# 1D SYMMETRY FOR SEMILINEAR PDES FROM THE LIMIT INTERFACE OF THE SOLUTION

ALBERTO FARINA AND ENRICO VALDINOCI

ABSTRACT. We study bounded, monotone solutions of $\Delta u = W'(u)$ in the whole of $\mathbb{R}^n$, where $W$ is a double-well potential. We prove that under suitable assumptions on the limit interface and on the energy growth, $u$ is 1D.

In particular, differently from the previous literature, the solution is not assumed to have minimal properties and the cases studied lie outside the range of $\Gamma$-convergence methods.

We think that this approach could be fruitful in concrete situations, where one can observe the phase separation at a large scale and whishes to deduce the values of the state parameter in the vicinity of the interface.

As a simple example of the results obtained with this point of view, we mention that monotone solutions with energy bounds, whose limit interface does not contain a vertical line through the origin, are 1D, at least up to dimension $4$.

## CONTENTS



2010 *Mathematics Subject Classification.* 35J61, 35J15.

*Key words and phrases.* Phase transitions, symmetry results, limit interface.

*To appear in Communications in Partial Differential Equations.*

It is a pleasure to thank Yoshihiro Tonegawa for a very instructive discussion held in Banff. This work has been supported by the ERC grant 277749 "EPSILON Elliptic Pde's and Symmetry of Interfaces and Layers for Odd Nonlinearities".

## NOTATION

We take $n \geqslant 2$. A point $x \in \mathbb{R}^n$ will be often written as $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

For any $1 \leqslant i \leqslant n$, the partial derivatives with respect to $x_i$ will be denoted by

$$\partial_i = \partial_{x_i} = \frac{\partial}{\partial x_i}.$$

Also, given an ambient domain $\Omega \subseteq \mathbb{R}^n$ and a set $F \subseteq \Omega$, we denote by $\chi_F$ its characteristic function, i.e.

$$\chi_F(x) := \begin{cases} 1 & \text{if } x \in F, \\ 0 & \text{if } x \in \mathscr{C}F, \end{cases}$$

where $\mathscr{C}F := \Omega \setminus F$ is the complement of $F$ in its ambient space.

We denote by $W$ the classical "double-well potential" $W(r) = (1 - r^2)^2$ (more general type of double-well potentials may be treated in the same way), and we will study solutions

$$\begin{aligned} & u \in C^2(\mathbb{R}^n, [-1, 1]) \text{ of} \\ & \Delta u(x) = W'(u(x)) \text{ for any } x \in \mathbb{R}^n \end{aligned} \tag{1}$$

under the monotonicity condition

$$\frac{\partial u}{\partial x_n}(x) > 0 \text{ for any } x \in \mathbb{R}^n. \tag{2}$$

After [**?**], condition (2) has become classical in the study of semilinear equations. From the variational point of view, it implies that

$$u \text{ is a stable solution,} \tag{3}$$

i.e. the second variation of the energy is nonnegative, see e.g. Corollary 4.3 in [**?**].

For such $u$ and any $\varepsilon > 0$, we define the rescaled solution

$$u_\varepsilon(x) := u(x/\varepsilon).$$

We observe that, by the Maximum Principle and (2), we have that $|u(x)| < 1$ for every $x \in \mathbb{R}^n$ and so, in particular,

$$u_\varepsilon(0) = u(0) \in (-1, 1). \tag{4}$$

Given a bounded open set $\Omega \subset \mathbb{R}^n$, we also consider the energy functional associated to (1), namely

$$\mathscr{E}(u, \Omega) := \int_\Omega \frac{|\nabla u(x)|^2}{2} + W(u(x))\, dx.$$

We say that $u$ is a (local) minimizer if, for any bounded open set $\Omega$ and any $\varphi \in C_0^\infty(\Omega)$, we have that

$$\mathscr{E}(u, \Omega) \leqslant \mathscr{E}(u + \varphi, \Omega).$$

Similarly, one says that $u$ is quasiminimal if, for some $Q \geqslant 1$, one has that

$$\mathscr{E}(u, \Omega) \leqslant Q\, \mathscr{E}(u + \varphi, \Omega)$$

for any bounded open set $\Omega$ and any $\varphi \in C_0^\infty(\Omega)$.

## 1. INTRODUCTION

The study of the PDE in (1) under the monotonicity assumption (2) is a classical topic in semilinear elliptic equations and it goes back, at least, to the study of the Ginzburg-Landau-Allen-Cahn phase segregation model, in connection with the theory of hypersurfaces with minimal perimeter. In particular, the following striking problem was posed in [**?**]:

**Question 1.1.** *Let $u$ be a solution of* (1) *satisfying* (2)*. Is it true that $u$ is* 1*D, i.e. that all the level sets of $u$ are hyperplanes, at least if $n \leqslant 8$?*

We refer to [**?**, **?**] and also [**?**, **?**] for the proof that Question 1.1 has a positive answer in dimension $2 \leqslant n \leqslant 3$. See also [**?**], where it is shown that Question 1.1 also has a positive answer in dimension $4 \leqslant n \leqslant 8$ provided that

$$\lim_{x_n \to -\infty} u(x', x_n) = -1 \quad \text{and} \quad \lim_{x_n \to +\infty} u(x', x_n) = 1. \tag{5}$$

When $n \geqslant 9$, an example of a solution $u$ satisfying (1), (2) and (5) that is not 1D was constructed in [**?**], showing that the dimensional constraint in Question 1.1 cannot be removed.

See also [**?**] for some symmetry results that hold under conditions at infinity which are weaker than (5). In particular, it is shown in [**?**] that condition (5) can be relaxed to suitable symmetry assumptions on the asymptotic profiles

$$\begin{aligned} \overline{u}(x') &:= \lim_{x_n \to -\infty} u(x', x_n), \\ \underline{u}(x') &:= \lim_{x_n \to +\infty} u(x', x_n). \end{aligned} \tag{6}$$

More precisely, in [**?**] it is proved that solutions of (1) satisfying (2) are 1D if:

- $2 \leqslant n \leqslant 4$ and at least one between $\overline{u}$ and $\underline{u}$ are 2D,
- $2 \leqslant n \leqslant 8$ and both $\overline{u}$ and $\underline{u}$ are 2D.

It is also proved in [**?**] that solutions of (1) that satisfy (2) are 1D if $2 \leqslant n \leqslant 8$, provided that at least one level set is a complete graph.

Other symmetry results are obtained in [**?**] for quasilinear equations and[1] for quasiminimal solutions

We also refer to [**?**] for further motivation and a review on Question 1.1, and to [**?**] for its connection with an important problem posed by [**?**]. We also mention the very recent contribution in [**?**].

In the light of the above mentioned results, we have that Question 1.1 is still open in dimension $4 \leqslant n \leqslant 8$, and our paper would like to be a further step towards this direction.

