{\mathcal{M}})$ (as showed in Theorem 3.3, Chapter 3, for the case the odd dimension is equal to 2) and if this is the case, then the supermanifold is split.
In the following theorem we use the same method as above to write down the generic form of the transition functions of $\mathbb{P}^n_\Pi$: we will see that a certain nilpotent correction appears in the even transition functions.

Theorem 4.1. *Let* $\mathbb{P}^n_\Pi := (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n_\Pi})$ *be the* n*-dimensional* Π*-projective space and let* $\tilde{\mathcal{U}_i} = (\mathcal{U}_i, \mathbb{C}[z_{ji}, \theta_{ji}]) \cong \mathbb{C}^{n|n}$ *for* $i = 0, \ldots, n$, $j \neq i$ *be the affine supermanifolds covering* $\mathbb{P}^n_\Pi$*. In the intersections* $\mathcal{U}_i \cap \mathcal{U}_j$ *for* $0 \leqslant i < j \leqslant n+1$ *the transition functions characterising* $\mathcal{O}_{\mathbb{P}^n_\Pi}$ *have the following form:*

$$\ell \neq i : \qquad z_{\ell j} = \frac{z_{\ell i}}{z_{ji}} + \frac{\theta_{ji}\theta_{\ell i}}{z_{ji}^2}, \qquad \theta_{\ell j} = \frac{\theta_{\ell i}}{z_{ji}} - \frac{z_{\ell i}}{z_{ji}^2}\theta_{ji}; \tag{4.4}$$

$$\ell = i : \qquad z_{ij} = \frac{1}{z_{ji}}, \qquad \theta_{ij} = -\frac{\theta_{ji}}{z_{ji}^2}. \tag{4.5}$$

Proof. $\mathbb{P}^n_\Pi$ is covered by $n+1$ affine charts, whose coordinates are given in the super big cell notation by

$$\mathcal{Z}_{\mathcal{U}_i} = \left(\begin{array}{ccccc||ccccc} z_{1i} & \cdots & 1 & \cdots & z_{ni} & \theta_{1i} & \cdots & 0 & \cdots & \theta_{ni} \\ \hline -\theta_{1i} & \cdots & 0 & \cdots & -\theta_{ni} & z_{1i} & \cdots & 1 & \cdots & z_{ni} \end{array} \right), \tag{4.6}$$

where the 1's and 0's sit at the i-th positions. Considering the super big cell $\mathcal{Z}_{\mathcal{U}_j}$ for $j \neq i$ one can find the transition functions by bringing $\mathcal{Z}_{\mathcal{U}_i}$ in the form of $\mathcal{Z}_{\mathcal{U}_j}$ by means of allowed rows and column operations (and remembering that it is not possible to divide by a nilpotent element) as done above in the case of $\mathbb{P}^1_\Pi$. It is easily checked that this yields the claimed result. □

In the following section we will see that the same transition functions characterising $\mathbb{P}^n_\Pi$ arise naturally upon the choice of the cotangent sheaf as the fermionic sheaf for a supermanifold over $\mathbb{P}^n$.

4.3 Cotangent Sheaf and Π-Projective Spaces

We now enter the main section of the chapter: here we will provide a construction of $\mathbb{P}^n_\Pi$ as a non-projected supermanifold over $\mathbb{P}^n$ having ferminic sheaf given by the cotangent sheaf on ordinary projective space.

We keep the notation we have employed in the previous chapter when working on ordinary projective spaces $\mathbb{P}^n$. We consider the usual covering by $n+1$ open sets $\{\mathcal{U}_i\}_{i=0}^n$ characterised by the condition $\mathcal{U}_i := \{[X_0 : \ldots : X_n] \in \mathbb{P}^n : X_i \neq 0\}$. Defining the affine coordinates to be

$$z_{ji} := \frac{X_j}{X_i}, \tag{4.7}$$

we have that $\mathbb{P}^n$ gets covered by standard $n+1$ affine charts isomorphic to $\mathbb{C}^n$. This allows to easily write down the transition functions for two sheaves of interest, the tangent and the cotangent sheaf.

- **Tangent Sheaf** $\mathcal{T}_{\mathbb{P}^n}$: on the intersection $\mathcal{U}_i \cap \mathcal{U}_j$ one finds:

$$\partial_{z_{ji}} = -z_{ij} \sum_{k \neq j} z_{kj} \partial_{z_{kj}} \tag{4.8}$$

$$\partial_{z_{ki}} = z_{ij} \partial_{z_{kj}} \qquad k \neq j \tag{4.9}$$

- **Cotangent Sheaf** $\Omega^1_{\mathbb{P}^n}$: on the intersection $\mathcal{U}_i \cap \mathcal{U}_j$ one finds:

$$dz_{ji} = -\frac{dz_{ij}}{z_{ij}^2} \tag{4.10}$$

$$dz_{ki} = -\frac{z_{kj}}{z_{ij}^2} dz_{ij} + \frac{dz_{kj}}{z_{ij}} \qquad k \neq j \tag{4.11}$$

We now consider a supermanifold of dimension $n|n$ having reduced space given by $\mathbb{P}^n$ and a fermionic sheaf $\mathcal{F}_{\mathcal{M}}$ given by $\Pi\Omega^1_{\mathbb{P}^n}$. As already stressed in the previous chapter for the specific case of $\Pi\Omega^1_{\mathbb{P}^2}$, that gave an example of non-projected supermanifolds of the kind $\mathbb{P}^2_\omega(\mathcal{F}_{\mathcal{M}})$ having

a non-decomposable homogeneous fermionic sheaf, we have that $\Pi\Omega^1_{\mathbb{P}^n}$ is a sheaf of $\mathcal{O}_{\mathbb{P}^n}$-modules of rank $0|n$, that is locally-generated on $\mathcal{U}_i$ by n odd elements $\{\theta_{1i}, \ldots, \theta_{ni}\}$ that transform on the intersections $\mathcal{U}_i \cap \mathcal{U}_j$ as the local generators of the cotangent sheaf, $\{dz_{1i}, \ldots, dz_{ni}\}$: in other words, the correspondence is $dz_{ki} \leftrightarrow \theta_{ki}$ for $k \neq i$.
The crucial observation is that *the fermionic sheaf* $\mathcal{F}_{\mathcal{M}} = \Pi\Omega^1_{\mathbb{P}^n}$ *determined by the cotangent sheaf* $\Omega^1_{\mathbb{P}^n}$ *reproduces exactly the odd transition functions of* $\mathbb{P}^n_{\Pi}$. It is then natural to ask whether the whole $\mathcal{O}_{\mathbb{P}^n_{\Pi}}$ is determined someway by $\Pi\Omega^1_{\mathbb{P}^n}$. We will see that this question has an affermative answer, by realising $\mathbb{P}^n_{\Pi}$ as the non-projected supermanifold whose even part of the structure sheaf $\mathcal{O}_{\mathbb{P}^n_{\Pi}}$ is determined by $\mathcal{O}_{\mathbb{P}^n}$ and the fundamental obstruction class $\omega_{\mathcal{M}}$.
First we need to prove that the choice $\mathcal{F}_{\mathcal{M}} = \Pi\Omega^1_{\mathbb{P}^n}$ can actually give rise to a non-projected supermanifold. For this to be true it is enough that $H^1(\mathcal{T}_{\mathbb{P}^n} \otimes Sym^2\Pi\Omega^1_{\mathbb{P}^n}) \neq 0$: this is achieved in the following

Lemma 4.2. $H^1(\mathcal{T}_{\mathbb{P}^n} \otimes Sym^2\Pi\Omega^1_{\mathbb{P}^n}) \cong \mathbb{C}$

Proof. Due to parity reason, one has $Sym^2\Pi\Omega^1_{\mathbb{P}^n} \cong \bigwedge^2 \Omega^1_{\mathbb{P}^n}$, therefore it amounts to evaluate $H^1(\mathcal{T}_{\mathbb{P}^n} \otimes \bigwedge^2 \Omega^1_{\mathbb{P}^n})$: this can be done using the Euler exact sequence tensored by $\Omega^2_{\mathbb{P}^n} := \bigwedge^2 \Omega^1_{\mathbb{P}^n}$, this reads

$$0 \longrightarrow \Omega^2_{\mathbb{P}^n} \longrightarrow \Omega^2_{\mathbb{P}^n}(+1)^{\oplus n+1} \longrightarrow \mathcal{T}_{\mathbb{P}^n} \otimes \Omega^2_{\mathbb{P}^n} \longrightarrow 0. \tag{4.12}$$

