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Duality invariance in Fayet-Iliopoulos gauged supergravity

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Abstract: We propose a geometric method to study the residual symmetries in $N = 2$, $d = 4$ U(1) Fayet-Iliopoulos (FI) gauged supergravity. It essentially involves the stabilization of the symplectic vector of gauge couplings (FI parameters) under the action of the U-duality symmetry of the ungauged theory. In particular we are interested in those transformations that act non-trivially on the solutions and produce scalar hair and dyonic black holes from a given seed. We illustrate the procedure for finding this group in general and then show how it works in some specific models. For the prepotential $F = -iX^0X^1$, we use our method to add one more parameter to the rotating Chow-Compère solution, representing scalar hair.

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1 Introduction

Duality transformations have played, and continue to play, an important role in fundamental developments in string theory, supergravity, quantum field theory as well as in the physics of black holes. Perhaps the most relevant example for this is the fact that the five known string theories are actually all related by a web of dualities, and correspond just to perturbative expansions of a single underlying theory about a distinct point in the moduli space of quantum vacua, cf. e.g. [1] for a review. This web contains in particular weak/strong coupling dualities, of which the celebrated AdS/CFT correspondence [2] is another famous example.

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Duality transformations have been instrumental also in the construction of black hole solutions in string theory. Typically one reduces a higher-dimensional theory (in presence of Killing directions) to lower dimensions, in particular to $d = 3$, where all vector fields can be dualized to become scalars. One gets then three-dimensional gravity coupled to a nonlinear sigma model, and employs the global symmetries of the latter to obtain new black holes from a given seed. This technique was used by Cvetič and Youm [3] to construct the most general rotating five-dimensional black hole solution to toroidally compactified heterotic string theory, specified by 27 charges, two rotational parameters and the ADM mass. In a similar way, Chow and Compère [4] obtained the most general asymptotically flat, stationary, rotating, nonextremal, dyonic black hole of four-dimensional $N = 2$ supergravity coupled to 3 vector multiplets (the so-called stu model). It generates through U-dualities the most general asymptotically flat, stationary black hole of $N = 8$ supergravity.

Note that this typical structure of getting, after a Kaluza-Klein reduction, three-dimensional gravity coupled to a nonlinear sigma model, is also crucial to prove full integrability in some particular cases, cf. e.g. [5, 6].

When (part of the) global symmetries of some given supergravity theory are gauged, as it typically happens in AdS supergravity, the sigma model target space isometries are generically broken by the presence of a scalar potential, so that the powerful solution-generating techniques described above seem to break down. An instructive example is the timelike dimensional reduction of four-dimensional Einstein-Maxwell gravity down to three dimensions, which gives Euclidean gravity coupled to an $\mathrm{SU}(2,1)/\mathrm{S}(\mathrm{U}(1,1)\times\mathrm{U}(1))$ sigma model [7, 8]. Adding a cosmological constant to the Einstein-Maxwell theory leads to a scalar potential in three dimensions, that breaks three of the eight $\mathrm{SU}(2,1)$ generators, corresponding to the generalized Ehlers and the two Harrison transformations. This leaves merely a semidirect product of a one-dimensional Heisenberg group and a translation group $\mathbb{R}^2$ as residual symmetry [9]. Although in this concrete example the surviving symmetries cannot be used to generate new solutions from known ones, they may nevertheless be useful in more general settings.

The aim of this paper is thus to provide a systematical and thorough investigation of the residual symmetries in $N = 2$, $d = 4$ U(1) Fayet-Iliopoulos (FI) gauged supergravity, elaborating on [10], where a particular stu model was considered. To this end, we shall use a geometric method, whose underlying idea is the following: the on-shell global symmetry group of the ungauged theory is called U-duality, and consists of the isometries of the special Kähler non-linear sigma model that act linearly also on the field strengths via the symplectic embedding [8]. For purely electric gaugings, the scalar potential generically spoils this invariance, but allowing also for dyonic gaugings one can recover the whole U-duality invariance, at the price of changing the vector of gauge couplings and so the physical theory. We will call this group U_{fi}, that stands for fake internal symmetry group, which acts on a solution by mapping it to other solutions of other theories. Given U_{fi}, we fix a generic choice of the coupling constants $\mathcal{G}$. The true internal symmetry group U_{i} of the gauged supergravity theory is then $S_{\mathcal{G}}$, the stabilizer of $\mathcal{G}$ under the action of U_{fi}.[1]

[1] As we will see later, this is true up to possible U(1) factors.

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The remainder of this paper is organized as follows: in the next section, we briefly review the theory we are interested in, namely $N = 2$, $d = 4$ U(1) FI-gauged supergravity, and explain more in detail the general idea outlined above. In section 3 we explicitely determine the residual symmetry group for four different prepotentials that are frequently used, but we stress that our method is general, and can be applied to arbitrary prepotentials and extended to $N = 4$ and $N = 8$ gauged supergravity theories as well. After that, in section 4, it is shown how to apply the residual symmetries to generate new black hole solutions from a given seed in each of the four cases. In section 5 we comment on a possible extension of our work to include also gauged hypermultiplets. Section 6 contains our conclusions and some final remarks. Some supplementary material is deferred to two appendices.

2 General strategy

2.1 $N = 2$, $d = 4$ FI-gauged supergravity

The bosonic sector of $N = 2$, $d = 4$ supergravity coupled to n_V vector multiplets consists of the vierbein $e^a{}_\mu$, $n_V + 1$ vector fields A^Λ_μ with $\Lambda = 0, \dots n_V$ (the graviphoton plus n_V other fields from the vector multiplets), and n_V complex scalar fields z^i ($i = 1, \dots, n_V$). The latter parametrize an n_V-dimensional special Kähler manifold, i.e., a Kähler-Hodge manifold, with Kähler metric $g_{i\bar{\jmath}}(z, \bar{z})$, which is the base of a symplectic bundle with the covariantly holomorphic sections[2]

$$\mathcal{V} = \begin{pmatrix} L^\Lambda \\ M_\Lambda \end{pmatrix}, \qquad D_{\bar{\imath}}\mathcal{V} \equiv \partial_{\bar{\imath}}\mathcal{V} - \frac{1}{2}\left(\partial_{\bar{\imath}}\mathcal{K}\right)\mathcal{V} = 0\,, \tag{2.1}$$

where $\mathcal{K}$ is the Kähler potential. $\mathcal{V}$ obeys the constraint

$$\langle \mathcal{V} | \bar{\mathcal{V}} \rangle \equiv \bar{L}^\Lambda M_\Lambda - L^\Lambda \bar{M}_\Lambda = -i\,. \tag{2.2}$$

Alternatively one can introduce the explicitly holomorphic sections of a different symplectic bundle,

$$v \equiv e^{-\mathcal{K}/2}\mathcal{V} \equiv \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix}. \tag{2.3}$$

In appropriate symplectic frames it is possible to choose a homogeneous function $F(X)$ of second degree, called prepotential, such that $F_\Lambda = \partial_\Lambda F$. In terms of the sections v the constraint (2.2) becomes

$$\langle v | \bar{v} \rangle \equiv \bar{X}^\Lambda F_\Lambda - X^\Lambda \bar{F}_\Lambda = -ie^{-\mathcal{K}}. \tag{2.4}$$

The couplings of the vector fields to the scalars are determined by the $(n_V + 1) \times (n_V + 1)$ period matrix $\mathcal{N}$, defined by the relations

$$M_\Lambda = \mathcal{N}_{\Lambda\Sigma} L^\Sigma\,, \qquad D_{\bar{\imath}}\bar{M}_\Lambda = \mathcal{N}_{\Lambda\Sigma}\, D_{\bar{\imath}}\bar{L}^\Sigma\,. \tag{2.5}$$

[2] We use the conventions of [11].

If the theory is defined in a frame in which a prepotential exists, $\mathcal{N}$ can be obtained from

$$\mathcal{N}_{\Lambda\Sigma} = \bar{F}_{\Lambda\Sigma} + 2i\frac{(N_{\Lambda\Gamma}X^{\Gamma})(N_{\Sigma\Delta}X^{\Delta})}{X^{\Omega}N_{\Omega\Psi}X^{\Psi}}\,, \tag{2.6}$$

where $F_{\Lambda\Sigma} = \partial_{\Lambda}\partial_{\Sigma}F$ and $N_{\Lambda\Sigma} \equiv \mathrm{Im}(F_{\Lambda\Sigma})$. Introducing the matrix[3]

$$\mathcal{M} = \begin{pmatrix} I + RI^{-1}R & -RI^{-1} \\ -I^{-1}R & I^{-1} \end{pmatrix}, \tag{2.7}$$

we have the important relation between the symplectic sections and their derivatives,

$$\frac{1}{2}(\mathcal{M} - i\Omega) = \Omega\bar{\mathcal{V}}\mathcal{V}\Omega + \Omega D_i\mathcal{V}g^{i\bar{\jmath}}D_{\bar{\jmath}}\bar{\mathcal{V}}\Omega\,, \tag{2.8}$$

with

$$\Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.9}$$

The bosonic Lagrangian reads

$$\sqrt{-g}^{-1}\mathscr{L} = \frac{R}{2} - g_{i\bar{\jmath}}\,\partial_{\mu}z^{i}\partial^{\mu}\bar{z}^{\bar{\jmath}} + \frac{1}{4}I_{\Lambda\Sigma}F^{\Lambda\mu\nu}F^{\Sigma}{}_{\mu\nu} + \frac{1}{4}R_{\Lambda\Sigma}F^{\Lambda\mu\nu}\,\star F^{\Sigma}{}_{\mu\nu} - V(z,\bar{z})\,. \tag{2.10}$$

In the case of dyonic U(1) FI-gauging, the scalar potential has the form [12]

$$V = g^{i\bar{\jmath}}D_i\mathcal{L}D_{\bar{\jmath}}\bar{\mathcal{L}} - 3\mathcal{L}\bar{\mathcal{L}}\,, \tag{2.11}$$

where $\mathcal{L} = \langle\mathcal{G},\mathcal{V}\rangle$, and $\mathcal{G} = (g^{\Lambda}, g_{\Lambda})^{t}$ denotes the symplectic vector of gauge couplings (FI parameters).

2.2 Fake internal symmetries, stabilization and solutions

The kinetic part of (2.10) corresponds to the action of the ungauged theory, whose on-shell global symmetry group is called U-duality, consisting of the isometries of the non-linear sigma model that act linearly also on the field strengths via the symplectic embedding [8]. For purely electric gaugings, the scalar potential generically spoils this invariance, but, as is clear from (2.11), for dyonic gauging one recovers the whole U-duality invariance, at the price of changing the vector of gauge couplings and so the physical theory. We will call this group U_{fi}, that stands for fake internal symmetry group.[4] The action of U_{fi} on a solution is the mapping to other solutions of other theories, in the same way in which some elements of the symplectic group map solutions of theories with different prepotential into each other [12], cf. e.g. (B.2), (B.3).

