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The Journal of Symbolic Logic
Volume 00, Number 0, XXX 0000

# STRICT COHERENCE ON MANY-VALUED EVENTS

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**Abstract.** We investigate the property of *strict coherence* in the setting of many-valued logics. Our main results read as follows: (i) a map from an MV-algebra to $[0, 1]$ is strictly coherent if and only if it satisfies Carnap's regularity condition and, (ii) a $[0, 1]$-valued book on a finite set of many-valued events is strictly coherent if and only if it extends to a faithful state of an MV-algebra that contains them. Remarkably this latter result allows us to relax the rather demanding conditions for the Shimony-Kemeny characterisation of strict coherence put forward in the mid 1950s in this Journal.

**§1. Introduction and motivation.** This paper contributes to the logical foundations of probability by investigating strict coherence on many-valued events. The notion of *strict* coherence was introduced in this Journal by Abner Shimony and John Kemeny as a logically inspired refinement of the notion of *coherence* used by Bruno de Finetti to ground his subjective interpretation of probability. Informally, coherence demands that a rational agent avoids the logical possibility of "sure loss" in suitably specified betting situations. Its strict counterpart, in addition, demands that each prospect of losing should be balanced by a prospect of gaining.

Interest in the condition of strict coherence was prompted by Carnap's analysis of what he termed "regular" probability functions in [4] (see also [37, Chapter 10 ]). Informally those functions arise by strengthening the usual normalisation axiom of probability in the right-to-left direction. That is to say that 1 (respectively, 0) is assigned only to tautologies (respectively, contradictions). The rationale for Carnap-regular functions is that, however unlikely, possible events *may* happen.

In [38] Shimony proved that imposing strict coherence to a map from *finite boolean algebras* to $[0, 1]$ was sufficient to single out Carnap-regular functions. Shortly after, the converse was established by Kemeny in [21], a result proved independently in [24]. The Shimony-Kemeny characterisation is obtained under far more restrictive conditions than those yielding de Finetti's theorem, which

1991 *Mathematics Subject Classification.* Primary 03B48, Secondary 60A05, 06D35, 06F20, 52B11.

*Key words and phrases.* Probability logic, strict coherence, MV-algebras, faithful states, many-valued logics.

Sadly Franco Montagna passed away after the first version of this paper was completed.

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0022-4812/00/0000-0000/$00.00

holds for *books* on arbitrary sets of events (see Section 2 below for precise details). This may suggest that the refinement of coherence to its strict counterpart is obtained at the price of giving up the full generality of de Finetti's foundation.

By investigating the problem of strict coherence for many-valued events we conclude, as a welcome side effect, that this is not the case. More centrally, this paper shows that taking many-valued events as elements of MV-algebras [5, 6] allows us to investigate the notion of coherence in a framework which is both general and methodologically perspicuous. In this respect the present paper complements the groundbreaking contributions of [35, 31, 23].

Our main results read as follows. Theorem 5.2 shows that a map from an MV-algebra to $[0,1]$ is strictly coherent if and only if it satisfies Carnap's regularity condition. Theorem 6.4 establishes an extension result for strictly coherent *books* on many-valued events. This result is obtained for *finite-dimensional* MV-algebras (see Section 3 below). Despite being a particular case, Corollary 6.5 shows that finite-dimensional MV-algebras are general enough to provide de Finetti-like extensions for *all* boolean events. In other words, the refinement to strict coherence does not come at the expense of generality.

The next Section collects the basic concepts and results on (strict) coherence. Readers who are familiar with this material may still wish to flip through it to get acquainted with the notation and terminology used in the paper.

**§2. Coherence and strict coherence.** In a series of seminal contributions, starting with [8] and culminating in [9], Bruno de Finetti provided a rather general justification for the probabilistic representation of rational beliefs. To this end he identifies degrees of belief with the price of gambles in a suitably defined betting situation. In this setting *incoherent* behaviour is defined as the disposition to choose prices in a way which may lead to *sure loss.* (See [13] for a logical account of the topic in the notation and spirit of the preset paper.)

Let $a_1, \ldots, a_n$ denote events and suppose a bookmaker B publishes a *book* $\Phi : a_1 \mapsto \alpha_1, \ldots, a_n \mapsto \alpha_1$, where $\alpha_1, \ldots, \alpha_n$ are real numbers in $[0,1]$. A gambler G then chooses real-valued stakes $\rho_1, \ldots, \rho_n$ and for $i = 1, \ldots, n$, pays $\rho_i\alpha_i$ to B. When a (classical propositional) valuation $v$ determines $a_i$, B gains $\rho_i$, if $v(a_i) = 1$ and 0 otherwise. Note that $\rho_i$ may be negative, in which case, paying $\rho_i\alpha_i$ means receiving $-\rho_i\alpha_i$ and receiving $\rho_i a_i$ means paying $-\rho_i\alpha_i$. The book $\Phi$ is said to be *coherent* if there is no choice of stakes $\rho_1, \ldots, \rho_n$ such that for every valuation $v$,

$$\sum_{i=1}^{n} \rho_i(\alpha_i - v(a_i)) < 0. \tag{1}$$

The left hand side of (1) captures the bookmaker's payoff, or *balance*, relative to book $\Phi$ under valuation $v$. De Finetti's theorem then reads as follows.

THEOREM 2.1 ([8]). *Let $a_1, \ldots, a_n$ be an arbitrary set of events of a boolean algebra $\mathbf{B}$ and let $\Phi : a_i \mapsto \alpha_i$, $i = 1, \ldots, n$ be a book. The following are equivalent:*

(1) $\Phi$ *is coherent.*
(2) $\Phi$ *is extended by a finitely additive and normalised measure on* $\mathbf{B}$.

This result, which de Finetti had no inclination to express logically, turns out to be remarkably robust to the variation of the underlying logic of events. This was first illustrated by the generalisation of the "Dutch Book Method" obtained by Paris [35], and reinforced by the further developments in the MV-algebraic domain initiated by Mundici [31, 23]. In the latter papers, events are regarded as elements of an MV-algebra $\mathbf{A}$, and valuations are regarded as MV-homomorphisms $v$ from $\mathbf{A}$ to the MV-algebra whose carrier is the real unit interval $[0,1]$. Except for the fact that valuations range on $[0,1]$ (rather than on $\{0,1\}$) the notion of coherence which underpins this research thread does not differ from the condition captured by inequality (1) above. Indeed this literature led to the generalisation of de Finetti's theorem to *states of MV-algebras*, which will be introduced in Section 4.

De Finetti's coherence guards bookmakers against the possibility of sure loss by simultaneously barring them from what they should reasonably aim to, namely making profit. The condition of *strict* coherence has been put forward as a natural reaction to this rather odd feature. As a terminological aside, [18] pointed out that Shimony termed his refinement of de Finetti's notion simply "coherence", whilst Kemeny dubbed his own "strict fairness", but the term "strict coherence" quickly stuck, and so we conform to this usage.

