Nekhoroshev theorem for perturbations of the central motion

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Abstract

In this paper we prove a Nekhoroshev type theorem for perturbations of Hamiltonians describing a particle subject to the force due to a central potential. Precisely, we prove that under an explicit condition on the potential, the Hamiltonian of the central motion is quasi-convex. Thus, when it is perturbed, two actions (the modulus of the total angular momentum and the action of the reduced radial system) are approximately conserved for times which are exponentially long with the inverse of the perturbation parameter.

1 Introduction

In this paper we study the applicability of Nekhoroshev's theorem [Nek77, Nek79] (see also [BGG85, Loc92, GM97, Nie04, Nie06, GCB16]) to the central motion. The main point is that Nekhoroshev's theorem applies to perturbations of integrable systems whose Hamiltonian is a *steep* function of the actions. Even if such a property is known to be generic, it is very difficult (and not at all explicit) to verify it. Here we prove that, under an explicit condition on the potential (see eq. (6)), the Hamiltonian of the central motion is a quasi-convex function of the actions and thus it is steep, so that Nekhoroshev's theorem applies. Actually, the form of Nekhoroshev's theorem used here is not the original one, but that for degenerate systems proved by Fassò in [Fas95]. This is due to the fact that the Hamiltonian of the central motion is a function of two actions only, namely, the modulus of the total angular momentum and the action of the effective one dimensional Hamiltonian describing the motion of the radial variable.

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On the one hand, as pointed out in [Fas95], this feature creates some problems for the proof of Nekhoroshev's theorem, but these problems were solved in [Fas95]. On the other hand, degeneracy reduces the difficulty for the verification of steepness or of the strongest property of being quasi-convex, since, in the two-dimensional case, quasi-convexity is generic and equivalent to the nonvanishing of the Arnold determinant

$$
\mathcal{D} := \det \begin{pmatrix} \dfrac{\partial^2 h_0}{\partial I^2} & \left(\dfrac{\partial h_0}{\partial I} \right)^t \\ \dfrac{\partial h_0}{\partial I} & 0 \end{pmatrix} \, ,
\tag{1}
$$

a property that it is not too hard to verify.

Indeed, since (1) is an analytic function of the actions, it is enough to show that it is different from zero at one point in order to ensure that it is different from zero almost everywhere. Here we explicitly compute the expansion of $h_0(I)$ at a circular orbit and we show that, provided the central potential $V(r)$ does not satisfy identically a fourth order differential equation that we explicitly write, the Hamiltonian $h_0(I)$ is quasi-convex on an open dense domain (whose complementary is possibly empty).

The rest of the paper is organized as follows: In sect. 2 we introduce the central motion problem and state the main results. Sect. 3 contains all the proofs. In the Appendix we prove that in the two dimensional case quasi-convexity is equivalent to Arnold isoenergetic nondegeneracy condition.

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2 Preliminaries and statement of the main results

We first recall the structure of the action angle variables for the central motion. Introducing polar coordinates, the Hamiltonian takes the form

$$
h_0(p_r, r, p_\varphi, \varphi, p_\vartheta, \vartheta) = \frac{p_r^2}{2} + \frac{p_\vartheta^2}{2r^2} + \frac{p_\varphi^2}{2r^2 \sin^2 \vartheta} + V(r) \, ,
\tag{2}
$$

and the actions on which h_0 depends are

$$
I_2 := \sqrt{p_\vartheta^2 + \frac{p_\varphi^2}{\sin^2 \vartheta}} \, ,
$$

and the action I_1 of the effective one dimensional Hamiltonian system

$$
h_0^* = \frac{p_r^2}{2} + V_{I_2}^*(r) \, , \quad V_{I_2}^*(r) = \frac{I_2^2}{2r^2} + V(r) \, .
\tag{3}
$$

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By construction h_0 turns out to be a function of the two actions only. We still write

$$h_0 = h_0(I_1, I_2) \ .$$

According to Fassò's theorem, if h_0 depends on (I_1, I_2) in a steep way, then Nekhoroshev's estimate applies. We recall that steepness is actually implied by quasi-convexity, the property that we are now going to verify.

