Proof-search in Hilbert calculi

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It is well-known [7] that the standard formalizations of classical and intuitionistic logic based on Hilbert calculi, sequent calculi and natural deduction are equivalent. In spite of this, proof-search has been mainly developed around the notion of sequent calculi almost neglecting the cases of natural deduction and Hilbert calculi. This is primarily motivated by the fact that the latters lack the "deep symmetries" of sequent calculi which can be immediately exploited to control and reduce the search space (see, e.g., [2,3,6] for an accurate discussion). However, as we have shown in [3], in the case of natural deduction it is possible to design proof-search procedures with structural properties and complexity comparable with those based on sequent calculi. In this paper we begin an analogous investigation concerning Hilbert calculi. In particular, we consider the $\{\to, \neg\}$-fragment of the Hilbert calculus for Classical Propositional Logic defined in [5] and we describe a terminating proof-search procedure for it.

1 The Hilbert calculus Hc

We consider the propositional language $\mathcal{L}$ of Classical Propositional Logic (CPL) based on a denumerable set of propositional variables $\mathcal{V}$ and the connectives $\to$ and $\neg$. The logical connectives $\wedge$ and $\vee$ can be introduced by setting $A \wedge B = \neg(A \to \neg B)$ and $A \vee B = \neg A \to B$. A *literal* is a formula of the form p or $\neg p$ with $p \in \mathcal{V}$; the set of literals is denoted by Lit.

We call **Hc** the Hilbert calculus for CPL introduced by Kleene in [5]. Logical axioms of **Hc** are:

(Ax1)	$A \to (B \to A)$
(Ax2)	$(A \to B) \to ((A \to (B \to C)) \to (A \to C))$
(Ax3)	$(A \to B) \to ((A \to \neg B) \to \neg A)$
(Ax4)	$\neg\neg A \to A$

The only rule of **Hc** is MP (*Modus Ponens*): from $A \to B$ and A infer B. Let Γ be a set of formulas and A a formula. A *deduction* of A from assumptions Γ is a finite sequence of formulas $\langle A_1, \ldots, A_n \rangle$ such that $A_n = A$ and, for every A_i in the sequence, one of the following conditions holds:

(a) $A_i \in \Gamma$ (namely, A_i is an assumption);
(b) A_i is an instance of a logical axiom;
(c) A_i is obtained by applying MP to A_j and A_k, with $j < i$ and $k < i$.

By $\mathcal{D} : \Gamma \vdash A$ we mean that $\mathcal{D}$ is a deduction of A from assumptions Γ. By $\mathrm{Fm}(\mathcal{D})$ we denote the set of formulas occurring in the sequence $\mathcal{D}$. In the following proposition we introduce some deductions:

Proposition 1. *For every formula A, B and K, the following deductions can be constructed:*

(1) $\mathcal{D}_{\mathrm{MP}}(A \to B, A) : A \to B, A \vdash B$;
(2) $\mathcal{D}_{\neg\neg\mathrm{E}}(A) : \neg\neg A \vdash A$;
(3) $\mathcal{D}_{\mathrm{EF}}(A, B) : A, \neg A \vdash B$;
(4) $\mathcal{D}_{\mathrm{EF}\to}(A, K) : \neg A \to K, \neg A \to \neg K \vdash A$;
(5) $\mathcal{D}_{\mathrm{EF}\to\neg}(A, K) : A \to K, A \to \neg K \vdash \neg A$. □

Let $\mathcal{D}$ and $\mathcal{E}$ be two deductions, where $\mathcal{E} = \langle A_1, \ldots, A_n \rangle$. The *concatenation* of $\mathcal{D}$ and $\mathcal{E}$, denoted by $\mathcal{D} \circ \mathcal{E}$, is defined as follows:

– let $\mathcal{E}'$ be obtained by removing from $\mathcal{E}$ the formulas A_j such that $1 \leq j < n$ and A_j is in $\mathcal{D}$; then, $\mathcal{D} \circ \mathcal{E}$ is the concatenation of the sequences $\mathcal{D}$ and $\mathcal{E}'$.