---

[1]The symmetry results in [**?**] come from the control of the profiles at infinity, from the graph properties of the level sets, from the minimal and quasiminimal properties of the solution or from the uniform limits at infinity.

The proofs of the results of [**?**] combine and extend several techniques, such as ODE analysis, a careful study of the limit profiles of the solution and sharp energy and density estimates. In addition, some of the results of [**?**] make use of previous work developed in [**?**, **?**, **?**, **?**, **?**].

In the literature, most of the research related to Question 1.1 heavily relies on the analysis of minimizers of the energy functional. This approach was also inspired by the classical $\Gamma$-convergence results (see e.g. [**?**]), which established an important link between the level sets of the solutions and the study of hypersurfaces with minimal perimeter.

On the other hand, from the point of view of pure mathematics, it is interesting to consider the case of solutions that do not necessarily have minimizing properties, or whose minimizing properties are not completely known a-priori. As a matter of fact, these solutions also appear in concrete situations, since there is numerical evidence that some solutions show unstable patterns before settling down to more stable configurations (see e.g. [**?**]).

Therefore, the scope of this paper is to investigate symmetry results without assuming any minimality condition on the solution, but only monotonicity assumptions, energy bounds and some geometric information on the limit interface.

For this, we will derive rigidity and symmetry results from either the behavior of the limit level set or the one of the limit varifold, using some results in [**?**] in order to introduce and describe the limit interface without minimizing assumptions (notice that in this case the $\Gamma$-convergence theory cannot be applied).

More precisely, the cornerstone to describe the limit interface lies in the assumption that $u$ satisfies the following energy bound: for any $R > 1$,

$$\mathscr{E}(u, B_R) \leqslant CR^{n-1}, \tag{7}$$

for some $C > 0$ independent of $R$. Such energy bound is a classical assumption in the setting of semilinear elliptic equations (see e.g. [**?**]) and it is satisfied by minimizers and monotone solutions with some conditions at infinity (see e.g. Theorem 5.2 in [**?**]). As a matter of fact, it is valid for quasiminimal solutions too (see Lemma 10 in [**?**]) and it is also implied by the weaker condition on the potential energy

$$\int_{B_R} W(u(x))\, dx \leqslant CR^{n-1},$$

see [**?**]. By scaling, the energy bound in (7) implies that the rescaled energy with density

$$\frac{\varepsilon|\nabla u_\varepsilon(x)|^2}{2} + \frac{W(u_\varepsilon(x))}{\varepsilon}\, dx$$

is locally bounded uniformly in $\varepsilon > 0$. Under this condition, and recalling (3), the results of [**?**] come into play. First of all, we fix a domain (for convenience a cylinder) and we look at the asymptotics of the rescaled level sets in such domain. That is, given $d > 0$ and $h > 0$ we denote by $\mathrm{C}(d, h)$ the (open) cylinder of base radius $d$ and height $2h$, i.e. we set

$$\mathrm{C}(d, h) := \{x \in \mathbb{R}^n \text{ s.t. } |x'| < d \text{ and } |x_n| < h\}. \tag{8}$$

Then we fix (once and for all in this paper) the quantities $d_o$, $h_o > 0$ and (see e.g. page 52 in [**?**]) we have that, up to a subsequence, $u_\varepsilon$ converges a.e. in $\mathrm{C}(d_o, h_o)$ to $\pm 1$, i.e. we can define $L_-$ as the set of points $p \in \mathrm{C}(d_o, h_o)$ such that $u_\varepsilon(p) \to -1$, and then $u_\varepsilon$ approaches a.e. the step function $\chi_{\mathscr{C}L_-} - \chi_{L_-}$.

We observe that $\chi_{\mathscr{C}L_-} - \chi_{L_-}$ is a measurable function, since it is obtained by limit of measurable (and, in fact, continuous) functions, and therefore $L_-$ is a measurable set. Nevertheless, it is worth to point out that we do not identify $L_-$ with the set of its points of Lebesgue density 1, since we have defined it directly via the pointwise convergence of the rescaled solution $u_\varepsilon$.

It is also interesting to notice that

$$0 \in \mathscr{C}L_-, \tag{9}$$

thanks to (4).

In a symmetric way, we also define $L_+$ as the set of points $p \in \mathrm{C}(d_o, h_o)$ such that $u_\varepsilon(p) \to 1$.

Having clearly stated the notion of the limit level set, now we introduce a more general object which encodes further asymptotic and geometric properties of the solution. Namely (see Theorem 1 and Proposition 4.2. of [**?**]) we have that there exists a varifold $V$ (which we will call "limit varifold" and whose geometric support will be denoted by $V$ as well) such that, up to subsequences:

- $u_\varepsilon \to \pm 1$ uniformly on each connected compact subset of $\mathrm{C}(d_o, h_o) \setminus V$;
- for any $\tilde{U} \Subset \mathrm{C}(d_o, h_o)$ and for any $c \in (-1, 1)$ the set $\{|u| \leqslant c\} \cap \tilde{U}$ converges uniformly to $V \cap \tilde{U}$, i.e., if we set

$$V_\delta := \bigcup_{p \in V} B_\delta(p), \tag{10}$$

  then for any $\delta > 0$ there exists $\varepsilon_o > 0$ such that for any $\varepsilon \in (0, \varepsilon_o)$ we have that

$$\{|u_\varepsilon| \leqslant c\} \cap \tilde{U} \subseteq V_\delta \cap \tilde{U}. \tag{11}$$

We stress that the limit procedure above is only taken *up to a subsequence*. In particular, in this paper we are *not* assuming any uniqueness of the limit varifold. Any assumption that will be taken on the limit varifold in what follows is therefore intended as "there exists a subsequence for which the corresponding limit varifold satisfy such assumption", but different subsequences may produce different limit varifolds without affecting our arguments.

We also remark that the limit set $L_-$ and the limit varifold $V$ that we have discussed here have a simple geometric explanation in terms of the physical interpretation of the Allen-Cahn equation: namely, they represent the limit interface that separate two coexisting phase states.

We will present a series of symmetry results for monotone transitions under the assumption that the transition set is not "too vertical". More precisely, in the following subsections, we will present: some symmetry results arising from the geometric control of the limit level set, other symmetry results which are a consequence of a suitable control in measure of the limit level set, a result which depends on the confinement of a level set below a graph, others which come from a geometric control of the limit varifold, and a characterization of the symmetry in terms of non-vertical hyperplanes as limit sets.