Shimony's intuitive rendering of his criterion is the following. A book is *strictly incoherent* when

> there exists a choice of stakes [...] such that [...] no matter what the actual truth values of [of the events] may be, [the bookmaker] can at best lose nothing and in at least one possible eventuality will suffer a positive loss. [38]

Hence a bookmaker's book is *strictly coherent* if every possibility of loss is paired with a possibility of *gain*. This intuition motivates the following definition.

DEFINITION 2.2. Let $\Phi : a_1 \mapsto \alpha_1, \ldots, a_n \mapsto \alpha_n$ be a book (either boolean or MV). We say that $\Phi$ is *strictly coherent* if for every choice of real-valued stakes $\rho_1, \ldots, \rho_n$, the existence of a valuation $v$ such that

$$\sum_{i=1}^{n} \rho_i \left(\alpha_i - v(a_i)\right) < 0$$

implies the existence of a valuation $w$ such that

$$\sum_{i=1}^{n} \rho_i(\alpha_i - w(a_i)) > 0.$$

To illustrate the idea, suppose a book contains $a \mapsto 0$ for some non-contradictory event $a$. Then the book is *not* strictly coherent. For, if the gambler bets 1 on $a$, then her balance is as follows: she pays $1 \cdot 0 = 0$ and gets back 0 if $v(a) = 0$ and 1 if $v(a) = 1$. Hence, the bookmaker never wins and possibly loses, i.e, she violates strict coherence.

In spite of its intuitive appeal, the notion of strict coherence has attracted a number of criticisms over the decades. De Finetti himself was not persuaded the strengthening of his coherence condition for reasons analogous to those which led him to argue against countable additivity [9, Section 3.11, Appendix 18.2].

As recently pointed out in [33], the Kroupa-Panti theorem ([22, Corollary 29], [34, Proposition 1.1]) shows that de Finetti goes to great lengths for no reason of substance. Finite additivity on the boolean algebra of events is in fact sufficient for $\sigma$-additivity on the dual Stone space. In light of this we can avoid potential terminological confusion by referring to finitely additive and normalised measure which satisfy Carnap's condition as *Carnap-regular measures.* The Shimony-Kemeny characterisation then reads as follows.

THEOREM 2.3 (Shimony, Kemeny). *Let $\Phi$ be a $[0,1]$-valued map on a finite boolean algebra. The following are equivalent:*

(1) $\Phi$ *is strictly coherent.*
(2) $\Phi$ *is a Carnap-regular measure.*

We end this Section by pointing out explicitly the relation between the setting of Theorem 2.3 and that of Theorem 2.1, a comparison which will be clarified significantly by our main results.

REMARK 2.4. The algebraic setting of de Finetti's theorem is more general than the one used by Shimony and Kemeny. Logically speaking, the conditions of de Finetti's theorem apply to events expressed in a possibly infinite language, whereas Theorem 2.3 effectively requires events to be expressed in a finite language. Even more restrictive is this theorem's hypothesis that the map $\Phi$ is defined over the algebra of events. No such assumption is required for Theorem 2.1.

**§3. MV-algebras, abelian $\ell$-groups and Riesz spaces.** This section introduces the algebraic framework for many-valued events, namely MV-algebras [5, 6]. After recalling their definition and key properties, we focus on how those structures relate Riesz MV-algebras [10], lattice-ordered abelian groups [17] and Riesz spaces [26]. Section 5 will illustrate their central role in the characterisation of strict coherence for many-valued events.

DEFINITION 3.1 ([5, 6]). An *MV-algebra* is an algebra $\mathbf{A} = (A, \oplus, ', 0)$ where:

(a) $(A, \oplus, 0)$ is a commutative monoid,
(b) the equations $x'' = x$, $x \oplus 0' = 0'$ and $y \oplus (x' \oplus y)' = x \oplus (y' \oplus x)'$ hold.

In any MV-algebra, we set $x \odot y = (x' \oplus y')'$ and $1 = 0'$. Obviously, MV-algebras form a variety.

DEFINITION 3.2 ([17]). A *lattice ordered abelian group* (*$\ell$-group* for short) is an algebra $\mathbf{G} = (G, +, -, \vee, \wedge, 0)$ such that:

(a) $(G, +, -, 0)$ is an abelian group.
(b) $(G, \vee, \wedge)$ is a lattice.
(c) The equations $x+(y\vee z) = (x+y)\vee(x+z)$ and $x+(y\wedge z) = (x+y)\wedge(x+z)$ hold.

A *unital $\ell$-group* is an algebra $(\mathbf{G}, u)$ where $\mathbf{G}$ is an $\ell$-group and $u \in G$ is a constant such that for every $x \in G$ there is a natural number $n$ such that $x \leq nu$.

Without risk of confusion, we shall denote unital $\ell$-groups by $(\mathbf{G}, \leq, u)$, emphasizing, in this way, the lattice order $\leq$.

Let us denote by $\mathcal{G}_u$ the category whose objects are unital $\ell$-groups and whose morphisms are $\ell$-group homomorphisms preserving the strong unit, and let $\mathcal{MV}$ be the algebraic category whose objects are MV-algebras. Given a unital $\ell$-group $(\mathbf{G}, \leq, u)$, the structure

$$\Gamma(\mathbf{G}, \leq, u) = (A, \oplus, ', 0), \tag{2}$$

where $A = \{x \in G \mid 0 \leq x \leq u\}$, $x' = u - x$ and $x \oplus y = (x + y) \wedge u$ is an MV-algebra. For every morphism $h$ of unital $\ell$-groups, say from $(\mathbf{G}, \leq, u)$ to $(\mathbf{G}', \leq', u')$, the map $\Gamma(h)$ obtained by restricting the domain of $h$ to $\Gamma(\mathbf{G}, u)$, is an MV-homomorphism. Indeed, $\Gamma$ is a functor from $\mathcal{G}_u$ to $\mathcal{MV}$. By a famous result by Mundici [29], $\Gamma$ has an adjoint, $\Gamma^{-1}$, and the pair $(\Gamma, \Gamma^{-1})$ constitutes a categorical equivalence between $\mathcal{G}_u$ and $\mathcal{MV}$.

Example 3.3. (1) Every boolean algebra is an MV-algebra. Moreover, for every MV-algebra $\mathbf{A}$, the set of its idempotent elements $B(\mathbf{A}) = \{x \mid x \oplus x = x\}$ is the domain of the largest boolean subalgebra of $\mathbf{A}$: its *boolean skeleton*.

(2) $(\mathbf{R}, 1)$ (the additive group of reals with the constants 1) is a unital $\ell$-group and $\Gamma(\mathbf{R}, 1)$ is an MV-algebra, denoted by $[0, 1]$ in the sequel. The MV-algebra $[0, 1]$ stands to the variety of MV-algebras as the two-elements boolean algebra stands to the variety of boolean algebras. In particular, the variety of MV-algebras is generated by $[0, 1]$ (see [5]).