One can easily check that:

Lemma 1. *Let $\mathcal{D} : \Gamma \vdash A$ and $\mathcal{E} : \Delta \vdash B$. Then, $\mathcal{D} \circ \mathcal{E} : \Gamma \cup (\Delta \setminus \mathrm{Fm}(\mathcal{D})) \vdash B$ and $\mathrm{Fm}(\mathcal{D} \circ \mathcal{E}) = \mathrm{Fm}(\mathcal{D}) \cup \mathrm{Fm}(\mathcal{E})$.* □

A distinguishing feature of Hilbert calculi is that there are no rules to close assumptions. Thus, to prove the Deduction Lemma we have to rearrange a deduction $\mathcal{D}$ of $A, \Gamma \vdash B$, as shown in next lemma.

Lemma 2 (Deduction Lemma). *Let $\mathcal{D} : A, \Gamma \vdash B$. Then, there exists a deduction $\mathcal{E}_{\mathrm{DL}}(\mathcal{D}) : \Gamma \vdash A \to B$ such that, for every $C \in \mathrm{Fm}(\mathcal{D})$, $A \to C \in \mathrm{Fm}(\mathcal{E}_{\mathrm{DL}}(\mathcal{D}))$.* □

In the next lemma we introduce the deductions $\mathcal{E}_{\neg\mathrm{E}}(\mathcal{D})$ and $\mathcal{E}_{\mathrm{I}\neg}(\mathcal{D})$.

Lemma 3.

(i) Let $\mathcal{D} : \neg A, \Gamma \vdash K$ such that $\neg K \in \mathrm{Fm}(\mathcal{D})$. Then, there exists a deduction $\mathcal{E}_{\neg\mathrm{E}}(\mathcal{D}) : \Gamma \vdash A$.
(ii) Let $\mathcal{D} : A, \Gamma \vdash K$ such that $\neg K \in \mathrm{Fm}(\mathcal{D})$. Then, there exists a deduction $\mathcal{E}_{\mathrm{I}\neg}(\mathcal{D}) : \Gamma \vdash \neg A$.

Proof. We prove Point (i). By the Deduction Lemma, there exists a deduction $\mathcal{E}_{\mathrm{DL}}(\mathcal{D}) : \Gamma \vdash \neg A \to K$ such that $\neg A \to \neg K \in \mathrm{Fm}(\mathcal{D})$. Let us consider the derivation $\mathcal{D}_{\mathrm{EF}\to}(A, K) : \neg A \to K, \neg A \to \neg K \vdash A$ defined in Proposition 1.(4). We can set $\mathcal{E}_{\neg\mathrm{E}}(\mathcal{D}) = \mathcal{E}_{\mathrm{DL}}(\mathcal{D}) \circ \mathcal{D}_{\mathrm{EF}\to}(A, K)$ which, by Lemma 1, is a deduction of $\Gamma \vdash A$. The definition of the deduction $\mathcal{E}_{\mathrm{I}\neg}(\mathcal{D})$ of Point (ii) is similar using $\mathcal{D}_{\mathrm{EF}\to\neg}(A, K)$ instead of $\mathcal{D}_{\mathrm{EF}\to}(A, K)$. □

A *(classical) interpretation* $\mathcal{I}$ is a subset of $\mathcal{V}$; the validity of a formula A in $\mathcal{I}$, denoted by $\mathcal{I} \models A$, is defined as usual. Given a set of formulas Γ, $\mathcal{I}$ is a *model* of Γ, denoted by $\mathcal{I} \models \Gamma$, iff $\mathcal{I} \models C$, for every $C \in \Gamma$. A set of literals is *consistent* if it does not contain a complementary pair of literals $\{p, \neg p\}$. Given a consistent set of literals Γ, the interpretation $\Gamma \cap \mathcal{V}$ is a model of Γ.