Definition 2.1. *A function h_0 of the actions is said to be* quasi-convex *at a point $\bar{I}$ if the system*

$$\begin{cases} dh_0(\bar{I})\eta = 0 \\ d^2 h_0(\bar{I})(\eta, \eta) = 0 \end{cases}$$

admits only trivial solutions. Here we denoted by $d^2 h_0(\bar{I})(\eta, \eta)$ the second differential of h_0 at $\bar{I}$ applied to the two vectors η, η.

To define the set $\mathcal{A}$ in which the actions vary we first assume that there exists an interval (r_2, r_1) such that, for $r_2 < r < r_1$ one has

$$V'(r) > 0 \ , \tag{4}$$

$$V''(r) + \frac{3V'(r)}{r} > 0 \ . \tag{5}$$

Then we define

$$\Gamma_1 := \sqrt{r_1^3 V'(r_1)} \ , \quad \Gamma_2 := \sqrt{r_2^3 V'(r_2)} \ ,$$

and, in order to fix ideas, we assume that $\Gamma_1 < \Gamma_2$. Then for $\Gamma_1 < I_2 < \Gamma_2$, the effective potential $V_{I_2}^*$ has a non degenerate minimum at some $r_0 = r_0(I_2)$.

Then, there exists a curve $E(I_2)$ such that for $h_0^* < E(I_2)$, all the orbits of the Hamiltonian (3) are periodic. Correspondingly, their action $I_1(E, I_2)$ vary in some interval $(0, F(I_2))$. Thus, the domain $\mathcal{A}$ of the actions I has the form

$$\mathcal{A} := \{(I_1, I_2) : \Gamma_1 < I_2 < \Gamma_2, \ 0 < I_1 < F(I_2)\} \ .$$

We remark that $\mathcal{A}$ is simply connected, a property that will play an important role in the following.

Our main result is the following.

Theorem 2.1. *Consider the Hamiltonian*

$$h_0(p_r, r, p_\varphi, \varphi, p_\vartheta, \vartheta) = \frac{p_r^2}{2} + \frac{p_\vartheta^2}{2r^2} + \frac{p_\varphi^2}{2r^2 \sin^2 \vartheta} + V(r) \ ,$$

with $V(r)$ analytic on $\mathbb{R}^3 \setminus \{0\}$. Assume that there exists a value $r_0 \in (r_2, r_1)$ of the radius such that the following fourth order equation

$$
\begin{aligned}
V^{(4)}(r_0) = &- \frac{84 V'(r_0)}{r_0^3} + \frac{32 V''(r_0)}{r_0^2} + \frac{16 V'''(r_0)}{r_0} - \frac{8(V''(r_0))^2}{r_0 V'(r_0)} \\
&+ \frac{240(V'(r_0))^2}{r_0^3(3V'(r_0) + r_0 V''(r_0))} - \frac{40 V'(r_0) V'''(r_0)}{r_0(3V'(r_0) + r_0 V''(r_0))} \\
&+ \frac{5 r_0 (V'''(r_0))^2}{3(3V'(r_0) + r_0 V''(r_0))}
\end{aligned}
\tag{6}
$$

is not *satisfied.*

Then, there exists a subset $\mathcal{S} \subset \mathcal{A}$ of the action space, with the property that its intersection with any compact set is composed by at most a finite number of lines, and such that h_0 restricted to $\mathcal{A} \setminus \mathcal{S}$ is quasi-convex.

Remark 2.1. *The fourth order equation* (6) *can be rewritten as a second order ordinary differential equation in terms of the variable $g(r) = \frac{rV''(r)}{V'(r)}$, namely,*

$$
\begin{aligned}
g''(r_0) =& \frac{(14 + g(r_0))g'(r_0)}{3r_0} + \frac{(g(r_0) - 1)(g(r_0) + 2)(g(r_0) + 3)}{3r_0^2} \\
&+ \frac{5g'(r_0)^2}{3(3 + g(r_0))}
\end{aligned}
\tag{7}
$$

Remark 2.2. *It is interesting to see what are the homogeneous potentials which do not give rise to steep Hamiltonians. Thus take*

$$
V(r) = kr^\alpha \, ,
$$

with $\alpha, k \in \mathbb{R}$, then the assumptions of Theorem 2.1 are fulfilled if

$$
k\alpha > 0 \, , \quad \alpha + 2 > 0 \, , \quad \alpha \neq -1, 2 \, ,
\tag{8}
$$

thus the excluded cases are the Kepler and the Harmonic potentials.