Given U_{fi}, we fix a choice of the coupling constants $\mathcal{G}$ and, at least at the beginning, we suppose that they are generic. We want to underline that for abelian dyonic gaugings, the Maxwell equations remain homogeneous and so the action (2.10) doesn't have topological terms [13].

[3] We defined $R = \mathrm{Re}\mathcal{N}$ and $I = \mathrm{Im}\mathcal{N}$.

[4] When the special Kähler manifold is symmetric we define the Lie algebra $\mathfrak{u}_{\mathrm{fi}}$ of U_{fi} through the equations (A.3). The corresponding definition for nonsymmetric special Kähler manifolds requires more care.

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The true internal symmetry group $U_{\rm i}$ of the gauged supergravity theory is $S_{\mathcal{G}}$, the stabilizer of $\mathcal{G}$ under the action of $U_{\rm fi}$, up to possible U(1) factors. This is obvious from the definition of the stabilizer,

$$S_{\mathcal{G}} = \{g \in U_{\rm fi} \,|\, g\mathcal{G} = \mathcal{G}\}\,, \tag{2.12}$$

which means that we impose to stay in the same theory, and this restricts of course the group of internal symmetries.

By acting with $S \in S_{\mathcal{G}}$ on a given seed solution $(\mathcal{V}, \mathcal{G}, \mathcal{F}_{\mu\nu})$[5] of the equations of motion, we can generate another configuration via the map

$$(\mathcal{V}, \mathcal{G}, \mathcal{F}_{\mu\nu}) \mapsto (\tilde{\mathcal{V}}, \tilde{\mathcal{G}}, \tilde{\mathcal{F}}_{\mu\nu}) := (S\mathcal{V}, S\mathcal{G}, S\mathcal{F}_{\mu\nu}) = (S\mathcal{V}, \mathcal{G}, S\mathcal{F}_{\mu\nu})\,. \tag{2.13}$$

The transformed fields solve the field equations by construction.[6] In general, the scalars transform nonlinearly under the corresponding isometry, the field strengths are rotated and the metric is functionally invariant.

Technically, in order to determine $S_{\mathcal{G}}$, it is simpler to work with the corresponding algebra

$$\mathfrak{s}_{\mathcal{G}} = \{a \in \mathfrak{u}_{\rm fi} \,|\, a\mathcal{G} = 0\}\,. \tag{2.14}$$

There are some cases in which $U_{\rm i}$ strictly contains $S_{\mathcal{G}}$, and this depends on some particular symmetric structures of the model under consideration. Typically, this happens because the symmetry of the model allows to act with some symplectic matrices in a more general way than (2.13), leaving nevertheless the theory invariant.

3 Stabilization and symmetries for some prepotentials

Now we want to apply these techniques to some specific prepotentials. Each of them exhibits different peculiar features related to the geometry of the underlying special Kähler manifold, namely to the symplectic embedding of the isometry group of the non-linear sigma model (cf. appendix B).

3.1 Prepotential $F = -iX^0X^1$

This prepotential encodes a particular special Kähler structure on the symmetric manifold SU(1, 1)/U(1). The symplectic section is $\mathcal{V} = (X^0, X^1, -iX^1, -iX^0)^t$, and we fix the couplings in a completely electric frame, $\mathcal{G} = (0, 0, g_0, g_1)^t$. The solution to (A.3) defines the algebra $\mathfrak{u}_{\rm fi}$,

$$b_1t_1 + b_2t_2 + b_3t_3 + b_4t_4 = \begin{pmatrix} b_4 & 0 & b_1 & b_2 \\ 0 & -b_4 & b_2 & b_3 \\ -b_3 & -b_2 & -b_4 & 0 \\ -b_2 & -b_1 & 0 & b_4 \end{pmatrix}\,,$$

[5] Actually we should write $(\mathcal{V}, \mathcal{G}, \mathcal{F}_{\mu\nu}, g_{\mu\nu})$, but since $S_{\mathcal{G}}$ does not act on the metric, we shall suppress the dependence on $g_{\mu\nu}$.

[6] As is clear from the formalism introduced in [12], the application of $S \in S_{\mathcal{G}}$ on a static solution of the BPS flow preserves the same amount of supersymmetry as the original configuration. In the rotating case, the same is true if one considers electric gaugings only [14].

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to be the U-duality su(1,1) plus a u(1), generated by t_2, which acts trivially on the z^i, as we will see shortly. From the stability equation (2.14) one finds that $\mathfrak{s}_{\mathcal{G}}$ is generated by

$$s = t_2 - \frac{g_1}{g_0} t_1 - \frac{g_0}{g_1} t_3 \,, \tag{3.1}$$

so that $S_{\mathcal{G}} \subseteq \mathrm{U}(1,1)$ is the 1-parameter subgroup

$$S = e^{\beta s} = \begin{pmatrix} \cos^2\beta & \frac{g_1}{g_0}\sin^2\beta & -\frac{g_1}{g_0}\cos\beta\sin\beta & \cos\beta\sin\beta \\ \frac{g_0}{g_1}\sin^2\beta & \cos^2\beta & \cos\beta\sin\beta & -\frac{g_0}{g_1}\cos\beta\sin\beta \\ \frac{g_0}{g_1}\sin\beta\cos\beta & -\cos\beta\sin\beta & \cos^2\beta & \frac{g_0}{g_1}\sin^2\beta \\ -\cos\beta\sin\beta & \frac{g_1}{g_0}\cos\beta\sin\beta & \frac{g_1}{g_0}\sin^2\beta & \cos^2\beta \end{pmatrix} . \tag{3.2}$$

On the other hand, the U(1) generated by t_2 is given by

$$T_\alpha = e^{\alpha t_2} = \begin{pmatrix} \cos\alpha & 0 & 0 & \sin\alpha \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ -\sin\alpha & 0 & 0 & \cos\alpha \end{pmatrix} , \tag{3.3}$$

and it transforms the section $\mathcal{V}$ according to

$$T_\alpha \mathcal{V} = e^{-i\alpha}\mathcal{V} \,. \tag{3.4}$$

The projective special Kähler coordinates are thus insensible to its action. The matrix $\mathcal{M}$ defined in (2.7) transforms as

$$T_\alpha^t \mathcal{M} T_\alpha = \mathcal{M} \,. \tag{3.5}$$

One can thus act with T_α on $\mathcal{F}_{\mu\nu}$ only, leaving the equations of motion still invariant. T_α is an example for a 'field rotation matrix' that is commonly used to generate non-BPS solutions, a technique first introduced in [15, 16] and subsequently applied to gauged supergravity in [17, 18]. In conclusion, the internal symmetry group of this model is $U_{\mathrm{i}} = \mathrm{U}(1) \times \mathrm{U}(1) \supset S_{\mathcal{G}}$, with the two U(1) factors identified respectively with S and T_α.

3.2 Prepotential $F = \frac{i}{4} X^\Lambda \eta_{\Lambda\Sigma} X^\Sigma$

The prepotential $F = \frac{i}{4} X^\Lambda \eta_{\Lambda\Sigma} X^\Sigma$, with $\eta_{\Lambda\Sigma} = \mathrm{diag}(-1,1,\ldots,1)$, describes a special Kähler structure on the symmetric manifolds $\mathrm{SU}(1,n_{\mathrm{V}})/(\mathrm{U}(1)\times\mathrm{SU}(n_{\mathrm{V}}))$. The symplectic section reads

$$\mathcal{V} = \left(X^\Lambda, \frac{i}{2}\eta_{\Lambda\Sigma} X^\Sigma \right)^t . \tag{3.6}$$

Due to the linearity of $\mathcal{V}$ in the coordinates X^Λ, one can easily construct the one-parameter subgroup

$$L_\alpha = \begin{pmatrix} \cos\alpha & 0 & 2\sin\alpha & 0 \\ 0 & I_{n_{\mathrm{V}}}\cos\alpha & 0 & -2I_{n_{\mathrm{V}}}\sin\alpha \\ -\frac{1}{2}\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & \frac{1}{2}I_{n_{\mathrm{V}}}\sin\alpha & 0 & I_{n_{\mathrm{V}}}\cos\alpha \end{pmatrix}$$

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of $\mathrm{Sp}(2n_{\mathrm{V}}+2,\mathbb{R})$, under which the section $\mathcal{V}$ transforms as

$$L_\alpha \mathcal{V} = e^{-i\alpha}\mathcal{V}\,. \tag{3.7}$$

Since

$$L_\alpha^t \mathcal{M} L_\alpha = \mathcal{M}\,, \tag{3.8}$$

we can add a new parameter to all the solutions of this model by acting with L_α on $\mathcal{F}_{\mu\nu}$ only.

The stability equation is slightly more involved. Notice that the case with only one vector multiplet is symplectically equivalent to $F = -iX^0X^1$, and thus the results for $n_{\mathrm{V}} = 1$ can be obtained from the previous subsection by an appropriate symplectic rotation, cf. appendix B.

Let us discuss the general case of $n_{\mathrm{V}} = n$ vector multiplets. Eq. (A.3) defining the algebra $\mathfrak{u}_{\mathrm{fi}}$ is equivalent to

$$Q^t = -\eta Q \eta\,, \qquad S = -\frac{1}{4}\eta R \eta\,. \tag{3.9}$$

These equations define an embedding of $\mathrm{U}(1,n)$ into $\mathrm{Sp}(2n+2,\mathbb{R})$. To see this, let $z = A + iB \in \mathfrak{u}(1,n)$. Then, $z^t\eta + \eta z = 0$ implies

$$A^t = -\eta A \eta\,, \qquad B^t\eta = \eta B\,, \tag{3.10}$$

so ηB is symmetric. This suggests an embedding

$$\iota_\alpha : \mathfrak{u}(1,n) \longrightarrow \mathfrak{sp}(2n+2,\mathbb{R})\,, \qquad A + iB \longmapsto \begin{pmatrix} A & \alpha B\eta \\ -\frac{1}{\alpha}\eta B & -A^t \end{pmatrix}, \tag{3.11}$$

for any real $\alpha \neq 0$. This is indeed an injective Lie algebra morphism, and its image consists of the elements of $\mathfrak{sp}(2n+2,\mathbb{R})$ which solve (A.3) with $F_\Lambda = \frac{i}{\alpha}\eta_{\Lambda\Sigma}X^\Sigma$. In particular, (3.9) selects ι_2.