2 The procedure Hp

We present the procedure **Hp** to search for a deduction in **Hc**. Let Γ be a set of formulas, let A be a formula or the special symbol $\bot$. We define the procedure **Hp** satisfying the following properties:

(H1) If $A \in \mathcal{L}$, $\mathbf{Hp}(\Gamma, A)$ returns either a deduction $\mathcal{D} : \Gamma \vdash A$ or a model of $\Gamma \cup \{\neg A\}$.
(H2) $\mathbf{Hp}(\Gamma, \bot)$ returns either a deduction $\mathcal{D} : \Gamma \vdash K$ such that $\neg K \in \mathrm{Fm}(\mathcal{D})$ or a model of Γ.

In Proposition 2 we prove that **Hp** is *correct*, namely, properties (H1) and (H2) hold. The procedure is presented in Fig. 1. In the presentation of **Hp**, we use a high-level formalism, discarding inessential details. The computation of $\mathbf{Hp}(\Gamma, A)$ is defined by cases on Γ and A. The first case among (1)–(9) matching the values of Γ and A is executed; if none of the conditions in (1)–(9) holds, case (10) is performed. We implicitly assume that the recursive calls are correct.

The rest of this section is devoted to the proof of correctness of **Hp**. Given a formula A, we denote by $\mathcal{V}(A)$ the set of propositional variables occurring in A and by $|A|$ the number of symbols occurring in A. Given a finite set Γ of formulas, we set:

$$\mathcal{V}(\Gamma) \;=\; \bigcup_{A \in \Gamma} \mathcal{V}(A) \qquad |\Gamma| = \sum_{A \in \Gamma} |A| \qquad \mathrm{Lit}(\Gamma) \;=\; \Gamma \cap \mathrm{Lit}$$

Let Γ_1 and Γ_2 be two finite sets of formulas and let A_1 and A_2 be either formulas or $\bot$. We define the ordering relation $(\Gamma_1, A_1) \prec (\Gamma_2, A_2)$ iff the following conditions hold (where $|\bot| = 1$):

(1) $\mathcal{V}(\Gamma_1 \cup \{A_1\}) \;\subseteq\; \mathcal{V}(\Gamma_2 \cup \{A_2\})$;
(2) either $\mathrm{Lit}(\Gamma_1) \supsetneq \mathrm{Lit}(\Gamma_2)$ or $\mathrm{Lit}(\Gamma_1) = \mathrm{Lit}(\Gamma_2)$ and $|\Gamma_1| + |A_1| \;<\; |\Gamma_2| + |A_2|$.

One can easily prove that $\prec$ is *well-founded*, namely: every $\prec$-chain of the form $\cdots \prec (\Gamma_2, A_2) \prec (\Gamma_1, A_1) \prec (\Gamma_0, A_0)$ has finite length. Indeed, by Point (1) of the definition of $\prec$, along a $\prec$-chain no new propositional variable is added, hence we cannot apply Point (2) infinitely many times. In particular, the length of every chain starting with (Γ, A) (and hence the number of nested recursive calls of **Hp**) is bounded by $O(|\Gamma| + |A|)$.

Proposition 2 (Correctness of Hp). *The procedure* **Hp** *is correct.*

Proof. Let us consider the call $\mathbf{Hp}(\Gamma, A)$. We have to prove that $\mathbf{Hp}(\Gamma, A)$ satisfies properties (H1) and (H2). One can easily check that every recursive call $\mathbf{Hp}(\Gamma', A')$ performed in the computation of $\mathbf{Hp}(\Gamma, A)$ satisfies $(\Gamma', A') \prec (\Gamma, A)$. In particular, note that no new variables are added, hence $\mathcal{V}(\Gamma' \cup \{A'\}) \subseteq \mathcal{V}(\Gamma \cup \{A\})$; moreover, literals are never deleted from the set of assumptions, hence $\mathrm{Lit}(\Gamma') \supseteq \mathrm{Lit}(\Gamma)$. Since the relation $\prec$ is well-founded, we can assume by