1.1. **Symmetry from the geometric control of the limit level set.** Using the setting mentioned above, the first result that we present deals with the rigidity properties of the solutions which are inherited from the structure of the limit level set $L_-$:

**Theorem 1.2.** *Let $u$ be a solution of* (1) *satisfying* (2) *and* (7). *Suppose that there exist $d \in (0, d_o]$, and $\overline{K}$, $K \in (0, h_o]$, with $\overline{K} > K$, such that*

$$L_- \cap \mathrm{C}(d, \overline{K}) \neq \varnothing, \qquad L_+ \cap \mathrm{C}(d, \overline{K}) \neq \varnothing, \tag{12}$$

*and*

$$\Big( (\partial L_-) \cup (\partial L_+) \Big) \cap \mathrm{C}(d, \overline{K}) \subseteq \{|x_n| < K\}. \tag{13}$$

*Then:*

(i) *The limits in* (5) *hold true,*
(ii) *The solution $u$ is a local minimizer,*
(iii) *If $2 \leqslant n \leqslant 8$, then $u$ is* 1*D.*

We observe that condition (12) is quite natural, since it is consistent with the physical interpretation of the model describing the coexistence of two phases separated by an interface. Moreover, it is always satisfied if the solution $u$ is minimal or, more generally, quasiminimal: indeed, in this case, it follows from (4) and Corollary 13 in [**?**] that $\min\{|B_\delta \cap L_-|, |B_\delta \cap L_+|\} \geqslant c\delta^n$, for a suitable $c > 0$.

We also point out that condition (12) is also directly implied by (2) and suitable geometric constraints on the zero level sets of the solution (for instance, by the condition that $\{u = 0\} \subseteq \{|x_n| \leqslant \kappa\}$ for some $\kappa \geqslant 0$).

The geometric restriction on the limit level set $L_-$ in (13) is depicted in Figure 1 (the picture also remarks that, in principle, condition (13) allows $\partial L_-$ to have vertical parts outside the origin, though, of course, some further restrictions on $L_-$ are imposed by (2)).

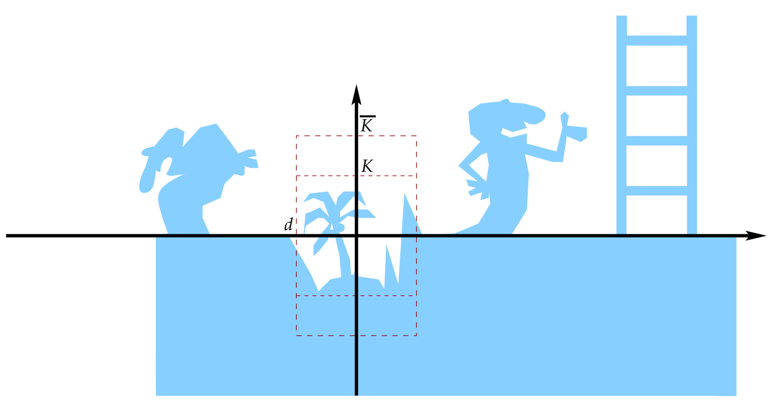

*Figure 1. The trapping condition on the set $L_-$ (which is the colored region) given by* (13).

As a matter of fact, we will derive Theorem 1.2 from the following more general result, in which the asymptotic profiles of the solutions are determined only by a one-side constraint on the limit level set:

**Theorem 1.3.** *Let $u$ be a solution of* (1) *satisfying* (2) *and* (7). *Suppose that there exist $d \in (0, d_o]$, and $\overline{K}$, $K \in (0, h_o]$, with $\overline{K} > K$, such that*

$$L_- \cap \mathrm{C}(d, \overline{K}) \neq \varnothing \tag{14}$$

*and*

$$(\partial L_-) \cap \mathrm{C}(d, \overline{K}) \subseteq \{x_n > -K\}. \tag{15}$$

*Then*

$$\lim_{x_n \to -\infty} u(x', x_n) = -1. \tag{16}$$

*Moreover, if $2 \leqslant n \leqslant 4$, then $u$ is 1D.*

*Similarly, if*

$$L_+ \cap \mathrm{C}(d, \overline{K}) \neq \varnothing \tag{17}$$

*and*

$$(\partial L_+) \cap \mathrm{C}(d, \overline{K}) \subseteq \{x_n < K\}, \tag{18}$$

*then*

$$\lim_{x_n \to +\infty} u(x', x_n) = 1.$$

*Moreover, if $2 \leqslant n \leqslant 4$, then $u$ is 1D.*

It is worth to point out that condition (14) is quite natural: indeed, we already know from (9) that $\mathscr{C} L_- \cap \mathrm{C}(d, \overline{K}) \neq \varnothing$, thus condition (14) may be seen as the symmetric counterpart of it. Also, by (9), we have that (14) is equivalent to

$$(\partial L_-) \cap \mathrm{C}(d, \overline{K}) \neq \varnothing.$$

1.2. **Symmetry from the control in measure of the limit level set.** We stress that Theorem 1.3 (and so Theorem 1.2) possesses a measure theoretic version, in which one is allowed to modify $L_-$ or $L_+$ by sets of measure zero. We give a detailed statement for completeness:

**Theorem 1.4.** *Let $u$ be a solution of* (1) *satisfying* (2) *and* (7). *Suppose that there exist $d \in (0, d_o]$, and $\overline{K}$, $K \in (0, h_o]$, with $\overline{K} > K$, and a set $\Lambda_- \subseteq L_-$ such that*

$$|L_- \setminus \Lambda_-| = 0, \tag{19}$$

$$|\Lambda_- \cap \mathrm{C}(d, \overline{K})| > 0 \tag{20}$$

*and*

$$(\partial \Lambda_-) \cap \mathrm{C}(d, \overline{K}) \subseteq \{x_n > -K\}. \tag{21}$$

*Then*

$$\lim_{x_n \to -\infty} u(x', x_n) = -1. \tag{22}$$

*Moreover, if* $2 \leqslant n \leqslant 4$*, then* $u$ *is* 1*D.*

*Similarly, if there exists a set* $\Lambda_+ \subseteq L_+$ *such that*

$$|L_+ \setminus \Lambda_+| = 0, \tag{23}$$

$$|\Lambda_+ \cap \mathrm{C}(d, \overline{K})| > 0 \tag{24}$$

*and*

$$(\partial \Lambda_+) \cap \mathrm{C}(d, \overline{K}) \subseteq \{x_n < K\}, \tag{25}$$

*then*

$$\lim_{x_n \to +\infty} u(x', x_n) = 1.$$

*Moreover, if* $2 \leqslant n \leqslant 4$*, then* $u$ *is* 1*D.*

*In particular, if there are sets* $\Lambda_- \subseteq L_-$ *and* $\Lambda_+ \subseteq L_+$ *satisfying* (19)*,* (20)*,* (21)*,* (23)*,* (24) *and* (25)*, then*

(i) *The limits in* (5) *hold true,*
(ii) *The solution* $u$ *is a local minimizer,*
(iii) *If* $2 \leqslant n \leqslant 8$*, then* $u$ *is* 1*D.*

The reader may compare (20) with (14) and (21) with (15).