A basis for $\mathfrak{u}(1,n)$ is given by the matrices

$$\{A_a\}_{a=1}^{n(n+1)/2}\,, \qquad \{iB_k\}_{k=0}^{n(n+3)/2}\,, \tag{3.12}$$

where A_a are a basis for the space of $(n+1)\times(n+1)$ real matrices A such that ηA is antisymmetric, and B_k generate the space of $(n+1)\times(n+1)$ real matrices B such that ηB is symmetric, with $B_0 = I$, the identity matrix. The embedding extends obviously to the group level via the exponential map, and, in particular, notice that

$$\exp(\alpha\iota_2(iB_0)) = L_\alpha\,. \tag{3.13}$$

Let us now consider the symmetry group $S_{\mathcal{G}}$. If we set

$$\mathcal{G} = (\underline{0}, \underline{g})^t = (0, \vec{0}, g_0, \vec{g})^t\,, \tag{3.14}$$

with $\vec{g} = (g_1, \ldots, g_n)$, then we see that the invariance of $\mathcal{G}$ is defined by the equations

$$A^t\underline{g} = 0\,, \qquad B\eta\underline{g} = 0\,, \tag{3.15}$$

which define a maximal compact subgroup[7] $\mathrm{U}(n)$ of $\mathrm{U}(1,n)$. To see this, let us first put[8]

$$\hat{g} := \sqrt{-\underline{g}^2}\,, \tag{3.16}$$

and define $\Lambda_{\underline{g}} \in \mathrm{SO}(1,n)$ by

$$(g_0, \vec{g}) = (\hat{g}, \vec{0})\Lambda_{\underline{g}}\,. \tag{3.17}$$

Thus, A (or ηB^t) has $\underline{g}$ in the cokernel if and only if $\Lambda_{\underline{g}} A \Lambda_{\underline{g}}^{-1}$ (or $\Lambda_{\underline{g}} \eta B^t \Lambda_{\underline{g}}^{-1}$) has $(\hat{g}, \vec{0})$ in the cokernel. From this we immediately get that $\mathfrak{s}_{\mathcal{G}}$ is generated by the elements of $\mathfrak{u}(1,n)$ of the form

$$z_{\underline{g}} = \Lambda_{\underline{g}}^{-1} z \Lambda_{\underline{g}}\,, \tag{3.18}$$

where $z \in \mathfrak{u}(1,n)$ has vanishing first row and first column. Thus, $z_{\underline{g}} \in \mathrm{U}(n)$.

This provides also a way to realize an explicit construction of the group elements of $S_{\mathcal{G}}$. One can choose e.g. a generalized Gell-Mann basis [19] for $\mathfrak{su}(n)$, add the identity matrix I_n and then embed the basis into $\mathfrak{u}(1,n)$ by adding a first row and column of zeros. If we call $\{z_I\}_{I=0}^{n^2-1}$ such a basis for the compact subalgebra $\mathfrak{u}(n)$ of $\mathfrak{su}(1,n)$, then

$$\{\iota_2(z_I)\}_{I=0}^{n^2-1}$$

is a basis for $\mathfrak{s}_{\mathcal{G}_0}$, where $\mathcal{G}_0 \equiv (0, \vec{0}, \hat{g}, \vec{0})$. Then we can explicitly construct the group elements by means of the Euler construction of $S_{\mathcal{G}_0}$,[9] as in [19, 21]. Finally we have

$$S_{\mathcal{G}} = \tilde{\Lambda}_{\underline{g}}^{-1} S_{\mathcal{G}_0} \tilde{\Lambda}_{\underline{g}}\,, \tag{3.19}$$

with

$$\tilde{\Lambda}_{\underline{g}} = \begin{pmatrix} \Lambda_{\underline{g}} & 0 \\ 0 & \Lambda_{\underline{g}}^{-1} \end{pmatrix}. \tag{3.20}$$

For practical purposes we can take $\Lambda_{\underline{g}}$ defined by

$$\Lambda_{\underline{g}}{}^0{}_0 = \frac{g_0}{\hat{g}}\,, \qquad \Lambda_{\underline{g}}{}^i{}_0 = \Lambda_{\underline{g}}{}^0{}_i = \frac{g_i}{\hat{g}}\,, \qquad \Lambda_{\underline{g}}{}^i{}_j = \frac{g_0 - \hat{g}}{\hat{g}\vec{g}^2} g_i g_j + \delta^i{}_j\,, \tag{3.21}$$

whose inverse is obtained by the replacement $\vec{g} \to -\vec{g}$.

Let us focus on the first nontrivial case $\mathrm{SU}(1,2)/(\mathrm{U}(1) \times \mathrm{SU}(2))$. We fix the couplings in a completely electric frame, $\mathcal{G} = (0,0,0,g_0,g_1,g_2)^t$. A basis for $\mathfrak{u}(2)$ (relative to the vector $\mathcal{G}_0 = (0, \vec{0}, \hat{g}, \vec{0})$) is

$$t_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}, \quad t_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad t_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}, \tag{3.22}$$

[7] To be precise, this is the subgroup $\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))$.

[8] We assume $\underline{g}$ to be timelike future-directed, i.e., $\eta^{\Lambda\Sigma} g_\Lambda g_\Sigma < 0$, $g_0 > 0$.

[9] In a similar way one can use the Iwasawa construction to obtain the whole group U_{fi}, whose compact part is just $S_{\mathcal{G}}$ [20].

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which, by means of ι_2, defines the basis of $\mathfrak{s}_{\mathcal{G}_0}$

$$T_0 = \begin{pmatrix} 0 & 0 & 0 & 0\,0\,0 \\ 0 & 0 & 0 & 0\,2\,0 \\ 0 & 0 & 0 & 0\,0\,2 \\ 0 & 0 & 0 & 0\,0\,0 \\ 0 & -\frac{1}{2} & 0 & 0\,0\,0 \\ 0 & 0 & -\frac{1}{2} & 0\,0\,0 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} 0 & 0 & 0 & 0\,0\,0 \\ 0 & 0 & 0 & 0\,0\,2 \\ 0 & 0 & 0 & 0\,2\,0 \\ 0 & 0 & 0 & 0\,0\,0 \\ 0 & 0 & -\frac{1}{2} & 0\,0\,0 \\ 0 & -\frac{1}{2} & 0 & 0\,0\,0 \end{pmatrix},$$

$$T_2 = \begin{pmatrix} 0\,0 & 0 & 0\,0 & 0 \\ 0\,0 & -1 & 0\,0 & 0 \\ 0\,1 & 0 & 0\,0 & 0 \\ 0\,0 & 0 & 0\,0 & 0 \\ 0\,0 & 0 & 0\,0 & -1 \\ 0\,0 & 0 & 0\,1 & 0 \end{pmatrix}, \qquad T_3 = \begin{pmatrix} 0 & 0 & 0\,0\,0 & 0 \\ 0 & 0 & 0\,0\,2 & 0 \\ 0 & 0 & 0\,0\,0 & -2 \\ 0 & 0 & 0\,0\,0 & 0 \\ 0 & -\frac{1}{2} & 0\,0\,0 & 0 \\ 0 & 0 & \frac{1}{2}\,0\,0 & 0 \end{pmatrix}. \tag{3.23}$$

Note that

$$T_0^2 = -\Delta\,, \qquad [T_i, T_j]_+ = -\delta_{ij}\Delta\,, \quad 1 \le i \le j \le 3\,,$$

with

$$\Delta = \begin{pmatrix} 0\,0\,0\,0\,0\,0 \\ 0\,1\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0 \\ 0\,0\,0\,0\,0\,1 \end{pmatrix}, \tag{3.24}$$

from which we immediately get the expression for a generic element of $S_{\mathcal{G}_0}$,

$$\begin{aligned} S_0(x^0, \vec{x}) &= e^{x^0 T_0} e^{\vec{x}\cdot\vec{T}} \\ &= \left(I_6 - 2\sin^2\frac{x^0}{2}\Delta + \sin x^0 T_0\right)\left(I_6 - 2\sin^2\frac{|\vec{x}|}{2}\Delta + \sin|\vec{x}|\ \vec{x}\cdot\vec{T}\right), \end{aligned} \tag{3.25}$$

where $\vec{x} = (x^1, x^2, x^3)$, $|\vec{x}| = \sqrt{\vec{x}\cdot\vec{x}}$, $\vec{T} = (T_1, T_2, T_3)$ and $\vec{x}\cdot\vec{T} = \sum_{i=1}^3 x^i T_i$.

Finally, after setting

$$T_\mu^g = \tilde{\Lambda}_{\underline{g}}^{-1} T_\mu \tilde{\Lambda}_{\underline{g}}\,, \quad \mu = 0,1,2,3\,, \qquad \Delta_{\underline{g}} = \tilde{\Lambda}_{\underline{g}}^{-1}\Delta\tilde{\Lambda}_{\underline{g}}\,, \tag{3.26}$$

we get for a generic element of $S_{\mathcal{G}}$

$$\begin{aligned} S_{\underline{g}}\left(x^0, \vec{x}\right) &= \tilde{\Lambda}_{\underline{g}}^{-1} S_0\left(x^0, \vec{x}\right)\tilde{\Lambda}_{\underline{g}} \\ &= \left(I_6 - 2\sin^2\frac{x^0}{2}\Delta_{\underline{g}} + \sin x^0 T_0^g\right)\left(I_6 - 2\sin^2\frac{|\vec{x}|}{2}\Delta_{\underline{g}} + \sin|\vec{x}|\ \vec{x}\cdot\vec{T}^g\right). \end{aligned} \tag{3.27}$$

In order to have even more manageable expressions for the matrices, it may be convenient to change to the basis R_μ defined by

$$R_0 = T_0^g\,, \quad R_1 = \frac{g_1^2 - g_2^2}{g_1^2 + g_2^2}T_1^g - \frac{2g_1g_2}{g_1^2 + g_2^2}T_3^g\,, \quad R_2 = T_2^g\,, \quad R_3 = \frac{g_1^2 - g_2^2}{g_1^2 + g_2^2}T_3^g + \frac{2g_1g_2}{g_1^2 + g_2^2}T_1^g\,.$$

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3.3 Prepotential $F = -X^1X^2X^3/X^0$

This prepotential describes a special Kähler structure on the symmetric manifold $(\mathrm{SU}(1,1)/\mathrm{U}(1))^3$, the well-known stu model. This is symplectically equivalent to the model with $F = -2i(X^0X^1X^2X^3)^{1/2}$, for which supersymmetric black holes with purely electric gaugings are known analytically [22]. After a symplectic transformation to $F = -X^1X^2X^3/X^0$, the electric gaugings considered in [22] become $\mathcal{G} = (0, g^1, g^2, g^3, g_0, 0, 0, 0)^t$, so we shall concentrate on this case in what follows. The symplectic section reads

$$\mathcal{V} = (X^0, X^1, X^2, X^3, X^1X^2X^3/(X^0)^2, -X^2X^3/X^0, -X^1X^3/X^0, -X^2X^1/X^0)^t\,.$$

