UNIVERSITÀ DEGLI STUDI DI MILANO
SCUOLA DI DOTTORATO IN SCIENZE MATEMATICHE
DIPARTIMENTO DI MATEMATICA F. ENRIQUES

CORSO DI DOTTORATO DI RICERCA IN MATEMATICA
XXIX CICLO
TESI DI DOTTORATO DI RICERCA

Long time dynamics of the Klein-Gordon equation in the non-relativistic limit

settore scientifico disciplinare: MAT/07

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A.A. 2016-2017

To my parents.

Ringraziamenti

Ringrazio innanzitutto il mio relatore, il professor Dario Bambusi, per la sua guida durante il mio dottorato, per il tempo impiegato in numerose discussioni tutte le volte in cui ho avuto bisogno, e per la sua pazienza. Lo ringrazio anche per avermi insegnato il rigore, la pazienza e la dedizione necessari per diventare un matematico migliore rispetto a quando ho iniziato.

Uno speciale ringraziamento va ai miei genitori, Claudio e Franca. Non smetterò mai di ringraziarli abbastanza per avermi cresciuto ed educato nel migliore dei modi, e spero che siano orgogliosi del loro figlio.

Un ringraziamento anche alla mia famiglia: allo zio Paolo e a Cristiane, allo zio Mauro, allo zio Gianni, al cugino Erik e a tutta la sua famiglia ... vi ringrazio per il vostro supporto durante la mia avventura milanese.

Un ringraziamento va ai dottorandi dei vari cicli che ho incontrato in via Saldini in tutti questi anni, ed in particolare a quelli con cui ho condiviso lo studio: un grazie a Lara, che mi ha dato i primi consigli su come muovermi in dipartimento; un grazie a Matteo, campione di buone maniere e di eleganza; un grazie a Luca, per il suo sobrio buonumore e per aver portato in studio il Bang; un grazie a Francesco, per avermi mostrato come si può sopportare stoicamente la lontananza dal sole, dal mare e dai cibi buoni; un grazie a Cinzia, iron woman forgiata dal thriatlon e letale praticante di Krav Maga; un grazie a Lorenzo, mascotte dello studio ex-MASSC, per averci insegnato il verso del Manzoni e per averci istruito a non essere politicamente corretti; un grazie a Pietro, campione indiscusso nel tiro da tre; un grazie a Giulio, sempre pronto a consigliare mostre e spettacoli teatrali; un grazie a Tommaso, rappresentante che, pur stando all'estero, spesso era più aggiornato di noi che stavamo lì a Milano. Un grazie a Davide, con cui ho imperversato a Ingegneria Gestionale. Un grazie ai miei fratellini accademici, Alessandra e Luca, con i quali ho condiviso interessi e conferenze. E ancora un grazie a Guglielmo, a Federico, a Davide, a Claudia, ad Andrea, a Jacopo e a Claudia, a Mattia e a Laura, a Francesco, a Simone e a tutti più i giovani che ho conosciuto negli ultimi mesi. Un grazie anche ad Alberto e a Marco, che hanno più volte pranzato con i più giovani.

Un grazie a Roberta e a Chiara, che ci sono più volte venute a trovare quando erano studentesse; un grazie a Beatrice, Costanza, Eleonora, Francesco, Luca, Mai e tutti gli altri studenti del tavolo al primo piano che mi mettevano di buonumore ogni volta che passavo per di là. E un grazie di cuore alle fanciulle polemiche che non ho nominato in precedenza.

Vorrei ringraziare anche tutti i ragazzi e le ragazze con cui ho condiviso numerose conferenze in Italia e all'estero, per i bei momenti trascorsi insieme: un grazie ad Alberto, ad Annalaura, a Biagio, a Chiara, ad Emanuele, a Felice, a Filippo, a Jessica, a Livia, a Paolo, a Raffaele, a Riccardo, a Roberto e a tutti gli altri.

Non dimenticherò mai i miei coinquilini della Maison, ragazzi e ragazze che mi hanno sopportato e supportato durante la mia permanenza a Milano: un grazie a Francesca, ad Andrea, a Cristiano, a Riccardo, a Mattia e a tutti gli altri.

Un grazie agli amici e alle amiche conosciuti quando ero studente universitario, perché riusciamo a trascorrere dei bei momenti insieme ogni volta che ci ritroviamo, anche se le nostre vite ci fanno intraprendere strade diverse.

E un grazie a tutti gli amici e le amiche conosciuti a Milano in frangenti non accademici, per avermi fatto compagnia in questi ultimi anni. Il buon ricordo che avrò di questo triennio è anche merito vostro.

Contents

1 Introduction **1**
- 1.0.1 The NLKG equation on $\mathbb{R}^3$ . . . 1
- 1.0.2 The case of $x \in [0, \pi]$ . . . 7

2 Dispersive estimates **9**
- 2.1 Dispersive properties of the Klein-Gordon equation . . . 9
- 2.2 Dispersive properties of the Spinless Salpeter Equation with a potential . . . 14
- 2.3 Dispersive properties of the Klein-Gordon equation with a potential . . . 22

3 Galerkin Averaging **25**
- 3.1 Galerkin Averaging Method . . . 25
- 3.2 Proof of Theorem 3.1.3 . . . 28
- 3.3 Application to the nonlinear Klein-Gordon equation . . . 33
  - 3.3.1 The real nonlinear Klein-Gordon equation . . . 33
  - 3.3.2 The complex nonlinear Klein-Gordon equation . . . 37

4 Approximation of the NLKG equation in the non-relativistic limit **41**
- 4.1 Dynamics . . . 41
- 4.2 Long time approximation . . . 43

5 The non-relativistic limit of KG equation on $\mathbb{T}$ **53**
- 5.1 The NLKG Equation with a potential . . . 54
- 5.2 Proof of Theorem 5.1.1 . . . 60

A Proof of Lemma 3.2.3 **69**

B Properties of the Klein-Gordon and Spinless Salpeter equations **75**
- B.1 Formula for the action of the $e^{itc\langle\nabla\rangle_c}$ . . . 75
- B.2 Resolvent estimates . . . 76
  - B.2.1 Free resolvent . . . 76
  - B.2.2 Asymptotic behavior of the resolvent of $\mathcal{H}_0$ . . . 82
  - B.2.3 Resolvent of equation (4.1.6) with a time-independent potential . . . 89

C Interpolation theory for relativistic Sobolev spaces **95**

D Analytical tools **101**
D.1 Fourier Transform 101
D.2 Symbols 103
D.3 Action of pseudo-differential operators in $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ 106
D.4 Adjoints and products of pseudo-differential operators 109
D.5 Action of pseudo-differential operators on Sobolev spaces 111
D.6 L^p estimates 113
D.7 Pseudodifferential operators on a manifold 115

Bibliography **118**

Abstract

In this thesis I study the non-relativistic limit ($c \to \infty$) of the nonlinear Klein-Gordon (NLKG) equation on a manifold M, namely

$$\frac{1}{c^2}u_{tt} - \Delta u + c^2 u + \lambda |u|^{2(l-1)}u = 0, \quad t \in \mathbb{R}, x \in M \tag{0.0.1}$$

where $\lambda = \pm 1$, $l \geq 2$. The aim of the present work is to discuss the convergence of solutions of the NLKG to solutions of a suitable nonlinear Schrödinger (NLS) equation, and to study whether such convergence may hold for large (namely, of size $\mathcal{O}(c^r)$ with $r \geq 1$) timescales.

In particular I obtain the following results: (1) when M is a general manifold, I show that the solution of NLS describes well the solution of the original equation up to times of order $\mathcal{O}(1)$; (2) when $M = \mathbb{R}^d$, $d \geq 3$, I consider higher order approximations of NLKG and prove that small radiation solutions of the approximating equation describe well solutions of NLKG up to times of order $\mathcal{O}(c^{2r})$ for any $r \geq 1$; (3) when $M = [0, \pi] \subset \mathbb{R}$ I consider the NLKG equation with a convolution potential and prove existence for long times of solutions in H^s uniformly in c, which however has to belong to a set of large measure.

I also get some new dispersive estimates for a Klein Gordon type equation with a potential.

Riassunto della Tesi

In questa tesi si studia il limite non-relativistico ($c \to \infty$) dell'equazione di Klein-Gordon non lineare (NLKG) su una varietà M,

$$\frac{1}{c^2} u_{tt} - \Delta u + c^2 u + \lambda |u|^{2(l-1)} u = 0, \quad t \in \mathbb{R}, x \in M \tag{0.0.2}$$

dove $\lambda = \pm 1$, $l \geq 2$. L'obiettivo del presente lavoro è di discutere la convergenza delle soluzioni della NLKG alle soluzioni di un'opportuna equazione di Schrödinger non lineare (NLS), e di studiare quando questa convergenza possa valere su scale di tempo lunghe (più precisamente, dell'ordine di $\mathcal{O}(c^r)$, $r \geq 1$).

In particolare si possono ottenere i seguenti risultati: (1) quando M è una generica varietà, si può mostrare che le soluzioni della NLS approssimano bene le soluzioni dell'equazione originale, fino a tempi dell'ordine di $\mathcal{O}(1)$; (2) quando $M = \mathbb{R}^d$, $d \geq 3$, considerando approssimazioni ad ordini più alti di NLKG si può mostrare che le soluzioni di radiazioni piccole dell'equazione approssimata approssimano soluzioni di NLKG fino a tempi dell'ordine di $\mathcal{O}(c^{2r})$ per ogni $r \geq 1$; (3) quando $M = [0, \pi] \subset \mathbb{R}$, considerando la NLKG con un potenziale di convoluzione, si può dimostrare l'esistenza per tempi lunghi delle soluzioni in H^s uniformemente in c, che deve però appartenere ad un insieme di misura grande.

Si sono inoltre dimostrate delle stime dispersive per un'equazione di tipo Klein-Gordon con potenziale.

Chapter 1

Introduction

In this thesis we consider the nonrelativistic limit (namely, the limit in which the speed of light $c \to \infty$) of the nonlinear Klein-Gordon (NLKG) equation. Formal computations going back to the first half of the last century suggest that, up to corrections of order $\mathcal{O}(c^{-2})$, the system should be described by the nonlinear Schrödinger (NLS) equation. Subsequent mathematical results have shown that the NLS describes the dynamics over time scales of order $\mathcal{O}(1)$.

In the present thesis we obtain some results for the dynamics of NLKG over longer time scales. Actually we get two kinds of results: (i) results for NLKG uniform as $c \to \infty$ and (ii) approximation results showing that solutions of NLKG can be approximated by solutions of suitable higher order NLS equations.

The theory is completely different in the case where the equation lives on $\mathbb{R}^3$ or in a compact manifold. We are now going to present the results splitting these two cases.

1.0.1 The NLKG equation on $\mathbb{R}^3$

The NLKG equation describes the motion of a spinless particle with mass $m > 0$. Consider first the real NLKG

$$\frac{\hbar^2}{2mc^2} u_{tt} - \frac{\hbar^2}{2m}\Delta u + \frac{mc^2}{2} u + \lambda |u|^{2(l-1)} u = 0, \tag{1.0.1}$$

where $c > 0$ is the speed of light, $\hbar > 0$ is the Planck constant, $\lambda \in \mathbb{R}$, $l \geq 2$, $c > 0$.
In the following we will take $m = 1$, $\hbar = 1$. As anticipated above, we are interested in the behaviour of solutions as $c \to \infty$.

First it is convenient to reduce equation (1.0.1) to a first order system, by making the following symplectic change variables

$$\psi := \frac{1}{\sqrt{2}} \left[\left(\frac{\langle \nabla \rangle_c}{c} \right)^{1/2} u - i \left(\frac{c}{\langle \nabla \rangle_c} \right)^{1/2} u_t \right].$$

where

$$\langle \nabla \rangle_c := (c^2 - \Delta)^{1/2}, \tag{1.0.2}$$

which reduces (1.0.1) to the form

$$-i\psi_t = c\langle\nabla\rangle_c\psi + \frac{\lambda}{2^l}\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2}\left[\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2}(\psi+\bar{\psi})\right]^{2l-1}, \tag{1.0.3}$$

which is hamiltonian with Hamiltonian function given by

$$H(\bar{\psi},\psi) = \langle\bar{\psi}, c\langle\nabla\rangle_c\psi\rangle + \frac{\lambda}{2l}\int\left[\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2}\frac{\psi+\bar{\psi}}{\sqrt{2}}\right]^{2l}\mathrm{d}x. \tag{1.0.4}$$

To state our first result, we introduce for any $k \in \mathbb{R}$ and for any $1 < p < \infty$ the following relativistic Sobolev spaces

$$\mathscr{W}_c^{k,p}(\mathbb{R}^3) := \left\{u \in L^p : \|u\|_{\mathscr{W}_c^{k,p}} := \left\|c^{-k}\langle\nabla\rangle_c^k u\right\|_{L^p} < +\infty\right\}, \tag{1.0.5}$$

$$\mathscr{H}_c^{k}(\mathbb{R}^3) := \left\{u \in L^2 : \|u\|_{\mathscr{H}_c^{k}} := \left\|c^{-k}\langle\nabla\rangle_c^k u\right\|_{L^2} < +\infty\right\}, \tag{1.0.6}$$

and remark that the energy space is $\mathscr{H}_c^{1/2}$. We also remark that for finite $c > 0$ such spaces coincide with the standard Sobolev spaces, while for $c = \infty$ they are equivalent to the Lebesgue spaces L^p.

We first begin with a global existence result for the NLKG (1.0.3) in the cubic case, $l = 2$, for small initial data.

Theorem 1.0.1. *Consider Eq.* (1.0.3) *with* $l = 2$.
There exists $\epsilon_* > 0$ *such that, if the norm of the initial datum* ψ_0 *fulfills*

$$\|\psi_0\|_{\mathscr{H}_c^{1/2}} \leq \epsilon_*, \tag{1.0.7}$$

then the corresponding solution of (1.0.3) *exists globally in time*

$$\|\psi(t)\|_{L_t^\infty\mathscr{H}_c^{1/2}} \preceq \|\psi_0\|_{\mathscr{H}_c^{1/2}}, \tag{1.0.8}$$

All the constants do not depend on c.

Remark 1.0.2. *For finite* c *this is the standard result for small amplitude solution, while for* $c = \infty$ *it becomes the standard result for the NLS. Thus Theorem 1.0.1 interpolates between these apparently completely different situations.*

Remark 1.0.3. *We also remark that the lack of a priori estimates for the solutions of NLKG in the limit* $c \to \infty$ *was the main obstruction in order to obtain global existence results uniform in* c *in standard Sobolev spaces.*

We are now interested in discussing the approximation of the solutions of NLKG with NLS-type equations. Before giving the result we describe the general strategy we use to get them.

We remark that Eq. (1.0.1) is Hamiltonian with Hamiltonian function (1.0.4). If we divide the Hamiltonian by a factor c^2 (which corresponds to a rescaling of time) and we expand in powers of c^{-2} it takes the form

$$\langle\psi,\bar{\psi}\rangle + \frac{1}{c^2}P_c(\psi,\bar{\psi}) \tag{1.0.9}$$

with a suitable funtion P_c. One can notice that this Hamiltonian is a perturbation of $h_0 := \langle\psi,\bar{\psi}\rangle$, which is the generator of the standard Gauge transform, and which in particular admits a flow

that is periodic in time.
Thus the idea is to exploit canonical perturbation theory in order to conjugate such a Hamiltonian system to a system in normal form, up to remainders of order $\mathcal{O}(c^{-2r})$, for any given $r \geq 1$. The problem is that the perturbation P_c has a vector field which is small only as an operator extracting derivatives. One can Taylor expand P_c and its vector field, but the number of derivatives extracted at each order increases. This is typical in singular perturbation problems. Problems of this kind have already been studied with canonical perturbation theory, but the price to pay to get a normal form is that the remainder of the perturbation turns out to be an operator that extracts a large number of derivatives. The standard way to exploit such a "singular" normal form is to use it just to construct some approximate solution of the original system, and then to apply Gronwall Lemma in order to estimate the difference with a true solution with the same initial datum.

This strategy works also here, but it only leads to a control of the solutions over times of order $\mathcal{O}(c^2)$, that, when scaled back to the physical time, turns out to be of order $\mathcal{O}(1)$.

The idea we use here in order to improve the time scale of the result is that of substituting Gronwall Lemma with a more sophisticated tool, namely dispersive estimates and the retarded Strichartz estimate. This can be done each time one can prove a dispersive or a Strichartz estimate (in the spaces $\mathscr{W}_c^{k,p}$ or $W^{k,p}$) for the linearization of equation (1.0.3) on the approximate solution uniformly in c.

It turns out that this is often a quite hard task. Actually we were able to accomplish it only for radiation solutions. For solutions of other kind we have some preliminary results that could have some interest in themselves, and that will be described later on.

In order to state the approximation result for radiation solutions, we consider the approximate equation given by the Hamilton equations of the normal form truncated at order $\mathcal{O}(c^{-2r})$, and let ψ_r be a solution of such a normalized equation.

Of course, in order to produce some solution of the normal form equation one has to know the equation itself. In Sect. 3.3 we compute it at order 4. It is given by:

$$\begin{aligned} -i\psi_t \;&=\; c^2\psi - \frac{1}{2}\Delta\psi + \frac{3}{4}\lambda|\psi|^2\psi \\ &+ \frac{1}{c^2}\left[\frac{51}{8}\lambda^2|\psi|^4\psi + \frac{3}{16}\lambda\left(2|\psi|^2\,\Delta\psi + \psi^2\Delta\bar{\psi} + \Delta(|\psi|^2\bar{\psi})\right) - \frac{1}{8}\Delta^2\psi\right]. \end{aligned} \tag{1.0.10}$$

We remark that it is a singular perturbation of a Gauge-transformed NLS equation. If one, after a gauge transformation, only considers the first order terms, one has the NLS, for which radiation solution exist (for example in the defocusing case all solutions are of radiation type). For higher order NLS nothing is known. There are some preliminary results by Kim, Arnold and Yao (see [48]) and by Carles, Lucha and Moulay (see [23]), who proved dispersive estimates and local-in-time Strichartz estimates for solutions of the linearized normal form equation (which actually do not involve any normal form transformation)

$$-i\psi_t \;=\; c^2\psi - \sum_{j=1}^{r}\frac{a_j}{c^{2(j-1)}}\Delta^j\psi, \tag{1.0.11}$$

where $a_j = \frac{(2j-1)!}{j!(j-1)!2^{2j-1}}$ for any $j \geq 1$.

Before stating the result, we introduce the following set of admissible exponents:

$$\Delta_r := \{(p,q) : (1/p, 1/q) \text{ lies in the closed quadrilateral ABCD,}\}$$

where

$$A=\left(\frac{1}{2},\frac{1}{2}\right),\quad B=\left(1,\frac{1}{\tau_r}\right),\quad C=(1,0),\quad D=\left(\frac{1}{\tau_r'},0\right),\quad \tau_r=\frac{2r-1}{r-1},\quad \frac{1}{\tau_r}+\frac{1}{\tau_r'}=1.$$

Now, the aforementioned authors proved the following dispersive estimates

Proposition 1.0.4. *Let $r\geq 1$, and denote by $\psi_r(t)$ the solution of the linearized normal form equation of order r (1.0.11). Then we have the following local-in-time dispersive estimate*

$$\|\psi_r(t)\|_{L^\infty(\mathbb{R}^3)}\preceq c^{3\left(1-\frac{1}{r}\right)}|t|^{-3/(2r)}\,\|\psi_r(0)\|_{L^1(\mathbb{R}^3)}\,,\quad 0<|t|\leq c^{2(r-1)}. \tag{1.0.12}$$

Furthermore, $\|\psi_r(t)\|_{L^2}=\|\psi_r(0)\|_{L^2}$ for any $t\in\mathbb{R}$.
Therefore for any $(p,q)\in\Delta_r\setminus\{(2,2),(1,\tau_r),(\tau_r',\infty)\}$

$$\|\psi_r(t)\|_{L^q(\mathbb{R}^3)}\preceq c^{3\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)}|t|^{-\frac{3}{2r}\left(\frac{1}{q}-\frac{1}{p}\right)}\,\|\psi_r(0)\|_{L^p(\mathbb{R}^3)}\,,\quad 0<|t|\leq c^{2(r-1)}. \tag{1.0.13}$$

We have the following theorems

Theorem 1.0.5. *Let $r>1$, and fix $k_1\gg 1$. Let $1\leq p\leq 2$ be such that $(p,3)\in\Delta_r\setminus\{(1,\tau_r)\}$. Then $\exists\ k_0=k_0(r)>0$ such that for any $k\geq k_1$ the following holds: consider the solution $\psi_r(t)$ of the nonlinear normal form equation with initial datum $\psi_{r,0}\in W^{k+k_0,p}$. Assume also that $\psi_r(t)$ satisfies the decay estimate (1.0.13) for Eq. (1.0.11).*

Then there exists $\alpha^*:=\alpha^*(l,r,p)>0$ and there exists $c^*:=c^*(r,k,p)>1$, such that for any $\alpha>\alpha^*$ and for any $c>c^*$, if $\psi_{r,0}$ satisfies

$$\|\psi_{r,0}\|_{W^{k+k_0,p}}\preceq c^{-\alpha}, \tag{1.0.14}$$

then

$$\sup_{t\in[0,T]}\|\psi(t)-\psi_r(t)\|_{H^k_x}\preceq\frac{1}{c^2},\quad T\preceq c^{2(r-1)}, \tag{1.0.15}$$

where $\psi(t)$ is the solution of (1.0.3) with initial datum $\psi_r(0)$.

Theorem 1.0.6. *Let $r>1$, and fix $k_1\gg 1$. Let $1\leq p\leq 2$ be such that $(p,3)\in\Delta_r\setminus\{(1,\tau_r)\}$, and let $1\leq p_1\leq 2$ be such that $(p_1,6(l-1))\in\Delta_r$. Then $\exists\ k_0=k_0(r)>0$ such that for any $k\geq k_1$ the following holds: consider the solution $\psi_r(t)$ of the nonlinear normal form equation with initial datum $\psi_{r,0}\in\mathscr{W}_c^{k+k_0,p}\cap L^{p_1}$. Assume also that $\psi_r(t)$ satisfies the decay estimate (1.0.13) for Eq. (1.0.11).*

Then there exist $\alpha^*:=\alpha^*(l,r,p)>0$ and $\alpha_1^*:=\alpha_1^*(l,r,p_1)>0$ and there exists $c^*:=c^*(r,k,p)>1$, such that for any $\alpha>\max(\alpha^*,\alpha_1^*)$ and for any $c>c^*$, if $\psi_{r,0}$ satisfies

$$\|\psi_{r,0}\|_{\mathscr{W}_c^{k+k_0,p}\cap L^{p_1}}\preceq c^{-\alpha},$$

then

$$\sup_{t\in[0,T]}\|\psi(t)-\psi_r(t)\|_{\mathscr{H}_c^k}\preceq\frac{1}{c^2},\quad T\preceq c^{2(r-1)},$$

where $\psi(t)$ is the solution of (1.0.3) with initial datum $\psi_{r,0}$.

The nonrelativistic limit for the Klein-Gordon equation on $\mathbb{R}^d$ has been extensively studied over more then 30 years, and essentially all the known results only show convergence of the solutions of NLKG to the solutions of the approximate equation for times of order $\mathcal{O}(1)$. The typical statement ensures convergence locally uniformly in time. We mention a first series of results (see [85], [61] and [54]) in which it was shown that, if the initial data are in a certain smoothness class, then the solutions converge in a weaker topology to the solutions of the approximating equation. These are informally called "results with loss of smoothness". Although we can prove a longer time convergence, our results also fill in this group.

Some other results, essentially due to Machihara, Masmoudi, Nakanishi and Ozawa, ensure convergence without loss of regularity in the energy space, again over time scales of order $\mathcal{O}(1)$ (see [55], [57] and [63]).

Concerning radiation solutions there is a remarkable result (see [62]) by Nakanishi, who considered the complex NLKG in the defocusing case, in which it is known that all solutions scatter (and thus the scattering operator exists), and proved that the scattering operator of the NLKG equation converges to the scattering operator of the NLS.

We remark that this result is not contained in our one and does not contain it. Indeed, the scattering operator involves the backward flow of the free equation, that for the considered class of solutions has some contracting properties.

We also mention the recent result proved by Lu and Zhang in [53], which concerns the NLKG with a quadratic nonlinearity. Here the problem is that the typical scale over which the standard approach allows to control the dynamics is $\mathcal{O}(c^{-1})$, while the dynamics of the approximating equation takes place over time scales of order $\mathcal{O}(1)$. In that work the authors are able to use a normal form transformation (in a spirit quite different from ours) in order to extend the time of validity of the approximation over the $\mathcal{O}(1)$ time scale. We did not try to reproduce or extend that result.

We remark that there are ome other well known solutions of NLS which would be interesting to study; indeed, it is well known that in the case of mixed-type nonlinearity

$$i\psi_t = -\Delta\psi - (|\psi|^2 - |\psi|^4)\psi,$$

such an equation admits linearly stable solitary wave solutions; it can also be proved that the standing waves of NLS can be modified in order to obtain standing wave solutions of the normal form of order r, for any r. It would be of clear interest to prove that true solutions starting close to such standing wave remain close to them for long times (remark that the NLKG does not admit stable standing wave solutions). In order to get this result one should prove a Strichartz estimate for NLKG close to the approximate solution and uniformly in c. For the moment we did not succeeed in obtaining such a result.

The thesis contains a preliminary result which can have some independent interest: it is a dispersive estimate for the linear equation

$$\begin{aligned} -i\,\psi_t &= \mathcal{H}(x)\psi := c\langle\nabla\rangle_c\,\psi + V(x)\,\psi, \\ \psi(0) &= \psi_0, \end{aligned} \tag{1.0.16}$$

where $V \in C(\mathbb{R}^3, \mathbb{R})$ satisfies

$$|V(x)| + |\nabla V(x)| \preceq \langle x\rangle^{-\beta}, \;\; x \in \mathbb{R}^3, \tag{1.0.17}$$

for some $\beta > 0$.

Theorem 1.0.7. *Let us assume that V satisfies* (1.0.17) *for some $\beta > 9$, and that the point c^2 is neither an eigenvalue nor a resonance for the operator $\mathcal{H}(x)$. Then for any $\sigma > 9/2$, for any $\psi_0 \in L^2_\sigma$ and for $c \geq 1$ sufficiently large one has*

$$\|e^{it\mathcal{H}(x)} P_c(\mathcal{H})\psi_0\|_{L^2_{-\sigma}} \preceq \langle t\rangle^{-3/2} \|\psi_0\|_{L^2_\sigma}, \quad |t| \to \infty.$$

Before closing the subsection, we add a few technical comments. The first one is that, in order to exploit Strichartz estimates after the normal form, we need to develop normal form in the framework of the spaces $W^{k,p}$, while known results in Galerkin averaging theory only allow to deal with the spaces H^k. This is due to the fact that the Fourier analysis is used in order to approximate the derivatives operators with bounded operators. Thus the first technical step needed in order to be able to exploit dispersion is to reformulate Galerkin averaging theory in terms of dyadic decompositions. This is done in Theorem 3.1.3.

At this point we also mention that actually the Galerkin averaging result proved in the thesis is of abstract form and has a further new application: it allows to justify the approximation of the solutions of NLKG by solutions of the NLS over time scales of order $\mathcal{O}(1)$, *on any manifold* admitting a Littlewood-Paley decomposition (such as Riemannian compact manifolds without borders, or $\mathbb{R}^d$; see the introduction of [19] for the construction of Littlewood-Paley decomposition on manifolds).

Proposition 1.0.8. *Let M be a manifold which admits a Littlewood-Paley decomposition, and consider Eq.* (1.0.1) *on M.*
Fix $r \geq 1$, $R > 0$, $k_1 \gg 1$, $1 < p < +\infty$. Then $\exists\ k_0 = k_0(r) > 0$ with the following properties: for any $k \geq k_1$ there exists $c_{l,r,k,p,R} \gg 1$ such that for any $c > c_{l,r,k,p,R}$, if we assume that

$$\|\psi_0\|_{k+k_0,p} \leq R$$

and that there exists $T = T_{r,k,p} > 0$ such that the solution of the equation in normal form up to order r satisfies

$$\|\psi_r(t)\|_{k+k_0,p} \leq 2R, \quad \text{for } 0 \leq t \leq T,$$

then

$$\|\psi(t) - \psi_r(t)\|_{k,p} \leq C_{k,p}\, c^{-2r}, \quad \text{for } 0 \leq t \leq T. \tag{1.0.18}$$

A similar result has been obtained for the case $M = \mathbb{T}^d$ by Faou and Schratz, who aimed to construct numerical schemes which are robust in the nonrelativistic limit (see [35]; we refer also to [12], [13] and to [14] for some numerical analysis of the nonrelativistic limit of the NLKG equation).

Actually the present thesis is part of a research program in qualitative theory of Hamiltonian PDEs in which canonical perturbation theory is used together with the theory of dispersive equations in order to understand the dynamics of some system. In this context, the nonrelativistic limit of the NLKG is a relevant example of a singular limit.

The issue of nonrelativistic limit has been studied also in the more general Maxwell-Klein-Gordon system ([15], [58]), in the Klein-Gordon-Zakharov system ([59], [60]), in the Hartree equation ([25]) and in the pseudo-relativistic NLS ([26]). However, all these results proved the convergence of the solutions of the limiting system in the energy space ([25] studied also the convergence in H^k), *locally uniformly in time*; no information could be obtained about the convergence of solutions for larger (in the case of NLKG, that would mean c-dependent) timescales.

Other examples of singular perturbations that have been studied either with canonical perturbation theory or with other techniques (typically multiscale analysis) are the problem of the continuous approximation of lattice dynamics (see e.g. [11], [76] and [75]) and the semiclassical analysis of Schrödinger operators (see e.g. [69], [40], [2]). In the framework of lattice dynamics, the time scale covered by all known results is that typical of averaging theorems, which corresponds to our $\mathcal{O}(1)$ time scale. We hope that the methods developed in the present thesis could allow to extend the time of validity of those results.

1.0.2 The case of $x \in [0, \pi]$

We consider here (1.0.1) on $M = [0, \pi] \subset \mathbb{R}$ with a convolution potential, namely

$$\frac{1}{c^2} u_{tt} - u_{xx} + c^2 u + V * u = f(u), \tag{1.0.19}$$

with $c \geq 1$, $x \in \mathbb{T}$, $f \in C^\infty(\mathbb{R})$ a real-valued function, with Dirichlet boundary condition. The potential has the form

$$V(x) = \sum_{j \geq 1} v_j \, cos(jx);$$

having fixed a positive s, for any $R > 0$ we consider the probability space

$$\mathcal{V} := \mathcal{V}_{s,R} = \left\{ (v_j)_{j \geq 1} \ : \ v_j' \ := \ R^{-1} j^s v_j \in \left[-\frac{1}{2}, \frac{1}{2} \right] \right\},$$

and we endow the space $(1, +\infty) \times \mathcal{V} \ni (c, (v_j)_j)$ with the product probability measure.
By introducing the following change of coordinates,

$$\psi := \frac{1}{\sqrt{2}} \left[\left(\frac{(c^2 - \Delta + \widetilde{V})^{1/2}}{c} \right)^{1/2} u - i \left(\frac{c}{(c^2 - \Delta + \widetilde{V})^{1/2}} \right)^{1/2} u_t \right],$$

where $\widetilde{V}$ is the operator that maps u to $V * u$, the Hamiltonian of (1.0.19) now takes the form

$$H(\psi, \bar{\psi}) = H_0(\psi, \bar{\psi}) + N(\psi, \bar{\psi}),$$

where

$$H_0(\psi, \bar{\psi}) = \left\langle \bar{\psi}, c(c^2 - \Delta + \widetilde{V})^{1/2} \psi \right\rangle,$$

$$N(\psi, \bar{\psi}) = \frac{\lambda}{2^4} \int f \left(\left(\frac{c}{(c^2 - \Delta + \widetilde{V})^{1/2}} \right)^{1/2} (\psi + \bar{\psi}) \right) \mathrm{d}x.$$

By generalizing the techniques developed in [10] we are able to prove the following long-time existence result.

Theorem 1.0.9. *Consider the equation* (1.0.19) *and fix* $\gamma > 0$*, and* $\tau > 1$*. Then for any* $r \geq 1$ *there exists a set* $\mathcal{R}_\gamma := \mathcal{R}_{\gamma,s,r} \subset]1,+\infty[\times\mathcal{V}$ *satisfying*

$$|\mathcal{R}_\gamma \cap ([n,n+1]\times\mathcal{V})| = \mathcal{O}(\gamma) \ \ \forall n \in \mathbb{N}_0,$$

and $s^* > 0$ *s.t.,* $\forall s > s^*$*, there exists* $\epsilon_{s,r,\gamma,\tau}$ *such that for any* $(c,(v_j)_j) \in (]1,+\infty[\times\mathcal{V})\setminus\mathcal{R}_\gamma$ *and for any initial datum fulfilling*

$$\epsilon := \|\psi_0\|_{H^s} \leq \epsilon_{s,r,\gamma,\tau} \ , \tag{1.0.20}$$

one has

$$\|\psi(t)\|_{H^s} \leq 2\epsilon \ , \quad |t| \leq \epsilon^{-r} \ . \tag{1.0.21}$$

Finally, there exists a smooth torus $\mathbb{T}_c$ *such that for any* $s_1 < s - 1/2$

$$d_{s_1}((\psi(t),\bar{\psi}(t)),\mathbb{T}_c) \preceq \epsilon^{\frac{r_1}{2}+1}, \ \ |t| \preceq \epsilon^{-(r-r_1+1/2)},$$

where $r_1 \leq r$*, and* d_{s_1} *is the distance in* H^{s_1}*. All the constants are independent of* c*.*

An immediate corollary of Theorem 1.0.9 allows us to show that for any $\alpha > 0$ any solution of Eq. (1.0.19) in H^s with initial datum of size $\mathcal{O}(c^{-\alpha})$ remains of size $\mathcal{O}(c^{-\alpha})$ up to times of order $O(c^{\alpha(r+1/2)})$ for any $r \geq 1$, uniformly in c; however, we have to assume that both the parameter c and the coefficients of the potential belong to a set of large measure. The main limitation of such a result is that it holds only for solutions with initial data which are small with respect to c.

For what concerns the result on $[0,\pi]$, the new ingredient with respect to [10] is a diophantine type estimate for the frequencies, which holds uniformly when $c \to \infty$.

A further comment is that it would of interest to study the dependence of the torus $\mathbb{T}_c$ on c. Of course one expects that it should converge to an invariant torus of the NLS (with a convolution potential). However this is a quite subtle property that we expect to be true, but needs further investigation for a proof. This is due to the fact that the NLS is the *singular* limit of NLKG and to the fact that c is only allowed to vary in Cantor like sets, so that one can only expect Whitney-smooth dependence on it.

A further aspect that would deserve future work is the study of the nonrelativistic limit of the NLKG without potential. This is expected to be a quite subtle problem since, for $c \neq 0$ the frequencies of NLKG are typically non resonant, while the limiting frequencies are resonant.

Chapter 2

Dispersive estimates

In this chapter we will discuss the dispersive properties of Klein-Gordon type equations. First we will study these properties in the free case, namely for the Klein-Gordon equation, for which we will derive both Strichartz estimates and a weighted-norm decay; we will later use these Strichartz estimates in order to approximate the NLKG equation for long times. These are the first examples of dispersive estimates for the KG equation that are uniform with respect to c.
Next we will study the dispersive properties of the Spinless Salpeter equation with a potential: we will generalize the weighted-norm decay obtained in the first part to the case of a time-independent potential, and we will obtain local-in-time a priori estimates for some time-dependent potentials.
Finally we will present Strichartz estimates for the Klein-Gordon equation with a time-independent potential: again this is the first result of this kind that is uniform with respect to c.

2.1 Dispersive properties of the Klein-Gordon equation

At the beginning we will obtain dispersive estimates for the linear equation

$$-i\,\psi_t \;=\; c\langle\nabla\rangle_c\,\psi, \quad x\in\mathbb{R}^3. \tag{2.1.1}$$

Proposition 2.1.1. *For any Schrödinger admissible couples (p,q) and (r,s), namely such that*

$$\frac{2}{p}+\frac{3}{q}=\frac{3}{2},\;\; \frac{2}{r}+\frac{3}{s}=\frac{3}{2},$$

one has

$$\|\langle\nabla\rangle_c^{\frac{1}{q}-\frac{1}{p}}\; e^{it\,c\langle\nabla\rangle_c}\;\psi_0\|_{L^p_tL^q_x} \;\preceq\; c^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}\;\|\langle\nabla\rangle_c^{1/2}\psi_0\|_{L^2}, \tag{2.1.2}$$

$$\left\|\langle\nabla\rangle_c^{\frac{1}{q}-\frac{1}{p}}\int_0^t e^{i(t-s)\,c\langle\nabla\rangle_c}\,F(s)\,\mathrm{d}s\right\|_{L^p_tL^q_x} \;\preceq\; c^{\frac{1}{q}-\frac{1}{p}+\frac{1}{s}-\frac{1}{r}-1}\;\|\langle\nabla\rangle_c^{\frac{1}{r}-\frac{1}{s}+1}F\|_{L^{r'}_tL^{s'}_x}. \tag{2.1.3}$$

Remark 2.1.2. *By choosing $p=+\infty$ and $q=2$, we get the following a priori estimate for finite energy solutions of* (2.1.1),

$$\|c^{1/2}\langle\nabla\rangle_c^{1/2}\;e^{it\,c\langle\nabla\rangle_c}\;\psi_0\|_{L^\infty_tL^2_x}\;\preceq\;\|c^{1/2}\langle\nabla\rangle_c^{1/2}\psi_0\|_{L^2}.$$

We also point out that, since the operators $\langle\nabla\rangle$ and $\langle\nabla\rangle_c$ commute, the above estimates in the spaces $L^p_t L^q_x$ extend to estimates in $L^p_t W^{k,q}_x$ for any $k \geq 0$.

Proof. We recall a result reported by D'Ancona-Fanelli in [30] for the operator $\langle\nabla\rangle := \langle\nabla\rangle_1$.

Lemma 2.1.3. *For all (p, q) Schrödinger-admissible exponents (ie, s.t. $\frac{2}{p} + \frac{3}{q} = \frac{3}{2}$)*

$$\|e^{i\tau\,\langle\nabla\rangle}\ \phi_0\|_{L^p_\tau\ W_y^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2},q}} = \|\langle\nabla\rangle^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}\ e^{it\,\langle\nabla\rangle}\ \phi_0\|_{L^p_\tau\ L^q_y} \leq \|\phi_0\|_{L^2_y}.$$

Now, the solution of equation (2.1.1) satifies $\hat{\psi}(t, \xi) = e^{ic\langle\xi\rangle_c t}\hat{\psi}_0(\xi)$. We may define $\eta := \xi/c$, in order to have that

$$\hat{\phi}(c^2 t, \eta) := \hat{\psi}(t, c\eta) = \hat{\psi}(t, \xi),$$

and in particular that $\hat{\phi}_0(\eta) = \hat{\psi}_0(\xi)$.
Since

$$\langle\xi\rangle_c = \sqrt{c^2 + |\xi|^2} = c\sqrt{1 + |\xi|^2/c^2}, \tag{2.1.4}$$

we get

$$\begin{aligned}\hat{\phi}(t, \eta) &= e^{it\,c^2\langle\xi/c\rangle}\,\hat{\phi}_0(\xi/c)\\ &= e^{i\,tc^2\,\langle\eta\rangle}\,\hat{\phi}_0(\eta)\\ &= e^{i\,\tau\,\langle\eta\rangle}\,\hat{\phi}_0(\eta)\end{aligned}$$

if we set $\tau := c^2 t$. Now, by setting $y := cx$ a simple scaling argument leads to

$$\|e^{i\tau\,\langle\nabla\rangle}\ \phi_0\|_{L^p_\tau\ L^q_y} \preceq \|\langle\nabla\rangle^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}\ \phi_0\|_{L^2} = \|\,\langle\eta\rangle^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}\ \hat{\phi}_0\|_{L^2}$$

and since

$$\|\,\langle\eta\rangle^k\,\hat{\phi}_0\|^2_{L^2} = \int_{\mathbb{R}^3}\langle\eta\rangle^{2k}\ |\hat{\phi}_0(\eta)|^2\,\mathrm{d}\eta = \int_{\mathbb{R}^3}\left\langle\frac{\xi}{c}\right\rangle^{2k}\ |\hat{\phi}_0(\eta/c)|^2\,\frac{\mathrm{d}\xi}{c^3} = \frac{1}{c^{2k+3}}\int_{\mathbb{R}^3}\langle\xi\rangle_c^{2k}\ |\hat{\psi}_0(\xi)|^2\,\mathrm{d}\xi,$$

we get

$$\|\,\langle\eta\rangle^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}\ \hat{\phi}_0\|_{L^2} = \frac{1}{c^{\frac{3}{2}-\frac{1}{q}+\frac{1}{p}+\frac{1}{2}}}\ \|\langle\nabla\rangle_c^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}\ \psi_0\|_{L^2}, \tag{2.1.5}$$

while on the other hand

$$\begin{aligned}\psi(t, x) &= (2\pi)^{-d/2}\int_{\mathbb{R}^3} e^{i\langle\xi,x\rangle}\ \hat{\psi}(t, \xi)\,\mathrm{d}\xi = (2\pi)^{-d/2}\int_{\mathbb{R}^3} e^{i\langle\eta,cx\rangle}\ \hat{\psi}(t, c\eta)\,c^3\mathrm{d}\eta\\ &= (2\pi)^{-d/2}\,c^3\int_{\mathbb{R}^3} e^{i\langle\eta,cx\rangle}\ \hat{\phi}(c^2 t, \eta)\,\mathrm{d}\eta = c^3\,\phi(c^2 t, cx),\end{aligned}$$

yields

$$\|\psi\|_{L^p_t L^q_x} = c^{3-\,3/q-\,2/p}\,\|\phi\|_{L^p_\tau L^q_y}. \tag{2.1.6}$$

Hence we can deduce (2.1.2); via a scaling argument we can also deduce (2.1.3). $\square$

One important application of the Strichartz estimates for the free Klein-Gordon equation is the following global existence result *uniform with respect to* c for the NLKG equation (1.0.3) with cubic nonlinearity (this means $l = 2$), with small initial data.

Theorem 2.1.4. *Consider Eq.* (1.0.3) *with* $l = 2$ *on* $\mathbb{R}^3$.
There exists $\epsilon_* > 0$ *such that, if the norm of the initial datum* ψ_0 *fulfills*

$$\|\psi_0\|_{\mathscr{H}_c^{1/2}} \leq \epsilon_*, \tag{2.1.7}$$

then the corresponding solution $\psi(t)$ *of* (1.0.3) *exists globally in time:*

$$\|\psi(t)\|_{L_t^\infty \mathscr{H}_c^{1/2}} \preceq \|\psi_0\|_{\mathscr{H}_c^{1/2}}, \tag{2.1.8}$$

All the constants do not depend on c.

Proof. It just suffices to apply Duhamel formula,

$$\psi(t) = e^{itc\nabla_c}\psi_0 + i\frac{\lambda}{2^l}\int_0^t e^{i(t-s)c\nabla_c}\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2}\left[\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2}(\psi+\bar{\psi})\right]^{2l-1},$$

and Proposition 2.1.1 with $p = +\infty$, in order to get that

$$\|\psi(t)\|_{L_t^\infty \mathscr{H}_c^{1/2}} \preceq \|\psi_0\|_{\mathscr{H}_c^{1/2}} + c^{1/s-1/r}\left\|\nabla_c^{1/r-1/s}\left[\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2}(\psi+\bar{\psi})\right]^3\right\|_{L_t^{r'}L_x^{s'}},$$

but by choosing $r = +\infty$ and by Hölder inequality we get

$$\begin{aligned}
\|\psi(t)\|_{L_t^\infty \mathscr{H}_c^{1/2}} &\preceq \|\psi_0\|_{\mathscr{H}_c^{1/2}} + \left\|\left[\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2}(\psi+\bar{\psi})\right]^3\right\|_{L_t^1 L_x^2} \\
&\preceq \|\psi_0\|_{\mathscr{H}_c^{1/2}} + \left\|\left[\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2}(\psi+\bar{\psi})\right]^2\right\|_{L_t^1 L_x^3}\left\|\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2}(\psi+\bar{\psi})\right\|_{L_t^\infty L_x^6} \\
&\preceq \|\psi_0\|_{\mathscr{H}_c^{1/2}} + \left\|\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2}(\psi+\bar{\psi})\right\|_{L_t^2 L_x^6}^2\left\|\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2}(\psi+\bar{\psi})\right\|_{L_t^\infty L_x^6} \\
&\preceq \|\psi_0\|_{\mathscr{H}_c^{1/2}} + \|\psi\|^2_{L_t^2 \mathscr{W}_c^{-1/2,6}}\|\psi\|_{L_t^\infty \mathscr{W}_c^{-1/2,6}} \\
&\preceq \|\psi_0\|_{\mathscr{H}_c^{1/2}} + \|\psi\|^2_{L_t^2 \mathscr{W}_c^{-1/3,6}}\|\psi\|_{L_t^\infty \mathscr{H}_c^{1/2}},
\end{aligned}$$

and one can conclude by a standard continuation argument. □

We also establish the weighted norm decay for the free Klein-Gordon equation; we will prove later that this decay can be extended to the spinless Salpeter equation with time-independent potential.
This result is the first weighted-norm decay uniform with respect to c for the KG equation. Some classical results for the KG equation may be found in the literature (see [20], [79], [64]), but they are not uniform with respect to c. The proof we give is inspired by the one for the weighted energy decay established in [49] and in [50]. For any $\rho \in \mathbb{R}$ we will denote by L^2_ρ the Hilbert space of functions $\psi \in L^2_{loc}(\mathbb{R}^3)$ with the finite norm

$$\|\psi\|_{L^2_\rho} := \left(\int_{\mathbb{R}^3} |\langle x\rangle^\rho \psi(x)|^2 \mathrm{d}^3x\right)^{1/2}.$$

Proposition 2.1.5. *Let $\sigma > 3/2$. Then for $\psi_0 \in L^2_\sigma$ and for any $c \geq 1$*

$$\|e^{it\, c\langle\nabla\rangle_c}\, \psi_0\|_{L^2_{-\sigma}} \preceq \langle t\rangle^{-3/2}\, \|\psi_0\|_{L^2_\sigma}, \quad |t| > 1. \tag{2.1.9}$$

Proof. It suffices to consider $t > 0$. Then the action of the dynamical group $e^{itc\langle\nabla\rangle_c}$ is the following (the proof of this fact can be found in sec. B.1)

$$e^{itc\langle\nabla\rangle_c}\psi_0(x) = \int_{\mathbb{R}^3} \mathcal{U}_0(x-y,t)\psi_0(y)\mathrm{d}^3y, \tag{2.1.10}$$

$$\mathcal{U}_0(z,t) = \frac{c^2}{2\pi}\frac{(ct+i0)H_2^{(2)}(c[ct+i0-|z|]^{1/2}[ct+i0+|z|]^{1/2})}{(ct+i0-|z|)(ct+i0+|z|)}, \tag{2.1.11}$$

where $H_2^{(2)}$ is the Hankel function of second kind of order 2. Now, let us fix $0 < \delta < 1$: by exploiting the classical asymptotics for the function $H_2^{(2)}$ (see formula 9.2.4 in [1]),

$$|H_2^{(2)}(z)| \preceq z^{-1/2}, \quad z \to \infty$$

we have that

$$\begin{aligned} |\mathcal{U}_0(z,t)| &\preceq \frac{c^2ctc^{-1/2}(1-\delta^2)^{-1/4}c^{-1/2}t^{-1/2}}{(1-\delta^2)^2c^2t^2} \\ &\preceq (1-\delta^2)^{-9/4}\langle t\rangle^{-3/2}, \quad |z| < \delta ct, t > 1, c \geq 1. \end{aligned} \tag{2.1.12}$$

Now consider an arbitrary $\bar{t} > 1$. Let us split the initial function ψ_0, $\psi_0 = \psi'_{0,c,\bar{t}} + \psi''_{0,c,\bar{t}}$, such that

$$\|\psi'_{0,c,\bar{t}}\|_{L^2_\sigma} + \|\psi''_{0,c,\bar{t}}\|_{L^2_\sigma} \preceq \|\psi_{0,c,\bar{t}}\|_{L^2_\sigma}, \quad \bar{t} > 1, \tag{2.1.13}$$

$$\psi'_{0,c,\bar{t}}(x) = 0 \text{ for } |x| \geq \delta c\bar{t}/2, \quad \psi''_{0,c,\bar{t}}(x) = 0 \text{ for } |x| \leq \delta c\bar{t}/4, \tag{2.1.14}$$

(for example, choose $\zeta \in C_c^\infty(\mathbb{R})$ s.t. $\zeta(s) = 1$ for $|s| \leq \delta/4$, $\zeta(s) = 0$ for $|s| \geq \delta/2$, and set $\psi'_{0,c,\bar{t}} = \zeta(|\cdot|/(c\bar{t}))\psi_0$, $\psi''_{0,c,\bar{t}} = [1-\zeta(|\cdot|/(c\bar{t}))]\psi_0$; notice that $\forall\alpha\ |\partial_x^\alpha\zeta(|\cdot|/(c\bar{t}))| \preceq 1$ for $\bar{t} > 1$). The estimate for $e^{i\bar{t}c\langle\nabla\rangle_c}\psi''_{0,c,\bar{t}}$ follows from the energy estimates for the Klein-Gordon equation, and (2.1.13):

$$\begin{aligned} \|e^{i\bar{t}c\langle\nabla\rangle_c}\psi''_{0,c,\bar{t}}\|_{L^2_{-\sigma}} &\overset{\sigma>0}{\leq} \|e^{i\bar{t}c\langle\nabla\rangle_c}\psi''_{0,c,\bar{t}}\|_{L^2} \preceq \|\psi''_{0,c,\bar{t}}\|_{L^2} \\ &\overset{\sigma\geq 0}{\preceq} \left(1+\frac{\delta^2c^2\bar{t}^2}{16}\right)^{-\sigma/2} \|\psi''_{0,c,\bar{t}}\|_{L^2_\sigma} \\ &\overset{\delta<1,c\geq 1}{\leq} \frac{2^{2\sigma}}{\delta^\sigma}\langle\bar{t}\rangle^{-\sigma}\|\psi''_{0,c,\bar{t}}\|_{L^2_\sigma} \\ &\overset{\sigma>3/2}{\preceq} \frac{2^{2\sigma}}{\delta^\sigma}\frac{\|\psi_0\|_{L^2_\sigma}}{\langle\bar{t}\rangle^{3/2}}. \end{aligned} \tag{2.1.15}$$

Now split $e^{i\bar{t}c\langle\nabla\rangle_c} = (1-\zeta(|\cdot|/(c\bar{t})))e^{i\bar{t}c\langle\nabla\rangle_c} + \zeta(|\cdot|/(c\bar{t}))e^{i\bar{t}c\langle\nabla\rangle_c}$, and recall that $1-\zeta(|x|/(c\bar{t})) = 0$ for $|x| < \delta c\bar{t}/4$. Then

$$\left\|[1-\zeta(|\cdot|/(c\bar{t}))]\, e^{i\bar{t}c\langle\nabla\rangle_c}\psi'_{0,c,\bar{t}}\right\|_{L^2_{-\sigma}} \preceq \left(1+\frac{\delta^2c^2\bar{t}^2}{16}\right)^{-\sigma/2}\left\|[1-\zeta(|\cdot|/(c\bar{t}))]\, e^{i\bar{t}c\langle\nabla\rangle_c}\psi'_{0,c,\bar{t}}\right\|_{L^2} \tag{2.1.16}$$

$$\begin{aligned} &\preceq \frac{2^{2\sigma}}{\delta^\sigma}\langle t\rangle^{-\sigma}\|e^{i\bar{t}c\langle\nabla\rangle_c}\psi'_{0,c,\bar{t}}\|_{L^2} \\ &\preceq \frac{2^{2\sigma}}{\delta^\sigma}\langle t\rangle^{-\sigma}\|\psi'_{0,c,\bar{t}}\|_{L^2} \\ &\preceq \frac{2^{2\sigma}}{\delta^\sigma}\langle t\rangle^{-\sigma}\|\psi'_{0,c,\bar{t}}\|_{L^2_\sigma} \\ &\preceq \frac{2^{2\sigma}}{\delta^\sigma}\langle t\rangle^{-3/2}\|\psi_0\|_{L^2_\sigma}. \end{aligned} \tag{2.1.17}$$

Finally, in order to estimate $\zeta(|\cdot|/(c\bar{t}))e^{i\bar{t}\langle\nabla\rangle_c}\psi'_{0,c,\bar{t}}$ we notice that

$$\begin{aligned} \mathcal{U}'_0(\cdot,\bar{t})\psi'_{0,c,\bar{t}} &:= \zeta(|\cdot|/(c\bar{t}))\mathcal{U}_0(\cdot,\bar{t})\psi'_{0,c,\bar{t}} \\ &= \zeta(|\cdot|/(c\bar{t}))\mathcal{U}_0(\cdot,\bar{t})\zeta(|\cdot|/(c\bar{t}))\psi'_{0,c,\bar{t}}. \end{aligned}$$

The kernel of the operator $[1-\zeta(|\cdot|/(c\bar{t}))]\mathcal{U}_0(\cdot,\bar{t})[1-\zeta(|\cdot|/(c\bar{t}))]$ is equal to

$$\mathcal{U}'_0(x-y,\bar{t}) = [1-\zeta(|x|/(c\bar{t}))]\mathcal{U}_0(x-y,\bar{t})[1-\zeta(|y|/(c\bar{t}))].$$

Since $1-\zeta(|x|/(c\bar{t}))=0$ for $|x|<\delta c\bar{t}/4$, the estimate (2.1.12) implies that

$$|\mathcal{U}'_0(x-y,\bar{t})| \preceq (1-\delta^2)^{-9/4}\langle\bar{t}\rangle^{-3/2}, \quad ,\bar{t}>1, c\geq 1. \tag{2.1.18}$$

Now, the norm of the operator $\mathcal{U}'_0(\cdot,\bar{t}) : L^2_\sigma \to L^2_{-\sigma}$ is equivalent to the norm of the operator

$$A_{c,\sigma,\bar{t}}(x,y) := \langle x\rangle^{-\sigma}[1-\zeta(|x|/(c\bar{t}))]\mathcal{U}_0(x-y,\bar{t})[1-\zeta(|y|/(c\bar{t}))]\langle y\rangle^{-\sigma} : L^2(\mathbb{R}^3_y)\to L^2(\mathbb{R}^3_x).$$

However, since

$$A_{c,\sigma,\bar{t}}(x,y) = \langle x\rangle^{-\sigma}\mathcal{U}'_0(x-y,\bar{t})\langle y\rangle^{-\sigma}, \tag{2.1.19}$$

we can estimate the norm of $A_{c,\sigma,\bar{t}}$ as follows

$$\|\langle x\rangle^{-\sigma}\mathcal{U}'_0(x-y,\bar{t})\langle y\rangle^{-\sigma}\|_{L^2\to L^2} = \sup_{\|f\|_{L^2}=1}\left[\int_{\mathbb{R}^3_x}\langle x\rangle^{-2\sigma}\left|\int_{\mathbb{R}^3_y}\mathcal{U}'_0(x-y,\bar{t})\langle y\rangle^{-\sigma}f(y)\mathrm{d}^3y\right|^2\mathrm{d}^3x\right]^{1/2}$$

$$= \sup_{\|f\|_{L^2}=1}\left[\int_{\mathbb{R}^3_x}\langle x\rangle^{-2\sigma}\left|\int_{\mathbb{R}^3_y}\xi(|x|/(c\bar{t}))\mathcal{U}'_0(x-y,\bar{t})\xi(|y|/(c\bar{t}))\langle y\rangle^{-\sigma}f(y)\mathrm{d}^3y\right|^2\mathrm{d}^3x\right]^{1/2}$$

$$\leq \|\mathcal{U}_0(\cdot,\bar{t})\|_{L^\infty(|\cdot|<\delta c\bar{t})}\sup_{\|f\|_{L^2}=1}\left[\int_{|x|\leq\delta c\bar{t}/2}\langle x\rangle^{-2\sigma}\left|\int_{|x|\leq\delta c\bar{t}/2}\xi(|x|/(c\bar{t}))\xi(|y|/(c\bar{t}))\langle y\rangle^{-\sigma}f(y)\mathrm{d}^3y\right|^2\mathrm{d}^3x\right]^{1/2}$$

$$\overset{(2.1.12)}{\preceq} (1-\delta^2)^{-9/4}\langle\bar{t}\rangle^{-3/2}\|\langle\cdot\rangle^{-\sigma}\|_{L^2_x}\sup_{\|f\|_{L^2}=1}\|\langle\cdot\rangle^{-\sigma}f\|_{L^1_y}$$

$$\overset{\sigma>3/2}{\leq} (1-\delta^2)^{-9/4}\|\langle\cdot\rangle^{-\sigma}\|^2_{L^2}\ \langle\bar{t}\rangle^{-3/2},$$

which leads to the thesis. □

Remark 2.1.6. *Like the Strichartz estimates* (2.1.2) *and* (2.1.3)*, also the weighted time-decay* (2.1.9) *can be extended to a* $H^k_\sigma - H^k_{-\sigma}$ *estimate, by exploiting a simple argument of pseudo-differential calculus.*

Remark 2.1.7. *Again by a simple argument of pseudo-differential calculus, one can also show that for any* $\sigma > 3/2$*, for any* $\psi_0 \in \langle\nabla\rangle_c^{-1/2} L^2_\sigma$ *and for any* $c \geq 1$ *the following energy decay holds*

$$\| \langle\cdot\rangle^{-\sigma} \langle\nabla\rangle_c^{1/2} e^{it\, c\langle\nabla\rangle_c} \psi_0\|_{L^2} \preceq \langle t\rangle^{-3/2} \| \langle\cdot\rangle^{\sigma} \langle\nabla\rangle_c^{1/2}\psi_0\|_{L^2}, \ \ |t| > 1. \tag{2.1.20}$$

2.2 Dispersive properties of the Spinless Salpeter Equation with a potential

Now consider the following equation

$$\begin{aligned} -i\,\psi_t &= \mathcal{H}(x)\psi := c\langle\nabla\rangle_c\,\psi + V(x)\,\psi, \\ \psi(0) &= \psi_0, \end{aligned} \tag{2.2.1}$$

where $V \in C(\mathbb{R}^3, \mathbb{R})$ satisfies

$$|V(x)| + |\nabla V(x)| \preceq \langle x\rangle^{-\beta}, \ \ x \in \mathbb{R}^3, \tag{2.2.2}$$

for some $\beta > 0$.