We also remark that the equation (6) *has also the solution $\alpha = -2$, which however is excluded by the second of* (8).

The theory of [Nek72] and [Fas95] is needed in order to apply Fassò's version of Nekhoroshev's theorem. We recall that the theory of [Nek72, Fas95] applyes provided $\mathcal{A}$ is simply connected and $\forall a \in \mathcal{A}$, the set $I^{-1}(a)$ is compact. This second property follows from the remark that in our case

$$
I^{-1}(a) = \{(p_r, r, p_\varphi, \varphi, p_\vartheta, \vartheta) : h_0^*(p_r, r, p_\varphi, \varphi, p_\vartheta, \vartheta) \leq h_0(a_1, a_2)\} \, ,
\tag{9}
$$

where h_0^* is constructed using $I_2 := a_2$. Then the set (9) is obviously compact. It follows from [Nek72, Fas95] that the set

$$
I^{-1}(\mathcal{A}) := \{(p_r, r, p_\varphi, \varphi, p_\vartheta, \vartheta) \ : \ I(p_r, r, p_\varphi, \varphi, p_\vartheta, \vartheta) \in \mathcal{A}\}
$$

can be covered by charts defining generalized action angle coordinates of the form (I, α, x, y) with $\alpha \in \mathbb{T}^2$.

Definition 2.2. *Fix a positive parameter ρ and denote by $B_\rho(I) \subset \mathbb{R}^2$ the open ball of radius ρ and center I and define*

$$
\mathcal{S}_\rho := \cup_{I \in \mathcal{S}} B_\rho(I) \, .
\tag{10}
$$

We now consider a small perturbation εf of h_0, with f a function of the original cartesian coordinates (p, q) in $T\mathbb{R}^3 \simeq \mathbb{R}^6$ which is analytic.

Theorem 2.2. *Fix a positive small parameter ρ and consider the Hamiltonian $h := h_0 + \varepsilon f$. Then, for every compact set $\mathcal{K} \subset \mathcal{A}$, there exist positive constants $\varepsilon^*, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ such that if the initial value I_0 of the actions fulfills $I_0 \in \mathcal{K} \smallsetminus \mathcal{S}_\rho$ and $|\varepsilon| < \varepsilon^*$ one has*

$$|I(t) - I_0| \leq \mathcal{C}_1 \varepsilon^{1/4} \ ,$$

for all times t satisfying

$$|t| \leq \mathcal{C}_3 exp(\mathcal{C}_2 \varepsilon^{-1/4}) \ . \tag{11}$$

Remark 2.3. *The main dynamical consequence is that, as in the central motion, for any initial datum as above, there exist r_m, r_M such that*

$$r_m \leq r \leq r_M$$

for the times (11).

Remark 2.4. *Actually, one can weaker the analyticity requirement on f, since it would be enough to have that it is analytic on the set of p and q's such that the action is close to I_0.*

Remark 2.5. *The Theorem holds also if one couples the system to an other system with a dynamic taking place over a much faster time scale. For example, this occurs in the case of a soliton interacting with radiation in the NLS equation as in [BM16].*

3 Proof of Theorem 2.1

As anticipated above, in the case of *two* actions, quasi-convexity is equivalent to the nonvanishing of the Arnold determinant $\mathcal{D}$ (cf. (1)).

In order to compute $\mathcal{D}$, we have to compute $h_0(I_1, I_2)$. To do this we proceed as follows.