Let us now look at the solutions of (A.3). To this end, we define

$$\boldsymbol{X} \equiv \begin{pmatrix} X^{0^3} \\ X^{0^2}X^1 \\ X^{0^2}X^2 \\ X^{0^2}X^3 \end{pmatrix}, \qquad \boldsymbol{F} \equiv \begin{pmatrix} X^1X^2X^3 \\ -X^0X^2X^3 \\ -X^0X^1X^3 \\ -X^0X^1X^2 \end{pmatrix}, \tag{3.28}$$

so that (A.3) becomes

$$\boldsymbol{X}S\boldsymbol{X} - \boldsymbol{F}R\boldsymbol{F} - 2\boldsymbol{X}Q^t\boldsymbol{F} = 0\,. \tag{3.29}$$

Since the l.h.s. is a homogeneous polynomial of degree 6 in (X^0, X^1, X^2, X^3), the coefficients of each monomial must be zero. The simplest way to get the general solutions is then to look at the powers of X^0. The possible powers of X^0 in $p_S \equiv \boldsymbol{X}S\boldsymbol{X}$, $p_R \equiv \boldsymbol{F}R\boldsymbol{F}$ and $p_Q \equiv \boldsymbol{X}Q^t\boldsymbol{F}$ are $(6,5,4)$, $(2,1,0)$ and $(4,3,2)$ respectively. Since S and R are symmetric, p_S and p_R can vanish only if S and R are zero. Thus, we are left with the following three possibilities:

1. $R = 0$ and p_Q cancels p_S. The only common power for X^0 is 4, so we have to take matrices which generate only this power and equal degrees for the remaining variables. A quick inspection gives the solutions[10]

$$S_1 = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ -1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&1&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ -1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \end{pmatrix},$$

$$U_1 = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ -1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \end{pmatrix}. \tag{3.30}$$

[10] To avoid confusion, note that S denotes the 4×4 matrix in (3.29), while S_1, S_2 and S_3 defined below are 8×8 matrices.

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2. $S = 0$ and p_Q cancels p_R. The only common power for X^0 is 2, so we have to take matrices generating only this and equal degrees for the remaining variables. The solution is

$$S_2 = S_1^t\,, \qquad T_2 = T_1^t\,, \qquad U_2 = U_1^t\,. \tag{3.31}$$

3. $R = S = 0$ and Q satisfies $p_Q = 0$. This implies that Q must be diagonal and that the space of such solutions is 3-dimensional. The simplest way to fix a basis of this space is to choose

$$S_3 = [S_1, S_2]\,, \qquad T_3 = [T_1, T_2]\,, \qquad U_3 = [U_1, U_2]\,. \tag{3.32}$$

In this way the nine matrices $\vec{S}$, $\vec{T}$ and $\vec{U}$ generate the group $U_{\rm fi} = (\mathrm{SL}(2,\mathbb{R}))^3$.

In order to determine the symmetry algebra $\mathfrak{s}_{\mathcal{G}}$ we have to consider the equation (using the same notation as in the previous subsection)

$$(\vec{x}\cdot\vec{S} + \vec{y}\cdot\vec{T} + \vec{z}\cdot\vec{U})\mathcal{G} = 0\,, \tag{3.33}$$

whose general solution is given by

$$\mathcal{U}(x,z) = g_0 g^3 x S_1 + g^1 g^2 x S_2 - g_0 g^2 (x+z) T_1 - g^1 g^3 (x+z) T_2 + g_0 g^1 z U_1 + g^2 g^3 z U_2\,,$$

for arbitrary $x, z \in \mathbb{R}$. A convenient basis is

$$\mathcal{U}_1 = \mathcal{U}(1,-1)\,, \qquad \mathcal{U}_2 = \mathcal{U}(1,0)\,, \tag{3.34}$$

which defines a two-dimensional abelian algebra. Notice that

$$\mathrm{tr}\,\mathcal{U}_1^2 = \mathrm{tr}\,\mathcal{U}_2^2 = 8 g_0 g^1 g^2 g^3\,, \tag{3.35}$$

so that the algebra is compact (and thus defines the group $\mathrm{U}(1) \times \mathrm{U}(1)$) if and only if $g_0 g^1 g^2 g^3 < 0$. One can easily verify that, unfortunately, none of these continuous symmetries survives for the truncation to the t^3 model [23, 24] with prepotential $F = -(X^1)^3/X^0$.

It is worth noting that a particular situation arises for $g^1 = g^2 = g^3 = -g_0 \equiv g$. As was shown in [10], there is an enhancement of the internal symmetry group in this case. This happens because the scalar potential V can be written in terms of fundamental objects that define the nonlinear sigma model of the non-homogeneous projective coordinates $z^i = x^i + i y^i$ [8, 10], namely

$$V = g^2 \sum_{i=1}^{3} \mathrm{tr} M_i\,, \qquad M_i = \begin{pmatrix} y^i + \frac{x^{i2}}{y^i} & \frac{x^i}{y^i} \\ \frac{x^i}{y^i} & \frac{1}{y^i} \end{pmatrix}. \tag{3.36}$$

In fact, the transformation property of M_i,

$$M_i \longmapsto \mathcal{T}^t M_i \mathcal{T}\,, \tag{3.37}$$

implies the invariance of the potential only if $\mathcal{T}\mathcal{T}^t = 1$. Going back to the symplectic formalism we see that this condition is equivalent to require for the symmetry group to be

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orthogonal, which, in terms of the elements of $\mathfrak{u}_{\rm fi}$ amounts to consider just the subspace of antisymmetric matrices. Thus, the symmetry algebra is generated by

$$W_1 = S_1 - S_2\,, \qquad W_2 = T_1 - T_2\,, \qquad W_3 = U_1 - U_2\,, \tag{3.38}$$

while the subalgebra leaving $\mathcal{G}$ fixed is generated by $W_2 - W_1$ and $W_3 - W_2$. The full symmetry group is therefore an extension $U_{\rm i} = \mathrm{U}(1)^3$ of $S_{\mathcal{G}} = \mathrm{U}(1)^2$.

3.4 Prepotential $F = X^1X^2X^3/X^0 - \frac{A}{3}(X^3)^3/X^0$

The base manifold for this prepotential is neither symmetric nor homogeneous and it has been studied in [25]. The symplectic section is given by $\mathcal{V} = (X^\Lambda, F_\Lambda)^t$, with

$$X^{\Lambda t} = \begin{pmatrix} X^0 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix}\,, \qquad F^t_\Lambda = \begin{pmatrix} -X^1X^2X^3/(X^0)^2 + \frac{A}{3}(X^3)^3/(X^0)^2 \\ X^2X^3/X^0 \\ X^1X^3/X^0 \\ X^1X^2/X^0 - A(X^3)^2/X^0 \end{pmatrix}\,. \tag{3.39}$$

The solution to (A.3) is obtained by proceeding exactly like in the previous subsection. After introducing the vectors

$$\boldsymbol{X} = \begin{pmatrix} X^{0\,3} \\ X^{0\,2}X^1 \\ X^{0\,2}X^2 \\ X^{0\,2}X^3 \end{pmatrix}\,, \qquad \boldsymbol{F} = \begin{pmatrix} \frac{A}{3}X^{3\,3} - X^1X^2X^3 \\ X^0X^2X^3 \\ X^0X^1X^3 \\ X^0X^1X^2 - AX^0X^{3\,2} \end{pmatrix}\,, \tag{3.40}$$

we reduce the equations to a polynomial identity, and looking at the coefficients we get a five-dimensional space of solutions generated by the symplectic matrices

$$S_1 = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&-1 \\ 0&0&1&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&-2A&0&0&0&0 \end{pmatrix}\,, \quad S_2 = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&-1&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \end{pmatrix}\,, \quad S_3 = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&-1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \end{pmatrix}\,,$$

$$D_1 = \begin{pmatrix} 3&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&-3&0&0&0 \\ 0&0&0&0&0&-1&0&0 \\ 0&0&0&0&0&0&-1&0 \\ 0&0&0&0&0&0&0&-1 \end{pmatrix}\,, \quad D_2 = \begin{pmatrix} 0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&-1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&-1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0 \end{pmatrix}\,. \tag{3.41}$$

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A direct comparison with the results of [25] shows that this algebra strictly contains the U-duality algebra. This is due to the fact that the group of symmetries of the scalar potential is larger than the symmetry group of the whole Lagrangian. Indeed the generator D_2 does not leave the metric invariant. Thus, the U-duality group is generated by the algebra

$$\langle S_1, S_2, S_3, D_1 \rangle_{\mathbb{R}} \,. \tag{3.42}$$

Notice that the S_i are nilpotent of order 4 for $i = 1$ and order 2 for $i = 2, 3$. They are indeed eigenmatrices for the adjoint action of D_1, all with eigenvalue -2. The stability equation (2.14) has a nontrivial solution only if $A = -g^1 g^2/(g^3)^2$. With this choice for A one gets a one-dimensional algebra $\mathfrak{s}_{\mathcal{G}}$ generated by

$$s = S_1 - \frac{g^1}{g^3} S_3 - \frac{g^2}{g^3} S_2 \,. \tag{3.43}$$

It is nilpotent of order 4 so that $U_{\rm i} = \mathcal{S}_{\mathcal{G}}$ is a unipotent group of order 4. It is worthwhile to note that for $g^1 = g^2 = g^3$ one gets $A = -1$, which is the physically most interesting case, since the corresponding prepotential arises in the context of type IIA string theory compactifed on Calabi-Yau manifolds [26].

4 Scalar hair and dyonic solutions

We shall now use the results of the previous section in order to generate new supergravity solutions from a given seed. The transformations in $U_{\rm i}$ add new parameters to a given solution and leave not only the equations of motion invariant, but also some potential first-order flow equations (if these are satisfied by the seed). The transformed field configuration preserves thus the same amount of supersymmetry as the one from which we started.

As was stressed in [10], the latter statement is not true in the stu model for the additional U(1) that arises for equal couplings, whose action generically leads to a non-BPS solution. The same story holds also in the quadratic models for T_α and L_α, due to the properties (3.5) and (3.8) [18].

In what follows we will consider several relevant examples for some well-studied prepotentials, but there is no obstacle to extending this method to other solutions and prepotentials as well. We underline that in the static case, owing to the existence of the black hole potential $V_{\rm BH}$ [27, 28], one can directly rotate the charges $\mathcal{Q}$ instead of the field strengths $\mathcal{F}_{\mu\nu}$.