Next we use the approach that we developed in [**?**], in which symmetry results and qualitative properties of the solutions to the considered problem are obtained when at least one level set of the solution is a complete graph. More precisely, we have the following results:

**Theorem 1.5.** *Let* $u$ *be a solution of* (1) *satisfying* (2) *and* (7)*. Suppose that there exist* $d \in (0, d_o]$ *and* $\overline{K} \in (0, h_o]$ *such that*

$$L_- \cap \mathrm{C}(d, \overline{K}) \neq \varnothing \text{ and } L_+ \cap \mathrm{C}(d, \overline{K}) \neq \varnothing. \tag{26}$$

*Suppose also that there exist* $K \in (0, \overline{K})$*, a set* $\Lambda \subseteq \mathrm{C}(d_0, h_0)$ *and a value* $c \in (-1, 1)$ *such that*

$$\text{the level set } \{u_\varepsilon = c\} \cap \mathrm{C}(d, \overline{K}) \text{ converges uniformly to } \partial\Lambda \tag{27}$$

*and*

$$(\partial \Lambda) \cap \mathrm{C}(d, \overline{K}) \subseteq \{|x_n| < K\}. \tag{28}$$

*Then:*

(i) *The level set* $\{u = c\}$ *is a complete graph in the vertical direction (i.e., for any* $x' \in \mathbb{R}^{n-1}$ *there exists a unique* $x_n(x')$ *such that* $u(x', x_n(x')) = c$*),*
(ii) *The limits in* (5) *hold true,*
(iii) *The solution* $u$ *is a local minimizer,*
(iv) *If* $2 \leqslant n \leqslant 8$*, then* $u$ *is* 1*D.*

We observe that condition (27) is somehow natural. For instance, if $u$ is a quasi-minimal solution, condition (27) is satisfied by choosing a set $\Lambda$ which differs from $L_-$ by a set of measure zero (cf. Corollary 2 in [**?**]). The same remark applies to the next Theorem 1.6

A variant of Theorem 1.5 that replaces assumption (26) with some measure theoretic conditions on the set $\Lambda$ goes as follows:

**Theorem 1.6.** *Let $u$ be a solution of (1) satisfying (2) and (7). Suppose that there exist $d \in (0, d_o]$, $\overline{K}$, $K \in (0, h_o]$, with $\overline{K} > K$, and a set $\Lambda \subseteq \mathrm{C}(d_0, h_0)$ such that*

$$|\Lambda \setminus L_-| = |L_- \setminus \Lambda| = 0, \tag{29}$$

$$|\Lambda \cap \mathrm{C}(d, \overline{K})| > 0, \qquad |(\mathscr{C}\Lambda) \cap \mathrm{C}(d, \overline{K})| > 0, \tag{30}$$

*for some $c \in (-1, 1)$*

*the level set $\{u_\varepsilon = c\} \cap \mathrm{C}(d, \overline{K})$ converges uniformly to $\partial\Lambda$,*

*and*

$$(\partial\Lambda) \cap \mathrm{C}(d, \overline{K}) \subseteq \{|x_n| < K\}. \tag{31}$$

*Then:*

(i) *The level set $\{u = c\}$ is a complete graph in the vertical direction,*
(ii) *The limits in (5) hold true,*
(iii) *The solution $u$ is a local minimizer,*
(iv) *If $2 \leqslant n \leqslant 8$, then $u$ is 1D.*

We notice that the difference between Theorems 1.5 and 1.6 is that assumption (26), which involves the sets $L_-$ and $L_+$, is replaced by assumptions (29) and (30), which are similar but only involve the set $\Lambda$. Also, in Theorem 1.5, one does not need to assume a-priori that the set $\Lambda$ coincides with $L_-$ up to sets of measure zero.

We also remark that assumption (29) may be replaced by the similar one that involves $L_+$ instead of $L_-$, namely

$$|\Lambda \setminus L_+| = |L_+ \setminus \Lambda| = 0.$$

1.3. **Symmetry from the confinement below a graph.** Now we present further rigidity and symmetry results that follow, at least in dimension $4$, under the assumption that the level set $\{u = 0\}$ is confined below (or above) a complete graph. We emphasize that no energy assumption is needed in this case. Indeed (6), in this case, is a byproduct of the other hypotheses. Namely, we have the following result:

**Theorem 1.7.** *Let $u$ be a solution of (1) satisfying (2).*

*Suppose that $\{u = 0\}$ is confined below a complete graph, i.e. suppose that there exists $\gamma \in C(\mathbb{R}^{n-1})$ such that*

$$\{u = 0\} \subseteq \{x_n \leqslant \gamma(x')\}. \tag{32}$$

*Then*

$$\lim_{x_n \to +\infty} u(x', x_n) = 1. \tag{33}$$

*Similarly, if* $\{u = 0\}$ *is confined above a complete graph, then*

$$\lim_{x_n \to -\infty} u(x', x_n) = -1.$$

*In any case, the energy bound in* (7) *holds true.*

*Moreover, if* $2 \leqslant n \leqslant 4$*, then* $u$ *is* 1*D.*

Symmetry results (as the one presented in Theorem 1.7) which rely on a trapping condition of the level set (in this case, below an entire graph) can be seen as the counterpart in low dimension of the example constructed in [**?**] for high dimension: namely, in high dimension the level sets may shadow a minimal surface with a graph property, and neverteless the solution is not necessarily 1D.

1.4. **Symmetry from the geometric control of the limit varifold.** While in Theorem 1.2, 1.3, 1.4 and 1.7 we have deduced symmetry results from the structure of the limit level set $L_-$, now we will prove further rigidity results under some geometric control on the limit varifold $V$:

**Theorem 1.8.** *Let* $u$ *be a solution of* (1) *satisfying* (2) *and* (7) *and let* $V$ *be the associated limit varifold.*

*Suppose that there exist two points* $\overline{x} = (0, \dots, 0, \overline{x}_n)$ *and* $\underline{x} = (0, \dots, 0, \underline{x}_n)$ *that do not belong to* $V$*, with*

$$h_o > \overline{x}_n > 0 > \underline{x}_n > -h_o.$$

*Then:*

(i) *The limits in* (5) *hold true,*
(ii) *The solution* $u$ *is a local minimizer,*
(iii) *If* $2 \leqslant n \leqslant 8$*, then* $u$ *is* 1*D.*

Concerning the assumptions of Theorem 1.2 and 1.8, it is worth to remark that, in general, condition (2) is not enough to imply (13), nor the existence of the points $\overline{x}$ and $\underline{x}$ in Theorem 1.8. The reason is, roughly speaking, that the limit interface could be "vertical". For instance, let

$$u(x_1, x_2) := -\frac{2}{\pi} \operatorname{arctg}(x_1 - \operatorname{arctg} x_2).$$

Since the derivative of the function $t \mapsto \operatorname{arctg} t$ is strictly positive, one readily checks that $\partial_{x_2} u > 0$, hence condition (2) is satisfied. Nevertheless, we have that

$$u_\varepsilon(x_1, x_2) = -\frac{2}{\pi} \operatorname{arctg} \left( \frac{x_1}{\varepsilon} - \operatorname{arctg} \frac{x_2}{\varepsilon} \right),$$

therefore, for any $c \in (-1, 1)$,

$$\{u_\varepsilon = c\} = \left\{ x_1 = \varepsilon \left[ \operatorname{arctg} \frac{x_2}{\varepsilon} - \operatorname{tg} \frac{c\,\pi}{2} \right] \right\}.$$

Accordingly, $\{u_\varepsilon = c\}$ approaches $\{x_2 = 0\}$, which shows that the geometric condition in Theorem 1.8 is not satisfied in this case.