(3) Let $X$ be a compact Hausdorff space and let $\mathcal{C}_X = C(X, \mathbb{R})$ be the $\ell$-group of all continuous functions from $X$ into $\mathbb{R}$, with the operations defined pointwise. Let $\mathbf{1}$ denote the constantly 1 function from $X$ into $\mathbb{R}$. Then $(\mathcal{C}_X, \leq, \mathbf{1})$ is a unital $\ell$-group, and $\Gamma(\mathcal{C}_X, \leq, \mathbf{1})$ is an MV-algebra which is a subalgebra of $[0, 1]^X$. When $X$ is finite, say $|X| = k$, we will denote by $[0, 1]^k$ the MV-algebra of all functions form $X$ to $[0, 1]$ and we shall call it a *finite-dimensional MV-algebra*. We anticipate that these structure will play a key role in Section 6 below.

We will henceforth denote by $\mathcal{C}_X$ the $\ell$-group of all continuous functions from a compact Hausdorff space $X$ into $\mathbb{R}$. We write $\mathcal{H}(\mathbf{A})$ for the set of MV-homomorphisms of an MV-algebra $\mathbf{A}$ in $[0, 1]$. For every MV-algebra $\mathbf{A}$, $\mathcal{H}(\mathbf{A})$ is a closed subset of the product space $[0, 1]^A$ equipped with the Tychonoff topology. It can be shown that $\mathcal{H}(\mathbf{A})$ is a compact Hausdorff space (cf. [30, Theorem 2.5]).

An MV-algebra $\mathbf{A}$ is said to be *semisimple* if it is so in the usual universal algebraic sense, i.e., $\mathbf{A}$ is isomorphic to a subdirect product of simple MV-algebras (see [3, Definition 12.1]).

Proposition 3.4. *For an MV-algebra* $\mathbf{A}$*, the following are equivalent:*

(i) $\mathbf{A}$ *is semisimple,*

(ii) $\mathbf{A}$ *is isomorphic to an algebra* $\hat{\mathbf{A}}$ *of separating and continuous* $[0, 1]$*-valued functions over the compact Hausdorff space* $\mathcal{H}(\mathbf{A})$*,*

(iii) *for every* $a \in A$*, if* $a > 0$*, then there is* $v \in \mathcal{H}(\mathbf{A})$ *such that* $v(a) > 0$*.*

Proof. $(i) \Leftrightarrow (ii)$ is proved in [5], while the equivalence between $(ii)$ and $(iii)$ is immediate. $\dashv$

Definition 3.5 ([10]). A *Riesz MV-algebra* is an MV-algebra $\mathbf{A}$ equipped with a family $\{\alpha(\cdot)\}_{\alpha\in[0,1]}$ of unary operations on $A$ so that the following conditions hold for every $\alpha, \beta \in [0,1]$ and every $x, y \in A$:

(i) If $\alpha+\beta \leq 1$, then $\alpha(x) \oplus \beta(x) = (\alpha+\beta)(x)$.
(ii) If $x \odot y = 0$, then $\alpha(x \oplus y) = \alpha(x) \oplus \alpha(y)$.
(iii) $\alpha(\beta(x)) = \alpha\beta(x)$.
(iv) $1(x) = x$.
(v) $\alpha(x) = 0$ iff either $\alpha = 0$ or $x = 0$.

For the sake of a lighter notation, in the sequel, we shall write $\alpha a$ instead of $\alpha(a)$, for every $\alpha \in [0,1]$ and every element $a$ of a Riesz MV-algebra $\mathbf{A}$.

Definition 3.6 ([26, 10]). A *unital Riesz space* is an algebra $(\mathbf{V}, \leq, u)$ where $\mathbf{V}$ is a vector space over the real field, $\leq$ is a lattice order and $u$ is a constant such that, denoting by $\mathbf{V}^-$ the group reduct of $\mathbf{V}$, $(\mathbf{V}^-, \leq, u)$ is a unital $\ell$-group.

The equivalence between the categories of MV-algebras and unital $\ell$-groups extends in the same fashion to Riesz MV-algebras and unital Riesz spaces. Indeed, given a unital Riesz space $(\mathbf{V}, \leq, u)$, let

$$\Gamma_{\mathbf{R}}(\mathbf{V}, \leq, u) = (\mathbf{A}, \{\alpha(\cdot)\})$$

where, denoting by $\mathbf{V}^-$ the group reduct of $\mathbf{V}$, $\mathbf{A} = \Gamma(\mathbf{V}^-, \leq, u)$ as in (2) and, for every $\alpha \in [0,1]$, $\alpha(\cdot) : A \to A$ is the scalar product of $\mathbf{V}$ restricted to scalars in $[0,1]$. Now, if we denote by $\mathcal{RS}_u$ the category of unital Riesz spaces whose morphisms are vector-lattices homomorphisms preserving the unit, and by $\mathcal{RMV}$ the algebraic category of Riesz MV-algebras, $\Gamma_{\mathbf{R}}$ is a functor between $\mathcal{RS}_u$ and $\mathcal{RMV}$. Furthermore, $\Gamma_{\mathbf{R}}$ has an adjoint $\Gamma_{\mathbf{R}}^{-1}$ and the pair $(\Gamma_{\mathbf{R}}, \Gamma_{\mathbf{R}}^{-1})$ establishes a categorical equivalence between $\mathcal{RS}_u$ and $\mathcal{RMV}$ (cf. [10]).

For any unital Riesz space $(\mathbf{V}, \leq, u)$, we shall henceforth denote by $\mathcal{H}(\mathbf{V})$ the set of all morphisms of $(\mathbf{V}, \leq, u)$ in $(\mathbf{R}, \leq, 1)$.

Lemma 3.7. *Let $(\mathbf{V}, \leq, u)$ be a unital Riesz space. Then every element $x \in V$ may be represented as $x = ma + ku$ for some integer $k \leq 0$, for some natural number $m > 0$ and for some $a \in \Gamma_{\mathbf{R}}(\mathbf{V}, \leq, u)$. Moreover, the restriction to $\Gamma_{\mathbf{R}}(\mathbf{V}, \leq, u)$ of a morphism $w \in \mathcal{H}(\mathbf{V})$ is a morphism in $\mathcal{H}(\Gamma_{\mathbf{R}}(\mathbf{V}, \leq, u))$, and a morphism $v$ in $\mathcal{H}(\Gamma_{\mathbf{R}}(\mathbf{V}, \leq, u))$ has a unique extension to a morphism $w$ in $\mathcal{H}(\mathbf{V})$, defined by $w(ma + ku) = mv(a) + k$.*

Proof. Since $(\mathbf{V}, \leq, u)$ is unital, we can find a natural number $n$ such that $-x \leq nu$. Hence, $0 \leq x + nu$. Moreover, there is a natural number $m > 0$ such that $x+nu \leq mu$. Now let $a = \frac{1}{m}(x+nu)$. Then $0 \leq a \leq u$ and $a \in \Gamma_{\mathbf{R}}(\mathbf{V}, \leq, u)$. Finally $x = ma - nu$ and letting $k = -n$ we get the result.