(1) If A is an instance of a logical axiom or $A \in \Gamma$.
Return the deduction only consisting of A.
(2) If there exists B such that $\{B, \neg B\} \subseteq \Gamma$.
If $A = \bot$, return the deduction $\langle \neg B, B \rangle : \Gamma \vdash B$.
If $A \neq \bot$, return the deduction $\mathcal{D}_{\mathrm{EF}}(B, A) : \Gamma \vdash A$.
(3) If there exists $B \to C$ such that $\{B \to C, B\} \subseteq \Gamma$.
Let $\mathcal{D}_1 = \mathbf{Hp}(\,(\Gamma \setminus \{B \to C\}) \cup \{C\}\,,\, A)$.
If $\mathcal{D}_1$ is an interpretation, then return $\mathcal{D}_1$.
Otherwise, return the deduction $\mathcal{D}_{\mathrm{MP}}(B \to C, B) \circ \mathcal{D}_1 : \Gamma \vdash A$.
(4) If there exists B such that $\neg\neg B \in \Gamma$
Let $\mathcal{D}_1 = \mathbf{Hp}(\,(\Gamma \setminus \{\neg\neg B\}) \cup \{B\}\,,\, A)$.
If $\mathcal{D}_1$ is an interpretation, then return $\mathcal{D}_1$.
Otherwise, return the deduction $\mathcal{D}_{\neg\neg\mathrm{E}}(B) \circ \mathcal{D}_1 : \Gamma \vdash A$.
(5) If $A = p$ with $p \in \mathcal{V}$ and $\neg p \notin \Gamma$.
Let $\mathcal{D}_1 = \mathbf{Hp}(\Gamma \cup \{\neg p\}, \bot)$.
If $\mathcal{D}_1$ is an interpretation, then return $\mathcal{D}_1$.
Otherwise, $\mathcal{D}_1$ is a deduction of $\neg p, \Gamma \vdash K$ with $\neg K \in \mathrm{Fm}(\mathcal{D}_1)$;
return the deduction $\mathcal{E}_{\neg\mathrm{E}}(\mathcal{D}_1) : \Gamma \vdash p$.
(6) If $A = B \to C$.
Let $\mathcal{D}_1 = \mathbf{Hp}(\Gamma \cup \{B\}, C)$.
If $\mathcal{D}_1$ is an interpretation, then return $\mathcal{D}_1$.
Otherwise, $\mathcal{D}_1$ is a deduction of $B, \Gamma \vdash C$;
return $\mathcal{E}_{\mathrm{DL}}(\mathcal{D}_1) : \Gamma \vdash B \to C$.
(7) If $A = \neg B$.
Let $\mathcal{D}_1 = \mathbf{Hp}(\Gamma \cup \{B\}, \bot)$.
If $\mathcal{D}_1$ is an interpretation, then return $\mathcal{D}_1$.
Otherwise, $\mathcal{D}_1$ is a deduction of $B, \Gamma \vdash K$ with $\neg K \in \mathrm{Fm}(\mathcal{D}_1)$;
return the deduction $\mathcal{E}_{\mathrm{I}\neg}(\mathcal{D}) : \Gamma \vdash \neg B$.

Remark 1. Hereafter $A = \bot$ or ($A \in \mathcal{V}$ and $\neg A \in \Gamma$).

(8) If there exists $\neg K \in \Gamma$ such that $K = B \to C$.
Let $\mathcal{D}_1 = \mathbf{Hp}(\Gamma \setminus \{\neg K\}, K)$.
If $\mathcal{D}_1$ is an interpretation, then return $\mathcal{D}_1$.
Otherwise, $\mathcal{D}_1$ is a deduction of $\Gamma \setminus \{\neg K\} \vdash K$.
If $A = \bot$, then return the deduction $\langle \neg K \rangle \circ \mathcal{D}_1 : \Gamma \vdash K$.
If $A \neq \bot$, then return the deduction $\mathcal{D}_1 \circ \mathcal{D}_{\mathrm{EF}}(K, A) : \Gamma \vdash A$.
(9) If there exists $B \to C$ such that $B \to C \in \Gamma$
Let $\mathcal{D}_1 = \mathbf{Hp}(\Gamma \setminus \{B \to C\}, B)$ and $\mathcal{D}_2 = \mathbf{Hp}((\Gamma \setminus \{B \to C\}) \cup \{C\}, A)$.
If, for some $i \in \{1, 2\}$, $\mathcal{D}_i$ is an interpretation, then return $\mathcal{D}_i$.
Otherwise, return $\mathcal{D}_1 \circ \mathcal{D}_{\mathrm{MP}}(B \to C, B) \circ \mathcal{D}_2 : \Gamma \vdash A$.