Using Bott formulas (see for example [49]) to evaluate the cohomology of $\Omega^2_{\mathbb{P}^n}$ and $\Omega^2_{\mathbb{P}^n}(+1)$ one is left with the isomorphism $H^1(\mathcal{T}_{\mathbb{P}^n} \otimes \Omega^2_{\mathbb{P}^n}) \cong H^2(\Omega^2_{\mathbb{P}^n}) \cong \mathbb{C}$, again by Bott formulas. $\square$

A consequence of the lemma is that each choice of a class $0 \neq \omega_{\mathcal{M}} \in H^1(\mathcal{T}_{\mathbb{P}^n} \otimes Sym^2\Pi\Omega^1_{\mathbb{P}^n})$ gives rise to a non-projected supermanifold having reduced space $\mathbb{P}^n$ and fermionic sheaf $\Pi\Omega^1_{\mathbb{P}^n}$.
Making use of the Bott formulas, as in the proof of the previous lemma, is certainly the briefest and easiest way to show the non vanishing of the cohomology group. Anyway, there is another more instructive way to achieve the same result: this also has the merit to allow to find the representative of $H^1(\mathcal{T}_{\mathbb{P}^n} \otimes Sym^2\Pi\Omega^1_{\mathbb{P}^n})$. Keeping in mind that $Sym^2\Pi\Omega^2_{\mathbb{P}^n} \cong \bigwedge^2 \Omega^1_{\mathbb{P}^n}$, one starts from the dual of the Euler exact sequence, that reads

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow 0. \tag{4.13}$$

Taking its second exterior power one gets

$$0 \longrightarrow \bigwedge^2 \Omega^1_{\mathbb{P}^n} \longrightarrow \bigwedge^2 \left(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1}\right) \longrightarrow \Omega^1_{\mathbb{P}^n} \longrightarrow 0. \tag{4.14}$$

Notice that clearly $\bigwedge^2 \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \cong \mathcal{O}_{\mathbb{P}^n}(-2)^{\oplus\binom{n+1}{2}}$, and, more important, that the existence of this short exact sequence depends on the fact that $\mathcal{O}_{\mathbb{P}^n}$ is of rank 1. A more careful discussion of the general framework for second exterior powers of short exact sequences of locally-free sheaf of $\mathcal{O}_{\mathcal{M}_{red}}$-modules is deferred to the Appendix.
This short exact sequence can be in turn tensored by $\mathcal{T}_{\mathbb{P}^n}$ as to yield

$$0 \longrightarrow \mathcal{T}_{\mathbb{P}^n} \otimes \bigwedge^2 \Omega^1_{\mathbb{P}^n} \longrightarrow \mathcal{T}_{\mathbb{P}^n}(-2)^{\oplus\binom{n+1}{2}} \longrightarrow \mathcal{T}_{\mathbb{P}^n} \otimes \Omega^1_{\mathbb{P}^n} \longrightarrow 0. \tag{4.15}$$

Upon using the Euler exact sequence for the tangent sheaf twisted by $\mathcal{O}_{\mathbb{P}^n}(-2)$, one sees that for $n > 1$ the cohomology groups $H^0(\mathcal{T}_{\mathbb{P}^n}(-2)^{\oplus\binom{n+1}{2}})$ and $H^1(\mathcal{T}_{\mathbb{P}^n}(-2)^{\oplus\binom{n+1}{2}})$ are zero, and therefore the long exact cohomology sequence gives the isomorphism $H^0(\mathcal{T}_{\mathbb{P}^n} \otimes \Omega^1_{\mathbb{P}^n}) \cong H^1(\mathcal{T}_{\mathbb{P}^n} \otimes \bigwedge^2 \Omega^1_{\mathbb{P}^n})$.
The global section generating $H^0(\mathcal{T}_{\mathbb{P}^n} \otimes \Omega^1_{\mathbb{P}^n})$ is easily identified in $C^0(\mathcal{T}_{\mathbb{P}^n} \otimes \Omega^1_{\mathbb{P}^n})$. In particular, it is easy to verify that $H^0(\mathcal{T}_{\mathbb{P}^n} \otimes \Omega^1_{\mathbb{P}^n})$ has basis given by $\{\eta_i\}_{i=0}^n \in \prod_{i=0}^n(\mathcal{T}_{\mathbb{P}^n} \otimes \Omega^1_{\mathbb{P}^n})(\mathcal{U}_i)$, with

$$\eta_i = \sum_{j \neq i} \partial_{z_{ji}} \otimes dz_{ji}. \tag{4.16}$$

We aim to lift this element to the generator of $H^1(\mathcal{T}_{\mathbb{P}^n} \otimes \bigwedge^2 \Omega^1_{\mathbb{P}^n})$, making the isomorphism explicit: this will be the key step of our construction of Π-projective spaces as non-projected supermanifolds.

To achieve this, we need to study carefully the homomorphisms of sheaves entering the exact sequence (4.14). First we consider the injective map $\wedge^2\iota : \bigwedge \Omega^1_{\mathbb{P}^n} \to \bigwedge^2 \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1}$. This is given by

$$\begin{aligned} \wedge^2\iota : \textstyle\bigwedge^2 \Omega^1_{\mathbb{P}^n} &\longrightarrow \textstyle\bigwedge^2 \left(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1}\right) \\ df \wedge dg &\longmapsto \iota(df) \wedge \iota(dg) \end{aligned} \tag{4.17}$$

where $\iota : \Omega^1_{\mathbb{P}^n} \to \mathcal{O}(-1)^{\oplus n+1}$ is the map from the dual of the Euler exact sequence, that is, working for example in the chart $\mathcal{U}_i$,

$$\begin{aligned} \iota : \Omega^1_{\mathbb{P}^n} &\longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \\ df = \textstyle\sum_{j\neq i} f_{ji} dz_{ji} &\longmapsto \left(\tfrac{f_{0i}}{X_i}, \dots, -\tfrac{1}{X_i^2}\textstyle\sum_{j\neq i} X_j f_{ji}, \dots, \tfrac{f_{nj}}{X_i}\right). \end{aligned} \tag{4.18}$$

Getting back to (4.14) and working in the chart $\mathcal{U}_i$, the map $\Phi_2 : \bigwedge^2 \left(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1}\right) \to \Omega^1_{\mathbb{P}^n}$ is defined as follows

$$\begin{aligned} \Phi_2 : \textstyle\bigwedge^2 \left(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1}\right) &\longrightarrow \Omega^1_{\mathbb{P}^n} \\ (f_0, \dots, f_n) \wedge (g_0, \wedge, g_n) &\longmapsto X_i \textstyle\sum_{j=0}^n \sum_{k\neq i} X_j \left(f_j \otimes g_k - g_j \otimes f_k\right) d\left(\tfrac{X_k}{X_i}\right), \end{aligned}$$

where $d(X_k/X_i) = dz_{ki}$. The reader can check that these maps give rise to an exact sequence of locally-free $\mathcal{O}_{\mathbb{P}^n}$-modules. Clearly, the maps entering the exact sequence (4.15) are just the same tensored by identity on the tangent sheaf.
Knowing these map, we prove the following lemma.

Lemma 4.3 (Lifting)**.** *The cohomology group $H^1(\mathcal{T}_{\mathbb{P}^n} \otimes \bigwedge^2 \Omega^1_{\mathbb{P}^n})$ has basis $\{\omega_{ij}\}_{i<j} \in \prod_{i<j}\left(\mathcal{T}_{\mathbb{P}^n} \otimes \bigwedge^2 \Omega^1_{\mathbb{P}^n}\right)(\mathcal{U}_i \cap \mathcal{U}_j)$, with*

$$\omega_{ij} = \sum_{k\neq j} \partial_{z_{kj}} \otimes \frac{dz_{ij} \wedge dz_{kj}}{z_{ij}}. \tag{4.19}$$