Proposition 2.2.1. *Let* $\sigma > 3/2$*, and assume that* $\beta \geq 2\sigma$*.*
Then Eq. (2.2.1) *admits a unique solution* $\psi \in L^\infty(\mathbb{R})L^2(\mathbb{R}^3)$*. Furthermore*

$$\|\psi(t)\|_{L^2_x} = \|\psi_0\|_{L^2_x}$$

Proof. One may argue by writing the Duhamel formula and by using a perturbative argument, by exploiting the boundedness of V, and the dispersive estimates (2.1.2) and (2.1.9) for the free KG equation. □

Theorem 2.2.2. *Let us assume that* V *satisfies* (2.2.2) *for some* $\beta > 9$*, and that the spectral condition* (B.2.52) *holds. Then for any* $\sigma > 9/2$*, for any* $\psi_0 \in L^2_\sigma$ *and for* $c \geq 1$ *sufficiently large one has*

$$\|e^{it\mathcal{H}(x)} P_c(\mathcal{H})\psi_0\|_{L^2_{-\sigma}} \preceq \langle t\rangle^{-3/2} \|\psi_0\|_{L^2_\sigma}, \ \ |t| \to \infty, \tag{2.2.3}$$

where P_c *denotes the projection onto the continuous spectrum of* $\mathcal{H}$*.*

Indeed, if we denote by $\mathcal{R}_{0,c}(z) := (c\langle\nabla\rangle_c - z)^{-1}$ the free resolvent and by $\mathcal{R}_c(z) := (\mathcal{H}(x) - z)^{-1}$ the perturbed resolvent, one can relate these two operators through the Born perturbation series,

$$\begin{aligned} \mathcal{R}_c(z) &= \mathcal{R}_{0,c}(z) - \mathcal{R}_{0,c}(z)V\,\mathcal{R}_c(z) \\ &= \mathcal{R}_{0,c}(z) - \mathcal{R}_{0,c}(z)\ \ V\,\mathcal{R}_{0,c}(z) \\ &\quad + \mathcal{R}_{0,c}(z)\ \ V\,\mathcal{R}_{0,c}(z)\ \ V\,\mathcal{R}_c(z). \end{aligned}$$

Next, by taking the inverse Fourier-Laplace transform, we can deduce the corresponding expansion for the evolution operator for (2.2.1),

$$e^{it\mathcal{H}(x)} = e^{itc\langle\nabla\rangle_c} + i\int_0^t e^{i(t-s)c\langle\nabla\rangle_c} V(x) e^{isc\langle\nabla\rangle_c} \mathrm{d}s - iF^{-1}_{z\to t}\left[\mathcal{W}_c(z)\mathcal{R}_c(z)\right], \tag{2.2.4}$$

where $\mathcal{W}_c(z) := \mathcal{R}_{0,c}(z)\ V\ \mathcal{R}_{0,c}(z)\ V$.

However we stress the fact that for the KG equation, unlike the Schrödinger equation, we cannot exploit the classical Jensen-Kato technique reported in [46] to deduce the L^2_σ - $L^2_{-\sigma}$ decay from the Born expansion: indeed, as pointed out by (B.2.35), the free resolvent of the KG equation does not decay for large $|z|$. Hence, for the KG equation the integration by parts does not provide the long-time decay.
As pointed out in [49], the fact that the multiplication by t^N (for large N) improves the smoothness of the solution is not only a technical difference between the KG equation and the Schrödinger equation; it corresponds to the different behaviour of wave propagation for relativistic and non-relativistic equations.

Therefore, to get the weighted energy decay we will deal with the terms in (2.2.4) as in [49]: by exploiting (2.1.9) for the first term, (2.1.9) and the decay of the potential for the second term, and Jensen-Kato technique combined with the asymptotics of $W_c(z)$ for large $|z|$ for the last term. Indeed, we can write

$$\begin{aligned} e^{it\mathcal{H}(x)} P_c(\mathcal{H})\psi_0 &= \frac{1}{2\pi i}\int_{c^2}^{+\infty} e^{-izt}\left[\mathcal{R}_c(z+i0) - \mathcal{R}_c(z-i0)\right]\psi_0 \mathrm{d}z \\ &=: \frac{1}{2\pi i}\left[\psi_1(t)+\psi_2(t)+\psi_3(t)\right], \end{aligned}$$

where

$$\begin{aligned} \psi_1(t) &:= \int_{c^2}^{+\infty} e^{-izt}\left[\mathcal{R}_{0,c}(z+i0) - \mathcal{R}_{0,c}(z-i0)\right]\psi_0\mathrm{d}z, \\ \psi_2(t) &:= \int_{c^2}^{+\infty} e^{-izt}\mathcal{R}_{0,c}(z+i0)V(x)\mathcal{R}_{0,c}(z+i0)\psi_0\mathrm{d}z + \\ &\quad - \int_{c^2}^{+\infty} e^{-izt}\mathcal{R}_{0,c}(z-i0)V(x)\mathcal{R}_{0,c}(z-i0)\psi_0\mathrm{d}z, \\ \psi_3(t) &:= \int_{c^2}^{+\infty} e^{-izt}\left[\mathcal{W}_c(z+i0)\mathcal{R}_{0,c}(z+i0) - \mathcal{W}_c(z-i0)\mathcal{R}_{0,c}(z-i0)\right]\psi_0\mathrm{d}z. \end{aligned}$$

Now, the first term $\psi_1(t) = e^{itc\langle\nabla\rangle_c}\psi_0$ by the LAP for the free resolvent. Hence, Proposition 2.1.5 implies that for $\sigma > 3/2$ and for any $c \geq 1$

$$\|\psi_1(t)\|_{L^2_{-\sigma}} \preceq \langle t\rangle^{-3/2}\|\psi_0\|_{L^2_\sigma}. \tag{2.2.5}$$

Lemma 2.2.3. *The following convolution representation holds*

$$\psi_2(t) = i\int_0^t e^{i(t-\tau)c\langle\nabla\rangle_c} V(x)\psi_1(\tau)\mathrm{d}\tau,\ \tau\in\mathbb{R}, \tag{2.2.6}$$

where the integral converges in $L^2_{-\sigma}$ for $\sigma > 3/2$.

Proof. We first recall that $\psi_2(t) = \psi_{21}(t) - \psi_{22}(t)$, where

$$\psi_{21}(t) := \int_{c^2}^{+\infty} e^{-izt}\mathcal{R}_{0,c}(z+i0)V(x)\mathcal{R}_{0,c}(z+i0)\psi_0 \mathrm{d}z,$$
$$\psi_{22}(t) := \int_{c^2}^{+\infty} e^{-izt}\mathcal{R}_{0,c}(z-i0)V(x)\mathcal{R}_{0,c}(z-i0)\psi_0 \mathrm{d}z.$$

If we denote by

$$\mathcal{U}_0^{\pm}(t) := \theta(\pm t)e^{itc\langle\nabla\rangle_c},$$
$$\psi_1^{\pm}(t) := \theta(\pm t)\psi_1(t).$$

We know that the Fourier-Laplace transform of ψ_1,

$$\tilde{\psi}_1^{+}(z) := \int_{\mathbb{R}} \theta(t)e^{izt}\psi_1(t)\mathrm{d}t,$$

solves the stationary equation $z\tilde{\psi}_1^{+}(z) = c\langle\nabla\rangle_c \tilde{\psi}_1^{+}(z) - i\psi_0$, and therefore may be rewritten as $\tilde{\psi}_1^{+}(z) = i\mathcal{R}_{0,c}(z)\psi_0$. Hence the term ψ_{21} satisfies also

$$\begin{aligned}\psi_{21}(t) &= -i\int_{\mathbb{R}} e^{-izt}\mathcal{R}_{0,c}(z+i0)V(x)\tilde{\psi}_1^{+}(z)\mathrm{d}z \\ &= -i\int_{\mathbb{R}} e^{-izt}\mathcal{R}_{0,c}(z+i0)V(x)\left(\int_{\mathbb{R}} e^{iz\tau}\psi_1^{+}(\tau)\mathrm{d}\tau\right)\mathrm{d}z \\ &= -i(i\partial_t+i)^2\int_{\mathbb{R}} \frac{e^{-izt}}{(z+i)^2}\mathcal{R}_{0,c}(z+i0)V(x)\left(\int_{\mathbb{R}} e^{iz\tau}\psi_1^{+}(\tau)\mathrm{d}\tau\right)\mathrm{d}z.\end{aligned}$$

The last double integral converges in $L^2_{-\sigma}$ with $\sigma > 3/2$ by the decay for the free equation (2.1.9), by the LAP for the free resolvent (B.2.11) and by the asymptotics (B.2.35). Hence, we can apply Fubini theorem to change the order of integration

$$\psi_{21}(t) = -i\int_0^t \mathcal{U}_0^{+}(t-\tau)V(x)\psi_1^{+}(\tau)\mathrm{d}\tau, \quad t > 0,$$

since $\mathcal{U}_0^{+}(t-\tau)$ may be rewritten by exploiting the spectral-Fourier representation for the solution of the free KG equation,

$$\theta(t)\psi(t) = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-i(z+i\epsilon)t}\mathcal{R}_{0,c}(z+i\epsilon)\psi_0\mathrm{d}z, \quad \epsilon > 0.$$

Similarly, one can show that

$$\psi_{22}(t) = -i\int_0^t \mathcal{U}_0^{-}(t-\tau)V\psi_1^{-}(\tau)\mathrm{d}\tau, \quad t < 0.$$

□

Now, let us consider $\sigma \in (3/2, \beta/2]$. Applying (2.1.9) to the integrand in (2.2.6) we get for $c \geq 1$

$$\begin{aligned} \|e^{i(t-\tau)c\langle\nabla\rangle_c} V\psi_1(\tau)\|_{L^2_{-\sigma}} &\overset{\sigma>3/2}{\preceq} \frac{1}{\langle t-\tau\rangle^{3/2}} \|V\psi_1(\tau)\|_{L^2_\sigma} \\ &\overset{\sigma\leq\beta/2}{\preceq} \frac{1}{\langle t-\tau\rangle^{3/2}} \|\psi_1(\tau)\|_{L^2_{-\sigma}} \\ &\preceq \frac{1}{\langle t-\tau\rangle^{3/2}} \frac{1}{\langle \tau\rangle^{3/2}} \|\psi_0\|_{L^2_\sigma}. \end{aligned}$$

Integrating in τ we obtain

$$\|\psi_2(t)\|_{L^2_{-\sigma}} \preceq \langle t\rangle^{-3/2} \|\psi_0\|_{L^2_\sigma}, \quad \sigma > 3/2. \tag{2.2.7}$$

Finally, we rewrite the term $\psi_3(t)$ as

$$\psi_3(t) = \int_{c^2}^{+\infty} e^{-izt}\mathcal{N}(z)\psi_0 \mathrm{d}z, \tag{2.2.8}$$

where $\mathcal{N}(z) := \mathcal{M}(z+i0) - \mathcal{M}(z-i0)$ for $z \in \Gamma := \mathbb{C} \setminus [c^2, +\infty)$, and

$$\mathcal{M}(z) := \mathcal{W}_c(z)\mathcal{R}_c(z), \quad \Gamma \setminus \Sigma(V).$$

By the asymptotic of $\mathcal{R}_{0,c}(z)$ and $\mathcal{R}_c(z)$ we can deduce that for sufficiently large c

Lemma 2.2.4.

$$\|\mathcal{N}''(z)\|_{L^2_\sigma \to L^2_{-\sigma}} = \mathcal{O}\left(c^{-5/2}|z-c|^{-3/2}\right), \quad z \to c^2, \ \sigma > 9/2, \ z \in \Gamma. \tag{2.2.9}$$

Proof. The asymptotic follows from

$$\mathcal{M}''(z) = \mathcal{W}_c''(z)\mathcal{R}_c(z) + 2\mathcal{W}_c'(z)\mathcal{R}_c'(z) + \mathcal{W}_c(z)\mathcal{R}_c''(z),$$

combined with (B.2.15), (B.2.16), (B.2.56) and (B.2.57). Indeed, we want to estimate terms of the form

$$\mathcal{R}_{0,c}^{(k_1)}(z)V\mathcal{R}_{0,c}^{(k_2)}(z)V\mathcal{R}_c^{(k_3)}(z),$$

with $k_1, k_2, k_3 \geq 0$, $k_1 + k_2 + k_3 = 2$. We provide the estimate for the term with $k_1 = k_2 = 1$ and for the term with $k_1 = k_2 = 0$, the others being similar.

Fixed $\sigma > 9/2$, choose $\delta \in (5/2, \min(\sigma, \beta - 5/2)]$; then for $c \geq 1$

$$\begin{aligned} &\|\mathcal{R}_{0,c}'(z)V\mathcal{R}_{0,c}'(z)V\mathcal{R}_c(z)\|_{L^2_\sigma \to L^2_{-\sigma}} \leq \\ &= \mathcal{O}\left(|z-c^2|^{-1/2}\right) \|V\mathcal{R}_{0,c}'(z)V\mathcal{R}_c(z)\|_{L^2_\sigma \to L^2_\delta} \\ &\overset{\beta\geq 2\delta}{=} \mathcal{O}\left(|z-c^2|^{-1/2}\right) \|\mathcal{R}_{0,c}'(z)V\mathcal{R}_c(z)\|_{L^2_\sigma \to L^2_{-\delta}} \\ &= \mathcal{O}\left(|z-c^2|^{-1}\right) \|V\mathcal{R}_c(z)\|_{L^2_\sigma \to L^2_\delta} \\ &= \mathcal{O}\left(|z-c^2|^{-1}\right) \|\mathcal{R}_c(z)\|_{L^2_\sigma \to L^2_{-\delta}} \\ &\leq \mathcal{O}\left(|z-c^2|^{-1}\right) \|\mathcal{R}_c(z)\|_{L^2_\sigma \to L^2_{-\sigma}} = \mathcal{O}\left(c^{-2}|z-c^2|^{-1}\right), \quad z \to c^2. \end{aligned}$$

On the other hand, if one chooses $\delta \in (9/2, \min(\sigma, \beta - 3/2)]$,

$$\begin{aligned}
&\|\mathcal{R}_{0,c}(z)V\mathcal{R}_{0,c}(z)V\mathcal{R}_c''(z)\|_{L^2_\sigma \to L^2_{-\sigma}} \leq \\
&\preceq \|\mathcal{R}_{0,c}(z)V\mathcal{R}_{0,c}(z)V\mathcal{R}_c''(z)\|_{L^2_\sigma \to L^2_{-\delta}} \\
&= \mathcal{O}\left(c^{-2}\right) \|V\mathcal{R}_{0,c}(z)V\mathcal{R}_c''(z)\|_{L^2_\sigma \to L^2_\delta} \\
&\overset{\beta \geq 2\delta}{=} \mathcal{O}\left(c^{-2}\right) \|\mathcal{R}_{0,c}(z)V\mathcal{R}_c''(z)\|_{L^2_\sigma \to L^2_{-\delta}} \\
&= \mathcal{O}\left(c^{-4}\right) \|V\mathcal{R}_c''(z)\|_{L^2_\sigma \to L^2_\delta} \\
&= \mathcal{O}\left(c^{-4}\right) \|\mathcal{R}_c''(z)\|_{L^2_\sigma \to L^2_{-\delta}} \\
&\leq \mathcal{O}\left(c^{-4}\right) \|\mathcal{R}_c''(z)\|_{L^2_\sigma \to L^2_{-\sigma}} = \mathcal{O}\left(c^{-4}|z-c|^{-3/2}\right), \quad z \to c^2.
\end{aligned}$$

□

Similarly we can show

Lemma 2.2.5. *Let $k = 0, 1, 2$; then for sufficiently large c*

$$\|\mathcal{N}^{(k)}(z)\|_{L^2_\sigma \to L^2_{-\sigma}} = \mathcal{O}\left(|z|^{-2}\right), \ \ |z| \to \infty, \ \sigma > \frac{3(k+1)}{2}, \ z \in \Gamma. \tag{2.2.10}$$

Proof. We just show the case $k = 2$. Differentiating $\mathcal{M}(z)$ twice we get

$$\mathcal{M}''(z) = \mathcal{W}_c''(z)\mathcal{R}_c(z) + 2\mathcal{W}_c'(z)\mathcal{R}_c'(z) + \mathcal{W}_c(z)\mathcal{R}_c''(z);$$

for a fixed $\sigma > 9/2$, choose $\delta \in (9/2, \min(\sigma, \beta - 3/2)]$, then for the first term we have

$$\begin{aligned}
\|\mathcal{W}_c''(z)\mathcal{R}_c(z)f\|_{L^2_{-\sigma}} &\leq \|\mathcal{W}_c''(z)\mathcal{R}_c(z)f\|_{L^2_{-\delta}} \\
&\overset{(B.2.60)}{\preceq} \mathcal{O}\left(|z|^{-2}\right) \ \|\mathcal{R}_c(z)f\|_{L^2_{-\delta}} \\
&\overset{(B.2.55)}{\preceq} \mathcal{O}\left(|z|^{-2}\right) \ \|f\|_{L^2_\delta}, \ |z| \to \infty, \ z \in \Gamma.
\end{aligned}$$

Other terms may be estimated similarly, by choosing a suitable value of δ. □

Now we can prove the decay of $\psi_3(t)$ by the usual Jensen-Kato technique. First, we split $\psi_3(t)$ into the low and high-energy components: we choose ϕ_1,$\phi_2 \in C_0^\infty(\mathbb{R})$, such that

- $\text{supp}(\phi_1) \subseteq [c^2/2, c^2 + 1]$,
- $\text{supp}(\phi_2) \subseteq [c^2 - 1, +\infty)$,
- $\phi_1(z) + \phi_2(z) = 1 \ \forall z \geq c^2$.

Then $\psi_3(t) = \psi_{31}(t) + \psi_{32}(t)$, where

$$\begin{aligned}
\psi_{31}(t) &= \int_{c^2}^{c^2+1} e^{-izt}\phi_1(z)\mathcal{N}(z)\psi_0 \mathrm{d}z, \\
\psi_{32}(t) &= \int_{c^2+1}^{+\infty} e^{-izt}\phi_2(z)\mathcal{N}(z)\psi_0 \mathrm{d}z.
\end{aligned}$$

By (2.2.9), we can apply to the Fourier integral $\psi_{31}(t)$ the corresponding version of the Lemma B.1 in [49] (which is based on Lemma 10.2 in [46]), in order to get

$$\|\psi_{31}(t)\|_{L^2_{-\sigma}} \preceq \langle t\rangle^{-3/2} \|\psi_0\|_{L^2_\sigma}, \quad t\to\infty, \quad \sigma > 9/2. \tag{2.2.11}$$

Furthermore, since $\mathrm{supp}(\phi_2\mathcal{N}) \subseteq [c^2+1,+\infty)$ and since $(\phi_2\mathcal{N})'' \in L^1([c^2+1,+\infty), L(L^2_\sigma, L^2_{-\sigma}))$ with $\sigma > 9/2$ by (2.2.10), by integrating by parts twice we get

$$\|\psi_{32}(t)\|_{L^2_{-\sigma}} \preceq \langle t\rangle^{-2} \|\psi_0\|_{L^2_\sigma}, \quad t\to\infty, \quad \sigma > 9/2. \tag{2.2.12}$$

Finally, the decay (2.2.3) (and thus the proof of Theorem 2.2.2) follows from (2.2.5), (2.2.7), (2.2.11) and (2.2.12).

Remark 2.2.6. *Unfortunately, one cannot derive from* (2.2.3) *the corresponding Strichartz estimates for the operator $\mathcal{H}(x)$. However, one can deduce the following weighted decay: since by Duhamel formula*

$$\psi(t) = e^{it\,c\langle\nabla\rangle_c}\psi_0 + \int_0^t e^{i(t-s)\,c\langle\nabla\rangle_c} V(x)\delta(s)\mathrm{d}s,$$

one has that by Proposition 2.1.1 for any $\sigma > 9/2$

$$\begin{aligned}
\|\psi(t)\|_{L^\infty_t L^2_x} &\preceq \|\psi_0\|_{L^2} + \|V(x)\psi(t)\|_{L^1_t L^2_x} \\
&\overset{(2.1.9)}{\preceq} \|\psi_0\|_{L^2_x} + \|\,\langle t\rangle^{-3/2}\|\psi_0\|_{L^2_\sigma}\|_{L^1_t} \\
&\leq \|\psi_0\|_{L^2} + \|\psi_0\|_{L^2_\sigma} \\
&\leq 2\|\psi_0\|_{L^2_\sigma}.
\end{aligned} \tag{2.2.13}$$

Similarly, one can prove that for any $\sigma > 9/2$ and for any $F \in L^\infty_t L^2_{\sigma,x}$

$$\begin{aligned}
\left\|\int_0^t e^{i(t-s)\mathcal{H}(x)}F(s)\mathrm{d}s\right\|_{L^\infty_t L^2_{-\sigma,x}} &\leq \left\|\int_0^t \|e^{i(t-s)\mathcal{H}(x)}F(s)\|_{L^2_{-\sigma}}\right\|_{L^\infty_t} \\
&\preceq \left\|\int_0^t \|\,\langle t-s\rangle^{-3/2}\|F(s)\|_{L^2_\sigma}\right\|_{L^\infty_t} \\
&= \|\,\langle\cdot\rangle^{-3/2} * \|F(\cdot)\|_{L^2_\sigma}\|_{L^\infty_t} \leq \|F(\cdot)\|_{L^1_t\cap L^\infty_t L^2_{\sigma,x}}.
\end{aligned} \tag{2.2.14}$$

The issue of proving dispersive estimates for PDEs with time-dependent potentials is absolutely non-trivial, and often requires refined estimates (see for example [70], which deals with the NLS equation).
However, by a simple adaptation of the argument used in the proof of Theorem 1.1 in [31], we can get local-in-time a priori estimates for potentials in the $L^1_t L^\infty_x$ class: we emphasise that the potentials in this case may be both large and may also change sign.

Proposition 2.2.7. *Let $I = [0,T]$ be a bounded time interval, and assume that $V \in L^1_t L^\infty_x$ is a real-valued potential. Assume also that $\psi_0 \in L^2$ and that $F \in L^1_t L^2_x$. Then the integral equation*

$$\psi(t,x) = e^{itc\langle\nabla\rangle_c}\psi_0(x) + \int_0^t e^{i(t-s)c\langle\nabla\rangle_c}[F(s) + V(s)\psi(s)]\mathrm{d}s \tag{2.2.15}$$

admits a unique solution $\psi \in C(I)L^2_x$ that satisfies the following a priori estimate

$$\|\psi\|_{L^\infty_t L^2_x} \preceq \|\psi_0\|_{L^2} + \|F\|_{L^1_t L^2_x}. \tag{2.2.16}$$

Proof. Consider a small time interval $J = [0, \delta]$, and for any $v \in C(J)L^2_x$ define the mapping

$$\Phi(v) := e^{itc\langle\nabla\rangle_c}\psi_0(x) + \int_0^t e^{i(t-s)c\langle\nabla\rangle_c}[F(s) + V(s)v(s)]\mathrm{d}s. \tag{2.2.17}$$

A direct application of Proposition 2.1.1 gives

$$\begin{aligned}\|\Phi(v)\|_{L^\infty_t L^2_x} &\preceq \|\psi_0\|_{L^2} + \|V\, v\|_{L^1_t L^2_x} + \|F\|_{L^1_t L^2_x} \\ &\preceq \|\psi_0\|_{L^2} + \|V\|_{L^1_t L^\infty_x}\|v\|_{L^\infty_t L^2_x} + \|F\|_{L^1_t L^2_x}.\end{aligned}$$

Thus we have construct a mapping $\Phi : C(J)L^2_x \to C(J)L^2_x$. Assume now that the interval J is so small that

$$\|V\|_{L^1_t L^\infty_x} \preceq \frac{1}{2}; \tag{2.2.18}$$

in this case we have that Φ is a contraction on $C(J)L^2_x$, and hence has a unique fixed point v, which is the required solution. Furthermore, we have that

$$\begin{aligned}&\|v\|_{L^\infty_t L^2_x} \preceq \|\psi_0\|_{L^2} + \frac{1}{2}\|v\|_{L^\infty_t L^2_x} + \|F\|_{L^1_t L^2_x}; \\ &\|v\|_{L^\infty_t L^2_x} \preceq 2\|\psi_0\|_{L^2} + 2\|F\|_{L^1_t L^2_x}.\end{aligned} \tag{2.2.19}$$

One can clearly apply (2.2.19) on any subinterval $J = [t_0, t_1] \subseteq I$ on which (2.2.18) holds; we will get an estimate of the form

$$\|v\|_{L^\infty(J)L^2_x} \preceq 2\|\psi(t_0)\|_{L^2} + 2\|F\|_{L^1(J)L^2_x}, \tag{2.2.20}$$

which implies that

$$\|v(t_1)\|_{L^2_x} \preceq 2\|\psi(t_0)\|_{L^2} + 2\|F\|_{L^1(J)L^2_x}; \tag{2.2.21}$$

by partitioning the interval I into a finite number of subintervals on which (2.2.18) holds, and by inductively applying (2.2.20) and (2.2.21) we can deduce (2.2.16). □

Remark 2.2.8. *Proposition 2.2.7 can also be generalized to an unbounded time interval $I = [0, +\infty)$, by partitioning the interval I in a finite number of subintervals in which condition* (2.2.18) *holds.*

Definition 2.2.9. *Let $V : \mathbb{R}^3 \to \mathbb{R}$ be a real-valued function such that*

$$\mathcal{H}(x) = c\langle\nabla\rangle_c + V(x)$$

admits a self-adjoint extension. We say that V is of Strichartz type if for any bounded time interval $I = [0, T]$, for any $\psi_0 \in L^2$ and for any $F \in L^1_t \cap L^\infty_t L^2_x$, the integral equation

$$h(t, x) = e^{it\mathcal{H}(x)}h_0 + \int_0^t e^{i(t-s)\mathcal{H}(x)}F(s)\mathrm{d}s$$

has a unique solution $h \in L^\infty_t L^2_x$ that satisfies the estimate

$$\|h\|_{L^\infty_t L^2_x} \leq K(I, V)\|h_0\|_{L^2} + K(I, V)\|F\|_{L^1_t \cap L^\infty_t L^2_x}. \tag{2.2.22}$$

Remark 2.2.10. *Consider a real-valued potential* $V \in C_t L_x^\infty$, *and assume that there exists* $\beta > 9$ *such that for each fixed* $t > 0$

$$|V(t,x)| \preceq \langle x \rangle^{-\beta}, \quad \forall x \in \mathbb{R}^3.$$

Then V *is of Strichartz type. Indeed, for any arbitrary* t_0, *we have that* $V(t_0,\cdot)$ *is of Strichartz type by Proposition 2.2.7 .*

Proposition 2.2.11. *Let* $I = [0,T]$ *be a bounded time interval, and let* $V \in C_t L_x^\infty$. *Assume that for each* $t \in I$ *the potential* $V(t,\cdot)$ *is of Strichartz type, and that* $h_0 \in L^2$, $F \in L_t^1 \cap L_t^\infty L_x^2$. *Then the local-in-time a priori estimate* (2.2.16) *holds.*
Moreover, if there exists $T_0 > 0$ *such that* $\|V(t,\cdot)\|_{L_x^\infty}$ *is sufficiently small for* $t > T_0$, *the results holds also in the case* $I = [0,+\infty)$ *(global-in-time a priori estimate).*

Proof. The proof follows the lines as the one of (2.2.16): indeed, for any fixed $t_0 > 0$ the continuity in time of the potential allows one to consider $V(t,x)$ as a small perturbation of $V(t_0,x)$ for t near t_0.
Let $J = [0,\delta]$ be a small time interval, and construct the following mapping on the space $C(J)L^2(\mathbb{R}^3)$,

$$\Phi(v) := e^{it\mathcal{H}(0,x)}\psi_0(x) + \int_0^t e^{i(t-s)\mathcal{H}(0,x)}[F(s) + W(s)v(s)]\mathrm{d}s, \tag{2.2.23}$$

where $\mathcal{H}(0,x) = c\langle\nabla\rangle_c + V(0,x)$, and $W(s,x) = V(s,x) - V(0,x)$ (formula (2.2.23) is meaningful because $V(0,x)$ is of Strichartz type). Hence, the following a priori estimate holds

$$\begin{aligned}\|\Phi(v)\|_{L_t^\infty L_x^2} &\preceq \|\psi_0\|_{L^2} + \|W\,v\|_{L_t^1 L_x^2} + \|F\|_{L_t^1 L_x^2} \\ &\preceq \|\psi_0\|_{L^2} + \|W\|_{L_t^1 L_x^\infty}\|v\|_{L_t^\infty L_x^2} + \|F\|_{L_t^1 L_x^2},\end{aligned}$$

and if δ is so small that

$$\|V\|_{L^1(J)L_x^\infty} \preceq \frac{1}{2},$$

we have that Φ is a contraction on $C(J)L_x^2$, and hence has a unique fixed point v, which satisfies the local-in-time a priori estimate (2.2.16) with some constant $K(0)$ for some bounded time interval $[0,\delta]$.
The same argument can be appled in a small time interval around each point $t_0 \in I$. More precisely, let $J = [t_0 - \delta, t_0 + \delta] \cap I$, and assume that $\delta > 0$ is so small that

$$W(t,x) = V(t,x) - V(t_0,x)$$

satisfies

$$\|W\|_{L^1(J)L_x^\infty} \leq \frac{1}{2K(t_0)}, \tag{2.2.24}$$

where $K(t_0)$ is the constant that appears in the a priori estimate for the potential $V(t_0,\cdot)$ for the bounded time interval $[0, t_0 + 1]$. Then one can argue as above, and we obtain that for any given initial time $t_1 \in J$, and for any $\phi_0 \in L^2$ the integral equation

$$\psi(t,x) = e^{it\mathcal{H}(t_0,x)}\phi_0(x) + \int_0^t e^{i(t-s)\mathcal{H}(t_0,x)}[F(s) + W(s)\psi(s)]\mathrm{d}s$$

where $\mathcal{H}(t_0, x) = c\langle\nabla\rangle_c + V(t_0, x)$ admits a unique solution in $C(J)L^2$, which satisfies the estimate

$$\|\Phi(v)\|_{L^\infty(J)L^2_x} \leq 2K(t_0)\|\psi_0\|_{L^2} + 2K(t_0)\|F\|_{L^1_t L^2_x},$$

for some constant $K(t_0)$ depending on the point t_0, but not on the initial time $t_1 \in J$.
Now we can argue via a continuation argument, as follows. Extend the local solution constructed on $[0, \delta]$ to a maximal interval $[0, T^*)$, namely consider the union on all intervals $[0, \delta]$ on which the solution $\psi \in C([0, \delta])L^2$ exists and sarisfies the Strichartz estimate with some constant K_δ. Assume by contradiction that $T^* < T$. Then the above local argument applied to $t_0 = T^*$ on an interval of the form $J = [T^* - \epsilon, T^* + \epsilon]$ (where $\epsilon > 0$ is sufficiently small) allow us to extend this maximal solution to $[0, T^* + \epsilon)$. Moreover, this extended solution satisfies the Strichartz estimate on $[0, T^* + \epsilon)$. Indeed, choose t_1 such that $T^* - \epsilon < t_1 < T^*$: then by construction we have that the a priori estimate (2.2.16) holds both on $I_1 = [0, t_1]$ with initial datum at $t = 0$

$$\|\psi\|_{L^\infty(I_1)L^2} \preceq \|\psi(t_0)\|_{L^2} + \|F\|_{L^1(I_1)L^2_x}, \tag{2.2.25}$$

and on $J = [T^* - \epsilon, T^* + \epsilon]$ with initial data at $t = t_1$,

$$\|\psi\|_{L^\infty(J)L^2} \preceq \|\psi(t_1)\|_{L^2} + \|F\|_{L^1(J)L^2_x}. \tag{2.2.26}$$

But since $\|\psi(t_1)\|_{L^2}$ can be estimated via (2.2.25), we can deduce the a priori estimate on $[0, T^* + \epsilon)$. This contradicts the assumption $T^* < T$, and we get that $T^* = T$.
The extension to the unbounded time interval $I = [0, +\infty)$ is analogous. □

2.3 Dispersive properties of the Klein-Gordon equation with a potential

As in [8], we can deduce Strichartz estimates for the operator $\mathcal{H}$ by exploiting the boundedness of the wave operators for the Schrödinger equation.

Theorem 2.3.1. *Let $c \geq 1$, and consider the operator*

$$\mathcal{H}(x) := c(c^2 - \Delta + V(x))^{1/2} = \mathcal{H}_0(1 + \langle\nabla\rangle_c^{-2}V)^{1/2}, \tag{2.3.1}$$

where $V \in C(\mathbb{R}^3, \mathbb{R})$ is a potential such that

$$|V(x)| + |\nabla V(x)| \preceq \langle x\rangle^{-\beta}, \quad x \in \mathbb{R}^3,$$

for some $\beta > 5$, and that 0 is neither an eigenvalue nor a resonance for the operator $-\Delta + V(x)$. Let (p, q) be a Schrödinger admissible couple, and assume that $\psi_0 \in \langle\nabla\rangle_c^{-1/2}L^2$ is orthogonal to the bound states of $-\Delta + V(x)$. Then

$$\|\langle\nabla\rangle_c^{\frac{1}{q}-\frac{1}{p}} e^{it\mathcal{H}(x)}\psi_0\|_{L^p_t L^q_x} \preceq c^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}\|\langle\nabla\rangle_c^{1/2}\psi_0\|_{L^2}. \tag{2.3.2}$$

In order to prove Theorem 2.3.1 we recall Yajima's result on wave operators [89] (where we denote by $P_c(-\Delta + V)$ the projection onto the continuous spectrum of the operator $-\Delta + V$).

Theorem 2.3.2. *Assume that*

- *0 is neither an eigenvalue nor a resonance for $-\Delta + V$;*

- $|\partial^{\alpha}V(x)| \preceq \langle x\rangle^{-\beta}$ *for* $|\alpha|\leq k$, *for some* $\beta>5$.

Consider the strong limits

$$\mathcal{W}_{\pm} := \lim_{t\to\pm\infty} e^{it(-\Delta+V)}e^{it\Delta},\ \ \mathcal{Z}_{\pm} := \lim_{t\to\pm\infty} e^{-it\Delta}e^{it(\Delta-V)}P_c(-\Delta+V).$$

Then $\mathcal{W}_{\pm} : L^2 \to P_c(-\Delta+V)L^2$ *are isomorphic isometries which extend into isomorphisms* $\mathcal{W}_{\pm} : W^{k,p} \to P_c(-\Delta+V)W^{k,p}$ *for all* $p\in[1,+\infty]$, *with inverses* $\mathcal{Z}_{\pm}$. *Furthermore, for any Borel function* $f(\cdot)$ *we have*

$$f(-\Delta+V)P_c(-\Delta+V) = \mathcal{W}_{\pm}f(-\Delta)\mathcal{Z}_{\pm},\ \ f(-\Delta) = \mathcal{Z}_{\pm}f(-\Delta+V)P_c(-\Delta+V)\mathcal{W}_{\pm}. \quad (2.3.3)$$

Now, in the case $c=1$ one can derive Strichartz estimates for $\mathcal{H}(x)$ from the Strichartz estimates for the free KG equation, just by applying the aforementioned Theorem by Yajima in the case $k=1$ (since $1/p-1/q+1/2\in[0,5/6]$ for all Schrödinger admissible couples (p,q)). In the general case, this will follow from the following remark.

Remark 2.3.3. *Estimates* (2.3.2) *clearly follow from Proposition 2.1.1 if we can prove that for any* $\alpha\in[-1/3,1/2]$ *and for any* $q\in[2,6]$

$$\|\langle\nabla\rangle_c^{\alpha}\mathcal{W}_{\pm}\langle\nabla\rangle_c^{-\alpha}\|_{L^q\to L^q} \preceq 1, \quad (2.3.4)$$

$$\|\langle\nabla\rangle_c^{\alpha}\mathcal{Z}_{\pm}\langle\nabla\rangle_c^{-\alpha}\|_{L^q\to L^q} \preceq 1. \quad (2.3.5)$$

Indeed in this case one would have

$$\|\langle\nabla\rangle_c^{1/q-1/p}\, e^{it\mathcal{H}(x)}P_c(-\Delta+V)\psi_0\|_{L^p_tL^q_x} = \|\langle\nabla\rangle_c^{1/q-1/p}\,\mathcal{W}_{\pm}e^{it\langle\nabla\rangle_c}\mathcal{Z}_{\pm}\psi_0\|_{L^p_tL^q_x},$$

but

$$\|\langle\nabla\rangle_c^{1/q-1/p}\,\mathcal{W}_{\pm}e^{it\langle\nabla\rangle_c}\mathcal{Z}_{\pm}\psi_0\|_{L^q_x} \preceq \|\langle\nabla\rangle_c^{1/q-1/p}\, e^{it\langle\nabla\rangle_c}\mathcal{Z}_{\pm}\psi_0\|_{L^q_x},$$

hence

$$\|\langle\nabla\rangle_c^{1/q-1/p}\, e^{it\mathcal{H}(x)}P_c(-\Delta+V)\psi_0\|_{L^p_tL^q_x} \preceq c^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}\|\langle\nabla\rangle_c^{1/2}\mathcal{Z}_{\pm}\psi_0\|_{L^2} \preceq c^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}\|\langle\nabla\rangle_c^{1/2}\psi_0\|_{L^2}.$$

To prove (2.3.5) *we first show that it holds for* $\alpha=2k$, $k\in\mathbb{N}$. *We argue by induction. The case* $k=0$ *is true by Theorem 2.3.2. Now, suppose that* (2.3.5) *holds for* $\alpha=2(k-1)$, *then*

$$\begin{aligned}
&\|(c^2-\Delta)^k\mathcal{Z}_{\pm}(c^2-\Delta)^{-k}\|_{L^q\to L^q} = \|(c^2-\Delta)(c^2-\Delta)^{k-1}\mathcal{Z}_{\pm}(c^2-\Delta)^{-(k-1)}(c^2-\Delta)^{-1}\|_{L^q\to L^q}\\
&\leq c^2\|(c^2-\Delta)^{k-1}\mathcal{Z}_{\pm}(c^2-\Delta)^{-(k-1)}(c^2-\Delta)^{-1}\|_{L^q\to L^q}\\
&\quad +\|-\Delta(c^2-\Delta)^{k-1}\mathcal{Z}_{\pm}(c^2-\Delta)^{-(k-1)}(c^2-\Delta)^{-1}\|_{L^q\to L^q}\\
&\leq c^2\|(c^2-\Delta)^{k-1}\mathcal{Z}_{\pm}(c^2-\Delta)^{-(k-1)}(c^2-\Delta)^{-1}\|_{L^q\to L^q}\\
&\quad +\|-\Delta(c^2-\Delta)^{-1}\,(c^2-\Delta)^{k-1}\mathcal{Z}_{\pm}(c^2-\Delta)^{-(k-1)}\|_{L^q\to L^q}\\
&+\|-\Delta(c^2-\Delta)^{k-1}[\mathcal{Z}_{\pm},(c^2-\Delta)^{-1}](c^2-\Delta)^{-(k-1)}\|_{L^q\to L^q}\\
&\preceq c^2\|(c^2-\Delta)^{-1}\|_{L^q\to L^q}+\|-\Delta(c^2-\Delta)^{-1}\|_{L^q\to L^q}\preceq 1,
\end{aligned}$$

since

$$\|[\mathcal{Z}_{\pm},(c^2-\Delta)^{-1}]\|_{L^2\to L^2} \preceq \frac{|\xi|}{(c^2+|\xi|^2)^2} \leq (c^2+|\xi|^2)^{-3/2}.$$

Similarly we can show (2.3.5) *for* $\alpha = -2k$, $k \in \mathbb{N}$.
By interpolation theory one can extend the result to any $\alpha \in \mathbb{R}$ *via the following result. Recall that we denote by*

$$\mathscr{W}_c^{k,p}(\mathbb{R}^3) := \left\{ u \in L^p : \|u\|_{\mathscr{W}_c^{k,p}} := \left\| c^{-k} \langle \nabla \rangle_c^k u \right\|_{L^p} < +\infty \right\}, \quad k \in \mathbb{R}, \quad 1 < p < +\infty,$$

the relativistic Sobolev space *of exponents* k *and* p.

Proposition 2.3.4. *Let* $k_0 \neq k_1$, $1 < p < +\infty$, *and assume that* $T : \mathscr{W}_c^{k_0,p} \to \mathscr{W}_c^{k_0,p}$ *has norm* M_0, *and that* $T : \mathscr{W}_c^{k_1,p} \to \mathscr{W}_c^{k_1,p}$ *has norm* M_1. *Then*

$$T : \mathscr{W}_c^{k,p} \to \mathscr{W}_c^{k,p}, \ k = (1-\theta)k_0 + \theta k_1,$$

with norm $M \leq M_0^{1-\theta} M_1^{\theta}$.

The above proposition is a consequence of Corollary C.0.8; we defer the statement and the proof of Corollary C.0.8 to the appendix, Ch. C.

Chapter 3

Galerkin Averaging

In this chapter we will state an abstract Normal Form Theorem, and we will prove it in Sec. 3.2. In Sec. 3.3 we will apply the Normal Form theorem to the nonlinear Klein-Gordon equation, and we will show that the normalized equation will be given by a NLS equation plus higher-order corrections for the real case, and by a system of two coupled NLS equations for the complex NLKG.

3.1 Galerkin Averaging Method

Consider the scale of Banach spaces $W^{k,p}(M, \mathbb{C}^n \times \mathbb{C}^n) \ni (\psi, \bar{\psi})$ ($k \geq 1$, $1 < p < +\infty$, $n \in \mathbb{N}_0$) endowed by the standard symplectic form. Having fixed k and p, and $U_{k,p} \subset W^{k,p}$ open, we define the gradient of $H \in C^\infty(U_{k,p}, \mathbb{R})$ w.r.t. $\bar{\psi}$ as the unique function s.t.

$$\left\langle \nabla_{\bar{\psi}} H, \bar{h} \right\rangle = \mathrm{d}_{\bar{\psi}} H \bar{h}, \quad \forall h \in W^{k,p},$$

so that the Hamiltonian vector field of a Hamiltonian function H is given by

$$X_H(\psi, \bar{\psi}) = (i\nabla_{\bar{\psi}} H, \; -i\nabla_{\psi} H).$$

The open ball of radius R and center 0 in $W^{k,p}$ will be denoted by $B_{k,p}(R)$.

We briefly recall some classical notion of Fourier analysis on $\mathbb{R}^d$. We first recall the definition of the space of Schwartz (or rapidly decreasing) functions,

$$\mathcal{S} := \{ f \in C^\infty(\mathbb{R}^d, \mathbb{R}) | \sup_{x \in \mathbb{R}^d} (1 + |x|^2)^{\alpha/2} |\partial^\beta f(x)| < +\infty, \quad \forall \alpha \in \mathbb{N}^d, \forall \beta \in \mathbb{N}^d \}.$$

In the following we will denote by $\langle x \rangle := (1 + |x|^2)^{1/2}$.
Now, for any $f \in \mathcal{S}$ we introduce the *Fourier transform* of f, $\hat{f} : \mathbb{R}^d \to \mathbb{R}$,

$$\hat{f}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i\langle x, \xi \rangle} \mathrm{d}x, \quad \forall \xi \in \mathbb{R}^d,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^d$.