In dimension $n \leqslant 4$, Theorem 1.8 can be strengthened: namely, it is enough that the limit varifold does not contain the vertical line in order to deduce symmetry properties, as stated in next result:

**Theorem 1.9.** *Let $2 \leqslant n \leqslant 4$ and $u$ be a solution of* (1) *satisfying* (2) *and* (7) *and let $V$ be the associated limit varifold.*

*Suppose that there is a point of the vertical line $r_\star := \{(0,\dots,0,t),\ t \in \mathbb{R}\}$ which is not contained in $V$.*

*Then $u$ is 1D.*

We remark that the results obtained are valid also for more general bistable nonlinearities than the classical Allen-Cahn equation (as a matter of fact, only in Theorem 1.7 one needs the nonlinearity to grow linearly at the origin, in order to exploit the results of [**?**, **?**]).

Also, it is worth to point out that Theorems 1.2, 1.3, 1.4 and 1.7 do not use the limit varifold $V$, but only the limit level sets $L_-$ and $L_+$.

We also observe that the assumptions used in this paper are geometric and related to the asymptotic behavior of the interface of the problem: we think that these kind of hypotheses are somehow natural from the viewpoint of the physical applications and they may be easier to check in concrete cases than the usual variational assumptions (such as the one requiring minimality of the solution).

Furthermore, the limit interface possesses somehow a geometric and physical evidence from the experimental viewpoint, therefore the conditions on the limit interface are perhaps more feasible to be checked in concrete applications, when one "sees" in practice the interface, and then can deduce from the symmetry results of the theory that the phase state depends, at a large scale, only on the distance from the phase separation.

1.5. **An equivalent formulation of the one-dimesional symmetry.** As a final observation, we point out that the symmetry of the solution is, in the end, somehow equivalent to the flatness of its interface, according to the following result:

**Corollary 1.10.** *Let $u$ be a solution of* (1) *satisfying* (2) *and* (7)*. Then $u$ is 1D if and only if $\partial L_- = \partial L_+$ is a non-vertical hyperplane containing the origin.*

The rest of the paper is organized as follows. In Section 2 we prove the rigidity results based on the geometric behavior of the limit set $L_-$ (namely, one after the other, Theorems 1.3, 1.2, 1.4, 1.7, 1.5 and 1.6, as well as Corollary 1.10, which follows from Theorem 1.2). Then, in Section 3 we deal with the rigidity results coming from the geometric behavior of the limit varifold $V$ (namely Theorems 1.8 and 1.9). More precisely, in Subsection 3.1 we describe the influence of the limit varifold on the asymptotic behavior of the solution, and Theorems 1.8 and 1.9 will be proved in Subsection 3.2.

## 2. Rigidity and symmetry from the limit level set

Here we prove the rigidity results that rely on the geometric structure of the limit sets $L_-$ and $L_+$ (namely, Theorems 1.3, 1.2, 1.4 and 1.7).

*Proof of Theorem 1.3.* We prove only the first part of Theorem 1.3 since the second part follows from the first one by replacing $u = u(x)$ with $-u(x', -x_n)$.

We start by showing that, if (14) and (15) holds true, then the limit in (16) holds true as well. For this scope, we let

$$\begin{aligned} Q_+ &:= \{|x'| < d\} \times \{x_n \in [K, \overline{K})\} \\ \text{and } Q_- &:= \{|x'| < d\} \times \{x_n \in (-\overline{K}, -K]\}. \end{aligned} \tag{34}$$

Notice that

$$\mathrm{C}(d, \overline{K}) = Q_- \cup Q_+ \cup \mathrm{C}(d, K). \tag{35}$$

We claim that

$$\text{either } \; Q_- \cap L_- = \varnothing \; \text{ or } \; Q_- \setminus L_- = \varnothing\,. \tag{36}$$

To prove it, we notice that if (36) were false, the set $Q_-$ would have to contain a point of the boundary of $L_-$. This gives a contradiction with (15) and so we have proved (36).

Now we improve (36) by showing that

$$Q_- \subseteq L_-. \tag{37}$$

Suppose not. Then (36) implies that $Q_- \cap L_- = \varnothing$, hence for any $x \in Q_-$ we have that $u_\varepsilon(x) \not\to -1$ as $\varepsilon \to 0^+$. Then, fix any point $y \in \mathrm{C}(d, \overline{K})$ and recall (35) to find a point $\tilde{y} \in Q_-$ such that $\tilde{y}' = y'$ and $\tilde{y}_n \leqslant y_n$. From (2), we know that $\partial_{x_n} u_\varepsilon > 0$. By collecting these pieces of information, we see that

$$u_\varepsilon(y) \geqslant u_\varepsilon(\tilde{y}) \not\to -1$$

as $\varepsilon \to 0^+$. In particular we have that $u_\varepsilon(y) \not\to -1$, that is $y \notin L_-$. Since this is valid for any $y \in \mathrm{C}(d, \overline{K})$, we have shown that $\mathrm{C}(d, \overline{K}) \cap L_- = \varnothing$. This is in contradiction with (14) and so the proof of (37) is complete.

Now, we set $\vartheta := (K + \overline{K})/2$ and $P := (0, \ldots, 0, -\vartheta)$. We have that $P \in Q_-$, thus, by (37), we conclude that $P \in L_-$ and so

$$-1 = \lim_{\varepsilon \to 0^+} u_\varepsilon(P) = \lim_{\varepsilon \to 0^+} u\left(0, \ldots, 0, -\frac{\vartheta}{\varepsilon}\right) = \underline{u}(0).$$

This and the Maximum Principle implies that $\underline{u}$ is constantly equal to $-1$, which establishes (16).