Proof. We need to lift the element $(\sum_{j\neq i} \partial_{z_{ji}} \otimes dz_{ji})_{i=0,\dots,n} \in Z^0(\mathcal{T}_{\mathbb{P}^n} \otimes \Omega^1_{\mathbb{P}^n})$ to $Z^1(\mathcal{T}_{\mathbb{P}^n} \otimes \bigwedge^2 \Omega^1_{\mathbb{P}^n})$ as in the following diagram:

$$\begin{tikzcd} C^1(\mathcal{T}_{\mathbb{P}^n} \otimes \bigwedge^2 \Omega^1_{\mathbb{P}^n}) \arrow[r, tail] & C^1(\mathcal{T}_{\mathbb{P}^n} \otimes \bigwedge^2 \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1}) & \\ & C^0(\mathcal{T}_{\mathbb{P}^n} \otimes \bigwedge^2 \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1}) \arrow[u, "\delta"] \arrow[r, two heads] & C^0(\mathcal{T}_{\mathbb{P}^n} \otimes \Omega^1_{\mathbb{P}^n}) \end{tikzcd}$$

where the maps are induced by those defining the short exact sequence 4.15. The first step is to find the pre-image of the element $\sum_{j\neq i} \partial_{z_{ji}} \otimes dz_{ji}$ in $C^0(\mathcal{T}_{\mathbb{P}^n} \otimes \bigwedge^2 \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1})$. We work, for simplicity, in the chart $\mathcal{U}_i$, and we look for elements f's and g's such that

$$\sum_{j\neq i} \partial_{z_{ji}} \otimes dz_{ji} \stackrel{!}{=} \left(\sum_{j\neq i} \partial_{z_{ji}}\right) \otimes \left(\sum_{\ell\neq i} X_i \sum_{k=0}^{n} X_k \left[f_k^{(j)} \otimes g_\ell^{(j)} - g_k^{(j)} \otimes f_\ell^{(j)}\right] dz_{\ell i}\right).$$

The condition is satisfied by the choice

$$f_k^{(j)} = \frac{\delta_{ki}}{X_i} \qquad\qquad g_\ell^{(j)} = \frac{\delta_{\ell j}}{X_i}, \tag{4.20}$$

so that one finds that the pre-image in the chart $\mathcal{U}_i$ reads

$$\Phi_2^{-1}\left(\sum_{j\neq i} \partial_{z_{ji}} \otimes dz_{ji}\right) = \sum_{k\neq i} \partial_{z_{ki}} \otimes \frac{e_i \wedge e_k}{X_i^2} \tag{4.21}$$

where we have denoted $\{e_i \wedge e_k\}_{i\neq k}$ a basis for the second exterior power $\bigwedge^2 \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1}$.
Now we lift this element to a Čech 1-cochain by means of the Čech coboundary map δ as to get on an intersection $\mathcal{U}_i \cap \mathcal{U}_j$

$$s_{ij} := (s_i - s_j)\lfloor_{\mathcal{U}_i\cap\mathcal{U}_j} = \frac{1}{X_j^2}\sum_{k\neq i,j} \partial_{z_{kj}} \otimes \left(\frac{z_{kj}}{z_{ij}} e_j \wedge e_i + \frac{1}{z_{ij}} e_i \wedge e_k + e_j \wedge e_k\right) \tag{4.22}$$

It is not hard to verity that $(s_{ij})_{i\neq j}$ is the image through the injective map $\wedge^2\iota$ of elements

$$\omega_{ij} := \sum_{k\neq j} \frac{dz_{ij}\wedge dz_{kj}}{z_{ij}} \otimes \partial_{z_{kj}} \in Z^1(\mathcal{U}, \mathcal{T}_{\mathbb{P}^n} \otimes \bigwedge^2 \Omega^1_{\mathbb{P}^n}), \tag{4.23}$$

which represents the lifting of $\sum_{j\neq i} \partial_{z_{ji}} \otimes dz_{ji}$, and generates the cohomology group $H^1(\mathcal{T}_{\mathbb{P}^n} \otimes \bigwedge^2 \Omega^1_{\mathbb{P}^n})$. □

Notice that in (4.23) one finds the actual elements that enter the transition functions of the non-projected supermanifold in the identification $dz_{ij} \wedge dz_{kj} \leftrightarrow \theta_{ij}\theta_{kj}$, where the second is to be understood as the symmetric product, giving an element in $Sym^2\Pi\Omega^1_{\mathbb{P}^n}$.
The previous lemma gives all the elements we need in order to recognise the Π-projective space $\mathbb{P}^n_\Pi$ as the non-projected supermanifold associated to the cotangent sheaf on $\mathbb{P}^n$. In particular, we have the following

Theorem 4.2 (Π-Projective Spaces)**.** *The Π-projected space $\mathbb{P}^n_\Pi := (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n_\Pi})$ is the non-projected supermanifold uniquely identified by the triple $(\mathbb{P}^n, \Pi\Omega^1_{\mathbb{P}^n}, \lambda)$, where $\lambda \neq 0$ is the representative of $\omega_{\mathcal{M}} \in H^1(\mathcal{T}_{\mathbb{P}^n} \otimes Sym^2\Pi\Omega^1_{\mathbb{P}^n}) \cong \mathbb{C}$.*

Proof. It is enough to proof that the sheaf $\mathcal{O}_{\mathbb{P}^n_\Pi}$ can be determined out of the structure sheaf $\mathcal{O}_{\mathbb{P}^n}$ of $\mathbb{P}^n$, the fermionic sheaf $\Pi\Omega^1_{\mathbb{P}^n}$ and the non-zero representative $\lambda \in \mathbb{C}\backslash\{0\}$ of $\omega_{\mathcal{M}} \in H^1(\mathcal{T}_{\mathbb{P}^n} \otimes Sym^2\Pi\Omega^1) \cong \mathbb{C}$. We have already observed that the transition functions of $(\mathcal{O}_{\mathbb{P}^n_\Pi})_1$ do coincide with those of $\Pi\Omega^1_{\mathbb{P}^n}$. Moreover, up to a change of coordinate or a scaling, λ can be chosen equal to 1. Then, we see that the transition functions of $(\mathcal{O}_{\mathbb{P}^n_\Pi})_0$ are determined by (3.10) as a non-projected extension of $\mathcal{O}_{\mathbb{P}^n}$ by $Sym^2\Pi\Omega^1_{\mathbb{P}^n}$, as follows

$$z_{ki} = \frac{z_{kj}}{z_{ij}} + \left(\sum_{k\neq j} \frac{\theta_{ij}\theta_{kj}}{z_{ij}} \partial_{z_{kj}}\right) z_{ki} = \frac{z_{kj}}{z_{ij}} + \left(\sum_{k\neq j} \frac{\theta_{ij}\theta_{kj}}{z_{ij}^2} \partial_{z_{ki}}\right) z_{ki} \tag{4.24}$$

$$= \frac{z_{kj}}{z_{ij}} + \frac{\theta_{ij}\theta_{kj}}{z_{ij}^2} \tag{4.25}$$

and clearly, $z_{ji} = 1/z_{ij}$. Here, we have used the result of the previous lemma 4.3 to write the representatives of the fundamental obstruction class $\{\omega_{ij}\}_{i<j}$. This completes the proof. □

Now that we have constructed Π-projective spaces as non-projected supermanifolds, we investigate a property that all of the Π-projective spaces share, regardless their dimensions: they have trivial Berezinian sheaf, and as such, they are Calabi-Yau supermanifolds (see section 1.3, in particular Definition 1.20 and the discussion thereof).

Before we go into the proof of the theorem, we recall that given a supermanifold $\mathcal{M}$, the Berezinian sheaf $\mathcal{B}er(\mathcal{M})$ is, by definition the sheaf $\mathcal{B}er(\Omega^1_{\mathcal{M}})$, as in Definition 1.18. That is, given an open covering $\{\mathcal{U}_i\}_{i\in\mathcal{I}}$ of the underlying topological space of $\mathcal{M}$, the Berezinian sheaf $\mathcal{B}er_{\mathcal{M}}$ is the sheaf whose transition functions $\{g_{ij}\}_{i\neq j\in\mathcal{I}}$ are obtained by taking the Berezinian of the super Jacobian of a change of coordinates in $\mathcal{U}_i \cap \mathcal{U}_j$.