The other claims are a direct consequence of the categorical equivalence between $\mathcal{RS}_u$ and $\mathcal{RMV}$, see [10, Theorem 4.2]. $\dashv$

Notation 3.8. Let $\mathbf{A}$ is a semisimple MV-algebra and let $\hat{\mathbf{A}}$ be as in Proposition 3.4 (ii). In what follows we shall adopt the following notations:

- $\mathbf{R}(\mathbf{A})$ stands for the Riesz MV-subalgebra of $[0,1]^{\mathcal{H}(\mathbf{A})}$ generated by $\hat{A}$.
- $(\mathbf{VL}(\mathbf{A}), \leq, \mathbf{1})$ denotes the unital Riesz subspace of $(\mathbb{R}^{\mathcal{H}(\mathbf{A})}, \leq, \mathbf{1})$ generated by $\hat{A}$.

As usual, we shall denote by $R(A)$ and $VL(A)$ the carriers of $\mathbf{R}(\mathbf{A})$ and $(\mathbf{VL}(\mathbf{A}), \leq, \mathbf{1})$ respectively.

Let $\mathbf{A}$ be a semisimple MV-algebra, let $(\mathbf{G}, \leq, u) = \Gamma^{-1}(\mathbf{A})$ and let

$$H = \left\{ \frac{g}{2^n} \mid g \in G, n \in \mathbb{N} \right\}. \tag{3}$$

As proved in [12, Corollary 3.7], $H$ is the domain of a dense unital $\ell$-subgroup of (the unital $\ell$-group reduct of) $(\mathcal{C}_{\mathcal{H}(\mathbf{A})}, \leq, \mathbf{1})$. Hence, any element of $(\mathbf{VL}(\mathbf{A}), \leq, \mathbf{1})$ can be approximated by the elements of $H$. The next lemma provides a slightly stronger statement.

Lemma 3.9. *For every semisimple MV-algebra $\mathbf{A}$, for every $z \in VL(A)$ and for every real number $\varepsilon > 0$, there are $z^+_\varepsilon, z^-_\varepsilon \in H$ such that $z^-_\varepsilon \leq z \leq z^+_\varepsilon$ and for every $v \in \mathcal{H}(\mathbf{A})$, $z(v) - z^-_\varepsilon(v) < \varepsilon$ and $z^+_\varepsilon(v) - z(v) < \varepsilon$.*

Proof. We shall prove the claim by induction on the structural complexity of $z \in VL(A)$. To this end, let us recall that $(\mathbf{VL}(\mathbf{A}), \leq, \mathbf{1})$ is generated by $\hat{A}$, and every $a \in \hat{A}$ is a separating and continuous function from $\mathcal{H}(\mathbf{A})$ in $[0,1]$, Proposition 3.4 (ii).

Clearly, if $z \in \hat{A}$, then we take $z^+_\varepsilon = z^-_\varepsilon = z$ and we are done.

Let us assume that $z = z_1 + z_2$. Then, by the inductive hypothesis, for $i = 1,2$, there are $z^+_{i\frac{\varepsilon}{2}}, z^-_{i\frac{\varepsilon}{2}} \in H$ such that $z^-_{i\frac{\varepsilon}{2}} \leq z_i \leq z^+_{i\frac{\varepsilon}{2}}$ and for every $v \in \mathcal{H}(\mathbf{A})$, $z_i(v) - z^-_{i\frac{\varepsilon}{2}}(v) < \frac{\varepsilon}{2}$ and $z^+_{i\frac{\varepsilon}{2}}(v) - z_i(v) < \frac{\varepsilon}{2}$. Then it suffices to take $z^+ = z^+_{1\frac{\varepsilon}{2}} + z^+_{2\frac{\varepsilon}{2}}$ and $z^- = z^-_{1\frac{\varepsilon}{2}} + z^-_{2\frac{\varepsilon}{2}}$.

If $z = z_1 \circ z_2$, where $\circ \in \{\vee, \wedge\}$, then, the inductive hypothesis ensures that for $i = 1,2$ we have $z^+_{i\varepsilon}, z^-_{i\varepsilon} \in H$ such that $z^-_{i\varepsilon} \leq z_i \leq z^+_{i\varepsilon}$ and for every $v \in \mathcal{H}(\mathbf{A})$, $z_i(v) - z^-_{i\varepsilon}(v) < \varepsilon$ and $z^+_{i\varepsilon}(v) - z_i(v) < \varepsilon$. Then it suffices to take $z^+ = z^+_{1\varepsilon} + z^+_{2\varepsilon}$ and $z^- = z^-_{1\varepsilon} + z^-_{2\varepsilon}$.

If $z = -u$, then by the inductive hypothesis for $i = 1,2$ there are $u^+_\varepsilon, u^-_\varepsilon \in H$ such that $u^-_\varepsilon \leq u \leq u^+_\varepsilon$ and for every $v \in \mathcal{H}(\mathbf{A})$, $z(v) - u^-_\varepsilon(v) < \varepsilon$ and $u^+_\varepsilon(v) - z(v) < \varepsilon$. Hence, it suffices to define $z^+_\varepsilon = -u^-_\varepsilon$ and $z^-_\varepsilon = -u^+_\varepsilon$.

Finally, suppose $z = \alpha u$, $\alpha$ a real number. Since we have already treated the case $z = -u$, we can assume without loss of generality that $\alpha \geq 0$. Let $M$ be a natural number such that $M \geq \alpha + 1$ and, for every $v \in \mathcal{H}(\mathbf{A})$, $-M + \varepsilon \leq u(v) \leq M - \varepsilon$. By the inductive hypothesis, there are $u^+, u^- \in H$ such that $u^- \leq u \leq u^+$ and, for every $v \in \mathcal{H}(\mathbf{A})$,

$$u(v) - u^-(v) < \tfrac{\varepsilon}{2M} \text{ and } u^+(v) - u(v) < \tfrac{\varepsilon}{2M}.$$

Moreover, $-M \leq u^-(v) \leq u^+(v) \leq M$, and $u^+(v) - u^-(v) < \frac{\varepsilon}{M}$. Let $n$ be such that $\frac{1}{2^n} < \frac{\varepsilon}{2M}$ and let $k$ be the maximum integer such that $\frac{k}{2^n} \leq \alpha$. Then $\frac{k+1}{2^n} > \alpha$. Let us consider

$$r = \tfrac{k}{2^n} \text{ and } t = \tfrac{k+1}{2^n}.$$

Now let

$$z^+ = t(u^+ \vee 0) + r(u^+ \wedge 0) \text{ and } z^- = r(u^- \vee 0) + t(u^- \wedge 0).$$

We verify that $z^- \leq z \leq z^+$. Let $v \in \mathcal{H}(\mathbf{A})$ and let us consider the following cases:

(i) If $u^-(v) \geq 0$, then $u^+(v) \geq 0$ as well, whence
$$z^+(v) = tu^+(v) \geq \alpha u(v) \geq ru^-(v) = z^-(v).$$
(ii) If $u^+(v) \leq 0$, then also $u^-(v) \leq 0$ and hence
$$z^+(v) = ru^+(v) \geq \alpha u(v) \geq tu^-(v) = z^-(v).$$
(iii) If $u^+(v) \geq 0$ and $u^-(v) \leq 0$, then
$$z^+(v) = tu^+(v) \geq \alpha u(v) \geq tu^-(v) = z^-(v).$$

This settles the claim. ⊣

**§4. States and faithful states of MV-algebras.** The purpose of this section is to present *states of an MV-algebra*, a notion which was introduced in [30] and further investigated in [34, 22, 31, 23].