First, as explained in sect. 2, it is easy to introduce the action I_2 which coincides with the modulus of the total angular momentum. Then, I_1 is the action of the effective one dimensional system (3) in which I_2 plays the role of a parameter. In order to have an explicit formula for the Hamiltonian as a function of the actions, we work at circular orbits. Precisely, we exploit the remark that in one dimensional systems Birkhoff normal form converges in a neighborhood of a nondegenerate minimum. Indeed, Birkhoff normal form allows to construct an analytic canonical transformation which, in a neighborhood of the critical point, conjugates the Hamiltonian h_0 to a function of the form

$$h_0 \left(\frac{p_r^2 + r^2}{2}, I_2 \right) \ , \tag{12}$$

which moreover is explicitly constructed as a power series in $\frac{p_r^2 + r^2}{2}$. Thus, one can define the first action I_1 by $I_1 := \frac{p_r^2 + r^2}{2}$. Remark that, since (12),

as a function of (p_r, r) is analytic in a whole *complex* neighborhood of zero, then $h_0(I_1, I_2)$ is analytic for I_1 in a whole neighborhood of zero. Then, from uniqueness of the actions in one dimensional systems, one has that the expression one gets is actually the expression of h_0 as a function of the actions as defined in sect. 2. It also follows that $\mathcal{D}(I_1, I_2)$ as a function of I_1 extends to a complex analytic function in whole neighborhood of zero, and such a function can be computed using the expression of h_0 obtained from (12).

We use this remark in order to compute the Arnold determinant at the equilibrium point r_0 of the effective one dimensional system described by h_0, which coincides with a circular orbit of the original system. This can be done by computing explicitly the second order Taylor expansion of h_0 which coincides with the fourth order Birkhoff normal form of h_0^* at r_0. Actually this was already done in [FK04] getting

$$h(I_1, I_2) = V^*(I_2) + \sqrt{A(I_2)}I_1 + \frac{-5B(I_2)^2 + 3C(I_2)A(I_2)}{48A(I_2)^2}I_1^2 + o(I_1^2) , \quad (13)$$

where

$$V^*(I_2) = \frac{I_2^2}{2r_0^2} + V(r_0) , \qquad A(I_2) = \frac{3I_2^2}{r_0^4} + V''(r_0) ,$$

$$B(I_2) = -\frac{12I_2^2}{r_0^5} + V'''(r_0) , \quad C(I_2) = \frac{60I_2^2}{r_0^6} + V^{(4)}(r_0) .$$

Fix a point I_2^* and let $r_0(I_2^*)$ be the corresponding critical point of the effective potential.

Inserting into the Arnold determinant $\mathcal{D}$, the first and second derivatives of the Hamiltonian (13) evaluated at the point $I^* := (0, I_2^*)$, one gets

$$\mathcal{D} = \det \begin{pmatrix} t(I_2^*) & \frac{1}{2\sqrt{A(I_2^*)}}\left(\frac{6I_2^*}{r_0^4} + B(I_2^*)\frac{\partial r_0}{\partial I_2}\right) & \sqrt{A(I_2^*)} \\ \frac{1}{2\sqrt{A(I_2^*)}}\left(\frac{6I_2^*}{r_0^4} + B(I_2^*)\frac{\partial r_0}{\partial I_2}\right) & \frac{1}{r_0^2} - \frac{2(I_2^*)^2}{r_0^3}\cdot\frac{\partial r_0}{\partial I_2} & \frac{I_2^*}{r_0^2} \\ \sqrt{A(I_2^*)} & \frac{I_2^*}{r_0^2} & 0 \end{pmatrix} ,$$

where we denoted by

$$t(I_2^*) = \frac{-5B(I_2^*)^2 + 3C(I_2^*)A(I_2^*)}{24A(I_2^*)^2} .$$

Thus, $\mathcal{D} = 0$ is equivalent to

$$\frac{6(I_2^*)^2}{r_0^6} + \left(\frac{BI_2^*}{r_0^2} + \frac{2AI_2^*}{r_0^3}\right)\frac{\partial r_0}{\partial I_2} - \frac{A}{r_0^2} - \frac{t(I_2^*)^2}{r_0^4} = 0 .$$