Theorem 4.3 ($\mathbb{P}^n_\Pi$ are Calabi-Yau Supermanifolds)**.** *Π-projective spaces $\mathbb{P}^n_\Pi$ have trivial Berezianian sheaf. That is, $\mathcal{B}er(\mathbb{P}^n_\Pi) \cong \mathcal{O}_{\mathbb{P}^n_\Pi}$.*

Proof. Let us consider the generic case $n \geqslant 2$. By symmetry, it is enough to prove the triviality of the Berezianian sheaf in a single intersection. One starts computing the super Jacobian of the transition functions in 4.4 and 4.5: this actually gives the transition functions of the cotangent sheaf to $\mathbb{P}^n_\Pi$ in a certain intersection $\mathcal{U}_i \cap \mathcal{U}_j$, that can be represented in a super matrix of the form

$$[\mathcal{J}ac]_{ij} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \tag{4.26}$$

for some A and D even and B and C odd sub-matrices depending on the intersection $\mathcal{U}_i \cap \mathcal{U}_j$. Then one computes $\mathrm{Ber}\,[\mathcal{J}ac]_{ij}$ by means of the formula $\mathrm{Ber}(X) = \det(A)\det(D - CA^{-1}B)^{-1}$ that reduces the computation of the Berezinian of a super matrix X to a computation of determinants of ordinary matrices. It can be easily checked that for every non-empty intersection $\mathcal{U}_i \cap \mathcal{U}_j$ one gets

$$\det A = -\frac{1}{z_{ij}^{n+1}}, \qquad \det(D - CA^{-1}B)^{-1} = -z_{ij}^{n+1}, \tag{4.27}$$

so that $\mathrm{Ber}\,[\mathcal{J}ac]_{ij} = 1$, proving triviality of $\mathcal{B}er(\mathbb{P}^n_\Pi)$. The interested reader finds some explicit computation performed in the intersection $\mathcal{U}_0 \cap \mathcal{U}_1$ in the Appendix.
Finally, the case $n = 1$ is trivial, as $\mathbb{P}^1_\Pi$ is split: in general, as shown in Theorem 1.1, for a *split* supermanifold $\mathcal{B}er(\mathcal{M}) \cong \pi^*(\mathcal{K}_{\mathcal{M}_{red}} \otimes \det \mathcal{F}^*_{\mathcal{M}})$ where $\pi : \mathcal{M} \to \mathcal{M}_{red}$ is the projection onto the reduced manifold and $\mathcal{K}_{\mathcal{M}_{red}}$ is the canonical sheaf of the reduced manifold, so that in the case of $\mathbb{P}^1_\Pi$ one gets

$$\mathcal{B}er(\mathbb{P}^1_\Pi) \cong \pi^*\left(\mathcal{K}_{\mathbb{P}^1} \otimes (\Omega^1_{\mathbb{P}^1})^*\right) \cong \pi^*\left(\mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}_{\mathbb{P}^1}(+2)\right) \cong \pi^*\mathcal{O}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1_\Pi}. \tag{4.28}$$

thus concluding the proof. □

Note that we did already know examples of *split* Calabi-Yau supermanifolds in every bosonic/even dimension: these are the well-known projective superspaces of the kind $\mathbb{P}^{n|n+1}$ for every $n \geqslant 1$ (see section 1.3 in the first chapter). The previous theorem characterises Π-projective spaces as relatively simple examples of *non-projected* Calabi-Yau supermanifold for every bosonic/even dimension. In this context, the Π-projective line $\mathbb{P}^1_\Pi$ is actually the only Calabi-Yau supermanifold of dimension $1|1$ which has $\mathbb{P}^1$ as reduced space.

4.4 A Glimpse at Π-Grassmannians

In the previous section we have shown that Π-projective spaces arise as certain non-projected supermanifolds whose fermionic sheaf is related to the cotangent sheaf of their reduced manifold $\mathbb{P}^n$ and, as such, they have a very simple structure.

Remarkably, something very similar happens more in general for Π-Grassmannians (see [41]), supporting the idea of a close connection between Π-symmetry in supergeometry and the ordinary geometry of cotangent sheaves of the ordinary reduced variety. Indeed we claim

"all of the Π-Grassmannians $G_\Pi(n;m)$ can be constructed as higher-dimensional non-projected supermanifolds whose fermionic sheaf is given precisely by the cotangent sheaf of their reduced manifold, the ordinary Grassmannian $G(n;m)$".

The difference that makes things trickier compared to the case of the Π-projective spaces, is that also higher obstruction classes - not only the fundamental one - might appear, leading to non-projected *and* non-split supermanifolds.
The construction of Π-Grassmannians as non-projected supermanifolds related to the cotangent sheaf $\Omega^1_{G(n;m)}$ of the underlying Grassmannian $G(n;m)$, the relation between their dimension, structure and the presence of higher obstructions to splitting are topics that the author wish to address in the near future.

For the time being, in support of the above claim and as an illustrative example, we analyse the structure of the transition functions in certain big-cells of the Π-Grassmannian $G_\Pi(2;4)$: notice that, as in the ordinary context, this is the first Π-Grassmannian that is *not* a Π-projective space. We start considering the reduced manifold, the ordinary Grassmannian $G(2;4)$ and look at the change of coordinates between the big-cells

$$\mathcal{Z}_{\mathcal{U}_1} = \begin{pmatrix} 1 & 0 & x_{11} & x_{21} \\ 0 & 1 & y_{11} & y_{21} \end{pmatrix} \qquad \mathcal{Z}_{\mathcal{U}_2} = \begin{pmatrix} 1 & x_{12} & 0 & x_{22} \\ 0 & y_{12} & 1 & y_{22} \end{pmatrix}. \tag{4.29}$$

By row-operations one easily finds that

$$x_{12} = -\frac{x_{11}}{y_{11}}, \qquad x_{22} = x_{21} - \frac{x_{11}y_{21}}{y_{11}}, \tag{4.30}$$

$$y_{12} = \frac{1}{y_11}, \qquad y_{22} = \frac{y_{21}}{y_{11}}. \tag{4.31}$$

In the correspondence $\{dx_{ij} \leftrightarrow \theta_{ij}, dy_{ij} \leftrightarrow \xi_{ij}\}$ of the local frames of the cotangent sheaf with those of its parity-reversed version $\Pi\Omega^1_{G(2;4)}$ we are concerned with, one has the following transition functions

$$\theta_{12} = -\frac{\theta_{11}}{y_11} + \frac{x_{11}}{y_{11}^2}\xi_{11}, \qquad \theta_{22} = \theta_{21} - \frac{y_{21}}{y_{11}}\theta_{11} - \frac{x_{11}}{y_{11}}\xi_{21} + \frac{x_{11}y_{21}}{y_{11}^2}\xi_{11}, \tag{4.32}$$

$$\xi_{12} = -\frac{\xi_{11}}{y_{11}^2}, \qquad \xi_{22} = \frac{\xi_{21}}{y_11} - \frac{y_{21}}{y_{11}^2}\xi_{11}. \tag{4.33}$$

Now look at the corresponding change of coordinates in $\mathcal{U}_1 \cap \mathcal{U}_2$ for $G_\Pi(2;4)$: the super big-cells then look like

$$\mathcal{Z}_{\mathcal{U}_1} = \left(\begin{array}{cccc||cccc} 1 & 0 & x_{11} & x_{21} & 0 & 0 & \theta_{11} & \theta_{21} \\ 0 & 1 & y_{11} & y_{21} & 0 & 0 & \xi_{11} & \xi_{21} \\ \hline\hline 0 & 0 & -\theta_{11} & -\theta_{21} & 1 & 0 & x_{11} & x_{21} \\ 0 & 0 & -\xi_{11} & -\xi_{21} & 0 & 1 & y_{11} & y_{21} \end{array}\right) \tag{4.34}$$

$$\mathcal{Z}_{\mathcal{U}_2} = \left(\begin{array}{cccc||cccc} 1 & x_{12} & 0 & x_{22} & 0 & \theta_{12} & 0 & \theta_{22} \\ 0 & y_{12} & 1 & y_{22} & 0 & \xi_{12} & 0 & \xi_{22} \\ \hline\hline 0 & -\theta_{12} & 0 & -\theta_{22} & 1 & x_{12} & 0 & x_{22} \\ 0 & -\xi_{12} & 0 & -\xi_{22} & 0 & y_{12} & 1 & y_{22} \end{array}\right). \tag{4.35}$$

Again, by acting with row-operations on $\mathcal{U}_1$ one finds the following change of coordinates for $G_\Pi(2;4)$ in $\mathcal{U}_1 \cap \mathcal{U}_2$

$$\begin{aligned} x_{12} &= -\frac{x_{11}}{y_{11}} - \frac{\theta_{11}\xi_{11}}{y_{11}^2}, & x_{22} &= x_{21} - \frac{x_{11}y_{21}}{y_{11}} + \frac{\theta_{11}\xi_{21}}{y_11} - \frac{x_11}{y_11^2}\xi_{11}\xi_{21} - \frac{y_{21}}{y_{11}^2}\theta_{11}\xi_{11}, \\ y_{12} &= \frac{1}{y_{11}}, & y_{22} &= \frac{y_{21}}{y_{11}} + \frac{\xi_{11}\xi_{21}}{y_{11}^2}, \\ \theta_{12} &= -\frac{\theta_{11}}{y_{11}} + \frac{x_{11}}{y_{11}^2}\xi_{11}, & \theta_{22} &= \theta_{21} - \frac{y_{21}}{y_{11}}\theta_{11} - \frac{x_{11}}{y_{11}}\xi_{21} + \frac{x_{11}y_{21}}{y_{11}^2}\xi_{11} - \frac{\theta_{11}\xi_{11}\xi_{21}}{y_{11}^2} \\ \xi_{12} &= -\frac{\xi_{11}}{y_{11}^2}, & \xi_{22} &= \frac{\xi_{21}}{y_{11}} - \frac{y_{21}}{y_{11}^2}\xi_{11}. \end{aligned}$$