Next, we call an *admissible family of cut-off (pseudo-differential) operators* a sequence $(\pi_j(D))_{j \geq 0}$, where $\pi_j(D) : W^{k,p} \to W^{k,p}$ for any $j \geq 0$, such that

- for any $j \geq 0$ and for any $f \in W^{k,p}$

$$f = \sum_{j \geq 0} \pi_j(D) f;$$

- for any $j \geq 0$ $\pi_j(D)$ can be extended to a self-adjoint operator on L^2, and there exist constants K_1, $K_2 > 0$ such that

$$K_1 \left(\sum_{j \geq 0} \|\pi_j(D) f\|_{L^2}^2 \right)^{1/2} \leq \|f\|_{L^2} \leq K_2 \left(\sum_{j \geq 0} \|\pi_j(D) f\|_{L^2}^2 \right)^{1/2} ;$$

- for any $j \geq 0$, if we denote by $\Pi_j(D) := \sum_{l=0}^{j} \pi_l(D)$, there exist positive constants K', (possibly depending on k and p) such that

$$\|\Pi_j f\|_{k,p} \leq K' \, \|f\|_{k,p} \quad \forall f \in W^{k,p};$$

- there exist positive constants K_1'', K_2'' (possibly depending on k and p) such that

$$K_1'' \|f\|_{W^{k,p}} \leq \left\| \left[\sum_{j \in \mathbb{N}} 2^{2jk} |\pi_j(D) f|^2 \right]^{1/2} \right\|_{L^p} \leq K_2'' \|f\|_{W^{k,p}}.$$

Remark 3.1.1. *Let $k \geq 0$, M be either $\mathbb{R}^d$ or the d-dimensional torus $\mathbb{T}^d$, and consider the Sobolev space $H^k = H^k(M)$. One can readily check that Fourier projection operators on H^k*

$$\pi_j \psi(x) := (2\pi)^{-d/2} \int_{j-1 \leq |k| \leq j} \hat{\psi}(k) e^{ik \cdot x} \mathrm{d}k, \quad j \geq 1$$

form an admissible family of cut-off operators. In this case we have

$$\Pi_N \psi(x) := (2\pi)^{-d/2} \int_{|k| \leq N} \hat{\psi}(k) e^{ik \cdot x} \mathrm{d}k, \quad N \geq 0.$$

Remark 3.1.2. *Let $k \geq 0$, $1 < p < +\infty$, we now introduce the Littlewood-Paley decomposition on the Sobolev space $W^{k,p} = W^{k,p}(\mathbb{R}^d)$ (see [84], Ch. 13.5).*

In order to do this, define the cutoff operators in $W^{k,p}$ in the following way: start with a smooth, radial nonnegative function $\phi_0 : \mathbb{R}^d \to \mathbb{R}$ such that $\phi_0(\xi) = 1$ for $|\xi| \leq 1/2$, and $\phi_0(\xi) = 0$ for $|\xi| \geq 1$; then define $\phi_1(\xi) := \phi_0(\xi/2) - \phi_0(\xi)$, and set

$$\phi_j(\xi) := \phi_1(2^{1-j}\xi), \quad j \geq 2. \tag{3.1.1}$$

Then $(\phi_j)_{j \geq 0}$ is a partition of unity,

$$\sum_{j \geq 0} \phi_j(\xi) = 1.$$

Now, for each $j \in \mathbb{N}$ and each $f \in W^{k,2}$, we can define $\phi_j(D) f$ by

$$\mathcal{F}(\phi_j(D) f)(\xi) := \phi_j(\xi) \hat{f}(\xi).$$

It is well known that for $p \in (1, +\infty)$ *the map* $\Phi : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d, l^2)$,

$$\Phi(f) := (\phi_j(D)f)_{j\in\mathbb{N}},$$

maps $L^p(\mathbb{R}^d)$ *isomorphically onto a closed subspace of* $L^p(\mathbb{R}^d, l^2)$, *and we have compatibility of norms ([84], Ch. 13.5, (5.45)-(5.46)),*

$$K'_p \|f\|_{L^p} \leq \|\Phi(f)\|_{L^p(\mathbb{R}^d, l^2)} := \left\| \left[\sum_{j\in\mathbb{N}} |\phi_j(D)f|^2 \right]^{1/2} \right\|_{L^p} \leq K_p \|f\|_{L^p},$$

and similarly for the $W^{k,p}$*-norm, i.e. for any* $k > 0$ *and* $p \in (1, +\infty)$

$$K'_{k,p} \|f\|_{W^{k,p}} \leq \left\| \left[\sum_{j\in\mathbb{N}} 2^{2jk} |\phi_j(D)f|^2 \right]^{1/2} \right\|_{L^p} \leq K_{k,p} \|f\|_{W^{k,p}}. \tag{3.1.2}$$

We then define the cutoff operator Π_N *by*

$$\Pi_N \psi := \sum_{j\leq N} \phi_j(D)\psi. \tag{3.1.3}$$

Hence, according to the above definition, the sequence $(\phi_j(D))_{j\geq 0}$ *is an admissible family of cut-off operators.*
We point out that the Littlewood-Paley decomposition, along with equality (3.1.2), *can be extended to compact manifolds (see [21]), as well as to some particular non-compact manifolds (see [19]).*

Now we consider a Hamiltonian system of the form

$$H = h_0 + \epsilon\, h + \epsilon\, F, \tag{3.1.4}$$

where $\epsilon > 0$ is a parameter. We fix an admissible family of cut-off operators $(\pi_j(D))_{j\geq 0}$ on $W^{k,p}(\mathbb{R}^d)$. We assume that

PER h_0 generates a linear periodic flow Φ^t with period 2π,

$$\Phi^{t+2\pi} = \Phi^t \quad \forall t.$$

We also assume that Φ^t is analytic from $W^{k,p}$ to itself for any $k \geq 1$, and for any $p \in (1, +\infty)$;

INV for any $k \geq 1$, for any $p \in (1, +\infty)$, Φ^t leaves invariant the space $\Pi_j W^{k,p}$ for any $j \geq 0$. Furthermore, for any $j \geq 0$

$$\pi_j(D) \circ \Phi^t = \Phi^t \circ \pi_j(D);$$

NF h is in normal form, namely

$$h \circ \Phi^t = h.$$

Next we assume that both the Hamiltonian and the vector field of both h and F admit an asymptotic expansion in ϵ of the form

$$h \sim \sum_{j\geq 1} \epsilon^{j-1} h_j, \;\; F \sim \sum_{j\geq 1} \epsilon^{j-1} F_j, \tag{3.1.5}$$

$$X_h \sim \sum_{j\geq 1} \epsilon^{j-1} X_{h_j}, \;\; X_F \sim \sum_{j\geq 1} \epsilon^{j-1} X_{F_j}, \tag{3.1.6}$$

and that the following properties are satisfied

HVF There exists $R^* > 0$ such that for any $j \geq 1$

- X_{h_j} is analytic from $B_{k+2j,p}(R^*)$ to $W^{k,p}$;
- X_{F_j} is analytic from $B_{k+2(j-1),p}(R^*)$ to $W^{k,p}$.

Moreover, for any $r \geq 1$ we have that

- $X_{h-\sum_{j=1}^{r} \epsilon^{j-1} h_j}$ is analytic from $B_{k+2(r+1),p}(R^*)$ to $W^{k,p}$;
- $X_{F-\sum_{j=1}^{r} \epsilon^{j-1} F_j}$ is analytic from $B_{k+2r,p}(R^*)$ to $W^{k,p}$.

The main result of this section is the following theorem.

Theorem 3.1.3. *Fix $r \geq 1$, $R > 0$, $k_1 \gg 1$, $1 < p < +\infty$. Consider (3.1.4), and assume PER, INV (with respect to the Littlewood-Paley decomposition), NF and HVF. Then $\exists\ k_0 = k_0(r) > 0$ with the following properties: for any $k \geq k_1$ there exists $\epsilon_{r,k,p} \ll 1$ such that for any $\epsilon < \epsilon_{r,k,p}$ there exists $\mathcal{T}_\epsilon^{(r)} : B_{k,p}(R) \to B_{k,p}(2R)$ analytic canonical transformation such that*

$$H_r := H \circ \mathcal{T}_\epsilon^{(r)} = h_0 + \sum_{j=1}^{r} \epsilon^j \mathcal{Z}_j + \epsilon^{r+1} \mathcal{R}^{(r)},$$

where $\mathcal{Z}_j$ are in normal form, namely

$$\{\mathcal{Z}_j, h_0\} = 0, \tag{3.1.7}$$

and

$$\sup_{B_{k+k_0,p}(R)} \|X_{\mathcal{Z}_j}\|_{W^{k,p}} \leq C_{k,p},$$

$$\sup_{B_{k+k_0,p}(R)} \|X_{\mathcal{R}^{(r)}}\|_{W^{k,p}} \leq C_{k,p}, \tag{3.1.8}$$

$$\sup_{B_{k,p}(R)} \|\mathcal{T}_\epsilon^{(r)} - id\|_{W^{k,p}} \leq C_{k,p}\, \epsilon. \tag{3.1.9}$$

In particular, we have that

$$\mathcal{Z}_1(\psi,\bar{\psi}) = h_1(\psi,\bar{\psi}) + \langle F_1 \rangle (\psi,\bar{\psi}),$$

where $\langle F_1 \rangle (\psi,\bar{\psi}) := \int_0^{2\pi} F_1 \circ \Phi^t(\psi,\bar{\psi}) \frac{\mathrm{d}t}{2\pi}$.

3.2 Proof of Theorem 3.1.3

We first make a Galerkin cutoff through the Littlewood-Paley decomposition (see [84], Ch. 13.5).

In order to do this, fix $N \in \mathbb{N}$, $N \gg 1$, and introduce the cutoff operators Π_N in $W^{k,p}$ by

$$\Pi_N \psi := \sum_{j \leq N} \phi_j(D) \psi,$$

where $\phi_j(D)$ are the operators we introduced in Remark 3.1.2.

We notice that by assumption INV the Hamiltonian vector field of h_0 generates a continuous flow Φ^t which leaves $\Pi_N W^{k,p}$ invariant.
Now we set $H = H_{N,r} + \mathcal{R}_{N,r} + \mathcal{R}_r$, where

$$H_{N,r} := h_0 + \epsilon\, h_{N,r} + \epsilon\, F_{N,r}, \tag{3.2.1}$$

$$h_{N,r} := \sum_{j=1}^{r} \epsilon^{j-1} h_{j,N}, \quad h_{j,N} := h_j \circ \Pi_N, \tag{3.2.2}$$

$$F_{N,r} := \sum_{j=1}^{r} \epsilon^{j-1} F_{j,N}, \quad F_{j,N} := F_j \circ \Pi_N, \tag{3.2.3}$$

and

$$\mathcal{R}_{N,r} := h_0 + \sum_{j=1}^{r} \epsilon^j h_j + \sum_{j=1}^{r} \epsilon^j F_j - H_{N,r}, \tag{3.2.4}$$

$$\mathcal{R}_r := \epsilon \left(h - \sum_{j=1}^{r} \epsilon^{j-1} h_j \right) + \epsilon \left(F - \sum_{j=1}^{r} \epsilon^{j-1} F_j \right). \tag{3.2.5}$$

The system described by the Hamiltonian (3.2.1) is the one that we will put in normal form.

In the following we will use the notation $a \preceq b$ to mean: there exists a positive constant K independent on N and R (but dependent on r, k and p), such that $a \leq Kb$.

We exploit the following intermediate results:

Lemma 3.2.1. *For any $k \geq k_1$ and $p \in (1, +\infty)$ there exists $B_{k,p}(R) \subset W^{k,p}$ s.t. $\forall\ \sigma > 0$, $N > 0$*

$$\sup_{B_{k+\sigma+2(r+1),p}(R)} \|X_{\mathcal{R}_{N,r}}(\psi, \bar{\psi})\|_{W^{k,p}} \preceq \frac{\epsilon}{2^{\sigma(N+1)}}, \tag{3.2.6}$$

$$\sup_{B_{k+2(r+1),p}(R)} \|X_{\mathcal{R}_r}(\psi, \bar{\psi})\|_{W^{k,p}} \preceq \epsilon^{r+1}. \tag{3.2.7}$$

Proof. We recall that $\mathcal{R}_{N,r} = h_0 + \sum_{j=1}^{r} \epsilon^j h_j + \sum_{j=1}^{r} \epsilon^j F_j - H_{N,r}$.
Now, $\|id - \Pi_N\|_{W^{k+\sigma,p} \to W^{k,p}} \preceq 2^{-\sigma(N+1)}$, since

$$\begin{aligned} \left\| \sum_{j \geq N+1} \phi_j(D) f \right\|_{W^{k,p}} &\preceq \left\| \left[\sum_{j \geq N+1} |2^{jk} \phi_j(D) f|^2 \right]^{1/2} \right\|_{L^p} \\ &\preceq 2^{-\sigma(N+1)} \left\| \left[\sum_{j \geq N+1} |2^{j(k+\sigma)} \phi_j(D) f|^2 \right]^{1/2} \right\|_{L^p} \\ &\preceq 2^{-\sigma(N+1)} \|f\|_{W^{k+\sigma,p}}, \end{aligned}$$

hence

$$\begin{aligned}
&\sup_{\psi\in B_{k+2(r+1)+\sigma,p}(R)} \|X_{\mathcal{R}_{N,r}}(\psi,\bar\psi)\|_{W^{k,p}} \\
&\preceq \|dX_{\sum_{j=1}^r \epsilon^j(h_j+F_j)}\|_{L^\infty(B_{k+2(r+1),p}(R),W^{k,p})}\|id-\Pi_N\|_{L^\infty(B_{k+2(r+1)+\sigma,p}(R),B_{k+2(r+1),p})} \\
&\preceq \epsilon\, 2^{-\sigma(N+1)}.
\end{aligned}$$

The estimate of $X_{\mathcal{R}_r}$ follow from the hypothesis HVF.

□

Lemma 3.2.2. *Let $j\geq 1$. Then for any $k\geq k_1+2(j-1)$ and $p\in(1,+\infty)$ there exists $B_{k,p}(R)\subset W^{k,p}$ such that*

$$\begin{aligned}
\sup_{B_{k,p}(R)} \|X_{h_{j,N}}(\psi,\bar\psi)\|_{k,p} &\leq K^{(h)}_{j,k,p}2^{2jN}, \\
\sup_{B_{k,p}(R)} \|X_{F_{j,N}}(\psi,\bar\psi)\|_{k,p} &\leq K^{(F)}_{j,k,p}2^{2(j-1)N},
\end{aligned}$$

where

$$\begin{aligned}
K^{(h)}_{j,k,p} &:= \sup_{B_{k,p}(R)} \|X_{h_j}(\psi,\bar\psi)\|_{k-2j,p}, \\
K^{(F)}_{j,k,p} &:= \sup_{B_{k,p}(R)} \|X_{F_j}(\psi,\bar\psi)\|_{k-2(j-1),p}.
\end{aligned}$$

Proof. It follows from

$$\sup_{\psi\in B_{k,p}(R)} \left\|\sum_{h\leq N}\phi_h(D)X_{F_{j,N}}(\psi,\bar\psi)\right\|_{W^{k,p}} \preceq \sup_{\psi\in B_{k,p}(R)} \left\|\left[\sum_{h\leq N}|2^{hk}\phi_h(D)X_{F_{j,N}}(\psi,\bar\psi)|^2\right]^{1/2}\right\|_{L^p} \tag{3.2.8}$$

$$\leq 2^{2(j-1)N}\sup_{\psi\in B_{k,p}(R)} \left\|\left[\sum_{h\leq N}|2^{h[k-2(j-1)]}\phi_h(D)X_{F_{j,N}}(\psi,\bar\psi)|^2\right]^{1/2}\right\|_{L^p} \tag{3.2.9}$$

$$\preceq 2^{2(j-1)N}\sup_{\psi\in B_{k,p}(R)} \|X_{F_{j,N}}(\psi,\bar\psi)\|_{k-2(j-1),p} \tag{3.2.10}$$

$$= K^{(F)}_{j,k,p}\,2^{2(j-1)N}, \tag{3.2.11}$$

and similarly for $X_{h_{j,N}}$. □

Next we have to normalize the system (3.2.1). In order to do this we need a slight reformulation of Theorem 4.4 in [4]. Here we report a statement of the result adapted to our context.

Lemma 3.2.3. *Let $k\geq k_1+2r$, $p\in(1,+\infty)$, $R>0$, and consider the system* (3.2.1). *Assume that $\epsilon<2^{-4Nr}$, and that*

$$(K^{(F,r)}_{k,p}+K^{(h,r)}_{k,p})r2^{2Nr}\epsilon < 2^{-9}e^{-1}\pi^{-1}R, \tag{3.2.12}$$

where

$$K_{k,p}^{(F,r)} := \sup_{1\le j\le r} \sup_{\psi\in B_{k,p}(R)} \|X_{F_j}(\psi,\bar\psi)\|_{k-2(j-1),p},$$
$$K_{k,p}^{(h,r)} := \sup_{1\le j\le r} \sup_{\psi\in B_{k,p}(R)} \|X_{h_j}(\psi,\bar\psi)\|_{k-2j,p}.$$

Then there exists an analytic canonical transformation $\mathcal{T}_{\epsilon,N}^{(r)} : B_{k,p}(R) \to B_{k,p}(2R)$ *such that*

$$\sup_{B_{k,p}(R/2)} \|\mathcal{T}_{\epsilon,N}^{(r)}(\psi,\bar\psi) - (\psi,\bar\psi)\|_{W^{k,p}} \le 4\pi r K_{k,p}^{(F,r)} 2^{2Nr}\epsilon,$$

and that puts (3.2.1) *in normal form up to a small remainder,*

$$H_{N,r}\circ \mathcal{T}_{\epsilon,N}^{(r)} = h_0 + \epsilon h_{N,r} + \epsilon Z_N^{(r)} + \epsilon^{r+1}\mathcal{R}_N^{(r)}, \tag{3.2.13}$$

with $Z_N^{(r)}$ *is in normal form, namely* $\{h_{0,N}, Z_N^{(r)}\} = 0$*, and*

$$\sup_{B_{k,p}(R/2)} \|X_{Z_N^{(r)}}(\psi,\bar\psi)\|_{k,p} \le 4\, 2^{2Nr}\,\epsilon\, \left(rK_{k,p}^{(F,r)} + rK_{k,p}^{(h,r)}\right) r2^{2Nr}K_{k,p}^{(F,r)}$$
$$= 4r^2 K_{k,p}^{(F,r)}(K_{k,p}^{(F,r)} + K_{k,p}^{(h,r)})2^{4NR}\epsilon, \tag{3.2.14}$$

$$\sup_{B_{k,p}(R/2)} \|X_{\mathcal{R}_N^{(r)}}(\psi,\bar\psi)\|_{k,p} \tag{3.2.15}$$
$$\le 2^8 e\frac{T}{R}(K_{k,p}^{(F,r)} + K_{k,p}^{(F,r)})r2^{2Nr} \tag{3.2.16}$$
$$\left[\frac{4T}{R}\left(9\cdot 2^9 e\frac{T}{R}(K_{k,p}^{(F,r)} + K_{k,p}^{(F,r)})K_{k,p}^{(F,r)} r^2 2^{4Nr}\epsilon + 5K_{k,p}^{(h,r)}\, r2^{2Nr} + 5K_{k,p}^{(F,r)}\, r2^{2Nr}\right) r\right]^r \tag{3.2.17}$$

The proof of Lemma 3.2.3 is postponed to the Appendix, Chapter A.

Remark 3.2.4. *In the original notation of Theorem 4.4 in [4] we set*

$$\begin{aligned}
\mathcal{P} &= W^{k,p},\\
h_\omega &= h_0,\\
\hat h &= \epsilon h_{N,r},\\
f &= \epsilon F_{N,r},\\
f_1 &= r = g \equiv 0,\\
F &= K_{k,p}^{(F,r)}\, r2^{2Nr}\,\epsilon,\\
F_0 &= K_{k,p}^{(h,r)}\, r2^{2Nr}\,\epsilon.
\end{aligned}$$

Remark 3.2.5. *Actually, Lemma 3.2.3 would hold also under a weaker smallness assumption on* ϵ*: it would be enough that* $\epsilon < 2^{-2N}$*, and that*

$$\epsilon\left[K_{k,p}^{(F,r)}\frac{1-2^{2Nr}\epsilon^r}{1-2^{2N}\epsilon} + K_{k,p}^{(h,r)}\frac{2^{2N}(1-2^{2Nr}\epsilon^r)}{1-2^{2N}\epsilon}\right] < 2^{-9}e^{-1}\pi^{-1}R \tag{3.2.18}$$

is satified. However, condition (3.2.18) *is less explicit than* (3.2.12)*, that allows us to apply directly the scheme of [4]. The disadvantage of the stronger smallness assumption* (3.2.12) *is that it holds for a smaller range of* ϵ*, and that at the end of the proof it will force us to choose a larger parameter* $\sigma = 4r^2$*. By using* (3.2.18) *and by making a more careful analysis, it may be possible to prove Theorem 3.1.3 also by choosing* $\sigma = 2r$*.*

Now we conclude with the proof of the Theorem 3.1.3.

Proof. Now consider the transformation $\mathcal{T}^{(r)}_{\epsilon,N}$ defined by Lemma 3.2.3, then

$$(\mathcal{T}^{(r)}_{\epsilon,N})^* H = h_0 \ + \sum_{j=1}^{r} \epsilon^j h_{j,N} \ + \epsilon Z^{(r)}_N + \epsilon^{r+1} \mathcal{R}^{(r)}_N + \epsilon^r \mathcal{R}_{Gal}$$

where we recall that

$$\epsilon^r \mathcal{R}_{Gal} := (\mathcal{T}^{(r)}_{\epsilon,N})^* (\mathcal{R}_{N,r} + \mathcal{R}_r).$$

By exploiting the Lemma 3.2.3 we can estimate the vector field of $\mathcal{R}^{(r)}_N$, while by using Lemma 3.2.1 and (A.0.10) we get

$$\sup_{B_{k+\sigma+2(r+1),p}(R/2)} \|X_{\mathcal{R}_{Gal}}(\psi,\bar\psi)\|_{W^{k,p}} \preceq \left(\frac{\epsilon}{2^{\sigma(N+1)}} + \frac{\epsilon^{r+1}}{\sigma + 2(r+1)} \right). \tag{3.2.19}$$

To get the result choose

$$\begin{aligned} k_0 &= \sigma + 2(r+1), \\ N &= r\sigma^{-1} \log_2(1/\epsilon) - 1, \\ \sigma &= 4r^2. \end{aligned}$$

□

Remark 3.2.6. *The compatibility condition* $N \geq 1$ *and* (3.2.12) *lead to*

$$\epsilon \leq \left[2^{-9} e^{-1} \pi^{-1} R (K^{(F,r)}_{k,p} + K^{(h,r)}_{k,p})^{-1} r^{-1} 2^{-2r} \right]^{\frac{\sigma}{2r}} =: \epsilon_{r,k,p} \leq 2^{-2\sigma/r} \leq 2^{-8r}.$$

Remark 3.2.7. *We point out the fact that Theorem 3.1.3 holds for the scale of Banach spaces* $W^{k,p}(M, \mathbb{C}^n \times \mathbb{C}^n)$*, where* $k \geq 1$*,* $1 < p < +\infty$*,* $n \in \mathbb{N}_0$*, and where* M *is a smooth manifold on which the Littlewood-Paley decomposition can be constructed, for example a compact manifold (see sect. 2.1 in [21]),* $\mathbb{R}^d$*, or a noncompact manifold satisfying some technical assumptions (see [19]).*

If we restrict to the case $p = 2$, and we consider M as either $\mathbb{R}^d$ or the d-dimensional torus $\mathbb{T}^d$, we can prove an analogous result for Hamiltonians $H(\psi,\bar\psi)$ with $(\psi,\bar\psi) \in H^k := W^{k,2}(M, \mathbb{C}\times\mathbb{C})$. In the following we denote by $B_k(R)$ the open ball of radius R and center 0 in H^k. We recall that the Fourier projection operator on H^k is given by

$$\pi_j \psi(x) := (2\pi)^{-d/2} \int_{j-1 \leq |k| \leq j} \hat\psi(k) e^{ik\cdot x} \mathrm{d}k, \ \ j \geq 1.$$

Theorem 3.2.8. *Fix $r \geq 1$, $R > 0$, $k_1 \gg 1$. Consider (3.1.4), and assume PER, INV (with respect to Fourier projection operators), NF and HVF. Then $\exists\ k_0 = k_0(r) > 0$ with the following properties: for any $k \geq k_1$ there exists $\epsilon_{r,k} \ll 1$ such that for any $\epsilon < \epsilon_{r,k}$ there exists $\mathcal{T}_\epsilon^{(r)} : B_k(R) \to B_k(2R)$ transformation s.t.*

$$H_r := H \circ \mathcal{T}_\epsilon^{(r)} = h_0 + \sum_{j=1}^{r} \epsilon^j \mathcal{Z}_j + \epsilon^{r+1}\, \mathcal{R}^{(r)},$$

where $\mathcal{Z}_j$ are in normal form, namely

$$\{\mathcal{Z}_j, h_0\} = 0, \tag{3.2.20}$$

and

$$\sup_{B_{k+k_0}(R)} \|X_{\mathcal{R}^{(r)}}\|_{H^k} \leq C_k, \tag{3.2.21}$$

$$\sup_{B_k(R)} \|\mathcal{T}_\epsilon^{(r)} - id\|_{H^k} \leq C_k\, \epsilon. \tag{3.2.22}$$

In particular, we have that

$$\mathcal{Z}_1(\psi,\bar{\psi}) = h_1(\psi,\bar{\psi}) + \langle F_1 \rangle (\psi,\bar{\psi}),$$

where $\langle F_1 \rangle (\psi,\bar{\psi}) := \int_0^{2\pi} F_1 \circ \Phi^t(\psi,\bar{\psi}) \frac{\mathrm{d}t}{2\pi}$.

The only technical difference between the proofs of Theorem 3.1.3 and the proof of Theorem 3.2.8 is that we exploit the Fourier cut-off operator

$$\Pi_N \psi(x) := \int_{|k|\leq N} \hat{\psi}(k) e^{ik\cdot x} \mathrm{d}k,$$

as in [5]. This in turn affects (3.2.6), which in this case reads

$$\sup_{B_{k+\sigma+2(r+1)}(R)} \|X_{\mathcal{R}_{N,r}}(\psi,\bar{\psi})\|_{H^k} \preceq \frac{\epsilon}{N^\sigma}, \tag{3.2.23}$$

and (3.2.19), for which we have to choose a bigger cut-off, $N = \epsilon^{-r\sigma}$.

3.3 Application to the nonlinear Klein-Gordon equation

3.3.1 The real nonlinear Klein-Gordon equation

We first consider the Hamiltonian of the real non-linear Klein-Gordon equation with power-type nonlinearity on a smooth manifold M (M is such the Littlewood-Paley decomposition is well-defined; take, for example, a smooth compact manifold, or $\mathbb{R}^d$). The Hamiltonian is of the form

$$H(u,v) = \frac{c^2}{2} \langle v, v \rangle + \frac{1}{2} \langle u, \langle \nabla \rangle_c^2 u \rangle \; + \; \lambda \int \frac{u^{2l}}{2l}, \tag{3.3.1}$$

where $\langle\nabla\rangle_c := (c^2 - \Delta)^{1/2}$, $\lambda \in \mathbb{R}$, $l \geq 2$.
If we introduce the complex-valued variable

$$\psi := \frac{1}{\sqrt{2}} \left[\left(\frac{\langle\nabla\rangle_c}{c} \right)^{1/2} u - i \left(\frac{c}{\langle\nabla\rangle_c} \right)^{1/2} v \right], \tag{3.3.2}$$

(the associated symplectic 2-form becomes $i \mathrm{d}\psi \wedge \mathrm{d}\bar{\psi}$), the Hamiltonian (3.3.1) in the coordinates $(\psi, \bar{\psi})$ is

$$H(\bar{\psi}, \psi) = \langle \bar{\psi}, c\langle\nabla\rangle_c \psi \rangle + \frac{\lambda}{2l} \int \left[\left(\frac{c}{\langle\nabla\rangle_c} \right)^{1/2} \frac{\psi + \bar{\psi}}{\sqrt{2}} \right]^{2l} \mathrm{d}x. \tag{3.3.3}$$

If we rescale the time by a factor c^2, the Hamiltonian takes the form (3.1.4), with $\epsilon = \frac{1}{c^2}$, and

$$H(\psi, \bar{\psi}) = h_0(\psi, \bar{\psi}) + \epsilon\, h(\psi, \bar{\psi}) + \epsilon\, F(\psi, \bar{\psi}), \tag{3.3.4}$$

where

$$h_0(\psi, \bar{\psi}) = \langle \bar{\psi}, \psi \rangle, \tag{3.3.5}$$

$$h(\psi, \bar{\psi}) = \langle \bar{\psi}, \left(c\langle\nabla\rangle_c - c^2\right) \psi \rangle \sim \sum_{j \geq 1} \epsilon^{j-1} \ \langle \bar{\psi}, a_j \Delta^j \psi \rangle =: \sum_{j \geq 1} \epsilon^{j-1} h_j(\psi, \bar{\psi}), \tag{3.3.6}$$

$$\begin{aligned} F(\psi, \bar{\psi}) &= \frac{\lambda}{2^{l+1} l} \int \left[\left(\frac{c}{\langle\nabla\rangle_c} \right)^{1/2} (\psi + \bar{\psi}) \right]^{2l} \mathrm{d}x & (3.3.7) \\ &\sim \frac{\lambda}{2^{l+1} l} \int (\psi + \bar{\psi})^{2l} \mathrm{d}x \\ &- \epsilon b_2 \int \left[(\psi + \bar{\psi})^{2l-1} \Delta(\psi + \bar{\psi}) + \ldots + (\psi + \bar{\psi}) \Delta((\psi + \bar{\psi})^{2l-1}) \right] \mathrm{d}x \\ &+ \mathcal{O}(\epsilon^2) \\ &=: \sum_{j \geq 1} \epsilon^{j-1}\, F_j(\psi, \bar{\psi}), & (3.3.8) \end{aligned}$$

where $(a_j)_{j\geq 1}$ and $(b_j)_{j\geq 1}$ are real coefficients, and $F_j(\psi, \bar{\psi})$ is a polynomial function of the variables ψ and $\bar{\psi}$ (along with their derivatives) and which admits a bounded vector field from a neighborhood of the origin in $W^{k+2(j-1),p}$ to $W^{k,p}$ for any $1 < p < +\infty$.
This description clearly fits the scheme treated in the previous section, and one can easily check that assumptions PER, NF and HVF are satisfied.

Therefore we can apply Theorem 3.1.3 to the Hamiltonian (3.3.4).

Remark 3.3.1. *About the normal forms obtained by applying Theorem 3.1.3, we remark that in the first step (case $r = 1$ in the statement of the Theorem) the homological equation we get is of the form*

$$\{\chi_1, h_0\} + F_1 = \langle F_1 \rangle, \tag{3.3.9}$$

where $F_1(\psi, \bar{\psi}) = \frac{\lambda}{2^{l+1} l} \int (\psi + \bar{\psi})^{2l} \mathrm{d}x$. Hence the transformed Hamiltonian is of the form

$$H_1(\psi, \bar{\psi}) = h_0(\psi, \bar{\psi}) + \frac{1}{c^2} \left[-\frac{1}{2} \langle \bar{\psi}, \Delta\psi \rangle + \langle F_1 \rangle (\psi, \bar{\psi}) \right] + \frac{1}{c^4} \mathcal{R}^{(1)}(\psi, \bar{\psi}). \tag{3.3.10}$$

If we neglect the remainder and we derive the corresponding equation of motion for the system, we get

$$-i\psi_t = \psi + \frac{1}{c^2}\left[-\frac{1}{2}\Delta\psi + \frac{\lambda}{2^{l+1}}\binom{2l}{l}|\psi|^{2(l-1)}\psi\right], \tag{3.3.11}$$

which is the NLS, and the Hamiltonian which generates the canonical transformation is given by

$$\chi_1(\psi,\bar\psi) = \frac{\lambda}{2^{l+1}l}\sum_{\substack{j=0,\dots,2l\\ j\neq l}}\frac{1}{i\,2(l-j)}\binom{2l}{j}\int \psi^{2l-j}\bar\psi^j \mathrm{d}x. \tag{3.3.12}$$

Remark 3.3.2. *Now we iterate the construction by passing to the case $r=2$, and for simplicity we consider just the case $l=2$, which at the first step yields the cubic NLS. In this case one has that*

$$\begin{aligned}\chi_1(\psi,\bar\psi) &= \int_0^T \tau\left[F_1(\Phi^\tau(\psi,\bar\psi)) \;-\; \langle F_1\rangle(\Phi^\tau(\psi,\bar\psi))\right]\frac{\mathrm{d}\tau}{T}\\ &= \frac{\lambda}{16}\int_0^{2\pi}\tau\int\left[|e^{i\tau}\psi + e^{-i\tau}\bar\psi|^4 - 6|\psi|^4\right]\,\mathrm{d}x\frac{\mathrm{d}\tau}{2\pi}.\end{aligned}$$

Since

$$|e^{i\tau}\psi + e^{-i\tau}\bar\psi|^4 = e^{4i\tau}\psi^4 + 4e^{2i\tau}\psi^3\bar\psi + 6\psi^2\bar\psi^2 + 4e^{-2i\tau}\psi\bar\psi^3 + e^{-4i\tau}\bar\psi^4$$

and since $\int_0^{2\pi}\tau e^{in\tau}\mathrm{d}\tau = \frac{2\pi}{i\,n}$ for any non-zero integer n, we finally get

$$\chi_1(\psi,\bar\psi) = \frac{\lambda}{16}\int\frac{\psi^4-\bar\psi^4}{4i} + \frac{2}{i}(\psi^3\bar\psi - \psi\bar\psi^3)\,\mathrm{d}x.$$

If we neglect the remainder of order c^{-6}, we have that

$$\begin{aligned}H\circ\mathcal{T}^{(1)} &= h_0 + \frac{1}{c^2}h_1 + \frac{1}{c^4}\{\chi_1,h_1\} + \frac{1}{c^4}h_2 +\\ &+ \frac{1}{c^2}\langle F_1\rangle + \frac{1}{c^4}\{\chi_1,F_1\} + \frac{1}{2c^4}\{\chi_1,\{\chi_1,h_0\}\} + \frac{1}{c^4}F_2 \qquad (3.3.13)\\ &= h_0 + \frac{1}{c^2}\left[h_1 + \langle F_1\rangle\right] + \frac{1}{c^4}\left[\{\chi_1,h_1\} + h_2 + \{\chi_1,F_1\} + \frac{1}{2}\{\chi_1,\langle F_1\rangle - F_1\} + F_2\right], \qquad (3.3.14)\end{aligned}$$

where $h_1(\psi,\bar\psi) = -\frac{1}{2}\langle\bar\psi,\Delta\psi\rangle$.

Now we compute the terms of order $\frac{1}{c^4}$.

$$\begin{aligned}\{\chi_1,h_1\} = \mathrm{d}\chi_1 X_{h_1} &= \frac{\partial\chi_1}{\partial\psi}\cdot i\frac{\partial h_1}{\partial\bar\psi} - i\frac{\partial\chi_1}{\bar\psi}\frac{\partial h_1}{\partial\psi} \qquad (3.3.15)\\ &= -\frac{\lambda}{32}\int\left[\Delta\psi\left(\psi^3 + 6\psi^2\bar\psi - 2\bar\psi^3\right) - \Delta\bar\psi(2\psi^3 - 6\psi\bar\psi^2 - \bar\psi^3)\right], \qquad (3.3.16)\end{aligned}$$

$$h_2 = -\frac{1}{8}\langle\bar\psi,\Delta^2\psi\rangle, \tag{3.3.17}$$

$$\{\chi_1, F_1\} = \frac{\lambda^2}{32} \int (4\psi^3 + 12\psi^2\bar{\psi} + 12\psi\bar{\psi}^2 + 4\bar{\psi}^3)(\psi^3 + 6\psi^2\bar{\psi} - 2\bar{\psi}^3) + \tag{3.3.18}$$

$$- (4\psi^3 + 12\psi^2\bar{\psi} + 12\psi\bar{\psi}^2 + 4\bar{\psi}^3)(2\psi^3 - 6\psi\bar{\psi}^2 - \bar{\psi}^3)\, \mathrm{d}x, \tag{3.3.19}$$

$$\{\chi_1, \langle F_1\rangle\} = \frac{\lambda^2}{2} \int \left[|\psi|^2\psi\, (\psi^3 + 6\psi^2\bar{\psi} - 2\bar{\psi}^3) - |\psi|^2\bar{\psi}\, (2\psi^3 - 6\psi\bar{\psi}^2 - \bar{\psi}^3)\right]\, \mathrm{d}x, \tag{3.3.20}$$

$$F_2 = \frac{\lambda}{16} \int \left[(\psi^3 + 3\psi^2\bar{\psi} + 3\psi\bar{\psi}^2 + \psi^3)\, \Delta\psi + (\bar{\psi}^3 + 3\bar{\psi}^2\psi + 3\bar{\psi}\psi^2 + \psi^3)\, \Delta\bar{\psi}\right]\, \mathrm{d}x. \tag{3.3.21}$$

Now, one can easily verify that $\langle\{\chi_1, h_1\}\rangle = \langle\{\chi_1, \langle F_1\rangle\}\rangle = 0$*, and that*

$$\langle\{\chi_1, F_1\}\rangle = \frac{\lambda^2}{32} \int (-8|\psi|^6 + 72|\psi|^6 + 4|\psi|^6) + (4|\psi|^6 + 72|\psi|^6 - 8|\psi|^6)\, \mathrm{d}x \tag{3.3.22}$$

$$= \frac{17}{4}\lambda^2 \int |\psi|^6\, \mathrm{d}x, \tag{3.3.23}$$

$$\langle F_2\rangle = \frac{\lambda}{16} \int 3\psi\bar{\psi}^2\, \Delta\psi + 3\bar{\psi}\psi^2\, \Delta\bar{\psi}\, \mathrm{d}x \tag{3.3.24}$$

$$= \frac{\lambda}{16} \int 3|\psi|^2(\psi\, \Delta\psi + \psi\Delta\bar{\psi})\, \mathrm{d}x. \tag{3.3.25}$$

Hence, up to a remainder of order $O\left(\frac{1}{c^6}\right)$*, we have that*

$$H_2 = h_0 + \frac{1}{c^2} \int \left[-\frac{1}{2}\langle\bar{\psi}, \Delta\psi\rangle + \frac{3}{8}\lambda|\psi|^4\right]\, \mathrm{d}x$$

$$+ \frac{1}{c^4} \int \left[\frac{17}{8}\lambda^2|\psi|^6 + \frac{3}{16}\lambda|\psi|^2(\bar{\psi}\, \Delta\psi + \psi\, \Delta\bar{\psi}) - \frac{1}{8}\langle\bar{\psi}, \Delta^2\psi\rangle\right]\, \mathrm{d}x, \tag{3.3.26}$$

which, by neglecting h_0 *(that yields only a gauge factor) and by rescaling the time, leads to the following equations of motion*

$$-i\psi_t = -\frac{1}{2}\Delta\psi + \frac{3}{4}\lambda|\psi|^2\psi$$

$$+ \frac{1}{c^2}\left[\frac{51}{8}\lambda^2|\psi|^4\psi + \frac{3}{16}\lambda\left(2|\psi|^2\, \Delta\psi + \psi^2\Delta\bar{\psi} - \Delta(|\psi|^2\bar{\psi})\right) - \frac{1}{8}\Delta^2\psi\right]. \tag{3.3.27}$$

To the author's knowledge, Eq. (3.3.27) *has never been studied before. It is the nonlinear analogue of a linear higher-order Schrödinger equation that appears in [22] and [23] in the context of semi-relativistic equations. Indeed, the linearization of Eq.* (3.3.27) *is studied within the framework of relativistic quantum field theory, as an approximation of nonlocal kinetic terms; Carles, Lucha and Moulay studied the well-posedness of these approximations, as well as the convergence of the equations as the order of truncation goes to infinity, in the linear case, also when one takes into account the effects of some time-independent potentials (e.g. bounded potentials, the harmonic-oscillator potential and the Coulomb potential).*
To the author's knowledge, very little is known for the nonlinear equation (3.3.27)*: we just mention [24], in which the well-posedness of a higher-order Schrodinger equation has been studied.*

3.3.2 The complex nonlinear Klein-Gordon equation

Now we consider the Hamiltonian of the complex non-linear Klein-Gordon equation with power-type nonlinearity on a smooth manifold M (take, for example, a smooth compact manifold, or $\mathbb{R}^d$)

$$H(w,p_w) = \frac{c^2}{2}\langle p_w, p_w\rangle + \frac{1}{2}\left\langle w, \langle\nabla\rangle_c^2 w\right\rangle \; + \; \lambda\int \frac{|w|^{2l}}{2l}, \tag{3.3.28}$$

where $w:\mathbb{R}\times M\to\mathbb{C}$, $\langle\nabla\rangle_c := (c^2-\Delta)^{1/2}$, $\lambda\in\mathbb{R}$, $l\geq 2$.
If we rewrite the Hamiltonian in terms of $u := Re(w)$ and $v := Im(w)$, we have

$$H(u,v,p_u,p_v) = \frac{c^2}{2}(\langle p_u,p_u\rangle + \langle p_v,p_v\rangle) + \frac{1}{2}(|\nabla u|^2 + |\nabla v|^2) + \frac{c^2}{2}(u^2+v^2) + \lambda\int\frac{(u^2+v^2)^l}{2l}. \tag{3.3.29}$$

We will consider by simplicity only the cubic case, $l=2$, but the argument may be readily generalized to the other power-type nonlinearities.

If we introduce the variables

$$\psi := \frac{1}{\sqrt{2}}\left[\left(\frac{\langle\nabla\rangle_c}{c}\right)^{1/2} u - i\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2} p_u\right], \tag{3.3.30}$$

$$\phi := \frac{1}{\sqrt{2}}\left[\left(\frac{\langle\nabla\rangle_c}{c}\right)^{1/2} v + i\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2} p_v\right], \tag{3.3.31}$$

(the associated symplectic 2-form becomes $i\mathrm{d}\psi\wedge\mathrm{d}\bar\psi - i\mathrm{d}\phi\wedge\mathrm{d}\bar\phi$), the Hamiltonian (3.3.28) in the coordinates $(\psi,\phi,\bar\psi,\bar\phi)$ reads

$$H(\psi,\phi,\bar\psi,\bar\phi) = \left\langle\bar\psi, c\langle\nabla\rangle_c\psi\right\rangle + \left\langle\bar\phi, c\langle\nabla\rangle_c\phi\right\rangle \tag{3.3.32}$$

$$+\frac{\lambda}{16}\int_M\left[\left\langle\psi+\bar\psi, \frac{c}{\langle\nabla\rangle_c}(\psi+\bar\psi)\right\rangle + \left\langle\phi+\bar\phi, \frac{c}{\langle\nabla\rangle_c}(\phi+\bar\phi)\right\rangle\right]^2\mathrm{d}x, \tag{3.3.33}$$

with associated equations of motion

$$\begin{cases} -i\psi_t &= c\langle\nabla\rangle_c\psi + \frac{1}{4}\left[\left\langle\psi+\bar\psi, \frac{c}{\langle\nabla\rangle_c}(\psi+\bar\psi)\right\rangle + \left\langle\phi+\bar\phi, \frac{c}{\langle\nabla\rangle_c}(\phi+\bar\phi)\right\rangle\right]\frac{c}{\langle\nabla\rangle_c}(\psi+\bar\psi), \\ i\phi_t &= c\langle\nabla\rangle_c\phi + \frac{1}{4}\left[\left\langle\psi+\bar\psi, \frac{c}{\langle\nabla\rangle_c}(\psi+\bar\psi)\right\rangle + \left\langle\phi+\bar\phi, \frac{c}{\langle\nabla\rangle_c}(\phi+\bar\phi)\right\rangle\right]\frac{c}{\langle\nabla\rangle_c}(\phi+\bar\phi). \end{cases}$$

If we rescale the time by a factor c^2, the Hamiltonian takes the form (3.1.4), with $\epsilon = \frac{1}{c^2}$, and

$$H(\psi,\phi,\bar\psi,\bar\phi) = H_0(\psi,\phi,\bar\psi,\bar\phi) + \epsilon\, h(\psi,\phi,\bar\psi,\bar\phi) + \epsilon\, F(\psi,\phi,\bar\psi,\bar\phi), \tag{3.3.34}$$

where

$$
\begin{aligned}
H_0(\psi,\phi,\bar\psi,\bar\phi) &= \left\langle \bar\psi, \psi \right\rangle + \left\langle \bar\phi, \phi \right\rangle, && (3.3.35)\\
h(\psi,\phi,\bar\psi,\bar\phi) &= \left\langle \bar\psi, \left(c\langle\nabla\rangle_c - c^2\right)\psi \right\rangle - \left\langle \bar\phi, \left(c\langle\nabla\rangle_c - c^2\right)\phi \right\rangle \\
&\sim \sum_{j\geq 1} \epsilon^{j-1} \left(\left\langle \bar\psi, a_j \Delta^j \psi \right\rangle + \left\langle \bar\phi, a_j \Delta^j \phi \right\rangle\right) \\
&=: \sum_{j\geq 1} \epsilon^{j-1} (h_j(\psi,\phi,\bar\psi,\bar\phi)), && (3.3.36)\\
F(\psi,\phi,\bar\psi,\bar\phi) &= \frac{\lambda}{16} \int_{\mathbb{T}} \left[\left\langle \psi+\bar\psi, \frac{c}{\langle\nabla\rangle_c}(\psi+\bar\psi) \right\rangle + \left\langle \phi+\bar\phi, \frac{c}{\langle\nabla\rangle_c}(\phi+\bar\phi) \right\rangle \right]^2 \mathrm{d}x, \\
&\sim \frac{\lambda}{16} \int \left[|\psi+\bar\psi|^2 + |\phi+\bar\phi|^2 \right]^2 \mathrm{d}x \\
&+ \mathcal{O}(\epsilon) \\
&=: \sum_{j\geq 1} \epsilon^{j-1} F_j(\psi,\phi,\bar\psi,\bar\phi), && (3.3.37)
\end{aligned}
$$

where $(a_j)_{j\geq 1}$ are real coefficients, and $F_j(\psi,\phi,\bar\psi,\bar\phi)$ is a polynomial function of the variables ψ, ϕ, $\bar\psi$, $\bar\phi$ (along with their derivatives) and which admits a bounded vector field from a neighborhood of the origin in $W^{k+2(j-1),p}(\mathbb{R}^d, \mathbb{C}^2\times\mathbb{C}^2)$ to $W^{k,p}(\mathbb{R}^d, \mathbb{C}^2\times\mathbb{C}^2)$ for any $1 < p < +\infty$. This description clearly fits the scheme treated in sect. 3.1 with $n = 2$, and one can easily check that assumptions PER, NF and HVF are satisfied.
Therefore we can apply Theorem 3.1.3 to the Hamiltonian (3.3.34).

Remark 3.3.3. *About the normal forms obtained by applying Theorem 3.1.3, we remark that in the first step (case $r = 1$ in the statement of the Theorem) the homological equation we get is of the form*

$$
\{\chi_1, h_0\} + F_1 = \langle F_1 \rangle, \tag{3.3.38}
$$

where $F_1(\psi,\bar\psi) = \frac{\lambda}{16}\int \left[|\psi+\bar\psi|^2 + |\phi+\bar\phi|^2\right]^2 \mathrm{d}x$. Hence the transformed Hamiltonian is of the form

$$
\begin{aligned}
H_1(\psi,\phi,\bar\psi,\bar\phi) &= h_0(\psi,\phi,\bar\psi,\bar\phi) + \frac{1}{c^2}\left[-\frac{1}{2}\left(\left\langle\bar\psi,\Delta\psi\right\rangle + \left\langle\bar\phi,\Delta\phi\right\rangle\right) + \langle F_1\rangle(\psi,\phi,\bar\psi,\bar\phi)\right] \\
&+ \frac{1}{c^4}\mathcal{R}^{(1)}(\psi,\phi,\bar\psi,\bar\phi), && (3.3.39)
\end{aligned}
$$

where

$$
\begin{aligned}
\langle F_1\rangle &= \frac{\lambda}{16}\left[6\psi^2\bar\psi^2 + 6\phi^2\bar\phi^2 + 8\psi\bar\psi\phi\bar\phi + 2\psi^2\phi^2 + 2\bar\psi^2\bar\phi^2\right] \\
&= \frac{\lambda}{8}\left[3(|\psi|^2+|\phi|^2)^2 + 2(\psi\phi - \bar\psi\bar\phi)^2\right].
\end{aligned}
$$

If we neglect the remainder and we derive the corresponding equation of motion for the system, we get

$$\begin{cases} -i\psi_t &= \psi + \frac{1}{c^2}\left\{-\frac{1}{2}\Delta\psi + \frac{\lambda}{4}\left[3(|\psi|^2+|\phi|^2)\psi + 2(\psi\phi+\bar{\psi}\bar{\phi})\bar{\phi}\right]\right\}, \\ \\ i\phi_t &= \phi + \frac{1}{c^2}\left\{-\frac{1}{2}\Delta\phi + \frac{\lambda}{4}\left[3(|\psi|^2+|\phi|^2)\phi + 2(\psi\phi+\bar{\psi}\bar{\phi})\bar{\psi}\right]\right\}, \end{cases} \tag{3.3.40}$$

which is a system of two coupled NLS equations.

Chapter 4

Approximation of the NLKG equation in the non-relativistic limit

4.1 Dynamics

Now we want to exploit the result of the previous section in order to deduce some consequences about the dynamics of the NLKG equation (3.3.4) in the non-relativistic limit. Consider the *simplified system*, that is the Hamiltonian H_r in the notations of Theorem 3.1.3, where we neglect the remainder:

$$H_{simp} := h_0 + \epsilon(h_1 + \langle F_1 \rangle) + \sum_{j=2}^{r} \epsilon^j (h_j + Z_j).$$

We recall that in the case of the NLKG the simplified system is actually the NLS (given by $h_0 + \epsilon(h_1 + \langle F_1 \rangle)$), plus higher-order normalized corrections. Now let ψ_r be a solution of

$$-i\,\dot{\psi}_r \;=\; X_{H_{simp}}(\psi_r), \tag{4.1.1}$$

then $\psi_a(t,x) := \mathcal{T}^{(r)}(\psi_r(c^2 t, x))$ solves

$$\dot{\psi}_a = ic\langle \nabla \rangle_c \psi_a + \frac{\lambda}{2l}\left(\frac{c}{\langle \nabla \rangle_c}\right)^{1/2} \left[\left(\frac{c}{\langle \nabla \rangle_c}\right)^{1/2} \frac{\psi_a + \bar{\psi}_a}{\sqrt{2}}\right]^{2l-1} - \frac{1}{c^{2r}} X_{\mathcal{T}^{(r)*}\mathcal{R}^{(r)}}(\psi_a, \bar{\psi}_a), \tag{4.1.2}$$

that is, the NLKG plus a remainder of order c^{-2r} (in the following we will refer to equation (4.1.2) as *approximate equation*, and to ψ_a as the *approximate solution* of the original NLKG). We point out that the original NLKG and the approximate equation differ only by a remainder of order c^{-2r}, which is evaluated on the approximate solution. This fact is extremely important: indeed, if one can prove the smoothness of the approximate solution (which often is easier to check than the smoothness of the solution of the original equation), then the contribution of the remainder may be considered small in the non-relativistic limit. This property is rather general, and has been already applied in the framework of normal form theory (see for example [7]).

Now let ψ be a solution of the NLKG equation (3.3.4), and set $\delta := \psi - \psi_a$ the error between the solution of the approximate equation and the original one. One can check that δ fulfills

$$\dot{\delta} = ic\langle \nabla \rangle_c \delta + [P(\psi_a + \delta, \bar{\psi}_a + \bar{\delta}) - P(\psi_a, \bar{\psi}_a)] + \frac{1}{c^{2r}} X_{\mathcal{T}^{(r)*}\mathcal{R}^{(r)}}(\psi_a(t), \bar{\psi}_a(t)),$$

where

$$P(\psi,\bar{\psi}) = \frac{\lambda}{2l}\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2}\left[\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2}\frac{\psi+\bar{\psi}}{\sqrt{2}}\right]^{2l-1}. \tag{4.1.3}$$

Thus we get

$$\dot{\delta} = i\,c\langle\nabla\rangle_c\delta + dP(\psi_a(t))\delta + \mathcal{O}(\delta^2) + \mathcal{O}\left(\frac{1}{c^{2r}}\right);$$

$$\delta(t) = e^{itc\langle\nabla\rangle_c}\delta_0 + \int_0^t e^{i(t-s)c\langle\nabla\rangle_c}dP(\psi_a(s))\delta(s)\mathrm{d}s + \mathcal{O}(\delta^2) + \mathcal{O}\left(\frac{1}{c^{2r}}\right). \tag{4.1.4}$$

By applying Gronwall inequality to (4.1.4) we obtain

Proposition 4.1.1. *Fix $r \geq 1$, $R > 0$, $k_1 \gg 1$, $1 < p < +\infty$. Then $\exists\ k_0 = k_0(r) > 0$ with the following properties: for any $k \geq k_1$ there exists $c_{l,r,k,p,R} \gg 1$ such that for any $c > c_{l,r,k,p,R}$, if we assume that*

$$\|\psi_0\|_{k+k_0,p} \leq R$$

and that there exists $T = T_{r,k,p} > 0$ such that the solution of (4.1.1) satisfies

$$\|\psi_r(t)\|_{k+k_0,p} \leq 2R, \quad \text{for } 0 \leq t \leq T,$$

then

$$\|\delta(t)\|_{k,p} \leq C_{k,p}\, c^{-2r}, \quad \text{for } 0 \leq t \leq T. \tag{4.1.5}$$

Remark 4.1.2. *If we restrict to $p = 2$, and to $M = \mathbb{R}^d$ or $M = \mathbb{T}^d$, the above result is actually a reformulation of Theorem 3.2 in [35]. We remark, however, that the time interval $[0,T]$ in which estimate* (4.1.5) *is valid is* independent *of c.*

Remark 4.1.3. *By exploiting estimate* (3.1.9) *about the canonical transformation, Proposition 4.1.1 leads immediately to a proof of Proposition 1.0.8.*

In order to study the evolution of the error between the approximate solution and the solution of the NLKG over longer (namely, c-dependent) time scales, we observe that the error is described by

$$\dot{\delta}(t) = i\,c\langle\nabla\rangle_c\delta(t) + dP(\psi_a(t))\delta(t); \tag{4.1.6}$$

$$\delta(t) = e^{itc\langle\nabla\rangle_c}\delta_0 + \int_0^t e^{i(t-s)c\langle\nabla\rangle_c}dP(\psi_a(s))\delta(s)\mathrm{d}s, \tag{4.1.7}$$

up to a remainder which is small, if we assume the smoothness of ψ_a.
Equation (4.1.6) in the context of dispersive PDEs is known as *semirelativistic spinless Salpeter equation* with a time-dependent potential. This system was introduced as a first order in time analogue of the KG equation for the Lorentz-covariant description of bound states within the framework of relativistic quantum field theory, and, despite the nonlocality of its Hamiltonian, some of its properties have already been studied (see [81] for a study from a physical point of view; for a more mathematical approach see [51] and the more recent works [22] and [23], which are closer to the spirit of our approximation).