Definition 4.1 ([30]). A *state* of an MV-algebra $\mathbf{A}$ is a $[0,1]$-valued map $s$ on $A$ such that $s(1) = 1$ and $s(x \oplus y) = s(x) + s(y)$, whenever $x \odot y = 0$.

A state $s$ of $\mathbf{A}$ is said to be *faithful*, if $s(x) = 0$ implies $x = 0$.

If $\mathbf{A}$ is a Riesz MV-algebra, a (faithful) state of $\mathbf{A}$ is a (faithful) state of its MV-algebraic reduct [10].

Definition 4.2 ([17]). A *state* of a unital $\ell$-group $(\mathbf{G}, \leq, u)$ is a map $\sigma$ from $G$ to $\mathbb{R}$ which is additive (i.e., for all $x, y$, $\sigma(x + y) = \sigma(x) + \sigma(y)$), monotonic (i.e., $x \geq 0$ implies $\sigma(x) \geq 0$) and normalized (i.e., $\sigma(u) = 1$).

A state $\sigma$ on a unital $\ell$-group is said to be *faithful* if for all $x > 0$, $\sigma(x) > 0$.

A (faithful) state of a unital Riesz space $(\mathbf{V}, \leq, u)$ is a (faithful) state of its unital $\ell$-group reduct. From [10, Lemma 11], it is clear that every state of a unital Riesz space $(\mathbf{V}, \leq, u)$ is a linear functional, that is, for every $z \in V$ and every $\alpha \in \mathbb{R}$, $\sigma(\alpha z) = \alpha\sigma(z)$.

The following result has been proved in [12, Theorem 3.2].

Lemma 4.3. *Every state on a semisimple MV-algebra* $\mathbf{A}$ *has a unique extension to a state on* $(\mathcal{C}_{\mathcal{H}(\mathbf{A})}, \leq, \mathbf{1})$*, and hence to a state on* $\Gamma(\mathcal{C}_{\mathcal{H}(\mathbf{A})}, \leq, \mathbf{1})$*. Hence, any state on a semisimple MV-algebra* $\mathbf{A}$ *has a unique extension to a state of* $\mathbf{R}(\mathbf{A})$*.*

The problem of extending a faithful state of an MV-algebra $\mathbf{A}$ to a Riesz MV-algebra that contains $\mathbf{A}$ was studied in [25]. There, the author shows that every faithful state $s$ of $\mathbf{A}$ can be extended, in a unique way, to a faithful state of what was called the *Riesz completion* of $\mathbf{A}$ (see [25, Theorem 4.2]). The following theorem establishes a slightly stronger result with an alternative technique.

Theorem 4.4. *Every faithful state of an MV-algebra* $\mathbf{A}$ *can be extended to a unique faithful state of* $(\mathbf{VL}(\mathbf{A}), \leq, \mathbf{1})$*, and hence, to a unique faithful state on* $\mathbf{R}(\mathbf{A})$*.*

Proof. Let $s$ be a faithful state of $\mathbf{A}$, and let $\sigma$ be its unique extension to the (unique up to isomorphism) unital $\ell$-group $(\mathbf{G}, \leq, u)$ such that $\mathbf{A} = \Gamma(\mathbf{G}, \leq, u)$ (cf. [30, Theorem 2.4]). Then $\sigma$ is a faithful state on $(\mathbf{G}, \leq, u)$, because every positive element $g \in G$ is the sum of elements $a_i$ of $A$, $g = \sum_{i=1}^{n} a_i$ (see [6, §7] for details), and $\sigma(g) = \sum_{i=1}^{n} s(a_i) > 0$. Let $H$ be as in (3). Then $\sigma$ has a unique extension $\sigma'$ to $H$, defined by $\sigma'(\frac{g}{2^n}) = \frac{\sigma(g)}{2^n}$. Moreover, by Lemma 4.3, $s$ has a unique extension to $(\mathcal{C}_{\mathcal{H}(\mathbf{A})}, \leq, \mathbf{1})$ and hence, a unique extension $\sigma''$ to a state on $(\mathbf{VL}(\mathbf{A}), \leq, \mathbf{1})$.

We conclude by proving that $\sigma''$ is faithful. Let $z \in VL(A)$, $z > 0$. By the very definition of $(\mathbf{VL}(\mathbf{A}), \leq, \mathbf{1})$ and since $\hat{\mathbf{A}}$ is an algebra of $[0,1]$-valued continuous functions, $z$ is a continuous function on the compact space $\mathcal{H}(\mathbf{A})$, whence it admits a maximum point $v_0$. Let $M = z(v_0) > 0$. Now let, by Lemma 3.9, $z^- = \frac{g}{2^n}$ (with $g \in G$) be such that for every valuation $v$, $z^-(v) \leq z(v)$ and $z(v) - z^-(v) < \frac{M}{2}$. After replacing $g$ by $g \vee 0$, we may assume without loss of generality that $z^- = \frac{g}{2^n} \geq 0$. Moreover the conditions $z(v_0) - z^-(v_0) < \frac{M}{2}$ and $z(v_0) = M$ imply $z^- > 0$ and $g > 0$. Since $\sigma$ is faithful, then $\sigma(g) > 0$ and $\sigma'(z^-) = \frac{\sigma(g)}{2^n} > 0$. Finally, $\sigma''(z) \geq \sigma'(\frac{g}{2^n}) > 0$, whence $\sigma''$ is faithful. ⊣

**§5. Strictly coherent maps.** We are now in a position to address the main question of this paper, namely the investigation of strict coherence on many-valued events. In what follows, we will say that a $[0,1]$-valued map $\Phi$ defined on a semisimple MV-algebra $\mathbf{A}$ is *(strictly) coherent*, if for every finite subset $A'$ of $A$, the book obtained by restricting $\Phi$ to $A'$ is (strictly) coherent according to Definition 2.2.

We begin by extending to the general case of MV-algebras a result [31, Theorem 5.6] which was proved in the framework of free MV-algebras over arbitrary sets of generators.

LEMMA 5.1. *Let $\mathbf{A}$ be an MV-algebra and let $\Phi$ be a $[0,1]$-valued map defined on $A$. Then $\Phi$ is coherent iff it is a state.*

PROOF. The right-to-left direction is trivial. Indeed, if $\Phi$ is a state, its restriction to any finite subset of $A$ extends to a state, namely to $\Phi$. Thus $\Phi$ is coherent from [23, Theorem 3.2]

In order to prove the left-to-right direction, suppose $\Phi$ is not a state. Then either $\Phi(1) < 1$ or there are $x, y$ such that $x \odot y = 0$ and $\Phi(x \oplus y) \neq \Phi(x) + \Phi(y)$. In the former case, betting 1 on the certain event clearly causes a sure loss for the bookmaker. In the latter case, we need to consider the two possible inequalities. If $\Phi(x \oplus y) < \Phi(x) + \Phi(y)$, we can cause a sure loss to the bookmaker by betting 1 on $x \oplus y$ and $-1$ on both $x$ and $y$. Finally, if $\Phi(x \oplus y) > \Phi(x) + \Phi(y)$, a sure loss to the bookmaker is caused by betting $-1$ on $x \oplus y$ and 1 on both $x$ and $y$. Thus, $\Phi$ is not coherent. ⊣

We can now characterise strictly coherent maps.