4.1 Prepotential $F = -iX^0X^1$

For this prepotential, we have $U_{\rm i} = \mathrm{U}(1)^2$, whose action on the static and magnetic BPS seed solution of [22] is

$$(\mathcal{V}, \mathcal{G}, \mathcal{Q}) \quad \longmapsto \quad (\tilde{\mathcal{V}}, \tilde{\mathcal{G}}, \tilde{\mathcal{Q}}) = (S\mathcal{V}, \mathcal{G}, T_\alpha S \mathcal{Q}) \,. \tag{4.1}$$

Using the results of section 3.1 and the constraints on the seed parameters (cf. [22]), one gets

$$\begin{aligned} \tilde{\mathcal{Q}} &= (p^0 \cos\alpha, p^1 \cos\alpha, -p^1 \sin\alpha, -p^0 \sin\alpha)^t \,, \\ \tilde{z} = \frac{\tilde{X}^1}{\tilde{X}^0} &= \frac{g_0}{g_1} \cdot \frac{g_1 z \cos\beta + i g_0 \sin\beta}{g_0 \cos\beta + i g_1 z \sin\beta} \,, \qquad z \equiv \frac{X^1}{X^0} \,. \end{aligned} \tag{4.2}$$

The parameter β does not modify the supersymmetry of the solution; for $\alpha = 0$ the new configuration satisfies again the BPS flow equations of [12, 22]. For $\alpha \neq 0$ one gets a solution that still obeys a first-order flow, but this time a non-BPS one [18], driven by the fake superpotential

$$W = e^U |\langle T_{-\alpha}\tilde{\mathcal{Q}}, \tilde{\mathcal{V}}\rangle - ie^{2(\psi-U)}\tilde{\mathcal{L}}| \,, \tag{4.3}$$

where $U(r)$ and $\psi(r)$ are functions appearing in the metric

$$ds^2 = -e^{2U}dt^2 + e^{-2U}dr^2 + e^{2(\psi-U)}(d\theta^2 + \sinh^2\theta d\phi^2) \,, \tag{4.4}$$

and $\mathcal{L}$ was defined in section 2.1. The first-order equations following from (4.3) imply the equations of motion provided the Dirac-type charge quantization condition

$$\langle \mathcal{G}, \mathcal{Q}\rangle = 1 \tag{4.5}$$

holds [18]. From (4.2) we see that for $\alpha \neq 0$ one generates a dyonic solution from a purely magnetic one, while β adds scalar hair to the seed. Note that this result was first obtained in [10].

As another example for the action of $U_{\rm i}$ we consider the Chow-Compère solution [29], that solves the equations of motion following from the Lagrangian (2.12) of [29],

$$\begin{aligned} \mathscr{L} = R\star 1 - \frac{1}{2}\star d\varphi\wedge d\varphi - \frac{1}{2}e^{2\varphi}\star d\chi\wedge d\chi - e^{-\varphi}\star F^1\wedge F^1 + \chi F^1\wedge F^1 \\ - \frac{1}{1+\chi^2 e^{2\varphi}}\left(e^{\varphi}\star F^2\wedge F^2 + \chi e^{2\varphi}F^2\wedge F^2\right) + g^2\left(4 + e^{\varphi} + e^{-\varphi} + \chi^2 e^{\varphi}\right)\star 1 \,, \end{aligned} \tag{4.6}$$

which is obtained from (2.10) by setting

$$z = \frac{g_0}{g_1}\left(e^{-\varphi} - i\chi\right) \,, \qquad g_0 g_1 = g^2 \,, \tag{4.7}$$

and redefining[11]

$$F^0 \longrightarrow \sqrt{\frac{g_1}{g_0}}F^1 \,, \qquad F^1 \longrightarrow \sqrt{\frac{g_0}{g_1}}F^2 \,. \tag{4.8}$$

The dyonic rotating black hole solution of [29] is given by

$$ds^2 = -\frac{R}{W}\left(dt - \frac{a^2 - u_1u_2}{a}d\phi\right)^2 + \frac{W}{R}dr^2 + \frac{U}{W}\left(dt - \frac{r_1r_2 + a^2}{a}d\phi\right)^2 + \frac{W}{U}du^2 \,, \tag{4.9}$$

where

$$\begin{aligned} R(r) &= r^2 - 2mr + a^2 + g^2 r_1 r_2(r_1r_2 + a^2) \,, \\ U(u) &= -u^2 + 2nu + a^2 + g^2 u_1u_2(u_1u_2 - a^2) \,, \\ W(r,u) &= r_1r_2 + u_1u_2 \,, \qquad r_{1,2} = r + \Delta r_{1,2} \,, \qquad u_{1,2} = u + \Delta u_{1,2} \,, \end{aligned} \tag{4.10}$$

[11] We assume $g_0/g_1 > 0$.

and $\Delta r_{1,2}$, $\Delta u_{1,2}$ are constants defined by

$$\begin{aligned}
\Delta r_1 &= m[\cosh(2\delta_1)\cosh(2\gamma_2) - 1] + n\sinh(2\delta_1)\sinh(2\gamma_1)\,,\\
\Delta r_2 &= m[\cosh(2\delta_2)\cosh(2\gamma_1) - 1] + n\sinh(2\delta_2)\sinh(2\gamma_2)\,,\\
\Delta u_1 &= n[\cosh(2\delta_1)\cosh(2\gamma_2) - 1] - m\sinh(2\delta_1)\sinh(2\gamma_1)\,,\\
\Delta u_2 &= n[\cosh(2\delta_2)\cosh(2\gamma_1) - 1] - m\sinh(2\delta_2)\sinh(2\gamma_2)\,.
\end{aligned} \tag{4.11}$$

Below we shall also use the linear combinations

$$\begin{aligned}
\Sigma_{\Delta r} &= \frac{1}{2}(\Delta r_1 + \Delta r_2)\,, \qquad \Delta_{\Delta r} = \frac{1}{2}(\Delta r_2 - \Delta r_1)\,,\\
\Sigma_{\Delta u} &= \frac{1}{2}(\Delta u_1 + \Delta u_2)\,, \qquad \Delta_{\Delta u} = \frac{1}{2}(\Delta u_2 - \Delta u_1)\,.
\end{aligned} \tag{4.12}$$

The complex scalar field has the very simple form

$$z = \frac{g_0}{g_1}\frac{r_1 - iu_1}{r_2 - iu_2}\,, \tag{4.13}$$

while the gauge fields and their duals read

$$\begin{aligned}
A^1 &= \zeta^1(dt - ad\phi) + \frac{r_2u_2\tilde{\zeta}_1}{a}d\phi\,, \qquad A^2 = \zeta^2(dt - ad\phi) + \frac{r_1u_1\tilde{\zeta}_2}{a}d\phi\,,\\
\tilde{A}_1 &= \tilde{\zeta}_1(dt - ad\phi) - \frac{r_1u_1\zeta^1}{a}d\phi\,, \qquad \tilde{A}_2 = \tilde{\zeta}_2(dt - ad\phi) - \frac{r_2u_2\zeta^2}{a}d\phi\,,
\end{aligned} \tag{4.14}$$

where the three-dimensional electromagnetic scalars are

$$\begin{aligned}
\zeta^1 &= \frac{1}{2W}\frac{\partial W}{\partial\delta_1} = \frac{Q_1r_2 - P^1u_2}{W}\,, \qquad \tilde{\zeta}_1 = \frac{Q_1u_1 + P^1r_1}{W}\,,\\
\zeta^2 &= \frac{1}{2W}\frac{\partial W}{\partial\delta_2} = \frac{Q_2r_1 - P^2u_1}{W}\,, \qquad \tilde{\zeta}_2 = \frac{Q_2u_2 + P^2r_2}{W}\,.
\end{aligned} \tag{4.15}$$

Here, $Q_{1,2}$ and $P^{1,2}$ denote respectively the electric and magnetic charges given by [29]

$$Q_1 = \frac{1}{2}\frac{\partial r_1}{\partial\delta_1}\,, \qquad Q_2 = \frac{1}{2}\frac{\partial r_2}{\partial\delta_2}\,, \qquad P^1 = -\frac{1}{2}\frac{\partial u_1}{\partial\delta_1}\,, \qquad P^2 = -\frac{1}{2}\frac{\partial u_2}{\partial\delta_2}\,. \tag{4.16}$$

The solution is thus specified by the 7 parameters m, n, a, $\gamma_{1,2}$ and $\delta_{1,2}$ that are related to the mass, NUT charge, angular momentum, two electric and two magnetic charges. Notice that a similar class of rotating black holes containing one parameter less was constructed in [30].

Let us now consider the action of S defined in (3.2). For the transformed scalar we get

$$\tilde{z} = \frac{\tilde{X}^1}{\tilde{X}^0} = \frac{g_0}{g_1}\frac{r + \Delta r_1' - i(u + \Delta u_1')}{r + \Delta r_2' - i(u + \Delta u_2')}\,, \tag{4.17}$$

where

$$\begin{pmatrix} \Delta r_1' \\ \Delta r_2' \\ \Delta u_1' \\ \Delta u_2' \end{pmatrix} = \begin{pmatrix} \cos^2\beta & \sin^2\beta & -\cos\beta\sin\beta & \cos\beta\sin\beta \\ \sin^2\beta & \cos^2\beta & \cos\beta\sin\beta & -\cos\beta\sin\beta \\ \cos\beta\sin\beta & -\cos\beta\sin\beta & \cos^2\beta & \sin^2\beta \\ -\cos\beta\sin\beta & \cos\beta\sin\beta & \sin^2\beta & \cos^2\beta \end{pmatrix} \begin{pmatrix} \Delta r_1 \\ \Delta r_2 \\ \Delta u_1 \\ \Delta u_2 \end{pmatrix}. \tag{4.18}$$

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Note that the quantities $\Sigma_{\Delta r}$ and $\Sigma_{\Delta u}$ defined in (4.12) remain invariant under (4.18), while $\Delta_{\Delta r}$ and $\Delta_{\Delta u}$ transform as

$$\begin{pmatrix} \Delta'_{\Delta r} \\ \Delta'_{\Delta u} \end{pmatrix} = \begin{pmatrix} \cos 2\beta & -\sin 2\beta \\ \sin 2\beta & \cos 2\beta \end{pmatrix} \begin{pmatrix} \Delta_{\Delta r} \\ \Delta_{\Delta u} \end{pmatrix} . \tag{4.19}$$

The transformed gauge fields can be easily inferred from

$$\begin{pmatrix} A^1 + A^2 \\ \frac{g_1}{g_0}\tilde{A}_1 + \frac{g_0}{g_1}\tilde{A}_2 \\ A^2 - A^1 \\ \frac{g_0}{g_1}\tilde{A}_2 - \frac{g_1}{g_0}\tilde{A}_1 \end{pmatrix}' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos 2\beta & -\sin 2\beta \\ 0 & 0 & \sin 2\beta & \cos 2\beta \end{pmatrix} \begin{pmatrix} A^1 + A^2 \\ \frac{g_1}{g_0}\tilde{A}_1 + \frac{g_0}{g_1}\tilde{A}_2 \\ A^2 - A^1 \\ \frac{g_0}{g_1}\tilde{A}_2 - \frac{g_1}{g_0}\tilde{A}_1 \end{pmatrix} . \tag{4.20}$$

In conclusion, S adds one more parameter β to the solution of [29].