It seems reasonable to estimate the solution of Equation (4.1.6) by studying and by exploiting its dispersive properties, and this will be the aim of the following sections. From now on we will consider by simplicity only the three-dimensional case, $d = 3$, but the argument may also be applied to $M = \mathbb{R}^d$ for $d > 3$.

4.2 Long time approximation

Now we study the evolution of the the error between the approximate solution ψ_a, namely the solution of (4.1.2), and the original solution ψ of (3.3.4) for long (that means, c-dependent) time intervals.

We begin by taking $\psi_0 \in W^{k+k_0,q}$ such that the solution ψ_r of the normalized equation (4.1.1) with initial datum ψ_0 exists for all times. We want to estimate the space-time norm $L^p_t W^{k,q}_x$ (for some particular values of the couple (p,q), that we will specify later) of the solution of (4.1.6).

Remark 4.2.1. *The assumption of global existence for ψ_r is actually a delicate matter. For the case $r = 1$ Eq.* (4.1.1) *is the nonlinear Schrödinger equation, for which the question of global existence has been widely studied, and a lot is known.*

For the general case $r > 1$, in [22] and [23] the authors proved that the linearized system, namely the one associated to

$$h_0 + \sum_{j=1}^{r} \epsilon^j h_j \tag{4.2.1}$$

admits a unique solution in $L^\infty(\mathbb{R})H^k(\mathbb{R}^3)$ (this is a simple application of the properties of the Fourier transform), and by a perturbative argument they also proved the global existence also for the higher oder Schrödinger equation with a bounded time-independent potential.
In the nonlinear case little is known (see for example [24] for the well-posedness for a higher-order nonlinear Schrödinger equation, and also Remark 4.2.9 in the next subsection).

Remark 4.2.2. *We point out that the case of the one-dimensional defocusing NLKG is also interesting, since for $\lambda = 1$ the normalized equation at first step is the defocusing NLS, which is integrable. It would be interesting also to understand whether globally well-posedness and scattering hold also the normalized order 2 equation* (3.3.27)*, which we later exploit to approximate solutions of the NLKG up to times of order $\mathcal{O}(c^2)$.*

Even though there is a one-dimensional integrable 4NLS equation related to the dynamics of a vortex filament (see [77] and references therein),

$$i\psi_t + \psi_{xx} + \frac{1}{2}|\psi|^2\psi - \nu\left[\psi_{xxxx} + \frac{3}{2}|\psi|^2\psi_{xx} + \frac{3}{2}\psi_x^2\bar{\psi} + \frac{3}{8}|\psi|^4\psi + \frac{1}{2}(|\psi|^2)_{xx}\psi\right] = 0,\ \nu \in \mathbb{R} \tag{4.2.2}$$

apparently there is no obvious relation between the above equation and Eq. (3.3.27)*. Furthermore, while the issue of local well-posedness for one-dimensional fourth-order Nonlinear Schrödinger has been quite studied (see for example [43]), there is only a recent result (see [74]) about global well-posedness and scattering for small radiation solutions of 4NLS, which unfortunately does not cover Eq.* (3.3.27)*, due to technical reasons.*

Therefore it seems difficult to give an explicit condition for global well-posedness and scattering for the normalized equation also in the one-dimensional case.

By following the arguments of Theorem 4.1 in [48] and Lemma 4.3 in [23] we obtain the following dispersive properties for (4.2.1) which will be useful in the sequel.

Proposition 4.2.3. *Let $r \geq 1$, and denote by $\mathcal{U}_r(t)$ the evolution operator associated to* (4.2.1) *rescaled back to the original time. Then we have the following local-in-time dispersive estimate*

$$\|\mathcal{U}_r(t)\|_{L^1(\mathbb{R}^3)\to L^\infty(\mathbb{R}^3)} \preceq c^{3\left(1-\frac{1}{r}\right)}|t|^{-3/(2r)},\ \ 0 < |t| \leq c^{2(r-1)}. \tag{4.2.3}$$

On the other hand, $\mathcal{U}_r(t)$ is unitary on $L^2(\mathbb{R}^3)$.
Now let us introduce the following set of admissible exponent pairs:

$$\Delta_r := \{(p,q) : (1/p, 1/q) \text{ lies in the closed quadrilateral } ABCD, \} \tag{4.2.4}$$

where

$$A = \left(\frac{1}{2}, \frac{1}{2}\right), \quad B = \left(1, \frac{1}{\tau_r}\right), \quad C = (1,0), \quad D = \left(\frac{1}{\tau_r'}, 0\right), \quad \tau_r = \frac{2r-1}{r-1}, \quad \frac{1}{\tau_r} + \frac{1}{\tau_r'} = 1.$$

Then for any $(p,q) \in \Delta_r \setminus \{(2,2), (1,\tau_r), (\tau_r', \infty)\}$

$$\|\mathcal{U}_r(t)\|_{L^p(\mathbb{R}^3)\to L^q(\mathbb{R}^3)} \preceq c^{3\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)} |t|^{-\frac{3}{2r}\left(\frac{1}{q}-\frac{1}{p}\right)}, \quad 0 < |t| \leq c^{2(r-1)}. \tag{4.2.5}$$

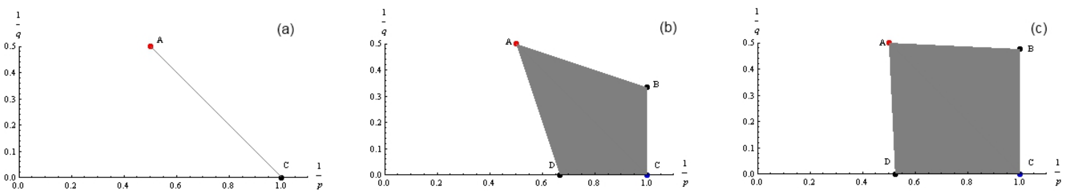

Figure 4.1: Set of admissible exponents Δ_r for different values of r: (a) r=1 (this is the Schrödinger case); (b) r=2; (c) r=11.

Let $r \geq 1$: in the following lemma we will say that (p,q) is an order-r admissible pair when $2 \leq q \leq +\infty$ for $r \geq 2$ ($2 \leq q \leq 6$ for $r = 1$), and

$$\frac{2}{p} + \frac{3}{rq} = \frac{3}{2r}. \tag{4.2.6}$$

Proposition 4.2.4. *Let $r \geq 1$, and denote by $\mathcal{U}_r(t)$ the evolution operator associated to* (4.2.1) *rescaled back to the original time. Let (p,q) and (r,s) be order-r admissible pairs, then for any $T \preceq c^{2(r-1)}$*

$$\|\mathcal{U}_r(t)\phi_0\|_{L^p([0,T])L^q(\mathbb{R}^3)} \preceq c^{3\left(1-\frac{1}{r}\right)\left(\frac{1}{2}-\frac{1}{q}\right)} \|\phi_0\|_{L^2(\mathbb{R}^3)} = c^{\left(1-\frac{1}{r}\right)\frac{2r}{p}} \|\phi_0\|_{L^2(\mathbb{R}^3)}. \tag{4.2.7}$$

Radiation solution

As an application of Proposition 2.1.1, we consider the following case. Fix $r > 1$, and let $\psi_r = \eta_{rad}$ be a radiation solution of (4.1.1), namely such that

$$\eta_{rad,0} := \eta_{rad}(0) \in W^{k+k_0,p}(\mathbb{R}^3), \tag{4.2.8}$$

where $k_0 > 0$ and $k \gg 1$ are the ones in Theorem 3.1.3, and such that $\eta_{rad}(c^2 t)$ satisfies (4.2.5) for any p such that $(p,3) \in \Delta_r \setminus \{(1,\tau_r)\}$, with $\mathcal{U}_r$ replaced by the evolution operator of (4.1.1) (rescaled back to the original time).

Remark 4.2.5. *The assumption $r > 1$ is due to* (4.2.3)*, and it reflects the fact that we want to study the behavior of the error δ for long (c-dependent) timescales.*

Let $\delta(t)$ be a solution of (4.1.6); then by Duhamel formula

$$\delta(t) := \mathcal{U}(t,0)\delta_0 = e^{itc\langle\nabla\rangle_c}\delta_0 + \int_0^t e^{i(t-s)c\langle\nabla\rangle_c}\mathrm{d}P(\psi_a(s))\mathcal{U}(s,0)\delta_0\mathrm{d}s. \tag{4.2.9}$$

Now fix $T \preceq c^{2(r-1)}$; we want to estimate the local-in-time norm in the space $L^\infty([0,T])H^k(\mathbb{R}^3)$ of the error $\delta(t)$.
By (2.1.2) we can estimate the first term. We can estimate the second term by (2.1.3): hence for any (p,q) Schrödinger-admissible exponents

$$\begin{aligned}
&\left\|\int_0^t e^{i(t-s)c\langle\nabla\rangle_c} dP(\psi_a(s))\delta(s)\mathrm{d}s\right\|_{L^\infty_t([0,T])H^k_x} \\
&\preceq c^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}\|\langle\nabla\rangle_c^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}\mathrm{d}P(\psi_a(t))\delta(t)\|_{L^{p'}_t([0,T])W^{k,q'}_x} \\
&\preceq c^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}\|\langle\nabla\rangle_c^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}\mathrm{d}P(\eta_{rad}(c^2t))\delta(t)\|_{L^{p'}_t([0,T])W^{k,q'}_x} \\
&\quad + c^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}}\|\langle\nabla\rangle_c^{\frac{1}{p}-\frac{1}{q}+\frac{1}{2}}[\mathrm{d}P(\psi_a(t)) - \mathrm{d}P(\eta_{rad}(c^2t))]\delta(t)\|_{L^{p'}_t([0,T])W^{k,q'}_x} \\
&=: I_p + II_p,
\end{aligned}$$

but recalling (4.1.3) one has that

$$I_p \preceq \frac{|\lambda|}{2^{l-1/2}(2l)(2l-1)}c^{\frac{1}{q}-\frac{1}{p}+\frac{1}{2}}\left\|\langle\nabla\rangle_c^{\frac{1}{p}-\frac{1}{q}-\frac{1}{2}}\left[\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2}(\eta_{rad}+\bar{\eta}_{rad})\right]^{2(l-1)}\delta(t)\right\|_{L^{p'}_t([0,T])W^{k,q'}_x},$$

and by choosing $p=2$, $q=6$ we get (since $\|(c/\langle\nabla\rangle_c)^{1/6}\|_{L^{6/5}\to L^{6/5}} \leq 1$)

$$I_2 \leq \frac{|\lambda|}{2^{l-1/2}(2l)(2l-1)}\left\|\left[\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2}(\eta_{rad}(c^2t)+\bar{\eta}_{rad}(c^2t))\right]^{2(l-1)}\delta(t)\right\|_{L^2_t([0,T])W^{k,6/5}_x}.$$

Now, since by Hölder inequality

$$\begin{aligned}
&\left\|\left[\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2}(\eta_{rad}(c^2t)+\bar{\eta}_{rad}(c^2t))\right]^{2(l-1)}\delta(t)\right\|_{L^2_t([0,T])W^{k,6/5}_x} \\
&\leq \left\|\left[\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2}(\eta_{rad}(c^2t)+\bar{\eta}_{rad}(c^2t))\right]^{2(l-1)}\right\|_{L^2_t([0,T])W^{k,3}_x}\|\delta(t)\|_{L^\infty_t([0,T])H^k_x},
\end{aligned}$$

and by Sobolev product theorem (recall that $l \geq 2$, and that $3k > 3$) we can deduce that

$$\begin{aligned}
&\left\| \left[\left(\frac{c}{\langle \nabla \rangle_c} \right)^{1/2} (\eta_{rad}(c^2 t) + \bar{\eta}_{rad}(c^2 t)) \right]^{2(l-1)} \right\|_{L^2_t([0,T]) W^{k,3}_x} \\
&\leq \left[\int_0^T \left\| \left[\left(\frac{c}{\langle \nabla \rangle_c} \right)^{1/2} (\eta_{rad}(c^2 t) + \bar{\eta}_{rad}(c^2 t)) \right] \right\|^{4(l-1)}_{W^{k,3}_x} \mathrm{d}t \right]^{1/2} \\
&\leq \left\| \eta_{rad}(c^2 t) + \bar{\eta}_{rad}(c^2 t) \right\|^{2(l-1)}_{L^{4(l-1)}_t([0,T]) W^{k,3}_x},
\end{aligned}$$

but for any $1 \leq p \leq 2$ such that $(p, 3) \in \Delta_r \setminus \{(1, \tau_r)\}$ we have

$$\begin{aligned}
\left\| \eta_{rad}(c^2 t) \right\|_{L^{4(l-1)}_t([0,T]) W^{k,3}_x} &\leq c^{3(1-1/r)\left(\frac{1}{p}-\frac{1}{3}\right)} \| \eta_{rad,0} \|_{W^{k,p}_x} \left\| |t|^{-\frac{3}{2r}\left(\frac{1}{3}-\frac{1}{p}\right)} \right\|_{L^{4(l-1)}_t([0,T])} \\
&\leq c^{3(1-1/r)\left(\frac{1}{p}-\frac{1}{3}\right)} \| \eta_{rad,0} \|_{W^{k+k_0,p}_x} \left\| |t|^{-\frac{3}{2r}\left(\frac{1}{3}-\frac{1}{p}\right)} \right\|_{L^{4(l-1)}_t([0,T])}, \quad (4.2.10)
\end{aligned}$$

which is finite and does not depend on c for

$$\| \eta_{rad,0} \|_{W^{k+k_0,p}_x} = c^{-\alpha} M, \quad (4.2.11)$$

$$\begin{aligned}
\alpha &\geq 3 \left(1 - \frac{1}{r} \right) \left(\frac{1}{p} - \frac{1}{3} \right) + \frac{(r-1)}{2(l-1)} + 3 \frac{r-1}{r} \left(\frac{1}{p} - \frac{1}{3} \right) \\
&= 6 \left(1 - \frac{1}{r} \right) \left(\frac{1}{p} - \frac{1}{3} \right) + \frac{(r-1)}{2(l-1)} := \alpha^*(l, r, p). \quad (4.2.12)
\end{aligned}$$

where M is independent of c. Indeed, under conditions (4.2.11) - (4.2.12) we obtain that for any $c \geq 1$

$$\left\| \eta_{rad}(c^2 t) \right\|_{L^{4(l-1)}_t([0,T]) W^{k,3}_x} \leq c^{-\alpha + \alpha^*(l,r,p)} M.$$

Furthermore, via (3.1.9) one can show that there exists $c_{r,k,p} > 0$ sufficiently large such that for $c \geq c_{r,k,p}$ the term II_2 can be bounded by $\frac{1}{c^2} I_2$.

This means that we can estimate the $L^\infty([0,T])H^k$ norm of the error only for a small (with respect to c) radiation solution.

Remark 4.2.6. *We notice that $\tau_r < 3$ for $r > 2$, hence the point $(1, 3)$ is contained in Δ_r for $r > 2$. The smallness conditions (4.2.11) - (4.2.12) are probably due to the fact that we had no loss of derivatives in the previous estimates, which in turn is based on the estimates (4.2.5) for the normalized equation. If one could find the analogue of (4.2.5) with loss of derivatives, we think that such conditions could be improved.*

To summarize, we get the following result

Proposition 4.2.7. *Consider (3.3.3), let $r > 1$, and fix $k_1 \gg 1$. Let $1 \leq p \leq 2$ be such that $(p, 3) \in \Delta_r \setminus \{(1, \tau_r)\}$ (where Δ_r and τ_r are defined as in (4.2.4)). Then $\exists\ k_0 = k_0(r) > 0$ such that for any $k \geq k_1$ the following holds: consider the solution η_{rad} of (4.1.1) with initial datum $\eta_{rad,0} \in W^{k+k_0,p}$, and assume also that η_{rad} satisfies the decay estimate (4.2.5) for (4.1.1).*

Call δ the difference between the solution of the approximate equation (4.1.2) *and the original solution of the Hamilton equation for* (3.3.3)*, and assume that $\delta_0 := \delta(0)$ satisfies*

$$\|\delta_0\|_{H^k_x} \preceq \frac{1}{c^2}.$$

Then there exist $\alpha^* := \alpha^*(l,r,p) > 0$ and there exists $c^* := c^*(r,k,p) > 1$, such that for any $\alpha > \alpha^*$ and for any $c > c^*$, if $\eta_{rad,0}$ satisfies

$$\|\eta_{rad,0}\|_{W^{k+k_0,p}} \preceq c^{-\alpha},$$

then

$$\sup_{t\in[0,T]} \|\delta(t)\|_{H^k_x} \preceq \frac{1}{c^2}, \;\; T \preceq c^{2(r-1)}.$$

By exploiting (3.1.9), we can rewrite Proposition 4.2.7 in terms of the solution of the normal form equation (4.1.1).

Theorem 4.2.8. *Consider* (3.3.3)*, let $r > 1$, and fix $k_1 \gg 1$. Let $1 \le p \le 2$ be such that $(p,3) \in \Delta_r \setminus \{(1,\tau_r)\}$ (where Δ_r and τ_r are defined as in* (4.2.4)*). Then $\exists\ k_0 = k_0(r) > 0$ such that for any $k \ge k_1$ the following holds: consider the solution ψ_r of* (4.1.1) *with initial datum $\psi_{r,0} \in W^{k+k_0,p}$. Assume also that ψ_r satisfies the decay estimate* (4.2.5) *for* (4.1.1)*.*
Then there exist $\alpha^* := \alpha^*(l,r,p) > 0$ and there exists $c^* := c^*(r,k,p) > 1$, such that for any $\alpha > \alpha^*$ and for any $c > c^*$, if $\psi_{r,0}$ satisfies

$$\|\psi_{r,0}\|_{W^{k+k_0,p}} \preceq c^{-\alpha},$$

then

$$\sup_{t\in[0,T]} \|\psi(t) - \psi_r(t)\|_{H^k_x} \preceq \frac{1}{c^2}, \;\; T \preceq c^{2(r-1)},$$

where $\psi(t)$ is the solution of (3.3.4) *with initial datum $\psi_{r,0}$.*

Remark 4.2.9. *For $l = 2$, which corresponds to the cubic NLKG, by taking $r = 2$ in Theorem 4.2.7, and this allows one to approximate small radiation solutions up to times of order $O(c^2)$, assuming that the decay* (4.2.5) *holds also for the simplified equation* (4.1.1)*.*

It would be interesting to study in detail Eq. (3.3.27)*, and to state explicitly some conditions that ensure scattering for solutions of the order-r normalized equation. Even though some results for the linearization of Eq.* (3.3.27) *have already been established (see [16] and [48] for dispersive estimates, and [23] for Strichartz estimates), the study of the fourth-order NLS-type (4NLS) equation is still open: while there are some papers dealing with the local well-posedness of 4NLS (see for example [77] for the one-dimensional case, [44] for the multidimensional case), global well-posedness and scattering results are much less known. The recent [74] gives the first global well-posedness and scattering result for small radiation solutions of 4NLS in any dimension $d \ge 1$, but unfortunately does not cover Eq.* (3.3.27)*, due to technical reasons. Therefore we cannot give a more explicit statement for the approximation up to times of order $\mathcal{O}(c^2)$ for the NLKG on $\mathbb{R}^d$, $d \ge 3$.*

Remark 4.2.10. *One may ask whether it is possible to prove an approximation result also in the relativistic Sobolev spaces $\mathscr{W}^{k,p}_c$. A modification of the argument used to prove Proposition 4.2.7 allows to state an approximation result in the space $L^\infty_t \mathscr{H}^k_c$. Indeed, by Proposition 2.1.1*

$$\begin{aligned}
&\left\| \int_0^t e^{i(t-s)c\langle\nabla\rangle_c} dP(\eta_{rad}(s))\delta(s)\mathrm{d}s \right\|_{L_t^\infty([0,T])\mathscr{H}_c^{k+1/2}} \\
&\preceq \left\| \left[\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2} (\eta_{rad}+\bar{\eta}_{rad}) \right]^{2(l-1)} \delta(t) \right\|_{L_t^2([0,T])\mathscr{W}_c^{k+1/3,6/5}} \\
&\overset{(C.0.7)}{\preceq} \left\| \left[\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2} (\eta_{rad}+\bar{\eta}_{rad}) \right]^{2(l-1)} \right\|_{L_t^2([0,T])\mathscr{W}_c^{k+1/3,3}} \|\delta\|_{L_t^\infty L_x^2} \\
&\quad + \left\| \left[\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2} (\eta_{rad}+\bar{\eta}_{rad}) \right]^{2(l-1)} \right\|_{L_t^2([0,T])L_x^3} \|\delta\|_{L_t^\infty \mathscr{H}_c^{k+1/3}} \\
&\preceq \left\| \left\| \left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2} (\eta_{rad}+\bar{\eta}_{rad}) \right\|_{\mathscr{W}_c^{k+1/3,3}}^{2(l-1)} \right\|_{L_t^2([0,T])} \|\delta\|_{L_t^\infty L_x^2} \\
&\quad + \left\| \left\| \left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2} (\eta_{rad}+\bar{\eta}_{rad}) \right\|_{L_x^{6(l-1)}}^{2(l-1)} \right\|_{L_t^2([0,T])} \|\delta\|_{L_t^\infty \mathscr{H}_c^{k+1/3}} \\
&\preceq \left(\| \|\eta_{rad}\|_{\mathscr{W}_c^{k-1/6,3}}^{2(l-1)} \|_{L_t^2} + \| \|\eta_{rad}\|_{\mathscr{W}_c^{-1/2,6(l-1)}}^{2(l-1)} \|_{L_t^2} \right) \|\delta\|_{L_t^\infty \mathscr{H}_c^{k+1/2}} \\
&\preceq \left(\|\eta_{rad}\|_{L_t^{4(l-1)}([0,T])\mathscr{W}_c^{k-1/6,3}}^{2(l-1)} + \|\eta_{rad}\|_{L_t^{4(l-1)}([0,T])\mathscr{W}_c^{-1/2,6(l-1)}}^{2(l-1)} \right) \|\delta\|_{L_t^\infty \mathscr{H}_c^{k+1/2}}.
\end{aligned}$$

Now, the term

$$\|\eta_{rad}\|_{L_t^{4(l-1)}([0,T])\mathscr{W}_c^{k-1/6,3}}^{2(l-1)},$$

can be bounded as in Proposition 4.2.7, namely by assuming the smallness conditions (4.2.11)-(4.2.12). *The term*

$$\|\eta_{rad}\|_{L_t^{4(l-1)}([0,T])\mathscr{W}_c^{-1/2,6(l-1)}}^{2(l-1)},$$

can also be bounded by exploiting the dispersive estimates (4.2.5)*. Indeed, for any* $p_1 \in [1,2]$ *such that* $(p_1, 6(l-1)) \in \Delta_r$ *one has*

$$\begin{aligned}
\|\eta_{rad}\|_{L_t^{4(l-1)}([0,T])\mathscr{W}_c^{-1/2,6(l-1)}} &\preceq \|\eta_{rad}\|_{L_t^{4(l-1)}([0,T])L_x^{6(l-1)}} \\
&\preceq c^{3(1-1/r)\left(\frac{1}{p_1}-\frac{1}{6(l-1)}\right)} \left\| |t|^{-\frac{3}{2r}\left(\frac{1}{6(l-1)}-\frac{1}{p_1}\right)} \right\|_{L_t^{4(l-1)}([0,T])} \|\eta_{rad,0}\|_{L^{p_1}},
\end{aligned}$$

which is finite and does not depend on $c \geq 1$ *for*

$$\|\eta_{rad,0}\|_{L_x^{p_1}} = c^{-\alpha} M, \tag{4.2.13}$$

$$\alpha \geq 6\left(1-\frac{1}{r}\right)\left(\frac{1}{p_1}-\frac{1}{6(l-1)}\right) + \frac{r-1}{2(l-1)} := \alpha_1^*(l,r,p_1). \tag{4.2.14}$$

where M *is independent of* c*. We obtain the following result*

Proposition 4.2.11. *Consider* (3.3.3)*, let* $r > 1$*, and fix* $k_1 \gg 1$*. Let* $1 \leq p \leq 2$ *be such that* $(p,3) \in \Delta_r \setminus \{(1,\tau_r)\}$*, and let* $1 \leq p_1 \leq 2$ *be such that* $(p_1, 6(l-1)) \in \Delta_r$ *(where* Δ_r *and* τ_r

are defined as in (4.2.4)*). Then* $\exists\ k_0 = k_0(r) > 0$ *such that for any* $k \geq k_1$ *the following holds: consider the solution* η_{rad} *of* (4.1.1) *with initial datum* $\eta_{rad,0} \in \mathscr{W}_c^{k+k_0,p} \cap L^{p_1}$*, and assume also that* η_{rad} *satisfies the decay estimate* (4.2.5) *for* (4.1.1)*.*
Call δ *the difference between the solution of the approximate equation* (4.1.2) *and the original solution of the Hamilton equation for* (3.3.3)*, and assume that* $\delta_0 := \delta(0)$ *satisfies*

$$\|\delta_0\|_{\mathscr{H}_c^k} \preceq \frac{1}{c^2}.$$

Then there exist $\alpha^* := \alpha^*(l, r, p) > 0$ *and* $\alpha_1^* := \alpha_1^*(l, r, p_1) > 0$ *and there exists* $c^* := c^*(r, k, p) > 1$*, such that for any* $\alpha > \max(\alpha^*, \alpha_1^*)$ *and for any* $c > c^*$*, if* $\eta_{rad,0}$ *satisfies*

$$\|\eta_{rad,0}\|_{\mathscr{W}_c^{k+k_0,p} \cap L^{p_1}} \preceq c^{-\alpha},$$

then

$$\sup_{t\in[0,T]} \|\delta(t)\|_{\mathscr{H}_c^k} \preceq \frac{1}{c^2}, \quad T \preceq c^{2(r-1)}.$$

By exploiting (3.1.9), we can rewrite Proposition 4.2.11 in terms of the solution of the normal form equation (4.1.1).

Theorem 4.2.12. *Consider* (3.3.3)*, let* $r > 1$*, and fix* $k_1 \gg 1$*. Let* $1 \leq p \leq 2$ *be such that* $(p, 3) \in \Delta_r \setminus \{(1, \tau_r)\}$*, and let* $1 \leq p_1 \leq 2$ *be such that* $(p_1, 6(l-1)) \in \Delta_r$ *(where* Δ_r *and* τ_r *are defined as in* (4.2.4)*). Then* $\exists\ k_0 = k_0(r) > 0$ *such that for any* $k \geq k_1$ *the following holds: consider the solution* ψ_r *of* (4.1.1) *with initial datum* $\psi_{r,0} \in \mathscr{W}_c^{k+k_0,p} \cap L^{p_1}$*. Assume also that* ψ_r *satisfies the decay estimate* (4.2.5) *for* (4.1.1)*.*
Then there exist $\alpha^* := \alpha^*(l, r, p) > 0$ *and* $\alpha_1^* := \alpha_1^*(l, r, p_1) > 0$ *and there exists* $c^* := c^*(r, k, p) > 1$*, such that for any* $\alpha > \max(\alpha^*, \alpha_1^*)$ *and for any* $c > c^*$*, if* $\psi_{r,0}$ *satisfies*

$$\|\psi_{r,0}\|_{\mathscr{W}_c^{k+k_0,p} \cap L^{p_1}} \preceq c^{-\alpha},$$

then

$$\sup_{t\in[0,T]} \|\psi(t) - \psi_r(t)\|_{\mathscr{H}_c^k} \preceq \frac{1}{c^2}, \quad T \preceq c^{2(r-1)},$$

where $\psi(t)$ *is the solution of* (3.3.4) *with initial datum* $\psi_{r,0}$*.*

Remark 4.2.13. *At the first step of Birkhoff Normal Form,* $r = 1$*, one can show with a similar argument (where one can exploit Strichartz estimates for NLS, instead of the stronger estimate* (4.2.3)*) that the approximation is valid up to* $\mathcal{O}(1)$*-timescales, hence only locally uniformly in time, but it does not need any smallness assumption as in* (4.2.12)*-*(4.2.12)*. An example of such a result for the cubic case* $l = 2$*, which is analogous to Proposition 4.1.1, is the following*

Proposition 4.2.14. *Consider* (3.3.3)*, and fix* $k_1 \gg 1$*. Then* $\exists\ k_0 > 0$ *such that for any* $k \geq k_1$ *the following holds: consider the solution* η_{rad} *of the cubic NLS* (4.1.1) *with initial datum* $\eta_{rad}(0) \in H^{k+k_0}$*.*
Call δ *the difference between the solution of the approximate equation* (4.1.2) *and the original solution of the Hamilton equation for* (3.3.3)*, and assume that* $\delta_0 := \delta(0)$ *satisfies*

$$\|\delta_0\|_{H_x^k} \preceq \frac{1}{c^2}.$$

Then there exists $c^* := c^*(k,p) > 0$, such that for any $c > c^*$ there exists $T := T(k,p) > 0$ independent of c such that

$$\sup_{t\in[0,T]} \|\delta(t)\|_{H^k_x} \preceq \frac{1}{c^2}.$$

Remark 4.2.15. *If one considers the linear KG equation* (2.1.1) *and applies the above argument, one obtains the following approximation result.*
Fix $r \geq 1$, $k_1 \gg 1$. Let $1 \leq p \leq 2$ be such that $(p,3) \in \Delta_r \setminus \{(1,\tau_r)\}$.
Then $\exists\ k_0 = k_0(r) > 0$ such that for any $k \geq k_1$ the solution η_{rad} of (4.2.1) *with initial datum $\eta(0) \in W^{k+k_0,p}$ satisfies the following property: call δ the difference between the solution of the approximate equation and the original solution of* (2.1.1), *and assume that $\delta_0 := \delta(0)$ satisfies*

$$\|\delta_0\|_{H^k_x} \preceq \frac{1}{c^2}.$$

Then there exists $c^* := c^*(r,k,p) > 0$, such that for any $c > c^*$

$$\sup_{t\in[0,T]} \|\delta(t)\|_{H^k_x} \preceq \frac{1}{c^2}, \quad T \preceq c^{2(r-1)}.$$

This result has been proved in the case $r = 1$ in Appendix A of [23].

Standing waves solutions

Now we consider the approximation of another important type of solutions, the so-called standing waves solutions. Fix $r \geq 1$, and let ψ_r be a standing wave solution of (4.1.1), namely of the form

$$\psi_r(t,x) = e^{it\omega}\eta_\omega(x), \tag{4.2.15}$$

where $\omega \in \mathbb{R}$, and $\eta_\omega \in \mathcal{S}(\mathbb{R}^3)$ solves

$$-\omega\eta_\omega = X_{H_{simp}}(\eta_\omega).$$

The issue of (in)stability of standing waves and solitons has a long history: for the NLS equation and the NLKG the orbital stability of standing waves has been discussed first in [78]; for the NLS the orbital stability of one soliton solutions has been treated in [39], while the asymptotic stability has been discussed in [29] for one soliton solutions, and in [71] and [72] for N-solitons. For the higher-order Schrödinger equation we mention [56], which deals with orbital stability of standing waves for fourth-order NLS-type equations. For the NLKG equation, the instability of solitons and standing waves has been studied in [79], [45] and [65].

As in the case of the radiation, if $\delta(t)$ is a solution of (4.1.6), then by Duhamel formula

$$\dot{\delta} = ic\langle\nabla\rangle_c\delta(t) + \mathrm{d}P(\psi_a(t), \bar{\psi}_a(t))\delta(t).$$

Since

$$P(e^{it\omega}\eta_\omega, e^{-it\omega}\bar{\eta}_\omega) = 2^{l-1/2}\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2}\left[\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2} Re(e^{it\omega}\eta_\omega)\right]^{2l-1},$$

we have that

$$\mathrm{d}P(\eta_\omega,\bar{\eta}_\omega)e^{it\omega}h = 2^{l-1/2}\left(\frac{c}{\langle\nabla\rangle_c}\right)\left[\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2}\cos(\omega t)\eta_\omega\right]^{2(l-1)}(e^{it\omega}h+e^{-it\omega}\bar{h}),$$

and by setting $\delta = e^{-it\omega}h$, one gets

$$-i\dot{h} = (c\langle\nabla\rangle_c+\omega)h + 2^{l-1/2}\cos^{2(l-1)}(\omega t)\left(\frac{c}{\langle\nabla\rangle_c}\right)\left[\left(\frac{c}{\langle\nabla\rangle_c}\right)^{1/2}\eta_\omega\right]^{2(l-1)}(h+e^{-2it\omega}\bar{h}) \tag{4.2.16}$$

$$+\left[\mathrm{d}P(\psi_a(s),\bar{\psi}_a(s)) - \mathrm{d}P(\eta_\omega,\bar{\eta}_\omega)\right]h. \tag{4.2.17}$$

Eq. (4.2.16) is a Salpeter spinless equation with a periodic time-dependent potential; therefore, in order to get some information about the error, one would need the corresponding Strichartz estimates for Eq. (4.2.16). Unfortunately, in the literature of dispersive estimates there are only few results for PDEs with time-dependent potentials, and the majority of them is of perturbative nature; for the Schrödinger equation we mention [31] and [38], in which Strichartz estimates are proved in a non-perturbative framework.

By using Proposition 4.1.1 one can show that the NLKG can be approximated by the simplified equation (4.1.1) locally uniformly in time, up to an error of order $\mathcal{O}(c^{-2r})$. One may think that arguing in a non-perturbative framework one could derive some almost-global-in-time Strichartz estimates for Eq. (4.2.16); however, since also Proposition (2.2.7) deals only with local-in-time Strichartz estimates, we are not able to exploit the techniques of [31] in order to get a result valid over the $\mathcal{O}(1)$-timescale.

Thus the result we get is the following one

Proposition 4.2.16. *Consider* (3.3.3)*, and fix $r \geq 1$ and $k_1 \gg 1$. Assume that $\omega \in \mathbb{R}$ and $\eta_\omega \in \mathcal{S}(\mathbb{R}^3)$ are such that* (4.2.15) *is a solution of the simplified equation* (4.1.1)*.*
Then there exists $k_0 = k_0(r) > 0$ such that for any $k \geq k_1$ the following holds. Call δ the difference between the solution of the approximate equation (4.1.2) *and the original solution of the Hamilton equation for* (3.3.3)*, and assume that $\delta_0 := \delta(0)$ satisfies*

$$\|\delta_0\|_{H^k_x} \preceq \frac{1}{c^{2r}}.$$

Then there exists $c^* := c^*(k,k_0,\|\eta_\omega\|_{k+k_0}) > 0$, such that for any $c > c^*$ there exists $T := T(k,k_0,\|\eta_\omega\|_{k+k_0}) > 0$ independent of c such that

$$\sup_{t\in[0,T]}\|\delta(t)\|_{H^k_x} \preceq \frac{1}{c^{2r}}.$$

Remark 4.2.17. *Of course the existence of a standing wave for the simplified equation* (4.1.1) *is a far from trivial question (see [39] for the NLS equation, and [56] for the fourth-order NLS-type equation).*
For $r = 1$ and $\lambda = 1$ (namely, the defocusing case), we can exploit the criteria in [39] for existence and stability of standing waves for the NLS: we recall that if we fix $\omega > 0$ and we consider η_ω to be the ground state of the corresponding equation, we have that the standing wave solution is orbitally stable for $\frac{1}{2} < l < \frac{7}{6}$, and unstable for $\frac{7}{6} < l < \frac{5}{2}$.

Remark 4.2.18. *One could ask whether one could get a similar result for more general (in particular, moving) soliton solution of* (4.1.1)*. Apart from the issue of existence and stability for such solutions, one can check that, provided that a moving soliton solution for* (4.1.1) *exists, then the error* $\delta(t)$ *must solve a* (4.2.16)*-type equation, namely a spinless Salpeter equation with a time-dependent moving potential. Unfortunately, since Eq.* (4.2.16)*, unlike KG, is not manifestly covariant, one cannot apparently reduce to an analogue equation, and once again one cannot justify the approximation over the* $\mathcal{O}(1)$*-timescale.*

Chapter 5

The non-relativistic limit of KG equation on $\mathbb{T}$

In this chapter we study the non-relativistic limit of the nonlinear Klein-Gordon (NLKG) equation on compact manifolds. To be definite, we consider

$$\frac{1}{c^2}\, u_{tt}(t,x) - \Delta u + \frac{m^2 c^2}{\hbar^2} u + \lambda f(u) = 0, \tag{5.0.1}$$

where $x \in \mathbb{T}$, $u = u(t,x)$ is a real-valued (or complex-valued) field, $\lambda \in \mathbb{R}$, $f(u)$ is a real-valued function (or $f(u) = \lambda g(|u|^2)u$ if u is complex-valued), and $m > 0$, $c > 0$, $\hbar > 0$ are respectively the mass, the speed of light and Planck's constant. In the following we will assume that $m = \hbar = 1$.
For fixed c, the well-posedness of the Klein-Gordon equation is well studied (see [36] and [37]). We mention also the papers [33], [32] and [9], in which the authors have discussed the time-existence beyond the timescale controlled by local existence theory for small solutions of (5.0.1): the main difficulty of such a problem, already when c is fixed, is the fact that dispersive estimates typically fail on a compact manifold, hence a more refined analysis is needed in order to obtain nontrivial results.
Here we focus on the non-relativistic limit ($c \to \infty$) of the NLKG with real initial data of the form $u(0) := u_0 \in H^s(\mathbb{T})$, $u_t(0) = v_0 \in H^{s-1}(\mathbb{T})$.

The nonrelativistic limit of (5.0.1) on the torus has recently gained a lot of interest, both from the analytical and from the numerical point of view: Faou and Schratz in [35] proved by using a normal form method the convergence of solutions in H^k, locally uniformly in time. We refer also to [14] for some numerical analysis of the nonrelativistic limit of NLKG.

Our aim is to show that for c sufficiently large, the solution of (5.0.1) with initial datum of size $\mathcal{O}(c^{-\alpha})$ ($\alpha > 0$) remains of size $\mathcal{O}(c^{-\alpha})$ for large (namely, $O(c^{\alpha(r+1/2)})$, with $r > 1$) timescales.

5.1 The NLKG Equation with a potential

Now consider the following equation:

$$\frac{1}{c^2}\, u_{tt} \; - \; u_{xx} \; + \; c^2 \; u \; + V * u = f(u), \tag{5.1.1}$$

with $c \geq 1$, $x \in \mathbb{T}$, $f \in C^\infty(\mathbb{R})$ a real-valued function ($f(u) = g(|u|^2)u$ where $g \in C^\infty(\mathbb{R})$ if u is complex-valued), with Dirichlet boundary condition. The potential has the form

$$V(x) = \sum_{j \geq 1} \; v_j \; cos(jx). \tag{5.1.2}$$

By using the same approach of [10], we fix a positive s, and for any $R > 0$ we consider the probability space

$$\mathcal{V} := \mathcal{V}_{s,R} = \left\{ (v_j)_{j\geq 1} \; : \; v_j' \; := \; R^{-1} j^s v_j \in \left[-\frac{1}{2}, \frac{1}{2} \right] \right\}, \tag{5.1.3}$$

and we endow the space $(1, +\infty) \times \mathcal{V} \ni (c, (v_j)_j)$ with the product probability measure.
We recall that in this case the frequencies are given by

$$\omega_j := \omega_j(c) = c\sqrt{c^2 + \lambda_j} \; = \; c^2 \; + \frac{\lambda_j}{1 + \sqrt{1 + \lambda_j / c^2}} \tag{5.1.4}$$

$$= \; c^2 \; + \frac{\lambda_j}{2} - \frac{\lambda_j^2}{2c^2} \frac{1}{(1 + \sqrt{1 + \lambda_j / c^2})^2}, \tag{5.1.5}$$

where $\lambda_j = j^2 + v_j$. Now, as we have done in sec. 3.3, we introduce the following change of coordinates,

$$\psi := \frac{1}{\sqrt{2}} \left[\left(\frac{(c^2 - \Delta + \widetilde{V})^{1/2}}{c} \right)^{1/2} u - i \left(\frac{c}{(c^2 - \Delta + \widetilde{V})^{1/2}} \right)^{1/2} u_t \right], \tag{5.1.6}$$

where $\widetilde{V}$ is the operator that maps u to $V * u$. The Hamiltonian of (5.1.1) now reads

$$H(\bar{\psi}, \psi) = \left\langle \bar{\psi}, c(c^2 - \Delta + \widetilde{V})^{1/2} \psi \right\rangle + \frac{\lambda}{4} \int_{\mathbb{T}} f \left(\left(\frac{c}{(c^2 - \Delta + \widetilde{V})^{1/2}} \right)^{1/2} \frac{\psi + \bar{\psi}}{\sqrt{2}} \right) \mathrm{d}x. \tag{5.1.7}$$

Therefore the Hamiltonian takes the form

$$H(\psi, \bar{\psi}) = H_0(\psi, \bar{\psi}) + N(\psi, \bar{\psi}), \tag{5.1.8}$$

where

$$H_0(\psi, \bar{\psi}) = \left\langle \bar{\psi}, c(c^2 - \Delta + \widetilde{V})^{1/2} \psi \right\rangle, \tag{5.1.9}$$

$$N(\psi, \bar{\psi}) = \frac{\lambda}{2^4} \int_{\mathbb{T}} f \left(\left(\frac{c}{(c^2 - \Delta + \widetilde{V})^{1/2}} \right)^{1/2} (\psi + \bar{\psi}) \right) \mathrm{d}x, \tag{5.1.10}$$

$$\sim \sum_{l \geq 4} \int_{\mathbb{T}} N_l(x) \left(\left(\frac{c}{(c^2 - \Delta + \widetilde{V})^{1/2}} \right)^{1/2} (\psi + \bar{\psi}) \right)^l \mathrm{d}x, \tag{5.1.11}$$

where $N_l \in C^\infty$ for each l (since $V \in C^\infty$), and

$$\left(\frac{c}{(c^2-\Delta+\widetilde{V})^{1/2}}\right)^{1/2} : H^s \to H^s$$

is a smoothing pseudodifferential operator, which can be estimated uniformly in $c \geq 1$.

Theorem 5.1.1. *Consider the equation (5.1.1) and fix $\gamma > 0$, and $\tau > 1$. Then for any $r \geq 1$ there exists $s^* > 0$ and, for any $s > s^*$, there exists a set $\mathcal{R}_\gamma := \mathcal{R}_{\gamma,s,r} \subset]1,+\infty[\times\mathcal{V}$ satisfying*

$$|\mathcal{R}_\gamma \cap ([n,n+1]\times\mathcal{V})| = \mathcal{O}(\gamma) \ \ \forall n \in \mathbb{N}_0,$$

and there exists $R_s > 0$ such that for any $(c,(v_j)_j) \in (]1,+\infty[\times\mathcal{V}) \setminus \mathcal{R}_\gamma$ and for any $R < R_s$ there exist $N := N(r,R) > 0$, and a canonical transformation

$$\mathcal{T}_c := \mathcal{T}_c^{(r)} : B_s(R/3) \to B_s(R)$$

such that

$$H_r := H \circ \mathcal{T}_c^{(r)} = H_0 + Z^{(r)} + R^{(r)},$$

where $Z^{(r)}$ is a polynomial of degree (at most) $r+2$, which is in (γ,τ,N) - normal form with respect to $\omega = (\omega_j(c))_{j\geq 1}$, namely such that

$$Z^{(r)}(\psi,\bar{\psi}) = \sum_{m,n\in\mathbb{N}^{\mathbb{N}}} Z_{m,n}\psi^m\bar{\psi}^n,$$

$$Z_{m,n} \neq 0 \Longrightarrow |\omega\cdot(n-m)| < \frac{\gamma}{N^\tau}, \quad \sum_{l\geq N+1} n_l + m_l \leq 2, \tag{5.1.12}$$

and such that

$$\sup_{B_s(R/3)} \|X_{R^{(r)}}(\psi,\bar{\psi})\|_{H^s} \leq K_s\, R^{r+3/2}, \tag{5.1.13}$$

$$\sup_{B_s(R/3)} \|\mathcal{T}_c^{(r)} - id\|_{H^s} \leq K_s\, R^2. \tag{5.1.14}$$

and we have that $Z_{m,n}$ depends on the actions $I = \psi\,\bar{\psi}$ only. Moreover, there exists $K_s^* > 0$ such that if the initial datum satisfies

$$\|(\psi_0,\bar{\psi}_0)\|_{H^s} \leq K\, R \tag{5.1.15}$$

with $K < K_s^*$, then

$$\|(\psi(t),\bar{\psi}(t))\|_{H^s} \preceq 2K\, R, \quad |t| \preceq R^{-(r+1/2)} \tag{5.1.16}$$

$$\|(I(t),\bar{I}(t))\|_{H^s} \preceq K\, R^3, \quad |t| \preceq R^{-(r+1/2)}. \tag{5.1.17}$$

Finally, there exists a smooth torus $\mathbb{T}_c$ such that for any $s_1 < s - 1/2$

$$d_{s_1}((\psi(t),\bar{\psi}(t)),\mathbb{T}_c) \preceq R^{\frac{r_1}{2}+1}, \quad |t| \preceq R^{-(r-r_1+1/2)}, \tag{5.1.18}$$

where $r_1 \leq r$, and d_{s_1} is the distance in H^{s_1}.

Remark 5.1.2. *The fact that Z depends only on the actions is a direct consequence of the non-resonance property established in Theorem 5.1.7.*

Remark 5.1.3. *By the same argument one can prove that if we fix $\alpha > 0$, then for any $r \geq 1$ there exists a set $\mathcal{R}_{\gamma,s,\alpha,r} \subset]1, +\infty[\times\mathcal{V}$ such that there exists $c^* > 0$ such that for any $(c, (v_j)_j) \in (]1, +\infty[\times\mathcal{V}) \setminus \mathcal{R}_{\gamma,s,\alpha,r}$ with $c > c^*$, if the initial datum satisfies*

$$\|(\psi_0, \bar{\psi}_0)\|_{H^s} \leq \frac{K}{c^\alpha}$$

for some $K > 0$, then

$$\|(\psi(t), \bar{\psi}(t))\|_{H^s} \preceq \frac{2K}{c^\alpha}, \quad |t| \preceq c^{\alpha(r+1/2)},$$
$$\|(I(t), \bar{I}(t))\|_{H^s} \preceq \frac{K}{c^{3\alpha}}, \quad |t| \preceq c^{\alpha(r+1/2)}.$$

Remark 5.1.4. *It would also be interesting to study the dependence of the torus $\mathbb{T}_c$ on c. One could expect that it should converge to an invariant torus of the NLS with a convolution potential. We expect this fact to be true, but it needs further investigation for a proof. This is due to the fact that the NLS is the singular limit of NLKG and to the fact that c is only allowed to vary in Cantor like sets, so that one can only expect a Whitney-smooth dependence.*

In order to get our result, we need to show some nonresonance properties of the frequencies $\omega = (\omega_j)_{j>0}$: it will be crucial that these properties hold uniformly (or at least, up to a set of small probability) in $(1, +\infty) \times \mathcal{V}$, since this will allow us to deduce a result which is valid in the non-relativistic limit regime.

Proposition 5.1.5. *Let $r \geq 1$, $c \geq 1$ be fixed. Then $\forall \gamma > 0$ $\exists \mathcal{V}'_{s,R,\gamma} \subset \mathcal{V}$ with $|\mathcal{V} \backslash \mathcal{V}'_{s,R,\gamma}| = \mathcal{O}(\gamma)$, and $\exists \tau > 1$ s.t. $\forall (v_j)_{j\geq 1} \in \mathcal{V}'_{s,R,\gamma}$ and $\forall$ $N \geq 1$*

$$|\omega \cdot k \ + \ n| \geq \frac{\gamma}{N^\tau} \tag{5.1.19}$$

for $0 < |k| \leq r$, $supp(k) \subseteq \{1, \ldots, N\}$, and $\forall$ $n \in \mathbb{Z}$.

Proof. Let $p_k((v_j)_{j\geq 1}) := \sum_{j=1}^N \omega_j k_j$, and assume that $k_h \neq 0$ for some h. Then

$$\left|\frac{\partial p_k}{\partial v'_h}\right| = \left|\frac{k_h h^s}{2\sqrt{1 + \lambda_h/c^2}}\right| \gtrsim \frac{1}{2\sqrt{1 + h^{max(s,2)}}} \geq \frac{1}{2\sqrt{1 + N^{max(s,2)}}} > 0,$$

hence by Lemma 17.2 of [73]

$$\left|\{(v'_j)_{j\geq 1} \ : \ |p_k((v_j)_{j\geq 1})| < \gamma\}\right| \leq \gamma \ N^{s+max(s,2)/2};$$

$$\left| \bigcup_{\substack{0<|k|\leq r \\ supp(k)\subseteq\{1,\ldots,N\}}} \{(v'_j)_{j\geq 1} \ : \ |p_k((v_j)_{j\geq 1})| < \gamma\} \right| \leq \gamma \ N^{s+r+max(s,2)/2} \leq \frac{\gamma}{N^\tau}$$

with $\tau \geq 1$ for $\gamma = \frac{\gamma_0}{N^{\tau+s+r+max(s,2)/2}}$. ☐

Proposition 5.1.6. *Let $r \geq 1$ be fixed. Then $\forall \gamma > 0$ $\exists \mathcal{R}_\gamma := \mathcal{R}_{\gamma,s,r} \subset]1,+\infty[\times\mathcal{V}$ with $|\mathcal{R}_\gamma| \to 0$ as $\gamma \to 0$, and $\exists \tau > 1$ such that $\forall (c,(v_j)_j) \in (]1,+\infty[\times\mathcal{V}) \setminus \mathcal{R}_\gamma$ and $\forall$ $N \geq 1$*

$$\left| \sum_{j=1}^{N} \omega_j k_j \ + \ \sigma\omega_l \right| \geq \frac{\gamma}{N^\tau} \tag{5.1.20}$$

for $0 < |k| \leq r$, supp$(k) \subseteq \{1,\dots,N\}$, $\sigma = \pm 1$, $l \geq N$.

Proof. Without loss of generality, we can choose $\sigma = -1$.
Now fix $k \in \mathbb{Z}^N$ with $0 < |k| \leq r$, and fix $l \geq N$. Set $p_{k,l}(c,(v_j)_{j\geq 1}) := \sum_{j=1}^N \omega_j k_j - \omega_l$. We can rewrite the function $p_{k,l}$ in the following way:

$$p_{k,l}(c,(v_j)_{j\geq 1}) \ = \alpha c^2 \ + \sum_{j=1}^{N} \frac{k_j\lambda_j}{1+\sqrt{1+\lambda_j/c^2}} \ - \frac{\lambda_l}{1+\sqrt{1+\lambda_l/c^2}},$$

where $\alpha := (\sum_{j=1}^N k_j) - 1 \in \{-r-1,\dots,r-1\}$. Now we have to distinguish some cases:

Case $\alpha = 0$: in this case we have that $p_{k,l}$ can be small only if $l^2 \leq 3(N^2+N^s)^2 r^2$. So to obtain the result we just apply Proposition 5.1.5 with $N' := \sqrt{3}(N^2+N^s)r$, $r' = r+1$.

Case $\alpha \neq 0$, $c \leq \lambda_N^{1/2} r^{1/2}$: we have that

$$\sum_{j=1}^{N} c\sqrt{c^2+\lambda_j}k_j \leq r\sqrt{c^4+c^2\lambda_N} \ \leq \ \sqrt{2}\, r^2\lambda_N,$$

so $|\sum_{j=1}^N \omega_j k_j - \omega_l|$ can be small only for $l^2 < rN^2$. Therefore, in order to get the thesis we apply Proposition 5.1.5 with $N' := \sqrt{r}N$, $r' := r+1$.