THEOREM 5.2. *Let $\mathbf{A}$ be a semisimple MV-algebra and $\Phi$ a $[0,1]$-valued map defined on $A$. Then $\Phi$ is strictly coherent iff it is a faithful state.*

PROOF. (Left-to-right). By contraposition, let us suppose that $\Phi$ is not a faithful state. By Lemma 5.1 it suffices to assume it is not faithful, i.e. $\Phi(x) = 0$ for some $x > 0$. Since $\mathbf{A}$ is semisimple, by Proposition 3.4 (iii), there exists $v \in \mathcal{H}(\mathbf{A})$ such that $v(x) > 0$. Hence, a bet of 1 on $x$ shows that $\Phi$ is not strictly coherent.

(Right-to-left). Suppose that $\Phi$ is a faithful state of $\mathbf{A}$ and let $\Phi'$ be its unique extenstion to $(\mathbf{VL}(\mathbf{A}), \leq, \mathbf{1})$ as ensured by Theorem 4.4. Assume, by way of contradiction, that $\Phi$ is not strictly coherent. Thus, there are a finite subset $A' = \{a_1, \ldots, a_n\}$ of $A$ and real numbers $\rho_1, \ldots, \rho_n$ such that for every $v \in \mathcal{H}(\mathbf{A})$, $\sum_{i=1}^n \rho_i(\alpha_i - v(a_i)) \leq 0$ and for some $w \in \mathcal{H}(\mathbf{A})$, $\sum_{i=1}^n \rho_i(\alpha_i - w(a_i)) < 0$.

Now, let us define

$$z = \sum_{i=1}^{n} \rho_i(\alpha_i \mathbf{1} - a_i)$$

and let, for every $v \in \mathcal{H}(\mathbf{A})$, $v^+$ be its unique extension in $\mathcal{H}(\mathbf{V})$ as ensured by Lemma 3.7. Clearly $z \in VL(A)$. In addition, the assumption that $\Phi$ is not strictly coherent implies that for every $v \in \mathcal{H}(\mathbf{A})$, $v^+(z) \leq 0$ and there is at least a $w \in \mathcal{H}(\mathbf{A})$ such that $w^+(z) < 0$. So $z < 0$, hence $-z > 0$. However,

$$\Phi'(-z) = \sum_{i=1}^{n} \rho_i(\Phi(a_i) - \alpha_i) = 0,$$

so $\Phi'$ is not faithful: a contradiction. $\dashv$

It is well known that any boolean algebra is a semisimple MV-algebra (this holds as a consequence of Example 3.3(1) plus the fact that any boolean algebra is semisimple [3]). Hence Theorem 5.2 yields immediately the following.

Corollary 5.3. *A $[0, 1]$-valued map $\Phi$ on a boolean algebra $\mathbf{A}$ is strictly coherent iff it is a Carnap-regular measure.*

In other words, the restriction concerning the finiteness of the boolean algebra of events assumed in the Shimony-Kemeny characterisation can be dispensed with. This addresses the first problem raised in Remark 2.4.

**§6. Strictly coherent books.** The main result of the previous section naturally prompts the question as to whether an extension result *à la* de Finetti (see Theorem 2.1 above) can be obtained for strict coherence. As a consequence of seminal results by Mundici [30, 3.2], Kelley [20] and Gaifman [16], an MV-algebra $\mathbf{A}$ may not have a faithful state. This implies that our desired result cannot be formulated, in general, as an equivalence between strict coherence and extensibility to a faithful state of $\mathbf{A}$. The main result of this section shows that such a characterisation can be obtained for the class of *finite-dimensional* MV-algebras. Corollary 6.5 shows that this restriction does not prevent us from encompassing the most general case of boolean algebras.

Our argument is geometric and draws heavily on the following presumably known Lemma, which is proved in the Appendix for the sake of clarity. Readers who are not familiar with elementary convex geometry may wish to consult [11, 27]. For $C$ a closed convex subset of $\mathbb{R}^n$, we will denote by $\mathrm{ext}(C)$ the set of its *extremal points*, by $\mathrm{relint}(C)$ its *relative interior* and by $\partial_r(C)$ its *relative boundary*. In this notation a *polytope* $C$ is a closed and convex subset of $\mathbb{R}^n$ such that $\mathrm{ext}(C)$ is finite.

Lemma 6.1. *Let $C \subseteq \mathbb{R}^n$ be a polytope. Then the following hold:*

(1) *If* $\mathrm{ext}(C) \subseteq \{x_1, \dots, x_t\}$, *then*
$$\mathrm{relint}(C) = \left\{ \sum_{i=1}^{t} \lambda_i x_i \mid \lambda_i > 0, \sum_{i=1}^{t} \lambda_i = 1 \right\}.$$
(2) *Let* $x \in C$. *Then* $x \notin \mathrm{relint}(C)$ *iff there is a* $p \in \mathbb{R}^n$ *properly supporting* $C$ *at* $x$, *i.e.,* $p \cdot x - p \cdot z \leq 0$ *for all* $z \in C$ *and* $p \cdot x - p \cdot y < 0$ *for some* $y \in C$.
(3) *Let* $x \in \partial_r(C)$. *Then there is a* $r \in \mathbb{R}^n$ *such that* $r \cdot z - r \cdot x \leq 0$ *for all* $z \in C$ *and* $r \cdot x - r \cdot y < 0$ *for all* $y \in \mathrm{relint}(C)$.

Proof. See Appendix. ⊣

Let $\mathcal{H}([0,1]^k)$ denote, as usual, the set of homomorphisms from $[0,1]^k$ to $[0,1]$. In the terminology of [7], the MV-algebra $[0,1]^k$ is *weakly finite* and hence [7, Lemma 2.7] shows that $\mathcal{H}([0,1]^k)$ is finite and its elements are the $k$ projections $\pi_i : [0,1]^k \to [0,1]$.

Fix $a_1, \dots, a_n \in [0,1]^k$ and identify a book $\Phi : a_i \mapsto \alpha_i$ with the $n$-tuple $(\alpha_1, \dots, \alpha_n) \in [0,1]^n$. Let us denote
$$\begin{array}{rcl} \mathcal{C}(a_1, \dots, a_n) & = & \{(\alpha_1, \dots, \alpha_n) \mid \Phi : a_i \mapsto \alpha_i \text{ is coherent}\} \\ \mathcal{Q}(a_1, \dots, a_n) & = & \{(\pi_i(a_1), \dots, \pi_i(a_n)) \mid \pi_i \in \mathcal{H}([0,1]^k)\}. \end{array}$$

It is well-known that $\mathcal{C}(a_1, \dots, a_n)$ is a polytope (see for instance [23, Proposition 2.4] and [35, Theorem 2]). In addition, $\mathcal{Q}(a_1, \dots, a_n)$ is finite and we will denote its elements by $q_1, \dots, q_k$.