Then, if additionally $2 \leqslant n \leqslant 4$, using (16) and Theorem 1.2 of [**?**], we obtain that $u$ is 1D. □

*Proof of Theorem 1.2.* We remark that (12) implies both (14) and (17). Moreover, condition (13) implies both (15) and (18). Accordingly, we can use Theorem 1.3 and obtain claim (i) of Theorem 1.2. Then, claims (ii) and (iii) follow from Theorem 1.3 of [**?**]. □

*Proof of Corollary 1.10.* Suppose that $u$ is 1D. Then there exist $\omega \in \mathrm{S}^{n-1}$ and a function of one variable $u_o$ such that $u(x) = u_o(\omega \cdot x)$ for every $x \in \mathbb{R}^n$. By solving the ODE satisfied by $u_o$ we obtain that

$$\lim_{t \to \pm\infty} u_o(t) = \pm 1.$$

We claim that

$$L_- = \{\omega \cdot x < 0\} \text{ and } L_+ = \{\omega \cdot x > 0\}. \tag{38}$$

Indeed,

$$\lim_{\varepsilon \to 0^+} u_\varepsilon(x) = \lim_{\varepsilon \to 0^+} u\left(\frac{x}{\varepsilon}\right) = \lim_{\varepsilon \to 0^+} u_o\left(\frac{\omega \cdot x}{\varepsilon}\right) = \begin{cases} 1 & \text{if } \omega \cdot x > 0, \\ u_o(0) & \text{if } \omega \cdot x = 0, \\ -1 & \text{if } \omega \cdot x < 0. \end{cases}$$

This and (4) prove (38). From (38), it follows that $\partial L_- = \partial L_+ = \{\omega \cdot x = 0\}$, which is a non-vertical hyperplane containing the origin.

Viceversa, let us now suppose that $\partial L_- = \partial L_+$ is a non-vertical hyperplane containing the origin. Then conditions (12) and (13) are satisfied and we infer from Theorem 1.2 that $u$ is a minimizer. In particular, we can use Corollary 7 in [**?**] and obtain that $u$ is 1D, without any restriction on the dimension of the ambient space. □

*Proof of Theorem 1.4.* The proof is a modification of the one Theorem 1.3, according to these lines. First, one replaces $L_-$ with $\Lambda_-$ in (36), i.e. instead of (36) one proves that

$$\text{either } Q_- \cap \Lambda_- = \varnothing \text{ or } Q_- \setminus \Lambda_- = \varnothing\ . \tag{39}$$

This follows easily from (21). Then, one replaces $L_-$ with $\Lambda_-$ in (37), i.e. one shows that

$$Q_- \subseteq \Lambda_-. \tag{40}$$

The proof is similar to the one of (37), but it makes use of (19) and (20). Namely, if (40) were false, we would deduce from (39) that $Q_- \cap \Lambda_- = \varnothing$. In particular, from (19), we obtain that for almost every $x \in Q_-$ we have that $u_\varepsilon(x) \not\to -1$. Thus, (2) implies $u_\varepsilon(y) \not\to -1$, for almost every $y \in \mathrm{C}(d, \overline{K})$. Hence $|\Lambda_- \cap \mathrm{C}(d, \overline{K})| = 0$, which is in contradiction with (20).

Having established (40), we use it to see that $P := (0, \dots, 0, -\vartheta) \in \Lambda_- \subseteq L_-$, where $\vartheta := (K + \overline{K})/2$, and so

$$-1 = \lim_{\varepsilon \to 0^+} u_\varepsilon(P) = \underline{u}(0),$$

which, by Maximum Principle implies that $\underline{u}$ is constantly equal to $-1$, which establishes (22). This and Theorem 1.2 of [**?**], we also obtain that $u$ is 1D if $2 \leqslant n \leqslant 4$.

This proves the first statement in Theorem 1.4. The second follows from the first, applied to the function $-u(x', -x_n)$. The third statement is then a combination of the first two ones: more precisely, claim (i) in the last statement of Theorem 1.4 follows from the previous two statements, and then this implies claims (ii) and (iii), by Theorem 1.3 of [**?**]. □

*Proof of Theorem 1.7.* Let us suppose that $\{u = 0\}$ is confined below a complete graph (the case in which it is confined above being analogous). We notice that

$$\{u = 0\} \neq \varnothing. \tag{41}$$

To prove it, we argue by contradiction and we suppose that $u > 0$ in the whole of $\mathbb{R}^n$ (the case in which $u < 0$ is analogous). Then, by Theorem 2.1 in [**?**], we have that $u$ is identically equal to 1. This is in contradiction with (2) and so (41) is proved.

From (41) and (2) we obtain that

$$\{u > 0\} \cap \{x_n > \gamma(x')\} \neq \varnothing.$$

Using this, (32) and the fact that the set $\{x_n > \gamma(x')\}$ is open and path-connected, we conclude that

$$\text{if} \quad x_n > \gamma(x') \quad \text{then} \quad u(x', x_n) > 0.$$

From this, we deduce that $\overline{u} > 0$. So we can apply once more Theorem 2.1 in [**?**] (this time to the solution $\overline{u}$ in $\mathbb{R}^{n-1}$) and deduce that $\overline{u}$ is identical to 1. This establishes (33).

Now assume in addition that $2 \leqslant n \leqslant 4$. Then (33) and Theorem 1.2 of [**?**] give that $u$ is 1D. □

### 2.1. **Geometric analysis of level sets.**

Here we collect some auxiliary geometric results of somehow elementary nature that will be useful for the proof of Theorems 1.5 and 1.6. For this, we use the following additional notation. Given $x' \in \mathbb{R}^{n-1}$, we denote by $\mathrm{r}(x')$ the vertical straight line through $x'$, i.e.

$$\mathrm{r}(x') := \{(x', t),\ t \in \mathbb{R}\}.$$

Also, for any $p \in \mathbb{R}^{n-1}$ and $r > 0$, we denote by $B_r^{n-1}(p)$ the $(n-1)$-dimensional ball of radius $r$ centered at $p$ and $B_r^{n-1} := B_r^{n-1}(0)$. Also, when no confusion arises, we implicitly identify $\mathbb{R}^{n-1} \times \{0\}$ and $\mathbb{R}^{n-1}$ (i.e. points of the form $(x', 0) \in \mathbb{R}^{n-1} \times \{0\}$ and points of the form $x' \in \mathbb{R}^{n-1}$).

**Lemma 2.1.** *Let $d > 0$, $\Psi \subset \mathbb{R}^n$ and $Z$ be the projection of $\Psi$ onto $\mathbb{R}^{n-1}$, that is*

$$Z := \{x' \in \overline{B_d^{n-1}} \text{ s.t. } \mathrm{r}(x') \cap \Psi \neq \varnothing\}. \tag{42}$$

*Suppose that*

$$\Psi \cap \{x \in \mathbb{R}^n \text{ s.t. } |x'| \leqslant d\} \text{ is compact.} \tag{43}$$

*Then $Z$ is closed.*

*Proof.* The geometric situation of Lemma 2.1 is described in Figure 2.