Under the action of T_α (cf. (3.3)) the scalar z does not change. It turns out that the new gauge fields can again be written in the form (4.14), but with the three-dimensional electromagnetic scalars replaced by

$$\begin{pmatrix} \sqrt{\frac{g_1}{g_0}}\zeta^1 \\ \sqrt{\frac{g_0}{g_1}}\zeta^2 \\ \sqrt{\frac{g_1}{g_0}}\tilde{\zeta}_1 \\ \sqrt{\frac{g_0}{g_1}}\tilde{\zeta}_2 \end{pmatrix} \longmapsto \begin{pmatrix} \cos\alpha & 0 & 0 & \sin\alpha \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ -\sin\alpha & 0 & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} \sqrt{\frac{g_1}{g_0}}\zeta^1 \\ \sqrt{\frac{g_0}{g_1}}\zeta^2 \\ \sqrt{\frac{g_1}{g_0}}\tilde{\zeta}_1 \\ \sqrt{\frac{g_0}{g_1}}\tilde{\zeta}_2 \end{pmatrix} . \tag{4.21}$$

In other words, they transform (up to prefactors) with the same matrix T_α. This invariance can be used to generate additional charges by starting from a given seed. Set e.g. $\gamma_2 = \delta_2 = 0$ in (4.11), which by (4.16) implies $P^2 = Q_2 = 0$. After acting with T_α one gets a solution with all four charges nonvanishing, namely

$$Q'_1 = Q_1 \cos\alpha\,, \qquad P^{1\prime} = P^1 \cos\alpha\,, \qquad Q'_2 = \frac{g_1}{g_0} P^1 \sin\alpha\,, \qquad P^{2\prime} = -\frac{g_1}{g_0} Q_1 \sin\alpha\,.$$

4.2 Prepotential $F = \frac{i}{4}((X^1)^2 + (X^2)^2 - (X^0)^2)$

In this case the most interesting feature of $U_{\rm i}$ is the non-abelianity of $\mathcal{S}_\mathcal{G}$, cf. section 3.2. As far as L_α is concerned, its effect is the same as the one of T_α for $F = -iX^0X^1$, namely the transformed configuration solves non-BPS first-order flow equations.

The nonabelian part acts nontrivially on the special scalars. With the 1-parameter subgroups $\exp(\alpha_\mu R_\mu)$ ($\mu = 0, \ldots, 3$, no summation over μ), where the R_μ are defined in section 3.2, one can describe the action of $\mathcal{S}_\mathcal{G}$ on a static seed solution with charge vector $\mathcal{Q}$ as

$$\begin{aligned} (\mathcal{V}, \mathcal{G}, \mathcal{Q}) \quad &\longmapsto \quad (\tilde{\mathcal{V}}, \tilde{\mathcal{G}}, \tilde{\mathcal{Q}}) = (e^{\alpha_0 R_0}\mathcal{V}, \mathcal{G}, e^{\alpha_0 R_0}\mathcal{Q})\,, \\ \tilde{z}^1 &= \frac{-g_1(g_0 + g_1 z^1 + g_2 z^2) + e^{i\alpha_0}(g_0 g_1 + (g_0^2 - g_2^2)z^1 + g_1 g_2 z^2)}{g_0(g_0 + g_1 z^1 + g_2 z^2) - e^{i\alpha_0}(g_1^2 + g_2^2 + g_0 g_2 z^2 + g_0 g_1 z^1)}\,, \\ \tilde{z}^2 &= \frac{-g_2(g_0 + g_1 z^1 + g_2 z^2) + e^{i\alpha_0}(g_0 g_2 + (g_0^2 - g_1^2)z^2 + g_1 g_2 z^1)}{g_0(g_0 + g_1 z^1 + g_2 z^2) - e^{i\alpha_0}(g_1^2 + g_2^2 + g_0 g_2 z^2 + g_0 g_1 z^1)}\,, \end{aligned}$$

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$$
\begin{aligned}
&(\mathcal{V},\mathcal{G},\mathcal{Q}) \quad \longmapsto \quad (\tilde{\mathcal{V}},\tilde{\mathcal{G}},\tilde{\mathcal{Q}}) = (e^{\alpha_1 R_1}\mathcal{V},\mathcal{G},e^{\alpha_1 R_1}\mathcal{Q})\,,\\
&\tilde{z}^1 = \frac{-g_1(g_0+g_1z^1+g_2z^2)+(g_0g_1+g_0^2z^1-g_2^2z^1+g_1g_2z^2)\cos\alpha_1-\hat{g}(g_2+g_0z^2)\sin\alpha_1}{g_0(g_0+g_1z^1+g_2z^2)-(g_1^2+g_0g_1z^1+g_2^2+g_0g_2z^2)\cos\alpha_1+\hat{g}(g_1z^2-g_2z^1)\sin\alpha_1}\,,\\
&\tilde{z}^2 = \frac{-g_2(g_0+g_1z^1+g_2z^2)+(g_0g_2+g_0^2z^2-g_1^2z^2+g_2g_1z^1)\cos\alpha_1+\hat{g}(g_1+g_0z^1)\sin\alpha_1}{g_0(g_0+g_1z^1+g_2z^2)-(g_1^2+g_0g_1z^1+g_2^2+g_0g_2z^2)\cos\alpha_1+\hat{g}(g_1z^2-g_2z^1)\sin\alpha_1}\,,\\
&(\mathcal{V},\mathcal{G},\mathcal{Q}) \quad \longmapsto \quad (\tilde{\mathcal{V}},\tilde{\mathcal{G}},\tilde{\mathcal{Q}}) = (e^{\alpha_2 R_2}\mathcal{V},\mathcal{G},e^{\alpha_2 R_2}\mathcal{Q})\,,\\
&\tilde{z}^1 = \frac{-g_1(g_0+g_1z^1+g_2z^2)+f(g_1,g_2,z^1,z^2)\cos\alpha_2-h(g_1,g_2,z^1,z^2)\sin\alpha_2}{g_0(g_0+g_1z^1+g_2z^2)-(g_1^2+g_0g_1z^1+g_2(g_2+g_0z^2))\cos\alpha_2+i\hat{g}(g_2z^1-g_1z^2)\sin\alpha_2}\,,\\
&\tilde{z}^2 = \frac{-g_2(g_0+g_1z^1+g_2z^2)+f(g_2,g_1,z^2,z^1)\cos\alpha_2+h(g_2,g_1,z^2,z^1)\sin\alpha_2}{g_0(g_0+g_1z^1+g_2z^2)-(g_1^2+g_0g_1z^1+g_2(g_2+g_0z^2))\cos\alpha_2+i\hat{g}(g_2z^1-g_1z^2)\sin\alpha_2}\,,\\
&(\mathcal{V},\mathcal{G},\mathcal{Q}) \quad \longmapsto \quad (\tilde{\mathcal{V}},\tilde{\mathcal{G}},\tilde{\mathcal{Q}}) = (e^{\alpha_3 R_3}\mathcal{V},\mathcal{G},e^{\alpha_3 R_3}\mathcal{Q})\,,\\
&\tilde{z}^1 = \frac{-g_1(g_1^2+g_2^2)(g_0+g_1z^1+g_2z^2)+e^{i\alpha_3}k(g_1,g_2,z^1,z^2)+e^{-i\alpha_3}g_2\hat{g}^2(g_2z^1-g_1z^2)}{(g_1^2+g_2^2)\left(g_0(g_0+g_1z^1+g_2z^2)-e^{i\alpha_3}(g_1^2+g_0g_1z^1+g_2^2+g_0g_2z^2)\right)}\,,\\
&\tilde{z}^2 = \frac{-g_2(g_1^2+g_2^2)(g_0+g_1z^1+g_2z^2)+e^{i\alpha_3}k(g_2,g_1,z^2,z^1)+e^{-i\alpha_3}g_1\hat{g}^2(g_1z^2-g_2z^1)}{(g_1^2+g_2^2)\left(g_0(g_0+g_1z^1+g_2z^2)-e^{i\alpha_3}(g_1^2+g_0g_1z^1+g_2^2+g_0g_2z^2)\right)}\,,
\end{aligned}
$$

where we used the definitions

$$
\begin{aligned}
&\hat{g} = \sqrt{g_0^2-g_1^2-g_2^2}\,, \qquad f(g_1,g_2,z^1,z^2) = g_0g_1+g_0^2z^1+g_1g_2z^2-g_2^2z^1\,,\\
&h(g_1,g_2,z^1,z^2) = \frac{i\hat{g}}{g_1^2+g_2^2}(2g_0g_1g_2z^1+g_1^2(g_2-g_0z^2)+g_2^2(g_2+g_0z^2))\,,\\
&k(g_1,g_2,z^1,z^2) = g_0g_1(g_1^2+g_0g_1z^1+g_2^2+g_0g_2z^2)\,.
\end{aligned}
\tag{4.22}
$$

The explicit expressions for $\tilde{\mathcal{Q}}$ are not particularly enlightening, so we don't report them here. One may apply the above transformations to the static and magnetic BPS seed given by eqs. (3.100) and (3.101) of [22] to generate dyonic and axionic solutions.

Note that the form of (3.27) splits the dependence of the group coordinates from the couplings. Defining the section $\mathcal{V}_{\underline{g}} = (\boldsymbol{X}_{\underline{g}},\boldsymbol{F}_{\underline{g}})^t \equiv \tilde{\Lambda}_{\underline{g}}\mathcal{V}$, the action of $\mathcal{S}_{\mathcal{G}}$ becomes $\tilde{\mathcal{V}}_{\underline{g}} = S_0(x^0,\vec{x})\mathcal{V}_{\underline{g}}$ that more explicitly reads

$$
\tilde{\boldsymbol{X}}_{\underline{g}} = \begin{pmatrix} X^0_{\underline{g}} \\ e^{ix^0}\left(X^1_{\underline{g}}\cos|\vec{x}| + i((x^1+ix^2)X^2_{\underline{g}}+ix^3X^1_{\underline{g}})\sin|\vec{x}|\right) \\ e^{ix^0}\left(X^2_{\underline{g}}\cos|\vec{x}| + i((x^1-ix^2)X^1_{\underline{g}}-ix^3X^2_{\underline{g}})\sin|\vec{x}|\right) \end{pmatrix}\,. \tag{4.23}
$$

This split is independent of the parametrization of the group and so one can also use that of [19, 21].