We observe the following facts: as in the the case of Π-projective spaces, the bosonic transition functions get nilpotent "corrections" taking values in $Sym^2\Pi\Omega^1_{G(2;4)}(\mathcal{U}_1 \cap \mathcal{U}_2)$.
More important, here is the difference: the fermionic transition functions are *almost* the same but *not* actually the same as those of $\Pi\Omega^1_{G(2;4)}$ above! Indeed, in the transition functions of θ_{22} appears a term taking values in $Sym^3\Pi\Omega^1_{G(2;4)}(\mathcal{U}_1 \cap \mathcal{U}_2)$ - the term $-\frac{\theta_{11}\xi_{11}\xi_{21}}{y_{11}^2}$ - that tells that $G_\Pi(2;4)$ will also be characterised geometrically by the presence of *higher (fermionic) obstructions*! This is a subtle issue, as these obstructions are actually not well-defined whenever the first obstruction is non-zero, as in this case!
Once again, this example exposes a crucial problem in the theory of supermanifolds and calls for the need of a careful study of higher obstruction classes, an issue the author intends to address in a near future.

4.5 Reprise: Homogeneous $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ and Embedding in $\mathbb{P}^n_\Pi$

In this section we make the connection with the previous chapter, by discussing in more detail the possible embeddings for homogeneous $\mathbb{P}^2_\omega(\mathcal{F}_\mathcal{M})$ in connection with Π-projective spaces $\mathbb{P}^n_\Pi$.

Again, as in the case of embeddings into projective superspaces discussed above, looking for embeddings in a Π-projective spaces calls for a search for suitable ample invertible sheaves, in particular we need to look for a special kind of invertible sheaf. With reference to Definition 4.1, we give the following

Definition 4.3 (Π-Invertible Sheaf). *Let $\mathcal{M}$ be a supermanifold. A Π-invertible sheaf $\mathcal{G}_\Pi$ is a pair $(\mathcal{G}, p_\Pi)$, where $\mathcal{G}$ is a locally-free sheaf of $\mathcal{O}_\mathcal{M}$-modules of rank $1|1$ and $p_\Pi : \mathcal{G} \to \Pi\mathcal{G}$ is a Π-symmetry.*

Locally, on an open set $\mathcal{U} \subset |\mathcal{M}|$ one has that the Π-symmetry $p_\Pi : \mathcal{G} \to \Pi\mathcal{G}$ exchanges the even and odd components of the sheaf, $\mathcal{G}|_\mathcal{U} \cong \mathcal{O}_\mathcal{M}(\mathcal{U}) \oplus \Pi\mathcal{O}_\mathcal{M}(\mathcal{U})$.

Without going into detail, we mention that an important Π-invertible sheaf studied in [36] and suggestively denoted with $\mathcal{O}_{\mathbb{P}^n_\Pi}(1)$, plays the same role of the hyperplane bundle in the context of Π-projective geometry, that is it governs the maps to Π-projective spaces. Indeed in [36] is proved a theorem analogous to the one that holds for the sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$ in relation with maps to projective spaces $\mathbb{P}^n$ in ordinary algebraic geometry and for the sheaf $\mathcal{O}_{\mathbb{P}^{n|m}}(1)$ in relations with maps to projective superspaces $\mathbb{P}^{n|m}$. More precisely, one finds that $\mathcal{O}_{\mathbb{P}^n_\Pi}(1)$ is globally generated by $n+1|n+1$ global sections, that we formally denote by $\{\widetilde{X}_0, \ldots, \widetilde{X}_n \,|\, \widetilde{\Theta}_0, \ldots, \widetilde{\Theta}_n\}$ (see [36]). These serve to prove that if $\mathcal{E}_\Pi$ is a Π-invertible sheaf on a supermanifold $\mathcal{M}$, having $n+1|n+1$ global sections $\{s_0, \ldots, s_n | \xi_0, \ldots, \xi_n\}$ that globally generate it, then there exists a (unique) morphism $\phi_{\mathcal{E}_\Pi} : \mathcal{M} \to \mathbb{P}^n_\Pi$ such that $\mathcal{E}_\Pi = \phi^*_{\mathcal{E}_\Pi}(\mathcal{O}_{\mathbb{P}^n_\Pi}(1))$ and such that $s_i = \phi^*_{\mathcal{E}_\Pi}(\widetilde{X}_i)$ and $\xi_j = \phi^*_{\mathcal{E}_\Pi}(\widetilde{\Theta}_j)$ for $i = 0, \ldots, n$ and $j = 0, \ldots, n$. The converse is also true: given a morphism $\phi : \mathcal{M} \to \mathbb{P}^n_\Pi$, then there exists a globally generated Π-invertible sheaf $\mathcal{G}_{\Pi,\phi}$ such that it is generated by the global sections $\phi^*(\widetilde{X}_i)$ and $\phi^*(\widetilde{\Theta}_i)$ for $i = 0, \ldots, n+1$. This result suggests the following

Definition 4.4 (Π-Projective Supermanifold). *We say that a complex supermanifold $\mathcal{M}$ is Π-projective if there exists a morphism $\phi : \mathcal{M} \to \mathbb{P}^n_\Pi$ such that ϕ is injective on $\mathcal{M}_{red}$ and its differential $d\phi$ is injective everywhere on $\mathcal{T}_\mathcal{M}$.*

In order to discuss Π-projectivity of supermanifolds, we are thus led to study the Π-invertible sheaves that can be defined on them. The most useful tools for this purpose are provided by Manin in [42]. There, it is noted that, giving an odd involution on a rank $1|1$ sheaf corresponds to reduce its structure group, that generically is the whole super Lie group $GL(1|1, \mathcal{O}_\mathcal{M})$, to the non-commutative multiplicative group $\mathbb{G}^{1|1}_m(\mathcal{O}_\mathcal{M})$. Likewise, the set of isomorphism classes of Π-invertible sheaves on a certain supermanifold $\mathcal{M}$, denoted with $\mathrm{Pic}_\Pi(\mathcal{M})$ by similarity with the usual Picard group, can be identified with the *pointed set* $H^1(\mathbb{G}^{1|1}(\mathcal{O}_\mathcal{M}))$.
The embedding $\mathbb{G}_m \hookrightarrow \mathbb{G}^{1|1}_m$, induces a map as follows

$$\begin{aligned} i : \mathrm{Pic}_0(\mathcal{M}) &\longrightarrow \mathrm{Pic}_\Pi(\mathcal{M}) \\ \mathcal{L}_\mathcal{M} &\longmapsto \mathcal{L}_\mathcal{M} \oplus \Pi\mathcal{L}_\mathcal{M}, \end{aligned} \tag{4.36}$$

where $\mathcal{L}_\mathcal{M}$ is a locally-free sheaf of $\mathcal{O}_\mathcal{M}$-modules of rank $1|0$ and the Π-invertible sheaf $\mathcal{L}_\mathcal{M} \oplus \Pi\mathcal{L}_\mathcal{M}$ is called the *interchange of summands*, to stress that it comes endowed with the odd involution p_Π.
We say that a Π-invertible $\mathcal{G}_\Pi$ sheaf splits if there exists a locally-free sheaf of $\mathcal{O}_\mathcal{M}$-module $\mathcal{L}_\mathcal{M}$ of rank $1|0$ such that $\mathcal{G}_\Pi$ is isomorphic to the interchange of summands $\mathcal{L}_\mathcal{M} \oplus \Pi\mathcal{L}_\mathcal{M}$. Analogously, we might have said that a Π-invertible sheaf splits if its structure group $\mathbb{G}^{1|1}_m$ can in turn be reduced to the usual $\mathbb{G}_m$.
The injective map $\mathbb{G}_m \to \mathbb{G}^{1|1}_m$ fits into an exact sequence as follows (see again [42])