Case $\alpha > 0$, $c > \lambda_N^{1/2} r^{1/2}$: first notice that if we set $f(x) := \frac{x^2}{2\sqrt{1+x}(1+\sqrt{1+x})^2}$, and we put $x_j := \lambda_j/c^2$, in this regime we get

$$\left| \sum_{j=1}^{N} k_j f(x_j) \right| \leq \frac{r}{2} f(x_N) \leq \frac{1}{2}.$$

Now define $\tilde{p}_{k,l}(c^2) := \alpha c^2 \ - \ \frac{\lambda_l}{1+\sqrt{1+\lambda_l/c^2}}$. One can verify that

$$\tilde{p}_{k,l}(c^2) = 0;$$
$$c^2 = \tilde{c}^2_{l,\alpha} := \frac{\lambda_l}{\alpha(\alpha+2)},$$

and that

$$\frac{\partial \tilde{p}_{k,l}}{\partial(c^2)}(\tilde{c}^2_{l,\alpha}) = \alpha - \frac{\alpha^2(\alpha+2)^2}{2\sqrt{1+\alpha(\alpha+2)}(1+\sqrt{1+\alpha(\alpha+2)})^2} > 0.$$

Besides, in an interval $\left[\tilde{c}^2_{l,\alpha} - \frac{\varrho}{\alpha(\alpha+2)}, c^2_{l,\alpha} + \frac{\varrho}{\alpha(\alpha+2)}\right] \ =: \ [\tilde{c}^2_{l,\alpha,-}, c^2_{l,\alpha,+}]$ we have that

$$\frac{\partial \tilde{p}_{k,l}}{\partial(c^2)}(c^2) > \left(\frac{1}{2} + \frac{1}{2(r+1)}\right)\alpha.$$

Then, by exploiting Lemma 17.2 of [73], we get that

$$\left|\left\{c^2 \in B\left(c^2_{l,\alpha}, \frac{\varrho}{\alpha(\alpha+2)}\right] : |\tilde{p}_{k,l}(c^2)| \leq \gamma\right\}\right| \leq \gamma \frac{2(r+1)}{(r+2)\alpha}$$

for any $\gamma > 0$ s.t. $\gamma \frac{2(r+1)}{(r+2)\alpha} < \frac{\varrho}{\alpha(\alpha+2)}$; $\gamma < \frac{(r+2)\varrho}{2(r+1)(\alpha+2)}$.

Now, since in this regime $\left|\frac{\partial(p_{k,l}-\tilde{p}_{k,l})}{\partial c^2}\right| \leq \frac{1}{2}$, we can deal with $p_{k,l}$ in a similar way as before, and we can conclude that

$$\lim_{\gamma\to 0}\left|\bigcup_{\substack{0<|k|\leq r \\ \text{supp}(k)\subseteq\{1,\dots,N}} \bigcup_{l\geq N} \{c^2 \in [c^2_{l,\alpha,-}, c^2_{l,\alpha,+}] \ : |p_{k,l}(c^2)| \leq \gamma\}\right| = 0. \tag{5.1.21}$$

Case $\alpha < 0$, $c > \lambda_N^{1/2} r^{1/2}$: since

$$\left|\sum_{j=1}^N \frac{k_j\lambda_j}{1+\sqrt{1+\lambda_j/c^2}}\right| \leq \frac{r\lambda_N}{2} \leq \frac{c^2}{2},$$

we have that $p_{k,l}$ can be small only if $\lambda_N < r\lambda_N$. So, in order to get the result, we apply Proposition 5.1.5 with $N' := r^{1/2}N$, $r' := r+1$. □

Theorem 5.1.7. *Let* $r \geq 1$ *be fixed. Then* $\forall\gamma > 0$ $\exists\mathcal{R}_\gamma := \mathcal{R}_{\gamma,s,r} \subset]1,+\infty[\times\mathcal{V}$ *with* $|\mathcal{R}_\gamma| \to 0$ *as* $\gamma \to 0$, *and* $\exists\tau > 1$ *such that* $\forall(c,(v_j)_j) \in (]1,+\infty[\times\mathcal{V}) \setminus \mathcal{R}_\gamma$ *and* $\forall$ $N \geq 1$

$$\left|\sum_{j=1}^N \omega_j k_j \ + \ \sigma_1\omega_l \ + \ \sigma_2\omega_m\right| \geq \frac{\gamma}{N^\tau} \tag{5.1.22}$$

for $0 < |k| \leq r$, *supp*$(k) \subseteq \{1,\dots,N\}$, $\sigma_1,\sigma_2 \in \{\pm 1\}$, $m > l \geq N$.

Proof. If $\sigma_i = 0$ for $i = 1,2$, then we can conclude by using Proposition 5.1.6.

Now, consider the case $\sigma_1 = -1$, $\sigma_2 = 1$, and denote

$$p_{k,l,m}(c^2) := \sum_{j=1}^N \omega_j(c^2)k_j \ - \omega_l(c^2) \ + \omega_m(c^2).$$

Now fix $\delta > 3$. If $m \lesssim N^\delta$, then we can conclude by applying Proposition 5.1.5 and 5.1.6. So from now on we will assume that $m, l > N^\delta$.

We have to distinguish several cases:

Case $c < \lambda_l^\alpha$: we point out that, since

$$c\sqrt{c^2+\lambda_l} = c\lambda_l^{1/2}\sqrt{1+\frac{c^2}{\lambda_l}} = c\lambda_l^{1/2}\left(1+\frac{c^2}{2\lambda_l}+O\left(\frac{1}{\lambda_l^2}\right)\right),$$

we get (denote $m = l + j$)

$$\omega_m - \omega_l = jc \ + \ \frac{1}{2}\left(\frac{v_m}{m} - \frac{v_l}{l}\right) \ + \ \frac{c^3}{2\lambda_l^{1/2}} - \frac{c^3}{2\lambda_m^{1/2}} + O\left(\frac{1}{m^3}\right) + O\left(\frac{1}{l^3}\right),$$

that is, the integer multiples of c are accumulation points for the differences between the frequencies as $l, m \to \infty$, provided that $\alpha < \frac{1}{6}$.

Case $c > \lambda_m$*:* in this case we have (again by denoting $m = l + j$) that $\lambda_m - \lambda_l = j(j+2l) + (v_m - v_l) = 2jl + j^2 + a_{lm}$, with $|a_{lm}| \leq \frac{C}{l}$, so that

$$p_{k,l,m} = \sum_{h=1}^{N} \omega_h k_h \ \pm 2jl \pm j^2 \pm a_{lm}.$$

If $l > 2\,CN^\tau/\gamma$ then the term a_{lm} represents a negligible correction and therefore we can conclude by applying Proposition 5.1.5. On the other hand, if $l \leq 2\,CN^\tau/\gamma$, we can apply the same Proposition with $N' := 2CN^\tau/\gamma$ and $r' := r + 2$.

Case $\lambda_l^{1/6} \leq c \lesssim \lambda_l^{1/2}$*:* if we rewrite the quantity to estimate

$$p_{k,l,m}(c^2) = \alpha c^2 \ + \sum_{h=1}^{N} \frac{\lambda_h k_h}{1 + \sqrt{1 + \frac{\lambda_j}{c^2}}} \ + \omega_m \ - \ \omega_l,$$

where $\alpha := \sum_{h=1}^{N} k_h$, we distinguish three cases:

- if $\alpha > 0$, then we notice that

$$\left| \sum_{h=1}^{N} \frac{\lambda_h k_h}{1 + \sqrt{1 + \frac{\lambda_j}{c^2}}} \right| \leq \frac{r\lambda_N}{1 + \sqrt{1 + \lambda_N/c^2}} \leq \frac{r\lambda_N}{1 + \sqrt{1 + \lambda_N/\lambda_l}} \leq \frac{r\lambda_N}{2},$$

$$\begin{aligned} |\omega_m - \omega_l| &= c\frac{\lambda_m - \lambda_l}{\sqrt{c^2 + \lambda_m} + \sqrt{c^2 + \lambda_l}} \overset{m>l}{\geq} \frac{c\lambda_l^{1/2}}{\sqrt{c^2 + \lambda_m} + \sqrt{c^2 + \lambda_l}} \\ &\gtrsim \frac{N^{\delta/3}\lambda^{1/2}}{\sqrt{N^{2\delta/3} + \lambda_m^{1/2}} + \sqrt{N^{2\delta/3} + \lambda_l^{1/2}}} > 0 \end{aligned}$$

  thus $|p_{k,l,m}| > |\lambda_l^{1/3} - \frac{r}{2}\lambda_N| > 0$, since $l > N^3$;

- if $\alpha = 0$, then we just notice that

$$|\omega_m - \omega_l| \geq \gamma(\lambda_m - \lambda_l) \overset{m>l}{\gtrsim} \ \gamma_0 \ \lambda_l^{1/2},$$

  which is greater than γ_0/N^τ for $\tau > -1$, since $l > N^3$;

- if $\alpha < 0$, then we just recall that $|\omega_m - \omega_l| > \gamma_0 \lambda_l^{1/2}$, and by choosing γ_0 sufficiently small (actually $\gamma_0 \leq |\alpha|$) we get that also in this case $p_{k,l,m}$ is bounded away from zero.

□

5.2 Proof of Theorem 5.1.1

The proof is based on the method of Lie transform. Let $s > s^*$ be fixed.
Given an auxiliary function χ analytic on H^s, we consider the auxiliary differential equation

$$\dot{\psi} = i\nabla_{\bar{\psi}}\chi(\psi,\bar{\psi}) =: X_\chi(\psi,\bar{\psi}) \tag{5.2.1}$$

and denote by Φ^t_χ its time-t flow. A simple application of Cauchy inequality gives

Lemma 5.2.1. *Let χ and its symplectic gradient be analytic in $B_s(\rho)$. Fix $\delta < \rho$, and assume that*

$$\sup_{B_s(R-\delta)} \|X_\chi(\psi,\bar{\psi})\|_s \leq \delta.$$

Then, if we consider the time-t flow Φ^t_χ of X_χ we have that for $|t| \leq 1$

$$\sup_{B_s(R-\delta)} \|\Phi^t_\chi(\psi,\bar{\psi}) - (\psi,\bar{\psi})\|_s \leq \sup_{B_s(R-\delta)} \|X_\chi(\psi,\bar{\psi})\|_s.$$

The map $\Phi := \Phi^1_\chi$ will be called the *Lie transform* generated by χ.
Given a homogeneous polynomial f of degree m, we denote, following [10], its modulus

$$\lfloor f \rceil(\psi,\bar{\psi}) := \sum_{|j|=r} |f_j|\, z^j, \tag{5.2.2}$$

where f_j is given by

$$f(\psi) = \sum_{|j|=r} f_j z^j,$$

$$z^j := \cdots z_{-l}^{j_{-l}} \cdots z_{-1}^{j_{-1}} z_1^{j_1} \cdots z_l^{j_l} \cdots, \quad z_l = \left\langle \psi, \mathrm{e}^{il\cdot} \right\rangle, \quad z_{-l} = \left\langle \bar{\psi}, \mathrm{e}^{-il\cdot} \right\rangle.$$

Furthermore, given a multivector

$$\phi := (\phi^{(1)}, \ldots, \phi^{(r)}) = (\psi^{(1)}, \bar{\psi}^{(1)} \ldots, \psi^{(r)}, \bar{\psi}^{(r)})$$

we introduce the following norm

$$\|\phi\|_{s,1} := \frac{1}{r}\sum_{l=1}^{r} \|\phi^{(1)}\|_1 \ldots \|\phi^{(l-1)}\|_1 \|\phi^{(l)}\|_s \|\phi^{(l+1)}\|_1 \ldots \|\phi^{(r)}\|_1. \tag{5.2.3}$$

Definition 5.2.2. *Let $X : H^s \oplus H^s \to H^s \oplus H^s$ be a homogeneous polynomial of degree r,*

$$X(\psi,\bar{\psi}) = \sum_{l\in\mathbb{Z}\setminus\{0\}} X_l(\psi,\bar{\psi}) e^{il\cdot}.$$

Consider the r-linear symmetric form $\tilde{X}_l$ such that $\tilde{X}_l(\psi,\bar{\psi},\ldots,\psi,\bar{\psi}) = X_l(\psi,\bar{\psi})$, and set

$$\tilde{X} := \sum_{l\in\mathbb{Z}\setminus\{0\}} \tilde{X}_l(\psi,\bar{\psi}) e^{il\cdot},$$

so that $\tilde{X}(\psi, \bar{\psi}, \ldots, \psi, \bar{\psi}) = X_l(\psi, \bar{\psi})$.
Let $s \geq 1$, *then we say that* X *is an* s*-tame map if there exists* $K_s > 0$ *such that*

$$\|\tilde{X}(\phi^{(1)}, \ldots, \phi^{(r)})\|_s \leq K_s \sum_{l=1}^{r} \|\phi^{(1)}\|_1 \ldots \|\phi^{(l-1)}\|_1 \|\phi^{(l)}\|_s \|\phi^{(l+1)}\|_1 \ldots \|\phi^{(r)}\|_1, \qquad (5.2.4)$$
$$\forall\, \phi^{(1)}, \ldots, \phi^{(r)} \in H^s \oplus H^s.$$

If a map is s*-tame for any* $s \geq 1$, *then it will be said to be tame.*

Definition 5.2.3. *Let us consider a vector field* $X : H^s \oplus H^s \to H^s \oplus H^s$, *and denote by* X_l *its* l*-th component. We define its modulus by*

$$\lfloor X \rceil(\psi, \bar{\psi}) := \sum_{l \in \mathbb{Z} \setminus \{0\}} \lfloor X_l \rceil(\psi, \bar{\psi}) e^{il\cdot}.$$

A polynomial vector field X *is said to hace* s*-tame modulus if its modulus* $\lfloor X \rceil$ *is an* s*-tame map. The set of polynomial functions* f, *whose Hamiltonian vector fields has* s*-tame modulus will be denoted by* T^s_M. *If* $f \in T^s_M$ *for any* $s > 1$, *we will write* $f \in T_m$, *and say that* f *has tame modulus.*

Remark 5.2.4. *The property of having tame modulus depends on the coordinate system.*

Definition 5.2.5. *Let* X *be an* s*-tame vector field homogeneous polynomial of degree* r. *The infimum of the constants* K_s *such that the inequality*

$$\|\tilde{X}(\phi^{(1)}, \ldots \phi^{(r)})\| \leq K_s \|(\phi^{(1)}, \ldots, \phi^{(r)})\|_{s,1}$$
$$\forall\, \phi^{(1)}, \ldots, \phi^{(r)} \in H^s \oplus H^s$$

holds will be called tame s *norm of* X, *and will be denoted by* $|X|^T_s$.

The tame s norm of a polynomial Hamiltonian f of degree $r+1$ is given by

$$|f|_s := \sup \frac{\left\|\tilde{X}_{\lfloor f \rceil}(\phi)\right\|_s}{\|\phi\|_{s,1}}, \qquad (5.2.5)$$

where the sup is taken over all multivectors $\phi = (\phi^{(1)}, \ldots, \phi^{(r)})$ such that $\phi^{(j)} \neq 0$ for any j.

Definition 5.2.6. *Let* $f \in T^s_M$ *be a non-homogeneous polynomial, and consider its Taylor expansion*

$$f = \sum_m f_m,$$

where f_m *is homogeneous of degree* m. *Let* $R > 0$, *then we denote*

$$\langle |f| \rangle_{s,R} := \sum_{m \geq 2} |f_r|_s R^{m-1}. \qquad (5.2.6)$$

Such a definition extends naturally to analytic functions such that (5.2.6) *is finite. The set of functions of class* T^s_M *for which* (5.2.6) *is finite will be denoted by* $T_{s,R}$.

With the above definitions,

$$\sup_{B_s(R)} \|X_f(\psi,\bar\psi)\|_s \le \langle |f| \rangle_{s,R}\,.$$

It is easy to check that the set $T_{s,R}$ endowed with the norm (5.2.6) is a Banach space.

Now we introduce the Fourier projection

$$\Pi_N\psi(x) := \int_{|k|\le N} \hat\psi(k) e^{ik\cdot x} dk,$$

and we split the variables $(\psi,\bar\psi)$ into

$$(\psi_l,\bar\psi_l) := (\Pi_N\psi, \Pi_N\bar\psi),$$
$$(\psi_h,\bar\psi_h) := ((id-\Pi_N)\psi, (id-\Pi_N)\bar\psi).$$

The use of Fourier projection is important in view of the following result, whose proof can be found in Appendix A of [10].

Lemma 5.2.7. *Fix N, and consider the decomposition $\psi = \psi_l + \psi_h$ as above. Let $f \in T^s_M$ be a polynomial of degree less or equal than $r+2$. Assume that f has a zero of order three in the variables $(\psi_h,\bar\psi_h)$, then one has*

$$\sup_{B_s(R)} \|X_f(\psi,\bar\psi)\|_s \preceq \frac{\langle |f| \rangle_{s,R}}{N^{s-1}}. \tag{5.2.7}$$

Lemma 5.2.8. *Let $f, g \in T^s_M$ be homogeneous polynomial of degrees $n+1$ and $m+1$ respectively. Then one has $\{f,g\} \in T^s_M$, and*

$$|\{f,g\}|_s \le (n+m)|f|_s|g|_s. \tag{5.2.8}$$

The proof of this lemma can be found again in Appendix A of [10].

Remark 5.2.9. *Given g analytic on $H^s \oplus H^s$, consider the differential equation*

$$\dot\psi = X_g(\psi,\bar\psi), \tag{5.2.9}$$

where by X_g we denote the vector field of g. Now define

$$\Phi^* g(\phi,\bar\phi) := g \circ \Phi(\psi,\bar\psi).$$

In the new variables $(\phi,\bar\phi)$ defined by $(\psi,\bar\psi) = \Phi(\phi,\bar\phi)$ equation (5.2.9) is equivalent to

$$\dot\phi = X_{\Phi^* g}(\phi,\bar\phi). \tag{5.2.10}$$

Using the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi^t_\chi)^* g = (\Phi^t_\chi)^*\{\chi, g\},$$

we formally get

$$\Phi^* g = \sum_{l=0}^{\infty} g_l, \tag{5.2.11}$$

$$g_0 := g, \tag{5.2.12}$$

$$g_l := \frac{1}{l}\{\chi, g_{l-1}\}, \quad l \ge 1. \tag{5.2.13}$$

In order to estimate the terms appearing in (5.2.11) we exploit the following results

Lemma 5.2.10. *Let $h, g \in T_{s,R}$, then for any $d \in (0,R)$ one has that $\{h,g\} \in T_{s,R-d}$, and*

$$\langle|\{h,g\}|\rangle_{s,R-d} \leq \frac{1}{d} \langle|h|\rangle_{s,R} \langle|g|\rangle_{s,R} \tag{5.2.14}$$

Proof. Write $h = \sum_j h_j$ and $g = \sum_k g_k$, with h_j homogeneous of degree j and similarly for g. Then we have

$$\{h,g\} = \sum_{j,k} \{h_j, g_k\},$$

where $\{h_j, g_k\}$ has degree $j+k-2$. Therefore

$$\begin{aligned}\langle|\{h_j,g_k\}|\rangle_{s,R-d} &= |\{h_j,g_k\}|_s (R-d)^{j+k-3} \\ &\leq |h_j|_s |g_k|_s (j+k-2)(R-d)^{j+k-3} \\ &\leq |h_j|_s |g_k|_s \frac{1}{d} R^{j+k-2} = \frac{1}{d} \langle|h_j|\rangle_{s,R} \langle|g_k|\rangle_{s,R},\end{aligned}$$

where we exploited the inequality $k(R-d)^{k-1} < R^k/d$, which holds for any positive R and $d \in (0,R)$. □

Lemma 5.2.11. *Let $g, \chi \in T_{s,R}$ be analityc functions; denote by g_l the functions defined recursively by* (5.2.11)*; then for any $d \in (0,R)$ one has that $g_l \in T_{s,R-d}$, and*

$$\langle|g_l|\rangle_{s,R-d} \leq \langle|g|\rangle_{s,R} \left(\frac{e}{d} \langle|\chi|\rangle_{s,R}\right)^l. \tag{5.2.15}$$

Proof. Fix l, and denote $\delta := d/l$. We look for a sequence $C_m^{(l)}$ such that

$$\langle|g_m|\rangle_{s,R-m\delta} \preceq C_m^{(l)}, \quad \forall m \leq l.$$

By (5.2.14) we can define the sequence

$$\begin{aligned}C_0^{(l)} &:= \langle|g|\rangle_{s,R}, \\ C_m^{(l)} &= \frac{2}{\delta m} C_{m-1}^{(l)} \langle|\chi|\rangle_{s,R} \\ &= \frac{2l}{dm} C_{m-1}^{(l)} \langle|\chi|\rangle_{s,R}.\end{aligned}$$

One has

$$C_l^{(l)} = \frac{1}{l!} \left(\frac{2l}{d} \langle|\chi|\rangle_{s,R}\right)^l \langle|g|\rangle_{s,R},$$

and using the inequality $l^l < l! e^l$ one can conclude. □

Lemma 5.2.12. *Let $f \in T_M^s$ be a polynomial which is at most quadratic in the variables $(\psi_h, \bar{\psi}_h)$. Then there exist $\chi, Z \in T_{s,R}$ in (γ, τ, N)-normal form such that*

$$\{H_0, \chi\} + Z = f. \tag{5.2.16}$$

Moreover, χ and Z satisfy the following estimates

$$\langle|\chi|\rangle_{s,R} \leq \frac{N^\tau}{\gamma} \langle|f|\rangle_{s,R}, \tag{5.2.17}$$

$$\langle|Z|\rangle_{s,R} \leq \langle|f|\rangle_{s,R}. \tag{5.2.18}$$

Proof. Expanding f in Taylor series, namely $f(\psi,\bar\psi)=\sum_{j,l} f_{j,l}\psi^j\bar\psi^l$, and similarly for χ and Z, equation (5.2.16) becomes an equation for the coefficients of f, χ and Z,

$$i\omega\cdot(j-l)\chi_{j,l}+Z_{j,l}=f_{j,l}.$$

Then we define

$$Z_{j,l}:=f_{j,l},\quad \text{when } |\omega\cdot(j-l)|<\frac{\gamma}{N^\tau}, \tag{5.2.19}$$

$$\chi_{j,l}:=\frac{f_{j,l}}{i\omega\cdot(j-l)},\quad \text{when } |\omega\cdot(j-l)|\geq\frac{\gamma}{N^\tau}. \tag{5.2.20}$$

By construction we get estimates (5.2.17) and (5.2.18). Furthermore, since f is at most quadratic in $(\psi_h,\bar\psi_h)$, we obtain that $\sum_{k>N}(j_k+l_k)\leq 2$, and thus Z is in (γ,τ,N)-normal form. □

Remark 5.2.13. *Let $s>s^*$, and assume that χ, F are analytic on $B_s(R)$. Fix $d\in(0,R)$, and assume also that*

$$\sup_{B_s(R)}\|X_\chi(\psi,\bar\psi)\|_s\leq d/3,$$

Then for $|t|\leq 1$

$$\sup_{B_s(R-d)}\|X_{(\Phi^t_\chi)^*F-F}(\psi,\bar\psi)\|_s=\sup_{B_s(R-d)}\|X_{F\circ\Phi^t_\chi-F}(\psi,\bar\psi)\|_s \tag{5.2.21}$$

$$\overset{(5.2.14)}{\leq}\frac{5}{d}\sup_{B_s(R)}\|X_\chi(\psi,\bar\psi)\|_s\sup_{B_s(R)}\|X_F(\psi,\bar\psi)\|_s \tag{5.2.22}$$

$$<2\sup_{B_s(R)}\|X_F(\psi,\bar\psi)\|_s. \tag{5.2.23}$$

Lemma 5.2.14. *Let $\chi\in T_{s,R}$ be the solution of the equation (5.2.16), with $f\in T^s_M$. Denote by $H_{0,l}$ the functions defined recursively via (5.2.11) from H_0. Then for any $d\in(0,R)$ one has that $H_{0,l}\in T_{s,R-d}$, and*

$$\langle|H_{0,l}|\rangle_{s,R-d}\leq 2\langle|f|\rangle_{s,R-d}\left(\frac{e}{d}\langle|\chi|\rangle_{s,R}\right)^l. \tag{5.2.24}$$

Proof. Using (5.2.16) one gets $H_{0,1}=Z-f\in T^s_M$. Then, arguing as for (5.2.15), one can conclude. □

The main step of the proof of Theorem 5.1.1 is the following result, that allows to increase by one the order of the perturbation. As a preliminary step, we take the Taylor series of $N(\psi,\bar\psi)$ up to order $r+2$,

$$N(\psi,\bar\psi)=\sum_{l=1}^r\hat N_l(x,\psi,\bar\psi) \tag{5.2.25}$$

$$+N(\psi,\bar\psi)-\sum_{l=1}^r\hat N_l(x,\psi,\bar\psi) \tag{5.2.26}$$

$$=:N^{(1)}(\psi,\bar\psi)+N^{(1,r)}(\psi,\bar\psi), \tag{5.2.27}$$

where N_l is a homogeneous polynomial in ψ and $\bar{\psi}$ of degree $l+2$ with variable C^∞-coefficients (since $V \in C^\infty$).
Now we consider the analytic Hamiltonian

$$H^{(0)} := H_0 + N^{(1)}. \tag{5.2.28}$$

Then for R sufficiently small one has that

$$\left\langle |N^{(1)}| \right\rangle_{s,R} \preceq R^2, \tag{5.2.29}$$

$$\left\langle |N^{(1,r)}| \right\rangle_{s,R} \preceq R^{r+2}. \tag{5.2.30}$$

Lemma 5.2.15. *Consider the Hamiltonian* (5.2.28)*, and fix $s > s^*$. Then for any $m \leq r$ there exists $R^*_m \ll 1$ and, for any $N > 1$ there exists an analytic canonical transformation*

$$\mathcal{T}^{(m)} : B_s\left(\frac{(2r-m)}{2N^\tau r} R^{*,2}_m\right) \to H^s$$

such that

$$H^{(m)} := H^{(0)} \circ \mathcal{T}^{(m)} = H^{(0)} + Z^{(m)} + f^{(m)} + \mathcal{R}^{(m)}_N + \mathcal{R}^{(m)}_T, \tag{5.2.31}$$

where for any $R < R^*_m/N^\tau$ the following properties are fulfilled

1. *the transformation $\mathcal{T}^{(m)}$ satisfies*

$$\sup_{B_s(R)} \|\mathcal{T}^{(m)} - id\|_s \preceq N^\tau\ R^2; \tag{5.2.32}$$

2. *$Z^{(m)}$ is a polynomial of degree (at most) $m+2$ in (γ, τ, N)-normal form; $f^{(m)}$ is a polynomial of degree (at most) $r+2$. Moreover*

$$\sup_{B_s((1-m/(2r))R)} \|X_{Z^{(m)}}(\psi, \bar{\psi})\|_s \preceq R^2, \quad \forall m \geq 1, \tag{5.2.33}$$

$$\sup_{B_s((1-m/(2r))R)} \|X_{f^{(m)}}(\psi, \bar{\psi})\|_s \preceq R^{m+2} N^{\tau m}, \quad \forall m \geq 1; \tag{5.2.34}$$

3. *the remainder terms $\mathcal{R}^{(m)}_N$ and $\mathcal{R}^{(m)}_T$ satisfy*

$$\sup_{B_s((1-m/(2r))R)} \|X_{\mathcal{R}^{(m)}_T}(\psi, \bar{\psi})\|_s \preceq R^{r+2} N^{\tau(r+2)}, \tag{5.2.35}$$

$$\sup_{B_s((1-m/(2r))R)} \|X_{\mathcal{R}^{(m)}_N}(\psi, \bar{\psi})\|_s \preceq \frac{R^2}{N^{s-1}} \tag{5.2.36}$$

Proof. We argue by induction. The theorem is trivial for the case $m = 0$, by setting $cT^{(0)} = id$, $Z^{(0)} = 0$, $f^{(0)} = N^{(1)}$, $\mathcal{R}^{(r)}_N = \mathcal{R}^{(r)}_T = 0$.
Then we split $f^{(m)}$ into two parts, an effective one and a remainder. Indeed, we perform a Taylor expansion of $f^{(m)}$ only in the variables $(\psi_h, \bar{\psi}_h)$, namely we write

$$f^{(m)} = f^{(m)}_0 + f^{(m)}_N,$$

where $f_0^{(m)}$ is the truncation of such a series at second order, and $f_0^{(m)}$ is the remainder. Since both $f_0^{(m)}$ and $f_N^{(m)}$ are truncations of $f^{(m)}$, one has that

$$\left\langle |f_0^{(m)}| \right\rangle_{s,(1-m/(2r))R} \preceq \left\langle |f^{(m)}| \right\rangle_{s,(1-m/(2r))R}$$
$$\left\langle |f_N^{(m)}| \right\rangle_{s,(1-m/(2r))R} \preceq \left\langle |f^{(m)}| \right\rangle_{s,(1-m/(2r))R} .$$

Now consider the truncated Hamiltonian $H_0 + Z^{(m)} + f_0^{(m)}$: we look for a Lie transform $\mathcal{T}_m$ that eliminates the non-normalized part of order $m+4$ of the truncated Hamiltonian. Let χ_m be the analytic Hamiltonian generating $\mathcal{T}_m$. Using (5.2.11) we have

$$(H_0 + Z^{(m)} + f_0^{(m)}) \circ \mathcal{T}_m = H_0 + Z^{(m)}$$
$$+ f_0^{(m)} + \{\chi_m, H_0\} \tag{5.2.37}$$
$$+ \sum_{l\geq 1} Z_l^{(m)} + \sum_{l\geq 1} f_{0,l}^{(m)} + \sum_{l\geq 2} H_{0,l}, \tag{5.2.38}$$

with $Z_l^{(m)}$ the l-th term in the expansion of the Lie transform of $Z^{(m)}$, and similarly for the other quantities. It is easy to see that the terms in the first line are already normalized, that the term in the (5.2.37) is the non-normalized part of order $m+3$ that will vanish through the choice of a suitable χ_m, and that the last lines contains all the terms having a zero of order $m+4$ at the origin.

Now we want to determine χ_m in order to solve the so-called "homological equation"

$$\{\chi_m, H_0\} + f_0^{(m)} \ = \ Z_{m+1},$$

with Z_{m+1} in (γ, τ, N)-normal form. The existence of χ_m and Z_{m+1} is ensured by Lemma 5.2.12, and by applying (5.2.18) and (5.2.33) we get

$$\langle |\chi_m| \rangle_{s,(1-m/(2r))R} \leq N^\tau \ R^2 \, (N^\tau \ R)^m \, , \tag{5.2.39}$$
$$\langle |Z_{m+1}| \rangle_{s,(1-m/(2r))R} \leq R^2 \ \ (N^\tau \ R)^m \, . \tag{5.2.40}$$

In particular, in view of (5.2.23), we can deduce (5.2.32) at level $m+1$. Now define $Z^{(m+1)} := Z^{(m)} + Z_{m+1}$, and $f_C^{(m+1)} :=$ (A.0.20). By 5.2.39, recalling that $R < R_m^*/N^\tau$, we can deduce (5.2.33) at level $m+1$. Moreover, provided that $R_m^* < 2^{-(m+1)/2}$, one has

$$\delta := e \, \frac{2r}{R} \ \langle |\chi_m| \rangle_{s,(1-m/(2r))R} \leq (N^\tau \ R)^{m+1} < \frac{1}{2}.$$

By (5.2.15) and (5.2.33) one thus gets

$$\left\langle |f_C^{(m+1)}| \right\rangle_{s,(1-(m+1)/(2r))R} \preceq \sum_{l\geq 1} R^2 \delta^l + \sum_{l\geq 1} R^2 \delta^l \, (N^\tau \ R)^m + \sum_{l\geq 2} R^2 \delta^{l-1} \, (N^\tau \ R)^m$$
$$\preceq R^2 \, (N^\tau \ R)^{m+1} \, .$$

Write now $f_C^{(m+1)} = f^{(m+1)} + \mathcal{R}_{m,T}$, where $f^{(m+1)}$ is the Taylor polynomial of order $r+2$ of $f_C^{(m+1)}$, and where $\mathcal{R}_{m,T}$ has a zero of order $r+3$ at the origin. Clearly $f^{(m+1)}$ satisfies (5.2.34) at level $m+1$, since it is a truncation of $f_C^{(m+1)}$. The remainder may be bounded by using Lagrange and Cauchy estimates,

$$\begin{aligned}\sup_{B_s((1-m/(2r))R)} \|X_{\mathcal{R}_{m,T}}(\psi,\bar{\psi})\|_s &\preceq \frac{1}{(r+2)!} R^{r+2} \sup_{B_s(R_m^*/(2N^\tau))} \|\partial^{r+2} X_{f_C^{(m+1)}}(\psi,\bar{\psi})\|_s \\ &\preceq R^{r+2}\left(\frac{2N^\tau}{R_m^*}\right)^{r+2} \sup_{B_s(R_m^*/N^\tau)} \|X_{f_C^{(m+1)}}(\psi,\bar{\psi})\|_s \\ &\preceq (N^\tau\, R)^{r+2}.\end{aligned}$$

Now define $\mathcal{R}_T^{(m+1)} := \mathcal{R}_T^{(m)} \circ \mathcal{T}_m + \mathcal{R}_{m,T}$. By (5.2.23) we can deduce (5.2.35) at level $m+1$. Then set $\mathcal{R}_N^{(m+1)} := (\mathcal{R}_N^{(m)} + f_N^{(m)}) \circ \mathcal{T}_m$. By (5.2.34) and (5.2.36), together with (5.2.23) we obtain (5.2.36) at level $m+1$. □

Now we conclude the proof of Theorem 5.1.1.
By taking the canonical transformation $\mathcal{T}^{(r)}$ defined in the iterative Lemma 5.2.15 we have that

$$H^{(r)} = H_0 + Z^{(r)} + \mathcal{R}_N^{(r)} + \mathcal{R}_T^{(r)} + N^{(1,r)} \circ \mathcal{T}^{(r)}, \tag{5.2.41}$$

where $Z^{(r)}$ is in (γ,τ,N)-normal form, and for any $R < R_m^*/N^\tau$ the following holds

$$\sup_{B_s(R)} \|\mathcal{T}^{(r)}(\psi,\bar{\psi}) - (\psi,\bar{\psi})\|_s \preceq N^{2\tau}\, R^3,$$

$$\sup_{B_s(R)} \|X_{\mathcal{R}_N^{(r)}}(\psi,\bar{\psi})\|_s \preceq \frac{R^2}{N^{s-1}},$$

$$\sup_{B_s(R)} \|X_{\mathcal{R}_T^{(r)}}(\psi,\bar{\psi})\|_s \preceq (N^\tau\, R)^{r+2},$$

$$\sup_{B_s(R)} \|X_{N^{(1,r)}\circ\mathcal{T}^{(r)}}(\psi,\bar{\psi})\|_s \preceq (N^\tau\, R)^{r+2}.$$

To conclude we have just to choose N and s sich that $\mathcal{R}_N^{(r)}$ and $\mathcal{R}_T^{(r)}$ are of the same order of magnitude. First take $N = R^{-a}$, with a still to be determined; then, in order to obtain that $\mathcal{R}_T^{(r)}$ is of order $\mathcal{O}(R^{r+3/2})$ we choose $a := \frac{1}{2\tau(r+2)}$. By taking $s > 2\tau r(r+2)+1$ we get that also $N^{(1,r)}$ is of the same order of magnitude.
Now take $K^* = 1/24$, and construct the canonical transformation $(\psi,\bar{\psi}) = \mathcal{T}^{(r)}(\psi',\bar{\psi}')$. Denote by I' the actions expressed in the variable $(\psi',\bar{\psi}')$, and define the function $\mathcal{N}(\psi',\bar{\psi}') := \|I'\|_s^2$. By (5.1.14) one has that $\mathcal{N}(\psi_0',\bar{\psi}_0') \le \frac{32}{31}R^2$, provided that R is sufficiently small. Since

$$\frac{\partial \mathcal{N}}{\partial t}(\psi',\bar{\psi}') = \{R^{(r)}, \mathcal{N}\}(\psi',\bar{\psi}'),$$

and therefore, as far as $\mathcal{N}(\psi',\bar{\psi}') < \frac{64}{9}R^2$,

$$\left|\frac{\partial \mathcal{N}}{\partial t}(\psi',\bar{\psi}')\right| \le K_s'\, R^{r+5/2}. \tag{5.2.42}$$

Denote by T_f the escape time of $(\psi',\bar{\psi}')$ from $B_s(R/3)$; observe that for all times smaller than T_f, (5.2.42) holds. So one has

$$\frac{64}{9}R^2 = \mathcal{N}(\psi'(T_f),\bar{\psi}'(T_f)) \le \mathcal{N}(\psi_0',\bar{\psi}_0') + K_s'\, R^{r+5/2} T_f,$$

which shows that T_f should be of order (at least) $R^{r+1/2}$. Going back to the original variables one gets (5.1.16).
To show (5.1.17), one has to recall that

$$|I(t)-I(0)| \leq |I(t)-I'(t)| + |I'(t)-I'(0)| + |I'(0)-I(0)|,$$

and that by (5.1.14) and (5.1.16) one can estimate the first and the third term; the second term can be bounded by computing the time derivative of $\|I'\|_s^2$ with the Hamiltonian, and observing that it is of order $\mathcal{O}(R^{r+5/2})$.
Now, consider the initial actions $(I_0, \bar{I}_0) := (I(0), \bar{I}(0))$. By passing to the Fourier transform,

$$I_j(t) := \widehat{I(t)}(j), \ \ j \geq 1,$$

we have that for any $r_1 \leq r$

$$|(I_j(t), \bar{I}_j(t)) - (I_j(0), \bar{I}_j(0))| \preceq \frac{R^{2r_1}}{j^{2s}}, \ \ |t| \preceq R^{-(r-r_1+1/2)}. \tag{5.2.43}$$

If we define the torus

$$\mathbb{T}_c := \{(\psi,\bar{\psi}) \in H^s : (I_j(\psi,\bar{\psi}), \bar{I}_j(\psi,\bar{\psi}) = (I_j(0), \bar{I}_j(0)), \ \text{for any} \ \ j \geq 1\},$$

we get

$$d_{s_1}((\psi(t),\bar{\psi}(t)), \mathbb{T}_c) \leq \left[\sum_j j^{2s_1}\left(|\sqrt{I_j(t)} - \sqrt{I_j(0)}|^2 + |\sqrt{\bar{I}_j(t)} - \sqrt{\bar{I}_j(0)}|^2\right)\right]^{1/2},$$

and by using (5.2.43) we obtain

$$d_{s_1}((\psi(t),\bar{\psi}(t)), \mathbb{T}_c)^2 \leq \left(\sup_j j^{2s}|I_j(t)-I_j(0)|^2 + j^{2s}|\bar{I}_j(t) - \bar{I}_j(0)|^2\right) \sum_j \frac{1}{j^{2(s-s_1)}},$$

which is convergent for $s_1 < s - 1/2$.

Appendix A

Proof of Lemma 3.2.3

In order to normalize system (3.2.1), we used an adaptation of Theorem 4.4 in [4]. The result is based on the method of Lie transform, that we will recall in the following.
Let $k \geq k_1$ and $p \in (1, +\infty)$ be fixed.
Given an auxiliary function χ analytic on $W^{k,p}$, we consider the auxiliary differential equation

$$\dot{\psi} = i\nabla_{\bar{\psi}}\chi(\psi, \bar{\psi}) =: X_\chi(\psi, \bar{\psi}) \tag{A.0.1}$$

and denote by Φ^t_χ its time-t flow. A simple application of Cauchy inequality gives

Lemma A.0.1. *Let χ and its symplectic gradient be analytic in $B_{k,p}(\rho)$. Fix $\delta < \rho$, and assume that*

$$\sup_{B_{k,p}(R-\delta)} \|X_\chi(\psi, \bar{\psi})\|_{k,p} \leq \delta.$$

Then, if we consider the time-t flow Φ^t_χ of X_χ we have that for $|t| \leq 1$

$$\sup_{B_{k,p}(R-\delta)} \|\Phi^t_\chi(\psi, \bar{\psi}) - (\psi, \bar{\psi})\|_{k,p} \leq \sup_{B_{k,p}(R-\delta)} \|X_\chi(\psi, \bar{\psi})\|_{k,p}.$$

Definition A.0.2. *The map $\Phi := \Phi^1_\chi$ will be called the* Lie transform *generated by χ.*

Remark A.0.3. *Given G analytic on $W^{k,p}$, consider the differential equation*

$$\dot{\psi} = X_G(\psi, \bar{\psi}), \tag{A.0.2}$$

where by X_G we denote the vector field of G. Now define

$$\Phi^* G(\phi, \bar{\phi}) := G \circ \Phi(\psi, \bar{\psi}).$$

In the new variables $(\phi, \bar{\phi})$ defined by $(\psi, \bar{\psi}) = \Phi(\phi, \bar{\phi})$ equation (A.0.2) *is equivalent to*

$$\dot{\phi} = X_{\Phi^* G}(\phi, \bar{\phi}). \tag{A.0.3}$$

Using the relation

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi^t_\chi)^* G = (\Phi^t_\chi)^* \{\chi, G\},$$

we formally get

$$\Phi^* G = \sum_{l=0}^{\infty} G_l, \tag{A.0.4}$$

$$G_0 := G, \tag{A.0.5}$$

$$G_l := \frac{1}{l}\{\chi, G_{l-1}\}, \quad l \geq 1. \tag{A.0.6}$$

In order to estimate the terms appearing in (A.0.4) we exploit the following results

Lemma A.0.4. *Let $R > 0$, and assume that χ, G are analytic on $B_{k,p}(R)$. Then, for any $d \in (0, R)$ we have that $\{\chi, G\}$ is analytic on $B_{k,p}(R - d)$, and*

$$\sup_{B_{k,p}(R-d)} \|X_{\{\chi,G\}}(\psi, \bar{\psi})\|_{k,p} \preceq \frac{2}{d}. \tag{A.0.7}$$

Lemma A.0.5. *Let $R > 0$, and assume that χ, G are analytic on $B_{k,p}(R)$. Let $l \geq 1$, and consider G_l as defined in* (A.0.4)*; for any $d \in (0, R)$ we have that G_l is analytic on $B_{k,p}(R - d)$, and*

$$\sup_{B_{k,p}(R-d)} \|X_{G_l}(\psi, \bar{\psi})\|_{k,p} \preceq \left(\frac{2e}{d}\right)^l. \tag{A.0.8}$$

Proof. Fix l, and denote $\delta := d/l$. We look for a sequence $C_m^{(l)}$ such that

$$\sup_{B_{k,p}(R-m\delta)} \|X_{G_m}(\psi, \bar{\psi})\|_{k,p} \preceq C_m^{(l)}, \quad \forall m \leq l.$$

By (A.0.7) we can define the sequence

$$C_0^{(l)} := \sup_{B_{k,p}(R)} \|X_G(\psi, \bar{\psi})\|_{k,p},$$

$$C_m^{(l)} = \frac{2}{\delta m} C_{m-1}^{(l)} \sup_{B_{k,p}(R)} \|X_\chi(\psi, \bar{\psi})\|_{k,p}$$

$$= \frac{2l}{dm} C_{m-1}^{(l)} \sup_{B_{k,p}(R)} \|X_\chi(\psi, \bar{\psi})\|_{k,p}.$$

One has

$$C_l^{(l)} = \frac{1}{l!}\left(\frac{2l}{d} \sup_{B_{k,p}(R)} \|X_\chi(\psi, \bar{\psi})\|_{k,p}\right)^l \sup_{B_{k,p}(R)} \|X_G(\psi, \bar{\psi})\|_{k,p},$$

and by using the inequality $l^l < l! e^l$ we can conclude. □

Remark A.0.6. *Let $k \geq k_1$, $p \in (1, +\infty)$, and assume that χ, F are analytic on $B_{k,p}(R)$. Fix $d \in (0, R)$, and assume also that*

$$\sup_{B_{k,p}(R)} \|X_\chi(\psi, \bar{\psi})\|_{k,p} \leq d/3,$$

Then for $|t| \leq 1$

$$\sup_{B_{k,p}(R-d)} \|X_{(\Phi^t_\chi)^* F - F}(\psi,\bar\psi)\|_{k,p} = \sup_{B_{k,p}(R-d)} \|X_{F\circ\Phi^t_\chi - F}(\psi,\bar\psi)\|_{k,p} \tag{A.0.9}$$

$$\stackrel{(A.0.7)}{\leq} \frac{5}{d} \sup_{B_{k,p}(R)} \|X_\chi(\psi,\bar\psi)\|_{k,p} \sup_{B_{k,p}(R)} \|X_F(\psi,\bar\psi)\|_{k,p}. \tag{A.0.10}$$

Lemma A.0.7. *Let* $k \geq k_1$, $p \in (1,+\infty)$, *and assume that* G *is analytic on* $B_{k,p}(R)$, *and that* h_0 *satisfies PER. Then there exists* χ *analytic on* $B_{k,p}(R)$ *and* Z *analytic on* $B_{k,p}(R)$ *with* Z *in normal form, namely* $\{h_0, Z\} = 0$, *such that*

$$\{h_0,\chi\} \; + \; G \; = \; Z. \tag{A.0.11}$$

Furthermore, we have the following estimates on the vector fields

$$\sup_{B_{k,p}(R)} \|X_Z(\psi,\bar\psi)\|_{k,p} \leq \sup_{B_{k,p}(R)} \|X_G(\psi,\bar\psi)\|_{k,p}, \tag{A.0.12}$$

$$\sup_{B_{k,p}(R)} \|X_\chi(\psi,\bar\psi)\|_{k,p} \preceq \sup_{B_{k,p}(R)} \|X_G(\psi,\bar\psi)\|_{k,p}. \tag{A.0.13}$$

Proof. One can check that the solution of (A.0.11) is

$$\chi(\psi,\bar\psi) = \frac{1}{T}\int_0^T t\, \left[G(\Phi^t(\psi,\bar\psi)) - Z(\Phi^t(\psi,\bar\psi))\right] \mathrm{d}t,$$

with $T = 2\pi$. Indeed,

$$\begin{aligned}
\{h_0,\chi\}(\psi,\bar\psi) &= \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0} \chi(\Phi^s(\psi,\bar\psi)) \\
&= \frac{1}{2\pi}\int_0^{2\pi} t \frac{\mathrm{d}}{\mathrm{d}s}_{|s=0} \left[G(\Phi^{t+s}(\psi,\bar\psi)) - Z(\Phi^{t+s}(\psi,\bar\psi))\right] \mathrm{d}t \\
&= \frac{1}{2\pi}\int_0^{2\pi} t \frac{\mathrm{d}}{\mathrm{d}t} \left[G(\Phi^{t}(\psi,\bar\psi)) - Z(\Phi^{t}(\psi,\bar\psi))\right] \mathrm{d}t \\
&= \frac{1}{2\pi}\left[tG(\Phi^t(\psi,\bar\psi)) - tZ(\Phi^t(\psi,\bar\psi))\right]_{t=0}^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} \left[G(\Phi^{t}(\psi,\bar\psi)) - Z(\Phi^{t}(\psi,\bar\psi))\right] \mathrm{d}t \\
&= G(\psi,\bar\psi) - Z(\psi,\bar\psi).
\end{aligned}$$

Finally, (A.0.12) follows from the fact that

$$X_\chi(\psi,\bar\psi) = \frac{1}{T}\int_0^T t\, \Phi^{-t} \circ X_{G-Z}(\Phi^t(\psi,\bar\psi)\mathrm{d}t$$

by applying property (3.1.2). □

Lemma A.0.8. *Let* $k \geq k_1$, $p \in (1,+\infty)$, *and assume that* G *is analytic on* $B_{k,p}(R)$, *and that* h_0 *satisfies PER. Let* χ *be analytic on* $B_{k,p}(R)$, *and assume that it solves* (A.0.11). *For any* $l \geq 1$ *denote by* $h_{0,l}$ *the functions defined recursively as in* (A.0.4) *from* h_0. *Then for any* $d \in (0,R)$ *one has that* $h_{0,l}$ *is analytic on* $B_{k,p}(R-d)$, *and*

$$\sup_{B_{k,p}(R-d)} \|X_{h_{0,l}}(\psi,\bar\psi)\|_{k,p} \leq 2 \sup_{B_{k,p}(R)} \|X_G(\psi,\bar\psi)\|_{k,p} \left(\frac{5}{d} \sup_{B_{k,p}(R)} \|X_\chi(\psi,\bar\psi)\|_{k,p}\right)^l. \tag{A.0.14}$$

Proof. By using (A.0.11) one gets that $h_{0,1} = Z - G$ is analytic on $B_{k,p}(R)$. Then by exploiting (A.0.10) one gets the result. □

Lemma A.0.9. *Let $k_1 \gg 1$, $p \in (1, +\infty)$, $R > 0$, $m \geq 0$, and consider the Hamiltonian*

$$H^{(m)}(\psi, \bar{\psi}) = h_0(\psi, \bar{\psi}) + \epsilon \hat{h}(\psi, \bar{\psi}) + \epsilon Z^{(m)}(\psi, \bar{\psi}) + \epsilon^{m+1} F^{(m)}(\psi, \bar{\psi}). \tag{A.0.15}$$

Assume that h_0 satisfies PER and INV, that $\hat{h}$ satisfies NF, and that

$$\sup_{B_{k,p}(R)} \|X_{\hat{h}}(\psi, \bar{\psi})\|_{k,p} \leq F_0,$$
$$\sup_{B_{k,p}(R)} \|X_{F^{(0)}}(\psi, \bar{\psi})\|_{k,p} \leq F.$$

Fix $\delta < R/(m+1)$, and assume also that $Z^{(m)}$ are analytic on $B_{k,p}(R - m\delta)$, and that

$$\sup_{B_{k,p}(R)} \|X_{Z^{(0)}}(\psi, \bar{\psi})\|_{k,p} = 0,$$
$$\sup_{B_{k,p}(R-m\delta)} \|X_{Z^{(m)}}(\psi, \bar{\psi})\|_{k,p} \leq F \sum_{i=0}^{m-1} \epsilon^i K_s^i, \quad m \geq 1,$$
$$\sup_{B_{k,p}(R-m\delta)} \|X_{F^{(m)}}(\psi, \bar{\psi})\|_{k,p} \leq F\, K_s^m, \quad m \geq 1, \tag{A.0.16}$$

with $K_s := \frac{2\pi}{\delta}(18F + 5F_0)$.
Then, if $\epsilon K_s < 1/2$ there exists a canonical transformation $\mathcal{T}_\epsilon^{(m)}$ analytic on $B_{k,p}(R-(m+1)\delta)$ such that

$$\sup_{B_{k,p}(R-m\delta)} \|\mathcal{T}_\epsilon^{(m)}(\psi, \bar{\psi}) - (\psi, \bar{\psi})\|_{k,p} \leq 2\pi \epsilon^{m+1} F, \tag{A.0.17}$$

$H^{(m+1)} := H^{(m)} \circ \mathcal{T}^{(m)}$ has the form (A.0.15) and satisfies (A.0.16) with m replaced by $m+1$.

Proof. The key point of the lemma is to look for $\mathcal{T}_\epsilon^{(m)}$ as the time-one map of the Hamiltonian vector field of an analytic function $\epsilon^{m+1}\chi_m$. Hence, consider the differential equation

$$(\dot{\psi}, \dot{\bar{\psi}}) = X_{\epsilon^{m+1}\chi_m}(\psi, \bar{\psi}); \tag{A.0.18}$$

by standard theory we have that, if $\|X_{\epsilon^{m+1}\chi_m}\|_{B_{k,p}(R-m\delta)}$ is sufficiently small and $(\psi_0, \bar{\psi}_0) \in B_{k,p}(R-(m+1)\delta)$, then the solution of (A.0.18) exists for $|t| \leq 1$. Therefore we can define $\mathcal{T}^t_{m,\epsilon} : B_{k,p}(R-(m+1)\delta) \to B_{k,p}(R - m\delta)$, and in particular the corresponding time-one map $\mathcal{T}_\epsilon^{(m)} := \mathcal{T}^1_{m,\epsilon}$, which is an analytic canonical transformation, ϵ^{m+1}-close to the identity. We have

$$\begin{aligned}(\mathcal{T}_\epsilon^{(m+1)})^* \, (h_0 + \epsilon\hat{h} + \epsilon Z^{(m)} + \epsilon^{m+1}F^{(m)}) &= h_0 + \epsilon\hat{h} + \epsilon Z^{(m)} \\ + \epsilon^{m+1}\left[\{\chi_m, h_0\} + F^{(m)}\right] + \\ + \left(h_0 \circ \mathcal{T}^{(m+1)} - h_0 - \epsilon^{m+1}\{\chi_m, h_0\}\right) + \epsilon(\hat{h} \circ \mathcal{T}^{(m+1)} - \hat{h}) + \epsilon\left(Z^{(m)} \circ \mathcal{T}^{(m+1)} - Z^{(m)}\right)\end{aligned} \tag{A.0.19}$$

$$+ \epsilon^{m+1}\left(F^{(m)} \circ \mathcal{T}^{(m+1)} - F^{(m)}\right). \tag{A.0.20}$$

It is easy to see that the first three terms are already normalized, that the term in the second line is the non-normalized part of order m+1 that will vanish through the choice of a suitable χ_m, and that the last lines contains all the terms of order higher than m+1.