4.3 Prepotential $F = -X^1X^2X^3/X^0$

This model is related to the one with $F = -2i(X^0X^1X^2X^3)^{1/2}$ by a symplectic rotation with the matrix (B.3). As a seed solution we shall thus take the static magnetic BPS black holes given by eqs. (3.31)-(3.34) of [22], transformed to $F = -X^1X^2X^3/X^0$. In this new frame, the vectors of charges and couplings are respectively given by

$$
\mathcal{Q} = (p^0,0,0,0,0,q_1,q_2,q_3)^t\,, \qquad \mathcal{G} = (0,g^1,g^2,g^3,g_0,0,0,0)^t\,. \tag{4.24}
$$

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Assuming $g_0 g^1 g^2 g^3 < 0$ and defining $A \equiv (-g_0 g^1 g^2 g^3)^{1/2}$, the finite transformations $\exp(\alpha_1 \mathcal{U}_1)$ and $\exp(\alpha_2 \mathcal{U}_2)$ generated by (3.34) act as

$$\begin{aligned}
&(\mathcal{V}, \mathcal{G}, \mathcal{Q}) \quad \longmapsto \quad (\tilde{\mathcal{V}}, \tilde{\mathcal{G}}, \tilde{\mathcal{Q}}) = (e^{\alpha_1 \mathcal{U}_1} \mathcal{V}, \mathcal{G}, e^{\alpha_1 \mathcal{U}_1} \mathcal{Q}) \,, \\
&\tilde{z}^1 = \frac{A z^1 \cos(A\alpha_1) + g_0 g^1 \sin(A\alpha_1)}{A \cos(A\alpha_1) + z^1 g^2 g^3 \sin(A\alpha_1)} \,, \\
&\tilde{z}^2 = z^2 \,, \\
&\tilde{z}^3 = \frac{A z^3 \cos(A\alpha_1) - g_0 g^3 \sin(A\alpha_1)}{A \cos(A\alpha_1) - z^3 g^1 g^2 \sin(A\alpha_1)} \,,
\end{aligned} \tag{4.25}$$

$$\begin{aligned}
&(\mathcal{V}, \mathcal{G}, \mathcal{Q}) \quad \longmapsto \quad (\tilde{\mathcal{V}}, \tilde{\mathcal{G}}, \tilde{\mathcal{Q}}) = (e^{\alpha_2 \mathcal{U}_2} \mathcal{V}, \mathcal{G}, e^{\alpha_2 \mathcal{U}_2} \mathcal{Q}) \,, \\
&\tilde{z}^1 = z^1 \,, \\
&\tilde{z}^2 = \frac{A z^2 \cos(A\alpha_2) + g_0 g^2 \sin(A\alpha_2)}{A \cos(A\alpha_2) + z^2 g^1 g^3 \sin(A\alpha_2)} \,, \\
&\tilde{z}^3 = \frac{A z^3 \cos(A\alpha_2) - g_0 g^3 \sin(A\alpha_2)}{A \cos(A\alpha_2) - z^3 g^1 g^2 \sin(A\alpha_2)} \,.
\end{aligned} \tag{4.26}$$

Again, the expressions for $\tilde{\mathcal{Q}}$ are not particularly enlightening, so we shall not report them here. Notice that the transformations (4.25), (4.26) preserve the supersymmetry of the seed.

As we pointed out in section 3.3, in the special case $\mathcal{G} = (0, g, g, g, -g, 0, 0, 0)^t$ there is an enhancement of the symmetry group to $\mathrm{U}(1)^3$ generated by (3.38). If we define $T = \exp[\frac{\alpha_3}{3}(W_1 + W_2 + W_3)]$, the action of the extra U(1) is

$$\begin{aligned}
&(\mathcal{V}, \mathcal{G}, \mathcal{Q}) \quad \longmapsto \quad (\tilde{\mathcal{V}}, \tilde{\mathcal{G}}, \tilde{\mathcal{Q}}) = (T\mathcal{V}, \mathcal{G}, T\mathcal{Q}) \,, \\
&\tilde{z}^1 = \frac{z^1 \cos\alpha_3 - \sin\alpha_3}{z^1 \sin\alpha_3 + \cos\alpha_3} \,, \\
&\tilde{z}^2 = \frac{z^2 \cos\alpha_3 - \sin\alpha_3}{z^2 \sin\alpha_3 + \cos\alpha_3} \,, \\
&\tilde{z}^3 = \frac{z^3 \cos\alpha_3 - \sin\alpha_3}{z^3 \sin\alpha_3 + \cos\alpha_3} \,,
\end{aligned} \tag{4.27}$$

plus an expression for the charges $\tilde{\mathcal{Q}}$. (4.25), (4.26) and (4.27) where first obtained in [10]. Note that T breaks supersymmetry, since it does not belong to the stabilizer $\mathcal{S}_{\mathcal{G}}$. In fact,

$$T\mathcal{G} \equiv \mathcal{G}_{\alpha_3} = g(\sin\alpha_3, \cos\alpha_3, \cos\alpha_3, \cos\alpha_3, -\cos\alpha_3, \sin\alpha_3, \sin\alpha_3, \sin\alpha_3)^t \,. \tag{4.28}$$

However, the transformed solution still satisfies first-order non-BPS flow equations driven by the fake superpotential [18][12]

$$W = e^U |\langle \tilde{\mathcal{Q}}, \tilde{\mathcal{V}} \rangle - i e^{2(\psi - U)} \langle \mathcal{G}_{\alpha_3}, \tilde{\mathcal{V}} \rangle| \,, \tag{4.29}$$

provided the charge quantization condition $\langle \mathcal{G}, \mathcal{Q} \rangle = -\kappa$ holds, where $\kappa = 0, 1, -1$ for flat, spherical or hyperbolic horizons respectively.

[12] Notice that this flow is a BPS flow for a theory with gaugings given by $\mathcal{G}_{\alpha_3}$.

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4.4 Prepotential $F = X^1X^2X^3/X^0 + \frac{g^1g^2}{3(g^3)^2}(X^3)^3/X^0$

In this case the only known solution with running scalars is that of [25], with static metric and purely imaginary scalar fields,

$$X^1/X^0 = z^1 = -i\lambda^1\,, \qquad X^2/X^0 = z^2 = -i\lambda^2\,, \qquad X^3/X^0 = z^3 = -i\lambda^3\,. \tag{4.30}$$

The charges and coupling constants are given by

$$\mathcal{Q} = (p^0, 0, 0, 0, 0, q_1, q_2, q_3)^t\,, \qquad \mathcal{G} = (0, g^1, g^2, g^3, g_0, 0, 0, 0)^t\,. \tag{4.31}$$

Applying the finite transformation generated by (3.43) yields for the scalars

$$\tilde{z}^1 = -i\lambda^1 - \frac{g^1}{g^3}c\,, \qquad \tilde{z}^2 = -i\lambda^2 - \frac{g^2}{g^3}c\,, \qquad \tilde{z}^2 = -i\lambda^3 + c\,, \tag{4.32}$$

and for the charges

$$\tilde{\mathcal{Q}} = \begin{pmatrix} p^0 \\ -(cg^1p^0)/g^3 \\ -(cg^2p^0)/g^3 \\ cp^0 \\ -(4c^3g^1g^2p^0)/(3g^{3^2}) + (g^1q_1 + g^2q_2 - g^3q_3)/g^3 \\ q_1 - c^2g^2p^0/g^3 \\ q_2 - c^2g^1p^0/g^3 \\ q_3 + 2c^2g^1g^2p^0/g^{3^2} \end{pmatrix}\,, \tag{4.33}$$

where c is a group parameter. This solution is again BPS but has also nontrivial (constant) axions turned on and all charges are nonvanishing.

5 Extension to hypermultiplets

In this section we briefly comment on a possible generalization of our work to include also hypermultiplets. In this case the situation is more involved, since the coupling constants are replaced by the moment maps $\mathcal{P}^x$. However, when only abelian isometries of the quaternionic hyperscalar target space are gauged, the scalar potential can be cast into the form [31]

$$V = \mathbb{G}^{AB}\mathbb{D}_A\mathcal{L}\,\mathbb{D}_B\bar{\mathcal{L}} - 3|\mathcal{L}|^2\,, \tag{5.1}$$

where we defined

$$\mathbb{G}^{AB} = \begin{pmatrix} g^{i\bar{\jmath}} & 0 \\ 0 & h^{uv} \end{pmatrix}\,, \quad \mathbb{D}_A = \begin{pmatrix} D_i \\ \mathsf{D}_u \end{pmatrix}\,, \quad \mathcal{L} = \mathcal{Q}^x\mathcal{W}^x, \quad \mathcal{Q}^x = \langle\mathcal{P}^x, \mathcal{Q}\rangle\,, \quad \mathcal{W}^x = \langle\mathcal{P}^x, \mathcal{V}\rangle\,.$$

Here, h_{uv} denotes the metric on the quaternionic manifold, and D_u is the covariant derivative acting on the hyperscalars.

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The most general symmetry transformation of the nonlinear sigma model is a linear combination of the isometries of the quaternionic and the special Kähler manifold. Let us define the formal operator

$$\delta = k^u \mathsf{D}_u + U\mathcal{V}\frac{\delta}{\delta\mathcal{V}} + U\bar{\mathcal{V}}\frac{\delta}{\delta\bar{\mathcal{V}}} + U\mathcal{A}_\mu\frac{\delta}{\delta\mathcal{A}_\mu} + k^i\partial_i + k^{\bar{\imath}}\partial_{\bar{\imath}}\,, \tag{5.2}$$

where k^u is a Killing vector of the quaternionic manifold, U an element of the U-duality algebra, k^i the corresponding holomorphic special Kähler Killing vector, and $\mathcal{A}_\mu$ is the symplectic vector of the gauge potentials [31]. Then it is clear from (5.1) that a sufficient condition for $\delta V = 0$ is $\delta\mathcal{L} = 0$,[13] that holds if and only

$$k^u \mathsf{D}_u \hat{\mathcal{P}}^x = U\hat{\mathcal{P}}^x\,, \tag{5.3}$$

where we added a hat to the quaternionic quantities that define the gaugings. Moreover the invariance of the kinetic term of the hyperscalars [11] leads to

$$(\mathcal{L}_k\hat{k})^v = U\hat{k}^v\,. \tag{5.4}$$

After choosing a specific model, these equations can in principle be solved for the parameters that define the linear combination of Killing vectors (5.2). In practice, (5.3) and (5.4) represent a highly constrained and very model-dependent system, and it is a priori not guaranteed that a nontrivial solution exists in general. In the FI limit, (5.3) boils down to the stabilization equation for the coupling constants $\mathcal{G}$ and (5.4) is trivially satisfied, as it must be.