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}^{1|1}_m \longrightarrow \mathbb{G}^{0|1}_a \longrightarrow 0, \tag{4.37}$$

that is useful to study whenever a Π-invertible sheaf splits as the interchange of summands. Indeed, as $\mathbb{G}_m$ is central in $\mathbb{G}^{1|1}_m$, the sequence of pointed sets corresponding to the first Čech cohomology groups associated to the short exact sequence above can be further extended to $H^2(\mathbb{G}_m(\mathcal{O}_{\mathcal{M}})) = H^2(\mathcal{O}^*_{\mathcal{M},0})$, giving

$$\cdots \longrightarrow \mathrm{Pic}_0(\mathcal{M}) \longrightarrow \mathrm{Pic}_\Pi(\mathcal{M}) \longrightarrow H^1(\mathcal{O}_{\mathcal{M},1}) \xrightarrow{\delta} H^2(\mathcal{O}^*_{\mathcal{M},0}). \tag{4.38}$$

Clearly, the obstruction to splitting of Π-invertible sheaves for a supermanifolds lies in the image of the map $\mathrm{Pic}_\Pi(\mathcal{M}) \to H^1(\mathcal{O}_{\mathcal{M},1})$ or, analogously, by exactness, in the kernel of $H^1(\mathcal{O}_{\mathcal{M},1}) \to H^2(\mathcal{O}^*_{\mathcal{M},0})$.

These considerations can be used to study the existence of embeddings into $\mathbb{P}^n_\Pi$ for the two homogeneous supermanifolds $\mathbb{P}^2_\omega(\mathcal{F}_{\mathcal{M}})$ in the case of decomposable and non-decomposable fermionic sheaf.

4.5.1 $\mathbb{P}^2_\omega(\Pi\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \Pi\mathcal{O}_{\mathbb{P}^2}(-2))$ is Non Π-Projective

In this subsection we study the existence of embeddings into $\mathbb{P}^n_\Pi$ for the non-projected homogeneous supermanifold $\mathbb{P}^2_\omega(\mathcal{F}_{\mathcal{M}})$ in the case one chooses the fermionic sheaf to be the decomposable sheaf $\mathcal{F}_{\mathcal{M}} = \Pi\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \Pi\mathcal{O}_{\mathbb{P}^2}(-2))$. In particular, we have the following theorem.

Theorem 4.4. *Let $\mathcal{M} = \mathbb{P}^2_\omega(\mathcal{F}_{\mathcal{M}})$ with fermionic sheaf $\mathcal{F}_{\mathcal{M}}$ given by $\mathcal{F}_{\mathcal{M}} = \Pi\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \Pi\mathcal{O}_{\mathbb{P}^2}(-2)$. Then $\mathrm{Pic}_\Pi(\mathcal{M})$ is just a point, representing the trivial Π-invertible sheaf $\mathcal{O}_{\mathcal{M}} \oplus \Pi\mathcal{O}_{\mathcal{M}}$. In particular $\mathcal{M}$ cannot be embedded in a Π-projective space.*

Proof. Remembering that $\mathcal{F}_{\mathcal{M}} \cong \mathcal{O}_{\mathcal{M},1}$, as the supermanifold has dimension $2|2$, one easily compute that

$$H^1(\mathcal{O}_{\mathcal{M},1}) \cong H^1(\mathcal{F}_{\mathcal{M}}) \cong H^1(\mathcal{O}_{\mathbb{P}^2}(-1)) \oplus H^1(\mathcal{O}_{\mathbb{P}^2}(-2)) = 0. \tag{4.39}$$

This tells that we have a surjection

$$\mathrm{Pic}_0(\mathcal{M}) \longrightarrow \mathrm{Pic}_\Pi(\mathcal{M}) \longrightarrow 0. \tag{4.40}$$

and therefore all the Π-invertible sheaves will be of the form $\mathcal{L}_{\mathcal{M}} \oplus \Pi\mathcal{L}_{\mathcal{M}}$. On the other hand we do already know that the even Picard group of $\mathcal{M}$ is actually trivial, and the only invertible sheaf of rank $1|0$ is actually the structure sheaf. This tells that the only Π-invertible sheaf that can be defined on $\mathcal{M}$ endowed with a decomposable fermionic sheaf as above is actually given by $\mathcal{G}_\Pi := \mathcal{O}_{\mathcal{M}} \oplus \Pi\mathcal{O}_{\mathcal{M}}$. We have

$$\mathrm{Pic}_\Pi(\mathcal{M}) = \{\mathcal{O}_{\mathcal{M}} \oplus \Pi\mathcal{O}_{\mathcal{M}}\}, \tag{4.41}$$

that is the pointed set $\mathrm{Pic}_\Pi(\mathcal{M})$ is given by its base point only. Clearly, as there are no non-trivial Π-invertible sheaf there is no hope for $\mathcal{M}$ to be embedded in a Π-projective space. □

This result shows that, under these circumstances, the notion of Π-invertible sheaves is not useful to get more geometrical knowledge of the supermanifold: the non-projected supermanifold $\mathbb{P}^2_\omega(\mathcal{F}_{\mathcal{M}})$ endowed with the decomposable fermionic sheaf $\mathcal{F}_{\mathcal{M}}$ is not only non-projective, but also non Π-projective and, so far, super Grassmannians proved to be the only suitable ambient space for it. The scenario is much different as one considers $\mathbb{P}^2_\omega(\mathcal{F}_{\mathcal{M}})$ endowed with the non-decomposable fermionic sheaf $\mathcal{F}_{\mathcal{M}} = \Pi\Omega^1_{\mathbb{P}^2}$, as we shall see in the next subsection.

4.5.2 $\mathbb{P}^2_\omega(\Pi\Omega^1_{\mathbb{P}^2})$ is $\mathbb{P}^2_\Pi$ and its Minimal Embedding

In this subsection we stress an obvious consequence of Theorem 4.2: the non-projected homogeneous supermanifold $\mathbb{P}^2_\omega(\Pi\Omega^1_{\mathbb{P}^2})$ *is* actually the Π-projective plane $\mathbb{P}^2_\Pi$, and as such it describes $1|1$-dimesional Π-symmetric subspaces of $\mathbb{C}^{3|3}$.

Corollary 4.1 ($\mathbb{P}^2_\omega(\Pi\Omega^1_{\mathbb{P}^2}) = \mathbb{P}^2_\Pi$ (Version 1))**.** *The non-projected homogeneous supermanifold $\mathbb{P}^2_\omega(\Pi\Omega^1_{\mathbb{P}^2})$ is the Π-projective plane $\mathbb{P}^2_\Pi$.*

Proof. This is nothing but a corollary of Theorem 4.2, that characterises Π-projective spaces $\mathbb{P}^n_\Pi$, by a triple $(\mathbb{P}^n, \Pi\Omega^1_{\mathbb{P}^n}, \omega)$, for ω a non-zero class in $H^1(\mathcal{T}_{\mathbb{P}^n} \otimes Sym^2 \Pi\Omega^1_{\mathbb{P}^n}) \cong \mathbb{C}$. In particular, for $\mathbb{P}^2_\Pi$ we have the triple $(\mathbb{P}^2, \Pi\Omega^1_{\mathbb{P}^2}, \omega)$ for ω a non-zero class in $H^1(\mathcal{T}_{\mathbb{P}^2}(-3)) \cong \mathbb{C}$. In view of Theorem 3.3 and Definition 3.2, this supermanifold is nothing but the non-projected homogeneous supermanifold $\mathbb{P}^2_\omega(\Pi\Omega^1_{\mathbb{P}^2})$. □

Clearly, the same result could also be achieved explicitly. Indeed the Π-projective plane $\mathbb{P}^2_\Pi$ is covered by three affine charts, whose coordinates are given in the super big-cell notation by

$$\mathcal{Z}_{\mathcal{U}_0} = \left(\begin{array}{ccc||ccc} 1 & z_{10} & z_{20} & 0 & \theta_{10} & \theta_{20} \\ \hline\hline 0 & -\theta_{10} & -\theta_{20} & 1 & z_{10} & z_{20} \end{array} \right) \qquad \mathcal{Z}_{\mathcal{U}_1} = \left(\begin{array}{ccc||ccc} z_{11} & 1 & z_{21} & \theta_{11} & 0 & \theta_{21} \\ \hline\hline -\theta_{11} & 0 & -\theta_{21} & z_{11} & 1 & z_{21} \end{array} \right)$$

$$\mathcal{Z}_{\mathcal{U}_2} = \left(\begin{array}{ccc||ccc} z_{12} & z_{22} & 1 & \theta_{12} & \theta_{22} & 0 \\ \hline\hline -\theta_{12} & -\theta_{22} & 0 & z_{12} & z_{22} & 1 \end{array} \right),$$

these make apparent that the reduced manifold of $\mathbb{P}^2_\Pi$ is actually $\mathbb{P}^2$ and can also be used to find the transition functions to be compared with the ones we have found in (3.139).