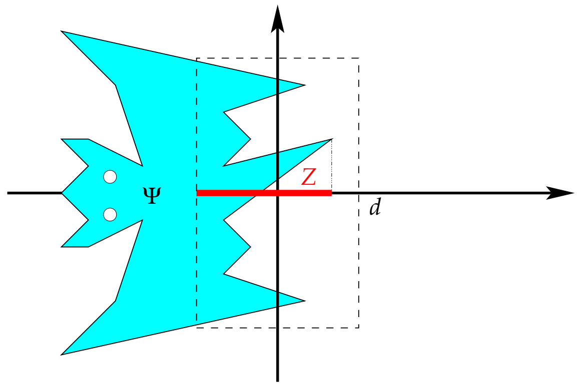


*Figure 2. The sets of Lemma 2.1.*

The proof of Lemma 2.1 goes as follows. Let $x'_k \in Z$ be a sequence approaching some $x'_\star \in \mathbb{R}^{n-1}$. Then, there exists $t_k$ such that $(x'_k, t_k) \in \mathrm{r}(x'_k) \cap \Psi$. By (43), we can take a converging subsequence, that is we can write

$$\lim_{j\to+\infty} (x'_{k_j}, t_{k_j}) = (x'_\star, t_\star) \in \Psi \cap \{x \in \mathbb{R}^n \text{ s.t. } |x'| \leqslant d\}..$$

That is $(x'_\star, t_\star) \in \mathrm{r}(x'_\star) \cap \Psi$ and so $\mathrm{r}(x'_\star) \cap \Psi \neq \varnothing$. This implies that $x'_\star \in Z$, and so $Z$ is closed. □

**Corollary 2.2.** *Let* $c \in \mathbb{R}$, $d > 0$, $\overline{K} > K > 0$, $\eta := (\overline{K} - K)/4$, $v \in C^1(\mathbb{R}^n)$,

$$\Gamma \subseteq \mathrm{C}(d, K) \tag{44}$$

*and*

$$Z_\star := \Big\{ x' \in \overline{B_d^{n-1}} \text{ s.t. } \mathrm{r}(x') \cap \{v = c\} \neq \varnothing \Big\}. \tag{45}$$

*Suppose that*

$$\{v = c\} \cap \mathrm{C}(d, \overline{K}) \neq \varnothing, \tag{46}$$

*that*

$$\{v = c\} \cap \mathrm{C}(d, \overline{K}) \subseteq \Gamma_\eta := \bigcup_{p\in\Gamma} B_\eta(p) \tag{47}$$

*and that*

$$\partial_n v(x) > 0 \text{ for any } x \in \mathbb{R}^n. \tag{48}$$

*Then,* $Z_\star = \overline{B_d^{n-1}}$.

*Proof.* The geometric situation of Corollary 2.2 is described in Figure 3.

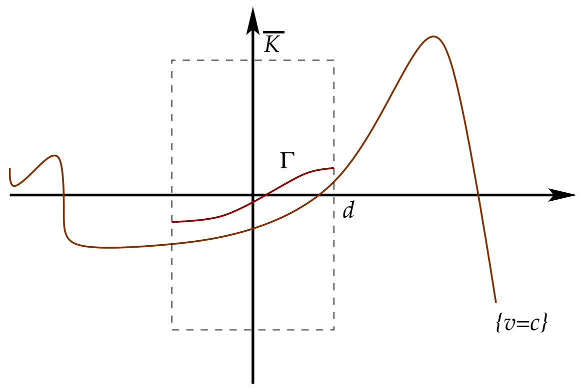


*Figure 3. The sets in Corollary 2.2.*

Its proof goes as follows. We set $\Psi := \{v = c\} \cap \{|x_n| \leqslant \overline{K}\}$ and we observe that $\Psi$ is closed since $v$ is continuous. We take $Z := \{x' \in \overline{B_d^{n-1}}$ s.t. $\mathrm{r}(x') \cap \Psi \neq \varnothing\}$. Notice that this definition is coherent with (42) and that

$$Z \neq \varnothing, \tag{49}$$

thanks to (46).

Accordingly, by Lemma 2.1, we have that

$$Z \text{ is closed.} \tag{50}$$

On the other hand, by (44) and (47),

$$\begin{aligned} \Psi \cap \{|x'| \leqslant d\} &= \{v = c\} \cap \mathrm{C}(d, \overline{K}) \\ &\subseteq \Gamma_\eta \cap \mathrm{C}(d, \overline{K}) \subseteq \mathrm{C}(d, K + \eta). \end{aligned} \tag{51}$$

Now let $p' \in Z$. Then, there exists $t \in \mathbb{R}$ such that $(p', t) \in \Psi$, hence $v(p', t) = c$ and, by (51), we have that $|t| \leqslant K + \eta$. Thus, by (48) and the Implicit Function Theorem, there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that for any $q' \in B_{\delta_1}^{n-1}(p') \cap \overline{B_d^{n-1}}$ there exists $t(q') \in (t - \delta_2, t + \delta_2)$ for which $v(q', t(q')) = c$. By possibly taking $\delta_1$ smaller, we may and do suppose that $\delta_2 < \eta$, therefore $|t(q')| \leqslant |t| + \delta_2 < K + 2\eta \leqslant \overline{K}$, which gives that $q' \in Z$. This says that

$$Z \text{ is also open in } \overline{B_d^{n-1}}. \tag{52}$$

Accordingly, by (50), (52) and (49), we have that $Z = \overline{B_d^{n-1}}$. As a consequence, using (42) and (45), we obtain

$$\begin{aligned} \overline{B_d^{n-1}} &= Z = \{x' \in \overline{B_d^{n-1}} \text{ s.t. } \mathrm{r}(x') \cap \Psi \neq \varnothing\} \\ &\subseteq \{x' \in \overline{B_d^{n-1}} \text{ s.t. } \mathrm{r}(x') \cap \{v = c\} \neq \varnothing\} = Z_\star \subseteq \overline{B_d^{n-1}}, \end{aligned}$$

hence we have proved the desired result. □

With this, we can now prove Theorems 1.5 and 1.6:

*Proof of Theorem 1.5.* We will show that, given $c \in (-1,1)$ as in the statement of Theorem 1.5,

$$\text{the level set } \{u = c\} \text{ is a complete graph, i.e. for any fixed } x'_o \in \mathbb{R}^{n-1}, \text{ we have that } r(x'_o) \cap \{u = c\} \neq \varnothing. \tag{53}$$

For this, we take $c_-, c_+ \in \mathbb{R}$ such that

$$-1 < c_- < c < c_+ < 1.$$

From (26) we know that there exist $p \in L_- \cap \mathrm{C}(d, \overline{K})$ and $q \in L_+ \cap \mathrm{C}(d, \overline{K})$ such that $u_\varepsilon(p)$ approaches 1 and $u_\varepsilon(q)$ approaches $-1$ as $\varepsilon \to 0^+$. In particular, if $\varepsilon$ is suitably small, $u_\varepsilon(p) > c_+$ and $u_\varepsilon(q) < c_-$. Since $\mathrm{C}(d, \overline{K})$ is convex and $u_\varepsilon$ continuous, this gives that there exists $x(\varepsilon) \in \mathrm{C}(d, \overline{K})$ such that $u_\varepsilon(x(\varepsilon)) = c$, that is

$$\{u_\varepsilon = c\} \cap \mathrm{C}(d, \overline{K}) \neq \varnothing.$$

We remark that this implies (46) with $v := u_\varepsilon$, while, with this setting, condition (48) comes from (2).