Now we want to determine χ_m in order to solve the so-called "homological equation"

$$\{\chi_m, h_0\} + F^{(m)} \; = \; Z_{m+1},$$

with Z_{m+1} in normal form. The existence of χ_m and Z_{m+1} is ensured by Lemma A.0.7, and by applying (A.0.12) and the inductive hypothesis we get

$$\sup_{B_{k,p}(R-m\delta)} \|X_{\chi_m}(\psi,\bar{\psi})\|_{k,p} \leq 2\pi F, \tag{A.0.21}$$

$$\sup_{B_{k,p}(R-m\delta)} \|X_{Z_{m+1}}(\psi,\bar{\psi})\|_{k,p} \leq 2\pi F. \tag{A.0.22}$$

Now define $Z^{(m+1)} := Z^{(m)} + \epsilon^m Z_{m+1}$, and notice that by Lemma A.0.1 we can deduce the estimate of $X_{Z^{(m+1)}}$ on $B_{k,p}(R-(m+1)\delta)$ and (A.0.17) at level $m+1$. Next, set $\epsilon^{m+2}F^{(m+1)} :=$ (A.0.19) + (A.0.20). Then we can use (A.0.10) and (A.0.14), in order to get

$$\sup_{B_{k,p}(R-(m+1)\delta)} \|X_{\epsilon^{m+2}F^{(m+1)}}(\psi,\bar{\psi})\|_{k,p} \tag{A.0.23}$$

$$\leq \left(\frac{10}{\delta}\epsilon^m K_s^m\,\epsilon F + \frac{5}{\delta}\epsilon F_0 + \frac{5}{\delta}\epsilon F \sum_{i=0}^{m-1} \epsilon^i K_s^i + \frac{5}{\delta}\epsilon F\,\epsilon^m K_s^m\right) \epsilon^{m+1} \sup_{B_{k,p}(R-m\delta)} \|X_{\chi_m}(\psi,\bar{\psi})\|_{k,p}$$

$$= \epsilon^{m+2}\left(\frac{10}{\delta}\epsilon^m K_s^m\,F + \frac{5}{\delta}F_0 + \frac{5}{\delta}F \sum_{i=0}^{m-1} \epsilon^i K_s^i + \frac{5}{\delta}F\,\epsilon^m K_s^m\right) \sup_{B_{k,p}(R-m\delta)} \|X_{\chi_m}(\psi,\bar{\psi})\|_{k,p}. \tag{A.0.24}$$

If $m = 0$, then the third term is not present, and (A.0.24) reads

$$\sup_{B_{k,p}(R-\delta)} \|X_{\epsilon^2 F^{(1)}}(\psi,\bar{\psi})\|_{k,p} \leq \epsilon^2 \left(\frac{15}{\delta}\,F + \frac{5}{\delta}F\right) 2\pi F < \epsilon^2 K_s F.$$

If $m \geq 1$, we exploit the smallness condition $\epsilon K_s < 1/2$, and (A.0.24) reads

$$\sup_{B_{k,p}(R-(m+1)\delta)} \|X_{\epsilon^{m+2}F^{(m+1)}}(\psi,\bar{\psi})\|_{k,p} < \left(\frac{18}{\delta}\epsilon F + \frac{5}{\delta}\epsilon F_0\right) 2\pi\,\epsilon F\,\epsilon^m K_s^m = \epsilon^{m+2}\,F K_s^{m+1}.$$

□

Now fix $R > 0$.

Proof. (of Lemma 3.2.3) The Hamiltonian (3.2.1) satisfies the assumptions of Lemma A.0.9 with $m = 0$, $F_{N,r}$ in place of $F^{(0)}$ and $h_{N,r}$ in place of $\hat{h}$, $F = K_{k,p}^{(F,r)}\, r 2^{2Nr}$, $F_0 = K_{k,p}^{(h,r)}\, r 2^{2Nr}$ (for simplicity we will continue to denote by F and F_0 the last two quantities). So we apply Lemma A.0.9 with $\delta = R/4$, provided that

$$\frac{8\pi}{R}(18F + 5F_0)\epsilon < \frac{1}{2},$$

which is true due to (3.2.12). Hence there exists an analytic canonical transformation $\mathcal{T}_{\epsilon,N}^{(1)}$: $B_{k,p}(3R/4) \to B_{k,p}(R)$ with

$$\sup_{B_{k,p}(3R/4)} \|\mathcal{T}_{\epsilon,N}^{(1)}(\psi,\bar{\psi}) - (\psi,\bar{\psi})\|_{k,p} \leq 2\pi F\,\epsilon,$$

such that

$$H_{N,r} \circ \mathcal{T}_{\epsilon,N}^{(1)} = h_0 + \epsilon h_{N,r} + \epsilon Z_N^{(1)} + \epsilon^2 \mathcal{R}_N^{(1)}, \tag{A.0.25}$$

$$Z_N^{(1)} := \langle F_{N,r} \rangle , \tag{A.0.26}$$

$$\begin{aligned} \epsilon^2 \mathcal{R}_N^{(1)} &:= \epsilon^2 F^{(1)} \\ &= \left(h_0 \circ \mathcal{T}_{\epsilon,N}^{(1)} - h_0 - \epsilon\{\chi_1, h_0\} \right) + \epsilon(\hat{h}_{N,r} \circ \mathcal{T}_{\epsilon,N}^{(1)} - \hat{h}_{N,r}) + \epsilon \left(Z_N^{(1)} \circ \mathcal{T}_{\epsilon,N}^{(1)} - Z_N^{(1)} \right) \\ &\quad + \epsilon^2 \left(F_{N,r} \circ \mathcal{T}_{\epsilon,N}^{(1)} - F_{N,r} \right), \end{aligned} \tag{A.0.27}$$

$$\sup_{B_{k,p}(3R/4)} \| X_{h_{N,r} + Z_N^{(1)}} (\psi, \bar{\psi}) \|_{k,p} \le F_0 + F =: \tilde{F}_0, \tag{A.0.28}$$

$$\sup_{B_{k,p}(3R/4)} \| X_{\mathcal{R}_N^{(1)}} (\psi, \bar{\psi}) \|_{k,p} \le \frac{8\pi}{R} (18F + 5F_0) F =: \tilde{F}. \tag{A.0.29}$$

Again (A.0.25) satisfies the assumptions of Lemma A.0.9 with $m = 0$, and $h_{N,r} + Z_N^{(1)}$ and $\mathcal{R}_N^{(1)}$ in place of $F^{(0)}$ and $\hat{h}$.
Now fix $\delta := \delta(R) = \frac{R}{4r}$, and apply r times Lemma A.0.9; we get an Hamiltonian of the form (3.2.13), such that

$$\sup_{B_{k,p}(R/2)} \| X_{Z_N^{(r)}} (\psi, \bar{\psi}) \|_{k,p} \le 2\tilde{F}, \tag{A.0.30}$$

$$\sup_{B_{k,p}(R/2)} \| X_{\mathcal{R}_N^{(r)}} (\psi, \bar{\psi}) \|_{k,p} \le \tilde{F}. \tag{A.0.31}$$

□

Appendix B

Properties of the Klein-Gordon and Spinless Salpeter equations

B.1 Formula for the action of the $e^{itc\langle\nabla\rangle_c}$

We now derive an explicit formula for the action of the unitary group generated by $c\langle\nabla\rangle_c$. To the author's knowledge, it is not easy to find in the literature such a formula. We proceed as in [68] for the corresponding massless case.
Consider $f \in C_c^\infty(\mathbb{R}^3)$, $g \in \mathcal{S}(\mathbb{R}^3)$ with $\hat{g} \in C_c^\infty(\mathbb{R}^3)$, and set

$$\zeta_\pm(z) := \int_{\mathbb{R}^3} e^{\pm izc\langle k\rangle_c} \hat{f}(k)\overline{\hat{g}(k)}\mathrm{d}^3k.$$

Notice that $\zeta_\pm$ is an entire function on $\mathbb{C}$, and one has $\zeta_\pm(\mp t) = \left\langle e^{-itc\langle\nabla\rangle_c} f, g\right\rangle$ for any $t \in \mathbb{R}$. By [52] (Sec. 7.11) one has that for any $t > 0$

$$\begin{aligned}\zeta_\pm(\pm it) &= \left\langle e^{-tc\langle\nabla\rangle_c} f, g\right\rangle \\ &= \int_{\mathbb{R}^3} \left[\frac{c^2}{2\pi^2}\int_{\mathbb{R}^3} \frac{ctK_2(c[c^2t^2+|x-y|^2]^{1/2})}{c^2t^2+|x-y|^2} f(y)\mathrm{d}^3y\right]\overline{g(x)}\mathrm{d}^3x,\end{aligned}$$

where K_2 denotes the modified Bessel function of the third kind of order 2. We recall that the modified Bessel functions of the third kind are defined in the following way (see [1], formulae 9.6.10 and 9.6.2)

$$I_\nu(z) := \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu+k+1)}\left(\frac{z}{2}\right)^{2k+\nu}, \tag{B.1.1}$$

$$K_\nu(z) := \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}, \tag{B.1.2}$$

when for integer ν the right hand side of (B.1.2) is replaced by its limiting value.
Now, set $\eta_\pm(z) := \int_{\mathbb{R}^3}\left[\frac{c^2}{2\pi^2}\int_{\mathbb{R}^3}\frac{\mp iczK_2(c[|x-y|^2-c^2z^2]^{1/2})}{|x-y|^2-c^2z^2} f(y)\mathrm{d}^3y\right]\overline{g(x)}\mathrm{d}^3x$.
By exploiting the following formula (see [34], section 7.2.2, formula (16)),

$$K_\nu(iz) = -\frac{i\pi}{2}e^{-i\nu\pi/2}H_\nu^{(2)}(z),$$

($H_\nu^{(2)}$ denotes the Hankel function of the second kind of order ν) we get

$$\eta_\pm(z) = \int_{\mathbb{R}^3} \left[\frac{c^2}{2\pi} \frac{\mp cz H_2^{(2)}(c[c^2z^2 - |x-y|^2]^{1/2})}{c^2z^2 - |x-y|^2} f(y)\mathrm{d}^3y \right] \overline{g(x)}\mathrm{d}^3x.$$

One can observe that, since the Hankel functions are analytic in the complex plane cut along the negative real axis, η_+ is holomorphic on $\mathcal{C}_+ := \mathbb{C} \setminus (\mathbb{R} \cup (0,i])$ (respectively η_- is holomorphic on $\mathcal{C}_- := \mathbb{C} \setminus (\mathbb{R} \cup [-i,0))$). Furthermore, since $\zeta_\pm(\pm it) = \eta_\pm(\pm it)$ for $t > 1$, one has that $\zeta_\pm = \eta_\pm$ on $\mathcal{C}_\pm$ by analytic continuation.
Therefore for $t > 1$

$$\begin{aligned}\left\langle e^{-it\langle\nabla\rangle_c} f, g \right\rangle &= \zeta_-(t) = \lim_{\epsilon\to0} \zeta_-(t - i\epsilon) = \lim_{\epsilon\to0} \eta_-(t-i\epsilon) \\ &= \frac{c^2}{2\pi} \lim_{\epsilon\to0} \int_{\mathbb{R}^3_x} \left[\int_{\mathbb{R}^3_y} \frac{c(t-i\epsilon)H_2^{(2)}(c[ct - ic\epsilon - |x-y|]^{1/2}[ct - ic\epsilon + |x-y|]^{1/2})}{(ct - ic\epsilon - |x-y|)(ct - ic\epsilon + |x-y|)} f(y)\mathrm{d}^3y \right] \cdot \\ &\cdot \overline{g(x)}\mathrm{d}^3x \\ &= \frac{c^2}{2\pi} \int_{\mathbb{R}^3_x} \int_{\mathbb{R}^3_y} \frac{(ct - i0)H_2^{(2)}(c[ct - i0 - |x-y|]^{1/2}[ct - i0 + |x-y|]^{1/2})}{(ct - i0 - |x-y|)(ct - i0 + |x-y|)} f(y)\mathrm{d}^3y\, \overline{g(x)}\mathrm{d}^3x\end{aligned}$$

where the distributions $s \to (s \pm i0)^a$ are defined for example in [41], Sec. 3.2. On the other hand, for $t < -1$

$$\begin{aligned}\left\langle e^{-itc\langle\nabla\rangle_c} f, g \right\rangle &= \lim_{\epsilon\to0} \eta(|t| + i\epsilon) \\ &= \frac{c^2}{2\pi} \int_{\mathbb{R}^3_x} \int_{\mathbb{R}^3_y} \frac{(ct + i0)H_2^{(2)}(c[ct + i0 - |x-y|]^{1/2}[ct + i0 + |x-y|]^{1/2})}{(ct + i0 - |x-y|)(ct + i0 + |x-y|)} f(y)\mathrm{d}^3y\, \overline{g(x)}\mathrm{d}^3x\end{aligned}$$

Corollary B.1.1. *For any $f \in C_c^\infty(\mathbb{R}^3)$, $g \in \mathcal{S}(\mathbb{R}^3)$ with $\hat{g} \in C_c^\infty(\mathbb{R}^3)$ and $t > 1$ (respectively $t < -1$), one has*

$$\left\langle e^{itc\langle\nabla\rangle_c} f, g \right\rangle = \frac{c^2}{2\pi} \int_{\mathbb{R}^3_x} \int_{\mathbb{R}^3_y} \frac{(ct \pm i0)H_2^{(2)}(c[ct \pm i0 - |x-y|]^{1/2}[ct \pm i0 + |x-y|]^{1/2})}{(ct \pm i0 - |x-y|)(ct \pm i0 + |x-y|)} f(y)\mathrm{d}^3y\, \overline{g(x)}\mathrm{d}^3x \tag{B.1.3}$$

Remark B.1.2. *In [52] (Sec. 7.11) one also can find the explicit formula for the action of $e^{-tc\langle\nabla\rangle_c}$ also for dimension $d > 3$. Since the expression is based on the modified Bessel functions $K_{(d+1)/2}$, the previous argument applies also to higher dimensions. By exploiting this formula one may derive the decay in weighted energy norm for the free equation arguing as in Proposition 2.1.5.*

B.2 Resolvent estimates

B.2.1 Free resolvent

Now we study the resolvent of the equation (2.1.1). Since we are interested in the non-relativistic limit, we assume throughout this section that $c \geq 1$. The resolvent of the operator $\mathcal{H}_0 := c\langle\nabla\rangle_c$

(which in the literature is sometimes called relativistic Schrödinger operator) will be denoted by

$$\mathcal{R}_{0,c}(z) = (\mathcal{H}_0 - z)^{-1}, \ \ z \in \mathbb{C} \setminus [c^2, +\infty).$$

Recall that we can write the resolvent through the Fourier transform $F : L^2 \to L^2$,

$$\mathcal{R}_{0,c}(z) = F^{-1}(c\,\langle\xi\rangle_c - z)^{-1}F$$

(in [86], [30], [49] and [6] the resolvent of the KG equation is handled by using the well known Jensen-Kato estimates for the Schrödinger resolvent); however, in order to write the resolvent kernel, we will not use this formula. Arguing as in [87], we will use the explicit expression of the semigroup generated by $-c\langle\nabla\rangle_c$ as a convolution with the following Poisson kernel ([52], sec. 7.11)

$$e^{-tc\langle\nabla\rangle_c}\psi(x) \ = \ P_{t,c} * \psi(x) = \int_{\mathbb{R}^3} P_{t,c}(x-y)\psi(y)\mathrm{d}^3y, \ \ t > 0, \psi \in L^2, \tag{B.2.1}$$

$$P_{t,c}(x) = \frac{c^2}{2\pi^2}\frac{ctK_2\left(c[|x|^2 + c^2t^2]^{1/2}\right)}{c^2t^2 + |x|^2}, \tag{B.2.2}$$

where K_ν is the modified Bessel function of third kind of order ν (see (B.1.2)). We also recall the following properties of modified Bessel functions (see formulae 9.6.6, 9.6.26 and 9.6.28 in [1]) that we will use in the sequel,

$$K_{\nu+1}(z) = K_{\nu-1}(z) + \frac{2\nu}{z}K_\nu(z), \tag{B.2.3}$$

$$K'_\nu(z) = -K_{\nu-1}(z) - \frac{\nu}{z}K_\nu(z), \tag{B.2.4}$$

$$\left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^k (z^\nu e^{i\pi\nu}K_\nu(z)) = z^{\nu-k}e^{i\pi(\nu-k)}K_{\nu-k}(z), \ k = 0, 1, 2, \ldots, \tag{B.2.5}$$

$$\left(\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^k (z^{-\nu} e^{i\pi\nu}K_\nu(z)) = z^{-\nu-k}e^{i\pi(\nu+k)}K_{\nu+k}(z), \ k = 0, 1, 2, \ldots. \tag{B.2.6}$$

We then take the Laplace transform of $e^{-tc\langle\nabla\rangle_c}$ to get the free resolvent ([47], ch. 9, (1.28)):

$$\mathcal{R}_{0,c}(z) = \int_0^{+\infty} e^{tz}e^{-tc\langle\nabla\rangle_c}\mathrm{d}t, \ \ \text{for } Re(z) < 0. \tag{B.2.7}$$

By integration by parts and by exploiting (B.2.6)-(B.2.4), we obtain that for any $a > 0$ and for $Re(z) < 0$

$$\begin{aligned} \frac{c^2}{2\pi^2}\int_0^{+\infty} e^{tz}\,\frac{ctK_2\left(c[a^2 + c^2t^2]^{1/2}\right)}{c^2t^2 + a^2}\,\mathrm{d}t &= \frac{c^2}{2\pi^2}\left[\frac{K_1(ca)}{c^2a} + \frac{z}{c^2}\int_0^{+\infty} e^{tz}\frac{K_1(c[a^2 + c^2t^2]^{1/2})}{[a^2 + c^2t^2]^{1/2}}\mathrm{d}t\right] \\ &= \frac{1}{2\pi^2}\left[\frac{K_1(ca)}{a} + z\int_0^{+\infty} e^{tz}\frac{K_1(c[a^2 + c^2t^2]^{1/2})}{[a^2 + c^2t^2]^{1/2}}\mathrm{d}t\right], \end{aligned} \tag{B.2.8}$$

where the last Laplace transform is well-defined for $Re(z) < c^2$. In order to study the dispersive properties of equation (4.1.6) we need the asymptotics of the free resolvent for $|z| \to \infty$ and for $|z| \to c^2$. These will be obtained by using the so-called Abel theorems for the Laplace transform (see for example [88], ch. 5, theorem 1 and corollary1a and corollary 1b).

First we define, for any $z \in \mathbb{C}$ with $Re(z) < c^2$, the function $g_{z,c} : \mathbb{R}^3 \to \mathbb{R}$,

$$g_{z,c}(x) := = \frac{1}{2\pi^2}\left[\frac{K_1(c|x|)}{|x|} + z\int_0^{+\infty} e^{tz}\frac{K_1(c[|x|^2+c^2t^2]^{1/2})}{[|x|^2+c^2t^2]^{1/2}}\mathrm{d}t\right],$$

and the associated operator $G_{z,c}$ on $L^2(\mathbb{R}^3)$ given by

$$G_{z,c}\psi(x) := g_{z,c} * \psi(x) = \int_{\mathbb{R}^3} g_{z,c}(x-y)\psi(y)\mathrm{d}^3y.$$

Notice that for $Re(z) < 0$ one has that $g_{z,c}(x) = \int_0^{+\infty} e^{tz}P_{t,c}(x)\mathrm{d}t$.

Proposition B.2.1. *If $z \in \mathbb{C}$ with $Re(z) < c^2$, then $\mathcal{R}_{0,c}(z)\psi = G_{z,c}\psi$ for all $\psi \in C_c^\infty(\mathbb{R}^3)$.*

Proof. Without loss of generality it suffices to show that

$$\langle \mathcal{R}_{0,c}(z)\psi, \phi\rangle_{L^2} = \langle G_{z,c}\psi, \phi\rangle_{L^2},$$

for all $z \in \mathbb{C} \setminus [c^2, +\infty)$, and all $\psi, \phi \in C_c^\infty(\mathbb{R}^3)$. Since

$$\begin{aligned}\langle \mathcal{R}_{0,c}(z)\psi, \phi\rangle_{L^2} &= \int_0^{+\infty} \left\langle e^{-t\mathcal{H}_0}\psi, \phi\right\rangle_{L^2}\mathrm{d}t \\ &= \int_0^{+\infty} e^{tz}\left[\int_{\mathbb{R}^3_x}\left(\int_{\mathbb{R}^3_y} P_{t,c}(x-y)\psi(y)\mathrm{d}^3y\right)\overline{\phi(x)}\mathrm{d}^3x\right]\mathrm{d}t \end{aligned} \tag{B.2.9}$$

for $Re(z) < 0$, in order to make a change of integration in the previous formula, we have to show that $e^{tz}P_{t,c}(x-y)\psi(y)\overline{\phi(x)}$ is absolutely integrable with respect to x, y and t for $Re(z) < 0$ and $\psi, \phi \in C_c^\infty(\mathbb{R}^3)$. Indeed, by integration by parts

$$\begin{aligned}\int_0^{+\infty} e^{t\,Re(z)}\frac{ctK_2\left(c[a^2+c^2t^2]^{1/2}\right)}{c^2t^2+a^2}\,\mathrm{d}t &= \frac{K_1(ca)}{c^2a} + \frac{Re(z)}{c^2}\int_0^{+\infty} e^{t\,Re(z)}\frac{K_1(c[a^2+c^2t^2]^{1/2})}{[a^2+c^2t^2]^{1/2}}\mathrm{d}t \\ &\preceq \frac{K_1(ca)}{c^2a} + \frac{Re(z)K_1(ca)}{c^2a}\int_0^{+\infty} e^{t\,Re(z)}\mathrm{d}t \\ &\preceq \frac{K_1(ca)}{c^2a}.\end{aligned}$$

Therefore, this implies

$$
\begin{aligned}
&\int_{\mathbb{R}^6\times(0,+\infty)} \left|e^{tz}P_{t,c}(x-y)\psi(y)\overline{\phi(x)}\right| \mathrm{d}^3x\,\mathrm{d}^3y\,\mathrm{d}t \preceq \frac{1}{2\pi^2}\int_{\mathbb{R}^6} \frac{|\psi(y)\phi(x)|}{|x-y|}K_1(c|x-y|)\mathrm{d}^3x\mathrm{d}^3y \\
&= \frac{1}{2\pi^2}\int_{\mathbb{R}^3}|\phi(x)|\left(\int_{|x-y|\le 1}\frac{|\psi(y)}{c|x-y|^2}\mathrm{d}^3y + \int_{|x-y|>1}\frac{|\psi(y)|e^{-c|x-y|}}{c^{1/2}|x-y|^{3/2}}\mathrm{d}^3y\right)\mathrm{d}^3x \\
&\preceq \frac{1}{2\pi^2}\int_{\mathbb{R}^3}|\phi(x)|\left(c^{-1}\int_{|x-y|\le 1}\frac{|\psi(y)|}{|x-y|^2}\mathrm{d}^3y + c^{-1/2}e^{-c}\|\psi\|_{L^1}\right)\mathrm{d}^3x \\
&\preceq \frac{1}{2\pi^2}\int_{\mathbb{R}^3}|\phi(x)|\left(c^{-1}\|\psi\|_{L^\infty}\int_{|y|\le 1}\frac{1}{|y|^2}\mathrm{d}^3y + c^{-1/2}e^{-c}\|\psi\|_{L^1}\right)\mathrm{d}^3x \\
&\overset{c\ge 1}{\preceq} \|\phi\|_{L^1}\ \left(\|\psi\|_{L^\infty} + \|\psi\|_{L^1}\right) < +\infty.
\end{aligned}
$$

Hence we can make a change of integration in (B.2.9),

$$
\langle \mathcal{R}_{0,c}(z)\psi,\phi\rangle_{L^2} = \int_{\mathbb{R}^3_x}\left[\int_{\mathbb{R}^3_y}\left(\int_0^{+\infty} e^{tz}P_{t,c}(x-y)\mathrm{d}t\right)\psi(y)\mathrm{d}^3y\right]\overline{\phi(x)}\mathrm{d}^3x,
$$

and if we apply (B.2.8) to the integral with respect to the t variable, we get that

$$
\langle \mathcal{R}_{0,c}(z)\psi,\phi\rangle_{L^2} = \langle G_{z,c}\psi,\phi\rangle_{L^2}\,,\ \text{ for } Re(z) < 0.
$$

Finally, differentiating

$$
\int_{\mathbb{R}^3_x}\int_{\mathbb{R}^3_y} g_{z,c}(x-y)\psi(y)\overline{\phi(x)}\mathrm{d}^3y\,\mathrm{d}^3x
$$

with respect to z under the integral sign, and by applying the properties of the Laplace transform, we obtain that $\langle G_{z,c}\psi,\phi\rangle_{L^2}$ is a holomorphic function for $Re(z) < c^2$. Since $\langle R_{0,c}(z)\psi,\phi\rangle_{L^2}$ is also a holomorphic function for $Re(z) < c^2$, we finally get

$$
\langle \mathcal{R}_{0,c}(z)\psi,\phi\rangle_{L^2} = \langle G_{z,c}\psi,\phi\rangle_{L^2}\,,\ \text{ for } Re(z) < c^2.
$$

□

Remark B.2.2. *By exploiting the definition of the free resolvent through the Fourier transform,*

$$
\mathcal{R}_{0,c}(z)f := F^{-1}[(c\langle\xi\rangle_c - z)^{-1}\hat{f}(\xi)],\ \ \forall\, f\in L^2,\ z\in\mathbb{C}\setminus[c^2,+\infty),
$$

we can show the boundedness of $\mathcal{R}_{0,c} : L^2 \to L^2$. Indeed, since the symbol of the free resolvent for $z \in \mathbb{C}\setminus[c^2,+\infty)$ satisfies

$$
|(c\langle\xi\rangle_c - z)^{-1}| \preceq N_{0,c}(z) := \begin{cases} |Im(z)|^{-1} & \text{if } Im(z) > 0, \\ |c^2 - z|^{-1} & \text{if } Im(z) \le 0, \end{cases} \tag{B.2.10}
$$

for any $\xi \in \mathbb{R}^3$, we can deduce that for $z \in \mathbb{C}\setminus[c^2,+\infty)$ the free resolvent $\mathcal{R}_{0,c}(z) : L^2 \to L^2$ exists and is continuous. Similarly one can show that also $\mathcal{R}_{0,c}(z) : L^2 \to H^1$ is continuous.

Furthermore, (B.2.10)*, combined with the previous proposition, implies that also $G_{z,c} : L^2 \to L^2$ is bounded for $Re(z) < c^2$. On the other hand, the function $x \mapsto K_1(c|x|)/|x|$ does not define a bounded operator on L^2, and it would have been difficult to verify directly from* (B.2.8)-(B.2.9) *the boundedness of $G_{z,c}$.*

Remark B.2.3. *By using the same technique of [17] one can prove that for any $c \geq 1$ and for any $\sigma > 1/2$ the resolvent $\mathcal{R}_{0,c}(z) : L^2_\sigma(\mathbb{R}^3) \to L^2_{-\sigma}(\mathbb{R}^3)$ for $z \in \mathbb{C} \setminus [c^2, +\infty)$ is a bounded operator, and that for $\sigma > 1/2$ the following limiting absorption principle (LAP) holds*

$$\lim_{\epsilon \to 0^+} \|\mathcal{R}_{0,c}(\lambda \pm i\epsilon) - \mathcal{R}_{0,c}(c^2 \pm i0)\|_{L^2_\sigma \to L^2_{-\sigma}} = 0, \ \lambda > c^2. \tag{B.2.11}$$

(Just set $f(\theta) := \sqrt{c^2 + \theta^2}$ in [17], Theorem 2A).

Remark B.2.4. *From the resolvent identities*

$$\mathcal{R}'_{0,c}(z) = F^{-1}[(c\langle\xi\rangle_c - z)^{-2}F] = \mathcal{R}_{0,c}(z)^2,$$
$$\mathcal{R}''_{0,c}(z) = F^{-1}[2(c\langle\xi\rangle_c - z)^{-3}F] = 2\mathcal{R}_{0,c}(z)^3$$

one can derive the boundedness of $\mathcal{R}'_{0,c}(z)$ and $\mathcal{R}''_{0,c}(z)$ from $L^2_\sigma(\mathbb{R}^3)$ to $L^2_{-\sigma}(\mathbb{R}^3)$ (for sufficiently large $\sigma > 0$) from the boundedness of $\mathcal{R}_{0,c}(z)$. Indeed, for $\sigma > 1$

$$\begin{aligned}
\|\mathcal{R}'_{0,c}(z)\|_{L^2_\sigma \to L^2_{-\sigma}} &= \| \langle\cdot\rangle^{-\sigma} \mathcal{R}_{0,c}(z)^2 \langle\cdot\rangle^{-\sigma} \|_{L^2 \to L^2} \\
&\leq \| \langle\cdot\rangle^{-\sigma/2} \mathcal{R}_{0,c}(z) \langle\cdot\rangle^{-\sigma/2} \langle\cdot\rangle^{-\sigma/2} \mathcal{R}_{0,c}(z) \langle\cdot\rangle^{-\sigma/2} \|_{L^2 \to L^2} && \text{(B.2.12)} \\
&+ \| \langle\cdot\rangle^{-\sigma/2} [\langle\cdot\rangle^{-\sigma/2}, \mathcal{R}_{0,c}(z)]\mathcal{R}_{0,c}(z) \langle\cdot\rangle^{-\sigma} \|_{L^2 \to L^2} && \text{(B.2.13)} \\
&+ \| \langle\cdot\rangle^{-\sigma/2} \mathcal{R}_{0,c}(z) \langle\cdot\rangle^{-\sigma/2} [\mathcal{R}_{0,c}(z), \langle\cdot\rangle^{-\sigma/2}] \langle\cdot\rangle^{-\sigma/2} \|_{L^2 \to L^2} && \text{(B.2.14)}
\end{aligned}$$

(notice that (B.2.12) *is finite for $\sigma > 1$) and since $[\mathcal{R}_{0,c}, \langle x\rangle^{-\sigma/2}] = Op(b(x,\xi))$, where*

$$b(x,\xi) = -\frac{\sigma}{2}(c\langle\xi\rangle_c - z)^{-2} \langle\xi\rangle_c^{-1} \xi \langle x\rangle^{-(\sigma+4)/2} x,$$

we have that $|b(x,\xi)| \leq N_{0,c}(z)^2 \langle x\rangle^{-1-\sigma/2}$, and that $Op(b(x,\xi))$ is an integral operator with kernel

$$k(x,y) := (F_\xi b)(x, y-x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{i(x-y)} b(x,\xi) \mathrm{d}^3\xi.$$

Furthermore, by the previous computation one may observe that the PDO in (B.2.13) *is a Hilbert-Schmidt operator if and only if $-2\sigma - 1 < -3$; $\sigma > 1$.*
A similar argument leads to the boundedness of $\|\mathcal{R}''_{0,c}(z)\|_{L^2_\sigma(\mathbb{R}^3) \to L^2_{-\sigma}(\mathbb{R}^3)}$ for $\sigma > 3/2$.

For the Schrödinger operator, estimates for the derivatives of the free resolvent were found in [46] by exploiting a Lavine-type identity that links the resolvent to its derivatives. To the best of our knowledge, there is no analogue of such an identity for the relativistic Schrödinger operator.

Lemma B.2.5. *For $z \in \mathbb{C}$ with $Re(z) < c^2$ we have*

$$\|\mathcal{R}_{0,c}(z)\|_{L^2_\sigma \to L^2_{-\sigma}} = \mathcal{O}(1/c^2), \ \ z \to c^2, \ \sigma > 3/2. \tag{B.2.15}$$

$$\|\mathcal{R}^{(k)}_{0,c}(z)\|_{L^2_\sigma \to L^2_{-\sigma}} = \mathcal{O}(|z - c^2|^{1/2-k}), \ \ z \to c^2, \ \sigma > 3(k+1)/2, \ k = 1, 2. \tag{B.2.16}$$

Proof. Again, observe that for any $\sigma > 0$, $k = 0, 1, 2$, the quantity

$$\|\mathcal{R}^{(k)}_{0,c}(z)\|_{L^2_\sigma \to L^2_{-\sigma}} = \|\partial_z^{(k)} \mathcal{R}_{0,c}(z)\|_{L^2_\sigma(\mathbb{R}^3) \to L^2_{-\sigma}(\mathbb{R}^3)},$$

is finite for $\sigma > 3(k+1)/2$, and that for $Re(z) < c^2$ the operator $\langle x\rangle^{-\sigma} \partial_z^{(k)} \mathcal{R}_{0,c}(z) \langle x\rangle^{-\sigma}$: $L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is the integral operator with kernel

$$\tilde{g}^{(k)}_{z,c}(x,y) := \langle x\rangle^{-\sigma} \partial_z^{(k)} g_{z,c}(x-y) \langle y\rangle^{-\sigma},$$

where $g_{z,c}$ is given by

$$g_{z,c}(x-y)=\frac{1}{2\pi^2}\left[\frac{K_1(c|x-y|)}{|x-y|}+z\int_0^{+\infty}e^{tz}\frac{K_1(c[|x-y|^2+c^2t^2]^{1/2})}{[|x-y|^2+c^2t^2]^{1/2}}\mathrm{d}t\right].$$

Furthermore, by simple computations we have that for $\sigma>3(k+1)/2$ and $Re(z)<c^2$

$$\|\mathcal{R}^{(k)}_{0,c}(z)\|_{L^2_\sigma\to L^2_{-\sigma}}=\|\langle x\rangle^{-\sigma}\mathcal{R}^{(k)}_{0,c}(z)\langle x\rangle^{-\sigma}\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)}=\sup_{\|f\|_{L^2}=1}\|\langle x\rangle^{-\sigma}\mathcal{R}^{(k)}_{0,c}(z)(\langle x\rangle^{-\sigma}f)\|_{L^2} \tag{B.2.17}$$

$$=\sup_{\|f\|_{L^2}=1}\left[\int_{\mathbb{R}^3_x}\langle x\rangle^{-2\sigma}\left|\int_{\mathbb{R}^3_y}\partial_z^{(k)}g_{z,c}(x-y)\langle y\rangle^{-\sigma}f(y)\mathrm{d}^3y\right|^2\mathrm{d}^3x\right]^{1/2}. \tag{B.2.18}$$

Now, one can relate the asymptotic of a function with the asymptotic of its Laplace transform via the so-called Abelian Theorems (read [88], chapter V, Theorem 1 and Corollary 1a and 1b); these results generalize the remark that if $f(s)$ is defined for real $s>0$ by the convergent integral

$$f(s)=\int_0^\infty e^{-st}\mathrm{d}\alpha(t),$$

then

$$\lim_{s\to 0^+}f(s)=\lim_{t\to\infty}\alpha(t),$$

provided that the limit on the right-hand side exists.
We apply the Abelian Theorems for the Laplace transform to $g_{z,c}$ and to its derivatives.

We exploit Corollary 1b in ([88], Chap. 5); since

$$\frac{K_1(c(|x|^2+c^2t^2)^{1/2})}{(|x|^2+c^2t^2)^{1/2}}\overset{t\to\infty}{\sim}c^{-1/2}(|x|^2+c^2t^2)^{-3/4}e^{-c(|x|^2+c^2t^2)^{1/2}},$$

we get

$$\int_0^{+\infty}e^{tz}\frac{K_1(c(|x|^2+t^2)^{1/2})}{(|x|^2+t^2)^{1/2}}\mathrm{d}t\overset{z\to c^2}{\sim}c^{-2}(z-c^2)^{1/2}.$$

Hence

$$g_{z,c}(x)\overset{z\to c^2}{\sim}\frac{1}{2\pi^2}\left[\frac{K_1(c|x|)}{|x|}+(z-c^2)^{1/2}\right]. \tag{B.2.19}$$

Since $\int_{\mathbb{R}^3}\frac{K_1(c|x|)}{|x|}\mathrm{d}x=\frac{\pi}{2c^2}$, for $\sigma>3/2$ we can apply the weighted Young inequality for convolution, in order to estimate (B.2.17) by a constant times

$$\sup_{\|f\|_{L^2}=1}\|g_{z,c}\|_{L^1}\|\langle\cdot\rangle^{-\sigma}f(y)\|_{L^2}\le\|g_{z,c}\|_{L^1}\sup_{\|f\|_{L^2}=1}\|f\|_{L^2}\overset{z\to c^2}{\sim}\mathcal{O}(1/c^2), \tag{B.2.20}$$

and this proves (B.2.15).

In the case $k = 1$, by exploiting the l.h.s. in (B.2.7) and through the same kind of argument of Proposition B.2.1 we have that for $Re(z) < c^2$

$$\begin{aligned}\partial_z g_{z,c}(x) &= \frac{c^2}{2\pi^2}\int_0^{+\infty} e^{tz} t \,\frac{ct}{c^2t^2+|x|^2} K_2(c(|x|^2+c^2t^2)^{1/2})\mathrm{d}t \\ &= \frac{1}{2\pi^2}\mathcal{L}\left[\frac{K_1(c[|x|^2+c^2\,\sharp^2]^{1/2})}{[|x|^2+c^2\,\sharp^2]^{1/2}}\right](z) + \frac{z}{2\pi^2}\mathcal{L}\left[\sharp\frac{K_1(c[|x|^2+c^2\,\sharp^2]^{1/2})}{[|x|^2+c^2\,\sharp^2]^{1/2}}\right](z),\end{aligned}$$

and that for $Re(z) < 0$

$$\begin{aligned}\int_0^{+\infty} e^{tz}\frac{tK_1(c(|x|^2+c^2t^2)^{1/2})}{(|x|^2+c^2t^2)^{1/2}}\mathrm{d}t &= \int_0^{+\infty} e^{tz}\frac{\mathrm{d}}{\mathrm{d}t}\left[-c^{-3}K_0(c(|x|^2+c^2t^2)^{1/2})\right]\mathrm{d}t \\ &= \frac{K_0(c|x|)}{c^3} + \frac{z}{c^3}\mathcal{L}\left[K_0(c(|x|^2+c^2\,\sharp^2)^{1/2})\right](z).\end{aligned}$$

By Proposition B.2.1 we get that

$$\partial_z g_{z,c}(x) = \frac{1}{2\pi^2}\mathcal{L}\left[\frac{K_1(c[|x|^2+c^2\,\sharp^2]^{1/2})}{[|x|^2+c^2\,\sharp^2]^{1/2}}\right](z) + \frac{z}{2\pi^2c^3}K_0(c|x|) + \frac{z^2}{2\pi^2c^3}\mathcal{L}\left[K_0(c(|x|^2+c^2\,\sharp^2)^{1/2})\right](z),$$

and similarly

$$\begin{aligned}\partial_z^{(2)} g_{z,c}(x) = &\frac{1}{2\pi^2}\mathcal{L}\left[\frac{\sharp K_1(c[|x|^2+\sharp^2]^{1/2})}{[|x|^2+\sharp^2]^{1/2}}\right](z) + \frac{K_0(c|x|)}{2\pi^2c^3} + \frac{z}{\pi^2c^3}\mathcal{L}\left[K_0(c(|x|^2+\sharp^2)^{1/2})\right](z) \\ &+ \frac{z^2}{2\pi^2c^3}\mathcal{L}\left[\sharp K_0(c(|x|^2+\sharp^2)^{1/2})\right](z).\end{aligned}$$

To prove (B.2.16) for the case $k = 1, 2$, we can argue as before, and Corollary 1b in ([88], Chap. 5) gives us that

$$\partial_z g_{z,c}(x) \overset{z\to c^2}{\sim} \mathcal{O}(c^{-2}|z-c^2|^{1/2}) + c^{-3}K_0(c|x|) + \mathcal{O}(|z-c^2|^{-1/2}), \tag{B.2.21}$$

$$\partial_z^{(2)} g_{z,c}(x) \overset{z\to c^2}{\sim} \mathcal{O}(c^{-2}|z-c^2|^{-1/2}) + c^{-3}K_0(c|x|) + \mathcal{O}(c^{-1}|z-c^2|^{-1/2}) + \mathcal{O}(|z-c^2|^{-3/2}), \tag{B.2.22}$$

and by replacing $g_{z,c}$ with $\partial_z^{(k)} g_{z,c}$ in (B.2.20) we can conclude. □

B.2.2 Asymptotic behavior of the resolvent of $\mathcal{H}_0$

In this section we want to prove the asymptotic of the resolvent of $\mathcal{H}_0$ as $|z| \to \infty$, by generalizing the approach of [86] and [67]. These allow us to deduce the asymptotic (B.2.35). Here we deal only with the 3-dimensional case, but a slight modification of the argument works also for $\mathbb{R}^d$ ($d \geq 3$).

Remark B.2.6. *In order to deduce the asymptotic for $|z| \to \infty$ one cannot apply the Abelian Theorems for the Laplace transform to $g_{z,c}$ and to its derivatives, as we did in Lemma B.2.5.*

Indeed, in the case $k=0$, since as $t\to 0^+$

$$\frac{K_1(c[|x|^2+c^2t^2]^{1/2})}{[|x|^2+c^2t^2]^{1/2}} \sim \frac{K_1(c|x|)}{|x|} - \left[\frac{c^3K_0(c|x|)}{|x|^2} + 2\frac{c^2K_1(c|x|)}{|x|^3} + \frac{c^3K_2(c|x|)}{|x|^2}\right]\frac{t^2}{4} + \mathcal{O}(t^4)$$
$$= \frac{K_1(c|x|)}{|x|} - c^3\frac{K_2(c|x|)}{2|x|^2}t^2 + \mathcal{O}(t^4),$$

by the Abel theorem for the Laplace transform we would get that

$$\mathcal{L}\left[\frac{K_1(c[|x|^2+t^2]^{1/2})}{[|x|^2+t^2]^{1/2}}\right] := \int_0^{+\infty} e^{tz}\frac{K_1(c[|x|^2+t^2]^{1/2})}{[|x|^2+t^2]^{1/2}}\mathrm{d}t$$
$$\overset{|z|\to+\infty}{\sim} -\frac{K_1(c|x|)}{|x|z} + c^3\frac{K_2(c|x|)}{|x|^2z^3} + \mathcal{O}(|z|^{-5}),$$

hence

$$g_{z,c}(x) \sim c^3z^{-2}\frac{K_2(c|x|)}{2\pi^2|x|^2}, \quad |z|\to\infty,\ Re(z)<c^2, \tag{B.2.23}$$

but the function $x\mapsto \frac{K_2(c|x|)}{|x|^2}$, unlike the functions we considered in Lemma B.2.5, is not integrable on $\mathbb{R}^3$. One can check that such a problem persists also for the derivatives of $\mathcal{R}_{0,c}(z)$.

First, we will prove the following:

Proposition B.2.7. *Let $c\geq 1$ and $z\in\mathbb{C}\setminus[c^2,+\infty)$. Then*

$$\liminf_{|z|\to\infty} \|\mathcal{R}^{(k)}_{0,c}(z)\|_{L^2_\sigma\to L^2_{-\sigma}} \geq \mathcal{O}(1/c^2), \quad \sigma > 3(k+1)/2, k=0,1,2. \tag{B.2.24}$$

The previous proposition will readily follow from

Lemma B.2.8. *For each $k=0,1,2$ and for $c\geq 1$ there exist sequences $(h_{k,c,j})_{j\in\mathbb{N}_0}\subset\mathcal{S}(\mathbb{R}^3)$, $(z_{k,c,j})_{j\in\mathbb{N}_0}\subset\mathbb{R}$ with $|z_{k,c,j}|\overset{j\to\infty}{\to}+\infty$ such that*

(i) $\sup_j\|h_{k,c,j}\|_{L^2_\sigma}<+\infty$ *for every $\sigma\geq 0$;*

(ii) $\lim_{j\to\infty}\left\langle \mathcal{R}^{(k)}_{0,c}(z_{k,c,j})h_{k,c,j}, h_{k,c,j}\right\rangle \neq 0.$

In the proof of this lemma we will need the following result of residue calculus (often found as Sokhotski-Plemelj theorem on the line):

Lemma B.2.9. *Let $a<0<b$, and let $\phi\in C([a,b],\mathbb{R})$. Then*

$$\lim_{\mu\to 0}\int_a^b \frac{\phi(\sigma)}{\sigma\mp i\mu}\mathrm{d}\sigma = \pm i\pi\phi(0) + \ p.v.\ \int_a^b\frac{\phi(\sigma)}{\sigma}\mathrm{d}\sigma,$$

where p.v. denotes the principal value of an integral.
Furthermore, if $\phi\in C^1([-1,1],\mathbb{R})$ and $\phi(0)=0$, then the last integral may be replaced by the term

$$\int_{-1}^1\left(\int_0^1\phi'(\sigma\theta)\mathrm{d}\theta\right)\mathrm{d}\sigma.$$

Finally, if $k > 0$ and $\phi \in C_c^k((a,b),\mathbb{R})$, then by integration by parts we get

$$\int_a^b \frac{\phi(\sigma)}{(\sigma \mp i\mu)^{k+1}} \mathrm{d}\sigma = \frac{1}{k}\int_a^b \frac{\phi'(\sigma)}{(\sigma \mp i\mu)^{k}} \mathrm{d}\sigma = \frac{1}{k!}\int_a^b \frac{\phi^{(k)}(\sigma)}{\sigma \mp i\mu} \mathrm{d}\sigma,$$

hence

$$\lim_{\mu\to 0}\int_a^b \frac{\phi(\sigma)}{(\sigma \mp i\mu)^{k+1}} \mathrm{d}\sigma = \pm\frac{i\pi}{k!}\phi^{(k)}(0) + \frac{1}{k!}\ p.v.\ \int_a^b \frac{\phi^{(k)}(\sigma)}{\sigma} \mathrm{d}\sigma.$$

Proof. (Lemma B.2.8) Choose an even function ϕ_0 such that

$$\mathrm{supp}(\phi_0) \subset (-1,1),$$
$$\phi_0(0) = 1.$$

Define the sequence $(z_j)_j \subset \mathbb{R}$ by

$$z_j := j+2, \quad j \in \mathbb{N}_0, \tag{B.2.25}$$

and the sequence of functions $(h_{0,j})_j := (h_{0,c,j})_j \in \mathcal{S}(\mathbb{R}^3)$ by

$$\widehat{h}_{0,j}(\xi) := c^{-1/2}|\xi|^{-1}\phi_0(c^{-1}\left\langle \xi\right\rangle_c - z_j), \quad j \in \mathbb{N}_0. \tag{B.2.26}$$

It is easy to check that

$$\mathrm{supp}(\widehat{h}_{0,j}) \subset \{\xi \in \mathbb{R}^3 : |\xi| \in (c\sqrt{z_j^2 - 2z_j}, c\sqrt{z_j^2+2z_j})\}.$$

Let $\alpha \in \mathbb{N}^3$ be a given multi-index: by the Plancherel theorem, we have

$$\begin{aligned}\|x^\alpha h_{0,j}\|_{L^2}^2 &= (2\pi)^{-3}\int_{\mathbb{R}^3}\left|\left(i\frac{\partial}{\partial\xi}\right)^\alpha \hat{h}_{0,j}(\xi)\right|^2 \mathrm{d}^3\xi \\ &\preceq K_{\alpha,\phi_0}\ c^{-1}\int_{c\sqrt{z_j^2-2z_j}<|\xi|<c\sqrt{z_j^2+2z_j}} |\xi|^{-2}\mathrm{d}^3\xi,\end{aligned}$$

where K_{α,ϕ_0} is a constant depending only on α and ϕ_0. Furthermore, a simple calculation shows that the last integral may be bounded by a constant independent both of j and c. Thus we obtain

$$\sup_{j\geq 1}\|x^\alpha h_{0,j}\|_{L^2} < +\infty,$$

and by the arbitrariness of α we get (i).

Next we show (ii). Passing to polar coordinates, we get

$$\langle \mathcal{R}_{0,c}(z)h_{0,j}, h_{0,j}\rangle = (2\pi)^{-3}\omega_3\, c^{-1}\int_0^\infty \frac{1}{c\sqrt{c^2+r^2}-z}\phi_0(c^{-1}\sqrt{c^2+r^2}-z_j)^2\mathrm{d}r,$$

where ω_3 denotes the surface area of the unit sphere in $\mathbb{R}^3$. By setting $\sigma := c^{-1}\sqrt{c^2+r^2} - z_j$ we get

$$\langle \mathcal{R}_{0,c}(z)h_{0,j}, h_{0,j}\rangle = (2\pi)^{-3}\omega_3\, c^{-1}\int_{-1}^{1} \frac{1}{c^2\sigma + c^2 z_j - z}\phi_0(\sigma)^2 c\frac{\sigma + z_j}{\sqrt{(\sigma+z_j)^2-1}}\mathrm{d}\sigma, \qquad \text{(B.2.27)}$$

where we have used that $\mathrm{supp}(\phi_0) \subset (-1,1)$. Now take $z = c^2(z_j + i\mu)$ with $\mu > 0$; taking the limit as $\mu \to 0$ and exploiting the prevoius lemma we obtain

$$\frac{(2\pi)^3}{\omega_3}\left\langle \mathcal{R}_{0,c}(c^2 z_j)h_{0,j}, h_{0,j}\right\rangle = c^{-2}\left[i\pi\phi_0(0)\frac{z_j}{\sqrt{z_j^2-1}} + \int_{-1}^{1}\left(\int_0^1 \psi_j'(\sigma\theta)\mathrm{d}\theta\right)\mathrm{d}\sigma\right],$$

where $\psi_j(\sigma) := \phi_0(\sigma)^2\frac{\sigma+z_j}{\sqrt{(\sigma+z_j)^2-1}}$.
Now, from Lebesgue dominated convergence theorem we have

$$\lim_{j\to\infty}\int_{-1}^{1}\left(\int_0^1 \psi_j'(\sigma\theta)\mathrm{d}\theta\right)\mathrm{d}\sigma = \int_{-1}^{1}\left(\int_0^1 (\phi_0^2)'(\sigma\theta)\mathrm{d}\theta\right)\mathrm{d}\sigma,$$

and since ϕ_0^2 is an even function, the integral on the r.h.s. vanishes. Hence

$$\lim_{j\to\infty}\frac{(2\pi)^3}{\omega_3}\left\langle \mathcal{R}_{0,c}(c^2 z_j)h_{0,j}, h_{0,j}\right\rangle = i\pi c^{-2}, \qquad \text{(B.2.28)}$$

and by setting $z_{0,c,j} := c^2 z_j$ we obtain the thesis for $k = 0$.

To discuss the case $k = 1$, we again exploit Lemma B.2.9. Indeed, choose an even function ϕ_1 such that

$$\begin{aligned} \mathrm{supp}(\phi_1) &\subset (-1,1), \\ \phi_1(0) &= 1, \\ \int_0^1 \phi_1(\sigma^2)\mathrm{d}\sigma &\neq 0, \end{aligned}$$

and define

$$\widehat{h}_{1,j}(\xi) := c^{-1/2}|\xi|^{-1}\phi_1((c^{-1}\langle\xi\rangle_c - z_j)^2), \ \ j \in \mathbb{N}_0. \qquad \text{(B.2.29)}$$

Arguing as before we get

$$\begin{aligned} \left\langle \mathcal{R}_{0,c}'(z)h_{1,j}, h_{1,j}\right\rangle &= (2\pi)^{-3}\omega_3\, c^{-1}\int_0^\infty \frac{1}{(c\sqrt{c^2+r^2}-z)^2}\phi_1((c^{-1}\sqrt{c^2+r^2}-z_j)^2))^2\mathrm{d}r \\ &= 2\pi\omega_3 c^{-1}\int_{-1}^{1}\frac{1}{(c^2\sigma + c^2 z_j - z)^2}\phi_1(\sigma^2)^2 c\frac{\sigma+z_j}{\sqrt{(\sigma+z_j)^2-1}}\mathrm{d}\sigma, \end{aligned}$$

and choosing as before $z := c^2(z_j + i\mu)$ we obtain

$$\begin{aligned}\left\langle \mathcal{R}'_{0,c}(c^2(z_j+i\mu))h_{1,j}, h_{1,j}\right\rangle &= (2\pi)^{-3}\omega_3\, c^{-1}\int_0^\infty \frac{1}{\sigma - i\mu}\psi_{1,j}(\sigma)\mathrm{d}\sigma \\ &=: I_{1,j}(\mu),\end{aligned}$$

where

$$\psi_{1,j}(\sigma) := 2\phi_1(\sigma^2)2\sigma\frac{\sigma+z_j}{\sqrt{(\sigma+z_j)^2-1}} + \phi_1(\sigma^2)^2\left(\frac{1}{\sqrt{(\sigma+z_j)^2-1}} - \frac{(\sigma+z_j)^2}{[(\sigma+z_j)^2-1]^{3/2}}\right).$$

Now,

$$\lim_{\mu\to 0} I_{1,j}(\mu) = (2\pi)^{-3}\omega_3\, c^{-2}\left[i\pi\phi_1(0)^2\frac{(-1)}{[z_j^2-1]^{3/2}} + \int_{-1}^{1}\sigma^{-1}\psi_{k,j}^{(1)}(\sigma)\mathrm{d}\sigma\right], \tag{B.2.30}$$

and by passing to $\lim_{j\to\infty}$ we get by Lebesgue dominated convergence

$$\lim_{j\to\infty}\left\langle \mathcal{R}'_{0,c}(c^2 z_j)h_{1,j}, h_{1,j}\right\rangle = \frac{\omega_3}{(2\pi)^3}c^{-2}\, 4\int_{-1}^{1}\phi_1(\sigma^2)\mathrm{d}\sigma = \frac{\omega_3}{c^2\pi^3}\int_0^1\phi_1(\sigma^2)\mathrm{d}\sigma, \tag{B.2.31}$$

which is non-zero by definition of ϕ_1.