Corollary 4.2 ($\mathbb{P}^2_\omega(\Pi\Omega^1_{\mathbb{P}^2})$ is $\mathbb{P}^2_\Pi$ (Version 2))**.** *The non-projected homogeneous supermanifold $\mathbb{P}^2_\omega(\Pi\Omega^1_{\mathbb{P}^2})$ is the Π-projective plane $\mathbb{P}^2_\Pi$.*

Proof. We have already seen that the topological space underlying $\mathbb{P}^2_\Pi$ is $\mathbb{P}^2$. To prove that the two spaces are the same supermanifold we consider the structure sheaf $\mathcal{O}_{\mathbb{P}^2_\Pi}$ of $\mathbb{P}^2_\Pi$ and we prove that the transition functions among its affine charts coincide with those of $\mathbb{P}^2_\omega(\Pi\Omega^1_{\mathbb{P}^2})$. To this end, by allowed rows operation we get

$$\left(\begin{array}{ccc||ccc} z_{11} & 1 & z_{12} & \theta_{11} & 0 & \theta_{12} \\ \hline\hline -\theta_{11} & 0 & -\theta_{12} & z_{11} & 1 & z_{12} \end{array} \right) \xrightarrow{R_0/z_{11}, R_1/z_{11}} \left(\begin{array}{ccc||ccc} 1 & 1/z_{11} & z_{21}/z_{11} & \theta_{11}/z_{11} & 0 & \theta_{12} \\ \hline\hline -\theta_{11}/z_{11} & 0 & -\theta_{12}/z_{11} & 1 & 1/z_{11} & z_{12}/z_{11} \end{array} \right) \rightarrow$$

$$\xrightarrow{R_0 - \theta_{11}/z_{11} R_1} \left(\begin{array}{ccc||ccc} 1 & 1/z_{11} & z_{21}/z_{11} + \theta_{11}\theta_{12}/(z_{11})^2 & 0 & -\theta_{11}/(z_{11})^2 & \theta_{12}/z_{11} - z_{12}\theta_{11}/(z_{11})^2 \\ \hline\hline -\theta_{11}/z_{11} & 0 & -\theta_{12}/z_{11} & 1 & 1/z_{11} & z_{12}/z_{11} \end{array} \right) \rightarrow$$

$$\xrightarrow{R_1 + \theta_{11}/z_{11} R_0} \left(\begin{array}{ccc||ccc} 1 & 1/z_{11} & z_{21}/z_{11} + \theta_{11}\theta_{12}/(z_{11})^2 & 0 & -\theta_{11}/(z_{11})^2 & \theta_{12}/z_{11} - z_{12}\theta_{11}/(z_{11})^2 \\ \hline\hline 0 & +\theta_{11}/(z_{11})^2 & -\theta_{12}/z_{11} + z_{12}\theta_{11}/(z_{11})^2 & 1 & 1/z_{11} & z_{12}/z_{11} + \theta_{11}\theta_{12}/(z_{11})^2 \end{array} \right).$$

So that one finds

$$z_{10} = \frac{1}{z_{11}}, \qquad z_{20} = \frac{z_{21}}{z_{11}} + \frac{\theta_{11}\theta_{21}}{(z_{11})^2}, \qquad \theta_{10} = -\frac{\theta_{11}}{(z_{11})^2}, \qquad \theta_{20} = -\frac{z_{21}\theta_{11}}{(z_{11})^2} + \frac{\theta_{21}}{z_{11}},$$

these indeed coincide with the transition functions we found in (3.139), once it is set $\lambda = 1$. By analogous calculations one checks that the same happens in the other intersections, thus showing $\mathbb{P}^2_\Pi = \mathbb{P}^2_\omega(\Pi\Omega^1_{\mathbb{P}^2})$. □

This result, in turn, has an obvious corollary, that gives the *minimal embedding of $\mathbb{P}^2_\omega(\Pi\Omega^1_{\mathbb{P}^2})$ into a super Grassmannian.*

Corollary 4.3 (Minimal Embedding of $\mathbb{P}^2_\omega(\Pi\Omega^1_{\mathbb{P}^2})$)**.** *The non-projected homogeneous supermanifold $\mathbb{P}^2_\omega(\Pi\Omega^1_{\mathbb{P}^2})$ embeds into $G(1|1;\mathbb{C}^{3|3})$.*

Proof. Since we have shown that $\mathbb{P}^2_\omega(\Pi\Omega^1_{\mathbb{P}^2}) = \mathbb{P}^2_\Pi$ and the Π-projective plane $\mathbb{P}^2_\Pi$ has been presented as a closed sub-supermanifold inside $G(1|1;\mathbb{C}^{3|3})$, the same holds true for $\mathbb{P}^2_\omega(\Pi\Omega^1_{\mathbb{P}^2})$ and we have a linear embedding of $\mathcal{M}$ into $G(1|1;\mathbb{C}^{3|3})$. □

Appendix A

Some Exact Sequences

As the construction is not readily available in literature, we clarify in what follows the structure of the maps entering the second exterior power of a short exact sequence of locally-free sheaf of $\mathcal{O}_X$-modules / vector bundles, where X is an ordinary complex manifold: these constructions are exploited in section 4.3 of the thesis. We will work in full generality, even if for our purposes it is enough to consider the (easier) special case in which the quotient sheaf is invertible.

We start looking at the following exact sequence of locally-free sheaf of $\mathcal{O}_X$-modules:

$$0 \longrightarrow \mathcal{F} \xrightarrow{\iota} \mathcal{G} \xrightarrow{\pi} \mathcal{H} \longrightarrow 0. \tag{A.1}$$

Then, in general, there is an exact sequence

$$0 \longrightarrow \bigwedge^2 \mathcal{F} \xrightarrow{\wedge^2 \iota} \bigwedge^2 \mathcal{G} \xrightarrow{\phi} \mathcal{Q} \longrightarrow 0. \tag{A.2}$$

where the map ϕ has yet to be defined and the quotient bundle fits into

$$0 \longrightarrow \mathcal{F} \otimes \mathcal{H} \longrightarrow \mathcal{Q} \longrightarrow \bigwedge^2 \mathcal{H} \longrightarrow 0. \tag{A.3}$$

Indeed, to get an idea, *locally*, the first exact sequence splits to give $\mathcal{G} \cong \mathcal{F} \oplus \mathcal{H}$. Taking the second exterior power one gets

$$\bigwedge^2 \mathcal{G} = \bigwedge^2 \mathcal{F} \oplus (\mathcal{F} \oplus \mathcal{H}) \oplus \bigwedge^2 \mathcal{H}, \tag{A.4}$$

therefore, keep working locally, taking the quotient by $\bigwedge^2 \mathcal{F}$ it gives

$$\mathcal{Q} \cong \bigwedge^2 \mathcal{G} / \bigwedge^2 \mathcal{F} \cong (\mathcal{F} \otimes \mathcal{H}) \oplus \bigwedge^2 \mathcal{H}, \tag{A.5}$$

that suggests why the second exact sequence is true.