Furthermore, if we take $\Gamma := (\partial\Lambda) \cap \mathrm{C}(d, \overline{K})$, we have that (28) implies (44), and that (27) implies (47). Consequently, we can apply Corollary 2.2 with $v := u_\varepsilon$ and $\Gamma := (\partial\Lambda) \cap \mathrm{C}(d, \overline{K})$, and we conclude that, for small $\varepsilon$,

$$\varepsilon x'_o \in \overline{B^{n-1}_d} = \left\{ x' \in \overline{B^{n-1}_d} \text{ s.t. } \mathrm{r}(x') \cap \{u_\varepsilon = c\} \neq \varnothing \right\}.$$

That is, $\mathrm{r}(\varepsilon x'_o) \cap \{u_\varepsilon = c\} \neq \varnothing$. This, by the definition of $u_\varepsilon$, proves (53) and claim (i) of Theorem 1.5.

Then, claims (ii)-(iv) of Theorem 1.5 follow from claim (i) and Theorem 1.3 of [**?**]. □

*Proof of Theorem 1.6.* Both $\Lambda \cap \mathrm{C}(d, \overline{K})$ and $(\mathscr{C}\Lambda) \cap \mathrm{C}(d, \overline{K})$ have positive measure by (30), hence using (29) and the a.e. convergence of $u_\varepsilon$, we conclude that both $L_- \cap \mathrm{C}(d, \overline{K})$ and $L_+ \cap \mathrm{C}(d, \overline{K})$ have positive measure. In particular, they are non-empty and (26) is satisfied. With this, Theorem 1.6 is now a direct consequence of Theorem 1.5. □

## 3. Rigidity and symmetry from the limit varifold

In this section we investigate the structure of the limit varifold, with the aim of proving Theorems 1.8 and 1.9.

### 3.1. **The limit varifold.**

Here we relate the structure of the limit varifold with the asymptotic properties of the solutions.

**Lemma 3.1.** *Let $u$ be a solution of* (1) *satisfying* (2) *and* (7) *and let $V$ be the associated limit varifold. Assume that there exists $\overline{x} = (0, \dots, 0, \overline{x}_n)$ with $\overline{x}_n \in (0, h_o)$ that does not belong to $V$. Then*

$$\lim_{x_n \to +\infty} u(x', x_n) = 1. \tag{54}$$

*Similarly, if there exists* $\underline{x} = (0, \ldots, 0, \underline{x}_n)$ *with* $\underline{x}_n \in (-h_o, 0)$ *that does not belong to* $V$*, then*

$$\lim_{x_n \to -\infty} u(x', x_n) = -1.$$

*Proof.* We prove the first claim since the second one is alike. Since $V$ is closed in $\mathrm{C}(d_o, h_o)$, the distance from $\overline{x}$ to $V$ is strictly positive, therefore there exists $\delta > 0$ such that $B_\delta(\overline{x}) \subseteq \mathrm{C}(d_o, h_o)$ and

$$B_\delta(\overline{x}) \cap V_\delta = \varnothing, \tag{55}$$

where the notation in (10) has been used. Using the uniform convergence of $u_\varepsilon$ on each connected compact subset of $C(d_o, h_o) \setminus V$ we immediately infer that either

$$0 = \lim_{\varepsilon \to 0^+} |u_\varepsilon(\overline{x}) - 1| = \lim_{\varepsilon \to 0^+} \left| u\left(0, \ldots, 0, \frac{\overline{x}_n}{\varepsilon}\right) - 1 \right| = |\overline{u}(0) - 1|$$

or

$$0 = \lim_{\varepsilon \to 0^+} |u_\varepsilon(\overline{x}) + 1| = \lim_{\varepsilon \to 0^+} \left| u\left(0, \ldots, 0, \frac{\overline{x}_n}{\varepsilon}\right) + 1 \right| = |\overline{u}(0) + 1|,$$

where the notation in (6) and the assumption that $\overline{x}_n > 0$ have been used. Hence either $\overline{u}(0) = 1$ or $\overline{u}(0) = -1$. This and the Maximum Principle implies that $\overline{u}$ is constantly equal to either $1$ or $-1$.

On the other hand, $\overline{u}$ cannot be constantly equal to $-1$ (otherwise, by (2), also $u$ would be constantly equal to $-1$, thus contradicting (2) itself). This says that $\overline{u}$ is constantly equal to 1, which is (54). □

### 3.2. **Proof of the symmetry results from the behavior of the limit varifold.**

Now we are ready to complete the proof of Theorems 1.8 and 1.9.

*Proof of Theorem 1.8.* By Lemma 3.1, we have that

$$\lim_{x_n \to +\infty} u(x', x_n) = 1 \ \text{ and } \lim_{x_n \to -\infty} u(x', x_n) = -1,$$

which is (i). Then, claims (ii) and (iii) follow from [**?**]. □

*Proof of Theorem 1.9.* Let $\tilde{x} = (0, \ldots, \tilde{x}_n) \in r_\star \setminus V$. We observe that $\tilde{x}_n \neq 0$, since, by (4) and (11), we know that $0 \in V$.

Combining this with Lemma 3.1, we see that

$$\text{either } \lim_{x_n \to +\infty} u(x', x_n) = 1 \ \text{ or } \lim_{x_n \to -\infty} u(x', x_n) = -1.$$

This and Theorem 1.2 of [**?**] give that $u$ is 1D. □

LAMFA – CNRS UMR 6140, UNIVERSITÉ DE PICARDIE JULES VERNE, FACULTÉ DES SCIENCES, 33, RUE SAINT-LEU, 80039 AMIENS CEDEX 1, FRANCE
*E-mail address*: alberto.farina@u-picardie.fr

WEIERSTRASS INSTITUT FÜR ANGEWANDTE ANALYSIS UND STOCHASTIK, MOHRENSTRASSE, 39, 10117 BERLIN, GERMANY, AND, DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI MILANO, VIA SALDINI 50, 20133 MILAN, ITALY
*E-mail address*: enrico@mat.uniroma3.it