Furthermore, [30, Theorem 2.5] and the fact that every $\pi_i \in \mathcal{H}([0,1]^k)$ is a state of $[0,1]^k$, show the following inclusions:
$$\mathrm{ext}(\mathcal{C}(a_1, \dots, a_n)) \subseteq \mathcal{Q}(a_1, \dots, a_n) \subseteq \mathcal{C}(a_1, \dots, a_n) \tag{4}$$

Lemma 6.2. *Let* $a_1, \dots, a_n \in [0,1]^k$ *and let* $\Phi \in \mathcal{C}(a_1, \dots, a_n)$. *Then the following are equivalent:*
(1) $\Phi$ *is strictly coherent,*
(2) $\Phi \in \mathrm{relint}(\mathcal{C}(a_1, \dots, a_n))$.

Proof. (1) ⇒ (2). Suppose $\Phi \notin \mathrm{relint}(\mathcal{C}(a_1, \dots, a_n))$, and let us prove that $\Phi$ is not strictly coherent. Since $\Phi$ is coherent and $\Phi \notin \mathrm{relint}(\mathcal{C}(a_1, \dots, a_n))$, then $\Phi \in \partial_r \mathcal{C}(a_1, \dots, a_n)$, and by Lemma 6.1 (3), there exists a $\rho \in \mathbb{R}^n$ such that, for all $\gamma \in \mathcal{C}(a_1, \dots, a_n)$, $\rho \cdot \Phi \leq \rho \cdot \gamma$ and for all $\sigma \in \mathrm{relint}(\mathcal{C}(a_1, \dots, a_n))$, $\rho \cdot \Phi < \rho \cdot \sigma$.

Since $\mathcal{C}(a_1, \dots, a_n)$ is closed and convex, by the Krein-Milman theorem for finite dimensional spaces [17, Theorem 5.17] and (4), its elements are convex combinations of the elements $q_1, \dots, q_k$ of $\mathcal{Q}(a_1, \dots, a_n)$. Take $\lambda_1, \dots, \lambda_k > 0$ such that $\sum_{i=1}^{k} \lambda_i = 1$ and let
$$b = \sum_{i=1}^{k} \lambda_i q_i.$$
Then, Lemma 6.1 (1) implies that $b \in \mathrm{relint}(\mathcal{C}(a_1, \dots, a_n))$ and hence, by Lemma 6.1 (3), there is $\rho = (\rho_1, \dots, \rho_n) \in \mathbb{R}^n$ such that $\rho \cdot \Phi \leq \rho \cdot q_i$ and $\rho \cdot \Phi < \rho \cdot b$. It follows that for at least a $q_{i_0} \in \mathcal{Q}(a_1, \dots, a_n)$ one has $\rho \cdot \Phi < \rho \cdot q_{i_0}$. For otherwise, if $\rho \cdot \Phi = \rho \cdot q_i$ for every $q_i$, we would have $\rho \cdot \Phi = \rho \cdot b$. Thus,

$$\sum_{j=1}^{n} \rho_j(\alpha_j - \pi_i(a_j)) \leq 0 \text{ and } \sum_{j=1}^{n} \rho_j(\alpha_j - \pi_{i_0}(a_j)) < 0.$$

Hence, $\Phi$ is not strictly coherent.

(2) $\Rightarrow$ (1). Suppose $\Phi = (\alpha_1, \ldots, \alpha_n)$ is not strictly coherent. Then, there exist $\rho = (\rho_1, \ldots, \rho_n) \in \mathbb{R}^n$ such that for every $\pi_i \in \mathcal{H}([0,1]^k)$,

$$\sum_{j=1}^{n} \rho_j(\alpha_j - \pi_i(a_j)) = \rho \cdot \Phi - \rho \cdot q_i \leq 0, \tag{5}$$

and for some $\pi_h \in \mathcal{H}([0,1]^k)$,

$$\sum_{j=1}^{n} \rho_j(\alpha_j - \pi_h(a_j)) = \rho \cdot \Phi - \rho \cdot q_h < 0. \tag{6}$$

Again by the Krein-Milman theorem for the finite dimensional case and (4), every element of $\mathcal{C}(a_1, \ldots, a_n)$ has the form $x = \sum_{i=1}^{k} \lambda_i q_i$ for some non-negative real numbers $\lambda_1, \ldots, \lambda_k$ such that $\sum_{i=1}^{k} \lambda_i = 1$. From (5-6), $\rho \cdot \Phi - \rho \cdot x \leq 0$ for all $x \in \mathcal{C}(a_1, \ldots, a_n)$ and $\rho \cdot \Phi - \rho \cdot y < 0$ for some $y \in \mathcal{C}(a_1, \ldots, a_n)$. Therefore, $\rho$ properly supports $\mathcal{C}(a_1, \ldots, a_n)$ at $\Phi$ and by Lemma 6.1 (2), $\Phi \notin \mathrm{relint}(\mathcal{C}(a_1, \ldots, a_n))$. ⊣

Remark 6.3. As an immediate consequence of [32, Theorem 10.5], the set of states of $[0,1]^k$ is affinely isomorphic to the simplex, $\Delta_k = \{(\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k \mid \sum_{i=1}^{k} \lambda_i = 1\}$ via the map $\lambda \mapsto s_\lambda$ which associates, to every $\lambda = (\lambda_1, \ldots, \lambda_k) \in \Delta_k$, the state $s_\lambda$ defined as follows: for every $a \in [0,1]^k$,

$$s_\lambda(a) = \sum_{i=1}^{k} \lambda_i \pi_i(a) \tag{7}$$

(cf. [14, Corollary 4.1.2]).

Now, the restriction of the above map $\lambda \mapsto s_\lambda$ to the relative interior of $\Delta_k$ determines an affine isomorphism between $\mathrm{relint}(\Delta_k)$ and the set of faithful states of $[0,1]^k$. As a matter of fact, $\lambda \in \mathrm{relint}(\Delta_k)$ iff $\lambda_i > 0$ for every $i = 1, \ldots, k$ (Lemma 6.1 (1)) whence, if $a \in [0,1]^k$ is strictly positive, a straightforward computation shows that (7) implies $s_\lambda(a) > 0$.

Theorem 6.4. *Let $a_1, \ldots, a_n \in [0,1]^k$ and let $\Phi : a_i \mapsto \alpha_i$, $i = 1, \ldots, n$ be a book. Then $\Phi$ is strictly coherent iff $\Phi$ extends to a faithful state of $[0,1]^k$.*

Proof. (Right-to-left). If $s$ is a faithful state that extends $\Phi$, then $\Phi$ is strictly coherent by Theorem 5.2.