An interesting class of these models are the $N = 2$ truncations of M-theory described in [32, 33]. In this case the solution of (5.3) and (5.4) could simplify the study of the attractor equations [31], necessary to work along the lines of [34], namely to compare the gravity side with the recent field theory results of [35–37].

6 Conclusions

In this paper we presented a geometric method to determine the residual symmetries in $N = 2$, $d = 4$ U(1) Fayet-Iliopoulos gauged supergravity. It involves the stabilization of the symplectic vector of gauge couplings, i.e., the FI parameters, under the action of the U-duality symmetry of the ungauged theory. We then applied this to obtain the surviving symmetry group for a number of prepotentials frequently used in the string theory literature, and showed how this group can be used to produce hairy and dyonic black holes from a given seed solution. Moreover, we pointed out how our method may be extended to a more general setting including also gauged hypermultiplets.

It would be very interesting to combine our results with dimensional reduction or oxidation as a solution-generating technique much like in the ungauged case discussed in the introduction. For instance one might think of starting from five-dimensional $N = 2$ gauged supergravity coupled to vector multiplets and then reduce to $d = 4$ along a Killing

[13] Note that, as in the FI case, $\delta\mathcal{L} = 0$ is in general sufficient but not necessary.

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direction to get one of the models discussed here. One can then apply the residual symmetry group of the four-dimensional theory and subsequently lift back to $d = 5$ to generate new solutions. Notice that, for a timelike dimensional reduction, the scalar manifold of the resulting Euclidean four-dimensional theory is para-Kähler rather than Kähler [38], so that our results can not be applied straightforwardly, but require some modifications. Another direction for future work could be to reduce gauged supergravity theories to three dimensions and study in general the surviving symmetry preserved by the scalar potential. Work along these directions is in progress [39].

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A Reparametrization and invariances

A symplectic reparametrization of the section $\mathcal{V}$ for a prepotential $F = F(X)$ is a transformation

$$\mathcal{V} = (X^\Lambda, F_\Lambda)^t \longmapsto \tilde{\mathcal{V}} = (\tilde{X}^\Lambda, \tilde{F}_\Lambda)^t \,. \tag{A.1}$$

In the new frame a prepotential does not necessarily exist. We are interested in the subgroup of $\mathrm{Sp}(2n_\mathrm{V} + 2, \mathbb{R})$ that leaves the prepotential invariant [40–42],

$$F(\tilde{X}) = \tilde{F}(\tilde{X}) \,. \tag{A.2}$$

Its algebra is determined by the equation

$$X^\Lambda S_{\Lambda\Sigma} X^\Sigma - F_\Lambda R^{\Lambda\Sigma} F_\Sigma - 2X^\Lambda Q^t{}_\Lambda{}^\Sigma F_\Sigma = 0 \,, \tag{A.3}$$

where Q, R and S parametrize the symplectic algebra,

$$U = \begin{pmatrix} Q & R \\ S & -Q^t \end{pmatrix} , \qquad R = R^t \,, \qquad S = S^t \,.$$

A reparametrization of this type, in special projective coordinates, leaves $\mathcal{V}$ invariant up to a Kähler transformation.

B Symplectic embedding

The choice of the symplectic embedding of the non-linear sigma model isometry group is necessary to completely specify the special Kähler structure over a manifold [11, 20, 23, 40, 41]. In what follows we shall summarize some properties used in the bulk of our paper.

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B.1 Symplectically equivalent embeddings

The way in which the isometry group is embedded in the symplectic group is fixed by supersymmetry, and in particular for $\mathrm{SU}(1,n_{\mathrm{V}})/(\mathrm{U}(1)\times\mathrm{SU}(n_{\mathrm{V}}))$ and $\mathrm{SU}(1,1)/\mathrm{U}(1)\times\mathrm{SO}(2,2)/(\mathrm{SO}(2)\times\mathrm{SO}(2))$ one has respectively [23]

$$(\mathbf{n_V}+\mathbf{1})\oplus(\mathbf{n_V}+\mathbf{1}) \qquad \text{and} \qquad \mathbf{2}\otimes(\mathbf{4}\oplus\mathbf{4})\,. \tag{B.1}$$

This embedding is not unique since one can always act by conjugation with a symplectic matrix to construct a symplectically equivalent embedding. There are choices for the section $\mathcal{V}$ such that the isometry group sits in the symplectic group in a simple way, but the existence of a prepotential in that frame is in general not guaranteed. On the other hand, many symplectically equivalent embeddings are encoded by different prepotentials. Two physically interesting examples are [43, 44]

$$\mathcal{S}_1=\begin{pmatrix} 1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & 0 & \frac{1}{2} & -\frac{1}{2}\end{pmatrix}, \qquad -iX^0X^1\longmapsto\frac{i}{4}(X^{1^2}-X^{0^2})\,, \tag{B.2}$$

$$\mathcal{S}_2=\begin{pmatrix} 1 & 0 & 0 & 0 & 0&0&0&0\\ 0 & 0 & 0 & 0 & 0&1&0&0\\ 0 & 0 & 0 & 0 & 0&0&1&0\\ 0 & 0 & 0 & 0 & 0&0&0&1\\ 0 & 0 & 0 & 0 & 1&0&0&0\\ 0 & -1 & 0 & 0 & 0&0&0&0\\ 0 & 0 & -1 & 0 & 0&0&0&0\\ 0 & 0 & 0 & -1 & 0&0&0&0\end{pmatrix}, \qquad -\frac{X^1X^2X^3}{X^0}\longmapsto -2i\sqrt{X^0X^1X^2X^3}\,. \tag{B.3}$$

A physically less important transformation, which is nevertheless useful for practical purposes, is for instance

$$\mathcal{S}_a=\begin{pmatrix} a & 0\\ 0 & \frac{1}{a}\end{pmatrix}, \qquad \frac{i}{4}X^{\Lambda}\eta_{\Lambda\Sigma}X^{\Sigma}\longmapsto\frac{i}{4a^2}X^{\Lambda}\eta_{\Lambda\Sigma}X^{\Sigma}\,. \tag{B.4}$$

One can also construct inequivalent embeddings over the same manifold, the simplest example being $\mathrm{SU}(1,1)/\mathrm{U}(1)$ [23]. Notice finally that symplectic equivalence does not mean physical equivalence. Even if it is possible to construct maps between the solutions of symplectically equivalent models, in general the solutions are physically different.

B.2 Special Kähler structure over $\mathrm{SU}(1,n_{\mathrm{V}})/(\mathrm{U}(1)\times\mathrm{SU}(n_{\mathrm{V}}))$

For this noncompact version of $\mathbb{C}\mathrm{P}^n$ a simple way to embed $\mathrm{SU}(1,n_{\mathrm{V}})$ into $\mathrm{Sp}(2n_{\mathrm{V}}+2,\mathbb{R})$ is obtained from the fact that

$$\mathrm{Sp}(2n_{\mathrm{V}}+2,\mathbb{R})\cong\mathrm{Usp}(1+n_{\mathrm{V}},1+n_{\mathrm{V}})=\mathrm{Sp}(2n_{\mathrm{V}}+2,\mathbb{C})\cap\mathrm{U}(1+n_{\mathrm{V}},1+n_{\mathrm{V}})\,. \tag{B.5}$$

This isomorphism is provided by conjugation with the Cayley matrix,

$$C_\alpha:\ \mathrm{Sp}(2n_{\mathrm{V}}+2,\mathbb{R})\longrightarrow\mathrm{Usp}(1+n_{\mathrm{V}},1+n_{\mathrm{V}})\,, \qquad U\longmapsto\hat{\mathcal{C}}_\alpha U\hat{\mathcal{C}}_\alpha^{-1}\,, \tag{B.6}$$

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where

$$\hat{\mathcal{C}}_\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{\alpha}} I_{n_V+1} & i\sqrt{\alpha}\eta \\ \frac{1}{\sqrt{\alpha}} I_{n_V+1} & -i\sqrt{\alpha}\eta \end{pmatrix} , \tag{B.7}$$

and η is the Minkowski metric in $n_V + 1$ dimensions. In fact $\mathrm{Usp}(1+n_V, 1+n_V)$ is defined by the conditions

$$\mathcal{U}\mathbb{H}\mathcal{U}^\dagger = \mathbb{H} , \qquad \mathcal{U}\tilde{\Omega}\mathcal{U}^t = \tilde{\Omega} . \tag{B.8}$$

If the invariant bilinear forms are chosen as

$$\mathbb{H} = \begin{pmatrix} \eta & 0 \\ 0 & -\eta \end{pmatrix} , \qquad \tilde{\Omega} = \begin{pmatrix} 0 & -\eta \\ \eta & 0 \end{pmatrix} , \tag{B.9}$$

(B.8) becomes

$$\mathcal{U} = \begin{pmatrix} A & C^* \\ C & A^* \end{pmatrix} , \qquad A\eta A^\dagger - C^*\eta C^t = \eta , \qquad A^*\eta C^t - C\eta A^\dagger = 0 . \tag{B.10}$$

The first of (B.1) is obtained by restricting the action of $\iota_\alpha \equiv C_\alpha^{-1}$ to the subgroup with $C = 0$. One can also explicitly verify that in this frame the prepotential exists and is given by $F = -\frac{i}{2\alpha} X^\Lambda \eta_{\Lambda\Sigma} X^\Sigma$.

B.3 Special Kähler structure over $\mathbf{SU(1,1)/U(1) \times SO(2,2)/(SO(2) \times SO(2))}$

This manifold belongs to the infinite sequence $\mathrm{SU}(1,1)/\mathrm{U}(1) \times \mathrm{SO}(2,n)/(\mathrm{SO}(2) \times \mathrm{SO}(n))$, which for $n = 2$ is isomorphic to $(\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2))^3$. To find the symplectic embedding it is useful to choose a frame [23, 45–47] in which the symplectic section cannot be integrated to have a prepotential. In this frame the Calabi-Visentini parametrization appears in a natural way. The embedding problem is solved by

$$\mathrm{SO}(2,2) \ni L \longmapsto \begin{pmatrix} L & 0 \\ 0 & L^{-1t} \end{pmatrix} \in \mathrm{Sp}(8,\mathbb{R}) , \tag{B.11}$$

$$\mathrm{SL}(2,\mathbb{R}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & b\hat{\eta} \\ c\hat{\eta} & d \end{pmatrix} \in \mathrm{Sp}(8,\mathbb{R}) , \tag{B.12}$$

where $\hat{\eta}$ is the metric preserved by $\mathrm{SO}(2,2)$. A symplectic transformation that leads to a frame in which a prepotential exists is highly nontrivial to find [23].

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