Remark 2. Here Γ is a consistent set of literals and ($A = \bot$ or $\neg A \in \Gamma$).

(10) Return the interpretation $\Gamma \cap \mathcal{V}$.

Figure 1. Procedure **Hp** (Γ, A)

induction hypothesis that $\mathbf{Hp}(\Gamma', A')$ satisfies (H1) and (H2). Using the induction hypothesis, the correctness of **Hp** easily follows. Note that, if none of the cases (1)–(7) is matched, then the property in Remark 1 holds. Thus, in case (8), if $A \in \mathcal{V}$ and $\mathcal{D}_1$ is an interpretation, by the induction hypothesis $\mathcal{D}_1 \models \Gamma \setminus \{K\}$. By Remark 1 $\neg A \in \Gamma$, hence $\mathcal{D}_1 \models \neg A$ as well. If none of the cases (1)–(9) is matched, then Remark 2 holds; thus, in Case (10) Γ is a consistent set of literals, hence $\Gamma \cap \mathcal{V}$ is a model of Γ. □

3 Future work

In this paper we have presented a work in progress on the design of terminating proof-search procedures for Hilbert calculi. Here we only discuss the $\{\to, \neg\}$-fragment of the Hilbert calculus [5] for CPL. We have implemented the procedure **Hc** defined in Sec. 2 in Prolog, extending it to the full language [1]. In the full language, the main problems arise from disjunctive goals and from the relationships between the implicative and negated assumptions and the goal to be proved. To get a complete strategy, we have to introduce some "reasoning by absurd" steps and some care is needed to avoid infinite loops.

The main drawback is that in general the obtained proofs are very huge. For instance, the deduction of $\neg(p \to q) \to p$ consists of 53 formulas and involves very large instances of the axioms. This comes from the fact that the number of lines of $\mathcal{E}_{\mathrm{DL}}(\mathcal{D})$ is three times the number of lines of $\mathcal{D}$. To reduce the size of deductions, we plan to develop techniques similar to those used in [4]. We also aim at investigating the extension to Intuitionistic Propositional Logic, using the machinery in [4] to get termination. Further possible extensions regard the intermediate logics, by exploiting the filtration techniques introduced in [1].

References

1. A. Avellone, P. Miglioli, U. Moscato, and M. Ornaghi. Generalized tableau systems for intermediate propositional logics. In D. Galmiche, editor, *TABLEAUX 1997*, volume 1227 of *LNAI*, pages 43–61. Springer-Verlag, 1997.
2. M. D'Agostino. Classical natural deduction. In S.N. Artëmov et al., editor, *We Will Show Them!*, pages 429–468. College Publications, 2005.
3. M. Ferrari and C. Fiorentini. Proof-search in natural deduction calculus for classical propositional logic. In H. De Nivelle, editor, *TABLEAUX 2015*, volume 9323 of *LNCS*, pages 237–252. Springer International Publishing, 2015.
4. M. Ferrari, C. Fiorentini, and G. Fiorino. An evaluation-driven decision procedure for G3i. *ACM Transactions on Computational Logic (TOCL)*, 6(1):8:1–8:37, 2015.
5. S.C. Kleene. *Introduction to Metamathematics*. Van Nostrand, New York, 1952.
6. W. Sieg and J. Byrnes. Normal natural deduction proofs (in classical logic). *Studia Logica*, 60(1):67–106, 1998.
7. A.S. Troelstra and H. Schwichtenberg. *Basic Proof Theory*, volume 43 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2nd edition, 2000.

[1] The Prolog prototype is available at `https://drive.google.com/open?id=0ByzJsfKRVFPMVUE2VzBtN2xpUEU`