(Left-to-right). By Lemma 6.2, $\Phi$ is strictly coherent iff $\Phi \in \mathrm{relint}(\mathcal{C}(a_1, \ldots, a_n))$ iff, by Lemma 6.1 (1), there is $\lambda = (\lambda_1, \ldots, \lambda_k) \in (0,1]^k$ such that $\sum_{i=1}^{k} \lambda_i = 1$ and $\Phi = \sum_{i=1}^{k} \lambda_i q_i$. Thus, the state $s_\lambda$ defined as in (7) is faithful (Remark 6.3) and

$$(s_\lambda(a_1), \ldots, s_\lambda(a_n)) = \sum_{i=1}^{n} \lambda_i q_i = \Phi.$$

It follows that $\Phi$ is extended by a faithful state, and the claim is settled. ⊣

We conclude this section by presenting a remarkable corollary of Theorem 6.4, which depends on the observation that boolean algebras form a locally finite variety (see for instance [1]).

Recall that by the above-mentioned results of Kelley and Gaifman, Carnap-regular measures may not exist for boolean algebras. As a consequence, the following is the most general de Finetti-like result for strict coherence on boolean events.

COROLLARY 6.5. *Let* $\mathbf{A}$ *be any boolean algebra, let* $\{a_1, \dots, a_n\}$ *be a finite subset of* $A$ *and let* $\mathbf{A}[a_1, \dots, a_n]$ *be the subalgebra of* $\mathbf{A}$ *generated by* $a_1, \dots, a_n$*. Then a book* $\Phi$ *on* $\{a_1, \dots, a_n\}$ *is strictly coherent iff it extends to a Carnap-regular measure of* $\mathbf{A}[a_1, \dots, a_n]$*.*

**§7. Conclusion.** Our last corollary adds to a series of recent results which clearly give MV-algebras a twofold prominent role in the foundations of probability. First the MV-algebraic setting allows naturally for greater generality compared to its boolean counterpart [35, 31, 23]. Second MV-algebras provide a rich framework capable of shedding new light on standard probability, i.e. on boolean valued events. As pointed out in [33] the Kroupa-Panti theorem paved the way to this, and recent results obtained by the authors in [15] point out the wide applicability of the method. Given that generality and applicability usually pull in opposite directions, this twofold role of MV-algebras is all the more remarkable.

Theorem 5.2 immediately implies that if a book on a semisimple MV-algebra is extensible to a faithful state then it is strictly coherent. Whether the converse holds as well remains an open question which is currently under investigation. The motivation for pursuing this goal is the fact that, in the setting of our paper, it would yield the most general characterisation of strict coherence.

As a consequence of results of [36, 2], $NP$ constitutes the lower bound on the computational complexity of deciding strict coherence. Ongoing research by the authors is investigating the upper bound for this problem.

Finally, further work will also tell us whether the characterisation of strict coherence for many-valued events provided in this paper has an interesting counterpart when probabilities are extended to the hyperreals along the lines of [28, 19]. As implied by [28, Theorem 4.2] this setting is immune to the limitations arising from the potential lack, in general, of faithful states of MV-algebras.

**Acknowledgments.** This paper has benefitted enormously from the unrestrained criticisms of an anonymous Reviewer to whom we are particularly grateful.

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**Appendix A. Proof of Lemma 6.1.** For the sake of keeping the paper self-contained we provide a proof of Lemma 6.1.

Proof. (1). If $\mathrm{ext}(C) = \{x_1, \dots, x_t\}$ the claim is known to be true by [27, Theorem 6]. Thus, assume $\mathrm{ext}(C) \subset \{x_1, \dots, x_t\}$ and suppose, without loss of generality, that $s < t$ and $\mathrm{ext}(C) = \{x_1, \dots, x_s\}$ so that, by [27, Theorem 6] for every $z \in \mathrm{relint}(C)$ there are $\lambda_1, \dots, \lambda_s \in (0, 1]$ such that $z = \sum_{i=1}^{s} \lambda_i x_i$. We prove by induction on $m$ that for all $s \le m \le t$, there are $\lambda_1^m, \dots, \lambda_m^m \in (0, 1]$ with $\sum_{i=1}^{m} \lambda_i^m = 1$ such that $z = \sum_{i=1}^{m} \lambda_i^m x_i$. The claim clearly holds for $m = s$. Suppose, by the induction hypothesis that the claim holds for some $m$ with $s \le m < t$, and let us prove that it holds for $m + 1$. Then $z = \sum_{i=1}^{m} \lambda_i^m x_i$ with $\lambda_i^m > 0$ and $\sum_{i=1}^{m} \lambda_i^m = 1$. Moreover, since $x_{m+1} \in C$, it can be written as $x_{m+1} = \sum_{i=1}^{s} \tau_i x_i$ where $\tau_i \ge 0$ and $\sum_{i=1}^{s} \tau_i = 1$. Let $M > 0$ be such that for $i = 1, \dots, s$, $\frac{\tau_i}{M} < \lambda_i$. Define

$$\lambda_i^{m+1} = \begin{cases} \lambda_i - \frac{\tau_i}{M} & \text{if} \quad i \le s \\ \lambda_i & \text{if} \quad s < i \le m \\ \frac{1}{M} & \text{if} \quad i = m + 1. \end{cases}$$

Then

$$\sum_{i=1}^{m+1} \lambda_i^{m+1} x_i = \sum_{i=1}^{m} \lambda_i x_i - \sum_{i=1}^{s} \frac{\tau_i}{M} x_i + \frac{x_{m+1}}{M} = z - \frac{x_{m+1}}{M} + \frac{x_{m+1}}{M} = z.$$

Moreover $\lambda_i^{m+1} > 0$ for all $i \le m + 1$ and

$$\sum_{i=1}^{m+1} \lambda_i^{m+1} = \sum_{i=1}^{m} \lambda_i - \sum_{i=1}^{s} \frac{\tau_i}{M} + \frac{1}{M} = 1.$$

(2). This is the usual Supporting Hyperplane Theorem, see [27, Theorem 14].

(3). Let $x \in \partial_r(C)$. By part (2), there is a $r \in \mathbb{R}^n$ that properly supports $C$ at $x$, that is, $r \cdot z \le r \cdot x$ for all $z \in C$. In particular, there is a $t \in C$ such that $r \cdot t < r \cdot x$. Let $y \in \mathrm{relint}(C)$ and assume by way of contradiction, that $r \cdot y = r \cdot x$. Since $y \in \mathrm{relint}(C)$, $y + \varepsilon(y - t) \in C$ for a sufficiently small $\varepsilon > 0$. Then,

$$\begin{array}{rcl} r \cdot (y + \varepsilon(y-t)) & = & r \cdot y + \varepsilon(r \cdot y - r \cdot t) \\ & = & r \cdot x + \varepsilon(r \cdot x - r \cdot t) \\ & = & r \cdot x + \varepsilon r \cdot (x - t) \\ & > & r \cdot x. \end{array}$$

and a contradiction is reached since $(y + \varepsilon(y - t)) \in C$, but $r \cdot z \leq r \cdot x$ for all $z \in C$. ⊣

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