Isolating the fourth order derivative of V one gets

$$V^{(4)}(r_0) = \frac{48A}{r_0^2} + \left(\frac{8ABr_0^2}{I_2^*} + \frac{16A^2r_0}{I_2^*}\right)\cdot\frac{\partial r_0}{\partial I_2} - \frac{8A^2r_0^2}{(I_2^*)^2} + \frac{5B^2}{3A} - \frac{60(I_2^*)^2}{r_0^6} . \quad (14)$$

Using that r_0 is a critical point of V^*, one can express I_2^* in terms of r_0, namely,

$$I_2(r_0) = \sqrt{r_0^3 V'(r_0)} \,,$$

and computing

$$\frac{\partial r_0}{\partial I_2} = \frac{1}{\frac{\partial I_2}{\partial r_0}} = \frac{2(r_0^3 V'(r_0))^{1/2}}{r_0^2 \left(3V'(r_0) + r_0 V''(r_0)\right)} \,,$$

one can rewrite the equation (14) in terms of the radius r_0, obtaining the equation (6). Thus, if there exists r_0 such that (6) is not satisfied, then at this point $\mathcal{D} \neq 0$. This concludes the proof of the Theorem.

$\square$

Proof of Theorem 2.2 From the previous result, one gets that the Hamiltonian is quasi-convex on the subset $\mathcal{K} \smallsetminus \mathcal{S}_\rho$ with uniform bounds on the quasi-convexity constants. Thus, if we choose an initial datum in such a set and a sufficiently small parameter ε, then Fassò's version of the Nekhoroshev theorem for degenerate Hamiltonians applies (see [Fas95] for details).

Furthermore, we remark that, in the case of quasi-convex Hamiltonians, one gets optimal Nekhoroshev's exponents which for two dimensional systems are $1/4$.

$\square$

4   Appendix

In this appendix we show that, in the two dimensional case, quasi-convexity is equivalent to the nonvanishing of the Arnold determinant $\mathcal{D}$. We start by recalling the Arnold condition.

Definition 4.1. *Let h_0 be a complete integrable Hamiltonian with n degrees of freedom and frequency ω. Then, h_0 is said to satisfy the* Arnold condition *at I^* if the following map*

$$(I, \lambda) \to (\lambda \omega(I), h_0(I))$$

has maximal rank at $(I^*, 1)$.

Explicitly, this condition can be written in the form

$$\mathcal{D}(I^*) = \det \begin{pmatrix} \dfrac{\partial \omega(I^*)}{\partial I} & \dfrac{\partial h_0(I^*)}{\partial I} \\ \omega(I^*) & 0 \end{pmatrix} \neq 0 \,.$$

Proposition 4.1. *Let $h_0 : \mathcal{A} \to \mathbb{R}$ with $\mathcal{A} \subset \mathbb{R}^2$ be an Hamiltonian. Then, h_0 is quasi-convex at $I^* \in \mathcal{A}$ if and only if $\mathcal{D}(I^*) \neq 0$.*

Proof. In the two dimensional case, $\mathcal{D} \neq 0$ takes the form

$$\omega_1 \left(\frac{\partial^2 h_0}{\partial I_1 \partial I_2} \omega_2 - \frac{\partial^2 h_0}{\partial I_2^2} \omega_1 \right) - \omega_2 \left(\frac{\partial^2 h_0}{\partial I_1^2} \omega_2 - \frac{\partial^2 h_0}{\partial I_1 \partial I_2} \omega_1 \right) \neq 0 \ ,$$

namely,

$$\frac{\partial^2 h_0}{\partial I_1^2} \omega_2^2 - 2 \frac{\partial^2 h_0}{\partial I_1 \partial I_2} \omega_1 \omega_2 + \frac{\partial^2 h_0}{\partial I_2^2} \omega_1^2 \neq 0$$

where all the quantities are evaluated at the point I^*.

Moreover, this condition can be explicitly written as

$$d^2 h_0(I^*)(\eta, \eta) \neq 0 \ ,$$

where we denoted by $\eta = (\omega_2, -\omega_1)$.

Thus, we conclude that, in the case $n = 2$, the Arnold condition is equivalent to the request of the second differential $d^2 h_0(I^*)$ to be different from zero on the hyperplane generated by the vector η normal to the gradient, namely, quasi-convexity. $\qquad\square$

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