Finally, for the case $k = 2$ choose an even function ϕ_2 such that

$$\begin{gathered}\operatorname{supp}(\phi_2) \subset (-1,1), \\ \phi_2(0) = 1, \\ \int_0^1 \phi_2(\sigma^3)\phi_2''(\sigma^3)\mathrm{d}\sigma \neq 0,\end{gathered}$$

and define

$$\widehat{h}_{2,j}(\xi) := c^{-1/2}|\xi|^{-1}\phi_2((c^{-1}\langle\xi\rangle_c - z_j)^3), \quad j\in\mathbb{N}_0. \tag{B.2.32}$$

By using the same approach as before we get

$$\begin{aligned}\left\langle \mathcal{R}''_{0,c}(c^2(z_j+i\mu))h_{2,j}, h_{2,j}\right\rangle &= (2\pi)^{-3}\omega_3\, c^{-2}\int_0^\infty \frac{1}{\sigma - i\mu}\psi_{2,j}(\sigma)\mathrm{d}\sigma \\ &=: I_{2,j}(\mu),\end{aligned}$$

where

$$\begin{aligned}\psi_{2,j}(\sigma) := &-\frac{2(\sigma+z_j)}{[(\sigma+z_j)^2-1]^{3/2}} + \frac{(\sigma+z_j)[1+2(\sigma+z_j)^2]}{[(\sigma+z_j)^2-1]^{5/2}}\phi_2(\sigma^3)^2 + \\ &- 12\sigma^2\frac{\phi_2(\sigma^3)\phi_2'(\sigma^3)}{[(\sigma+z_j)^2-1]^{3/2}} + \\ &+ \frac{(\sigma+z_j)[12\sigma\phi_2(\sigma^3)\phi_2'(\sigma^3) + 18\sigma^4(\phi_2'(\sigma^3))^2 + 18\sigma^4\phi_2(\sigma^3)\phi_2''(\sigma^3)]}{[(\sigma+z_j)^2-1]^{1/2}},\end{aligned}$$

hence

$$\lim_{j\to\infty}\left\langle \mathcal{R}''_{0,c}(c^2 z_j)h_{2,j},h_{2,j}\right\rangle = \frac{6\omega_3}{(2\pi)^3}c^{-2}\int_{-1}^{1}\left[2\phi_2(\sigma^3)\phi_2'(\sigma^3)+3\sigma^3(\phi_2'(\sigma^3))^2+3\sigma^3\phi_2(\sigma^3)\phi_2''(\sigma^3)\right]\mathrm{d}\sigma \tag{B.2.33}$$

$$= \frac{3\omega_3}{c^2\pi^3}\int_{-1}^{1}\phi_2(\sigma^3)\phi_2'(\sigma^3)\mathrm{d}\sigma \tag{B.2.34}$$

which is non-zero by definition of ϕ_2. □

At this stage it is easy to deduce (B.2.24). Fixed $k=0,1,2$ take the sequences $(z_{k,c,j})_{j\in\mathbb{N}_0}\subset\mathbb{R}$ and $(h_{k,c,j})_{j\in\mathbb{N}_0}\subset\mathcal{S}(\mathbb{R}^3)$; for $\sigma>3(k+1)/2$ we have

$$|\left\langle \mathcal{R}^{(k)}_{0,c}(z_{k,c,j})h_{k,c,j},h_{k,c,j}\right\rangle| \le \|\|\mathcal{R}^{(k)}_{0,c}(z_{k,c,j})\|_{L^2_\sigma\to L^2_{-\sigma}}\|h_{k,c,j}\|^2_{L^2_\sigma},$$

and the last inequality, combined with Lemma B.2.8, implies that

$$\liminf_{j\to\infty}\|\mathcal{R}^{(k)}_{0,c}(z_{k,c,j})\|_{L^2_\sigma\to L^2_{-\sigma}} \ge \mathcal{O}(1/c^2).$$

Inequality (B.2.24) gives us a lower bound to $\|\mathcal{R}_{0,c}(z)\|_{L^2_\sigma\to L^2_{-\sigma}}$ as $|z|\to\infty$. On the other hand, via an argument similar to the proof of Theorem 2.3 in [67], one can deduce the following upper bound.

Proposition B.2.10. *Let $c\ge 1$ and $z\in\mathbb{C}\setminus[c^2,+\infty)$. Then*

$$\|\mathcal{R}^{(k)}_{0,c}(z)\|_{L^2_\sigma\to L^2_{-\sigma}} \le \mathcal{O}(1/c),\ \ |z|\to\infty,\ \sigma>3(k+1)/2,\ k=0,1,2. \tag{B.2.35}$$

In order to prove Proposition B.2.10 we begin by noticing that

$$(c\langle\xi\rangle_c - z)(c\langle\xi\rangle_c + z) = (c^2\langle\xi\rangle_c^2 - z^2),$$

and that

$$\mathcal{R}_{0,c}(z) = F^{-1}\left[\frac{1}{c^2\langle\xi\rangle_c^2 - z^2}(c\langle\xi\rangle_c + z)\right]F,\ \ z\in\mathbb{C}\setminus((-\infty,-c^2]\cup[c^2,+\infty)).$$

Proposition B.2.10 is a consequence of the following lemma.

Lemma B.2.11. *Let $c\ge 1$ and $\sigma>3/2$, then*

$$\sup\{\|\mathcal{R}_{0,c}(z)\|_{L^2_\sigma\to L^2_{-\sigma}} : |Re(z)|\ge 2c^2, 0<|Im(z)|<1\} \le \mathcal{O}(1/c). \tag{B.2.36}$$

Proof. Let $c\ge 1$, and set $J_c := \{z\in\mathbb{C}||Re(z)|\ge 2c^2, 0<|Im(z)|<1\}$.
Now choose $\rho\in C^\infty(\mathbb{R})$ such that

$$\rho(t)=1,\ \text{for } |t|<1/2,$$
$$\rho(t)=0,\ \text{for } |t|>1.$$

Now, for each $z \in J_c$, we define a cutoff function $\gamma_{z,c} : \mathbb{R}^3 \to \mathbb{R}$,

$$\gamma_{z,c}(\xi) := \rho(c\langle\xi\rangle_c - Re(z)), \quad \text{for } Re(z) \geq 2c^2,$$
$$\gamma_{z,c}(\xi) := \rho(c\langle\xi\rangle_c + Re(z)), \quad \text{for } Re(z) \leq -2c^2.$$

Using the cutoff $\gamma_{z,c}$ we decompose the resolvent $\mathcal{R}_{0,c}(z)$ into three parts:

$$\mathcal{R}_{0,c}(z) = (-c^2\Delta + c^4 - z^2)^{-1} A_{z,c} + B_{z,c} + z(-c^2\Delta + c^4 - z^2)^{-1},$$

where

$$A_{z,c} = F^{-1}\left[\gamma_{z,c}(\xi)\, c\langle\xi\rangle_c\right] F,$$
$$B_{z,c} = F^{-1}\left[\frac{1-\gamma_{z,c}(\xi)}{c^2\langle\xi\rangle_c^2 - z^2}\, c\langle\xi\rangle_c\right] F.$$

Note that for $c \geq 1$, $z \in J_c$ and for $\xi \in \mathrm{supp}(\gamma_{z,c})$

$$|\xi|^2 \leq c^{-2}(Re(z)+1)^2 - c^2;$$

this implies that for any multi-index α $\exists K_\alpha > 0$ such that

$$\left|\left(\frac{\partial^\alpha}{\partial\xi^\alpha}\right)(\gamma_{z,c}(\xi)\, c\langle\xi\rangle_c)\right| \preceq K_\alpha |(c^4 - z^2)^{1/2}|,$$

hence

$$\|A_{z,c}\psi\|_{L^2_\sigma} \preceq |(z^2 - c^4)^{1/2}| \|\psi\|_{L^2_\sigma}. \tag{B.2.37}$$

On the other hand, by the well-known estimate for the Schrödinger resolvent, we have that for $z \in J_c$

$$\|(-c^2\Delta + c^4 - z^2)^{-1}\|_{L^2_\sigma \to L^2_{-\sigma}} \preceq c^{-2}\left(\frac{z^2}{c^2} - c^2\right)^{-1/2} = c^{-1}(z^2 - c^4)^{-1/2}, \tag{B.2.38}$$

and by combining (B.2.37) and (B.2.38), we get

$$\|(-c^2\Delta + c^4 - z^2)^{-1} A_{z,c}\|_{L^2_\sigma \to L^2_{-\sigma}} \preceq c^{-1}, \quad z \in J_c. \tag{B.2.39}$$

Furthermore, since for $c \geq 1$, $z \in J_c$, and $\xi \in \mathrm{supp}(1 - \gamma_{z,c})$

$$|c\langle\xi\rangle_c - (z^2 - c^4)| \geq \frac{1}{2} c\langle\xi\rangle,$$

we can deduce that also

$$\|B_{z,c}\|_{L^2_\sigma \to L^2_{-\sigma}} \preceq c^{-1}, \quad z \in J_c. \tag{B.2.40}$$

By combining (B.2.39) and (B.2.40) we finally get (B.2.36). □

Analogous estimates for the derivatives $\mathcal{R}^{(k)}_{0,c}(z)$ of the free resolvent follow by exploiting resolvent identities, and by arguing as before.

Remark B.2.12. *We just remark that the behavior of the resolvent of the pseudo-relativistic Schrödinger operator is quite different from the one of the resolvent of the Schrödinger operator, since the latter decays like $\mathcal{O}(|z|^{-1/2})$ as $|z| \to \infty$. This difference was already reported in the case $c = 1$ in [49].*

B.2.3 Resolvent of equation (4.1.6) with a time-independent potential

Now, consider the operator

$$\mathcal{H}(x) := c(c^2 - \Delta)^{1/2} + V(x) = \mathcal{H}_0(1 + c^{-1} \langle\nabla\rangle_c^{-1} V), \tag{B.2.41}$$

where $V \in C(\mathbb{R}^3, \mathbb{R})$ is a potential such that

$$|V(x)| + |\nabla V(x)| \preceq \langle x\rangle^{-\beta}, \;\; x \in \mathbb{R}^3, \tag{B.2.42}$$

for some $\beta > 0$ sufficiently large (we will specify more precise conditions on V later).

Let $V_0 := \min_{x\in\mathbb{R}^3} V(x)$. Then we can define the perturbed resolvent

$$\mathcal{R}_c(z) = (\mathcal{H}(x) - z)^{-1}, \;\; z \in \Gamma := \mathbb{C} \setminus [V_0, +\infty). \tag{B.2.43}$$

One can construct $\mathcal{R}_c$ from the free resolvent $\mathcal{R}_{0,c}$ using the decomposition formula

$$\begin{aligned}
\mathcal{H}(x) - z &= \mathcal{H}_0(1 + c^{-1} \langle\nabla\rangle_c^{-1} V(x)) - z \\
&= (\mathcal{H}_0 - z)\{1 + \mathcal{R}_{0,c}(z)c \langle\nabla\rangle_c \;\; [(1 + c^{-1} \langle\nabla\rangle_c^{-1} V(x)) - 1]\} && \text{(B.2.44)} \\
&= (\mathcal{H}_0 - z)\{1 + \mathcal{R}_{0,c}(z)V(x)\} && \text{(B.2.45)} \\
&= \{1 + V(x) \;\; \mathcal{R}_{0,c}(z)\}(\mathcal{H}_0 - z). && \text{(B.2.46)}
\end{aligned}$$

In order to deduce the boundedness of the perturbed resolvent we perform a variant of the Jensen-Kato approach for the Schrödinger operator (see [46]).

Theorem B.2.13. *For $z \in \Gamma$ the operators*

$$1 + \mathcal{R}_{0,c}(z)V(x), \;\; 1 + V(x)\mathcal{R}_{0,c}(z)$$

are invertible in L^2.

Proof. We show the thesis only for $1 + V(x) \;\; \mathcal{R}_{0,c}(z)$; the estimate for the other operator follow from the fact that

$$1 + \mathcal{R}_{0,c}(z)V(x) = \{1 + V(x) \;\; \mathcal{R}_{0,c}(\bar{z})\}^*.$$

First step: we prove that equation $(\mathcal{H} - z)\psi = 0$ for $\psi \in L^2$ admits only the trivial solution $\psi = 0$ for $z \in \tilde{\Gamma}$.
First, $(\mathcal{H} - z)\psi = 0$ implies

$$(c \langle\nabla\rangle_c + 1)\psi = (V(x) + z + 1)\psi.$$

Then $(c \langle\nabla\rangle_c + 1)\psi \in L^2$, hence $\psi \in D(\mathcal{H}_0)$. Therefore,

$$\langle(\mathcal{H} - z)\psi, \psi\rangle = \langle\mathcal{H}\psi, \psi\rangle - z\|\psi\|_{L^2}^2.$$

Consider the case $z \in \mathbb{C} \setminus \mathbb{R}$. Then

$$Im(\langle(\mathcal{H} - z)\psi, \psi\rangle) = -Im(z\|\psi\|_{L^2}^2) \neq 0$$

for $\psi \neq 0$, since the scalar product $\langle \mathcal{H}\psi, \psi \rangle$ is real (because $\langle \nabla \rangle_c$ is symmetric and $\psi \in D(\mathcal{H}_0)$).
Hence, $(\mathcal{H} - z)\psi \neq 0$ for $\psi \neq 0$.
Now consider the case $z \in \mathbb{R}$, $Re(z) < V_0$. Then

$$\begin{aligned} Re(\langle (\mathcal{H} - z)\psi, \psi \rangle) &= \langle \mathcal{H}_0\psi, \psi \rangle + \langle (V(x) - Re(z))\psi, \psi \rangle \\ &\geq \langle (V_0 - Re(z))\psi, \psi \rangle \neq 0, \end{aligned}$$

since $\langle \mathcal{H}_0\psi, \psi \rangle \geq 0 \ \forall \psi \in D(\mathcal{H}_0)$.
Second step: the above decomposition for the perturbed resolvent, together with the previous step, allows us to deduce that also the equation

$$[1 + V(x) \ \ \mathcal{R}_{0,c}(z)]\psi = 0, \quad z \in \Gamma$$

with $\psi \in L^2$ admits only the trivial solution $\psi = 0$.
Third step: one shows, via an approximation argument and Sobolev Embedding Theorem, that $V(x)\mathcal{R}_{0,c}(z)$ is compact in L^2 for $z \in \Gamma$.
Indeed, for any $\delta > 0$ we can write $V(x) = V_\delta(x) + r_\delta(x)$, with $V_\delta \in C_c^\infty(\mathbb{R}^3)$ and $\|r_\delta\|_{L^\infty} \leq \delta$.
Then, by (B.2.10) and pseudo-differential calculus we have

$$\lim_{\delta \to 0} \|V(x)\mathcal{R}_{0,c}(z) - V_\delta(x)\mathcal{R}_{0,c}(z)\|_{L^2 \to L^2} = 0,$$

i.e. $V\mathcal{R}_{0,c}(z)$ is the limit of the operators $V_\delta\mathcal{R}_{0,c}(z)$ in the operator norm, and therefore $V\mathcal{R}_{0,c}(z)$ is compact as an operator from L^2 to itself if $V_\delta\mathcal{R}_{0,c}(z)$ is compact. Since the multiplication by V_δ is continuous in L^2 and since $\mathcal{R}_{0,c}(z)$ is continuous from L^2 to H^1, we have that the composition $V_\delta\mathcal{R}_{0,c}(z)$ is bounded, and also

$$\mathrm{supp}(V_\delta\mathcal{R}_{0,c}(z)) \ \subseteq \ \mathrm{supp}(V_\delta), \tag{B.2.47}$$

and $\mathrm{supp}(V_\delta)$ is bounded because $V_\delta \in C_c^\infty(\mathbb{R}^3)$. Now, by using (B.2.47) and recalling that $\mathcal{R}_{0,c}(z) : L^2 \to H^1$ is bounded, we can deduce by compact Sobolev Embedding Theorem the compactness of $V_\delta\mathcal{R}_{0,c}(z)$.
Fourth step: We exploit Fredholm Theorem in order to invert the operator

$$V_\delta\mathcal{R}_{0,c}(z).$$

Indeed, Fredholm Theorem states that, given a Hilbert space X and a compact operator $K : X \to X$, then the operator $1 + K : X \to X$ is invertible iff the equation $(1 + K)\psi = 0$, $\psi \in X$, admits only the trivial solution $\psi = 0$.
The result follows by choosing $X = L^2$ and $K = V_\delta\mathcal{R}_{0,c}(z)$, for $z \in \Gamma$. □

Now, recalling (B.2.44), we have

$$\mathcal{R}_c(z) = [1 + \mathcal{R}_{0,c}(z)V(x)]^{-1} \ \ \mathcal{R}_{0,c}(z) \tag{B.2.48}$$

$$= \mathcal{R}_{0,c}(z) \ \ [1 + V(x) \ \ \mathcal{R}_{0,c}(z)]^{-1}. \tag{B.2.49}$$

This splitting, combined with the previous proposition, leads to

Corollary B.2.14. *Let $z \in \Gamma$, then*

$$\|\mathcal{R}_c(z)\|_{L^2 \to L^2} \preceq \mathcal{N}_c(z) := \frac{1}{dist(z, \Gamma)} \tag{B.2.50}$$

$$= \begin{cases} |Im(z)|^{-1} & \textit{if } Re(z) \geq V_0, \\ |V_0 - z|^{-1} & \textit{if } Re(z) \leq V_0. \end{cases} \tag{B.2.51}$$

Actually, we may argue by standard arguments from complex analysis in order to extend the resolvent $\mathcal{R}_c(z)$ as an holomorphic operator for $z \in \mathbb{C} \setminus ([c, +\infty) \cup \Sigma(V))$, where $\Sigma(V)$ is a discrete subset of $[V_0, c)$.

Using the estimate in the previous subsection, and the fact that the multiplication by

$$V(x) : L^2_{-\sigma} \to L^2_\sigma$$

is compact for $\beta \geq 2\sigma$, we can deduce the following

Lemma B.2.15. *Let* (B.2.42) *hold for some* $\beta > 2$. *Then for any* $c \geq 1$

(i) For $\lambda > c^2$ *the operators*

$$\mathcal{R}_{0,c}(\lambda \pm i0)\, V(x) : L^2_{-\sigma} \to L^2_{-\sigma}$$

are compact for $\sigma \in (1/2, \beta/2]$.

(ii) The operators

$$\mathcal{R}_{0,c}(c)\, V(x) : L^2_{-\sigma} \to L^2_{-\sigma},$$
$$V(x)\, \mathcal{R}_{0,c}(c) : L^2_\sigma \to L^2_\sigma,$$

are compact for $\sigma \in (1, \beta/2]$.

Now consider the space

$$\mathcal{M}_\sigma := \{\psi \in L^2_{-\sigma} : [1 + V(x)\ \ \mathcal{R}_{0,c}(c)]\, \psi = 0\}, \quad \sigma \in (1, \beta/2).$$

One can show that $\mathcal{M}_\sigma$ does not depend on σ (just consider the space defined through the adjoint operator $1 + \mathcal{R}_{0,c}(c)\, V(x)$), hence we can denote the space $\mathcal{M}_\sigma$ by $\mathcal{M}$. Functions in $\mathcal{M} \cap L^2$ are the zero eigenfunctions of the operator $\mathcal{H}$, while functions in $\mathcal{M} \setminus L^2$ are called the *zero resonances* of the operator $\mathcal{H}$.

As usual in this framework ([66], [49]), we will assume that the point $z = c^2$ is neither an eigenvalue nor a resonance for the operator $\mathcal{H}$, ie

$$\mathcal{M} = 0. \tag{B.2.52}$$

The above spectral condition (B.2.52) ensures the invertibility of

$$1 + \mathcal{R}_{0,c}(c)\, V(x) : L^2_{-\sigma} \to L^2_{-\sigma}, \quad \sigma \in (1, \beta/2),$$

and it holds for generic potentials V satisfying (B.2.42).
One can also deduce

Lemma B.2.16. *Let the potential* $V(x)$ *satisfy* (B.2.42) *for some* $\beta > 2$, *and let the spectral condition* (B.2.52) *hold. Then for any* $\sigma > 1$

$$\lim_{z \to c^2} \|\mathcal{R}_c(z) - \mathcal{R}_c(c^2)\|_{L^2_\sigma \to L^2_{-\sigma}} = 0, \quad z \in \mathbb{C} \setminus [c^2, +\infty). \tag{B.2.53}$$

Remark B.2.17. *By using* (B.2.50) *and pseudo-differential calculus, one can show that the perturbed resolvent is bounded not only from L^2 to L^2, but also (for any $c \geq 1$) from L^2_σ to $L^2_{-\sigma}$ for $\sigma > 1/2$. Indeed, just notice that for $z \in \Gamma$*

$$\begin{aligned}\|\mathcal{R}_c(z)\|_{L^2_\sigma \to L^2_{-\sigma}} &= \|\mathcal{R}_{0,c}(z)[1 + V(x)\ \mathcal{R}_{0,c}(z)]^{-1}\|_{L^2_\sigma \to L^2_{-\sigma}} \\ &\leq \|\mathcal{R}_{0,c}(z)\|_{L^2_\sigma \to L^2_{-\sigma}} \\ &+ \left\|\mathcal{R}_{0,c}(z)\left\{[1 + V(x)\ \ \mathcal{R}_{0,c}(z)]^{-1} - 1\right\}\right\|_{L^2_\sigma \to L^2_{-\sigma}}\end{aligned}$$

but for $c \geq 1$ the last term can be bounded (up to a remainder which is smoother) by

$$\begin{aligned}&\left\|\langle\cdot\rangle^{-\sigma}\mathcal{R}_{0,c}(z)\left[-V(x)\ \ \mathcal{R}_{0,c}(z)\right]\langle\cdot\rangle^{-\sigma}\right\|_{L^2 \to L^2} \\ &\preceq \left\|\langle\cdot\rangle^{-\sigma}\mathcal{R}_{0,c}(z)V(x)\mathcal{R}_{0,c}(z)\langle\cdot\rangle^{-\sigma}\right\|_{L^2 \to L^2},\end{aligned}$$

and the P.D.O. in the last norm has symbol

$$\langle x\rangle^{-\sigma}\left(c\langle\xi\rangle_c - z\right)^{-1}V(x)(c\langle\xi\rangle_c - z)^{-1}\langle x\rangle^{-\sigma} \overset{(B.2.10),(B.2.42)}{\preceq} N_{0,c}(z)^2\langle x\rangle^{-(2\sigma+\beta)}.$$

The above argument may be easily extended to $\mathcal{R}_c^{(k)}(z)$ $(k = 1,2)$; thus for any $c \geq 1$ one obtains

$$\|\mathcal{R}_c^{(k)}(z)\|_{L^2_\sigma \to L^2_{-\sigma}} < +\infty,\ \ z \in \Gamma,\ \sigma > k + 1/2,\ k = 0,1,2. \tag{B.2.54}$$

The previous remark allows us to deduce the asymptotics for the perturbed resolvent.

Corollary B.2.18. *For any $c \geq 1$ sufficiently large*

$$\|\mathcal{R}_c^{(k)}(z)\|_{L^2_\sigma \to L^2_{-\sigma}} \leq \mathcal{O}(1/c),\ \ |z| \to \infty,\ \sigma > 3(k+1)/2,\ k = 0,1,2; \tag{B.2.55}$$

$$\|\mathcal{R}_c(z)\|_{L^2_\sigma \to L^2_{-\sigma}} = \mathcal{O}(1/c^2),\ \ z \to c^2,\ \sigma > 3/2. \tag{B.2.56}$$

$$\|\mathcal{R}_c^{(k)}(z)\|_{L^2_\sigma \to L^2_{-\sigma}} = \mathcal{O}(|z-c|^{1/2-k}),\ \ z \to c^2,\ \sigma > 3(k+1)/2,\ k = 1,2. \tag{B.2.57}$$

Finally, we recall that the free resolvent $\mathcal{R}_{0,c}(z)$ and the perturbed resolvent $\mathcal{R}_c(z)$ are related through the Born perturbation series,

$$\begin{aligned}\mathcal{R}_c(z) &= \mathcal{R}_{0,c}(z) - \mathcal{R}_{0,c}(z)V\ \mathcal{R}_c(z) \\ &= \mathcal{R}_{0,c}(z) - \mathcal{R}_{0,c}(z)\ \ V\ \mathcal{R}_{0,c}(z) &\text{(B.2.58)} \\ &+ \mathcal{R}_{0,c}(z)\ \ V\ \mathcal{R}_{0,c}(z)\ \ V\ \mathcal{R}_c(z), &\text{(B.2.59)}\end{aligned}$$

which follows by iterating the formula (B.2.48). An important property in order to deduce dispersive estimates for $\mathcal{H}(x)$ is the asymptotics for large $|z|$ of the following operator which appears in (B.2.58),

$$\mathcal{W}_c(z) := \mathcal{R}_{0,c}(z)\ \ V\ \mathcal{R}_{0,c}(z)\ \ V.$$

Proposition B.2.19. *Let the potential V satisfy* (B.2.42) *with $\beta > 2\delta + 3(k+1)$, where $\delta > 0$ and $k = 0,1,2$. Then for any $c \geq 1$*

$$\|\mathcal{W}_c^{(k)}(z)\|_{L^2_{-\delta} \to L^2_{-\delta}} = \mathcal{O}(|z|^{-2}),\ \ |z| \to \infty,\ Re(z) < 0. \tag{B.2.60}$$

Proof. By recalling that $\mathcal{R}_{0,c}(z)$ commutes with powers of $\langle\nabla\rangle_c$, we have

$$\begin{aligned}&\|\mathcal{W}_c^{(k)}(z)\|_{L^2_{-\delta}\to L^2_{-\delta}} =\\ &= \left\|\partial_z^{(k)}\left[\mathcal{R}_{0,c}(z)\ \ V(x)\ \mathcal{R}_{0,c}(z)\ \ V(x)\right]\right\|_{L^2_{-\delta}\to L^2_{-\delta}},\end{aligned}$$

which reduces to estimate terms of the form

$$\|\mathcal{R}_{0,c}^{(k_1)}(z)V(x)\mathcal{R}_{0,c}^{(k_2)}(z)V(x)\|_{L^2_{-\delta}\to L^2_{-\delta}}$$

with $k_1,k_2\in\{0,1,2\}$, $k_1+k_2=2$; but these may be bounded by

$$\|\mathcal{R}_{0,c}^{(k_1)}(z)V(x)\|_{L^2_{-\delta}\to L^2_{-\delta}}\,\|\mathcal{R}_{0,c}^{(k_2)}(z)V(x)\|_{L^2_{-\delta}\to L^2_{-\delta}}.$$

The asymptotics (B.2.60) follows from

$$\|\mathcal{R}_{0,c}^{(k)}(z)V(x)\|_{L^2_{-\delta}\to L^2_{-\delta}} = \mathcal{O}\left(|z|^{-1}\right),\ |z|\to\infty, Re(z)<0. \tag{B.2.61}$$

Now we want to show (B.2.61) under the assumptions (B.2.42) for V, for $\delta>0$, $\beta>2\delta+3(k+1)$, $k=0,1,2$ and for $c\geq 1$. We have

$$\begin{aligned}\left\|\mathcal{R}_{0,c}^{(k)}(z)V\right\|_{L^2_{-\delta}\to L^2_{-\delta}} &\preceq \left\|\mathcal{R}_{0,c}^{(k)}(z)\left\langle\cdot\right\rangle^{-\beta}\right\|_{L^2_{-\delta}\to L^2_{-\delta}} \qquad\qquad \text{(B.2.62)}\\ &= \left\|\left\langle\cdot\right\rangle^{-\delta}\mathcal{R}_{0,c}^{(k)}(z)\left\langle\cdot\right\rangle^{-\beta}\left\langle\cdot\right\rangle^{\delta}\right\|_{L^2\to L^2}\\ &\overset{\beta\geq 2\delta}{=} \left\|\left\langle\cdot\right\rangle^{-\delta}\mathcal{R}_{0,c}^{(k)}(z)\left\langle\cdot\right\rangle^{-\beta/2+\delta}\left\langle\cdot\right\rangle^{-\beta/2-\delta}\left\langle\cdot\right\rangle^{\delta}\right\|_{L^2\to L^2}\\ &\leq \left\|\mathcal{R}_{0,c}^{(k)}(z)\right\|_{L^2_{\beta/2}\to L^2_{-\beta/2}}\\ &+ \left\|\left\langle\cdot\right\rangle^{-\delta}\left[\mathcal{R}_{0,c}^{(k)}(z),\left\langle\cdot\right\rangle^{-\beta/2+\delta}\right]\left\langle\cdot\right\rangle^{-\beta/2-\delta}\left\langle\cdot\right\rangle^{\delta}\right\|_{L^2\to L^2}\\ &=: I+II. \qquad\qquad \text{(B.2.63)}\end{aligned}$$

First we estimate I for the case $k=0$. In this case we exploit the trivial identity

$$\begin{aligned}c\left\langle\nabla\right\rangle_c\mathcal{R}_{0,c}(z) &= 1+z\mathcal{R}_{0,c}(z),\\ \mathcal{R}_{0,c}(z) &= z^{-1}c\left\langle\nabla\right\rangle_c\mathcal{R}_{0,c}(z)-z^{-1},\end{aligned}$$

in order to get

$$\|\mathcal{R}_{0,c}(z)\|_{L^2_{\beta/2}\to L^2_{-\beta/2}} = \mathcal{O}\left(|z|^{-1}\right),\ \ |z|\to\infty,\ Re(z)<0.$$

Next we estimate I for the cases $k=1,2$. This follows from

$$\begin{aligned}\|\mathcal{R}_{0,c}^{(k)}(z)\|_{L^2_{\beta/2}\to L^2_{-\beta/2}} &\preceq \|\mathcal{R}_{0,c}(z)^{k+1}\|_{L^2_{\beta/2}\to L^2_{-\beta/2}}\\ &= \|\mathcal{R}_{0,c}(z)\mathcal{R}_{0,c}(z)^{k}\|_{L^2_{\beta/2}\to L^2_{-\beta/2}}\\ &\preceq \|\mathcal{R}_{0,c}(z)\|_{L^2_{\beta/2}\to L^2_{-\beta/2}}\|\mathcal{R}_{0,c}(z)^{k}\|_{L^2_{\beta/2}\to L^2_{\beta/2}}\\ &\preceq \|\mathcal{R}_{0,c}(z)\|_{L^2_{\beta/2}\to L^2_{-\beta/2}}\|\mathcal{R}_{0,c}(z)\|^{k}_{L^2_{\beta/2}\to L^2_{\beta/2}},\end{aligned}$$

and

$$\|\mathcal{R}_{0,c}(z)\|_{L^2_{\beta/2}\to L^2_{\beta/2}} = \| \langle\cdot\rangle^{\beta/2} \mathcal{R}_{0,c}(z) \langle\cdot\rangle^{-\beta/2} \|_{L^2\to L^2} \tag{B.2.64}$$

$$\preceq \|\mathcal{R}_{0,c}(z)\|_{L^2\to L^2} + \left\| \langle\cdot\rangle^{\beta/2} \left[\mathcal{R}_{0,c}(z), \langle\cdot\rangle^{-\beta/2}\right] \right\|_{L^2\to L^2} \tag{B.2.65}$$

$$\preceq N_{0,c}(z) + N_{0,c}(z)^2 \tag{B.2.66}$$

because $\left[\mathcal{R}_{0,c}(z)^{(k)}, \langle\cdot\rangle^{-\beta/2}\right] = \mathrm{Op}(b(x,\xi;z))$, where

$$|b(x,\xi;z)| \preceq \langle x\rangle^{-\beta/2-1} N_{0,c}(z)^{k+2}.$$

Therefore, by recalling the definition of $N_{0,c}(z)$, we can conclude that

$$\|\mathcal{R}^{(k)}_{0,c}(z)\|_{L^2_{\beta/2}\to L^2_{-\beta/2}} = \mathcal{O}\left(|z|^{-1}\right), \quad |z| \to \infty, \; k = 1, 2. \tag{B.2.67}$$

Finally, we point out that term II in (B.2.63) may be bounded by $N_{0,c}(z)^{k+2}$, by using standard theorems of PDO calculus. □

Appendix C

Interpolation theory for relativistic Sobolev spaces

In this section we show an analogue of Theorem 6.4.5 (7) in [18] for the relativistic Sobolev spaces $\mathscr{W}_c^{k,p}$, $k \in \mathbb{R}$, $1 < p < +\infty$.
They have been used in Theorem 2.3.1, in order to get Strichartz estimates for (4.1.6).
We begin by reporting the so-called *Phragmén-Lindelöf principle* (see Chapter 4, Theorem 3.4 in [80]).

Proposition C.0.1. *Let F be a holomorphic function in the sector $S = \{\alpha < arg(z) < \beta\}$, where $\beta - \alpha = \pi/\lambda$. Assume also that F is continuous on $\bar{S}$, that*

$$|F(z)| \leq 1 \quad \forall z \in \partial S,$$

and that there exists $K > 0$ and $\rho \in [0, \lambda)$ such that

$$|F(z)| \leq e^{K|z|^\rho} \quad \forall z \in S.$$

Then $|F(z)| \leq 1$ $\forall z \in S$.

By Proposition C.0.1 one can prove the *3 lines theorem.*

Lemma C.0.2. *Let F be analytic on $\{0 < Re(z) < 1\}$ and continuous on $\{0 \leq Re(z) \leq 1\}$. If*

$$|F(it)| \leq M_0 \quad \forall t \in \mathbb{R},$$
$$|F(1+it)| \leq M_1 \quad \forall t \in \mathbb{R},$$

then $|F(\theta + it)| \leq M_0^{1-\theta} M_1^{\theta}$ for all $t \in \mathbb{R}$ and for any $\theta \in (0,1)$.

Proof. Let $\epsilon > 0$, $\lambda \in \mathbb{R}$. Set

$$F_\epsilon(z) = e^{\epsilon z^2 + \lambda z} F(z).$$

Then $F_\epsilon(z) \to 0$ as $|Im(z)| \to +\infty$, and

$$|F_\epsilon(it)| \leq M_0 \quad \forall t \in \mathbb{R},$$
$$|F_\epsilon(1+it)| \leq M_1 e^{\epsilon + \lambda} \quad \forall t \in \mathbb{R},$$

By Phragmén-Lindelöf principle we get that $|F_\epsilon(z)| \leq \max(M_0, M_1 e^{\epsilon+\lambda})$, namely

$$|F(\theta+it)| \leq e^{-\epsilon(\theta^2-t^2)} \max(M_0 e^{-\theta\lambda}, M_1 e^{(1-\theta)\lambda+\epsilon}), \quad \forall \theta, t.$$

By taking the limit $\epsilon \to 0$ we deduce that

$$|F(\theta+it)| \leq \max(M_0 e^{-\theta\lambda}, M_1 e^{(1-\theta)\lambda}).$$

The right-hand side is as small as possible for $M_0 e^{-\theta\lambda} = M_1 e^{(1-\theta)\lambda}$, i.e. for $e^\lambda = M_0/M_1$. Thus, if we choose $\lambda = log(M_0/M_1)$, we get

$$|F(\theta+it)| \leq M_0^{1-\theta} M_1^{\theta}.$$

□

Now we introduce some notation used in the framework of complex interpolation method (read [18], chapter 4).
Let $A = (A_0, A_1)$ be a couple of Banach spaces, and denote by $A_0 + A_1$ the space for which the following norm is finite,

$$\|a\|_{A_0+A_1} := \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + \|a_1\|_{A_1}).$$

The space $A_0 + A_1$, endowed with the above norm, is also a Banach space.
We then define the space $\mathscr{F}(A)$ of all functions $f : \mathbb{C} \to A_0 + A_1$ which are analytic on the open strip $S := \{z : 0 < Re(z) < 1\}$, continuous and bounded on $\bar{S} = \{z : 0 \leq Re(z) \leq 1\}$, such that the functions

$$t \mapsto f(j+it) \in C(\mathbb{R}, A_j), \quad j = 0, 1,$$

and such that $\lim_{|t|\to\infty} f(j+it) = 0$ for $j = 0, 1$.
The space $\mathscr{F}(A)$, endowed with the norm

$$\|f\|_{\mathscr{F}(A)} := \max(\sup_t \|f(it)\|_{A_0}, \sup_t \|f(1+it)\|_{A_1}),$$

is a Banach space (Lemma 4.1.1 in [18]).

Next we define the interpolation space

$$A_\theta := \{a \in A_0 + A_1 \,:\, a = f(\theta) \text{ for some } f \in \mathscr{F}(A)\},$$
$$\|a\|_\theta := \inf\{\|f\|_{\mathscr{F}(A)} : f \in \mathscr{F}(A), f(\theta) = a\}.$$

Now we show a classical result of complex interpolation theory (Theorem 5.1.1. in [18]).

Theorem C.0.3. *Let p_0, $p_1 \geq 1$, and $0 < \theta < 1$. Then*

$$(L^{p_0}, L^{p_1})_\theta = L^p \text{ for } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \tag{C.0.1}$$

Proof. We prove that $\|a\|_{(L^{p_0},L^{p_1})_\theta} = \|a\|_{L^p}$ for all $a \in C_c^\infty(\mathbb{R}^d)$. Set

$$f(z) := e^{\epsilon z^2 - \epsilon\theta^2} |a|^{p/p(z)} a/|a|,$$

where $1/p(z) = (1-z)/p_0 + z/p_1$.
Assume that $\|a\|_{L^p} = 1$, then $f \in \mathscr{F}(L^{p_0}, L^{p_1})$, and $\|f\|_{\mathscr{F}} \leq e^\epsilon$. Since $f(\theta) = a$, we conclude

that $\|a\|_{(L^{p_0},L^{p_1})_\theta} \le e^\epsilon$, hence $\|a\|_{(L^{p_0},L^{p_1})_\theta} \le \|a\|_{L^p}$.
On the other hand, since

$$\|a\|_{L^p} = \sup\{|\langle a,b\rangle| : \|b\|_{L^{p'}} = 1, b \in C_c^\infty(\mathbb{R}^d)\},$$

we can define

$$g(z) := e^{\epsilon z^2 - \epsilon\theta^2}|b|^{p'/p'(z)}b/|b|,$$

where $1/p'(z) = (1-z)/p'_0 + z/p'_1$. Writing $F(z) := \langle f(z), g(z)\rangle$, we have $|F(it)| \le e^\epsilon$ and $|F(1+it)| \le e^{2\epsilon}$, provided that $\|a\|_{(L^{p_0},L^{p_1})_\theta} = 1$. Hence, by the three line theorem it follows that $|\langle a,b\rangle| \le |F(\theta)| \le e^{2\epsilon}$. This implies that $\|a\|_{L^p} \le \|a\|_{(L^{p_0},L^{p_1})_\theta}$. □

In order to study the relativistic Sobolev spaces, we have to introduce the notion of Fourier multipliers.

Definition C.0.4. *Let $1 < p < +\infty$, and $\rho \in \mathcal{S}'$. We call ρ a* Fourier multiplier *on $L^p(\mathbb{R}^d)$ if the convolution $(\mathcal{F}^{-1}\rho) * f \in L^p(\mathbb{R}^d)$ for all $f \in L^p(\mathbb{R}^d)$, and if*

$$\sup_{\|f\|_{L^p}=1} \|(\mathcal{F}^{-1}\rho) * f\|_{L^p} < +\infty. \tag{C.0.2}$$

The linear space of all such ρ is denoted by M_p, and is endowed with the above norm (C.0.2).

One can check that for any $p \in (1,+\infty)$ one has $M_p = M_{p'}$ (where $1/p + 1/p' = 1$), and that by Parseval's formula $M_2 = L^\infty$. Furthermore, by Riesz-Thorin theorem one gets that for any $\rho \in M_{p_0} \cap M_{p_1}$ and for any $\theta \in (0,1)$

$$\|\rho\|_{M_p} \le \|\rho\|_{M_{p_0}}^{1-\theta}\|\rho\|_{M_{p_1}}^{\theta}, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}. \tag{C.0.3}$$

In particular, one can deduce that $\|\cdot\|_{M_p}$ decreases with $p \in (1,2]$, and that $M_p \subset M_q$ for any $1 < p < q \le 2$.

More generally, if H_0 and H_1 are Hilbert spaces, one can introduce a similar definition of Fourier multiplier.

Definition C.0.5. *Let $1 < p < +\infty$, let H_0 and H_1 be two Hilbert spaces, and consider $\rho \in \mathcal{S}'(H_0,H_1)$. We call ρ a* Fourier multiplier *if the convolution $(\mathcal{F}^{-1}\rho) * f \in L^p(H_1)$ for all $f \in L^p(H_0)$, and if*

$$\sup_{\|f\|_{L^p(H_0)}=1} \|(\mathcal{F}^{-1}\rho) * f\|_{L^p(H_1)} < +\infty. \tag{C.0.4}$$

The linear space of all such ρ is denoted by $M_p(H_0,H_1)$, and is endowed with the above norm (C.0.4).

Next we state the so-called *Mihlin multipier theorem* (Theorem 6.1.6 in [18]).

Theorem C.0.6. *Let H_0 and H_1 be Hilbert spaces, and assume that $\rho : \mathbb{R}^d \to L(H_0,H_1)$ be such that*

$$|\xi|^\alpha \|D^\alpha \rho(\xi)\|_{L(H_0,H_1)} \le K, \ \ \forall \xi \in R^d, |\alpha| \le L$$

for some integer $L > d/2$. Then $\rho \in M_p(H_0,H_1)$ for any $1 < p < +\infty$, and

$$\|\rho\|_{M_p} \le C_p K, \ \ 1 < p < +\infty.$$

Now, recall the Littlewood-Paley functions $(\phi_j)_{j\geq 0}$ defined in (3.1.1), and introduce the maps $\mathcal{J}:\mathcal{S}'\to\mathcal{S}'$ and $\mathcal{P}:\mathcal{S}'\to\mathcal{S}'$ via formulas

$$(\mathcal{J}f)_j := \phi_j * f, \quad j\geq 0, \tag{C.0.5}$$

$$\mathcal{P}g := \sum_{j\geq 0}\tilde{\phi}_j * g_j, \quad j\geq 0, \tag{C.0.6}$$

where $g=(g_j)_{j\geq 0}$ with $g_j\in\mathcal{S}'$ for all j, and

$$\tilde{\phi}_0 := \phi_0+\phi_1,$$
$$\tilde{\phi}_j := \phi_{j-1}+\phi_j+\phi_{j+1}, \quad j\geq 1.$$

One can check that $\mathcal{P}\circ\mathcal{J}f=f\ \forall f\in\mathcal{S}'$, since $\tilde{\phi}_j*\phi_j=\phi_j$ for all j. We then introduce for $c\geq 1$ and $k\geq 0$ the space

$$l^{2,k}_c := \{(z_j)_{j\in\mathbb{Z}}\ :\ c^{-k}\sum_{j\in\mathbb{Z}}(c^2+|j|^2)^k|z_j|^2<+\infty\}.$$

Theorem C.0.7. *Let $c\geq 1$, $k\geq 0$, $1<p<+\infty$. Then $\langle\nabla\rangle_c^k L^p$ is a retract of $L^p(l^{2,k}_c)$, namely that the operators*

$$\mathcal{J}:\mathscr{W}^{k,p}_c\to L^p(l^{2,k}_c)$$
$$\mathcal{P}:L^p(l^{2,k}_c)\to\mathscr{W}^{k,p}_c$$

satisfy $\mathcal{P}\circ\mathcal{J}=id$ on $\mathscr{W}^{k,p}_c$.

Proof. First we show that $\mathcal{J}:\mathscr{W}^{k,p}_c\to L^p(l^{2,k}_c)$ is bounded.
Since $\mathcal{J}f=(\mathcal{F}^{-1}\chi_c)*\mathcal{J}^k_c f$, where

$$(\chi_c(\xi))_j := (c^2+|\xi|^2)^{-k/2}\hat{\phi}_j(\xi), \quad j\geq 0$$
$$\mathcal{J}^k_c f := \mathcal{F}^{-1}((c^2+|\xi|^2)^{k/2}\hat{f}),$$

we have that for any $\alpha\in\mathbb{N}^d$

$$|\xi|^\alpha\|D^\alpha\chi_c(\xi)\|_{L(\mathbb{C},l^{2,k}_c)}\leq|\xi|^\alpha\sum_{j\geq 0}(2^{jk}c^k|D^\alpha(\chi_c(\xi))_j|\leq K_\alpha$$

because the sum consists of at most two non-zero terms for each ξ. Thus $\mathcal{J}\in M_p(\mathscr{W}^{k,p}_c,L^p(l^{2,k}_c))$ by Mihlin multiplier Theorem.

On the other hand, consider $\mathcal{P}:L^p(l^{2,k}_c)\to\mathscr{W}^{k,p}_c$.
Since $\mathcal{J}^k_c\circ\mathcal{P}g=(\mathcal{F}^{-1}\delta_c)*g_{(k)}$, where

$$g=(g_j)_{j\geq 0},$$
$$g_{(k)} := (2^{jk}g_j)_{j\geq 0},$$
$$\delta_c(\xi)g := \sum_{j\geq 0}2^{-jk}(c^2+|\xi|^2)^{k/2}\tilde{\phi}_j(\xi)g_j,$$

we have that for any $\alpha\in\mathbb{N}^d$

$$|\xi|^\alpha\|D^\alpha\delta_c(\xi)\|_{L(l^{2,k}_c,\mathbb{C})}\leq|\xi|^\alpha\left[\sum_{j\geq 0}(2^{-jk}c^{-k}|D^\alpha(c^2+|\xi|^2)^{k/2}\tilde{\phi}_j(\xi)|)^2\right]^{1/2}\leq K_\alpha,$$

because the sum consists of at most four non-zero terms for each ξ. Thus $\mathcal{P}\in M_p(L^p(l^{2,k}_c),\mathscr{W}^{k,p}_c)$ by Mihlin multiplier Theorem, and we can conclude. □

Corollary C.0.8. *Let $\theta \in (0,1)$, and assume that k_0, $k_1 \geq 0$ ($k_0 \neq k_1$) and p_0, $p_1 \in (1,+\infty)$ satisfy*

$$k = (1-\theta)k_0 + \theta k_1,$$
$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Then $(\mathscr{W}_c^{k_0,p}, \mathscr{W}_c^{k_1,p})_\theta = \mathscr{W}_c^{k,p}$.

Proof. It follows from the abstract result that if $B = (B_0, B_1)$ is a retract of $A = (A_0, A_1)$, then B_θ is a retract of A_θ (Theorem 6.4.2 in [18]). □

The previous corollary, combined with Lemma C.0.2, immediately gives a proof of Proposition 2.3.4.

We also give a formulation of the Kato-Ponce inequality for the relativistic Sobolev spaces.

Proposition C.0.9. *Let $f, g \in \mathcal{S}(\mathbb{R}^3)$, and let $c > 0$, $1 < r < \infty$ and $k \geq 0$. Then*

$$\|f\ g\|_{\mathscr{W}_c^{k,r}} \preceq \|f\|_{\mathscr{W}_c^{k,r_1}} \|g\|_{L^{r_2}} + \|f\|_{L^{r_3}} \|g\|_{\mathscr{W}_c^{k,r_4}}, \tag{C.0.7}$$

with

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3} + \frac{1}{r_4}, \quad 1 < r_1, r_4 < +\infty.$$

Remark C.0.10. *For $c = 1$ Eq.* (C.0.7) *reduces to the classical Kato-Ponce inequality.*

Proof. We follow an argument by Cordero and Zucco (see Theorem 2.3 in [28]).
We introduce the dilation operator $S_c(f)(x) := f(x/c)$, for any $c > 0$.
Then we apply the classical Kato-Ponce inequality to the rescaled product $S_c(fg) = S_c(f)\ S_c(g)$,

$$\|S_c(fg)\|_{W^{k,r}} \preceq \|S_c(f)\|_{W^{k,r_1}} \|S_c(g)\|_{L^{r_2}} + \|S_c(f)\|_{L^{r_3}} \|S_c(g)\|_{W^{k,r_4}}, \tag{C.0.8}$$

where

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3} + \frac{1}{r_4}, \quad 1 < r_1, r_4 < +\infty.$$

Now, combining the commutativity property

$$\langle\nabla\rangle^k S_c(f)(x) = c^{-k} S_c(\langle\nabla\rangle_c^k\ f)(x),$$

with the equality $\|S_c(f)\|_{L^r} = c^{-3/r}\|f\|_{L^r}$, we can rewrite (C.0.8) as

$$\|\langle\nabla\rangle^k(f\ g)\|_{L^r} \preceq \|\langle\nabla\rangle^k f\|_{L^{r_1}} \|g\|_{L^{r_2}} + \|f\|_{L^{r_3}} \|\langle\nabla\rangle^k g\|_{L^{r_4}},$$

and this leads to the thesis. □

Appendix D

Analytical tools

In this chapter we give an outline of the theory of pseudodifferential operators. This theory has been developed since 1960s to treat problems in linear and nonlinear PDEs. We will not prove all the results in detail; for the interested reader we will address to the literature ([42], [84]; see also [3] for a nonlinear PDEs-oriented approach).
We define pseudodifferential operators with symbols in Hörmander's classes $S^m_{\rho,\delta}$; then we derive some useful properties of their Schwartz kernels, and discuss their properties. We proceed to a discussion of mapping properties on L^2 and on the Sobolev spaces H^k; we also discuss L^p estimates, in particular some fundamental results of Calderon-Zygmund, and applications to Littlewood-Paley Theory of "dyadic decomposition". This decomposition, which is based on frequency space localization, allows one to rapidly obtain interesting properties of operators on Sobolev spaces.