Notice that if we consider the case $\operatorname{rank} \mathcal{H} = 1$, then one has $\bigwedge^2 \mathcal{H} = 0$, and then sequence for $\mathcal{Q}$ tells that $\mathcal{Q} \cong \mathcal{F} \otimes \mathcal{H}$, therefore one finds that

$$0 \longrightarrow \bigwedge^2 \mathcal{F} \xrightarrow{\wedge^2 \iota} \bigwedge^2 \mathcal{G} \xrightarrow{\phi} \mathcal{F} \otimes \mathcal{H} \longrightarrow 0. \tag{A.6}$$

Getting back to the general setting, in order to define the map ϕ we consider the alternating map

$$\begin{aligned} \Phi : \mathcal{G} \otimes \mathcal{G} &\longrightarrow \mathcal{H} \otimes \mathcal{G} \\ g_1 \otimes g_2 &\longmapsto \pi(g_1) \otimes g_2 - \pi(g_2) \otimes g_1. \end{aligned} \tag{A.7}$$

Notice that one has the commutative diagram

$$\begin{tikzcd} \mathcal{G} \otimes \mathcal{G} \arrow[r, "\Phi"] \arrow[d, "q"'] & \mathcal{H} \otimes \mathcal{G} \\ \bigwedge^2 \mathcal{G} \arrow[ur, "\phi_2"] & \end{tikzcd} \tag{A.8}$$

and, by the usual universal property, the map Φ factors over $\bigwedge^2 \mathcal{G}$ to induce a map

$$\begin{aligned} \Phi_2 : \mathcal{G} \otimes \mathcal{G} &\longrightarrow \mathcal{H} \otimes \mathcal{G} \\ g_1 \wedge g_2 &\longmapsto \pi(g_1) \otimes g_2 - \pi(g_2) \otimes g_1. \end{aligned} \tag{A.9}$$

We note that $\Phi_2 \circ \wedge^2 \iota = 0$, indeed:

$$\left(\Phi_2 \circ \wedge^2 \iota\right)(f_1 \wedge f_2) = \Phi_2(f_1 \wedge f_2) = \pi(f_1) \otimes f_2 - \pi(f_2) \otimes f_1 = 0, \tag{A.10}$$

since $\iota : \mathcal{F} \to \mathcal{G}$ is an inclusion and since $\ker \pi = \operatorname{im} \iota \cong \mathcal{F}$.
Now, since $\Phi_2 \circ \wedge^2 \iota = 0$, one has that $\Phi_2(\bigwedge^2 \mathcal{F}) = 0$, so that one has that, in turn Φ_2 factors through $\mathcal{Q}$, as to yield a well-defined map $\phi_2 : \mathcal{Q} \to \mathcal{F} \otimes \mathcal{H}$, as follows

$$\begin{tikzcd} \mathcal{G} \otimes \mathcal{G} \arrow[r, "\Phi"] \arrow[d, "q"'] & \mathcal{H} \otimes \mathcal{G} \\ \bigwedge^2 \mathcal{G} \arrow[ur, "\Phi_2"] \arrow[d, "\phi"] & \\ \mathcal{Q} \arrow[uur, "\phi_2"'] & \end{tikzcd} \tag{A.11}$$

Now, let us examine the following exact sequence obtained by tensoring with $\mathcal{H} \otimes -$ the first exact sequence:

$$0 \longrightarrow \mathcal{H} \otimes \mathcal{F} \xrightarrow{1 \otimes \iota} \mathcal{H} \otimes \mathcal{G} \xrightarrow{1 \otimes \pi} \mathcal{H} \otimes \mathcal{H} \longrightarrow 0, \tag{A.12}$$

where the maps are the obvious ones. We observe that

- $\operatorname{im} 1 \otimes \iota \subset \Phi_2(\bigwedge^2 \mathcal{G})$, indeed for $g \in \mathcal{G}$ such that $\pi(g) = h$ we have that

  $$(1 \otimes \iota)(h \otimes f) = h \otimes \iota(f) = \pi(g) \otimes f - \pi(f) \otimes g = \Phi_2(g \wedge f) \tag{A.13}$$

  as $\pi(f) = 0$ and confusing $\iota(f)$ with f (remember that ι is an immersion). This implies that $\mathcal{H} \otimes \mathcal{F} \cong \operatorname{im} 1 \otimes \iota \subset \phi_2(\mathcal{Q})$.

- $\operatorname{im}((1 \otimes \pi) \circ \Phi_2) \subset \bigwedge^2 \mathcal{H}$, indeed

  $$\begin{aligned} (1 \otimes \pi)\left(\Phi_2(g_1 \wedge g_2)\right) &= (1 \otimes \pi)\left(\pi(g_1) \otimes g_2 - \pi(g_2) \otimes g_1\right) \\ &= \pi(g_1) \otimes \pi(g_2) - \pi(g_2) \otimes \pi(g_1) = \pi(g_1) \wedge \pi(g_2). \end{aligned} \tag{A.14}$$

  Conversely, working on the fibers, one has that $\bigwedge^2 \mathcal{H} \subset \operatorname{im}((1 \otimes \pi) \circ \Phi_2)$, so that one concludes that $\operatorname{im}((1 \otimes \pi) \circ \Phi_2) = \bigwedge^2 \mathcal{H}$ that in turn implies that

  $$\operatorname{im}\left((1 \otimes \pi) \circ \phi_2\right) = \bigwedge^2 \mathcal{H}. \tag{A.15}$$

  This says that ϕ_2 maps onto $\bigwedge^2 \mathcal{H}$.

As we have shown that the map ϕ_2 is such that $\mathcal{F} \otimes \mathcal{H} \subset \operatorname{im} \phi_2$, and that ϕ_2 is onto $\bigwedge^2 \mathcal{H}$, by counting the dimensions, ϕ_2 is actually also injective, therefore $\mathcal{F} \otimes \mathcal{H} \subset \phi_2(\mathcal{Q})$ implies that $\mathcal{F} \otimes \mathcal{G} \subset \mathcal{Q}$, which establishes the second exact sequence for $\mathcal{Q}$.

Appendix B

The Berezinian Sheaf of $\mathbb{P}^n_\Pi$

In this Appendix we support the proof of Theorem 4.3, by explicitly working out the Berezinian of the super Jacobian in the intersection $\mathcal{U}_0 \cap \mathcal{U}_1$ of the usual covering of $\mathbb{P}^n$.
Given the transition functions of $\mathbb{P}^n_\Pi$, as in 4.4 and 4.5, in the intersection $\mathcal{U}_0 \cap \mathcal{U}_1$, the super Jacobian matrix reads

$$[\mathcal{J}ac]_{10} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \tag{B.1}$$

where one has

$$A = \begin{pmatrix} -\frac{1}{z_{10}^2} & 0 & \cdots & \cdots & 0 \\ -\frac{z_{20}}{z_{10}^2} - 2\frac{\theta_{10}\theta_{20}}{z_{10}^3} & \frac{1}{z_{10}} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ -\frac{z_{n0}}{z_{10}^2} - 2\frac{\theta_{10}\theta_{n0}}{z_{10}^3} & 0 & \cdots & 0 & \frac{1}{z_{10}} \end{pmatrix} \qquad B = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ \frac{\theta_{20}}{z_{10}^2} & -\frac{\theta_{10}}{z_{10}^2} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ \frac{\theta_{n0}}{z_{10}^2} & 0 & \cdots & 0 & -\frac{\theta_{10}}{z_{10}^2} \end{pmatrix} \tag{B.2}$$

$$C = \begin{pmatrix} +2\frac{\theta_{10}}{z_{10}^3} & 0 & \cdots & \cdots & 0 \\ -\frac{\theta_{20}}{z_{10}^2} + 2\frac{z_{20}}{z_{10}^3}\theta_{10} & -\frac{\theta_{10}}{z_{10}^2} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ -\frac{\theta_{n0}}{z_{10}^2} + 2\frac{z_{n0}}{z_{10}^3}\theta_{10} & 0 & \cdots & 0 & -\frac{\theta_{10}}{z_{10}^2} \end{pmatrix} \qquad D = \begin{pmatrix} -\frac{1}{z_{10}^2} & 0 & \cdots & \cdots & 0 \\ -\frac{z_{20}}{z_{10}^2} & \frac{1}{z_{10}} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ -\frac{z_{n0}}{z_{10}^2} & 0 & \cdots & 0 & \frac{1}{z_{10}} \end{pmatrix}. \tag{B.3}$$

Now, clearly $\det(A) = -\frac{1}{z_{10}^{n+1}}$ and one can compute that

$$D - CA^{-1}B = \begin{pmatrix} -\frac{1}{z_{10}^2} & 0 & \cdots & \cdots & 0 \\ -\frac{z_{20}}{z_{10}^2} - \frac{\theta_{10}\theta_{20}}{z_{10}^3} & \frac{1}{z_{10}} & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & \\ -\frac{z_{n0}}{z_{10}^2} - \frac{\theta_{10}\theta_{n0}}{z_{10}^3} & 0 & \cdots & 0 & \frac{1}{z_{10}} \end{pmatrix} \tag{B.4}$$

so that one finds $\det(D - CA^{-1}B)^{-1} = -z_{10}^{n+1}$. Putting the two results together, one has

$$\mathrm{Ber}\,[\mathcal{J}ac]_{10} = \det(A)\det(D - CA^{-1}B)^{-1} = \left(-\frac{1}{z_{10}^{n+1}}\right) \cdot \left(-z_{10}^{n+1}\right) = 1. \tag{B.5}$$

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