D.1 Fourier Transform

In this section we briefly recall some classical notion of Fourier analysis on $\mathbb{R}^d$. We first recall the definition of the space of Schwartz (or rapidly decreasing) functions,

$$\mathcal{S}(\mathbb{R}^d) := \{f \in C^\infty(\mathbb{R}^d, \mathbb{R}) | \sup_{x\in\mathbb{R}^d} (1+|x|^2)^{\alpha/2} |\partial^\beta f(x)| < +\infty, \ \ \forall \alpha \in \mathbb{N}^d, \forall \beta \in \mathbb{N}^d\}. \tag{D.1.1}$$

In the following we will denote by $\langle x \rangle := (1+|x|^2)^{1/2}$. The space $\mathcal{S}(\mathbb{R}^d)$, endowed with the above family of semi-norms for $\alpha, \beta \in \mathbb{N}^d$, is complete. Now, for any $f \in \mathcal{S}(\mathbb{R}^d)$ we introduce the *Fourier transform* of f, $\hat{f} : \mathbb{R}^d \to \mathbb{R}$,

$$\hat{f}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x) e^{-i\langle x, \xi\rangle} \mathrm{d}^d x, \ \ \forall \xi \in \mathbb{R}^d, \tag{D.1.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^d$. Using the above seminorms one can show that the linear mapping $\mathcal{F} : f \mapsto \hat{f}$ is continuous from $\mathcal{S}(\mathbb{R}^d)$ to itself. The map $\mathcal{F}$ is also continuous from $L^1(\mathbb{R}^d)$ to $L^\infty(\mathbb{R}^d)$, since

$$\|\hat{f}\|_{L^\infty(\mathbb{R}^d)} \le (2\pi)^{-d/2} \|f\|_{L^1(\mathbb{R}^d)}.$$

Now, introduce the operator $D_j := -i\frac{\partial}{\partial x_j}$ for $1 \leq j \leq d$, and more generally

$$D^\alpha := D_1^{\alpha_1} \dots D_d^{\alpha_d}, \ \ \alpha \in \mathbb{N}^d,$$

and let us introduce also the translation operator $\tau_y f(\cdot) := f(\cdot - y)$, $y \in \mathbb{R}^d$. We can easily check the fundamental properties of the Fourier transform,

$$\widehat{D_j f}(\xi) = \xi_j f(\xi), \ \ \forall \xi \in \mathbb{R}^d, \tag{D.1.3}$$

$$\widehat{\tau_y f}(\xi) := f(\xi - y) = e^{-i\,\langle y, \xi\rangle} \hat{f}(\xi), \tag{D.1.4}$$

$$\widehat{e^{i\,\langle y, \cdot\rangle} f}(\xi) = \tau_y \hat{f}(\xi), \tag{D.1.5}$$

$$\widehat{x_j u}(\xi) = -D_j \hat{f}(\xi). \tag{D.1.6}$$

A linear form on $\mathcal{S}(\mathbb{R}^d)$ which is continuous with respect to the semi-norms defined in (D.1.1) is called a *tempered distribution* on $\mathbb{R}^d$. The space of tempered distributions is is denoted by $\mathcal{S}'(\mathbb{R}^d)$. In particular, by defining the following endomorphism of $\mathcal{S}(\mathbb{R}^d)$ into itself

$$T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d),$$

$$T(u) : v \ \mapsto \ \langle u, v\rangle := \int_{\mathbb{R}^d} u(x)v(x)\,\mathrm{d}x, \forall v \in \mathcal{S}(\mathbb{R}^d), \forall u \in \mathcal{S}(\mathbb{R}^d),$$

we can deduce that $\mathcal{S}(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$ (actually, it is well known that $\mathcal{S}(\mathbb{R}^d)$ is dense in $\mathcal{S}'(\mathbb{R}^d)$). As for all linear maps, one can observe that by Fubini's theorem

$$\int_{\mathbb{R}^d} \hat{u}(\xi)v(\xi)\mathrm{d}\xi = \int_{\mathbb{R}^d} u(\xi)\hat{v}(\xi)\mathrm{d}\xi, \ \ \forall u, v \in \mathcal{S}(\mathbb{R}^d),$$

hence the formula

$$\langle \hat{u}, v\rangle = \langle u, \hat{v}\rangle, \ \ \forall u \in \mathcal{S}'(\mathbb{R}^d), v \in \mathcal{S}(\mathbb{R}^d)$$

defines a linear map $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$, which is the unique continuous extension of $\mathcal{F} : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. In particular, $\mathcal{F}$ satisfies (D.1.3) - (D.1.6).
Moreover, using (D.1.4) and (D.1.5), one gets

$$\hat{\hat{f}}(x) = f(-x), \ \ \forall f \in \mathcal{S}(\mathbb{R}^d).$$

Another well-known result is the *Fourier inversion formula*,

Lemma D.1.1. *The Fourier transform $\mathcal{F} : \phi \to \hat{\phi}$ is an isomorphism from $\mathcal{S}(\mathbb{R}^d)$ to itself, with inverse given by the following formula*

$$f(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \hat{f}(\xi) e^{i\,\langle x, \xi\rangle} \mathrm{d}\xi. \tag{D.1.7}$$

Proof. See the proof of Theorem 7.1.5 in [41], Ch. VII. □

By the Fourier inversion formula one can deduce the *Plancherel inequality*,

$$\left\langle \hat{f}, \hat{g} \right\rangle = \langle f, g\rangle, \ \ \forall f, g \in L^2(\mathbb{R}^d). \tag{D.1.8}$$

D.2 Symbols

We begin by noting that if one differentiates the above Fourier inversion formula (D.1.7), one gets

$$D^\alpha f(x) = (2\pi)^{-d/2} \int_{R^d} \xi^\alpha \hat{f}(\xi) e^{i\langle x,\xi\rangle} \mathrm{d}\xi, \quad \forall \alpha \in \mathbb{N}^d. \tag{D.2.1}$$

Hence, if

$$p_m(x, D) = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha \tag{D.2.2}$$

with $a_\alpha \in C^\infty(R^d)$ for all α, is a differential operator of order m, we have that

$$\begin{aligned} p_m(x, D) f(x) &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} p_m(x, \xi) \hat{f}(\xi) e^{i\langle x,\xi\rangle} \mathrm{d}\xi \\ &= \sum_{|\alpha| \le m} (2\pi)^{-d/2} \int_{\mathbb{R}^d} a_\alpha(x) \xi^\alpha \hat{f}(\xi) e^{i\langle x,\xi\rangle} \mathrm{d}\xi. \end{aligned} \tag{D.2.3}$$

The above computation suggests that if we consider a suitable class of functions, called symbols, we can write the action of the associated (pseudo-)differential operators acting on $L^2(\mathbb{R}^d)$ by using the Fourier integral representation (D.2.3).

Definition D.2.1. *Let $\rho, \delta \in [0,1]$, $m \in \mathbb{R}$. We define the class of symbols $S^m_{\rho,\delta}$ as the set of functions $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that for all $\alpha, \beta \in \mathbb{N}^d$ there exists $K_{\alpha,\beta} > 0$ such that*

$$|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \le K_{\alpha,\beta} \langle \xi \rangle^{m - \rho|\alpha| + \delta|\beta|}. \tag{D.2.4}$$

The real number m is called order *of the symbol.*
Furthermore, if there exist smooth functions $a_{m-j}(x, \xi)$ homogeneous in ξ of degree $m - j$ for $|\xi| \ge 1$, i.e.

$$a_{m-j}(x, r\xi) = r^{m-j} a_{m-j}(x, \xi), \quad \textit{for } r, |\xi| \ge 1,$$

and if

$$a(x, \xi) \sim \sum_{j \ge 0} a_{m-j}(x, \xi), \quad \textit{namely} \tag{D.2.5}$$

$$a - \sum_{j=0}^{N} a_{m-j} \in S^{m-N}_{1,0} \quad \forall N \in \mathbb{N}, \tag{D.2.6}$$

we will write that $a \in S^m := S^m_{1,0}$. We will also denote by $S^{-\infty} = \cap_{m \in \mathbb{R}} S^m$.

Remark D.2.2.
- *Any differential symbol of order m, $p_m(x, \xi) = \sum_{|\alpha| \le m} a_\alpha(x) \xi^\alpha$ with $a_\alpha \in C^\infty(R^d)$ for all α, is clearly in S^m.*

- *Let $a \in C^\infty(\mathbb{R}^d \setminus \{0\})$ be a positively homogeneous function of degree m. If $\chi \in C_0^\infty$, with $\chi \equiv 1$ in a neighborhood of 0, then the function*

  $$\tilde{a}(\xi) := (1 - \chi(\xi)) a(\xi)$$

  is a symbol of order m.

- *Let Ω be an open subset of $\mathbb{R}^d$, and let p_m be a differential symbol of order m defined on Ω. Assume that p_m is* elliptic *in Ω, namely*

$$p_m(x,\xi) \neq 0,\ \forall x \in \Omega, \forall \xi \in \mathbb{R}^d \setminus \{0\}.$$

Let $\phi \in C_0^\infty(\Omega)$, and $\chi \in C_0^\infty(\mathbb{R}^d)$ with $\chi \equiv 1$ in a neighborhood of 0, then for sufficiently large $K > 0$ the function

$$a(x,\xi) := \phi(x)(1 - \chi(\xi/K))/p_m(x,\xi)$$

is a symbol of order $-m$.

- *Let $a \in \mathcal{S}(\mathbb{R}^d)$, then $a(\xi) \in S^{-\infty}$.*

- *The function $a(x,\xi) = e^{i\langle x,\xi\rangle}$ is not a symbol.*

- *A symbol $a(x,\xi)$ with $x,\xi \in \mathbb{R}^d$ is not necessarily a symbol in the variables $y = (x,x')$, $\eta = (\xi,\xi')$ with $x',\xi' \in R^k$, $k \geq 1$, except when it is differential.*

The following properties follow readily from the definition of symbol.

Remark D.2.3.

- *Let $a \in S^m_{\rho,\delta}$, then $\partial_\xi^\alpha \partial_x^\beta a \in S^{m-\rho|\alpha|+\delta|\beta|}_{\rho,\delta}$.*

- *Let $a \in S^m_{\rho,\delta}$, $b \in S^n_{\rho,\delta}$, then $ab \in S^{m+n}_{\rho,\delta}$.*

Lemma D.2.4. *If $a_1,\dots,a_k \in S^0$, and $F \in C^\infty(\mathbb{C}^k)$, then $F(a_1,\dots,a_k) \in S^0$.*

Now let us define the following family of semi-norms on S^m,

$$|a|^m_{\alpha,\beta} := \sup_{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d} \langle\xi\rangle^{-(m-|\alpha|)}\, |\partial_\xi^\alpha \partial_x^\beta a(x,\xi)|,\quad \alpha,\beta \in \mathbb{N}^d. \tag{D.2.7}$$

Space S^m, endowed with the above family of seminorms, is complete; the convergence $a_n \overset{n\to\infty}{\to} a$ in S^m means that for every $\alpha,\beta \in \mathbb{N}^d$ we have that $|a_n - a|^m_{\alpha,\beta} \overset{n}{\to} 0$.

Lemma D.2.5. *Let $a \in S^0(\mathbb{R}^d \times \mathbb{R}^d)$, and set $a_\delta(x,\xi) = a(x,\delta\xi)$. Then a_δ is bounded in S^0, and $\lim_{\delta\to 0} a_\delta = a_0$ in S^m for all $m > 0$.*

Proof. We will show that for $0 \leq m \leq 1$, $0 \leq \delta \leq 1$, and for any $\alpha,\beta \in \mathbb{N}^d$ $|a_\delta - a_0|^m_{\alpha,\beta} \leq K_{\alpha,\beta,m}\delta^m$. Indeed, for $\alpha = 0$

$$\partial_x^\beta(a_\delta - a_0)(x,\xi) = \delta\xi \int_0^1 \partial_\xi \partial_x^\beta a(x,t\delta\xi)\,\mathrm{d}t,$$

thus

$$\begin{aligned}|\partial_x^\beta(a_\delta - a_0)(x,\xi)| &\leq K_\beta \int_0^{\delta s} \frac{\mathrm{d}s}{\langle s\rangle}\\ &\leq K_d K_\beta \int_0^{\delta s} \frac{\mathrm{d}s}{1+s} = K_d K_\beta \log(1+s\delta),\end{aligned}$$

and the estimate follows directly from the inequality $\log(1+x) \leq K_m x^m$ (which holds for any $x \geq 0$, and for any $m > 0$).

On the other hand, for $\alpha \neq 0$ we have that $\partial_\xi^\alpha \partial_x^\beta a_0 = 0$, while

$$|\partial_\xi^\alpha \partial_x^\beta a_\delta(x,\xi)| \leq K_{\alpha,\beta} \delta^{|\alpha|} \langle \delta\xi \rangle^{|\alpha|},$$

and the result follows, since $\langle \xi \rangle \geq \langle \delta\xi \rangle$. □

Just observe that the convergence $a_\delta \to a_0$ does not take place in S^0. In particular, if $\chi \in \mathcal{S}(\mathbb{R}^d)$, $\chi = 1$ in a neighbourhood of 0, and $a \in S^m$, then the symbols $a_\delta(x,\xi) = \chi(\delta\xi)a(x,\xi)$ are of order $-\infty$, and $a_\delta \to a$ in $S^{m'}$ for all $m' > m$.

In Definition D.2.1 we have already introduced the notion of asymptotic expansion of a symbol $a \in S^m$; we will say that a admits the asymptotic (in the sense of the behaviour of symbols as $|\xi| \to \infty$) expansion,

$$a(x,\xi) \asymp \sum_{j=0}^{\infty} a_j$$

if there exists a decreasing sequence $m_j \to -\infty$, and a sequence of symbols $(a_j)_{j\in\mathbb{N}}$ with $a_j \in S^{m_j}$ for all j such that for any $N \geq 0$

$$a - \sum_{j=0}^{N} a_j \in S^{m_{N+1}}.$$

In practice, one will often have that $m_j = m - \delta j$ for some $\delta > 0$. In order to beccome familiar with the notion of asymptotic sum, we first show the following result proved by Borel.

Lemma D.2.6. *Let $(b_j)_{j\in\mathbb{N}}$ be a sequence of complex numbers. Then there exists $f \in C^\infty(\mathbb{R})$ such that $f^{(j)}(0) = b_j$ for all j, or equivalently such that*

$$f(x) \asymp \sum_{j\geq 0} b_j \frac{x^j}{j!}, \quad x \to 0.$$

Proof. Let $\chi \in C_c^\infty(\mathbb{R})$ be such that $\chi = 1$ if $|x| \leq 1$, and such that $\operatorname{supp}(\chi) \subseteq B(0,2]$, and let $(\lambda_j)_{j\in\mathbb{N}}$ be a sequence of positive numbers such that $\lambda_j \to +\infty$. We will prove that we can choose $(\lambda_j)_{j\in\mathbb{N}}$ such that the function f defined by

$$f(x) := \sum_{j=0}^{\infty} b_j \frac{x^j}{j!} \chi(\lambda_j x)$$

has the properties of the statement. First note that the above series is simply convergent by properties of $(\lambda_j)_j$ and χ.

Let $k \in \mathbb{N}$; if $j \geq k$, then the k-th derivative of the term of rank j is given by

$$f_j^{(k)}(x) = \sum_{0\leq l\leq k} \binom{k}{l} b_j \frac{x^{j-l}}{(j-l)!} \chi^{k-l}(\lambda_j x) \lambda_j^{k-l}.$$

Since $\lambda_j x$ remains bounded on the support of χ and its derivatives, we get that there exists a constant $C_k > 0$ such that

$$|f_j^{(k)}(x)| \leq C_k |b_j| \frac{\lambda_j^{k-j}}{(j-k)!},$$

and if we choose $\lambda_j \geq 1 + |b_j|$, then the series $\sum_j |f_j^{(k)}(x)|$ converges uniformly for $x \in \mathbb{R}$, which ensures that f is of class C^∞, and that we can compute its derivatives by differentiating term by term. Hence for all k we have $f^{(k)}(0) = b_k$. □

By adapting the previous lemma, we get

Proposition D.2.7. *There exists $a \in S^m$ such that $a \sim \sum_j a_j$. Furthermore, we have $supp(a) \subseteq \bigcup_j supp(a_j)$.*

Proof. By taking the asymptotics for $\frac{1}{|\xi|} \to 0$, we take

$$a = \sum_j \tilde{a}_j = \sum_j (1 - \chi(\epsilon_j \xi))\, a_j,$$

with $\chi \in C_c^\infty(\mathbb{R}^d)$, $\chi = 1$ near 0, for a suitable $(\epsilon_j)_j \to 0$. More precisely, we require that

$$\left|\partial_\xi^\alpha \partial_x^\beta \tilde{a}_j\right| \leq 2^{-j} \langle\xi\rangle^{1+m_j-|\alpha|} \ \text{ if } \ |\alpha| + |\beta| \leq j;$$

the existence of such $(\epsilon_j)_j$ is ensured by the approximation Lemma D.2.5, since $1 - \chi(\epsilon\xi)$ tends to 0 in S^1. Then the sum is locally finite, and therefore $a \in C^\infty$. Now, given α, β and k, we have that for $N \geq |\alpha| + |\beta|$ and $m_N + 1 \leq m_{k+1}$

$$\left|\partial_\xi^\alpha \partial_x^\beta \left(a - \sum_{j \leq N-1} \tilde{a}_j\right)\right| \leq \langle\xi\rangle^{m_{k+1}-|\alpha|}.$$

Hence $a - \sum_{j\leq k} a_j = a - \sum_{j \leq N-1} \tilde{a}_j + \sum_{k+1 \leq j \leq N-1} \tilde{a}_j + \sum_{j \leq k}(a_j - \tilde{a}_j)$ satisfies

$$\left|\partial_\xi^\alpha \partial_x^\beta \left(a - \sum_{j \leq k} \tilde{a}_j\right)\right| \leq C_{\alpha,\beta,k} \langle\xi\rangle^{m_{k+1}-|\alpha|},$$

because $a_j - \tilde{a}_j \in S^{-\infty}$, and $\tilde{a}_j \in S^{m_j}$. □

The above proposition suggests the following definition.

Definition D.2.8. *A symbol $a \in S^m$ is called classical if $a \sim \sum_j a_j$, where the functions a_j are homogeneous of degree $m - j$ for $|\xi| \geq 1$, namely $a_j(x, \lambda\xi) = \lambda^{m-j} a_j(x, \xi)$ for $|\xi| \geq 1$, $\lambda \geq 1$.*

D.3 Action of pseudo-differential operators in $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$

Lemma D.3.1. *Let $\delta \in [0, 1)$ and $a \in S^m_{\rho,\delta}$, then $a(x, D) := Op(a) : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. Moreover, the linear mapping Op from $S^m_{\rho,\delta}$ to $\mathcal{S}'(\mathbb{R}^d)$ is injective, and satisfies*

$$[Op(a), D_j] = i\, Op(\partial_{x_j} a), \ \ 1 \leq j \leq d, \tag{D.3.1}$$

$$[Op(a), x_j] = -i\, Op(\partial_{\xi_j} a), \ \ 1 \leq j \leq d, \tag{D.3.2}$$

where x_j denotes the multiplication by the function $x \mapsto x_j$.

Proof. Given $u \in \mathcal{S}'(\mathbb{R}^d)$, $v \in \mathcal{S}(\mathbb{R}^d)$, we have formally

$$\langle v, a(\cdot, D)u \rangle = \langle a_v, \hat{u} \rangle \,, \tag{D.3.3}$$

where

$$a_v(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} v(x) a(x,\xi) e^{i\langle x,\xi \rangle} \mathrm{d}x. \tag{D.3.4}$$

Now, by integration by parts we obtain

$$\xi^\alpha a_v(\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} D_x^\alpha \left(v\, a(\cdot, \xi) \right) e^{i\langle x,\xi \rangle} \mathrm{d}x,$$

hence

$$|a_v(\xi)| \leq K_\alpha \, \langle \xi \rangle^{m+\delta|\alpha|-|\alpha|} \,,$$

thus for $\delta < 1$ we have that p_v is decreasing, and the same is true also for its derivatives. Therefore $a_v \in \mathcal{S}(\mathbb{R}^d)$, and the right-hand-side of (D.3.3) is well-defined.

Relations (D.3.1) follow from integration by parts: indeed,for example

$$\begin{aligned} Op(a)D_j u(x) &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\langle x,\xi \rangle} a(x,\xi) \widehat{D_j u}(\xi) \mathrm{d}\xi \\ &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\langle x,\xi \rangle} a(x,\xi) \xi_j \hat{u}(\xi) \mathrm{d}\xi, \end{aligned}$$

while

$$D_j(Op(a)u)(x) = -i \left[(2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\langle x,\xi \rangle} i\xi_j a(x,\xi) \hat{u}(\xi) \mathrm{d}\xi + Op(\partial_{x_j} a) u(x) \right],$$

so we get the first formula of (D.3.1). □

Arguing as before, one can show that if $a \in S^m_{\rho,\delta}$ with ρ, $\delta \in [0,1]$, then $a(x,D)$ is bounded as an operator from $\mathcal{S}(\mathbb{R}^d)$ to itself.

Definition D.3.2. *For $a \in S^m_{\rho,\delta}$ the operator $Op(a)$ is the pseudo-differential operator with symbol a. A pseudo-differential operator is called of order m if its symbol in $S^m_{\rho,\delta}$ for some ρ, δ. We denote the space of pseudo-differential operators with associated symbol in $S^m_{\rho,\delta}$ by $OPS^m_{\rho,\delta}$.*

An altenative representation for a pseudo-differential operator can be obtained via the family of unitary operators

$$e^{i\langle q,X \rangle} e^{i\langle p,D \rangle} u(x) = e^{i\langle q,x \rangle} u(x+p). \tag{D.3.5}$$

Indeed, given $a \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$, we have

$$\begin{aligned} &\int_{\mathbb{R}^{2d}} \hat{a}(q,p) e^{i\langle q,X \rangle} e^{i\langle p,D \rangle} u(x) \mathrm{d}q \mathrm{d}p \\ &= (2\pi)^{-d} \int_{\mathbb{R}^{4d}} a(y,\xi) e^{-i\langle q,y \rangle} e^{i\langle q,x \rangle} e^{-i\langle p,\xi \rangle} u(x+p) \mathrm{d}y \mathrm{d}\xi \mathrm{d}q \mathrm{d}p \\ &= (2\pi)^{-d/2} \int_{\mathbb{R}^{2d}} a(x,\xi) e^{-i\langle p,\xi \rangle} u(x+p) \mathrm{d}\xi \mathrm{d}p \\ &= (2\pi)^{-d/2} \int_{\mathbb{R}^d} a(x,\xi) e^{i\langle x,\xi \rangle} \hat{u}(\xi) \mathrm{d}\xi, \end{aligned}$$

hence

$$a(x,D)u(x) = \int_{\mathbb{R}^{2d}} \hat{a}(q,p)e^{i\langle q,X\rangle}e^{i\langle p,D\rangle}u(x)\mathrm{d}q\mathrm{d}p;$$

this is analogous to the Weyl calculus, where

$$\begin{aligned} a(x,D)u(x) &= \int_{\mathbb{R}^{2d}} \hat{a}(q,p)e^{i\langle q,X\rangle+\langle p,D\rangle}u(x)\mathrm{d}q\mathrm{d}p \\ &= (2\pi)^{-d/2}\int_{\mathbb{R}^{2d}} a\left(\frac{1}{2}(x+y),\xi\right)e^{i\langle x-y,\xi\rangle}u(y)\mathrm{d}y\mathrm{d}\xi. \end{aligned}$$

However, we will not use Weyl calculus: we defer to Ch. 18.5 of [42] for a more detailed exposition of this topic.

To an operator $a(x,D) \in OPS^m_{\rho,\delta}$ defined by

$$a(x,D)u(x) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} a(x,\xi)\hat{u}(\xi)e^{i\langle x,\xi\rangle}\mathrm{d}\xi \tag{D.3.6}$$

corresponds a Schwartz kernel $K \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$, satisfying

$$\begin{aligned} \langle u(x)v(y),K\rangle &= (2\pi)^{-d/2}\int_{\mathbb{R}^{2d}} u(x)a(x,\xi)\hat{v}(\xi)e^{i\langle x,\xi\rangle}\mathrm{d}\xi\mathrm{d}x \\ &= (2\pi)^{-d}\int_{\mathbb{R}^{3d}} u(x)a(x,\xi)e^{i\langle x-y,\xi\rangle}v(y)\mathrm{d}y\mathrm{d}\xi\mathrm{d}x. \end{aligned}$$

Thus, the kernel K of the operator $a(x,D)$ corresponds to the oscillatory integral

$$K(x,y) = (2\pi)^{-d}\int_{\mathbb{R}^d} a(x,\xi)e^{i\langle x-y,\xi\rangle}\mathrm{d}\xi. \tag{D.3.7}$$

Proposition D.3.3. *Let $\rho > 0$, then K is C^∞ off the diagonal in $\mathbb{R}^d \times \mathbb{R}^d$.*

Proof. For given $\alpha \geq 0$

$$(x-y)^\alpha K(x,y) = \int_{\mathbb{R}^d} e^{i\langle x-y,\xi\rangle}D^\alpha_\xi a(x,\xi)\mathrm{d}\xi. \tag{D.3.8}$$

The last integral is absolutely convergent for $|\alpha|$ so large such that $m - \rho|\alpha| < -d$. Similarly one can check that applying l derivatives to (D.3.8) gives an absolutely convergent integral, provided that $m + l - \rho|\alpha| < -d$, hence in that case $(x-y)^\alpha K \in C^l(\mathbb{R}^d \times \mathbb{R}^d)$. This allows us to conclude. □

More generally, if T has the mapping property

$$\begin{aligned} &T : \mathcal{D}(\mathbb{R}^d) := C^\infty_c(\mathbb{R}^d) \to \mathcal{E}(\mathbb{R}^d) := C^\infty(\mathbb{R}^d), \\ &T : \mathcal{E}'(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d), \end{aligned}$$

and its Schwartz kernel K is C^∞ off the diagonal, it follows that

$$\text{sing supp } Tu \subseteq \text{sing supp } u, \quad \text{for } u \in \mathcal{E}'(\mathbb{R}^d). \tag{D.3.9}$$

This is called *pseudolocal property*. By Lemma D.3.1 it holds for any $T \in OPS^m_{\rho,\delta}$ for $\rho > 0$ and $\delta \in (0,1]$. By arguing as in the proof of Proposition D.3.3 one gets also that

$$|D^\beta_{x,y}K(x,y)| \preceq |x-y|^{-k},$$

where k is any integer strictly greater than $\rho^{-1}(m + d + |\beta|)$.

Proposition D.3.4. *If $a(x,D) \in OPS^m_{1,\delta}$, then its Schwartz kernel K satisfies*

$$|D^\beta_{x,y} K(x,y)| \preceq |x-y|^{-m-d-|\beta|}. \tag{D.3.10}$$

provided that $m + |\beta| > -d$.

The results follows easily from the case $a(x,D) = a(D)$, for which its kernel is given by $K(x,y) = \hat{a}(y-x)$. It suffices to prove Proposition D.3.4 for such a case, for $\beta = 0$ and $m > -d$, by exploiting the following

Lemma D.3.5. *Let $p \in C^\infty(\mathbb{R}^d)$. Then $p \in S^m$ if and only if*

$$p_r(\xi) := r^{-m} p(r\xi)$$

is bounded in $C^\infty(1 \leq |\xi| \leq 2)$, for any $r \geq 1$.

For further details we defer to [83], ch. 7, §2.

D.4 Adjoints and products of pseudo-differential operators

Given $a \in S^m_{\rho,\delta}(\mathbb{R}^d \times \mathbb{R}^d)$, one readily obtains

$$a(x,D)^* u(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \bar{a}(y,\xi) e^{i\langle x-y,\xi\rangle} u(y) \mathrm{d}y \mathrm{d}\xi,$$

but this is not in the form (D.3.6), since $\bar{a}(y,\xi)$ is not a function of x and ξ. In order to tackle this problem, we study a more general class of operators of the form

$$Au(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^{2d}} a(x,y,\xi) e^{i\langle x-y,\xi\rangle} u(y) \mathrm{d}y \mathrm{d}\xi, \tag{D.4.1}$$

where we assume that

$$|D^\gamma_y D^\beta_x D^\alpha_\xi a(x,y,\xi)| \leq K_{\alpha,\beta,\gamma} \langle\xi\rangle^{m-\rho|\alpha|+\delta_1|\beta|+\delta_2|\gamma|};$$

we will denote this class by $S^m_{\rho,\delta_1,\delta_2}$. By exploiting the unitary operators (D.3.5) one gets

$$Au(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^{2d}} q(x,\xi) e^{i\langle x-y,\xi\rangle} u(y) \mathrm{d}y \mathrm{d}\xi,$$

where

$$\begin{aligned} q(x,\xi) &= (2\pi)^{-d/2} \int_{\mathbb{R}^{2d}} a(x,y,\eta) e^{i\langle x-y,\eta-\xi\rangle} \mathrm{d}y \mathrm{d}\eta \\ &= e^{iD_\xi \cdot D_y} a(x,y,\xi)|_{y=x}. \end{aligned}$$

Note that a formal expansion $e^{iD_\xi \cdot D_y} = Id + iD_\xi \cdot D_y - \frac{1}{2}(D_\xi \cdot D_y)^2 + \ldots$ leads to

$$q(x,\xi) \sim \sum_{\alpha \geq 0} \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi D^\alpha_y a(x,y,\xi)|_{y=x}. \tag{D.4.2}$$

Let $a \in S^m_{\rho,\delta_1,\delta_2}$ with $0 \leq \delta_2 < \rho \leq 1$, then the general term in (D.4.2) is in $S^{m-(\rho-\delta)|\alpha|}_{\rho,\delta}$ with $\delta = \min(\delta_1,\delta_2)$, hence the sum on the right is formally asymptotic.

Proposition D.4.1. *Let* $a \in S^m_{\rho,\delta_1,\delta_2}$ *with* $0 \le \delta_2 < \rho \le 1$, *then the operator* (D.4.1) *is in* $OPS^m_{\rho,\delta}$ *with* $\delta = \max(\delta_1, \delta_2)$. *Indeed,* $A = q(x, D)$, *where* q *admits the asymptotic expansion* (D.4.2).

Proof. To prove the proposition, one can show that the Schwartz kernel

$$K(x,y) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} a(x,y,\xi) e^{i\langle x-y,\xi\rangle} \mathrm{d}\xi$$

satisfies Proposition D.3.3, and therefore, up to an operator in $OPS^{-\infty}$, we can assume a is supported on $|x-y| \le 1$. Let

$$\hat{b}(x,\eta,\xi) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} a(x, x+y, \xi) e^{-i\langle y,\eta\rangle} \mathrm{d}y,$$

hence

$$a(x,\xi) = \int_{\mathbb{R}^d} \hat{b}(x,\eta,\xi+\eta) \mathrm{d}\eta.$$

Our assumptions on a imply that

$$|D^\beta_x D^\alpha_\eta \hat{b}(x,\eta,\xi)| \le K_{\alpha,\beta,\nu} \langle\xi\rangle^{m+\delta|\beta|+\delta_2\nu-\rho|\alpha|} \langle\eta\rangle^{-\nu},$$

where $\delta = \max(\delta_1, \delta_2)$, and for any $\nu > 0$. Since $\delta_2 < 1$, it follows that a and its derivatives can be bounded by some power of $\langle\xi\rangle$. Now, an asymptotic expansion of $\hat{b}(x,\eta,\xi+\eta)$ in the last argument about ξ gives that for any $N > 0$

$$\left| \hat{b}(x,\eta,\xi+\eta) - \sum_{|\alpha|<N} \frac{1}{\alpha!} (iD_\xi)^\alpha \hat{b}(x,\eta,\eta)\eta^\alpha \right| \le K_\nu |\eta|^N \langle\eta\rangle^{-\nu} \sup_{0\le t\le 1} \langle\xi + t\eta\rangle^{m+\delta_2\nu-N\rho}.$$

With $\eta = N$ the right-hand side is bounded by a constant times $\langle\xi\rangle^{m-(\rho-\delta_2)N}$ for $|\eta| \le |\xi|/2$, and if ν is large we get a bound by any power of $\langle\eta\rangle^{-1}$ for $|\xi| \le 2|\eta|$. Hence,

$$\left| a(x,\xi) - \sum_{|\alpha|<N} \frac{1}{\alpha!} (iD_\xi)^\alpha D^\alpha_y a(x, x+y, \xi)|_{y=0} \right| \preceq \langle\xi\rangle^{m+d-(\rho-\delta_2)N},$$

that leads to the thesis. ☐

If we apply Proposition D.4.1 to $a(x,D)^*$, one gets

Proposition D.4.2. *Let* $a(x,D) \in OPS^m_{\rho,\delta}$, $0 \le \delta < \rho \le 1$, *then*

$$a(x,D)^* = \bar{a}(x,D) \in OPS^m_{\rho,\delta},$$

with

$$a^*(x,\xi) \sim \sum_{\alpha\ge 0} \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi D^\alpha_x \bar{a}(x,\xi). \tag{D.4.3}$$

The results for products of PDOs is the following

Proposition D.4.3. *Let $a_j(x,D) \in OPS^m_{\rho_j,\delta_j}$ for $j = 1, 2$. Assume that*

$$0 \le \delta_2 < \rho \le 1,$$

where $\rho = \min(\rho_1, \rho_2)$. Then

$$a_1(x,D)a_2(x,D) = b(x,D) \in OPS^{m_1+m_2}_{\rho,\delta},$$

with $\delta = \max(\delta_1, \delta_2)$, and

$$b(x,\xi) \sim \sum_{\alpha \ge 0} \frac{i^{|\alpha|}}{\alpha!} D^\alpha_\xi a_1(x,\xi)\, D^\alpha_x a_2(x,\xi). \tag{D.4.4}$$

Also a result for the commutator of two PDOs holds.

Proposition D.4.4. *Let $a_j(x,D) \in OPS^m_{\rho_j,\delta_j}$ for $j = 1, 2$. Assume that*

$$0 \le \delta < \rho \le 1,$$

where $\delta = \max(\delta_1, \delta_2)$, $\rho = \min(\rho_1, \rho_2)$. Then

$$[a_1(x,D), a_2(x,D)] =: \{a_1, a_2\}^q(x,D) \in OPS^{m_1+m_2-1}_{\rho,\delta},$$

with

$$\{a_1, a_2\}^q(x,\xi) \sim -i\{a_1, a_2\}(x,\xi), \qquad \mathrm{mod}\ S^{m_1+m_2-2}_{\rho,\delta}, \tag{D.4.5}$$

where

$$\{a_1, a_2\}(x,\xi) = \sum_{j=1}^d \left(\frac{\partial a_1}{\partial \xi_j}(x,\xi) \frac{\partial a_2}{\partial x_j}(x,\xi) - \frac{\partial a_1}{\partial x_j}(x,\xi) \frac{\partial a_1}{\partial \xi_j}(x,\xi) \right)$$

is the Poisson bracket of the two symbols a_1 and a_2, and $\mathrm{mod}\ S^{m_1+m_2-2}_{\rho,\delta}$ denotes that such an expansion holds up to a symbol in $S^m_{\rho,\delta}$.

Remark D.4.5. *The above commutator of two PDOs, $\{a_1, a_2\}^q$, sometimes called the* Moyal bracket *between a_1 and a_2, has a very concrete physical interpretation. Indeed, if one considers a quantum particle in a box, then the canonical commutation relation between the position operator Q and the momentum operator P*

$$[Q, P] = i\hbar$$

can be simply regarded as the commutation relation between a PDO of order 0 (the operator Q) and an operator of order 1 (the operator P), which gives as a result a multiplication-by-a-constant operator, which clearly belongs to S^0. Similarly, asymptotic expansions of operators can be studied within the context of the semi-classical limit $\hbar \to 0$.

D.5 Action of pseudo-differential operators on Sobolev spaces

Here we want to obtain H^k-estimates for pseudo-differential operators. We begin with a simple estimates, which holds also for L^p spaces.

Proposition D.5.1. *Let $\rho > 0$, $m < -d + \rho(d-1)$, and let $a(x, D) \in OPS^m_{\rho,\delta}$. Then*

$$a(x, D) : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d),\ \ 1 \le p \le +\infty. \tag{D.5.1}$$

Furthermore, if $a(x, D) \in OPS^m_{1,\delta}$, then (D.5.1) *holds for any $m < 0$.*

Proposition D.5.1 follows from the following measure theory result.

Lemma D.5.2. *Let (X, μ) be a measure space. Assume that $k : X \times X \to \mathbb{R}$ is measurable, and that*

$$\int_X |k(x, y)| \mathrm{d}\mu(x) \le C_1,\ \ \forall y,$$
$$\int_X |k(x, y)| \mathrm{d}\mu(y) \le C_2,\ \ \forall x.$$

Then $Tu(x) := \int_X k(x, y)u(y)\mathrm{d}\mu(y)$ satisfies

$$\|Tu\|_{L^p} \le C_1^{1/p} C_2^{1/p'} \|u\|_{L^p},\ \ 1 \le p \le \infty,$$

where p and p' are conjugate exponents, namely $\frac{1}{p} + \frac{1}{p'} = 1$.

For the proof of the Lemma, we refer to section 5 of Appendix A in [82].
To prove Proposition D.5.1 apply the previous Lemma with $X = \mathbb{R}^d$, and $k = K$ as the Schwartz kernel of $a(x, D) \in OPS^m_{\rho,\delta}$, which by Proposition D.3.3 satisfies

$$|K(x, y)| \le C_N |x - y|^{-N},\ \ \text{for}\ \ |x - y| \ge 1,\ \ \forall N,$$

when $\rho > 0$, whereas

$$|K(x, y)| \preceq |x - y|^{-(d-1)},\ \ \text{for}\ \ |x - y| \le 1,$$

for $m < -d + \rho(d - 1)$.

Theorem D.5.3. *Let $0 \le \delta < \rho \le 1$, and let $a(x, D) \in OPS^0_{\rho,\delta}$. Then*

$$a(x, D) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d). \tag{D.5.2}$$

Proof. First step: we begin by proving (D.5.2) for a PDO $a(x, D) \in OPS^{-a}_{\rho,\delta}$, with $0 \le \delta < \rho \le 1$, and $a > 0$. Since $\|a(x, D)u\|^2_{L^2} = \langle a^*au, u\rangle$, it suffices to prove that some power of $p(x, D) := a(x, D)a(x, D)^*$ is bounded on L^2; but $p^k \in OPS^{-2ka}_{\rho,\delta}$, so for k large enough this follows from Proposition D.5.1.
Second step: in order to prove the Theorem, we now consider

$$p(x, D) = a(x, D)^* a(x, D) \in OPS^0_{\rho,\delta},$$

and assume that $|p(x, \xi)| \le M - b$, $b > 0$, so that

$$M - Re(p(x, \xi)) \ge b > 0.$$

In the matrix case, take $Re(p(x, \xi)) = \frac{1}{2}(p(x, \xi) + p(x, \xi)^*)$. Hence

$$A(x, \xi) = (M - Re\ p(x, \xi))^{1/2} \in S^0_{\rho,\delta},$$

and

$$A(x,D)^*A(x,D) = M - p(x,D) + r(x,D), \quad r(x,D) \in OPS^{-(\rho-\delta)}_{\rho,\delta}.$$

By applying the first step to $r(x,D)$, we have that there exists $K > 0$ such that

$$\begin{aligned} M\|u\|^2_{L^2} - \|a(x,D)u\|^2_{L^2} &= \|A(x,D)u\|^2_{L^2} - \langle r(x,D)u, u\rangle \\ &\geq -K\|u\|^2_{L^2}, \end{aligned}$$

or

$$\|a(x,D)u\|^2_{L^2} \preceq M\|u\|^2_{L^2}.$$

and we can conclude. □

Now PDO-calculus, namely (D.4.4) and Theorem D.5.3, gives

Theorem D.5.4. *Let $0 \leq \delta < \rho \leq 1$, let $k, m \in \mathbb{R}$, and let $a(x,D) \in OPS^m_{\rho,\delta}$. Then*

$$a(x,D) : H^k(\mathbb{R}^d) \to H^{k-m}(\mathbb{R}^d). \tag{D.5.3}$$

D.6 L^p estimates

As shown in Proposition D.3.4, if $0 \leq \delta < 1$ and $a(x,D) \in OPS^0_{1,\delta}$, then its Schwartz kernel K satisfies

$$|K(x,y)| \preceq |x-y|^{-d}, \tag{D.6.1}$$

$$|\nabla_{x,y}K(x,y)| \preceq |x-y|^{-d-1}, \tag{D.6.2}$$

$$\|a(x,D)u\|_{L^2} \preceq \|u\|_{L^2}; \tag{D.6.3}$$

also the smoothing of the PDO $a(x,D)$ have smooth Schwartz kernel satisfying (D.6.1)-(D.6.3). We want to prove the following result, due to Calderon and Zygmund.

Theorem D.6.1. *Assume that $a(x,D) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a weak limit of operators with smooth Schwartz kernel satisfying* (D.6.1)-(D.6.3) *uniformly. Then*

$$a(x,D) : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d), \ 1 < p < +\infty. \tag{D.6.4}$$

Actually, the hypotheses imply a stronger property, namely that $a(x,D)$ is of weak type $(1,1)$. An operator P is of weak type (p,p) if for any $\lambda > 0$

$$meas(\{x : |Pu(x)| > \lambda\}) \preceq \frac{\|u\|^q_{L^q}}{\lambda^q}.$$

Note that any bounded operator on L^p is of weak type (p,p), due to the Markov inequality

$$meas(\{x : |v(x)| > \lambda\}) \preceq \frac{\|v\|_{L^1}}{\lambda}.$$

Hence, in order to prove (D.6.4), we just prove

Proposition D.6.2. *Assume that $a(x,D) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a weak limit of operators with smooth Schwartz kernel satisfying* (D.6.1)-(D.6.3) *uniformly. Then $a(x,D)$ is of weak type* $(1,1)$.

Once Proposition D.6.2 is proved, then one can readily prove (D.6.4) via Macinkiewicz Interpolation Theorem (see ch. 1.3 of [18] for a proof of this result).

Theorem D.6.3. *Let $r < p < q$, and assume that T is both of weak type (r, r) and (q, q). Then $T : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$.*

Now we deal with the proof of Proposition D.6.2: we exploit the following intermediate results (whose proofs can be found in ch. 13.5 of [84]).

Lemma D.6.4. *Let $u \in L^1(\mathbb{R}^d)$, and $\lambda > 0$ be given. Then there exists $v, w_k \in L^1(\mathbb{R}^d)$, and disjoint cubes Q_k with centers x_k, $1 \leq k < +\infty$, such that*

$$u = v + \sum_{k=1}^{\infty} w_k,$$

$$\|v\|_{L^1} + \sum_{k=1}^{\infty} \|w_k\|_{L^1} \leq 3\|u\|_{L^1},$$

$$|v(x)| \preceq 2^d \lambda,$$

$$\int_{Q_k} w_k(x)\mathrm{d}x = 0, \quad \textit{supp } w_k \subseteq Q_k,$$

$$\sum_{k=1}^{\infty} \textit{meas}(Q_k) \leq \lambda^{-1}\|u\|_{L^1}.$$

Note that the function v of the previous lemma can be estimated by $\|v\|_{L^2}^2 \leq 2^d \lambda \|u\|_{L^1}$, hence

$$\|a(x, D)v|_{L^2}^2 \preceq \|v\|_{L^2}^2 \leq 2^{2d} \lambda \|u\|_{L^1}, \tag{D.6.5}$$

which gives

$$\left(\frac{\lambda}{2}\right)^2 \text{ meas}\left(\left\{x : |a(x, D)v(x)| > \frac{\lambda}{2}\right\}\right) \preceq \lambda \|u\|_{L^1}.$$

In order to estimate the action of $a(x, D)$ on $w = \sum_k w_k$, we exploit

Lemma D.6.5. *There exists a positive constant K_0 such that, for any $t > 0$, $|y| \leq t$, and $x_0 \in \mathbb{R}^d$*

$$\int_{|x| \geq 2t} |K(x, x_0 + y) - K(x, x_0)|\mathrm{d}x \leq K_0.$$

Note that

$$\begin{aligned} a(x, D)w_k &= \int_{\mathbb{R}^d} K(x, y)w_k(y)\mathrm{d}y \\ &= \int_{Q_k} [K(x, y) - K(x, x_k)]\, w_k(y)\mathrm{d}y. \end{aligned} \tag{D.6.6}$$

Now let Q_k^* be the cube concentric with Q_k, enlarged by a factor $2d^{1/2}$. For some $t_k > 0$, we have

$$\begin{aligned} Q_k &\subseteq \{x : |x - x_k| \leq t_k\}, \\ (Q_k^*)^c &\subseteq \{x : |x - x_k| > 2t_k\}. \end{aligned}$$

Furthermore, if we set $Q^* := \cup_k Q_k^*$, we have

$$meas \ \ Q^* \leq \frac{2^d d^{d/2}}{\lambda} \|u\|_{L^1}; \tag{D.6.7}$$

from (D.6.6) we can deduce

$$\begin{aligned}
&\int_{(Q_k^*)^c} |a(x,D)w_k(x)| \mathrm{d}x \\
&\leq \int_{|y|\leq t_k} \int_{|x|\geq 2t_k} |K(x+x_k, y+x_k) - K(x+x_k, x_k)| \, |w_k(y+x_k)| \mathrm{d}x \mathrm{d}y \\
&\preceq \|w_k\|_{L^1},
\end{aligned}$$

where the last inequality follows from Lemma D.6.5. Therefore

$$\int_{(Q_k^*)^c} |a(x,D)w(x)| \mathrm{d}x \preceq 3\|u\|_{L^1}.$$

The last inequality, combined with (D.6.7), gives

$$\frac{\lambda}{2} \ meas \ \left\{x : |a(x,D)w(x)| > \frac{\lambda}{2}\right\} \preceq \|u\|_{L^1},$$

which, along with (D.6.5), allows us to deduce the weak $(1,1)$ estimate

$$meas \ \{x : |a(x,D)u(x)| > \lambda\} \preceq \frac{\|u\|_{L^1}}{\lambda}.$$

We just point out that Theorem D.6.1 can be restated in a more general context. Indeed, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, and assume that

$$A(D) : L^2(\mathbb{R}^d, \mathcal{H}_1) \to L^2(\mathbb{R}^d, \mathcal{H}_2).$$

The operator A admits an $L(\mathcal{H}_1, \mathcal{H}_2)$-operator valued Schwartz kernel K. If one assumes the hypotheses of Theorem D.6.1, and replaces $|K(x,y)|$ with the $L(\mathcal{H}_1, \mathcal{H}_2)$ norm of $K(x,y)$, one obtains

Proposition D.6.6. *Let $A \in C^\infty(\mathbb{R}^d, L(\mathcal{H}_1, \mathcal{H}_2))$, and assume that*

$$\|D_\xi^\alpha A(\xi)\|_{L(\mathcal{H}_1,\mathcal{H}_2)} \leq K_\alpha \langle \xi \rangle^{-|\alpha|}$$

for all α. Then

$$A(D) : L^p(\mathbb{R}^d, \mathcal{H}_1) \to L^p(\mathbb{R}^d, \mathcal{H}_2), \ 1 < p < +\infty.$$

D.7 Pseudodifferential operators on a manifold

Now we consider a C^∞ manifold M, and a continuous linear operator $A : C_c^\infty(M) \to C^\infty(M)$. We want to extend the definition of pseudo-differential operator by imposing that the expression of A in any coordinate system is of the form $a(x,D)$, for some local symbol a. By simplicity we will consider only operators in $OPS^m := OPS^m_{1,0}$.
We first define the PDOs in an open subset of $\mathbb{R}^d$, and then we study how the expression of a PDO behaves under change of variables.

Definition D.7.1. *Let $\Omega \subset \mathbb{R}^d$ be an open set, then we define $S^m_{loc}(\Omega \times \mathbb{R}^d)$ to be the set of $A \in C^\infty(\Omega \times \mathbb{R}^d)$ such that $\phi A \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$ for all $\phi \in C^\infty_c(\Omega)$.*

Proposition D.7.2. *Let $A : C^\infty_c(\Omega) \to C^\infty(\Omega)$ be a continuous linear operator such that for all ϕ, $\psi \in C^\infty_c(\Omega)$ we have that $\phi A \psi \in OPS^m$. Then there exists $A' \in S^m_{loc}(\Omega \times \mathbb{R}^d)$ with $A(x,\xi) = A'(x,\xi) + R(x,\xi)$, where R is the symbol of an operator with kernel in $C^\infty(\Omega \times \Omega)$. The symbol A' is determined up to a remainder in $S^{-\infty}_{loc}(\Omega \times \mathbb{R}^d)$.*

Proof. Omitted (see, for example, Ch. I, § 6 of [3]). □

If A satisfies Proposition D.7.2, we say that A is a pseudo-differential operator of order m on Ω, and the class of A' in $S^m_{loc}/S^{-\infty}_{loc}$ is called the symbol of A.

Proposition D.7.3. *Let $\chi : \Omega \to \Omega'$, $y = \chi(x)$, be a smooth diffeomorphism between two open subsets of $\mathbb{R}^d$. Let us assume that the PDO $A(x,D)$ has kernel with compact support in $\Omega \times \Omega$. Then*

i. the function

$$A'(y,\eta) = A'(\chi(x),\eta) = e^{-i\chi(x)\eta} A(x,\eta) e^{i\chi(x)\eta}$$

($A' = 0$ for $y \notin \Omega'$) is a symbol in S^m. Moreover,

$$A'(\chi(x),\eta) \sim \sum_\alpha \frac{1}{\alpha!} \partial^\alpha_\xi A(x, \chi'(x)\eta) D^\alpha_y (e^{i\rho_x(y)\eta})|_{y=x},$$

where $\rho_x(y) = \chi(y) - \chi(x) - \chi'(x)(y-x)$;

ii. the kernel of $A'(x,D)$ has compact support in $\Omega' \times \Omega'$;

iii. for any $u \in \mathcal{S}'(\Omega')$ we have $A(x,D)(u \circ \chi) = (A'(x,D)u) \circ \chi$.

Proof. Omitted (see, for example, Ch. I, § 7 of [3]). □

Remark D.7.4. *We remark that if $A \in S^m$, then $A(x,D)e^{ix\xi} = e^{ix\xi}a(x,\xi)$. Indeed, if $\hat{u} \in C^\infty_c$,*

$$A(x,D)u(\delta x)e^{ix\xi} = e^{ix\xi}(2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\delta x\eta} A(x, \chi + \epsilon\eta)\hat{u}(\eta)\mathrm{d}\eta,$$

which, if $u(0) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \hat{u}(\eta)\mathrm{d}\eta = 1$, tends to

$$e^{ix\xi} A(x,\xi) \in \mathcal{S}'(\mathbb{R}^d);$$

as $\delta \to 0$; moreover, $u(\delta x)e^{ix\xi} \to e^{ix\xi}$ in $\mathcal{S}'(\mathbb{R}^d)$ as $\delta \to 0$.
Note also that in iii. $A(x,D)(u \circ \chi)$ is well-defined, since the kernel of $A(x,D)$ is compactly supported in $\Omega \times \Omega$.

Definition D.7.5. *The operator $A : C^\infty_c(M) \to C^\infty(M)$ is called a pseudo-differential operator of order m if for any coordinate system $\phi : V \to V' \subset \mathbb{R}^d$ the transported operator $\tilde{A} : u \mapsto [A(u \circ \phi)] \circ \phi^{-1}$ from $C^\infty_c(V')$ to $C^\infty(V')$ is pseudodifferential of order m in V', namely for any ϕ, $\psi \in C^\infty_c(V')$ we have that $\phi A \psi \in OPS^m$. In this case we write $A \in \Psi^m(M)$.*

Now we have defined the notion of PDO on M, but there is still an issue with the associated symbols: indeed, in each coordinate system, the transported operator has an associated symbol determined $\mod S^{-\infty}$, but this depends on the given coordinate system. It is therefore relevant to study whether for a PDO A of order m on M there exists an intrinsecally defined function, whose expression in local coordinates is the symbol of the operator.
We briefly show that it is possible to define such a function, but under the restriction that the associated symbols coincide $\mod S^{m-1}$, namely, only the principal symbol may be defined intrinsecally.

We recall that for a manifold M the cotangent bundle T^*M is the set of points (p,ω) such that $p \in M$, and $\omega \in (T_pM)^*$. The projection $\pi : T^*M \to M$ is given by $\pi(p,\omega) = m$, while we denote by $\pi^{-1}(p)$ the dual space of T_pM.
If $(x_1,\ldots,x_d)$ are the local coordinates on $V \subset M$, then the vector fields $(\partial_1,\ldots,\partial_d)$ form a basis of T_pM at any point $p \in V$, while the forms $(\mathrm{d}x_1,\ldots,\mathrm{d}x_d)$ form a basis for $\pi^{-1}(p)$. Denoting a 1-form by $\omega = \sum_{i=1}^d \xi_i \mathrm{d}x_i$, we obtain local coordinates (x,ξ) on $\pi^{-1}(V)$. In another coordinate system $x' = \chi(x)$, the point m will have coordinates $x'(p) = \chi(x(p))$, whereas the form $\omega = \sum_{i=1}^d \xi_i \mathrm{d}x_i$ wil be written as

$$\sum_i \xi_i \left(\sum_j \frac{\partial \chi_i}{\partial x'_j}(x) \partial x'_j \right),$$

hence

$$\xi'_j = \sum_{i=1}^d \frac{\partial \chi_i}{\partial x'_j}(x)\xi_i;$$

therefore, the same point (p,ω) will be written as $(x, \chi'(x)^T\eta)$ and $(\chi(x),\eta)$ in the coordinate systems (x,ξ) and (x',ξ') respectively.

Now, let $A = A_m \mod S^{m-1}$, where a_m is homogeneous of degree m; the same holds for A', and

$$A'_m(\chi(x),\eta) = A_m(x\chi'(x)^T\eta).$$

We say in this case that A has principal symbol A_m. If, in any local map, the representative of $A \in \Psi^m(M)$ admits a principal symbol, then the previous considerations allow us to deduce that these different principal symbols are the expressions in local coordinates of a unique function on T^*M, which we call the principal symbol of A.

Theorem D.7.6. *Let $A_i \in \Psi^{m_i}(M)$, $i = 1,2$, be properly supported and assume that they admit principal symbols a_i, $i = 1,2$, then $A = A_1A_2 \in \Psi^{m_1+m_2}(M)$ is properly supported and admits principal symbol a_1a_2.*
Furthermore, the commutator $[A_1,A_2]$ admits principal symbol $\{a_1,a_2\}$.

Proof. Omitted (see, for example, Ch. I, § 7 of [3]). □

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